On the moments of a magnetic dipole moving in a medium

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For several years there has been discussion on the type of electric dipole moment acquired by a magnetic dipole moving in a medium. Now it has been accepted that the relativistic formula for a magnetic dipole moving in a vacuum can also be used for the relation between the magnetic and electric moments in a medium. An additional argument in support of the above statement is that of a closed current loop moving in a vacuum, which is examined in this article. In the limit case, this loop is equivalent to a moving magnetic dipole. A comparison of the obtained results with known data for an electric dipole shows that in the case of a medium there are no reasons to change relativistic transformation formulas.

The question of the relativistic transformation of the moment of a magnetic dipole moving in a medium arose a long time ago, due to an examination of the Vavilov-Cherenkov radiation of electric and magnetic dipoles.[1] For an electric dipole, the result obtained was fully analogous to that which occurs for the radiation of an electric charge: the energy of the radiation is a sine squared function of the characteristic angle of radiation $\theta(\sin^2\theta = 1-(\beta^2n^2)^{-1})$. In contrast, for a magnetic dipole oriented perpendicular to the velocity, the energy depends in a quite complex manner on the index of refraction, n, and θ. Relativistic transformation formulas for the moments of electric and magnetic dipoles were used to obtain these results, and it was assumed that they were applicable for movement in a medium with an index of refraction n. Thus, it was taken that an electric dipole $\mathbf{p}'$ moving with a speed $\beta = v/c$ in a stationary coordinate system has the moment

$$\mathbf{p} = \mathbf{p}' - (1-\alpha)(\mathbf{p}'\mathbf{z}_1)\,\mathbf{z}_1, \tag{1}$$

where $\alpha = (1-\beta^2)^{1/2}$, and $\mathbf{z}_1$ is the vector of velocity, which is directed along the z axis. Consequently, the component of $\mathbf{p}'$ perpendicular to the velocity (let us say that it is oriented along the x axis) remains unchanged, and the component oriented along p'_z decreases, as it should, by a factor of $(1-\beta^2)^{1/2}$, that is

$$p_x = p'_x \quad p_z = (1-\beta^2)^{1/2}\,p'_z. \tag{2}$$

Moreover, it was assumed that, as in a vacuum, a moving electric dipole induces a magnetic moment, the magnitude of which is

$$\mathbf{m} = -\beta\,[\mathbf{z}_1\mathbf{p}'], \quad \text{i.e.}\ m_y = -\beta p'_x. \tag{3}$$

An analogous situation should also take place for the transformation of a magnetic dipole with components m'_y and m'_z:

$$m_y = m'_y \quad m_z = (1-\beta^2)^{1/2} m'_z, \tag{4}$$

$$\mathbf{p}_1 = \beta\,[\mathbf{z}_1\mathbf{m}'] \quad p_x = -\beta m'_y. \tag{5}$$

In the case of Vavilov-Cherenkov radiation by a magnetic dipole, the magnitude of the induced electric moment used in Ref. (1) led to paradoxical results, as was noted earlier. They can be eliminated (but not completely, as was explained later) if it is assumed that instead of Eq. (5) there should be[2]

$$p_x = -\beta n^2 m'_y. \tag{6}$$

There have been arguments[3] which support the correctness of Eq. (6). And yet the argument in support of Eq. (6) was the unfounded assumption that an elementary dipole produced by a circular current should be equivalent in radiation to a dipole consisting of two opposite hypothetical magnetic charges. Actually, in the case of an electric dipole, the same results are obtained for Vavilov-Cherenkov radiation, regardless of whether the moving dipole is exmained using Eq. (3) or whether one determines the result of the interference of two opposite electrical charges which are physically close and moving together.[2] By this, the suitability of Eq. (3) was established for a medium as well. It was natural to examine the radiation of magnetic charges and dipoles consisting of these charges using an analogy which requires the following substitutions in all formulas: ε for μ, μ for ε, E for H and H for E [see Ref. (2)]. The result obtained was different from the result for the usual magnetic dipole.[1] It was not understood until after Refs. (4) and (5) that possibly there was no contradiction. An attempt was made to correct the result for a usual dipole, by substituting Eq. (5) for Eq. (6). Although now there are, apparently, no doubts about the correctness of Eq. (5), it may useful to examine how the electric dipole moment arises in the case of movement of a closed current loop. This reduces the problem of magnetic dipole radiation to an examination of a system of moving electrical charges, from which nothing unusual should be expected. Let us assume that there is a closed rectangular loop $l'_1 \times l'_2$ in size, consisting of four rectilinear conductors a,b,c,d (see Fig. 1). The cross section of the conductors is σ'. The current flowing through the loop is $J' = \sigma' I'$, where I' is the current density, and its magnetic moment is

$$m' = \frac{1}{c} J' l'_1 l'_2 = \frac{1}{c}\,\sigma' I' l'_1 l'_2. \tag{7}$$

All these values are indicated for a coordinate system, K', associated with the loop. Let us suppose that the system is moving with the loop at a speed v in the direction of the z axis. We will be interested in the results which should be

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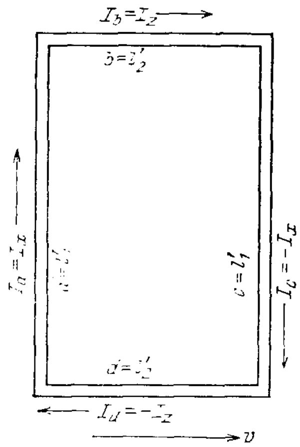

FIG. 1.

observed in the laboratory coordinate system K. They can be obtained in a completely elementary manner using the Lorentz formulas and known relativistic transformations for current density and charge density. Of course these same results can be obtained using the law of conservation of charge and the formula for the addition of velocities.[6]

Let us examine first the case where the current loop is perpendicular to the z axis, that is, it lies in the x, y plane. It is easy to be convinced that the current in the loop, measured in the stationary system K, is

$$J_K = (1 - \beta^2)^{1/2} J' = \sigma' I'. \tag{8}$$

According to the relativistic transformation of current density we have

$$I_x = I_y = I'. \tag{9}$$

As for the conductor itself, the size of its cross section in the direction of the z axis decreases by a factor of $(1-\beta^2)^{1/2}$; consequently,

$$\sigma = (1 - \beta^2)^{1/2} \sigma'. \tag{10}$$

If we consider Eqs. (9) and (10), we immediately get Eq. (8). The magnetic moment of the loop is directed in this case along the z axis or antiparallel to it, and since the length of the conductors, l'_1 and l'_2, remains unchanged, it is equal to

$$m = \frac{1}{c} J_K l'_1 l'_2 = (1 - \beta^2)^{1/2} m'. \tag{11}$$

As expected, this coincides with Eq. (4). Let us turn now to an examination of a more complex case, when the current loop lies in the x,z plane (see Fig. 1). We will assume that a is directed along the x axis, and b is directed along the z axis, that is, parallel to the velocity. The direction of the current in each of the sections of the loop is indicated by arrows in the figure. The magnetic moment, which is equal to Eq. (7) in system K', is in this case perpendicular to the plane of the figure. It is directed along the negative y axis:

$$m'_y = -m' = -\frac{1}{c} \sigma' I' l'_1 l'_2. \tag{12}$$

Let us now determine the current in each of the sections of the loop. Section a is perpendicular to the velocity. Thus, the current in section a is equal to Eq. (8), that is,

$$J_a = J_K = (1 - \beta^2)^{1/2} J'; \tag{13}$$

it is obvious that $J_c = -J_a$.

To determine J_b, in which the direction of the current coincides with the direction of the velocity, we use relativistic transformation formulas for current density and charge density. If in the K' system the charge density in the conductor is equal to zero, then in system K

$$I_z = \frac{1}{(1-\beta^2)^{1/2}} I'. \tag{14}$$

Thus, a charge density also arises in the conductor

$$\rho = \frac{1}{(1-\beta^2)^{1/2}} \frac{v}{c^2} I'. \tag{15}$$

The cross section of section b is perpendicular to the velocity; consequently, $\sigma_b = \sigma'$. As a result, we get

$$J_b = \frac{1}{(1-\beta^2)^{1/2}} \sigma' I'. \tag{16}$$

For section d the current and charge density have the opposite sign in relation to Eqs. (15) and (16). It should be no surprise that $J_b \neq J_a$. Indeed, conductor b now has an electric charge, and the charge density ρ moves along with it at a speed v, which is equivalent to the current density

$$I_\rho = \rho v = \frac{1}{(1-\beta^2)^{1/2}} \frac{v^2}{c^2} I' \tag{17}$$

and, consequently, to the current (since $\sigma_b = \sigma'$)

$$J_\rho = \frac{1}{(1-\beta^2)^{1/2}} \frac{v^2}{c^2} \sigma' I'. \tag{18}$$

Comparing Eqs. (13), (18), and (16), we get, as we should,

$$J_b = J_a + J_\rho. \tag{19}$$

If we turn to the limit case, it is easy to determine the dipole electric moment of the loop from Eq. (15). Indeed, the electric charge q^+ contained in the b section of conductor is equal to ρ, multiplied by the volume of the conductor $\sigma_b l_2$. Since, as a result of the Lorentz reduction, $l_2 = (1-\beta^2)^{1/2} l'_2$, we obtain

$$q^+ = \frac{v}{c^2} \sigma' l'_2 I'. \tag{20}$$

Section d contains obviously, the same charge, but of opposite sign. They are separated by a distance l'_1. If $l'_2 \ll l'_1$, then these charges may be considered an electric dipole directed along the x axis and equal to $q l'_1$. Consequently [see Ref. (7)],

$$p_x = \beta \frac{1}{c} \sigma' l'_1 l'_2 I' = \beta m'. \tag{21}$$

If we bear in mind that m' has a single component, $-m'_y = m'$ [see Ref. (12)], we find that Eq. (5) is satisfied. This, of course, is an obvious result, since Eqs. (5) and (21) are equally the result of relativistic transformations. However, in Eq. (21) the result is very obvious. Charges q^+ and q^- in current conductors b and d indeed arise from their movement in the direction of the z axis, and there is no reason to think that if the movement occurred in a medium, that it could be otherwise. As already stated in the beginning of the article, in the problem of Vavilov-Cherenkov radiation, we correctly considered two such charges separated by a distance l_1' as a moving dipole (under the condition, of course, that $l_1' \ll \lambda$, where λ is the wavelength of radiated light). This is a very significant argument for the applicability of Eq. (5) also to the case of movement in a medium. If this is so, then such a dipole should induce a magnetic moment. As a check, we determine its magnitude. From Eqs. (3) and (21), we get

$$m_y\ (p) = -\beta^2 m'. \tag{22}$$

This, however, is only that portion of the moment which is created by the charges moving with the conductor. In addition, we have the current $J_{\text{loop}} = (1-\beta^2)^{1/2}\sigma' I'$, flowing in the loop [see Eq. (13)]. The area of the loop in the K system is $(1-\beta^2)^{1/2}\, l_1' l_2'$. Comparing this with Eq. (17), we have

$$m_y\ (K) = -(1-\beta^2)\, m'. \tag{23}$$

Hence, the total magnetic moment induced by the current loop located in the plane which coincides with the direction of movement, is

$$m = m\ (p) + m\ (K) = m'. \tag{24}$$

Thus, as should have been expected, the magnetic moment of a dipole perpendicular to the velocity is the same in the K' coordinate system as in the K laboratory system. It is also clear that the electric dipole moment [Eq. (21)] and the magnetic moment [Eq. (24)] are strongly linked. If for the electric moment the transformation Eq. (6) is taken instead of Eq. (5), then this necessarily leads to a change in m in Eq. (24), which is related in a complex manner to the index of refraction. In principle, this can not be ruled out, but there is no basis for it. Indeed, the only reason for the transformation [Eq. (6)] was that in this case the formula for the energy of Vavilov-Cherenkov radiation becomes analogous to the formula for the radiation of the charge. If we consider the complication of the formula for m which would be caused by the transformation [Eq. (6)], then this argument ceases to hold. From what has been said, it follows that the same transformations of dipole moments should be used in the case of movement in a medium as are used in a vacuum.

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