Randomness, determinateness, and predictability

Yu. A. Kravtsov

Institute of General Physics, Academy of Sciences of the USSR
Usp. Fiz. Nauk **158**, 93–122 (May 1989)

The basic conventions regarding randomness employed in mathematics (set-theoretical approach, algorithmic approach) and in physics (decaying correlations, continuous spectrum, hyperbolicity, fractal nature, uncontrollability, nonrepeatability, nonreproducibility, nonpredictability, etc.) are analyzed. It is pointed out that phenomena that are random from one viewpoint may be determinate from another viewpoint. The concept of partially determinate processes, i.e., processes that admit prediction over bounded time intervals, is discussed. The theory of partially determinate processes is based on identifying randomness with unpredictability and establishes the interrelation between the real physical process $x(t)$, the observed process $y(t)$, and the model (predictive, hypothetical) process $t(t)$. In this theory the degree of determinateness, which is defined as the correlation coefficient between the observed process and prediction, is employed as a measure of the quality of predictability. Diverse theoretical, experimental, and numerical measures of partially determinate processes as well as examples of partially determinate fields are presented. It is emphasized that the time of determinate (i.e., predictable) behavior $\tau_{\rm det}$ of an observed process $y(t)$ can be much longer than the correlation time τ_c, and the degree of coherence is the worst estimate of the degree of determinateness. From the viewpoint expounded determinate chaos stands out as a completely determinate process over short time intervals ($\tau \ll \tau_{\rm det}$), as a completely random process over long intervals ($\tau \gg \tau_{\rm det}$), and as a partially determinate process over intermediate time intervals $\tau \sim \tau_{\rm det}$. It is significant that in the interval between τ_c and $\tau_{\rm det}$ chaotic and turbulent fields admit both a determinate and statistical (kinetic) description.

1. INTRODUCTION

Since the time of Laplace and until comparatively recently most physicists believed that given dynamic equations and initial conditions the behavior of any system can be satisfactorily predicted on the basis of classical physics.

Confidence in the potentially unlimited capabilities of classical physics to predict the behavior of complex systems amazingly coexisted with an enormous number of phenomena indicating the opposite: developed turbulence, different types of plasma instability, etc. This conviction gave rise over a long period of time to a paradigm in which cases of poor predictability are regarded as some kind of misunderstandings due to "insignificant" factors, such as an extremely large number of participants in the motion or uncertainty in the initial data. Though it was acknowledged that fundamentally unavoidable reasons for unpredictability exist unpredictability in the character of the equations of classical physics was by no means allowed.

In the meanwhile the idea that the predictive capabilities of classical physics are limited gradually acquired force, due largely, for example, to the remarkable works of H. Poincaré, in particular, his investigations of complex motions of the homoclinical structure type, which studies were several decades ahead of their time,[1] and the simple but important ideas of M. Born regarding the long-time unpredictability of classical motions owing to errors in the initial data. Eventually, as a result of the combined efforts of mathematicians, physicists, and experts in mechanics, at the beginning of the 1970s a qualitatively new conception of the nature of dynamic processes, namely, the idea of local instability of the behavior of the majority of the least bit complicated physical systems and of the very important role of chaotic and stochastic motions, which cannot be predicted over long time intervals,[2] was formulated.

The important concepts of mixing, local instability, topological entropy, strange attractors, fractal dimension, etc., that have become a part of physics and mathematics have already been repeatedly discussed in this journal (see the reviews of Refs. 3–6). The list of works on dynamic chaos is now increasing by several hundred publications per year, and no less than ten monographs devoted specially to this problem have already been published (aside from Refs. 1, 2, we also call attention to the books of Refs. 7–10).

The main feature of chaotic systems is that a small perturbation of the initial conditions for a dynamic variable or a small change in the parameters of the dynamic system itself causes the resulting motion to be unpredictable over a finite time, which J. Lighthill[11] aptly termed the predictability horizon.

In spite of the enormous interest in the problem of chaos one of its main aspects—the time-limited predictability of the behavior of dynamic systems—has not yet been exhaustively described. This review is devoted to a general approach to the problem of predictability, based on the idea of partially determinate processes, i.e., processes admitting dynamic prediction over bounded time intervals. This approach formalizes the unexpectedly complicated, even within the framework of classical physics, "interrelationships" between the observation $y(t)$ and the prediction $z(t)$ in terms of the joint probability density $w_2(y, z, t)$ and its moments.

The central problem addressed in this review is to determine what ultimately limits predictability: noise, interference, inexact initial data, or defects in the predictive model? This formulation of the problem is disturbing not only to physicists, but also to meteorologists, biologists, economists,

0038-5670/89/050434-16$01.80

and sociologists. In western countries the problem of prediction has already been addressed in an interdisciplinary manner (see, for example, the special edition of the Proceedings of the Royal Society of London, in which Lighthill's article[11] was published). Has not the time arrived for us physicists also to join forces with economists? Will we not be able to understand together how to avoid chaos where it can and should be avoided?

2. PHENOMENON, OBSERVATION, PREDICTION

2.1. The observed process. The role of measuring devices

In the study of a real physical process, which we denote by $x(t)$, let the observed process $y(t)$, which, generally speaking, has several components $y(t) = \{y_1(t),...,y_p(t)\}$, where p is the number of independent sensors (devices), be recorded. The observed process $y(t)$ differs from the process under study $x(t)$ in several respects.

First, the measuring devices in one way or another transform the process under study $x(t)$: they perform filtering (they alter the form of the spectrum), they introduce nonlinear distortions, and they even affect the dimension—the dimensipn p of the observed process $y(t)$ is always less than the dimension q of the process studied $x(t)$: $p<q$.

Second, the measuring devices always add to the result of the measurements an additional noise component $\nu(t)$, which we shall call the measurement noise.

Third, a measuring device also introduces distortions into the studied process itself $x(t)$. Until recently it was believed that the effect of a measuring device on the phenomenon under study is manifested only in measurements of microscopic quantities, when the quantum nature of the phenomena can no longer be ignored. It is obvious that the presence of a macroscopic device can radically affect the result of measurements in classical physics also,[3] when locally unstable processes, which react strongly and over a finite time interval even to small perturbations, are under study.

So as not to complicate the subsequent exposition we shall assume that none of the p recorded components of the signal is subjected to nonlinear and spectral distortions, but the components of the recorded signal are subjected to additive measurement noise $\nu(t)$, so that

$$y_j(t) = x_j(t) + \nu_j(t) \quad (j = 1, 2, \ldots, p, \ p < q). \quad (2.1)$$

2.2. Prediction as a model process

In striving to predict the behavior of a real process $x(t)$ we are actually obliged to predict only the observations $y(t)$, since we have no way to judge the state of the studied process $x(t)$ other than based on the indications of measuring devices. Let the process $y(t)$ be observed over a quite long time interval from the moment $t^0 - T$ in the past up to the running time t^0. It is obvious that any prediction of the behavior of $y(t)$ must be based on some hypotheses or models. For this reason a model process $z(t)$ (or in other words a predictive, idealized, hypothetical process), with respect to which the quality of predictability is to be evaluated, must be introduced into the analysis together with the real process $x(t)$ and the observed process $y(t)$.

The prediction must be based on some equation, rule, or algorithm. Since we are primarily interested in dynamic systems it is natural to attempt to conform the prediction $z(t)$ to a model (hypothetical) differential equation, which we shall write in the symbolic form

$$M\left(\frac{d}{dt},\ z;\ \alpha,\ 0\right)=0. \quad (2.2)$$

The symbol α in this equation denotes parameters of the model that are to be refined, while the symbol 0 as a fourth argument embodies everything that has not been taken into account in the chosen model: insignificant perturbations, unimportant degrees of freedom, noise, interference, etc.

The value of the observed process $y^0 = y(t^0)$ at the initial time $t = t^0$ can serve as a natural initial condition $z^0 = z(t^0)$ for the prediction $z(t)$:

$$z^0 = y^0,\ t = t^0. \quad (2.3)$$

More complicated formulations of this question are also possible. For example, to reduce the effect of strong additive noise $\nu(t)$ the observed process $y(t)$ can be filtered, and if $\tilde{y}(t)$ is the filtered signal, then the quantity $\tilde{y}^0 = \tilde{y}(t^0)$ should be used for z^0.

2.3. The process being studied

The description of the starting process $x(t)$ is dependent to the greatest extent on our understanding of the problem expressed in the choice of model. Taking Eq. (2.2) as the model description it is useful to represent the "real" equation for $x(t)$ in the form

$$M\left(\frac{d}{dt},\ x;\ a,\ f_k(t)\right)=0, \quad (2.4)$$

where a are the true (but unknown) parameters of the system, while the factors $f_k(t)$ symbolize the action of "the rest of the universe." These factors also satisfy definite equations. As the predictive model is refined (made more complicated) and when additional information appears some components $f_k(t)$ can be referred to $x(t)$ and the dimension is increased accordingly.

2.4. Relation between the studied, observed, and model processes

The interrelationship of these processes is illustrated in Fig. 1, where the processes $x(t)$, $y(t)$, and $z(t)$ are shown as if they had only one component. When the noise is comparatively weak the observed process $y(t)$ (the thick line) is virtually identical to the real process $x(t)$ (the broken line) for any value of t for which the recording instruments function properly. At the same time the prediction $z(t)$ (the thin line) is close to $y(t)$ only on a bounded interval, which it is

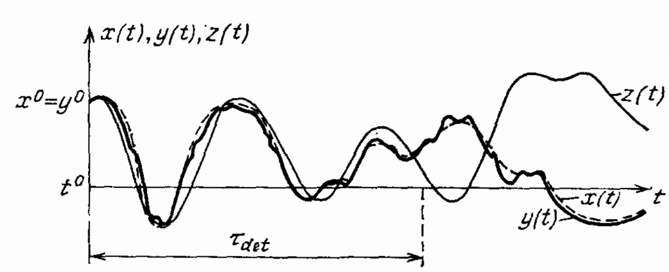

FIG. 1. Processes studied in the problem of predictability: $x(t)$ is the real process (broken line); $y(t)$ is the observed process (thick line); $z(t)$ is the model, hypothetical process (fine line).

natural to call the time of predicted behavior or, if the predictability is identified with determinateness, the time of determinate behavior $\tau_{\rm det}$.

In this paper I shall attempt to analyze the factors that limit this time, i.e., the factors responsible for narrowing the predictability horizon, and I shall present arguments supporting the fact that unpredictability is ultimately caused by the existence of noise, which is present in any physical system.

3. RANDOMNESS AS A CONVENTION

3.1. Set-theoretical approach

Attempts to give a logical interpretation of the concept of randomness have been made throughout the entire history of the development of science in general and the theory of probability in particular. The main thesis invariably consisted of regarding randomness as the "absence of laws" (A. N. Kolmogorov Ref. 12). It was found, however, that the idea of "absence of laws" is not unique and admits many different interpretations.

Several viewpoints regarding randomness, each of which emerges as a unique convention, have now been formulated. In most cases the different representations of randomness agree qualitatively with one another, but sometimes it turns out that the process (or phenomenon) of interest appears to be random from one viewpoint and nonrandom from a different viewpoint. This is possible because different conventions regard a phenomenon from seemingly different planes. As a characteristic example I call attention to the fact that the term "deterministic chaos" came into existence at the intersection of two planes: the adjective "deterministic" reflects a class of differential equations which describe chaos (these equations do not contain random functions in the sense adopted in the theory of probability), while the noun "chaos" corresponds to the character of the process (local instability, global boundedness, fractal dimension, limited predictability). In this connection it is useful to make a "separation," marking the boundaries between existing conventions. This procedure will help to clarify the meaning of many assertions about prediction.

I shall start with the set-theoretical approach on which the modern theory of probability is based. In this approach the concept of randomness is associated with the possibility of ascribing to a given quantity a probability measure, namely, a quantity is said to be random if it is determined by its probability distributions. The definition of a random variable formulated in the language of σ algebras and measurable functions essentially reduces to this (see, for example, Ref. 13, p. 132). The "absence of laws" is reflected here by the degree of spreading of the given quantity; determinate quantities correspond to distributions described by δ functions. This interpretation is also adhered to in theoretical physics.

In this connection recall that aside from random and determinate (in the sense indicated above) quantities there also exist indefinite quantities, for which probability measures are unknown or have not been determined with adequate experimental reliability (V. N. Tutubalin[14]).

3.2. Algorithmic (complexity) approach

Even the founders of the modern theory of probability were not completely satisfied with the set-theoretical interpretation of randomness and they attempted to develop alternative approaches.

The approach developed best thus far is that of A. N. Kolmogorov and his followers (Martin-Löf, Chaitin, and others), which is based on the interpretation of "absence of laws" as an algorithmic complexity (the status of the question is discussed in Ref. 15 by A. N. Kolmogorov and in the review of Ref. 16).

As a measure of the complexity of a given sequence of zeroes and ones $\{x_i\}$, $i = 1, 2, \ldots, N$ (any process can be represented with the help of such sequences). A. N. Kolmogorov proposed taking the length of the program l (in bits) which transforms a given sequence y_j into x_i. If the conversion algorithm is simple, then l is significantly shorter than the length of the sequence $\{x_i\}$, so that the sequence $\{x_i\}$ can be regarded as nonrandom. In the opposite case when $l \sim N$ the algorithm for the conversion $\{y_j\} \to \{x_i\}$ essentially reduces to recording the sequence $\{x_i\}$ itself symbol by symbol. The complexity of the corresponding algorithm can serve as a basis for classifying a given sequence as random. From the viewpoint of complexity almost all sequences $\{x_i\}$ turn out to be random, since simple algorithms form a set of measure zero. The situation here is identical to that of irrational numbers: almost all numbers are irrational, since the set of rational numbers has measure zero.

It is worthwhile to note also that the complexity approach admits several definitions of a random sequence, depending on the choice of admissible rules for selecting the elements: Mises-Church, Mises-Kolmogorov-Loveland, and Martin-Löf (see Ref. 16).

The complexity interpretation of randomness is certainly original, but it should still be noted that the convention of identifying randomness with algorithmic complexity does not completely correspond to the viewpoint regarding the nature of phenomena that is being developed in physics. The point is that algorithmic complexity in itself is not a fundamental obstacle to predicting a process over bounded time intervals: this obstacle is more likely of a technical or even psychological character. Additional difficulties are associated with the invariable presence of physical interference and noise, in the presence of which even algorithmically simple processes become algorithmically complex.

3.3. Randomness criteria employed in physics

In experimental physics specific randomness criteria that do not reduce to set-theoretical and algorithmic conventions are employed.

The most primitive randomness criterion is an irregular (aperiodic) form of the process. It is obvious that at a qualitative level this criterion juxtaposes randomness with periodicity.

The criterion of *decaying correlation* reduces to the requirement that the correlation function of the observed process $\psi_y(\tau) = \langle y(t) y(t-\tau)\rangle$ and the correlation coefficient

$$K_y(\tau) = \frac{\langle y(t)\, y(t-\tau)\rangle}{(\langle y^2(t)\rangle \langle y^2(t-\tau)\rangle)^{1/2}} \tag{3.1}$$

approach zero as τ increases. The purpose of this criterion is also to select periodic and aperiodic (chaotic) processes, but it realizes this selection quantitatively. The criterion has the obvious limitation that it cannot reveal periodicity with a period longer than the observation time T.

The *continuous spectrum* criterion juxtaposes randomness, interpreted as a chaotic process with a continuous spectrum, with periodic processes associated with a discrete spectrum. This criterion is obviously equivalent to the criterion of decaying correlations, since the spectral density and correlation function are related with one another via a Fourier transform. In spectral analysis the observation time T determines the spectral resolution $\Delta\omega_{\min} = 2\pi/T$. If the time T is not long enough and the interval between the discrete lines is less than $\Delta\omega_{\min} = 2\pi/T$, then a spectral instrument cannot distinguish a continuous spectrum for a discrete spectrum.

The continuous spectrum and decaying correlations criteria, though they reduce the idea of randomness to a primitive antithesis of periodicity, are still of practical value since they can be employed as indicators of a transition from a periodic regime to chaos. We note, however, that if these criteria are employed systematically, then as the observation time T is increased sooner or later we shall establish the finiteness of the width of discrete lines (in the language of correlation functions—decay of correlations) for any physical periodic process, and if we proceed in a linear fashion, then we shall be compelled to call all physical processes without exception chaos.

The randomness of objects demonstrating complex, chaotic behavior is characterized with the help of several indicators[1,2]: *fractal dimension* (metric, Lyapunov, correlation), *entropy* (rate of divergence of trajectories in phase space), and *degree of order* in phase space.[17]

In addition to these quantitative criteria other criteria that are more qualitative than the quantitative characteristics of the processes under study are employed, for example, *nonreproducibility* (impossibility of obtaining the same realization of a process under indentical external conditions), *nonrepeatability* (which can be interpreted both as nonreproducibility and the absence of periodicity in the given process), *noncontrollability* and *nonmonitorability* (impossibility of creating conditions under which the process would proceed in a prescribed manner.

The convention identifying randomness with *unpredictability* stands somewhat apart from the other conventions. This convention is implicitly adopted in most situations in life and it appears in the introductory sections of books on the theory of probability, but it cryptically vanishes from subsequent chapters. Strange as it may seem, however, it is precisely this aspect of randomness, which is important in life, that has thus far not been adequately formalized, in spite of the fact that it is directly relevant to the problem of interpretation of experimental data. In what follows I shall try to make such a formalization on the basis of the theory of partial determinateness, in which reliable predictability is interpreted as determinateness and poor predictability is interpreted as randomness. Preliminary sketches of this theory are given in Refs. 18–21, and more complete versions are given in Refs. 22 and 23.

4. RANDOMNESS AS UNPREDICTABILITY

4.1. The statistical pair "observation-prediction"

If at the initial time t^0 the quantities y and z assume the values y^0 and z^0, then the statistical description of the pair of quantities (y, z) at a time t is obtained with the help of the conditional two-dimensional probability density

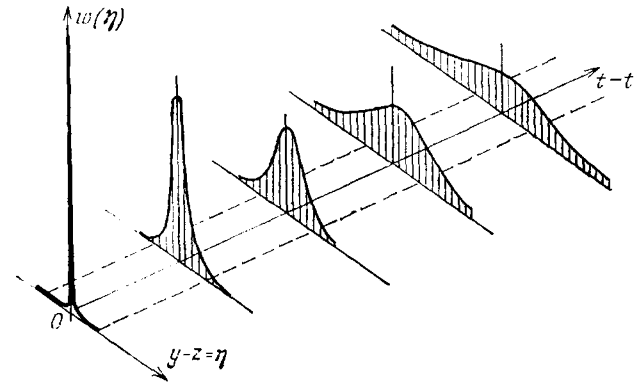

FIG. 2. Evolution of a two-dimensional, conditional probability density.

$$w_2 = w_2(y,\ z,\ t \mid y^0,\ z^0,\ t^0). \tag{4.1}$$

Taking for the model process $z(t)$ the natural initial condition $z^0 = y^0$ [see (2.3)], in the limit $t \to t^0$ we find that the two-dimensional probability density degenerates into a δ function:

$$w_2 \to \delta(y^0 - z^0)\, w_1(y^0), \tag{4.2}$$

where $w_1(y)$ is a one-dimensional, unconditional probability density for the observed process $y(t)$. The general character of the behavior of the probability density w_2 is shown in Fig. 2: the starting δ function spreads out with time, indicating that the quality of the prediction given by the model function $z(t)$ becomes worse.

In introducing an object of study that is new to theoretical physics it is useful to underscore how it differs from other probability characteristics of physical processes: the probability density w_2 characterizes not only the properties of the observed physical process $y(t)$, but also the properties of the hypothetical model process $z(t)$, whose behavior is determined not by the laws of nature, but rather exclusively by the models employed by the interpreter to represent the flow of the physical processes. Thus the hypothesis $z(t)$ is included in the statistics together with the studied process $y(t)$. Of course, this procedure is not absolutely new (it is sufficient to recall the statistics of the theory of distinguishing hypotheses, which was initially developed for the needs of radar and is now widely employed in many areas of physics, where a weak signal must be distinguished against the background of noise and interference), but the known equivalence between a hypothesis and an observation is not yet generally acknowledged. Only meteorology, where the correlation between prediction and observation has been employed for a long time as a criterion for the quality of prediction, is possibly an exception.

4.2. Measures of the quality of predictability

The most commonly employed measure for the quality of a prediction is the mean-square of the error $\eta = |y - z|$:

$$\sigma_\eta^2 \equiv \langle \eta^2 \rangle = \langle | y(t) - z(t) |^2 \rangle. \tag{4.3}$$

Based on the obvious initial condition (2.3) the variance σ_η^2 at the initial time $t = t^0$ equals zero: $\sigma_\eta^2\ (t^0) = 0$. Over sufficiently long time intervals, when positive and negative values of the product yz are encountered equally often, the processes $y(t)$ and $z(t)$ become statistically independent: $\langle yz\rangle = 0$. The variance σ_η^2 is then expressed as a sum of mean squares: $\sigma_\eta^2 = \langle y^2\rangle + \langle z^2\rangle$ (here and below we assume that the processes y and z are bounded; in particular, they can belong to an attractor). It is natural to call the quantity

$$E = \frac{\sigma_\eta^2}{\langle y^2\rangle + \langle z^2\rangle} = \frac{\langle |y-z|^2\rangle}{\langle y^2\rangle + \langle z^2\rangle} \qquad (4.4)$$

the relative error. As $t \to \infty$ it approaches unity. For simplicity we shall assume that $\langle y\rangle = \langle z\rangle = 0$.

The correlation measure of the quality of prediction is introduced as the normalized correlation function

$$D(\tau) = \frac{\langle y(t)\, z(t)\rangle}{\langle y^2(t)\rangle \langle z^2(t)\rangle^{1/2}}, \quad t = t^0 + \tau. \qquad (4.5)$$

The modulus of $D(\tau)$ is always less than unity

$$|D(\tau)| \leqslant 1, \qquad (4.6)$$

which follows from Bunyakovskiĭ's inequality. Using the identity

$$\langle yz\rangle = \tfrac{1}{2}\langle y^2\rangle + \langle z^2\rangle - \langle |y-z|^2\rangle$$

the correlation coefficient D can be expressed in terms of $\sigma_\eta^2 = \langle |y-z|^2\rangle$ and E:

$$D(\tau) = \frac{\langle y^2\rangle + \langle z^2\rangle - \sigma_\eta^2}{2(\langle y^2\rangle \langle z^2\rangle)^{1/2}} = \frac{\langle y^2\rangle + \langle z^2\rangle}{2(\langle y^2\rangle \langle z^2\rangle)^{1/2}}(1-E), \qquad (4.7)$$

so that all three quantities σ_η^2, E, and D can be equally employed to describe the quality of a prediction. But, as we shall verify below, D has definite advantages when randomness is interpreted as unpredictability.

The quantities σ_η^2, E, and D can be expressed "theoretically" in terms of the two-dimensional probability density (4.1), but these quantities can be determined directly, bypassing w_2, from experiment by collecting the required data and performing the standard empirical averaging.

The quality of predictability can also be characterized by a density function (the probability that y falls within a given interval ϵ near the prediction z, the *density measure of quality*) or by the information contained in the process $y(t)$ about the process z (information measure of quality). Both measures are more difficult to measure than the quantities σ_η^2, E, and D, and for this reason they are less useful.

5. PARTIALLY DETERMINATE PROCESSES

5.1. The concept of partial determinateness

The concept of partial determinateness is based on the convention that unpredictability (predictability) of an observed process based on a definite predictive model $z(t)$ or a class of models $z_\alpha(t)$ is used as an indicator of randomness (determinateness). In this approach randomness and determinateness are not juxtaposed with one another, but rather they are regarded as the opposite poles of the same property—*partial determinateness.*

Although the statistical properties of the pair "observation-prediction" are described in detail by the joint probability density w_2 it is, however, more convenient to introduce simpler characteristics of predictability, similarly to the manner in which the theory of coherence, together with the joint probability density of the values of the field at different points and different times, the moments of this density—the coherence functions of different order—are introduced. A simple and graphic characteristic of this type is the correlation coefficient (4.5) between observation and prediction, which we shall call in what follows the *degree of determinateness (predictability)*.

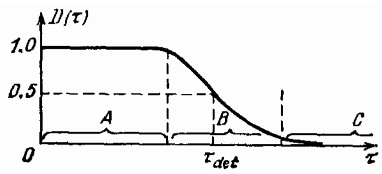

FIG. 3. Time dependence of the degree of determinateness. A—region of completely determinate behavior; B—region of partially determinate behavior; C—region of random (unpredictable) behavior. $\tau_{\rm det}$—determinate-behavior time.

Over short time intervals τ, when the model process $z(t)$ does not yet differ strongly from z^0, the quantity D is close to unity (Fig. 3). In this case one can talk about the completely determinate behavior of the process $y(t)$ relative to $z(t)$. Conversely, for sufficiently long times τ the quantity D approaches zero, indicating that the process $y(t)$ is weakly determinate (in the sense of weak predictability) with respect to the model $z(t)$.

From the viewpoint of the experimenter, who does not have at his disposal an excessively large number of models, it is natural to interpret weak predictability as randomness. This corresponds completely to the general tendency to regard randomness as the absence of laws, which led A. N. Kolmogorov to identify randomness with algorithmic complexity. In our opinion, however, the experimenter is more likely to have at his command not the "complexity" concept of randomness but rather a different concept that rests on predictive considerations and which we shall call the concept of partial determinateness. On the basis of this concept everything that does not agree with the given model $z(t)$ or with a given class of models $a_\alpha(t)$ is random (unpredictable).

On the basis of this approach the observed process $y(t)$ is a *determinate (predictable)* process if $|1-D| \ll 1$, a *random (unpredictable)* process if $|D| \ll 1$, and a *partially determinate (partially predictable)* process if $0 < |D| < 1$. The regions of determinate, partially determinate, and random behavior are shown in Fig. 3. The times during which the quantity D exceeds some value, say 1/2, is the *determinate behavior time* $\tau_{\rm det}$ (see Fig. 3). It can be found from the equation

$$D(\tau_{\rm det}) = \frac{1}{2}. \qquad (5.1)$$

5.2. Determinate behavior time. Predictability horizon

There are several reasons why the determinate behavior time (predictability time) of the observed process $y(t)$ is limited. First of all, the observed process $y(t)$ always differs from the real process $x(t)$, since measuring devices always introduce nonlinear, frequency and noise distortions.

Second, real processes $x(t)$ are always subjected to different perturbations, which are represented in the symbolic equation of motion (2.4) by the external forces $f_k(t)$. The external forces have both fluctuation components, which can be described only statistically, and all possible neglected and unknown actions, for which we retain the possibility (at least theoretical) of a determinate description determined by appropriate measurements and controls of the motion. This actually means that such components of the external actions can be converted into components of the model $z(t)$.

Finally, the third reason for prediction errors is associated with the uncertainty of the model operator itself. This uncertainty can be expressed both in the *a priori* uncertainty of the parameters α (parametric uncertainty) and in the structure of the model equation itself (structural uncertainty). These errors are always present in any idealized description of real processes. In the literature errors of this type are sometimes called *"lack of knowledge noise."*

Thus the determinate behavior time depends on many factors, of which the main ones are the measurement noise, fluctuation actions on the system, and defects in the model. This can be represented symbolically in the form of the dependence

$$\tau_{\rm det} = F(\nu, f, \Delta M), \tag{5.2}$$

where ΔM is interpreted as the uncertainty of the model.

To increase the predictability time we can try to reduce the measurement noise, for which low-noise sensors and signal detectors must be employed, and we can try to improve the quality of the model by decreasing the "discrepancy" ΔM as much as possible. In any physical system, even when all accessible means for isolating the system from the external world are employed unavoidable noise still remains: thermal noise, electromagnetic pickup, ageing processes, etc. It is precisely these unavoidable fluctuation perturbations that determine the potential limits of predictability.

Passing to the limits $\nu\to 0$ and $\Delta M\to 0$ in (5.2) we obtain the limiting value

$$\tau_{\rm lim} = \lim_{\nu\to 0,\ \Delta M\to 0} \tau_{\rm det} = F(0, f, 0), \tag{5.3}$$

which should be regarded as the *predictability horizon*. This limiting time fulfills several functions, which are physically close, but still have different interpretations.

First of all, the predictability horizon $\tau_{\rm lim}$ characterizes the *dynamic memory time of the system*. For times longer than $\tau_{\rm lim}$ the two-dimensional probability density w_2 no longer depends on the initial values y_0, z_0 and at the same time decomposes into a product of one-dimensional densities:

$$w_2(y, z, t \mid y^0, z^0, t^0) \to w_1(y)\, w_1(z) \quad (\tau \gg \tau_{\rm lim}). \tag{5.4}$$

As a result for $\tau\gg\tau_{\rm lim}$ we have $\langle yz\rangle \to \langle y\rangle\langle z\rangle = 0$, so that the degree of determinateness D vanishes. We note that the concept of the memory time of the system was introduced for control systems by N. Wiener, who, however, did not duly delineate the role of irremovable noise.

The dynamic memory time can be equally called the *time over which the system "forgets" the initial conditions.* This terminology, however, is unfortunately already "taken": in the kinetic theory this time is usually taken to mean the time τ_c over which the correlations become uncoupled: after this time is reached the correlation coupling between the values of the process $y(t)$ at neighboring times becomes weaker, and multidimensional distribution functions decompose into products of one-dimensional functions. The dynamic memory time is also associated with the decoupling of correlations, not between the values of the observed process $y(t)$, but rather between the values of the observations and the prediction.

Further the quantity $\tau_{\rm lim}$ characterizes the *reversible-behavior* time, i.e., the time during which the system can still return to its previous state upon the reversal (imagined, of course) of the velocities of all the particles. If the time interval Δt after the time t^* at which the velocities are reversed is short compared with $\tau_{\rm lim}$, then the dynamic system described by the classical equations can reproduce the state existing over a time Δt up to the moment of reversal. If, however, $\Delta t > \tau_{\rm lim}$, then even with ideally precise reversal of the velocities of all the particles the system will not return to its previous state owing to irremovable noise and interference, which are not time-reversed at the moment t^*.

Finally, the time $\tau_{\rm lim}$ is comparable to the *information memory time* $\tau_{\rm inf}$, which was introduced in the investigations of Refs. 24 and 25 to denote the time over which the observed process $y(t)$ loses information about noise that had previously acted on the system (in Refs. 24 and 25 systems of the strange attractor type were studied). The comparability of $\tau_{\rm lim}$ and $\tau_{\rm inf}$ in order of magnitude follows from the fact that both quantities can be expressed in terms of the same probability density w_2.

5.3. Partial determinateness and partial coherence. Degree of determinateness as a measure of internal couplings in the process

It is very important that the determinate behavior time $\tau_{\rm det}$ can be much longer than the coherence (correlation time τ_c, which serves as a characteristic scale for the decay of the degree of coherence (autocorrelation coefficient) (3.1).

Indeed the time τ_c is the typical interval between two neighboring maxima on the graph of the observed process or, which is equivalent, the inverse width of the spectrum $\tau_c \sim 1/\Delta\omega$. The predictability time $\tau_{\rm det}$, however, is determined by completely different factors: the noise level, the accuracy of the model adopted, etc. In other words the time $\tau_{\rm det}$ depends significantly on the *a priori* information about the dynamics of the system, and it is thus not surprising that in many cases $\tau_{\rm det}\gg\tau_c$ (Fig. 4).

It is also important that the degree of coherence (3.1) appears as the *worst degree of determinateness*. We have in mind the following: if we do not have a dynamic equation for the model $z(t)$, then the prediction must be constructed based only on the observed process $y(t)$. In the simplest case the prediction $z(t)$ over the time t can be taken as the value $y(t-\tau)$ at a preceding time (according to the principle "tomorrow will be the same as today").

$$z(t) = y(t-\tau), \tag{5.5}$$

and in this case the degree of determinateness (4.5) converts into the degree of coherence (3.1). Correspondingly the cor-

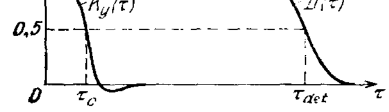

FIG. 4. Typical relation between the degree of determinateness $D(\tau)$ and the degree of coherence $K_y(\tau)$.

relation time appears as the lower limit of the determinate behavior time. We shall present several examples illustrating this result in the next section.

It is well known that the correlation coefficient characterizes the degree of linear coupling between random quantities. In particular, the correlation coefficient $K_{y(\tau)}$ reflects the linear statistical relation between the values of the observed process y at the times t and $t-\tau$. What then characterizes the degree of determinateness $D(\tau)$? By definition $D(\tau)$ describes the relation between the observation $y(t)$ and the prediction $z(t)$, i.e., the relation between what actually happens and what we think should happen.

This relation, which is revealed by means of the dynamic equations of the type (2.3), can be termed an *internal* (or *dynamic*) coupling and the quantity D can be termed the coefficient of internal (dynamic) correlation.[18–21]

5.4. Dynamic chaos as a partially determinate process

The question of the nature of dynamic chaos and the relation between randomness and determinateness in chaos has given rise to vigorous debate, reflected by the "randomness-determinateness of strange attractors" paradox as well as the confusing term "deterministic chaos," which is now firmly ensconced in the physical literature. These discussions are by no means finished, since many questions concerning randomness and determinateness in chaos have not yet been adequately resolved. It appears to us that the discussions presented here regarding the ambiguity of the concept of randomness and the idea of partially determinate processes permit casting new light on some aspects of chaos.

The "randomness-determinateness of strange attractors" paradox has become acute because not everyone participating in the discussion regards the question of randomness as a question of convention. How then does chaos appear from the viewpoint of the conventions that were mentioned in Sec. 3?

The overwhelming majority of conventions emphasizes in chaos the features of randomness. Indeed a strange attractor is characterized by a stationary probability distribution (or, which is the same thing, by an invariant measure) and therefore it is random in the set-theoretical sense. Further, chaos is random from the viewpoint of the theory of algorithmic complexity and from the viewpoint of many physical criteria: it has a continuous spectrum and decaying correlations; it can be assigned a fractal dimension; exponential divergence of trajectories is characteristic for chaos; etc.

At the same time "mathematical" chaos, i.e., chaos not subjected to noise, is described by a determinate equation which does not contain random forces or random coefficients. It is precisely this fact that has led to the term "deterministic chaos."

In spite of the fact that the concepts of "chaos" and "determinate equations" lie in different planes there have been many discussions attempting to reconcile randomness and determinateness. The best known viewpoint is that of J. Ford, who saw the resolution of the paradox in the theory of algorithmic complexity: if the process is represented by a sequence of symbols, as done in symbolic dynamics, then this sequence will be algorithmically complex and in this sense random, especially since almost all initial conditions are also algorithmically complex.[27] This approach is unsatisfactory in that it completely ignores noise that is present in any real dynamic system.

Finally the theory of partial determinateness based on interpretation of randomness as unpredictability presents one more point of view. On the basis of this convention randomness and determinateness are regarded from the very beginning as opposite poles of the same property—partial determinateness. This approach can evidently also be extended to dynamic chaos, which thus appears as a partially determinate process, i.e., as a process that is completely determinate for $\tau \ll \tau_{\rm det}$ and completely random for $\tau \gg \tau_{\rm det}$. Since deterministic chaos admits prediction only over bounded time intervals, $\tau \ll \tau_{\rm det}$, it is more correctly termed *partially determinate*.

There is no great difficulty in estimating the time $\tau_{\rm det}$ for locally unstable processes demonstrating chaotic behavior.[18–21]

If $\lambda_1 > 0$ is the largest of the Lyapunov indicators, then for the mean-square error σ_η^2 we have the estimate

$$\sigma_\eta^2 \sim (\sigma_\nu^2 + \sigma_f^2 + \sigma_{\Delta M}^2)\, e^{2\lambda_1 t}, \tag{5.6}$$

where the quantities σ_ν^2, σ_f^2, and $\sigma_{\Delta M}^2$ characterize the contribution of three basic factors determining the prediction error: the measurement noise $\nu(t)$, fluctuation processes $f(t)$, and the uncertainty ΔM of the model operator M.

Since the degree of determinateness $D(\tau)$ can be expressed in terms of σ_η^2 by the formula (4.7) we obtain from the condition $D(\tau_{\rm det}) = 1/2$ the estimate

$$\tau_{\rm det} = \frac{1}{2\lambda_1} \ln \frac{\langle y^2 \rangle}{\sigma_\nu^2 + \sigma_f^2 + \sigma_{\Delta M}^2}. \tag{5.7}$$

This expression implies that the predictability horizon, achieved in accurate measurements ($\sigma_\eta^2 \ll \sigma_f^2$) and based on a satisfactory model ($\sigma_\Delta^2\ M \ll \sigma_f^2$), is determined only by the action of fluctuations:

$$\tau_{\rm lim} \approx \frac{1}{2\lambda_1} \ln \frac{\langle y^2 \rangle}{\sigma_f^2}. \tag{5.8}$$

From here follows the important conclusion that the randomness of chaos, interpreted as unpredictability, is ultimately determined by the action of fluctuation forces.

We note that the estimate (5.7) agrees with the information analysis performed by Shaw[24] and Crutchfield and Packard.[25] It is true that in Refs. 24 and 25 only the measurement noise was taken into account, and then only in a mediated form—as the uncertainty of the measurements σ_ν^2. The information renewal time calculated in Refs. 24 and 25 is written in our notation as

$$\tau_{\rm inf} \sim \frac{1}{2\lambda_1} \ln \frac{\langle y^2 \rangle}{\sigma_\nu^2}. \tag{5.9}$$

The very slow logarithmic growth of the predictability horizon (5.7) as the noise level σ_f^2 decreases indicates that even microscopic fluctuations owing to thermal noise and quantum uncertainties can grow to macroscopic magnitudes over a finite time. For example, if $\langle y^2 \rangle^{1/2}$ is a measured voltage of the order of 1 volt while σ_f is the rms value of the fluctuations of order 10^{-6}V, then

$$\tau_{\rm lim} \approx \frac{1}{2\lambda_1} \ln 10^{12} \approx \frac{6 \ln 10}{\lambda_1} \approx \frac{14}{\lambda_1}. \tag{5.10}$$

A decrease in the noise level by six orders of magnitudes (which, of course, is not realizable in practice) will increase the predictability horizon by only a factor of 2:

$$\tau_{\lim} \approx \frac{28}{\lambda_1}.$$

In spite of the fact that the chaos predictability time is relatively short it still is longer than the correlation time τ_c, roughly estimated as $1/\lambda_1$. Thus for the numerical data employed in (5.10) we have $\tau_{\lim} \sim 14\tau_c$.

5.5. Combined average. Quasirandom processes

The degree of determinateness (4.5) is a local characteristic of the congruity of the processes $y(t)$ and $z(t)$, referring only to the time t. There is also a different variant of comparison, in which the behavior of processes over the entire observation interval $(t^0, t = t^0 + \tau)$ is analyzed. To this end we introduce the combined average, which combines statistical averaging, denoted here by angular brackets, with time averaging, i.e., time integration over the observation interval (t^0, t). We shall denote the latter by parentheses, for example,

$$(y(t)\, z(t)) = \frac{1}{\tau} \int_{t^0}^{t} y(t')\, z(t')\, \mathrm{d}t', \quad \tau = t - t^0. \tag{5.11}$$

Denoting the combined average by braces we obtain for the average of the product $y(t)z(t)$

$$\{yz\} = \langle (yz) \rangle = \frac{1}{\tau} \int_{t^0}^{t} \langle y(t')\, z(t') \rangle\, \mathrm{d}t'. \tag{5.12}$$

The corresponding degree of determinateness is introduced as

$$D_1(\tau) = \frac{\{y(t)\, z(t)\}}{\{y^2(t)\}\{z^2(t)\}^{1/2}} \tag{5.13}$$

and gives an integral measure of the congruity referring to the entire observation interval (t^0, t). Here lie the advantages of $D_1(\tau)$ over $D(\tau)$, since time averaging reduces the contribution of the measurement noise ν approximately by a factor of $(\Delta\omega_\nu \tau)^{1/2}$, where $\Delta\omega_\nu$ is the width of the noise spectrum in the measurement system.

It is known from the theory of signal processing that if the noise is Gaussian, then time-averaging of the observed signal $y(t)$ with the weighting function $z(t)$ is equivalent to optimal filtering of the signal $z(t)$. Thus $D_1(\tau)$ is an optimal (in the sense indicated above) measure of determinateness. For these reasons it can be expected that the corresponding determinate behavior time $\tau_{1\det}$ will be somewhat longer than $\tau_{\det}$, though it is still difficult to estimate by how much. It can also be expected that the measure of determinatensss $D_1(\tau)$ will be more sensitive to the choice of model than is $D(\tau)$.

We can give the following brief definition of quasirandom signals and processes based on the measure of determinateness (5.13): these are signals (processes) that satisfy all criteria for randomness (they have a continuous spectrum and a decaying autocorrelation function, they admit a statistical description, they are algorithmically complex, etc.) except one, viz., the degree of determinateness relative to the model signal (process) $z(t)$, constructed based on the same algorithm or simply copied from $y(t)$, equals unity.

A quasirandom signal can satisfy all possible tests for randomness of the type to which random-number generators in computers are subjected, but it does not satisfy the test for congruity with its own copy. If the copy or algorithm are unknown, however, then a quasirandom signal is indistinguishable from a random signal.

5.6. Randomness-determinateness transition. Do "truly random" processes exist?"

On the basis of the theory of partial determinateness "ignorance," i.e., the uncertainty of the model operator M, in the calculation of D is just as important as different fluctuation factors. The degree of determinateness D can increase both because one of the factors $f_k(t)$ is included in the expanded dynamic model $z_1(t)$ and because the model $z(t)$ is improved with the same fluctuation factors. In this respect the concept of partial determinateness only over long time intervals ($\tau \gg \tau_{\lim}$) agrees with the assertion that on the basis of classical physics stochasticity is wholly generated by dynamics, while over times shorter than $\tau_{\lim}$ the predictability-unpredictability boundary is not fixed and depends on the capabilities (or possibilities) of the observer to suggest successful hypotheses. This has the consequence that randomness of the process $y(t)$ with respect to one model $z_1(t)$ does not contradict the fact that the process $y(t)$ itself at the same distance τ from the starting point t^0 will be determinate with respect to a different, better model $z_2(t)$. The corresponding "transition" of a process from a random into a determinate one is shown in Fig. 5, which shows the degrees of determinateness $D_1(\tau)$ and $D_2(t)$ corresponding to the models $z_1(t)$ and $z_2(t)$. Of course, such a transition is possible only if $\tau \lesssim \tau_{\lim}$.

Since no system of hypotheses can be complete (this is a modification of Gödel's assertion) it is always possible to shift, at least slightly, the boundary of "ignorance."[4)] In connection with the mobility of the boundary of "ignorance" we note that an investigator who has a powerful computer has a decided advantage in transforming random processes into determinate processes over an investigator who does not have a computer, since the computer facilitates sorting of different models, differing, for example, by the values of several parameters.

Unlike the complexity approach, which can bring out arguments regarding randomness only after all imaginable tests for the absence of laws have been performed, the concept of partial determinateness permits checking only some hypotheses, namely those that the experimentor has at his disposal (or those which he can check). This agrees with the practice of experimentation in natural science, in which processes are divided into random (unpredictable) and determinate (predictable) precisely based on checking of different hypotheses.

The theory of partial determinateness is not limited by the complexity of the model, as in the algorithmic approach: in those cases when a prediction is vital one can even use very complex algorithms made accessible by modern computers.

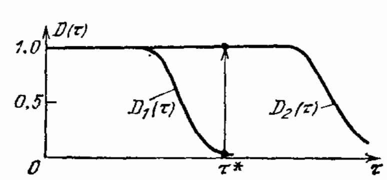

FIG. 5. The "randomness-determinateness" transition occurring as the model is refined.

As a result a process that appears to be random, "not conforming to law," for one investigator who does not have a computer may become determinate for an investigator who does have a powerful computer. In my opinion the role of computers in the aspect of the problem discussed here has not yet been fully acknowledged.

Of course, over times longer than the limiting time of determinateness τ_{lim} even the most powerful computer will be helpless: in this case any determinate forecasts are equivalent in the sense that no forecast can give reliable predictions, i.e., for all predictions $D \sim 0$.

In connection with the possibility of the transition "randomness $\rightarrow$ determinateness" I shall also discuss the question of "truly random" processes. It is obvious that the conventions regarding the division of processes into random and determinate can also be extended to judgments regarding "true randomness": the question of "true randomness" can be discussed only on the basis of some conventions, and in so doing it is desirable to have quantitative criteria for the concept "true."

This pertains to the set-theoretical approach, the algorithmic theory of probability, and the theory of partial determinateness discussed here. On the basis of this theory randomness of an observed process appears as a relative property manifested on comparison with the model process, and without indicating the class of predictive models it is impossible to distinguish a determinate (predictable) process from a random (unpredictable) process. Here one can talk about "true" or "real" randomness only conditionally, allowing for the process to cross the predictability horizon τ_{lim}: for $\tau > \tau_{\text{lim}}$ the behavior becomes completely unpredictable ($D \rightarrow 0$) on the set of all conceivable models.

The predictability viewpoint may turn out to be very constructive in attempts to define the concept of noise. It is easiest to formulate a definition of noise based on the converse: if the process admits prediction based on some algorithm, then it can be effectively subtracted out (compensated), and it will no longer be noise or interference. Therefore noise is a process for which we do not have a predictive model. In particular, dynamic chaos becomes noise when $\tau > \tau_{\text{det}}$, while for $\tau < \tau_{\text{det}}$ it is at least partially predictable and in this sense is no longer noise (of course, also partially).

5.7. Related questions

The idea of partial determinateness is relevant to a very wide range of questions in general physics. I shall mention some of them.

First, the naive idea of *a priori* separability of errors into random and systematic can be made more convincing if the systematic errors can be linked with imperfections of the model, hypothesis, or theory.

Second, by improving the predictive model $z(t)$ we solve the inverse problem of dynamics, and in so doing the condition for the degree of determinateness D to be maximum (or the predictability time τ_{det} to be maximum) can be employed as a criterion for matching of the observations $y(t)$ with the hypothesis $z(t)$.

Third, if the angular brackets $\langle yz \rangle$ are interpreted not as the average value of the product yz but rather the number of coincidences between the value of y and z within some range $\varepsilon/2$ of quantization of the data, then D will be the empirical probability that the difference $|y - z|$ does not exceed $\varepsilon/2$. This shows that the degree of determinateness can be given a probabilistic interpretation.

Fourth, recall the relation between the degree of determinateness [more precisely, the scalar product (y, z)] and the characteristic functional of the process y (this functional is expressed as $\Phi_y(z, t, t^0) = \langle \exp[i(y,z)] \rangle$). Two points are important here: the model processes $z(t)$ appear as trial functions (this is a new interpretation of trial functions), and the functional Φ_y exhibits singular behavior on the class of functions that approximate well the observed process $y(t)$.

6. EXAMPLES

6.1. Predictability in the case of a stretching piecewise-linear mapping

To illustrate the general propositions I shall present computational results demonstrating the exceptionally important role of noise and the no less important role of the model in the question of predictability of systems exhibiting complex behavior. Figures 7–10 below show the results of calculations of the degree of determinateness D and the correlation coefficient K_y, obtained in Ref. 21 for a system that is described by a one-dimensional, piecewise-linear mapping of the segment (0, 1) on itself (Fig. 6).

$$x_n = F(x_{n-1}) + f_{n-1}, \quad F(x) = \{Ax\}, \qquad y_n = x_n, \tag{6.1}$$

where n is the discrete time, $\{Ax\}$ is the fractional part of the number Ax, f_n is the fluctuation perturbation with a uniform probability density in the interval $10^{-p} \leqslant f \leqslant 10^{-p}$, and p is an index characterizing the strength of the fluctuations (in our example $2 < p < 6$). In this case I neglected the measurement noise ν, setting $y_n = x_n$, and for the model z I took the equation

$$z_n = F(z_{n-1}). \tag{6.2}$$

It follows from Fig. 7a, corresponding to the value $A = 2$ (see Fig. 6b), that for values of the index p in the interval from $p = 2$ to $p = 6$, i.e., for a noise level ranging from 10^{-6} to 10^{-2}, the degree of determinateness $D(\mu)$ is close to unity over time intervals $\mu = n - n^0$ of the order of 5–15 time steps, after which it drops quite rapidly to zero. The determinate behavior time μ_{det} depends strongly on the noise level and satisfies the logarithmic relations of the type (5.11) with $\lambda_1 = 1/A$. Since $\sigma_f = 10^{-p}/\sqrt{3}$, for $A = 2$ we have

$$\mu_{\text{det}} \sim \ln \frac{1}{\sigma_f^2} \propto p,$$

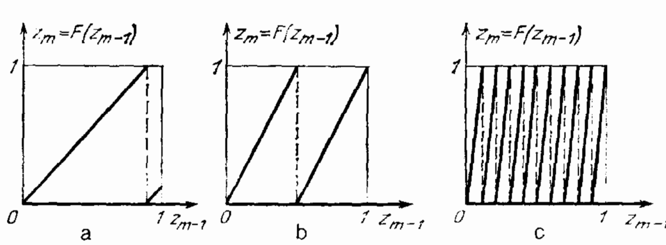

FIG. 6. The piecewise-linear mapping (6.1) with $A = 2$ (a), 1.1 (b), and 9.9 (c).

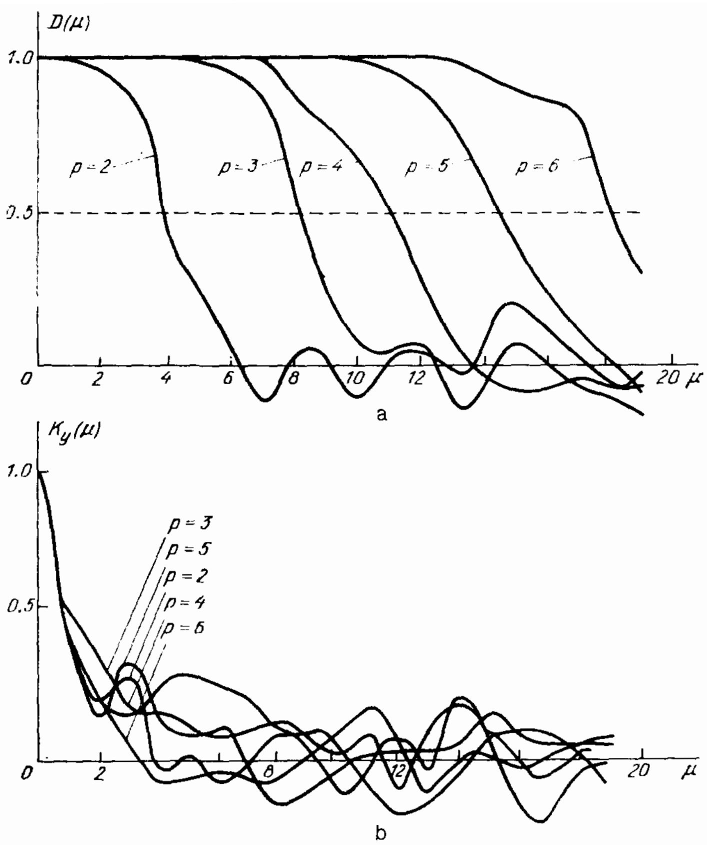

FIG. 7. Character of the dependence of the degree of determinateness (a) and the degree of coherence (b) on the discrete time μ for real values of the noise level ($p = 2$–6).

which agrees very well with the actual values of $\mu_{\rm det}$ extracted from Fig. 7a.[21] In particular, for $p = 6$ we obtain

$$\mu_{\rm det} \approx 16.$$

The correlation coefficient K_y drops to the level $\sim 1/2$ already at practically the first time step; in addition, K_y is insensitive to the noise level (Fig. 7b). This result indicates that there is a profound difference between the correlation time and the prediction time.

If in the model equation (6.2) the coefficient A is subjected to small variations ΔA, then the time $\mu_{\rm det}$ will be affected by the uncertainty of the model for $\Delta A \gtrsim \sigma_f$, since in this case

$$\mu_{\rm det} \sim \ln(\sigma_f^2 + \Delta A^2)^{-1}. \qquad (6.3)$$

Studying the dependence of $\mu_{\rm det}$ on the hypothetical value $\tilde{A} = A + \Delta A$ (Fig. 8) or, which in practice is more convenient, the dependence of the degree of determinateness D on A for fixed μ, two practically important problems can be solved: the parameter A in the physical system can be estimated, for example, by the value giving the maximum prediction time $\mu_{\rm det}$ and the noise level in the system σ_f can be determined from the width of the curve in Fig. 8. Both problems of this type always arise prominently in radar and sonar, but these methods are less often employed in physical studies of systems exhibiting complex behavior.

Changing the parameter A in the mapping (6.1) strongly affects the quality of prediction. Figures 9a and 10a show the dependences $D(\mu)$ for mappings with small ($A = 1.1$), average ($A = 2$), and large ($A = 9.9$) slope with $p = 2$ (Fig. 9a) and $p = 1$ (Fig. 10a). For $p = 1$ noise constitutes $\sim 10\%$ of the typical value $y \sim 1$, while for $p = 2$ it constitutes about 1%.

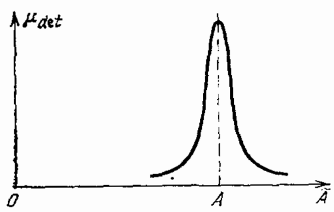

FIG. 8. The determinate-behavior time $\mu_{\rm det}$ as a function of the hypothetical parameter $\tilde{A}$ in a neighborhood of the true value A.

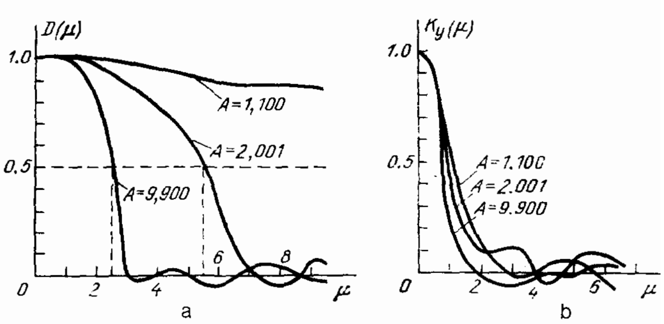

FIG. 9. Behavior of the degree of determinateness (a) and degree of coherence (b) for different values of the parameter A and for a moderate noise level ($p = 2$).

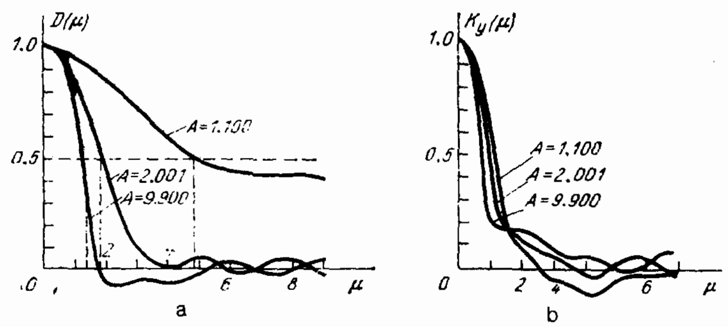

FIG. 10. Same as Fig. 9 but for a significant noise level $p = 1$.

These figures show how the predictability time increases as the slope A of the mapping (6.1) decreases. Thus for $A = 9.9$ and $p = 1$ the time $\mu_{\text{det}} = 1.4$, while for $A = 1.1$ it increases up to 4.7 (Fig. 10a). The time μ_{det} also increases similarly for $p = 2$ (Fig. 9a). Moreover it follows from Figs. 9b and 10b that on transferring from a small value $A = 1.1$ to a high value $A = 9.9$ and from a low noise level $\sigma_f \sim 10^{-2}$ to a high noise level $\sigma_f \sim 10^{-1}$ the correlation time remains virtually unchanged and equals approximately one time step.

Similar results can also be obtained for the mapping realized by the logistics curve $F(x) = rx(1-x)$. Here phenomena associated with "premature" onset of chaos owing to noise are interesting.[28,29]

6.2. A real generator with a tunnel diode

A generator with a tunnel diode in the oscillatory circuit for generating chaotic oscillations was proposed in Ref. 30. This generator produces trains of oscillations with increasing amplitude which are replaced each time by a damping process in the unblocking diode (Fig. 11). If the behavior of the generator is described by the mapping $y_n \to y_{n+1}$, where y_n is one of the sequential maxima of the signal, then in the (y_n, y_{n+1}) plane the experimental points $y_{n+1} = F(y_n)$ form characteristic configurations that are close to the theoretical configurations. A smooth polynomial approximation $\bar{F}(y)$, which later served as a theoretical prediction model $z_{n+1} = \bar{F}(z_n)$, was constructed in Ref. 31. The coefficients of the polynomials were determined from experimental data by the least-squares method.

The form of the experimental mappings is shown in Fig. 12: no external noise in Fig. 12a and with external noise in Fig. 12b. The degree of determinateness $D(\mu)$, shown in Fig. 13 by curve 1 (no noise) and curve 2 (with noise), was calculated according to the dynamic model and from the experimental data. As expected the introduction of noise reduced the prediction time μ_{det} (approximately from 10 to 5 discrete units).

The main value of the analysis performed in Ref. 31 lies in the fact that a method was tested in practice for solving the inverse problem of the theory of nonlinear oscillations using the idea of partial determinateness. An inverse problem of this type was studied by L. I. Gudzenko[32,33] who was studying regularity in the appearance of sun spots. He also depended on models, but he analyzed only the statistics of the experimental data themselves and not the joint observation-prediction pair statistics.

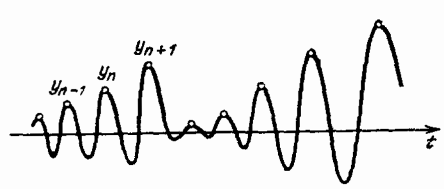

FIG. 11. The character of the process $y(t)$ for a generator with a tunnel diode in an oscillatory circuit.[30]

6.3. Prediction of low-dimension chaos based on similar events in the past

In the foregoing discussion I constantly emphasized that the predictability time τ_{det} can be much longer than the correlation time τ_c. This assertion can be made stronger by asserting that τ_{det} can also be longer than the prediction time based on autoregression models. This assertion was demonstrated successively by J. Farmer and J. Sidorovich[26], who compared for a number of systems (the logistics mapping, the Maki-Glass equation with retarded argument, Taylor-Couette flow, and Rayleigh-Bénard convection) the autoregression prediction $z_{\text{ar}}(t)$ with prediction based on similar events in the past.

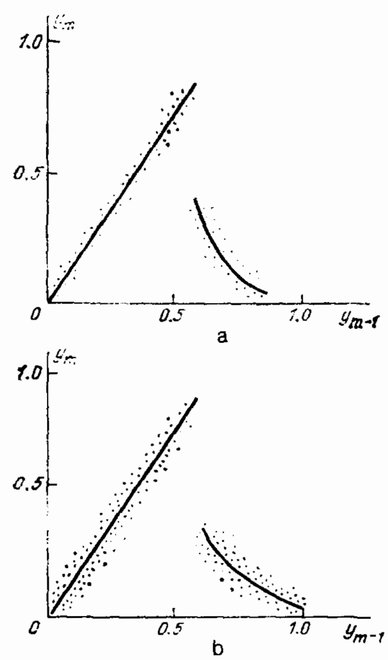

FIG. 12. The experimental (dots) and theoretical (lines) mapping for a generator with a tunnel diode.[31] The variance of the noise in b is approximately double that in a.

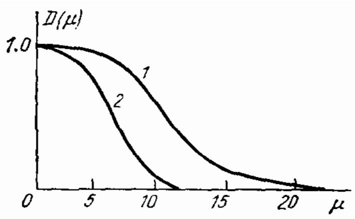

FIG. 13. The behavior of the degree of determinateness for a generator with a tunnel diode without noise (1) and with noise (2).

The prediction based on similar events in the past is based on finding preceding times t^* such that the value of $y(t)$ at $t = t^*$ equals the value of $y(t^0)$ at the initial moment of interest, while the behavior of $y(t)$ for $t < t^*$ is similar to that of the observed process for $t < t^0$. Then the value of the observed process $y(t)$ for $t > t^*$ in the past is taken as the prediction at the time $t > t^0$:

$$z(t^0 + \tau) = y(t^* + \tau). \tag{6.4}$$

In this prediction the behavior of the process $y(t)$ itself in the past is employed as a kind of analog computer. This approach is sometimes employed for long-term weather forecasting. The largest errors in prediction based on past behavior are associated with the fact that the behavior of the process $y(t^0 - \tau)$ prior to the moment of observation t^0 and of the process $y(t^* - \tau)$ prior to the reference time t^* by no means agree with one another as closely as they should. The implicit assumption that at the time t^* the system was "the same" as at the running time t^0 is another important source of error. As a rule this assumption is difficult to check, and it is yet another example of the application on faith of the forecasting principle "as it was, so it will be."[34]

As shown in Ref. 26 the maximum predictability time can be many times longer than the characteristic time $t_{\rm char}$, evaluated as the inverse of the frequency ω_p corresponding to the maximum of the energy spectrum. For a wide spectrum with bandwidth $\Delta\omega$ that is comparable to the "peak" frequency ω_p the characteristic time $t_{\rm char}$ is comparable to the correlation time τ_c. In Ref. 26 it was found that the typical ratio $\gamma = \tau_{\rm det}/t_{\rm char} = 5$–$10$, and tends to increase (sometimes up to a factor of 100 and more) as the dimension of the attractor decreases to $d = 2$; conversely, it tends to drop (to values $\gamma \sim 3$ and less) as the dimension increases to $d \sim 3$–10. Another valuable result obtained in Ref. 26 is the experimental confirmation of the fact that the error E grows exponentially with the time τ.

6.4. Degree of predictability in M. Born's example

In his time M. Born noted[35] that long-term unpredictability is inherent to not only quantum but also classical mechanics, though it has a different, nonquantum, nature associated with the unavoidable measurement errors. This important and, in general, old idea was illustrated in Ref. 35 by an example that elucidates very clearly the problem of unpredictability even in simple systems in which there is no local instability. The problem is that of a particle moving with a velocity v between two parallel walls $y = 0$ and $y = a$, alternately reflecting from them. If Δv is the error in determining the velocity, then after a time $\tau_a \sim a/\Delta v$ the uncertainty of the coordinate $\Delta y \sim \Delta v \cdot t$ will equal the distance a between the walls, so that for $t \gg \tau_a \sim a\Delta v$ the probability density for the particle coordinate will become uniform: $w(y) = \text{const} = 1/a$.

From the viewpoint of models the time $\tau_a \sim a/\Delta v$ must be regarded as a predictability horizon for the coordinate y of the particle. To calculate the degree of determinateness in this case the motion of the particle being studied $y(t)$ should be examined together with the motion of the model particle, whose velocity v_m is a parameter of the model and can differ from v by not more than Δv. Omitting the calculations and the obvious graph of the function $D(\tau)$ we note that for all its simplicity Born's example is a graphic model of many problems in radio physics and optics. Here are two examples of problems of this type. Over what time intervals does the uncertainty of the phase $\psi = \omega t$ becomes comparable (after subtraction of an integer number of 2π) with 2π? At what distances does it become impossible to determine the phase difference between two modes in a multimode waveguide?

6.5. The predictability of a quasisinusoidal signal with fluctuating phase

The problem of predicting a quasisinusoidal signal with a fluctuating phase leads to unexpected results. Neglecting the measurement noise and the amplitude fluctuations we write the signal generated by a self-oscillatory system of the Thompson type in the form $y(t) = \cos[\omega_0(t - t^0) + \varphi(t)]$, where $\varphi(t)$ is the random phase whose variance is characterized by the coefficient of diffusion D_φ: $\langle(\varphi(t) - \varphi(t^0))^2\rangle = D_\varphi \tau$. If the strictly sinusoidal signal $z(t) = \cos[(\omega_0 + \Omega)(t - t^0) + \varphi(t^0)]$ with a slightly different frequency but with the same starting phase $\varphi(t^0)$ is taken as the prediction for $y(t)$ and the combined average (5.12) is taken as the comparison criterion, then the degree of determinateness is given by

$$D(\tau) = \frac{D_\varphi 1 - e^{-D_\varphi \tau}\cos\Omega\tau + \Omega e^{-D_\varphi \tau}\sin\Omega\tau}{(D_\varphi^2 + \Omega^2)\tau}. \tag{6.5}$$

This expression is a good illustration of the relation between the fluctuation factors (in this case this is the coefficient of diffusion of the phase D_φ) and the model parameter Ω. The predictability time is estimated from (6.5) as

$$\tau_{\rm det} \sim \min\left(\frac{1}{\Omega}, \frac{1}{D_\varphi}\right) \sim \frac{1}{(D_\varphi^2 + \Omega^2)^{1/2}}. \tag{6.6}$$

To increase this time we can reduced Ω, i.e., we can refine the reference frequency. Obviously, for $\Omega \lesssim D_\varphi$ it becomes impossible to increase $\tau_{\rm det}$ further, so that the quantity $\tau_{\rm det} \sim 1/D_\varphi$ plays here the role of the predictability horizon. This time equals the correlation time $\tau_c \sim 1/D_\varphi$ of the complex envelope Y, since in this case $K_y(\tau) = \exp(-D_\varphi \tau)$. This problem has the characteristic feature that in the general case ($\Omega \gtrsim D_\varphi$) the predictability time does not exceed the correlation time ($\tau_{\rm det} \leqslant \tau_c$). This occurs because in a Thompson generator the correlations decay exclusively owing to noise, and not owing to the complex form of the oscillations, as in the case of dynamic chaos. Here the predictability horizon cannot be moved by improving the model, since the random fluctuations of the phase owing to thermal and shot noise do not admit deterministic modeling. This is one of the facts suggesting the following thought: even "the most chaotic" chaos leaves hope of obtaining a prediction over times longer than the correlation time, while the usual Thompson generator hardly permits reaching τ_c.

In this (and only in this!) sense the predictability of sinusoidal oscillations is worse than that of chaotic oscillations (of course, the achieved predictability time for chaos is much shorter in absolute magnitude than for generators of sinusoidal oscillations).

7. PARTIALLY DETERMINATE FIELDS

7.1. Degree of determinateness of scalar and vector fields

The idea of partially determinate processes can be extended naturally to fields, including wave fields[22,23]. In this case the observed field $Y(\mathbf{r}, t)$, which depends on three spatial coordinates and the time t, must be compared with the model field $Z(\mathbf{r}, t)$ in some space-time region.

In addition to the uncertainty factors already indicated above for partially determinate processes (fluctuation perturbations in the real process x; measurement noise, causing the observation y to differ from x; uncertainty of the idealized model equation for z), in the case of fields the errors associated with the fact that the observational data Y are obtained from a finite number of points $\mathbf{r}_k$, $k = 1,2,...,K$, must also be taken into account. As a result of this the starting data for intermediate points must be calculated with the help of some interpolation procedure, which, naturally, additionally degrades the predictability. For a scalar field $Y(\mathbf{r}, t)$ the degree of determinateness relative to the model field $Z(\mathbf{r}, t)$ is best introduced with the help of the relation

$$D_Q(\tau) = \frac{\{Y(\mathbf{r}, t)\, Z(\mathbf{r}, t)\}}{(\{Y^2(\mathbf{r}, t)\}\{Z^2(\mathbf{r}, t)\})^{1/2}}, \tag{7.1}$$

where the braces indicates simultaneous space-time and statistical averaging.[22] If $\mathbf{Y}(\mathbf{r}, t)$ and $\mathbf{Z}(\mathbf{r}, t)$ are vector fields, then the product $\mathbf{Y}(\mathbf{r}, t)\, \mathbf{Z}(\mathbf{r}, t)$ in (7.1) must be regarded as a scalar product. If the fields are complex, then the complex conjugate of the model field Z must be employed, as done in expressions for the complex coherence function.

The arguments regarding the finiteness of the determinate behavior time can obviously be extended now to spatial regions: three-, two-, or one-dimensional. The idea of partially determinate fields could be useful for many problems in hydrodynamics, acoustics, optics, plasma physics, etc. We shall examine some examples.

7.2. Spatial predictability of speckle-inhomogeneous fields

Speckle-inhomogeneous fields are formed as a result of the superposition of many waves and are characterized by a high degree of nonuniformity of the interference pattern. The scale of variation (correlation radius) of this pattern l_c is related with the width of the angular spectrum $\Delta\theta$ by the relation $l_c \sim \lambda/\Delta\theta$. The term "speckle fields" ("granular" fields) itself originated in optics, though complex interference fields of the same type are encountered in radiophysics and in acoustics, for example, in multimode oceanic waveguides.

It turns out that the random (in the sense of decaying correlations) character of speckle fields does not preclude predictability over distances greater than the correlation radius l_c. Such a prediction can be made in two stages[23]: first the sources of the field are determined from the speckle pattern measured on a finite aperture $-l_a/2 < x < l_a/2$ (assumed to be one-dimensional to simplify the discussions) and then the field $Z(\mathbf{r})$, which serves as the prediction outside the aperture, is reconstructed from the sources found. The possibility of reconstruction (extrapolation) of the field outside the aperture $|x| > l_a/2$ is itself based on the fact that dynamic equations (the wave equation or Maxwell's equations) and the measured values of the field on the aperture are available.

Of course, the size of the aperture l_a limits the possibility of resolving the sources, and for this reason the predictability (determinateness) length l_{det} usually does not exceed the aperture size l_a, but at the same time it is obviously larger than the correlation radius l_c. Both these features are reflected in Fig. 14, which shows the intensity of the starting speckle-inhomogeneous field $I(x)$ and the degree of determinateness $D(x)$, characterizing the quality of the reconstruction of the field outside the aperture.

These constructions make it tempting to use the reconstructed field to increase the resolution of the aperture. This possibility is, however, unrealizable in reality, since the reconstructed field does not contain any additional information aside from that existing on the starting aperture.

7.3. Partial predictability of turbulent flows. Region of compatibility of dynamic and kinetic descriptions

After the foregoing discussion the idea that the behavior of a turbulent flow can in principle be predicted will no longer seem strange. Of course, to make a prediction one must first acquire the necessary initial and boundary data on the flow with the help of a system of sensors, and the prediction itself extends only over limited spatial and time intervals, which, however, are larger than the spatial and time correlation radii of the turbulent field.

For anyone who is accustomed to stochastic models of turbulent flows such a prediction may seem unacceptable for purely psychological reasons: is it really possible to predict phenomena for which kinetic equations have been devised, equations for weak or strong turbulence, etc.? And yet, a determinate (or more precisely, partially determinate) description of turbulent fields based on equations of hydrodynamic type with corresponding initial and boundary data do not at all contradict the kinetic description, since they are compatible in some space-time region.[36]

In the time domain the region of compatability is the interval from the correlation time τ_c up to the predictability time τ_{det}:

$$\tau_c \leqslant \tau \leqslant \tau_{det}. \tag{7.2}$$

In this interval the correlations between the values of the

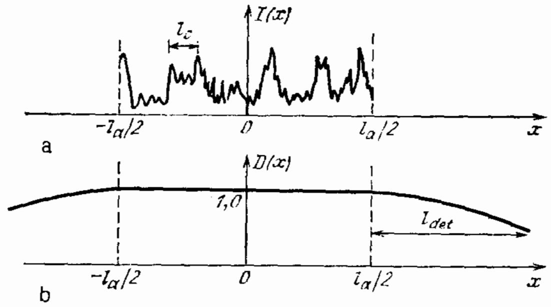

FIG. 14. The character of fluctuations of the amplitude of a speckle-nonuniform field on an aperture ($-l_a/2 < x < l_a/2$) (a) and the degree of determinateness $D(x)$ (b).

observed field have decayed (decoupled) to such an extent that the kinetic description is now applicable, but the determinate description has still not lost any force, since prediction and observation are still correlated. The compatibility of the two descriptions should not be surprising, because here one is dealing with two different methods for describing the same phenomenon: if the purpose of the determinate description is to predict the instantaneous and local values of the fields, which it is possible to do within the region of determinateness, then the kinetic approach is from the very beginning oriented toward determining the average characteristics.

7.4. Determination of nonuniformities in a waveguide from the interference pattern

As is well known, in multimode waveguides the field has a complicated, randomly-similar interference structure. Is it possible to determine based on this interference structure whether or not random nonuniformities are present in the waveguide?

Numerous calculations, performed predominantly for sound waves in oceanic waveguides, show that the spatial correlation functions of the field are virtually insensitive to nonuniformities of the medium. On the other hand the degree of determinateness, which depends on detailed agreement between the observed and model (computed) fields, is extremely sensitive to the presence of nonuniformities.[37] It should be noted that there is an asymmetry in the results for $D \neq 1$ and $D < 1$. If the equality $D = 1$ indicates absolutely that there are no nonuniformities, then for $D < 1$ any difference in the interference patterns $Y(\zeta)$ and $Z(\zeta)$ (ζ is the coordinate transverse to the axis of the waveguide) can indicate both the existence of nonuniformities and the existence of some other factors that have been ignored in the model (for example, unevenness of the walls).

7.5. Weather prediction. Value of a forecast

Weather forecasts are probably encountered more often than any other dynamic predictions. For the questions discussed in this paper a weather forecast is important in that it reflects all the most characteristic features of a physical prediction.

7.5.1. Existence of a predictability horizon

It was discovered in meteorology earlier than in other natural sciences that long-range predictions have a limit determined by the local instability of dynamic processes in the atmosphere. Strange attractors, a detailed analysis of which led to an understanding of the existence of fundamental limits of predictability, were first discovered in meteorology (E. N. Lorenz, 1963).[38]

7.5.2. Advantages of a dynamic prediction over a statistic prediction

The fundamental advantages of a dynamic prediction over statistic methods, based on a rectilinear correlation principle of the type $y(t^0+\tau) \approx y(t^0)$ ("tomorrow will be the same as today") or on more refined evaluations of the autoregression type, mentioned in Sec. 6.4, were clearly revealed in the example of weather prediction. Obviously the linear couplings of the type $<y(t)y(t-\tau)$ poorly reflect deep hydrodynamic processes, an adequate description of which is hardly possible on any basis other than a dynamic description with appropriate "input" information. In recent publications by the leaders of modern meteorology E. N. Lorenz[39] and J. Mason and R. S. Treas[40] the possibilities of dynamic prediction for Europe and North America are characterized by reliable predictions of up to seven to ten days[5)] while predictions based on the principle "tomorrow will be the same as today" hardly extend to two–three days. The two–three day interval is linked with the fact that for scales of the order of several hundreds of kilometers the doubling time of the perturbations under typical conditions in the earth's atmosphere equals precisely two-three days.

7.5.3. What actually limits the accuracy of a forecast?

Each of the three interfering factors—fluctuation perturbations, measurement noise, and defects in the model—can make a perceptible contribution to the limit of accuracy of a forecast. However the relative importance of separate factors decreases with time; this is characteristic for other physical phenomena also.

At the present time the most significant errors are associated with the uncertainty of the input data: the network of weather stations is too sparse, and systems for collecting information and feeding it into a computer are still too inefficient. One would think that as the number of ground stations increases, especially in regions outside of Europe and North America, the network of marine stations on ships in the ocean is expanded, and measurements from space are improved, the quality of forecasts will improve, and the errors associated with defects in the dynamic model will become predominant. Some significant failures in forecasts ("malfunctions") can already now be attributed to the fact that separate factors, most often moisture transfer, were neglected or not taken into account correctly in the dynamic equations. By expanding the measurement network and at the same time improving the model, meteorologists are taking into account atmospheric eddy motions on increasingly smaller scales, gradually transferring these motions from the category of fluctuations (from the viewpoint of the present level of description) into the category of the degrees of freedom taken into account in the model.

As increasingly smaller scales are incorporated the forecasts will become more accurate. However, will the "physical" predictability horizon, where the perturbations introduced by the sensors will start to have an appreciable affect on the forecast, be reached? The answer is obviously no, because limitations of an economic and technical character will appear much sooner.

7.5.4. Relation between the accuracy and cost of a forecast

It is obvious that expansion of the network of observation stations, gathering data from high masts and weather balloons, creation of telemetric networks, including the use of satellites, will all increase the accuracy of a forecast and at the same time forecasts will become more expensive. Assuming that the cost of a forecast increases in a power-law fashion we can write a symbolic equation of uncertainty of the type

$$VE^{\alpha} \sim \text{const}, \quad \alpha > 1, \tag{7.3}$$

where V is the cost of a forecast and E is the relative mean-square error. In the opinion of specialists, who take such limitations into account, the period of a forecast, achievable in our century with a quality sufficient for supporting civilian aircraft flights, will hardly reach 15 days. In any case predictability over 30 days is thought to be unfeasable. It is true that in individual cases a period of 15–20 days is achievable even today, but such cases are exceptions: they refer to periods with especially stable weather during the summer and winter. The forecast period can also be increased when only the *character* of the weather and not the exact values of the pressure and temperature is predicted, i.e., a more or less wide "corridor" of the parameters is indicated.

In concluding this section we note that uncertainty relations of the "cost-accuracy" type (7.3) appear in virtually any physical experiment: to reduce the measurement error we must inevitably spend more money.

8. THE NATURE OF RANDOMNESS

8.1. Hierarchy of models, levels of description, and degrees of determinateness

In the phrase "the nature of randomness" it is hardly possible to interpret randomness other than as unpredictability, so that the question of the nature of randomness essentially reduces to the question of how well our predictive models characterize the behavior of processes occurring in nature.

One must first stipulate the degree to which a problem is to be analyzed. This is important, since the same phenomenon can demonstrate completely determinate behavior on the basis of, say, the hydrodynamic description and be completely indeterminate at the molecular level. A determinate description of macrosystems obviously cannot be achieved at the molecular level because it is impossible to fix the starting state and record the change in the state of the system of molecules in time and space. There are even fewer possibilities of realizing a determinate description at the quantum-mechanical level. It would be tempting to introduce, by analogy to (7.1), the degree of determinateness for the wave function $\psi(\mathbf{r}_1,...,\mathbf{r}_N, t)$ of a system of N particles. However it would be difficult to give a physical content to this concept, since it is virtually impossible to prepare the starting (quantum) state ψ^0 or control the further behavior of the ψ function (it is premature to discuss the possibilities associated with hidden variables). For this reason, one can talk about deterministic predictability only starting with a definite level, namely, the level of effects recorded by macroscopic devices. Such effects admit either a classical description or a quantum-mechanical probabilistic description. Further we can adhere to different levels of description even within the framework of classical physics: this depends on the number of degrees of freedom taken into account, on the number of known participants in the interaction, and on the range of spatial and time scales taken into account. Thus there appears a *hierarchy of models*, differing by the degree of detail of the phenomena studied. There also arises a corresponding hierarchy of degrees of determinateness: $D_1 > D_2 > D_3 ...$, and the predictability is all the better ($D_j > D_{j+1}$) the larger the details the model operates with. Thus in the problem of weather forecasting studied above predictability for the smallest scales studied (tens of kilometers) is always worse than for scales of the order of 1000 km.

From the viewpoint of the indicated hierarchical structure of predictive models the nature of randomness must be sought in the factors that introduce the largest errors into the forecast claiming to give the *most detailed possible description*. In concrete cases these factors can belong to any of three basic classes of forecasting errors: fluctuation forces, measurement noise, and uncertainties of the model. This means that in studying the question of the nature of randomness the hypothesis of uncertainty and incompleteness of the model can be regarded on the same level as the assumptions about the action of fluctuation forces and measurement noise. In other words, ignorance emerges as a fully objective component of the nature of randomness.

8.2. Fundamental and practical limits of predictability

It is obvious (we started from this, introducing quantitatively the predictability horizon in Sec. 5.2) that as the model is refined and the measurement errors are reduced fluctuation forces become the most important and, moreover irremovable reason for long-term unpredictability. Ultimately these are thermal and quantum-mechanical fluctuations, which cannot be completely avoided under any circumstances, as well as different external fields (electromagnetic, gravitational, neutrino, etc.), from which the system of interest cannot be completely shielded. From the viewpoint of a classical description these fluctuations must be regarded as a constant source of perturbations for classical trajectories: they seemingly "scramble" the phase space. If the classical system is locally unstable (and this is true for most more or less complex systems), then it becomes a gigantic amplifier of thermal, quantum, and other fluctuations, i.e., essentially a generator of randomness. The combination of the two indicated factors—the presence of unavoidable microfluctuations on the one hand and local instability of macrosystems on the other—must evidently be regarded as a fundamental reason for randomness in our universe. Of course, if the resolution of the question of the nature of randomness is returned back to the practical level, then for most systems the limits of predictability will still be reached on the macroscopic level. Aside from real noise and real "ignorance noise" the perturbations introduced into the object under study by measuring devices can also play an important role here. For locally unstable systems the fundamental role of this restriction is yet to be studied. Does the connection of a sensor to an electric circuit or the insertion of an anemometer into a fluid flow not leave any trace? After all measurement devices can have a much stronger effect on the course of locally unstable processes under study than the flapping of the wings of a butterfly on the other side of the world from us (see Ref. 43 for a discussion of some important considerations).

9. ARS CONJECTANDI

The arguments presented here are an attempt to formalize the very nontrivial "relations" between observation and a predictive model on the basis of the concept of partial determinateness, which identifies unpredictability with randomness. This concept is convenient, intuitively acceptable, and de facto widely employed in natural science in the interpretation of experiments. Quantitative criteria which the theory of partial determinateness poses can be easily realized in practice, for example, with the help of correlators and in one form or another have already been employed in physics and allied areas of science.

The significant dependence of the accuracy of prediction on the quality of a hypothetical model of a phenomenon corresponds well with the version of the theory of probability that J. Bernoulli called "ars conjectandi"—"the art of conjecture."[41]

Dynamic prediction has greater possibilities than stochastic prediction,[6)] since the latter is oriented from the outset toward describing averaged and coarsened characteristics (probability distribution functions, statistical moments). However a dynamic forecast depends tremendously on a successful model. It is here that *ars conjectandi* comes into play.

There is no doubt that the idea of partial determinateness will also be useful in the solution of many other important problems not mentioned above, such as the problems of vision, associative memory, image recognition, directedness of evolution, self-organization of complex systems, and artificial intelligence.

In conclusion I am deeply grateful to V. L. Ginzburg and the participants of his seminar for their interest in the problem of predictability, V. I. Tatarskiĭ and V. S. Étkin for numerous useful discussions of the questions touched upon here, L. Sh. Il'kova for assistance in performing the calculations, and A. S. Chirkin for valuable critical remarks.

[1)]The history of the question and an exposition of the basic concepts of chaotic dynamics can be found in the recently published, excellent books of Refs. 1 and 2.

[2)]The re-examination of previous ideas regarding "predictable dynamics" was made largely owing to the works of V. I. Arnol'd, G. M. Zaglavskiĭ, A. N. Kolmogorov, E. N. Lorenz, D. Rouelle, Ya. G. Sianĭ, S. Smale, F. Takens, and B. V. Chirikov.

[3)]As far as I know the interesting question of the perturbing effect of measurement devices on the course of chaotic processes in classical physics has not yet been subjected to serious study.

[4)]At the time I was writing this paper the important idea of a link between randomness and incompleteness of any system of hypotheses appeared new to me. With some regret (and at the same time with satisfaction) I recently learned that this idea has already been used as a basis for a very serious study (Ref. 42).

[5)]The predictable quantity is usually taken as the pressure at a height of 500 *m* above land, and the forecast is regarded as correct if the deviation of the pressure from the predicted value does not exceed 50 mbar. Other conditions are given in Ref. 40 and in the literature cited there.

[6)]Ironically the meaning of the word "stochastic" in current usage is essentially opposite to its original meaning in Greek: $\sigma\tau o\chi\alpha\sigma\mu o\zeta$—guessable.

[1]P. S. Landa and Yu. I. Neĭmark, *Stochastic and Chaotic Oscillations* [in Russian], Nauka, M., 1987.
[2]H. G. Schuster, Deterministic Chaos, VCH Publishers, N.Y., 1984; 2nd ed. 1987. [Russ. transl., Nauka, M., 1988.
[3]G. M. Zaslavskiĭ and B. V. Chirikov, Usp. Fiz. Nauk **105**, 3 (1971) [Sov. Phys. Usp. **14**, 549 (1972)]; see also B. V. Chirikov, Phys. Rep. **56**, 263 (1979).
[4]M. I. Rabinovich, Usp. Fiz. Nauk **125**, 123 (1978) [Sov. Phys. Usp. **21**, 443 (1978)]
[5]A. V. Gaponov-Grekhov and M. I. Rabinovich, Usp. Fiz. Nauk 128, 579 (1979) [Sov. Phys. Usp. **22**, 590 (1979)]
[6]A. S. Monin, Usp. Fiz. Nauk **125**, 97 (1978); 150, 61 (1986) [Sov. Phys. Usp. **21**, 429 (1978); 29, **843** (1986)]
[7]A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion*, Springer-Verlag, N.Y., 1983 [Russ. transl., Mir. M., 1984.]
[8]G. M. Zaslavskiĭ, *Stochastic Dynamic Systems* [in Russian], Nauka, M., 1984.
[9]a) J. M. T. Thomson and H. B. Stewart, *Nonlinear Dynamics and Chaos*, Wiley, N. Y., 1986; b) A. B. Holden, *Chaos*, Princeton University Press, Princeton, 1986.
[10]F. C. Moon, *Chaotic Vibrations*, Wiley, N. Y., 1987.
[11]J. Lighthill, Proc. R. Soc., London A **407**, 35 (1986).
[12]A. N. Kolmogorov, Probl. Peredachi Inf. 1, 1 (1965) [Probl. Inf. Transm. 1965)]; see also: A. N. Kolmogorov, *Theory of Probability and Mathematical Statistics* [in Russian], Nauka, M., 1983, p. 267.
[13]Yu. V. Prokhorov and Yu. A. Rozanov, *Probability Theory*, Springer, Berlin, 1969 [Russ. original, Nauka, M., 1967 and 1973.]
[14]V. N. Tutubalin, *Theory of Probability* [in Russian], Moscow State University Press, M., 1972.
[15]A. N. Kolmogorov, Probl. Peredachi Inf. 5, 3 (1969) [Probl. Inf. Transm. (1969)]; see also: A. N. Kolmogorov, *Information Theory and the Theory of Algorithms* [in Russian], Nauka, M., 1987, p. 232.
[16]A. Kh. Shen', *Semiotics and Information Theory* [in Russian], VINITI, USSR Academy of Sciences, M., 1982, No. 18, p. 232.
[17]A. V. Gaponov-Grakhov and M. I. Rabinovich, in: *20th Century Physics: Results and Prospects* [in Russian], Nauka, M., 1984, p. 320.
[18]Yu. A. Kravtsov and V. S. Étkin, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 24, 992 (1981) [Radiophys. Quantum Electron. 24, 679 (1981)].
[19]Yu. A. Kravtsov and V. S. Etkin, in: *Nonlinear and Turbulent Processes in Physics*, (Ed.) R. Z. Sagdeev, Harwood Acad. Publ., N. Y., 1984, Vol. 3, p. 1411.
[20]Yu. A. Kravtsov and V. S. Étkin, Radiotekh, Elektron. **29**, 2358 (1984) [Sov. J. Commun. Technol. Electron. **30**, 1 (1985)]
[21]L. Sh. Il'kova, Yu. A. Kravtsov, O. S. Mergelyan, and V. S. Étkin, Izv. Vyssh. Uchebn. Zaved., Radiofiz. 28, 1035 (1985). [Radiophys. Quantum. Electron. 28, (1985)]
[22]Yu. A. Kravtsov and V. G. Petnikov, Dokl. Akad. Nauk SSSR 285, 871 (1985) [Sov. Phys. Dokl. **30**, 1039 (1985)].
[23]Yu. A. Kravtsov and V. G. Petnikov, J. Opt. Soc. Am. A **4**, 11 (1987).
[24]R. Shaw, Z. Naturforsch. A 36, 80 (1981).
[25]J. P. Crutchfield and N. H. Packard, Int. J. Theor. Phys. 21, 433 (1982).
[26]J. D. Farmer and J. Sidorovich, Phys. Rev. Lett. 59, 845 (1987).
[27]J. Ford, Phys. Today **36** (4), 40 (1983).
[28]B. Shraiman, C. E. Wayne, and P. C. Martin, Phys. Rev. Lett. **46**, 935 (1981).
[29]J. P. Crutchfield, J. D. Farmer, and B. A. Huberman, Phys. Rep. **92**, 45 (1982).
[30]S. V. Kiyashko, A. S. Pikovskiĭ, and A. S. Rabinovich, Radiotekh. Elektron. **25**, 336 (1980) [Radio Eng. Electron. (USSR) (1980)].
[31]L. Sh. Il'kova, Yu. A. Kravtsov, and A. S. Pikovskiĭ, Kr. Soobshch. Fiz. No. 11, 7 (1986) [Sov. Phys. Lebedev Inst. Rep. No. 11, 7 (1986)].
[32]L. I. Gudzenko, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **5**, 572 (1962) [Sov. Radiophys. 5, (1962)].
[33]L. I. Gudzenko and V. E. Chertoprud, ibid. **10**, 353, 363 (1967). [Radiophys. Quantum Electron. 10, (1967)].
[34]Yu. I. Alimov, *Alternatives to the Method of Mathematical Statistics* [in Russian], Znanie, M., 1980.
[35]M. Born, *Continuity, Determinism, and Reality*, Dan. Mat. Fys. Medd. **30**, No. 2, p. 1 (1955) [Russ. transl. Nauka, M., 1977].
[36]Yu. A. Kravtsov, 1987 International Conference on Plasma Physics, Kiev, April 6–12, 1987; Proceedings of Contributed Papers, Kiev (1987), Vol. 2, p. 161.
[37]Yu. A. Kravtsov, B. A. Niyazov, and V. G. Petnikov, Akust. Zh. **32**, 823 (1986) [Sov. Phys. Acoust. **32**, 512 (1986)].
[38]E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[39]E. N. Lorenz, Global Atm. Research Program Spec. Rep., Vol. 3, No. 43, 1 (1985).
[40]J. Mason and R. S. Treas, Proc. R. Soc. London A **407**, 51 (1986).
[41]J. Bernoulli, *On The Law of Large Numbers* [Russ. transl., Nauka, M., 1986], Part 4.
[42]H. De Long, *Randomness and Gödel's Incompleteness Theorem*, Addison-Wesley, Reading, 1987.
[43]B. V. Chirikov, *Nature of statistical laws of classical physics*, (In Russian) Preprint 78-66, Institute of Nuclear Physics, Siberian Branch of the USSR Academy of Sciences, Novosibirsk, 1978; Dynamical Mechanisms of Noise, Preprint 87-123, Institute of Nuclear Physics, Siberian Branch of the USSR Academy of Sciences, Novosibirsk, 1987.

Translated by M. E. Alferieff