Criteria for the degree of chaos

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The criteria for the degree of chaos are discussed by Ya. A. Kravtsov[1] and Yu. L. Klimontovich[2] elsewhere in this issue. Their respective points of view are compared below.

Before going any further, it is important to emphasize that the statistical description of a particular phenomenon (if we are not concerned with quantum physics) is only one of a number of possible methods of description. Occasionally (e.g., in statistical physics), the statistical approach is practically the only one, but deterministic and statistical descriptions can often be carried out in parallel. Let us illustrate this by a simple example. Suppose that we have a "black box" noise generator that produces a fluctuating voltage $U(t)$ at its output. This voltage can be described by specifying the probability distributions of different order (one-time, two-time, etc.). If we know the distributions, and measure the instantaneous voltages $U(t_1),...,U(t_n)$, we can use the statistical distribution laws to provide the most probable estimate of $U(t)$ for $t_k < t < t_{k+1}$ or $t > t_n$. The precision of this estimate will depend on the statistical properties of the signal. Let us now suppose that the noise generator is actually a computing machine that produces pseudorandom numbers which are then transformed into a continuous signal (e.g. the system produces pulses whose height is proportional to the successive random numbers, and these pulses are then passed through a frequency filter). If we know the algorithm used to generate the pseudorandom numbers and the characteristics of the system that processes the signal, we can predict the output voltage at any particular time with a high degree of precision.

In this particular example, it is clear that the question as to whether the signal $U(t)$ is random or not has no special meaning. There are problems in which it is sufficient to know the simple statistical characteristics of the signal and there is no need for a detailed description. However, the opposite situation is also possible.

We thus see that the answer to the question as to whether a particular variable is random or deterministic depends on the amount of information that we have at our disposal about the particular variable. The statistical description is the least detailed but the most universal: it is often valid in situations in which other methods of description are ineffective or impossible.

It is clear from the above considerations that the question of the degree of chaos is to some extent arbitrary: it has to be formulated within the framework of a particular model adopted for the phenomenon. This point of view is developed by Yu. A. Kravtsov[1] who estimates the degree of chaos in a random process from the degree of its predictability. If we compare two random processes, the less chaotic is that whose behavior can be predicted to a given degree of precision over a longer period of time. Of course, prediction in time is not the only possible one. The behavior of a random quantity can be estimated as a function of other parameters (for example, in terms of spatial coordinates). This definition can be generalized as follows: a random quantity U_1 is less chaotic than another U_2 with respect to a parameter α if the statistical estimate $U_1(\alpha)$ based on known values $U_1(\alpha_1),...,U_1(\alpha_n)$ is possible with greater relative precision than the corresponding estimate $U_2(\alpha)$ based on $U_2(\alpha_1),...,U_2(\alpha_n)$.

It is clear that if we change (improve) the model (as in the above example of a noise generator), the relation between the degrees of chaos of U_1 and U_2 may be reversed.

Of course, the definition given by Yu. A. Kravtsov, and the above generalization of it, are not the only possible ones. However, they have been tested in many interesting practical applications and are quite reasonable.

One of the central points of the paper by Yu. L. Klimontovich[2] is his definition of the degree of chaos. He defines the degree of chaos in relation to self-organization processes (see Section 2): the most chaotic state is the state that is the furthest (according to the value of the control parameter) from the state of "self-organization". The concept of "self-organization" thus becomes the basic, primary concept. In Section 7, Klimontovich formulates a universal quantitative criterion for comparing the degrees of chaos of two states. The probability density for a particular state, assumed the more chaotic, is then given by

$$f_0(x) = \exp(-H_{\text{eff}}(x)),$$

i.e., the "effective Hamiltonian" is $H_{\text{eff}}(x) = -\ln f_0(x)$. The next step is to introduce the "renormalized" distribution of the same form, but with a different (relative) temperature D:

$$\tilde{f}_0(x) = A\exp\left(-\frac{H_{\text{eff}}(x)}{D}\right) = A\,(f_0(x))^{1/D}.$$

This expression involves the two unknown quantities A and D, and two equations are then introduced to determine them. The first is the normalization condition

$$\int \tilde{f}_0(x)\,\mathrm{d}x = A\int \exp\left(-\frac{H_{\text{eff}}(x)}{D}\right)\mathrm{d}x = 1$$

and the second is the condition that the mean "energies" evaluated for the distribution $\tilde{f}_0$ and for the distribution $f(x)$ that is compared with f_0 according to the degree of chaos are equal:

$$\int H_{\text{eff}}(x)\,\tilde{f}_0(x)\,\mathrm{d}x = \int H_{\text{eff}}(x)\,f(x)\,\mathrm{d}x.$$

The final recipe for comparing the degrees of chaos is reduced to the following: if the value found from these equations is $D > 1$, the state with the distribution $f(x)$ is less chaotic than the state f_0 taken for comparison. In other words, if the initial system taken for comparison must be "heated" i.e., its temperature must be increased by the factor D in order that its mean energy be equal to the energy of the system described by the distribution $f(x)$, the state $f(x)$ is less chaotic (more ordered or "self-organized").

The application of this criterion often leads to results

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that are difficult to understand from the common sense point of view. Suppose, for example, that we have to compare the following two Gaussian distributions with different parameters:

$$f_0(x)=\frac{1}{\sqrt{2\pi}\,\sigma_0}\exp\left(-\frac{x^2}{2\sigma_0^2}\right),$$

$$f(x)=\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-a)^2}{2\sigma^2}\right].$$

The above procedure then readily yields the following expression for D:

$$D=(\sigma^2+a^2)\,\sigma_0^{-2}.$$

If $\sigma^2=\sigma_0^2$, then $D>1$ for any $a\neq 0$, i.e., we have "self-organization". This conclusion can be regarded as reasonable because we have a nonzero mean. However, when $a=0$, but $\sigma^2>\sigma_0^2$, we again have $D>1$. According to the above criterion, the latter case must also be looked upon as "self-organization" i.e., we must consider the state with the greater spread of values of x as the less chaotic. This result is difficult to comprehend.

The conclusion that the laminar flow of a liquid is more chaotic than the turbulent flow is one of the results of the application of Klimontovich's criterion that is difficult to interpret. Let us compare the two criteria put forward in the above papers. The question is: if turbulent flow is less chaotic than laminar flow, is there some flow variable that can be predicted more precisely for turbulent than for laminar flow? The answer to this question is as follows: since laminar flow is considered at a higher temperature than the turbulent flow, thermal fluctuations within it must be greater. Consequently, for the cooler turbulent flow, we can predict variables averaged over the thermal fluctuations with a greater precision than for the hotter laminar flow. This relies on the assumption that the averaging scale is the same in both cases and is small in comparison with typical macroscopic scales of the problem.

For example, suppose that the volume V over which the velocities of molecules in a gas are averaged is so small that, from the point of view of hydrodynamics, the mean velocity evaluated over this volume can be regarded as the instantaneous flow velocity at a point. (We note that, in Glukhovskii's paper entitled *Statistical description of the motion of a Brownian particle in a turbulent flow,*[3] the dimensions of the volume V were assumed small in comparison with the internal turbulent scale.) The mean velocity of a molecule within the volume can then be found for the cooler turbulent flow with a greater precision than for the hotter laminar flow. However, if we repeat this measurement (i.e., average over the volume V) a large number of times with a sufficiently large step in time, we find that, for the laminar flow, the additional averaging over the ensemble of all the measurements will lead to a further reduction in the uncertainty in the mean velocity, whereas, for the turbulent flow, the measurement uncertainty will not be reduced by the turbulent fluctuations.

The apparently paradoxical conclusion that turbulent flow is less chaotic than laminar flow is thus seen to refer to thermal and not to hydrodynamic fluctuations. Whenever this formulation of the problem is of interest in any particular situation, it cannot give rise to any objection. However, it must be remembered that the conclusion that the turbulent flow has a higher degree of order has nothing to do with hydrodynamic fluctuations.

The above paradox can also be examined by considering the following "gedanken" experiment. Suppose we have a closed ring channel in which turbulent flow is specified as the initial state. Viscosity will then damp out the turbulence, and the flow will become laminar after a certain interval of time. Turbulent energy will be transformed into heat, i.e., we shall have laminar flow with energy equal to that of the initial turbulent flow. Since this process occurs naturally in time, the final laminar state will have higher entropy than the initial turbulent state, which is in accordance with the formulation put forward by Klimontovich. However, the temporal development proceeds from the turbulent to the laminar flow, and not the other way, i.e., there is no sense in which we can speak of "self-organization" here.

There is no doubt that the criteria for estimating the degree of chaos in "self-organizing" systems present us with very interesting questions, and that Yu. L. Klimontovich has rendered valuable service with his formulation. However, the above examples show that his criterion is not unambiguous.

[1]Yu. L. Kravtsov, Usp. Fiz. Nauk **158**, 93 (1989) [Sov. Phys. **32** (1989), this issue].
[2]Yu. L. Klimontovich, Usp. Fiz. Nauk **158**, 59 (1989) [Sov. Phys. **32** (1989), this issue].
[3]A. B. Glukhovskii, Izv. Akad, Nauk SSSR, Fiz. Atmos. Okeana, **7**, 1039 (1971) [Izv. Acad. Sci. USSR, Atmos. Oceanic Phys. **7**, 687 (1971)].

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