Spin glasses and nonergodicity

I. Ya. Korenblit and E. F. Shender

B. P. Konstantinov Institute of Nuclear Physics, Academy of Sciences of the USSR, Leningrad Usp. Fiz. Nauk 157, 267–310 (February 1989)

This review is devoted to spin glasses, i.e., disordered magnets with randomly distributed ferroand antiferromagnetic exchange interactions. Extensive experimental data and the theory developed for a simple model with an infinite exchange interaction range show that, as the temperature is reduced, the competition between different types of disorder leads to a nonergodic (or quasi-nonergodic) state in spin glasses. The fundamentals of the statistical mechanics of nonergodic systems are established. The review also examines the connection between the problem of spin glasses and those of combinatorial optimization, as well as the use of spin-glass models in biology. Attempts to construct a theory of real spin glasses with a distance-dependent exchange interactions are discussed.

TABLE OF CONTENTS

1.	Introduction 1
2.	Basic experimental results 2
	2.1. Statistical properties. 2.2. Dynamic effects. 2.3. "Aging" of spin glasses.
	2.4. Spin glass and magnetic long-range order.
3.	Infinite-range model
4.	Parisi order parameter 8
5.	The Parisi anzatz. Solution near the transition temperature
6.	The equations of state of a spin glass at arbitrary temperature. Low-tempera-
	ture properties10
7.	Vector spin glass11
	7.1. Isotropic magnet. 7.2. Anisotropic glass. 7.3. Effect of random anisotro-
	py. 7.4. Experimental study of the H, T diagram
8.	Nonergodic magnetically-ordered systems13
9.	Ultrametric topology of the space of states. Non-self-averaging of the order
	parameter in spin glasses15
10.	Spin glasses with distance-dependent exchange interaction17
	10.1. Spatial dispersion in the Parisi scheme. 10.2. Other analytic approaches.
	10.3. Numerical simulation. Heuristic models.
11.	Spin-glass models in combinatorial optimization problems and in biology21
12.	Conclusion
Ref	erences

1. INTRODUCTION

Spin glasses are disordered magnets in which the exchange interaction energy varies randomly in magnitude and in sign. In contrast to ordinary magnets, as the temperature is reduced, long-range magnetic order does not arise in such systems with competing interactions. On the other hand, the slow gradual freezing of spins does not occur either. Below a certain experimentally well-known temperature, the magnet assumes a new state that has no analogs among ordered systems. The characteristic property of this state is its exceedingly slow relaxation. In any case, typical times for establishing equilibrium exceed 10^4-10^5 s. Phenomena associated with the irreversibility of statistical properties are observed at the same time.

Since the exchange interaction energy is very dependent on the disposition of magnetic and nonmagnetic atoms, most materials that can be classified as disordered magnets are spin glasses. An enormous number of spin glasses is now known. They include metals, dielectrics, semiconductors, dilute alloys (i.e., alloys with a low concentration of magnetic atoms) and concentrated alloys and crystalline and amorphous materials (see below for further details, and also the review articles in Refs. 1–4). Similar phenomena are observed in other systems (both magnetic and nonmagnetic), in which there are competing interactions. They include dielectrics containing dopants with electric dipole moments,⁵ mixtures of ferro- and antiferroelectrics,^{6,8} dipolar magnets,⁹ and disordered superconductors with Josephson junctions.^{10,11}

The first attempt to construct a theory of spin glasses was made by Edwards and Anderson¹² who assumed that a reduction in temperature was accompanied by a transition to a state with a random molecular-field distribution that uniquely determined the orientation of each spin. The quantity $(q = \overline{\langle S_r \rangle_T^2})$ plays the role of the order parameter in this theory (S_r) is the spin vector at the point **r**, the angle brackets represent thermodynamic averaging, and the bar represents averaging over the positions of the magnetic atoms). The Edward-Anderson theory explains the break on the temperature dependence of susceptibility, first reported by Cannella and Mydosh,¹³ but cannot explain the phenomena of irreversibility and long-term relaxation. A need therefore arose for a consistent theory, free from any *a priori* assumptions about the nature of the ground state.

A theory of this kind has been constructed for a model with an infinite interaction range (the interaction energy is a random quantity that does not depend on the separation between the spins). This theory plays the same role for spin glasses as the molecular-field theory for magnetically ordered media.

The properties of spin glass that could be described by the infinite-range model turned out to be unexpected and unlike much that had been encountered in solid state physics before. A reduction in temperature was found to be accompanied in this model by a phase transition from the paramagnetic or magnetically ordered state to a phase that could be called nonergodic. The ground state of this phase is degenerate, i.e., there is an infinite number of sets of spin orientations that correspond to the same energy in the macroscopic limit. These sets differ by the overturning of an infinite number of spins, so that the barriers between different realizations of the ground state are also infinite, i.e., averaging over time and over the ensemble must in general lead to different results. The properties of this type of nonergodic state are described not by one, but by an infinite number of order parameters, i.e., a function.

The degeneracy of the ground state is due to the fact that, in a magnet with competing exchange interactions, there are unavoidable frustrations (unsatisfied couplings), i.e., pairs of spins whose interaction energy is not a minimum. The ground state of the system has a complex and unusual hierarchical structure that is probably typical of all random systems in which imposed couplings cannot be satisfied simultaneously. It is therefore not surprising that the ideas employed in molecular-field theories and the results obtained thereby have been useful in biology, the theory of combinatorial optimization, and so on.¹⁴

The molecular-field theory provides a good qualitative decription of the static properties of real spin glasses, but the magnitude of the lowest critical dimension which, as usual in the theory of phase transitions, governs the validity of the infinite-range model for real media, remains an open question.

Studies of spin glasses have been advancing rapidly in the last 10–12 years. Up to 20% of all the papers given at international conferences on magnetism have been devoted to topics that are in some way related to spin glasses. Although the theory of real glasses with finite interaction range has not been constructed, so that the spin glass problem is still far from being solved, a picture has now emerged of the spin glass as a fundamentally new physical state.

The number of papers devoted to spin glasses is enormous. It will not be our aim in this review to cover all questions relating to this problem. Our task will be to elucidate the basic experimental results and new theoretical ideas, especially the foundations of the theoretical description of nonergodic states of random systems.

The review is constructed as follows. Section 2 presents the basic experimental properties that characterize spin glasses as a special class of magnetic materials. In Secs. 3–6, 9, and 11 we formulate the infinite-range model, proposed by Sherrington and Kirkpatrick for Ising spin glasses, and consider the properties of the nonergodic state that arises in this model. We also show how the methods developed by study-

140 Sov. Phys. Usp. 32 (2), February 1989

ing this model can be used to investigate optimization problems in biology. Readers interested in general problems in statistical physics and in applications that are not directly related to the physics of disordered systems may confine their attention to these sections.

In Secs. 7 and 8, the Sherrington-Kirkpatrick model is generalized to more complicated systems such as vector spin glasses and disordered antiferromagnets, and the results obtained in this way are used to interpret the statistical properties of disordered magnets.

Finally, Sec. 10 is devoted to attempts to extend the theory beyond the infinite-range model. In particular, heuristic models of the spin glass based on numerical simulations are discussed. The question of a phase transition to the spin glass state is discussed.

Because of lack of space, we shall not consider many interesting objects that are closely related to spin glasses, but having special properties. They include magnets with random anisotropy or random field. These topics in the physics of disordered systems deserve a separate review.

2. BASIC EXPERIMENTAL RESULTS

2.1 Statistical properties. Even very simple magnetic measurements show that spin glasses have unusual properties. It is found that, below a certain definite temperature T_{g} , the magnetic susceptibility of a specimen depends on its prehistory: cooling in a magnetic field and application of the field after cooling down to temperatures below T_g lead to different values of the magnetic moment (Fig. 1a). The isothermal magnetization M_{ZFC} obtained by cooling in zero field is irreversible, but the thermostatic magnetization $M_{\rm FC}$ (cooling in a field) is reversible. As a rule, the thermostatic susceptibility is almost independent of temperature below $T_{\rm g}$. The susceptibility is found to be irreversible in all spin glasses, and is one of the principal and simplest criteria for identifying the transition of a system to the spin glass state. The temperature T_{g} is then considered to be the corresponding transition temperature.

In an external magnetic field, the susceptibility remains irreversible, but the irreversibility begins at lower temperatures (Fig. 1b), and fields of the order of a few hundred gauss shift T_g by an amount of the order of a degree.

It is clear from Fig. 1a that the linear susceptibility has a discontinuity at $T = T_g$. It was suggested ^{16,17} in 1977 that the nonlinear susceptibility should diverge when the transition to the spin glass takes place. Soon after, a strong increase in the nonlinear susceptibility for $T \rightarrow T_g$ was indeed found¹⁸ when the nonlinear response was measured at the third harmonic of the alternating magnetic field. A rapid increase in nonlinear susceptibility $\chi_{nl} = \chi - (M/H)$ was subsequently discovered in many spin glasses.^{19–29} Figure 2 shows the temperature dependence of nonlinear susceptibilities χ_{2n+1} (n = 1,2,3), i.e., the coefficients in the expansion of magnetization in powers of H, in the case of manganese aluminosilicate containing 15% at. of manganese.²²

Measurements of the field and temperature dependence of the nonlinear susceptibility are usually analyzed in terms of the scaling formula of the type

$$\chi_{nl} \sim H^{2/\delta} f\left(\frac{H^2}{\epsilon^{\beta+\gamma}}\right), \quad \epsilon = \frac{T - T_{\rm g}}{T_{\rm g}}.$$
 (2.1)



At first sight, the resulting values of the indices are not universal and vary from medium to medium. Many workers have suggested^{30,31} that the difference between the indices may be explained by random anisotropy that gives rise to a transition (crossover) from Ising to Heisenberg critical behavior as the magnetic field increases.³²

The characteristic feature of spin glasses is the linear temperature dependence of magnetic specific heat C_M for $T \ll T_g$, and the smooth maximum of specific heat at temperatures just above T_g (Refs. 33–36). Fogle *et al.*³⁵ have reported singularities in the derivatives dC_M/dT and d^2C_M/dT^2 , which they ascribe to a transition to a spin glass.

2.2. Dynamic effects. Already very early work in this field showed that the real part of the susceptibility of spin glasses had an unexpectedly strong frequency dependence.^{1,3} Even at frequencies of the order of, or less than, 1 Hz, the maximum of the susceptibility χ_{ZFC} shifts toward higher temperature with increasing ω , and broadens (see Fig. 3a).³⁷ Frequency dispersion of $\chi'(\omega)$ is accompanied by the absorption of power, so that the imaginary part $\chi''(\omega)$ of the susceptibility (Fig. 3b) is nonzero in spin glasses. The rapid increase in χ'' with decreasing temperature (the knee on the $\chi''(T)$ curve) is usually found to begin at temperatures corresponding to the maximum of $\chi'(T)$. The function $\chi''(\omega)$ varies slowly with frequency. Even at the lowest frequencies ($\sim 10^{-2}$ Hz) at which these measurements were performed, $\chi''(\omega)$ was found to be nonzero.³⁷⁻³⁹

If $\chi''(\omega)$ is almost independent of frequency at low frequencies, and the fluctuation-dissipation theorem is satisfied, the magnetic noise intensity becomes



FIG. 2. Temperature dependence of the coefficients χ_{2n+1} in amorphous manganese alumosilicate containing 15 at.% of Mn (Ref. 22). l - n = 1, 2 - n = 2, 3 - n = 3.



FIG. 1. Magnetization as a function of temperature in AgMn alloys (2.6 at.% of Mn; Ref. 15). a—Cooling in zero magnetic field $(1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5)$ and in a field of 6 G $(5 \rightarrow 4 \rightarrow 6 \rightarrow 4 \rightarrow 5)$; b—magnetization as a function of temperature in different magnetic fields (in G): 80 (1), 130 (2), 260 (3), 340 (4), 500 (5); T_R , T_M , T_B are the temperatures corresponding to the onset of irreversibility, the maximum M, and the departure from Curie's law, respectively.

$$(M^2)_{\omega} \sim \frac{\chi''(\omega)}{\omega} \sim \frac{1}{\omega},$$
 (2.2)

i.e., 1/f noise should be observed. This is indeed the case in some spin glasses.³⁹⁻⁴² Figure 4 shows the results³⁹ for Eu_{0.4}Sr_{0.6}S. Direct measurements and calculations of $\chi''(\omega)$ from the noise intensity, using the fluctuation-dissipation theorem, are in good agreement. The application of a weak magnetic field produces a very large change in noise intensity. A field on only 0.3 G results in an increase in the steady value of the noise intensity by a factor of 10 (Ref. 41). We note, finally, that according to Ref. 41, the noise spectrum measured in a field applied in the case $T > T_g$, and also the magnetization, are independent of the time of observations.

The susceptibilities $\chi''(\omega)$ and $\chi'(\omega)$ are related by the following empirical formula^{37,43–45}:

$$\chi''(\omega) = -\frac{\pi}{2} \frac{\mathrm{d}\chi'}{\mathrm{d}\ln\omega} , \qquad (2.3)$$



FIG. 3. The susceptibility $\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$ of amorphous metallic spin glass $Fe_{0.15} Ni_{0.85})_{75} P_{16} B_6 Al_3$ as a function of temperature at different frequencies.⁴⁷ Curves 1–11 correspond to the following frequencies: $f = \omega/2\pi$) (Hz) = 0.51 (1), 1,7 (2), 5,1 (3), 17 (4), 56 (5), 170 (6), 510 (7), 1.7 \cdot 10^3 (8), 5.1 \cdot 10^3 (9), 17 \cdot 10^3 (10) and $51 \cdot 10^3 (11)$; χ_{FC} is the static susceptibility measured in zero magnetic field.



FIG. 4. Magnetic noise in $Eu_{0.4}$ Sr_{0.6} S³⁹. $T_g = 1.53$ K; T(K) = 1, 42 (1), T = 1.82 (2) and 1.42 (3) without the sample.

which can be obtained by assuming the spin glass has a wide spectrum of relaxation times.^{43,46}

Using the Kramers-Kronig theorem, and assuming that $\chi''(\omega) = \text{const}$, we find that the real part of the susceptibility is $\chi'(\omega) \sim \ln \omega$. This means that the nonequilibrium susceptibility is a logarithmic function of time. This was observed^{47,48} in spin glasses starting in 1974. When a magnetic field is applied to a sample that has been cooled in zero field down to $T < T_g$, the magnetization at first increase abruptly (in a microscopic time) and then grows gradually, approaching the equilibrium value in accordance with the logarithmic law (or a power-type law with a small exponent) over time intervals of the order of a few hours or days. Conversely, if the sample has been cooled in a field down to $T < T_{a}$, the removal of the field is accompanied by an initial abrupt fall in magentization, followed by a gradual reduction, but the magnetization remains finite even after time $t \approx 10^5 \text{s}$ (Fig. 5).⁴⁹

Like the irreversibility of static susceptibility, these dynamic properties of spin glasses show that they have a wide spectrum of relaxation times, which extends to macroscopic times $t \approx 10^5$ s and, possibly, even further, since there does not appear to be an upper limit for the relaxation time.



FIG. 5. Thermostatic magnetization of the alloy CuMn(5 at.% Mn). $T/T_g = 0.96$ (a), 0.89 (b) and 0.75 (c). $T_g = 28$ K.

142 Sov. Phys. Usp. 32 (2), February 1989



FIG. 6. Residual magnetization of AgMn((2.6) at.% Mn) at different temperatures⁵⁰ as a function of time. $T/T_g = 0.771$ (1), 0.856 (2), 0.897 (3) and 0.996 (4). The curves were calculated from (2.4).

Low-frequency absorption and the irreversibility of static susceptibility suggest that the ground state of spin glasses is degenerate or quasidegenerate for $T < T_g$. The energy minima ("valleys") corresponding to different sets of spin orientations are separated by macroscopic barriers, so that relaxation occurs over macroscopically long times.

It is clear from Fig. 5 that, strictly speaking, the dependence of residual magnetization $M_{\rm TRM}$ on time is not logarithmic. It has been shown^{50,51} that the time dependence of $M_{\rm TRM}$ is better described by the so-called stretched exponential (Kohlrausch's law)

$$M_{\rm TRM} = M_0 \exp\left[-C \left(\omega_0 t\right)^{1-n} \left(1-n\right)^{-1}\right], \qquad (2.4)$$

where C and ω_0 are constants and n is a function of temperature (Fig. 6).⁵⁰

Slow relaxation processes described by Kohlrausch's law are observed in many disordered systems, including ordinary glasses.^{52,53}

Of course, relaxation in accordance with Kohlrausch's law can be explained by suitably choosing the spectrum of relaxation times and by assuming that different relaxation processes occur simultaneously and independently, so that

$$M(t) = \int W(\tau) e^{-t/\tau} d\tau, \qquad (2.5)$$

where $W(\tau)$ is the relaxation-time distribution function.

To explain Kohlrausch's law, Palma *et al.*⁵⁴ have put forward the hierarchical relaxation model which assumes a series (rather than parallel) relaxation of the different de-



FIG. 7. Magnetization of CuMn (4 at.% Mn) cooled in zero magnetic field.⁵⁵ The cooling time (in minutes) prior to the application of the magnetic field is indicated against each curve. T = 23 K, $T_g = 26$ K.

grees of freedom: in order to include the higher-level degrees of freedom in the relaxation process, it is essential that the lower-level degrees of freedom have relaxed already. We shall see later that the hierarchical structure of degrees of freedom arises naturally in the simple model of a spin glass.

2.3 "Aging" of spin glasses. The reversibility of static susceptibility, measured by cooling in a field, suggests that the state produced in this way is the equilibrium state. This was indeed believed to be so until 1983 when it was shown⁵⁵ that the relaxation of magnetization after the field is removed depends on the time t_w spent by the system in the magnetic field at a given temperature $T < T_g$ (Fig. 7), although the magnetization does not depend on time prior to the removal of the field.⁵⁵⁻⁵⁷

The dependence of the rate of relaxation on the cooling time t_w can be observed up to $t_w \sim 10^5$ s (Ref. 58). It follows that long-term processes that lead to equilibrium, but do not affect magnetization, occur in the state produced by cooling in a field.⁵¹

Similar memory effects are also observed in the state produced by cooling in zero field^{37,55,56,59-61}: the establishment of equilibrium magnetization after the field is turned on in the case of $T < T_g$ depends on the cooling time prior to the application of the field.

The dynamical properties of the system for $t < t_w$ and $t > t_w$ are somewhat different. Different modifications of (2.4) have been proposed in the literature, ^{58,56,49} and take into account the effect of the aging process on dynamics. An interesting relation was noted in Ref. 57 between the logarithmic time derivative of the thermostatic S_R and isothermal S_1 susceptibilities:

$$-S_{\mathrm{R}}(T) = S_{\mathrm{I}}(T), \quad t \ll t_{\mathrm{w}},$$

$$-S_{\mathrm{R}}(T) = kS_{\mathrm{r}}(T), \quad t \gg t_{\mathrm{ex}}$$
(2.6)

where k is time-independent and greter than unity.

The effect of waiting time on the dynamics is very clearly seen when the noise spectrum is investigated.⁴¹ The frequency range in which the noise intensity is $\sim \omega^{-1}$ depends on the waiting time. When $t_w > 70$ h, the noise spectrum is a steady-state one and proportional to ω^{-1} in the frequency range between 2×10^{-4} and 10^{-2} Hz.

2.4 Spin glass and magnetic long-range order. It has frequently been shown experimentally that the transition to a spin glass as temperature is reduced is possible not only from the paramagnetic state, but also from the magneticallyordered ferro-or antiferromagnetic state. This type of transition is often referred to as recurrent. Nonergodicity manifests itself, as usual, in dynamic experiments, and in the irreversibility of static susceptibility.⁶²⁻⁸⁶ Such transitions are illustrated in Fig. 8 which shows the phase diagrams for the solid solutions $Eu_{1-x}Sr_xS$ and $Fe_xMg_{1-x}Cl_2$. The question as to whether or not long-range order continues in the nonergodic state is a more complicated one. For reasons that will be discussed later, this question is still difficult to answer in the case of the transition from the ferromagnetic state. Satisfactory results have so far been obtained for the dilute Ising antiferromagnet $Fe_{1-x}Mg_xCl_2$. The temperature dependence of susceptibility is shown in Fig. 9 for $Fe_{0.55}$ Mg_{0.45} Cl₂. The susceptibility maximum occurs at the Neel temperature, and the transition to the nonergodic state occurs at the temperature at which the frequency dispersion





FIG. 8. Phase diagrams of disordered solutions $\operatorname{Eu}_{x}\operatorname{Sr}_{1,\ldots,x} S(a)^{63}$ and $\operatorname{Fe}_{x}\operatorname{Mg}_{1,\ldots,x}\operatorname{Cl}_{2}(b)^{68}$. *P*—paramagnetic phase; *F*, *AF*—ferroand antiferromagnetic phases, respectively; *SG*—spin glass.

of $\chi'(\omega)$ appears. At the same time, the intensity of Braggscattered neutrons, which is proportional to the magnetization of the sublattices, smoothly increases with decreasing temperature in the nonergodic phase, without any apparent singularity at $T = T_g$ (Ref. 71), i.e., long-range antiferromagnetic order persists below T_g .

The susceptibility $\chi'(\omega)$ falls sharply near T_{e} (Fig. 10) during the recurrent transition from the ferromagnetic state.⁸⁵ This is the basis for assuming that the magnetization vanishes during the transition to the spin glass. A similar conclusion has been reported by several workers who measured the saturation magnetization, the intensity of Braggscattered neutrons, and the spin-wave stiffness coefficient (see Refs. 64, 72, 73, 74, and 87-91, and also the references cited in Refs. 1 and 3). However, more and more evidence has been accumulating in recent years in support of the conclusion that the magnetic moment persists during the transition to the spin glass. These data have been obtained as a result of different microscopic measurements, magnetic neutron-diffraction studies, and Mössbauer experiments.^{80,81,75-78,92-97} It has recently been reported ^{80,81,93,98} that the fall in susceptibility χ' for $T < T_g$ is related to the sharp increase in the anisotropy energy and coercive force, and does not necessarily indicate that the magnetization has vanished.



FIG. 9. Temperature dependence of the real part of the susceptibility of the solution FeMg_{0.5045} Cl₂ at the frequencies 11 Hz (1), 345 Hz (2) and 2785 Hz (3). $T_N = 7.5$ K, $T_g = 3.0$ (Ref. 71).



FIG. 10. Real and imaginary parts, χ' and χ'' , of the linear magnetic susceptibility of the alloys $Fe_{65}Ni_{25}Cr_{10}(a)$ and $Fe_{40}Ni_{40}Mn_{20}(b)$. The maxima of χ'' correspond to the temperatures T_c and T_a .

Significant information on the change of state of a system during the transition to the spin-glass state is provided by Mössbauer experiments indicating that the freezing of spin components at right angles to the magnetization begins at $T = T_g$. This is clearly seen in the temperature dependence of magnetization and the hyperfine field (Fig. 11). The two curves coincide above T_g , but the magnetization is lower at the lower temperatures at which the freezing of the transverse spin components begins.

3. INFINITE-RANGE MODEL

The simplest model that can be used to demonstrate the onset of nonergodicity in spin glasses is the infinite-range model proposed by Sherrington and Kirkpatrick.⁹⁹⁻¹⁰⁰ They assume that the energy associated with the exchange interactions between the spins J_{ij} is independent of the separation between them, and that the quantities J_{ij} are distributed in accordance with the Gaussian law

$$P(J_{ij}) = \frac{N^{1/2}}{(2\pi)^{1/2} J} \exp\left\{-\frac{[J_{ij} - (J_{\theta}/N)]^2}{2J^2}N\right\}; \quad (3.1)$$

where N is the number of spins in the system and $J_0/N > 0$ and $J/N^{1/2}$ are, respectively, the mean value and the variance of the interaction energy. They are defined so that the total energy of the system is proportional to N as $N \to \infty$.

We begin by considering an Ising magnet with the Hamiltonian¹⁾

$$\mathscr{H} = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j - H \sum_i \sigma_i; \qquad (3.2)$$

where $\sigma_i = \pm 1$ and *H* is the external field. Its free energy is given by

$$F = -T \overline{\ln Z}, \quad Z = \operatorname{Sp} \exp\left(-\frac{\mathcal{H}}{T}\right),$$
 (3.3)



FIG. 11. Mean hyperfine field (\overline{H}) in iron (full points) and the saturation magnetization M (open circles) of the amorphous alloy (Fe_{0.765} Mn_{0.235})₇₅ P₁₆B₆Al₃⁹³ as functions of temperature.

144 Sov. Phys. Usp. 32 (2), February 1989

where the bar represents configurational averaging with the Gaussian distribution function (3.1).

An exact evaluation of the free energy involving first the thermodynamic averaging over the system with given distributions of exchange integrals, and then averaging over their distribution, cannot be carried out. The free energy is in fact evaluated by the so-called replica method in which the configurational averaging is carried out first. With this in view, we note that (3.3) can be written in the form

$$F = -T \lim_{n \to 0} \overline{[(Z^n - 1)n^{-1}]}.$$
 (3.4)

When n is an integer, Z^n is the partition function of n independent systems (replicas) with the same distribution of exchange integrals:

$$Z^{n} = \operatorname{Sp} \exp\left(\frac{1}{2T} \sum_{\alpha=1}^{n} \sum_{i,j} J_{ij} \sigma_{i}^{\alpha} \sigma_{j}^{\alpha}\right).$$
(3.5)

The function Z^n is readily averaged over the distributions (3.1). The free energy can then be obtained by analytically continuing Z^n from integral *n* and then passing to the limit as $n \rightarrow 0$. This is a crucial point in the method of replicas because the analytic properties of Z^n as a function of *n* are not well-known. This high price has to be paid to ensure that the problem of the thermodynamics of the disordered system can be reduced to a problem with an effective Hamiltonian.

Integration of Z^n over all J_{ij} , using the distribution functions given by (3.1), eventually yields

$$F = -T \lim_{n \to 0} \frac{1}{n} \left\{ \left(\exp \frac{J^2 N n}{4T^2} \right) \operatorname{Sp} \exp \left[\sum_{\alpha \neq \beta} \left(\sum_{i, j} \sigma_i^{\alpha} \sigma_j^{\beta} \right)^2 \frac{J^2}{2NT^2} + \frac{J_0}{2NT} \sum_{\alpha} \left(\sum_{i} \sigma_i^{\alpha} \right)^2 + \frac{H}{T} \sum_{\alpha} \sum_{i} \sigma_i^{\alpha} \right] - 1 \right\}.$$
 (3.6)

in which we have neglected terms that vanish in the thermodynamic limit.

To reduce this problem to a single-site problem, we use the integral transformation

$$e^{\lambda x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{y^2}{2} + (2\lambda)^{1/2} yx\right] \mathrm{d}y.$$
 (3.7)

and tranform (3.6) so that

$$F = -T \lim_{n \to 0} \frac{1}{n} \left[\left(\exp \frac{J^2 N n}{4T^2} \right) \int_{\alpha} \prod_{\alpha} \left(\frac{N}{2\pi} \right)^{1/2} dx^{\alpha} \\ \times \prod_{\alpha\beta} \left(\frac{N}{2\pi} \right)^{1/2} dy^{\alpha\beta} \exp\left(-N\Phi\left(x^{\alpha}, y^{\alpha\beta}\right) \right) - 1 \right],$$
(3.8)

$$\Phi (x^{\alpha}, y^{\alpha\beta}) = \frac{1}{2} \sum_{\alpha > \beta} (y^{\alpha\beta})^2 + \frac{1}{2} \sum_{\alpha} (x^{\alpha})^2 - \ln \operatorname{Sp} \exp \left(\frac{J}{T} \sum_{\alpha > \beta} y^{\alpha\beta} \sigma^{\alpha} \sigma^{\beta} + \frac{J_0^{1/2}}{T^{1/2}} \sum_{\alpha} x^{\alpha} \sigma^{\alpha} + \frac{H}{T} \sum_{\alpha} \sigma^{\alpha} \right).$$
(3.9)

As $N \to \infty$, the integrals in (3.8) can be evaluated by the saddle-point method. At the stationary point, we have

$$y_0^{\alpha\beta} = \frac{J}{T} \langle \sigma^{\alpha}\sigma^{\beta} \rangle, \quad x_0^{\alpha} = \left(\frac{J_0}{T}\right)^{1/2} \langle \sigma^{\alpha} \rangle, \quad (3.10)$$

where $\langle ... \rangle$ denotes averaging with the effective Hamiltonian

$$\mathscr{H}_{eff} = -\sum_{\alpha} \left(J_0 m^{\alpha} + H \right) \sigma^{\alpha} - \frac{J^2}{T} \sum_{\alpha > \beta} q^{\alpha\beta} \sigma^{\alpha} \sigma^{\beta}; \quad (3.11)$$

in which

$$m^{\alpha} = \langle \sigma^{\alpha} \rangle, \quad q^{\alpha\beta} = \langle \sigma^{\alpha}\sigma^{\beta} \rangle, \quad \alpha \neq \beta.$$
 (3.12)

After taking the trace of \mathscr{H}_{eff} , we must pass to the limit as $n \to 0$.

Expressions (3.10)-(3.12) constitute a self-consistent set of equations which, in general, has many solutions. The simplest solution, for which the quantities m^{α} and $q^{\alpha\beta}$ do not depend on the replica indices (the replica symmetric solution) was investigated in Refs. 99 and 100:

 $m^{\alpha} = m, \quad q^{\alpha\beta} = q.$

A further transformation such as (3.7) can then be used to reduce the equation to the self-consistent form

$$q = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} \operatorname{th}^2 \frac{E(z)}{T} dz,$$

$$m = \int \frac{e^{-z^2/2}}{\sqrt{2\pi}} \operatorname{th} \frac{E(z)}{T} dz,$$

$$E(z) = J_0 m + H + J q^{1/2} z,$$

(3.13)

and the free energy is given by

$$f = \frac{F}{N} = T \left[\frac{J_0 m^2}{2T} - \frac{J^2 (1-q)^2}{4T^2} - \frac{1}{V^{2\pi}} \int e^{-z^2/2} \ln\left(2 \operatorname{ch} \frac{E(z)}{|T|}\right) \mathrm{d}z \right].$$
(3.14)

It is clear from (3.13) that *m* is the mean magnetization, and the averaging is carried out over the distribution of molecular fields, which is found to be Gaussian in this model. It is also clear from (3.13) that the parameter *q*, first introduced by Edwards and Anderson,¹² can be interpreted as the average taken over all sites of the square of magnetization at a site:

$$q = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle_{\mathrm{T}}^2; \qquad (3.15)$$

where $\langle ... \rangle_T$ represents thermodynamic averaging for a given en relaxation. When $H = J_0 = 0$, the magnetization *m* given by (3.13) is zero at all temperatures. The parameter *q* is zero for $T > T_g = J$, and differs from zero for $T < T_g$. It is thus clear that the replica-symmetric solution leads to a phase transition to the state with $q \neq 0$. It then follows from (3.13) that the susceptibility obeys the paramagnetic Curie law $\chi \sim T^{-1}$, and that a break occurs at the transition point. The nonlinear susceptibility diverges at this point:

$$\chi_{nl} \sim (T - T_g)^{-1}.$$

These two facts are in qualitative agreement with experimental data.

The parameter q is nonzero in a magnetic field at any temperature, and the phase transition does not occur in the replica-symmetric model.

However, Sherrington and Kirkpatrick originally showed that the entropy of the replica-symmetric solution becomes negative at low temperatures, and this has stimulated studies of the stability of the replica-symmetric solution with respect to fluctuations that break the replica symmetry.¹⁰¹ De Almeida and Thouless¹⁰¹ have investigated the eigenvalues of the stability matrix

$$\hat{M} = \lim_{n \to 0} \frac{\partial^2 \Phi}{\partial y^{\alpha \beta} \partial y^{\gamma \delta}} , \qquad (3.16)$$

and found that one of the eigenvalues changes sign at the temperature given by

$$T^{2} = J^{2} \int -\frac{e^{-z^{2}/2}}{\sqrt{2\pi}} \operatorname{ch}^{-4} \frac{E(z)}{T} dz, \qquad (3.17)$$

where *m* and *q* must be found from (3.13). Above this temperature, T_g , the replica-symmetric point of origin of the curve of steepest descent corresponds to an actual extremum of the functional Φ , whereas below this temperature we have a saddle point. All this means that the replica symmetry is broken at the temperature $T = T_g(J_0, H_0)$, and a new state appears.

The change in the sign of the eigenvalue of the stability matrix is due to a singularity in the generalized susceptibility $\chi^{(2)} = \overline{\chi_i^2}$ (Ref. 102), where the local susceptibility is given by

$$\chi_i = \sum_j \left(\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \right). \tag{3.18}$$

When $H = J_0 = 0$, it follows from (3.17) that $T_g = J$, i.e., the replica-symmetric solution is stable only in the paramagnetic phase, when q = 0. The magnetic field does not destroy the transition and only reduces T_g . The solution of (3.17) and (3.13) yields

$$T_{g}(H) = J \left[1 - \left(\frac{3}{4} \frac{H^{2}}{J^{2}} \right)^{1/3} \right], \quad H \ll J,$$
 (3.19)

$$= (3 \sqrt{2\pi})^{-1} J \exp\left(-\frac{H^2}{2J^2}\right), \quad H \gg J. \quad (3.20)$$

The phase diagram on the T, H plane, determined by (3.17), is shown in Fig. 12. The solid curve is the so-called Almeida-Thouless line and corresponds to the solution of (3.17). The broken curve represents the transition to the spin glass in the Heisenberg magnet (Gabay-Toulouse line), which will be discussed in Sec. 8. The reduction in T_g with increasing magnetic field has been observed in all "pure" $(J_0 < J)$ spin glasses. The experimental dat shown in Fig. 1 constitute an example of this. The precise form of the reduction in T_g is a more complicated problem, and will be discussed in Sec. 8.

In the replica method, we first average over $\{J_{ij}\}$ and then pass to the effective Hamiltonian without random variables. It is possible to evaluate the free energy for a given distribution of $\{J_{ij}\}$ by averaging over the disorder at the end. Thouless, Anderson, and Palma¹⁰³ (TAP) have summed the leading terms in 1/N in the high-temperature

i 11. Internationale



FIG. 12. The T, H phase diagram of a spin glass in the infinite-range model.

distribution and obtained the following expression for the free energy for $J_0 = H_0 = 0$:

$$F = Nf_0 + f_1, (3.21)$$

where above the transition

$$f_0 = -T \ln 2 - \frac{J^2}{4T},$$

$$f_1 = -\frac{1}{4} T \ln \left(1 - \frac{J^2}{T^2}\right) + \text{nonsingular terms.} \quad (3.22)$$

The N-independent term in the free energy diverges at the transition point. Below the transition point, the extensive part of F has the form

$$F = -\sum_{i>j} J_{ij} m_i m_j - \frac{1}{2T} \sum_{i>j} J_{ij}^2 (1 - m_i^2) (1 - m_j^2) + \frac{T}{2} \sum_i [(1 + m_i) \ln (1 + m_i) + (1 - m_i) \ln (1 - m_i)] m_i = \langle \sigma_i \rangle_{\rm T}.$$
(3.23)

The equations for the mean spin on site m_i are obtained by finding the stationary condition for F over all the m_i :

$$m_i = \text{th} \sum_j \frac{J_{ij}}{T} \left(m_j - m_i J_{ij} \frac{1 - m_j^2}{T} \right).$$
 (3.24)

The effective field

$$\sum_{j} J_{ij} m_{j} - \sum_{j} J_{ij}^{2} m_{i} \frac{1 - m_{j}^{2}}{T} , \qquad (3.25)$$

acting on the spins differs from the usual expression in the molecular-field approximation by the presence of the second term. In an ordered magnet, $J_{ij}^2 \sim N^{-2}$, so that the second term in the effective field is $\sim N^{-1}$, and can be discarded, whereas for spin glasses, the two terms are generally of the same order, and do not vanish as $N \rightarrow \infty$. Thus, the molecular-field theory does not reduce in the case of spin glasses to the replacement of one of the spins with its averge value, and correlation effects are always significant. This means that, in the paramagnetic phase, the specific heat is always different from zero.

The expressions given by (3.24) are the well-known Bethe equations. The second term in the effective field is the Onsager reaction term that describes the reduction in the moment at site *j* due to the reaction of the spin σ_i . In fact, the field acting at site *j* due to the spin σ_i is $J_{ij}m_i$, and the reduction in the moment at site *j* due to this field is $\chi_j J_{ij}m_i$. Since the local susceptibility is $\chi_j = 1 - m_j^2$, we thus arrive at the expression given by (3.25).

Above $T_g = J$, the free energy (3.22) becomes identical with that obtained by the replica method in Refs. 99 and 100.

It has not been possible to solve (3.24) below T_g , but it has been shown¹⁰⁴⁻¹⁰⁶ that, below T_g , there are many solutions of (3.24) that give the same free energy f_0 . The number of these solutions is exponentially large, i.e., it is proportional to $\exp[\alpha(T)N]$, where $\alpha(T)$ is a numerical coefficient that is equal to 0.2 at T = 0 and is proportional to $T_g - T)^6$ as $T \rightarrow T_g$. Their statistical weights are different because the extensive part of the free energy is a random quantity on the set of solutions. We shall see later that the entropy per spin must therefore vanish as $T \rightarrow 0$.

4. PARISI ORDER PARAMETER

We have seen that the replica-symmetric solution is invalid below the Almeida-Thouless temperature. The state of the system must be described by a set of quantities $q^{\alpha\beta}$ rather than the single parameter q. Parisi¹⁰⁷ was the first to put forward an internally consistent theory with a broken replica symmetry. This theory is heuristic in the sense that neither the method used to parametrize the matrix $q^{\alpha\beta}$ nor the procedure employed in the analytic continuation of this $n \times n$ matrix to n = 0 are forced by some general principles. Parisi's theory is mathematically somewhat complicated and unusual, and we shall present it in the next Section. However, we begin with Parisi's physical interpretation¹⁰⁸ of the order parameter found by him in Ref. 107.

Parisi's idea is that the order parameter of a spin glass must contain information on the infinite degeneracy of the ground state. Each state, which we shall label with the index α , is determined by the set of occupation numbers $m_i^{\alpha} = \langle \sigma_i^{\alpha} \rangle_{\rm T}$, where *i* is the site number. Let

$$q^{\alpha\beta} = \frac{1}{N} \sum_{i} m_{i}^{\alpha} m_{i}^{\beta}, \qquad (4.1)$$

which characterizes the overlap between different states. It is clear that $q^{\alpha\alpha} = 1/N \Sigma_i (m_i^{\alpha})^2$ is identical with the Edwards-Anderson parameter. When T = 0, the quantity $N(1 - q^{\alpha\beta})$ is twice the number of spins that must be turned over in a transition from state α to state β .

The random quantity $q^{\alpha\beta}$ is described by the distribution function

$$W(q) = \sum_{\alpha,\beta} P_{\alpha} P_{\beta} \delta(q - q^{\alpha\beta}), \qquad (4.2)$$

where P_{α} is the statistical weight of the state α . The probability that $q^{\alpha\beta}$ assumes a value not exceeding q is

$$x(q) = \int_{0}^{q} W(q') \, \mathrm{d}q'.$$
 (4.3)

The inverse function q(x), $0 \le x \le 1$ is the Parisi order parameter of a spin glass. By definition, q(x) increases monotonically with increasing x. If the ground state is not degenerate, $q^{\alpha\beta}$ has the single value $q^{\alpha\alpha}$, so that q(x) = const. If, on the other hand, the state is infinitely degenerate, the dependence of q(x) on the argument becomes significant on all intervals. The dependence of q(x) on x is thus signaled by the fact that the system has passed to a phase with a significantly degenerate ground state.

The function q(x) is directly related to the susceptibility of the pure spin glass $J_0 = 0$

$$\chi = \frac{1}{NT} \sum_{i, j} \left(\langle \sigma_i \sigma_j \rangle_{\mathrm{T}} - \langle \sigma_i \rangle_{\mathrm{T}} \langle \sigma_j \rangle_{\mathrm{T}} \right).$$
(4.4)

Because the exchange integrals J_{ij} in (4.4) are symmetrically distributed, only terms with i = j are nonzero, so that

$$\chi = \frac{1}{NT} \sum_{i} (1 - \langle \sigma_i \rangle_{\mathrm{T}}^2).$$
(4.5)

Subsequent calculations depend on whether or not we assume that the system is in one of the energy minima (or, as is often said, in one of the valleys) and cannot go over into the other minima, or whether we take the thermodynamic average in the usual way over all the states. The first method of averaging is equivalent to averaging over time. The definition of q(x) given above shows that, in the case of averaging over the states of one valley, we have

$$\frac{1}{N}\sum_{i}\langle\sigma_{i}\rangle_{\mathrm{T}}^{2}=\frac{1}{N}\sum_{i}(m_{i}^{\alpha})^{2}=q(1),$$

so that the corresponding susceptibility is

$$\chi_{\text{neij}} = \frac{1}{T} (1 - q (1)). \tag{4.6}$$

On the other hand, if we take the Gibbs average, we have

$$\frac{1}{N} \sum_{i} \langle \sigma_{i} \rangle_{\mathrm{T}}^{2} = \frac{1}{N} \sum_{i} \sum_{\alpha, \beta} P_{\alpha} P_{\beta} m_{i}^{\alpha} m_{i}^{\beta}$$
$$= \int W(q) q \, \mathrm{d}q = \int_{0}^{1} q(x) \, \mathrm{d}x,$$

and the equilibrium susceptibility is

$$\chi_{eq} = \frac{1}{T} \left(1 - \int_{0}^{1} q(x) \, \mathrm{d}x \right).$$
 (4.7)

The susceptibilities given by (4.6) and (4.7) are found to be different when q(x) depends on x.

5. THE PARISI ANSATZ

Solution near the transition point. In the limit as $n \to 0$, it follows¹⁰¹ from (3.9)-(3.12) that the free energy for $J_0 = 0$ is given by

$$f = T \max\left\{-\frac{J^2}{4T^2} + \lim_{n \to 0} \frac{1}{n} \left[\frac{1}{4} \sum_{\alpha > \beta} \frac{J^2}{T^2} (q^{\alpha\beta})^2 - \ln \operatorname{Sp} \exp\left(\frac{J^2}{T^2} \sum_{\alpha > \beta} q^{\alpha\beta} \sigma^{\alpha} \sigma^{\beta} + \frac{H}{T} \sum_{\alpha} \sigma^{\alpha}\right)\right]\right\}.$$
 (5.1)

Near $T_{\rm g}$, the exponential can be expanded in powers of $q^{\alpha\beta}$ so that

$$f = -T \lim \frac{1}{2n} \left(\epsilon \operatorname{Sp} \hat{q}^2 - \frac{1}{3} \operatorname{Sp} \hat{q}^3 + \frac{1}{6} \sum_{\{\alpha, \beta\}} \hat{q}^4 + \frac{H^2}{T^2} \sum_{\alpha, \beta} q^{\alpha\beta} \right),$$
(5.2)

where $\varepsilon = (T_g - T)/T_g$ and $T_g = J$.

Generally speaking, not all the terms of the order of q are written out in (5.2). Terms of the order of $\varepsilon^2 q^2$ have also been omitted. However, it is readily verified that the breaking of replica symmetry is due to the terms of the order of q^4 that have been retained in (5.2), and that the discarded terms cancel out,¹⁰² at least in the second order in ε . In accordance with the Parisi parametrization, the matrices \hat{q} are determined by k + 1 real numbers q_i (i = 0,...,k) and k integers m_i (i = 1,...,k) such that m_i/m_{i+1} is an integer and $m_0 = 1, m_{k+1} = n$. Thus, for $n = 8, k = 2, m_1 = 2, m_2 = 4$, the matrix \hat{q} takes the form

$$\begin{vmatrix} 0 & q_0 & q_1 & q_1 \\ q_0 & 0 & q_1 & q_1 \\ q_1 & q_1 & 0 & q_0 & q_2 \\ q_1 & q_1 & q_0 & 0 & & & \\ q_2 & q_1 & q_1 & 0 & q_0 \\ & & & & & & & \\ q_2 & q_1 & q_1 & 0 & q_0 \\ & & & & & & & & \\ q_1 & q_1 & q_1 & 0 & q_0 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ \end{array} \right)$$
(5.3)

Because of this structure, each element q_i is found $m_{i+1} = m_i$ times in a row (column). Hence, for example,

$$\sum_{\alpha>\beta} (q^{\alpha\beta})^2 = n \sum_{i=0}^{k} (m_{i+1} - m_i) q_i^2.$$
 (5.4)

According to Parisi, the procedure for going to the limit as $n \rightarrow 0$ is that the set of integers m_i is replaced with the set of quantities m_i (i = 1,...,k) lying within the interval (0,1), where

$$1 \ge m_1 \ge m_2 \dots \ge m_k \ge 0. \tag{5.5}$$

The index k is then allowed to tend to infinity, and the quanity m_i becomes a continuous variable on (0,1). We will denote it by x. The result for the sum in (5.4) is

$$\lim_{n \to 0} \frac{1}{n} \sum_{\alpha > \beta} (q^{\alpha \beta})^2 = - \int_0^1 q^2(x) \, \mathrm{d}x.$$
 (5.6)

The other terms in the free energy can be written down in a similar way, and the final result is

$$f = \frac{1}{2} T \max \left[\int_{0}^{1} \left(\epsilon q^{2} \left(x \right) + \frac{1}{6} q^{4} \left(x \right) - \frac{1}{3} q^{3} \left(x \right) \right) dx - q^{2} \left(x \right) \int_{x}^{1} q \left(y \right) dy + \frac{H^{2}}{T^{2}} \int_{0}^{1} q \left(x \right) dx \right].$$
(5.7)

By varying (5.7) with respect to the function q(x), we obtain the following equation for this function:

$$2q(x)(\varepsilon - \bar{q}) + \frac{2}{3}q^{3}(x) = \int_{0}^{1} (q(x) - q(y))^{2} dy - \frac{H^{2}}{T^{2}}, \quad (5.8)$$
$$\bar{q} = \int_{0}^{1} q(x) dx.$$

Differentiating this equation twice with respect to x, we obtain

$$q'(x)(2q(x) - x) = 0.$$
(5.9)

The nontrivial solution of (5.9) has the form

$$q(x) = q(0) = \frac{x_0}{2}, \quad 0 \le x \le x_0, \\ = \frac{x}{2}, \quad x_0 < x < x_1, \\ = q(1) = \frac{x_1}{2}, \quad x_1 \le x \le 1.$$
(5.10)

Substituting this in (5.8), and in the equation obtained by differentiating (5.8) with respect to x, we find that

$$x_0 = \left(6 \frac{J^2}{J^2}\right)^{1/3}, \quad x_1 = 2\varepsilon \ (1 + \varepsilon).$$
 (5.11)

The function q(x) is plotted in Fig. 13.

The shift of the transition temperature that is due to the magnetic field is determined by the condition $x_0 = x_1$ and is found to be $\varepsilon(H) = (3H^2/4J^2)^{1/3}$. We see that the shift obtained in the Parisi theory is the same as that obtained in the

......



FIG. 13. The Parisi function q(x) in an external magnetic field.

analysis of the stability of the replica-symmetric solution [see (3.19)].

From (5.7) we readily obtain the expression for the susceptibility given by (4.7), i.e., the first moments of the functions q(x) introduced in the valley model and those based on the replica-asymmetric Parisi ansatz are the same. Parisi has shown¹⁰⁸ that all the moments of these functions are equal, thus confirming the validity of the physical interpretation of the Parisi solution.

Substituting the expression for q(x) given by (5.10) into (4.7), we find that

$$\chi_{eq} = \frac{1}{J} \left[1 - \left(\frac{3}{4} \right)^{2/3} \left(\frac{II}{J} \right)^{4/3} \right].$$
 (5.12)

It is important to note that χ_{eq} is independent of ε in this approximation. We shall see later that, in zero magnetic fields, χ_{eq} is independent of temperature throughout the region below the transition point. It is clear from (5.12) that the nonlinear susceptibility diverges at the transition point as $H^{-2/3}$.

Since $q \sim \varepsilon$, the free energy is $f(\varepsilon) - f(0) \sim \varepsilon^3$ and hence, the transition to the spin glass is a third-order transition.

6. THE EQUATIONS OF STATE OF A SPIN GLASS AT ARBITRARY TEMPERATURE. LOW-TEMPERATURE PROPERTIES

The general scheme for studying the properties of a spin glass well away from its phase transition point is the same as for $\varepsilon \ll 1$: first, the Parisi ansatz is substituted into the free energy (5.1), and then, by going to the limit as $n \to 0$, the free energy is expressed in terms of q(x). Second, the equation for the function q(x) is obtained by varying the free energy functional.

An elegant method of deriving the functional f has been put forward by Duplantier.¹⁰⁹ We shall not go into the mathematical derivation and reproduce only the final result:

$$f = -\frac{J^2}{4T} - \frac{J^2}{4T} \left(\int_{0}^{1} q^2(x) \, \mathrm{d}x - 2q(1) \right) \\ - \frac{1}{(2\pi q(0))^{1/2}} \int \varphi(0, y) \exp\left[-\frac{(y-H)^2}{2J^2 q(0)} \right] \mathrm{d}y, \quad (6.1)$$

where $\varphi(x, y)$ is a solution of

$$\dot{\varphi} = -J^2 \frac{\dot{q}}{2} \varphi'' - \frac{J^2}{2T} \dot{q} x (\varphi')^2$$
(6.2)

subject to the boundary condition

$$\varphi(1, y) = T \ln \left(2 \operatorname{ch} \frac{y}{T} \right).$$
(6.3)

148 Sov. Phys. Usp. 32 (2), February 1989

The dot represents differentiation with respect to x and the prime with respect to y.

Variation of the function of (6.1), taking (6.2) and (6.3) into account, is conveniently performed by adding to f in (6.1) the term \tilde{f} (which is equal to zero)¹¹⁰:

$$\begin{split} \widetilde{f} &= \frac{1}{2J} \int_{-\infty}^{\infty} \mathrm{d}y \, P\left(1, \, y\right) \left[\varphi\left(1, \, y\right) - T \ln\left(2 \operatorname{ch} \frac{y}{T}\right) \right] \\ &- \frac{1}{2J} \int_{0}^{1} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \, P\left(x, \, y\right) \left[\varphi\left(x, \, y\right) \\ &+ \frac{\dot{q}}{2} \, \varphi'' + \frac{J}{2T} \, x \dot{q} \, (\varphi'\left(x, \, y\right))^2 \right], \end{split}$$

(6.4)

where P(x, y) are the undetermined Lagrange multipliers. By varying the functional $f + \tilde{f}$ with respect $\varphi(x, y)$ and $\varphi(0, y)$, we obtain the equation for the function P(x, y):

$$\vec{P}(x, y) = J^{2} \frac{\dot{q}}{2} P'' - \frac{J^{2}}{T} \dot{q} (\varphi' P)'$$
(6.5)

with the boundary condition

$$P(0, y) = (2\pi q(0))^{-1/2} \exp\left[-\frac{(y-H)^2}{2J^2 q(0)}\right].$$
 (6.6)

By varying $f + \tilde{f}$ with respect to q(x) and using (6.5), we obtain the relation between q(x) and $\varphi(x, y)$ and P(x, y):

$$q(x) = \frac{1}{J} \int m^2(x, y) P(x, y) \, \mathrm{d}y, \tag{6.7}$$

where we have introduced the notation $m(x, y) = \varphi'(x, y)$ which will be convenient later. In the paramagnetic phase, in which φ is independent of x, (6.3) shows that the function m(y) is equal to th (y/T), i.e., it is the spin magnetization in the field y.

Equations (6.2) and (6.5) and the boundary conditions (6.3) and (6.6) enable us, at least in principle, to find q(x) and the free energy at any temperature. The complete set of equations of this type was first given in Refs. 111–113. It is clear from (6.7) that P(x, y) can be interpreted as the distribution function of molecular fields y for valleys whose overlap does not exceed q(x).

Taking into account the fact that the free energy is timeindependent, we obtain the following simple form for the mean magnetic moment:

$$m = -\frac{\partial f}{\partial H} = \frac{1}{J} \int m(0, y) P(0, y) \, \mathrm{d}y.$$
 (6.8)

In the paramagnetic phase, (6.7) and (6.8) become the equation of state (3.13), obtained in the replica-symmetric approximation.

Differentiating (6.7) with respect to x, and using (6.5) and (6.2), we obtain the following important relation that is valid throughout the nonergodic phase:

$$1 = \int P(x, y) (\varphi''(x, y))^2 dy.$$
 (6.9)

When $T = T_g$, so that P(x, y) and $\varphi(x, y)$ are independent of x, the relation given by (6.9) becomes identical with the equation for the Almeida-Thouless line given by (3.17).

The physical interpretation of (6.9), which is referred to as the marginality condition, will be discussed below.

Equations (6.2)-(6.6) can be solved analytically, but only near T_g , in which case the results are the same as in the last Section. However, the marginality relation (6.9) can be

used to examine some of the general properties of spin glasses at low temperatures.

Since the functional $f + \tilde{f}$ is time-independent, we obtain the following expression for the entropy from (6.1) and (6.4):

$$S = -\frac{\partial (f+\tilde{f})}{\partial T} = -\frac{1}{4T^2} (1-q(1))^2 + \int P(1, y) \left[\ln \left(2 \operatorname{ch} \frac{y}{T} \right) - \frac{y}{T} \operatorname{th} \frac{y}{T} \right] \mathrm{d}y. \quad (6.10)$$

Taking (6.9) at x = 1, we have

$$\mathbf{1} = \frac{J}{T^2} \int P(1, y) \operatorname{ch}^{-1} \frac{y}{T} \, \mathrm{d}y.$$
 (6.11)

The main contribution to this integral is provided by $y \leq T$, so that (6.11) is satisfied for $T \rightarrow 0$ only if

$$P(\mathbf{1}, yT) T^{-1} \xrightarrow{T \to 0} \text{const.}$$

From (6.7) we then find that, as $T \rightarrow 0$, we have $1 - q(1) \sim T^2$, and it then follows from (6.7) that $S \sim T^2$ (Ref. 110). The Parisi theory is thus seen to be free from the contradiction encountered in replica-symmetric theory, namely, that at zero temperature, the entropy per spin of the spin glass is zero in the Parisi theory.

Differentiating (6.8) with respect H, and recalling that the function m(0, y) is odd, we find that

$$\chi_{\rm eq} = \lim_{H \to 0} \left(\frac{1}{J} \int P(0, y) q^*(0, y) \, \mathrm{d}y \right). \tag{6.12}$$

However, $q(0) \rightarrow 0$ as $H \rightarrow 0$, so that, according to (6.6),

$$\lim_{\mathcal{U}\to 0} P(0, y) = J\delta(y).$$

Hence

$$\chi_{eq} = \varphi''(0, 0).$$
 (6.13)

On the other hand, the marginality condition, taken at the point x = 0, shows that $\varphi''(0,0) = J^{-1}$. We thus find that, throughout the spin-glass phase, the equilibrium susceptibility is J^{-1} , and is independent of temperature.

Experiment shows that the susceptibility χ_{FC} obtained by cooling in a field is usually almost independent of temperature (see Fig. 1). Since the susceptibility is also reversible, and independent of the waiting time, one might think that, despite the aging effects, χ_{FC} is in many ways similar to the equilibrium susceptibility.

We thus see that the Parisi solution leads to reasonable physical results. However, the question arises as to whether the saddle point that corresponds to the Parisi parametrization is stable or not. This was investigated in Refs. 114–116 and it was shown that the eigenvalues of the corresponding matrix were nonnegative, and one of them was zero at all temperatures. This means that the Parisi solution is marginally stable. The vanishing of one of the eigenvalues is assured by (6.9), which therefore means that the excitation spectrum of the spin glass has a gapless mode.¹¹⁷ This mode exists because the ground state of the spin glass is degenerate.

The Parisi scheme also allows a more complex parametrization than that described above.¹¹⁸ The free energy functional then contains an additional arbitrary function $\Delta(x)$, which means that we can choose an arbitrary gauge in (6.2), (6.5), and (6.7). The function $\Delta(x)$ was first introduced by Sompolinsky^{119,120} who developed the dynamic theory of spin glasses. The Sompolinsky theory is based on the assumption that the spin glass is characterized by an infinite set of relaxation times t_x , $0 \le x \le 1$. In the thermodynamic limit, all the times t_x tend to infinity, so that $t_{x'}/t_x \rightarrow 0$ if x > x'. From this point of view, the function q(x) is defined as the correlator $\langle \sigma_i(0)\sigma_i(t_x) \rangle$ and is a measure of the correlation that remains at time t_x . The function q(x) is therefore a monotonically increasing function of x. The function $\Delta(x)$ is related to the local susceptibility measured at frequency $\omega_x = t_x^{-1}$. In particular, $\Delta(0) = T(\chi_{eq} - \chi_{neq})$, i.e., it is a measure of the irreversible response of the system. The Sompolinsky functional for the free energy becomes identical with the Parisi functional when we take the gauge $\dot{\Delta} = -(J/T)x\dot{q}$. Physical results are, of course, all independent of the gauge.

7. VECTOR SPIN GLASS

The infinite-range model can be naturally generalized to a magnet with the Heisenberg exchange interaction. The Hamiltonian then takes the form

$$\widetilde{\mathcal{AE}} = -\frac{1}{2} \sum_{i,j} J_{ij} \left(\mathbf{S}_i \mathbf{S}_j \right) - D \sum_{i} \left(S_i^{\sharp} \right)^2 - \sum_{i} \left(\mathbf{H}_i \mathbf{S}_i \right), \quad (7.1)$$

where S is the classical spin, whose length is conveniently taken to be $\sqrt{3}$, and D is the single-ion magnetic anisotropy energy. Generally speaking, both the direction of magnetic anisotropy axis in disordered magnets and the magnitude of the energy can fluctuate in space. The effect of the random anisotropy on the properties of spin glass will be discussed at the end of this Section. The exchange interaction energies J_{ij} will be distributed, as before, in accordance with the normal law (3.1).

The vector spin glass differs from the Ising and replicasymmetric approximations in that it is described by a number of parameters. In additioin to the Edwards-Anderson parameter, which in this case is the tensor

$$q_{\mu\nu} = \langle \langle S_{\mu} \rangle_{\mathbf{T}} \langle S_{\nu} \rangle_{\mathbf{T}} \rangle_{\mathbf{c}} \qquad (\mu, \ \gamma = x, \ y, \ z), \tag{7.2}$$

there is also the so-called quadrupole parameter

$$Q_{\mu\nu} = \langle \langle S_{\mu}S_{\nu} \rangle_{\rm T} \rangle_{\rm c}; \tag{7.3}$$

where $\langle ... \rangle_c$ represents the configurational average. Finally, the magnetization $\mathbf{m} = \langle \langle \mathbf{S} \rangle_T \rangle_c$ arises when $J_0 > J$.

When the magnetic field lies along the anisotropy axis, the tensors $q_{\mu\nu}$ and $Q_{\mu\nu}$ are diagonal and have two unequal components, namely, the longitudinal and the transverse. The condition $S^2 = 3$ relates Q_{\parallel} and Q_{\perp} , so that the magnet is described by the three parameter q_{\parallel},q_{\perp} and $X = (Q_{\parallel} - 1)/2$ when $J_0 = 0$. They are determined from the self-consistent set of equations given by (7.2) and (7.3), in which the thermodynamic average is evaluated for the effective Hamiltonian

$$\mathcal{H}_{\rm eff} = \mathbf{aS} + bS_{\mathbf{z}}^2,\tag{7.4}$$

where

$$a_{\mu} = (H\delta_{\mu z} + Jq_{\mu}^{1/2}t_{\mu}) T^{-1}.$$
(7.5)

$$b = J^2 T^{-2} (q_\perp - q_\perp + 3X) - DT^{-1}.$$
(7.6)

and the configurational average is taken over the normal distribution of the random fields t (kef. 121). We note that

....

i i angali u

the effective anisotropy is different from the crystallographic anisotropy. The additional term is due to disorder in the system, and is nonzero only in glasses that have a special axis or plane. In isotropic glasses (D = 0) we have $X = 0, q_{\perp} = q_{\parallel}$ in the absence of the magnetic field, and then b = 0.

7.1 Isotropic magnet, D = 0. When the magnetic field is H = 0, equations (7.2) and (7.3) have the nontrivial solution $q_{\perp} = q_{\parallel} \neq 0$ for $T < T_g = J$.

The parameter q_{\parallel} is nonzero at all temperatures in an external magnetic field. Analysis of (7.4)–(7.6) leads to the conclusion¹²¹ that a third order phase transition¹²² occurs in the system, i.e., a nonzero component q_{\perp} occurs on the line defined by

$$T_{g}(H) = J\left(1 - \frac{23}{100} \frac{H^{2}}{J^{2}}\right),$$
 (7.7)

when $H \ll J$. Analysis of the stability of the replica-symmetric solution¹²³⁻¹²⁶ has shown that nonergodicity arises at the same temperature, i.e, the parameters q_{\parallel} and q_{\perp} become functionals of x. Near $T_{g}(H),$ and when $JH^2 \gg (T_{g}(0) - T_{g}(H))^3$ the parameter q_{\parallel} is a slowlyvarying function of x, and $q_{\parallel}(1) - q_{\parallel}(0) \sim \varepsilon^2$. At the same time, $q_{\perp}(x)$ varies with x in roughly the same way as q(x) in the Ising model in zero magnetic field: q(0) = 0, $q_{\perp}(1) \sim \varepsilon$. If on the other hand $JH^2 \approx (T_g(0) - T_g(H))^3$, i.e., near the Almeida-Thouless line, both q_{\parallel} and $q_{\parallel}(x)$ depend significantly¹²⁶ on x. Because of the weak dependence of q_{\parallel} on x near the line defined by (7.7), which is called the Gabay-Toulouse line, this irreversibility of longitudinal susceptibility has not yet been observed, but it becomes appreciable at lower temperatures near the Almeida-Thouless line. It is important to emphasize that, strictly speaking, no phase transition occurs in the vector spin glass on the Almeida-Thouless line, and that this line provides only a conventional separation between the temperature region in which nonergodicity in the longitudinal direction is small and the region in which it becomes significant. The phase transition occurs on the Gabay-Toulouse line. However, even on this line, only the transverse susceptibility should exhibit appreciable irreversibility.

7.2 Anisotropic glass, $D \neq 0$. The longitudinal and transverse spin components are frozen at different temperatures in the anisotropic glass, and the temperature at which the transition to the "longitudinal" or "transverse" glass takes place obviously depends on the magnitude and sign of D. The phase diagram on the T, D plane is shown in Fig. 14 (Refs. 127–128). Two phase transitions occur in the system for $D^+ > D > D^-$ when the temperature is reduced: $q_{\parallel}(q_{\perp})$ appears first and is followed by $q_{\perp}(q_{\parallel})$.

Strictly speaking, replica-symmetric equations are valid only up to the temperature of the first-order transition. Since only q_{\parallel} (q_{\perp}) arises, the system undergoes a transition to the nonergodic state and, in order to understand what happens at lower temperatures, we have to use the Parisi ansatz and introduce the functions $q_{\parallel,\perp}(x)$. Olderfield and Sherrington¹²⁹ consider that the second transition occurs in the nonergodic state as well. They have also investigated the phase diagrams for $J_0 > J$ for which long range thermodynamic order occurs in the system.

A detailed experimental study of the phase diagram of anisotropic spin glasses is reported in Refs. 130 and 131. In



FIG. 14. The T,D diagram of a spin glass with uniaxial anisotropy. P paramagnetic phase $(q_{\parallel} = q_{\perp} = 0)$, L—longitudinal spin glass $(q_{\parallel} \neq 0, q_{\perp} = 0)$, T—transverse glass $(q_{\parallel} = 0, q_{\perp} \neq 0)$, L + T—mixed phase $(q_{\parallel} \neq 0, q_{\perp} \neq 0)$. D ⁺ = 0.32J, D ⁻ = -0.20J.

highly anisotropic alloys with an easy axis, the spin-glass properties are observed only for the longitudinal susceptibility, whereas in anisotropic alloys with an easy plane, they appear in the transverse susceptibility. In slightly anisotropic alloys, two successive transitions are observed¹³¹ as the temperature is reduced, which is in agreement with the phase diagram of Fig. 14.

7.3 Effect of random anisotropy. Both exchange and anisotropy are random in disordered magnets. Fluctuations in the internal electric field acting on d and f electrons lead to fluctuations in the single-ion anisotropy energy and in the direction of the anisotropy axis. The interaction between the spins can also be anisotropic. In dielectric spin glasses, random anisotropy is due, in the first instance, to the dipole or pseudodipole interaction between local magnetic moments. In semiconductors containing magnetic impurities, pseudodipole anisotropy occurs because the spectrum of electron and hole states is formed under the influence of a strong spinorbit interaction with the lattice.¹³² In a metallic spin glass, the Dzyaloshinsky-Moriya anisotropy plays the dominant role. Its origin was explained by Fert and Levy^{133,134} in terms of the influence of the spin-orbit interaction between electrons and nonmagnetic impurities on the indirect RKKY exchange interaction between local spins. The anisotropic addition to the RKKY exchange energy between spins S_A and S_B , due to the nonmagnetic impurity at a point C, is proportional to

$$\mathscr{H}_{\rm DM} \sim -\frac{\left[\mathbf{R}_{AC}\mathbf{R}_{BC}\right]\left[\mathbf{S}_{A}\mathbf{S}_{B}\right]}{R_{AC}R_{BC}R_{AC}z},\tag{7.8}$$

where R_{AB} , R_{AC} , R_{BC} are the sides of the triangle formed by the magnetic and nonmagnetic atoms. The corresponding formula has been derived¹³⁵ for the case of a high concentration of nonmagnetic impurities for which the mean free path of electrons is of the order of, or less than, the mean separation between the magnetic impurities.

The energy of the system is found to increase when the spins which, in equilibrium, are oriented in accordance with the distribution of exchange energy and the random anisotropy energy, are uniformly turned over. Since relaxation to a new position of equilibrium in a spin glass occurs quite slowly, the corresponding anisotropy energy can be measured by examining its effect on the width and shape of the hysteresis loop, the position of the electron spin resonance, and the data obtained in torsion experiments.^{136–138} The anisotropy energy of metallic glasses is found to increase sharply when nonmagnetic impurities with high spin-orbit coupling constant are added to them. This confirms the validity of the idea of Fert and Levy about the Dzyaloshinsky-Moriya (DM) random anisotropy mechanism.

Random anisotropy mixes the longitudinal and transverse spin components, and therefore has a significant effect on the phase diagram of spin glasses.^{32,139} In the infiniterange model, the DM interaction is simulated by the following term in the Hamiltonian:

$$= \frac{1}{2} \sum_{i,j} S^{\mu}_{i} D^{\mu\nu}_{ij} S^{\nu}_{j},$$

where the mean value of the tensor $D_{ij}^{\mu\nu}$ is zero and the variance is D^2/N . When the external magnetic field is not too strong,³² i.e., $(H/T)^{2/3} \ll D/T \ll 1$, the random anisotropy mixes the longitudinal and transverse spin components so that the transition is almost of the Ising type and occurs on the Almeida-Thouless line. In a strong field, for which $(H/T)^{5/2} \gg D/T$, transverse freezing on the Gabay-Toulouse line takes place. Finally, $(H/T)^{5/2} \ll D/T \ll (H/T)^{2/3}$ defines the intermediate region in which the shape of the phase boundary is neither the Almeida-Thouless nor the Gabay-Toulouse line.

7.4 Experimental study of the H, T diagram. The work of Almeida and Thouless¹⁰¹ and of Gabay and Toulouse¹²¹ led to a large number of experiments in with the H, T diagram of spin glasses was investigated. As already noted, a reduction in the transition temperature in an external magnetic field was observed for all "pure" spin glasses. The temperature corresponding to the susceptibility maximum, and the temperature at which irreversibility begins, do not coincide in the magnetic field, and it is therefore unclear which of them is to be identified with $T_g(H)$ (see, for example, the analysis given in Ref. 15). In most experiments, the temperature $T_g(H)$, determined in a particular way, is found to vary in accordance with the law $\varepsilon \sim H^{2/3}$, which should be valid for Ising spin glasses, but not Heisenberg spin glasses. There is also a small number of papers in which the Gabay-Toulouse line was observed for vector spin glasses.^{130,140-142} It is possible that this was due to the influence of random anisotropy. When the magnetic field is small in comparison with the random anisotropy then, as noted in Sec. 7.3, the glass is Ising-like. However, in glasses exhibiting weak anisotropy, in which the transition occurs on the Gabay-Toulouse line, the Almeida-Thouless dependence may be observed because of the strong nonergodicity of the longitudinal spin components, which begins precisely on the Almeida Thouless line. We note in this connection that the derivative of the magnetic part of the specific heat of CuMn has an anomaly on the Gabay-Toulouse line.³⁵ This line has also been observed in the course of measurements of the transverse magnetic susceptibility.143

To conclude this section, we note that quantum fluctuations in spin glasses contribute to the free energy and do not vanish at T = 0 even in the infinite-range model. However, these fluctuations do not affect the shape of the phase diagram.^{146,147}



FIG. 15. Phase diagram of a disordered Ising magnet with $\overline{J}_{ij} = J_0/N$. The nonergodic region is shown shaded. The vertical line separates the $J_0 < J$ region in which m = 0 from the $J_0 > J$ region in which $m \neq 0$.

8. NONERGODIC MAGNETICALLY-ORDERED SYSTEMS

It is clear from the Sherrington-Kirkpatrick equations given by (3.13) that the spontaneous magnetization is zero in the absence of the magnetic field if $J_0 < J$. When $J_0 > J$, ferromagnetic order becomes possible. The corresponding phase diagram is shown in Fig. 15 for the Ising magnet. The replica-symmetric phase of the frustrated ferromagnet described by the equations of state (3.13) lies between the $T = J_0$ line and the Almeida-Thouless line. Replica symmetry is broken on the Almeida-Thouless line, and the entire shaded region has the properties of a spin glass. We note that T_g does not vanish for any J_0 . When $J_0 \gg J$, the temperature T_g is exponentially small and is given by (3.20) in which H must be replaced with J_0 .

Since the suceptibility does not become infinite on the Almeida-Thouless line, the magnetization does not vanish in the transition from the frustrated ferromagnet to the nonergodic state. The phase in which the ordinary magnetic order parameter and the Parisi function coexist is often referred to as mixed. The mixed phase in which $m \neq 0$ will be referred to as a ferroglass.

As already noted, the equilibrium susceptibility of a pure spin glass with m = 0 is independent of T. It follows that temperature phase transition from the ferromagnetic to the mixed phase is impossible, i.e., the boundary between the "pure" spin glass and the mixed phase is a vertical line¹²¹ (Fig. 15). It follows that, whatever the relationship between the parameters, the magnetization of the mixed phase is nonzero right down to T = 0.

As already noted, the existence of the mixed phase was demonstrated experimentally for the frustrated antiferromagnet. The rich phase diagram of an antiferromagnet in an external field, in which a first or second order transition can occur, depending on the relationship between the parameters, can in principle be used to investigate the effects of frustration and nonergodicity on different phase transitions and on multicritical points or, conversely, to examine the influence of a change in long-range order on the transition to the spin glass. In the media that we have considered so far (pure spin glass and ferroglass), the magnetic field reduces monotonically the size of the temperature range in which the nonergodic phase exists. In antiferromagnetic spin glasses, in which the field suppresses both the long-range order and nonergodicity, a more complex field dependence may be expected.

A natural generalization¹⁴⁸ of the Sherrington-Kirkpatrick model to nonergodic antiferromagnets involves the assumption that the spins σ_{pi} are assumed to be distributed over two sublattices p = 1,2, and in the Ising exchange Hamiltonian of the system

$$\widetilde{\mathscr{H}} = \sum_{i, j} J_{ij} \sigma_{1i} \sigma_{2j} - \frac{1}{2} \sum_{p; i, j} V_{ij} \sigma_{pi} \sigma_{pj} - H \sum_{p, i} \sigma_{pi} \quad (8.1)$$

the energies of intra- and inter-sublattice interaction V_{ij} and J_{ij} are distributed normally with mean values V_0/N and J_0/N and variances $VN^{-1/2}$ and $JN^{-1/2}$ where N is the number of spins in the sublattice.

The set of equations for the sublattice magnetizations $m_{1,2}$ and the Edwards-Anderson parameters $q_{1,2}$, which describes the system above T_g is¹⁴⁹

$$m_{1,2} = \langle \operatorname{th} E_{1,2} \rangle_{c}, \quad q_{1,2} = \langle \operatorname{th}^{2} E_{1,2} \rangle_{c}, \quad (8.2)$$

$$E_{1,2} = H - J_{0} m_{2,1} + V_{0} m_{1,2} + 2 (J^{2} q_{2,1} + V^{2} q_{1,2})^{1/2},$$

and the equation for the Almeida-Thouless temperature is

$$(T_{g}^{2} - V^{2} \langle ch^{-4} E_{1} \rangle_{c}) (T_{g}^{2} - V^{2} \langle ch^{-4} E_{2} \rangle_{c})$$

- J⁴ \lappa ch^{-4} E_{4} \rangle_{c} \lappa ch^{-4} E_{2} \rangle_{c} = 0. (8.3)

Configurational averaging over the random fields z is performed as in (3.13) with the Gaussian distribution function. In the ergodic state, the free energy is

$$\begin{split} f &= -\frac{J_0}{2} m_1 m_2 + \frac{V_0}{4} (m_1^2 + m_2^2) - \frac{J^2}{4T} (1 - q_1) (1 - q_2) \\ &- \frac{V^2}{8T} \left[(1 - q_1)^2 + (1 - q_2)^2 \right] - \frac{T}{2} \langle \ln \langle 4 \operatorname{ch} E_1 \cdot \operatorname{ch} E_2 \rangle \rangle_{\text{c}}. \end{split}$$

$$(8.4)$$

In the absence of the field, for $V_0 = 0$, the phase diagram in terms of the coordinates $T(J^2 + V^2)^{-1/2}$, $J_0(J^2 + V^2)^{-1/2}$ is identical with that shown in Fig. 15 in which long-range order is not ferromagnetic but antiferromagnetic. However, the disordered antiferromagnet does not at all behave like a ferromagnet in the external field.

In the ordered antiferromagnet, the line corresponding to second order phase transitions on the T, H plane is known to transform at the tricritical point (whose position depends on the ratio of J_0 and V_0) to the line corresponding to first order transitions. To begin with, let us suppose that V_0 and Vare small, so that the intra-sublattice interaction can be neglected and antiferromagnetic order arises only via the second order transition.

Analysis of (8.3) then shows that, in this case, and whatever the actual relationships between the parameters, the typical effect is an increase in the temperature of transition to the mixed phase with increasing field. This is most simply seen in the weakly frustrated antiferromagnet with-



152 Sov. Phys. Usp. 32 (2), February 1989

out the intrasublattice interaction. According to (8.3), we have in this case

$$T_g^4 = J^4 \langle \operatorname{ch}^{-4} E_1(z) \rangle_c \langle \operatorname{ch}^{-4} E_2(z) \rangle_c.$$
(8.5)

When the external field does not overcompensate the molecular field, so that $|H - J_0 m_{1,2}| \ge J$, the temperature T_g is described by analogy with (3.20) by the formula

$$T_{z}(H) = \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} J \exp\left\{-\frac{1}{4J^2} \left[(H - J_0 m_1)^2 + (H - J_0 m_2)^2\right]\right\}.$$
 (8.6)

In the magnetically ordered phase $(H < J_0)$, in which $m_1 = -m_2 = 1$, the temperature T_g (H) decreases with increasing field, and on the boundary of the region $|H - J_0 m_{1,2}| \ge J$ we have

$$T_{g}(H \approx J_{0}) \approx \exp\left(-\frac{J_{0}^{2}}{J^{2}}\right).$$
 (8.7)

When $H > J_0$, we have $m_1 = m_2 = 1$ and

$$T_{\rm g}(H) = \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} J \exp\left[-\frac{1}{2J^2} (H - J_0)^2\right]. \quad (8.8)$$

Even when $H - J_0 \gg J$, the transition point given by (8.8) can be exponentially greater when T_g for $H < J_0$ [see (8.7)]. This means that, when $|H - J_0| \approx J$, so that the order parameter vanishes rapidly, the temperature T_g increases exponentially with increasing field.

The increase in T_g with increasing field is due to the reduction in the resultant field on one of the sublattices and, correspondingly, to the increase in local susceptibility. When the degree of disorder is low, the increase in $T_g(H)$ occurs in a narrow interval of field values $\Delta H \approx J$ near $H \approx J_0$, but this interval expands with increasing degree of frustration. It is clear from Fig. 16a that, when $J_0/J = 1.7$, which corresponds to a fully realistic situation, we have $T_N(0)/T_g(0) = 9.4$, and T_g increases monotonically with increasing fields.

After crossing the $T_N(H)$ line, the temperature $T_g(H)$ falls with increasing field in accordance with the Almeida-Thouless formula (3.17). At the triple point (T_0, H_0) , at which the line representing the second order phase transition crosses the $T_g(H)$ curve, the latter exhibits a break. The ratio $T_g(H_0)/T_g(0)$ increases with the decreasing J/J_0 . It follows from (8.3) that fluctuations in the intra-sublattice exchange V can only expand the range of fields for which $T_g(H)$ increases with H, and enhance this rise (we are assuming that V_0/J_0 is small, so that neither the metamagnetic transition nor the tricritical point are present).

If the mean intra-sublattice exchange is not too small, the $T_g(H)$ line crosses not the $T_N(H)$ line, as in Fig. 16a,

F16. 16. Phase diagram of a disordered Ising antiferromagnet in a magnetic field. The thick curve shows the second order transition. The nonergodic regions are shown shaded. $a-V_0 = V = 0$. The dashed line shows the slope of the $T_N(H)$ curve in the nonergodic phase; b— $V_0 = 1.25$; $J_0 = 0.25$. $V^2 = J^2 = 0.36$. The tricritical point can be clearly seen. Dot-dash line—loss of stability by the paramagnetic (1) and antiferromagnetic (2) phases; dashed lines—possible behavior of the phase transition lines below $T_g(H)$.

but the first order phase transition line. The phase diagram always has the form shown in Fig. 16b (Refs. 150 and 151). Its main feature is that, over the segment AB, the magnetic transition is accompanied by an abrupt appearance (disappearance) of the spin glass order parameter q(x), i.e., we have a second order phase transition to the nonergodic state. On the remainder of the curve bounding the nonergodic state, we have the usual third order transition to the spin glass.

It is important to emphasize that, even in the ergodic state, frustrations can sharply alter the usual phase transition picture. For example, a complete "spreading" of the jump by frustrations and the disappearance of the metamagnetic transition are possible.^{150,151}

Equations (8.2) and (8.3) enable us to construct the magnetic phase diagram in the ergodic state, and to determine its stability boundary (the Almeida-Thouless line). To construct the magnetic phase diagram below $T_g(H)$, we have to use the Parisi equations of state (6.1)–(6.6) for the spin glass, instead of the Sherrington-Kirkpatrick equations. These equations were given in Ref. 149 for the antiferromagnetic spin glass, and were solved near T_g . In particular, the solution shows that the $T_g(H)$ line continues into the nonergodic phase with a slight break at the triple point (T_0, H_0) , i.e., as usual, phase transitions are possible in the nonergodic phase on the field (temperature) axis, and are accompanied by the evolution (disappearance) of long-range magnetic order.

The equations obtained in Ref. 149 have been used to investigate some of the magnetic properties observed below T_g . It was found that, whatever the relationship between the parameters, the sublattice magnetization changes smoothly at T_g , without showing a break. We have already noted that precisely this type of behavior was observed experimentally.⁷¹ The reversible magnetic susceptibility does exhibit a break at T_g , and decreases into the nonergodic phase.

Heisenberg antiferromagnets with easy-axis anisotropy typically exhibit an initial increase in T_g with increasing field, applied along the easy axis, and then a reduction in $T_g(H)$ on the line of first order phase transition (spin flop), followed by an increase in T_g in the overturned phase.¹⁵²

A jump in the nonergodic order parameter occurs on the line representing the first order transition, just as it does for the metamagnet.

Frustrations not only modify the phase diagram in the ergodic state, which we have already discussed above, but they can also modify the usual temperature dependence in an



FIG. 17. The H, T diagrams for the alloys $Fe_{65}Ni_{14}Co_{20}$ (a) and $Fe_{68}Ni_{12}Cr_{20}$ (b) (Ref. 154).

153 Sov. Phys. Usp. 32 (2), February 1989

antiferromagnet. For example, frustrations always lead to an increase in transverse susceptibility below the Neel point. The behavior of longitudinal susceptibility is more complicated. Depending on the relationship between V and J, it can either increase into the antiferromagnetic phase or, as usual, it can decrease.^{150,152,153} The strong increase in T_g with H in the Ising and Heisenberg antiferromagnetic spin glasses (sometimes by a factor of several times) has been observed experimentally^{68,73,154} (Fig. 17). According to the theory presented above, the relative increase in T_g (H) becomes greater as the degree of frustration decreases. An increase in the transverse susceptibility with decreasing temperature below T_g was reported in Ref. 69.

9. ULTRAMETRIC TOPOLOGY OF THE SPACE OF STATES. NON-SELF-AVERAGING OF THE ORDER PARAMETER IN SPIN GLASSES

Studies of the structure of the infinitely degenerate ground state, undertaken after the physical significance of the Parisi order parameter became clear, have shown¹⁵⁵ that the space of states (valleys) in the Parisi solution has certain very general (and possibly universal) properties for systems with frustrations and disorder.

The so-called triangle rule was established first. This states that, if in the space of states, we introduce the distance between valleys α and β in accordance with the formula

$$d(\alpha, \beta) = \frac{1}{N} \sum_{i} (m_{i}^{\alpha} - m_{i}^{\beta})^{2} = 2 (q(1) - q^{\alpha\beta}).$$
(9.1)

then all triangles in the space of states are equilateral, i.e., d(1,2) = d(1,3) = d(2,3), or isosceles with the base shorter than the other two sides, d(1,2) = d(1,3) > d(2,3). The triangle rule was established in Ref. 155 as a result of an analysis of the distribution function $P(q_1,q_2,q_3)$ that gives the probability, averaged over the distribution $\{J_{ij}\}$, of simultaneously finding three valleys with overlaps q_1, q_2, q_3 :

$$P(q_{1}, q_{2}, q_{3}) = \langle P_{J}(q_{1}, q_{2}, q_{3}) \rangle_{c}$$

$$= \lim_{n \to 0^{n}} \frac{1}{(n-1)(n-2)} \sum_{\substack{a \neq b, h \neq c, \\ a \neq c}} \delta(q_{nc} - q_{1}) \,\delta(q_{ab} - q_{2}) \,\delta(q_{bc} - q_{2})$$
(9.2)

where P_J is the distribution function for a given realization. The algebra of Parisi matrices leads to the relation

$$P(q_1, q_2, q_3) = \frac{1}{2} P(q_1) x(q_1) \delta(q_1 - q_2) \delta(q_2 - q_3)$$

+ $\frac{1}{2} (P(q_1) P(q_2) \theta(q_1 - q_2) \delta(q_2 - q_3)$
+ permutations). (9.3)

which is actually the triangle rule.

The simplest way of confirming that the triangle rule is a direct consequence of the method used to parametrize the matrices q is to verify that it is satisfied for matrices such as (5.3) even before going to the limit $n \rightarrow 0$, in the case that n, k, $\{m_k\}$ are integers.

If follows from the triangle rule that all the valleys can be distributed over nonoverlapping groups (clusters), each of which is determined by the maximum separation d



FIG. 18. Hierarchic structure of the states of a spin glass. Points at the ends of the branches represent states; the branches represent clusters. The separation between two sets is determined by the height of the level (1, 2, or 3) occuapied by the immediate progenitor.

between valleys (minimum overlap q). Actually, if a valley γ belongs to two clusters $I_1(d)$ and $I_2(d)$, its distance from the two valleys α and β belonging to clusters $I_1(d)$ and $I_2(d)$ does not exceed $d: d(\gamma, \alpha) \leq d, d(\gamma, \beta) \leq d$. However, according to the triangle rule, this means that $d(\alpha, \beta) \leq d$. Hence, clusters formed at the same "level of overlap", i.e., with the same values of d and q, must either coincide or not overlap at all.

This property leads to the hierarchic structure of the space of states in a spin glass. We shall use points on a given horizontal level to represent noncrossing clusters of states with the same maximum distance $d(\alpha,\beta) < d$ (Fig. 18).

Each of the clusters can in turn be subdivided into clusters with $d(\alpha,\beta) < d' < d$. These are represented on a lower horizontal level, and the procedure can be continued indefinitely in steps as small as desired, right down to clusters with zero maximum separation d'. Spaces with this hierarchic structure (Fig. 18) are called ultrametric (a detailed review of the applications of ultrametry in physics can be found in Ref. 14).

The strong correlation between valleys, which is typical for ultrametric space, shows that there should be a specific correlation between microscopic distributions of magnetization in different valleys. This very interesting question was examined in Ref. 156. Calculations of correlation functions for magnetizations in different valleys show that, for any valleys with a common progenitor on the genealogic tree of Fig. 18, it is possible to subdivide the entire system of N spins into a certain number of subsystems in such a way that the magnetization of each subsystem is the same for all valleys. On the other hand, if any pair of chosen valleys has the same immediate progenitor, the magnetizations inside the subsystems have uncorrelated distributions for all valleys. It follows that the valley overlap (and other analogous averages) are simply related to subsystem magnetizations. For example,

$$q^{\alpha\beta} = \frac{1}{N} \sum_{i} m_i^{\alpha} m_i^{\beta} = \sum_{l} \frac{c_l}{N} M_l^2; \qquad (9.4)$$

where M_i is the subsystem magentization and c_i the relative number of spins in each of the subsystems. More detailed information on magnetization distributions also can be found in Ref. 156.

We now return to the description of a spin glass as a system of valleys in ultrametric space. To develop this description, we must elucidate the statistical properties of the quantities

$$P_{\alpha} = e^{-\beta f_{\alpha}} \left(\sum_{\gamma} e^{-\beta f_{\gamma}} \right)^{-1}, \quad \beta = T^{-1}.$$
(9.5)

154 Sov. Phys. Usp. 32 (2), February 1989

The point is that the free energies of the valleys are equal in the thermodynamic limit (i.e., the free energies per spin $\lim (F_{\alpha}/N)$ are equal as $(N \rightarrow \infty)$, but corrections of order N^{-1} to the free energies are random quantities.^{155,157,158} It was shown in Refs. 158 and 159 that, if we consider the free energies of valleys as independent random variables distributed in accordance with the law

$$P(f_{\alpha}) = \rho \exp \rho (f_{\alpha} - f_{c}) \theta (f_{c} - f_{\alpha}); \qquad (9.6)$$

$$\rho = \beta x (q_{\max}),$$

the free energy and the correlation functions of this set of independent valleys are the same as the corresponding quantities in the Parisi solutions, where q_{\max} is the maximum value of q(x). The cut off energy f_c and the number L of valleys must be allowed to tend to infinity at the end of the calculation, letting the density of states of given energy remain finite:

 $\lim Le^{-\rho f_{\rm C}} = v. \tag{9.7}$

All the results are independent of v.

The ultrametric topology of the space of states and the distribution function (9.6) provide a complete characterization of the space of valleys. The Parisi solution can be reproduced¹⁵⁹ by assuming this property and without introducing any further hypotheses.

The exponential increase in the distribution function (9.6) with increasing f_{α} is an indication of strong fluctuations in the free energy of valleys. The valley distribution can be transformed into the distribution I(q) of clusters that unite valleys with $q^{\alpha\beta} > q$. We now introduce the statistical weights W_I and free cluster energies f_I , given by

$$W_{I} = \sum_{\alpha \in I} P_{\alpha} = e^{-\beta f_{I}} \left(\sum_{I'} e^{-\beta f_{I'}} \right)^{-1}, \qquad (9.8)$$

It can be shown^{155,158} that the distribution function $P(f_I)$ differs from (9.6) only by the replacement of q_{\max} with q. We thus see that the distribution function depends on the parameters of the field, the temperature, and q in a universal manner via the function ρ .

From (9.6) we can obtain the statistical-weight distribution f(W,y) averaged over configurations with given y, which determines the distribution of all the physical quantities. It is found to be¹⁵⁵

$$f(W, y) = \frac{W^{y-2}(1-W)^{-y}}{\Gamma(y)\Gamma(1-y)} , \qquad (9.9)$$

where

$$y = 1 - x (q) < 1, \tag{9.10}$$

and $\Gamma(y)$ is the gamma function.

Equations (9.5)-(9.10) enable us to establish a clear picture of the cluster distribution in ultrametric space. The divergence of the integral $\int_0^1 f(W,y) dW$ at the lower limit shows that the total number of clusters for any $y \neq 1$ is infinite because of the large number of clusters with small W.

The function Wf(W,y) determines the probability, averaged over $\{J_{ij}\}$, that a randomly chosen valley belongs to a cluster with statistical weight W. It is clear from (9.9) that clusters with a small W provide a negligible contribution to the integral Wf(W,y) i.e., the number of states in them is small. As $y \to 1$, the function Wf(W,y) diverges near W = 1. Since the sum $\sum f W_I$ is equal to unity independently of y, this means that one cluster plays the dominant part in the limit as $y \rightarrow 1$. This is not an unexpected conclusion because the limit as $y \rightarrow 1$ corresponds to a transition to the ergodic state.

Corrections of order N^{-1} to the free energy give rise to an additional contribution to entropy, which can be expressed in terms P_{α} in the usual way:

$$\Delta S = -\sum_{\alpha} P_{\alpha} \ln P_{\alpha}. \tag{9.11}$$

The average of ΔS over the configuration is

$$\Delta S = -\int_{0}^{17} f(P) P \ln P \, \mathrm{d}P = \psi(1) - \psi(y_1), \qquad (9.12)$$

where $f(P) = f(W, y_1)$ and $y_1 = 1 - x(q_{\max})$ (Ref. 155); ψ is the digamma function. This quantity diverges as $T \rightarrow 0$ (we recall that the intensive part of entropy, which is proportional to N, tends ot zero as $T \rightarrow 0$). In principle, (9.9) can be used to construct the entropy distribution function.

The strong fluctuations in the free energy of valleys, determined by (9.6), suggest configurational fluctuations in the Parisi order parameter. The most convenient to investigate are fluctuations in the function

$$Y_{J}(q) = \int_{0}^{1} P_{J}(q') \, \mathrm{d}q'$$
 (9.13)

whose mean is y(q) (the index J indicates that the quantity is evaluated for a given distribution of exchange integrals). Since

$$Y_{J}(q) = \sum_{\alpha, \beta} P_{\alpha} P_{\beta} \theta \left(q^{\alpha \beta} - q \right) = \sum_{I} W_{I}^{2}, \qquad (9.14)$$

the distribution $\prod_{J} (Y)$ of the quantities $Y_{J}(q)$ can be written in the form

$$\Pi_{J}(Y) = \langle \delta(Y_{J} - Y) \rangle_{c} = \langle \delta(\sum_{I} W_{I}^{2} - Y) \rangle_{c}.$$
(9.15)

For Y approaching 1, the main contribution to $\Pi_J(Y)$ is provided by one cluster, just as in the case of f(W,y), and it then follows from (9.9) that, as $Y \rightarrow 1$,

$$\Pi_{T}(Y) \sim (1 - Y)^{-y}. \tag{9.16}$$

The most probable value is Y = 1 which, generally speaking, has nothing in common with the mean $\langle Y_J(q) \rangle_c = y(q)$. It is clear that, even in the thermodynamic limit $N \to \infty$, the order parameter of the spin glass is found to be non-selfaveraging. Both the valley overlap distribution function $P_J(q)$ and the distribution function for $P_J(q)$ appear at the transition point. Calculations show¹⁵⁵ that, for example, the variance of $P_I(q)$ is

$$\langle P_{J}(q_{1}) P_{J}(q_{2}) \rangle_{c} - P(q_{1}) P(q_{2})$$

= $\frac{1}{3} \langle P(q_{1}) \delta(q_{1} - q_{2}) - P(q_{1}) P(q_{2}) \rangle.$ (9.17)

Young, Bray and Moore¹⁶⁰ were the first to investigate the self-averaging of observables. They showed that quantities such as energy, magnetization, and the Edwards-Anderson parameter do not fluctuate in the macroscopic limit, and that the Gibbs susceptibility is non-self-averaging. The different fluctuation properties of m and $\chi = dm/dh$ can be qualitatively explained by the fact that, although the m(h)curve for each particular realization approaches the line

 $\langle m(h) \rangle_{c} = \chi h$ as N increases, its slope at h = 0 is, in general, unrelated to $\chi = \langle \chi_J \rangle_c$. However, measurements of this susceptibility, which fluctuates strongly from sample to sample, require fields smaller than the characteristic fluctuation scale Δh . Fields greater than Δh tend to "smear out" the fluctuation and restore self-averaging. Since $\Delta h \rightarrow 0$ as $N \rightarrow \infty$ (the authors of Ref. 160 consider that $\Delta h \sim N^{-1/2}$), a realistic experiment demonstrating non-self-averaging in spin glasses cannot unfortunately be formulated. On the other hand, a verification of the non-self-averaging of physical quantities in the Sherrington-Kirkpatrick model and of the ultrametric distribution should be very important because it would comprehensively resolve the question as to whether the extremum of the functional Φ , found by Parisi, yields the true free energy of spin glasses. Numerical experiments have therefore been carried out and have produced qualitative agreement with the Parisi theory, at first for small¹⁶¹ and later for relatively large¹⁶² systems, so that the qualitative conclusions reached so far can be regarded as reliable.

We note that non-self-averaging of the order parameter is also found to arise in the so-called random energy model,¹⁶³ which in many respects is similar to the Sherrington-Kirkpatrick model, but is mathematically simpler.

The molecular field theory provides a satifactory description of many of the statistical properties of spin glasses. However, a spin glass is then an absolutely nonergodic system in which barriers between different states are infinite, so that the dynamics of the system can be related only to intravalley transitions. It was precisely this dynamics that was essentially investigated in the pioneering work of Sompolinsky and Zippelius.^{119,120} However, experiments have shown that transitions between valleys do occur and are indeed responsible for the wide spectrum of relaxation times that leads to the specific dynamic properties of spin glasses (Sec. 1). Attempts have therefore been undertaken to construct a phenomenological theory of the dynamic properties of spin glasses.^{164,171} It was assumed that the ulatrametric structure of metastable states persisted in real glasses with short-range interaction, but that barriers between different states were finite. By specifying some particular barrier distribution or transition probability distribution, and then solving the kinetic equation for the probability $P_{\alpha}(t)$ of finding the system in a state α at time t, it was possible to reproduce the experimentally observed time dependences, including the stretched exponential law.²⁾

10. SPIN GLASSES WITH DISTANCE-DEPENDENT EXCHANGE INTERACTION

10.1 Spatial dispersion in the Parisi scheme. It is now reasonably well-established that the Parisi ansatz provides the correct solution in the infinite-range model. However, it still remains unclear to what extent the ultrametric structure of the ground state describes the properties of real spin glasses in which the exchange interaction energy decreases with distance in a particular way.

In principle, this question can be resolved by developing a perturbation theory in which the small parameter is the reciprocal of the exchange interaction range, and the Parisi theory is the zero-order approximation. Attempts of this type were undertaken in Refs. 173–178.

In spin glasses with a short-range interaction, the free

energy functional assumes the following form 176,179 [instead of (6.1)]:

$$f\left(Q^{\alpha\beta}\left(\mathbf{r}\right)\right) = \int d\mathbf{r} \left[\frac{J^{2}}{T^{2}} \sum_{\alpha \neq \beta} \left(Q_{\alpha\beta}\left(\mathbf{r}\right)\right)^{2} + \frac{J^{2}}{4zT^{2}} \sum_{\alpha \neq \beta} \left(\nabla Q^{\alpha\beta}\left(\mathbf{r}\right)\right)^{2} - \ln \operatorname{Sp} \exp\left(\frac{1}{2} \frac{J^{2}}{T^{2}} \sum_{\alpha \neq \beta} Q^{\alpha\beta}\left(\mathbf{r}\right) \sigma^{\alpha}\left(\mathbf{r}\right) \sigma^{\beta}\left(\mathbf{r}\right)\right)\right],$$

$$(10.1)$$

where z is the number of neighbors within the interaction range and the matrix $Q^{\alpha\beta}(\mathbf{r}) = \langle \langle \sigma^{\alpha}(\mathbf{r}) \sigma^{\beta}(\mathbf{r}) \rangle_{T} \rangle_{c}$ depends on the position coordinate **r** in the short-range interaction model. In the lowest order in the fluctuations relative to the Parisi solution

$$R_{\mathbf{r}}^{\alpha\beta} = Q_{\mathbf{r}}^{\alpha\beta} - q^{\alpha\beta}$$

the effective Lagrangian is then found to be

$$L(R_{\mathbf{r}}^{\alpha\beta}) = -\frac{J}{2} \int d\mathbf{r} \left[\sum_{\alpha \neq \beta} (\nabla Q_{\mathbf{r}}^{\alpha\beta})^2 + \sum R_{\mathbf{r}}^{\alpha\beta} M^{\alpha\beta, \gamma\nu} R_{\mathbf{r}}^{\gamma\nu} \right],$$
(10.2)

where \widehat{M} is the stability matrix (3.16). The Fourier transform on the Green's function for the fluctuations $R^{\alpha\beta}$ given by

$$G_{\mathbf{p}}^{\alpha\beta,\ \gamma\delta} = \langle R_{\mathbf{p}}^{\,\beta\beta} R_{-\mathbf{p}}^{\delta\gamma} \rangle, \tag{10.3}$$

where p is the momentum, is found from the equation

$$\hat{G}_{p}(p^{2}+\hat{M})=\hat{I},$$
(10.4)

i.e., the properties of the Green's function are intimately related to the properties of the eigenvalues of the stability matrix \hat{M} . In particular, the existence of the gapless mode is responsible for the fact the G_p diverges for $p \rightarrow 0$.

Altogether, there are seven independent functions $G_p^{\alpha\beta,\gamma\delta}$, the simplest of them being $G_p^{\alpha\beta,\alpha\beta} = G_p(x)$, where x is determined from the condition $q^{\alpha\beta} = q(x)$. This function exhibits the following behavior for small p and $\varepsilon = (T_g - T)/T_g$:

$$G_p(x) \sim p^{-4}, \quad x \ll p,$$

$$G_p(x) \sim p^{-2}, \quad x_1 \gg x \gg p,$$
(10.5)

where $x_1 \sim \varepsilon$ is found from the condition $q(x_1) = q(1)$. When $x \gg x_1$, the Green's function becomes $G(p) \sim p^{-2}$.

The nature of the singularity in G_p for small p is thus seen to depend on x.

In addition to $G_{p}(x)$, there are six independent functions that depend on two or three arguments such as x. Their behavior can be more singular than that of (7.5). The most singular function behaves¹⁷⁷ as p^{-6} . To find the spin-glass susceptibility

$$\chi_{\rm SG} = N^{-1} \sum_{i, \ j} (\langle \sigma_i \sigma_j \rangle_{\rm T} - \langle \sigma_i \rangle_{\rm T} \, \langle \sigma_j \rangle_{\rm T})^2,$$

we must know the behavior of all the Green's functions for all values of the different arguments because χ_{SG} is obtained from $G^{\alpha\beta,\gamma\delta}$ by summing over the replica indices, i.e., by integrating with respect to an argument such as x for $n \to 0$. However, so far, no-one has succeeded in overcoming the mathematical difficulties encountered in the execution of this program. It has only shown¹⁷⁵ that the correlator χ_{SG} , obtained by averaging over one valley, i.e., for $x \to 1$, behaves as p^{-2} .

156 Sov. Phys. Usp. 32 (2), February 1989

10.2. Other analytic approaches. An attempt to constuct a renormalization-group scheme for the effective Hamiltonian (10.1) was undertaken in Ref. 180.

An effective Hamiltonian, which allows one to reduce the problem of the phase transition in a spin glass to the problem of the Anderson localization, has been introduced in Ref. 181 using an expansion in terms of the reciprocal of the range of the exchange interaction. As a result the critical behavior does not admit scaling, and the nonlinear susceptibility lies in a region bounded by two enveloping curves. However, such behavior is not observed experimentally, the values of the critical indices characterizing various quantities differ significantly from those predicted in Ref. 181.

The difficulties encountered in the theory of spin glasses with a finite interaction range, which have not been overcome, have stimulated interest in special models, e.g., models³⁾ with short-range interaction on the Bethe lattice¹⁸²⁻¹⁸⁴ and the dilute spin glass model proposed by Vianna and Bray.^{186,189} However, the significance of their results in relation to real spin glasses remains unclear.

10.3. Numerical simulation. Heuristic models. It is well-known¹⁹⁰ that the molecular field theory predicts correctly the order parameter and the character of the phase transition if the dimension of space is greater than the so-called critical dimension d_c . Since, as we have already noted, a consistent analytic theory of spin glasses with short-range interaction has not as yet been constructed, extensive studies of spin glasses have been undertaken by numerical simulation.

10.3.1 Ising model. The most extensively investigated model is the simple Ising magnet with nearest-neighbor interaction in the simple quadratic or cubic lattice with a symmetric distribution of exchange interactions. Most of this work relies on the Monte Carlo method. Because of the slow of relaxation of spin glasses, exceedingly long computations are necessary to achieve reliable data. For example, in the work reported in Ref. 191, six months of machine time on a specially constructed fast computer were spent in an investigation of the behavior of a spin glass over long intervals of time.

The problem of phase transitions in two-dimensional glasses was solved a long time ago. Precise calculations of the partition function for finite systems with particle number $N \leq 250$ (Ref. 192), calculations of the energy defect, ¹⁹³ and Monte Carlo computer simulations¹⁹⁴ have shown that phase transitions do not occur in two-dimensional magnets. The Edwards-Anderson parameter is zero at all temperatures, an the correlation lengths of the nonlinear susceptibility associated with the Edwards-Anderson parameter remains finite, although it does increase with decreasing temperature.

The situation is more complicated in the case of the three-dimensional glass. Machine simulation was used in Refs. 192 and 193 as a basis for the conclusion that there are no phase transitions in the three-dimensional case and $d_c = 4$. This conclusion was in agreement with the result obtained by using the high-temperature expansion.¹⁹⁵ However, these results were subsequently reconsidered.

The first powerful arguments in favor of $d_c \leq 3$ were put forward by Bray and Moore^{196,197} and Macmillian^{198,199} who used the methods proposed in Ref. 192. Let us split the systems into blocks of size L, and introduce the characteristic energy of interaction between blocks \tilde{J} , whose dependence on L is determined by the critical index v: $\tilde{J}(L) = JL^{-1/\nu}$. When $\nu > 0$, the energy \tilde{J} tends to zero as $L \to \infty$, i.e., there is no correlation between distant spins in the system and, hence, there is no phase transition at a finite temperature. The index ν determines in this case the temperature dependence of the correlation length $\xi: \xi \sim (J/T)^{\nu}$. On the other hand, $\nu < 0$, a phase transition does occur at a finite temperature.

The quantity $\tilde{J}(L)$ is equal to the so-called defect energy which can be calculated on a computer. Let us specify periodic boundary conditions along all axes except one, for which we use random boundary conditions.

This means that spin directions in the first and last rows (d = 2) or the first and last planes (d = 3) are fixed randomly either upward or downward. If we look upon the sample under investigation as part of an infinite system, the random boundary conditions will simulate the coupling of the finite sample to the remainder of the system. The defect energy E_{def} is the difference between the energies for the given boundary conditions and the situation when the orientation of the spins has been reversed on one of the hypersurfaces with random boundary conditions.

Calculations show that the distribution of the random quantity E_{def} very rapidly reaches its asymptotic form with increasing sample size L, and is probably independent of the form of the distribution of exchange integrals. All the characteristic parameters of the distribution, namely, its mean, variance, and so on, then have the same dependence on L. We can therefore take $\tilde{J}(L)$ to be any of these quantities, e.g. $|\vec{E}_{def}|$.

Calculations have shown that v > 0 for d = 2, i.e., there is no phase transition in the two-dimensional case for $T \neq 0$, whereas in the three-dimensional case v < 0 and there *is* a phase transition. This conclusion is also confirmed by the high-temperature expansion of the Edwards-Anderson susceptibility²⁰⁰ χ_{SG} . The critical index of this susceptibility for d = 3 was found to be $\gamma = 2.9 \pm 0.5$, which is in good agreement with Monte Carlo calculations.¹⁹¹ The incorrect result reported in the former paper on high-temperature expansions was due to the fact that the number of terms in the series that were analyzed was to small.

The most complete picture of a phase transition in a spin glass with short-range interaction was reported by Ogielski¹⁹¹ who investigated long-term relaxation in large spin arrays up to $64 \times 64 \times 64$, and concluded that there was a phase transition in the three-dimensional case. For a non-zero temperature T_g , the correlation length ξ was found to diverge in accordance with the expression $\xi \sim (T - T_g)^{-\nu}$, $\nu = 1.3$. For the same temperature, the characteristic relaxation time was also found to diverge (see below). However, the order parameter was zero throughout the low temperature phase. The correlation length is probably infinite everywhere below T_g , i.e., any temperature $T < T_g$ is critical in the three-dimensional case, and d_c is either equal to or close to 3 (Refs. 201 and 202). Figure 19 shows q(t) defined by

$$q(t) = \frac{1}{N} \sum_{i} \langle \langle \sigma_i(0) \sigma_i(t) \rangle_{\mathrm{T}} \rangle_{\mathrm{c}}.$$
 (10.6)

It is related to the Edwards-Anderson parameter by

$$q = \lim q(t). \tag{10.7}$$

157 Sov. Phys. Usp. 32 (2), February 1989



FIG. 19. The function q(t) at temperatures $T/T_g = 1.30$ (1), 1.25 (2), 1.20 (3), 1.10 (4), 1.00 (5), 0.60 (6) and 0.70 (7). Lattice size 32³. The data are shown together with the corresponding uncertainties. The unit of time measurement is the Monte Carlo step (MCS).

The nature of the relaxation process is quite different above and below the transition point $T_g = 1.1275$ (model in which the exchange interaction energy between nearest neighbors in the lattice is ± 1). Above T_g , the relaxation is exponential

$$q(t) = At^{-x}e^{-\omega t^{\beta}}$$
(10.8)

where β and x are functions of temperature: $\beta(T_g) = 0.3$, $\beta(T) \rightarrow 1$ for $T \rightarrow \infty$, whereas at lower temperatures q(t) is a function of a power of t ($\omega = 0$). Despite the fact that the variation described by (10.8) becomes very slow as the temperature decreases (x(T) < 0.05 for T = 0.7) the overall shapen of the curves for all $T < T_g$ shows that q ($t \rightarrow \infty$) = 0. Generalizing (10.8), we can write the expression for q(t) in the scaling form

$$q(t) = t^{-x}Q\left(\frac{t}{\tau}\right). \tag{10.9}$$

The relaxation time τ is then satisfactorily described by the expression $\tau \sim (T - T_g)^{-zv}$ with $zv = 7.9 \pm 1.0$, $z = 6.1 \pm 0.3$.

Attempts have been reported^{203,204} to establish whether the ground state of spin glasses with a short-range interaction can be described as a set of valleys in ultrametric space. However, the size of the samples was too small for reliable conclusions to be drawn.

Computer simulations have thus shown that the Ising models has the basic properties observed in spin glass experiments, namely, there is a phase transition, long-term relaxation occurs below T_g , and although $q(t) \rightarrow 0$ (for $t \rightarrow \infty$), we have $q \neq 0$ below T_g for realistically attainable times.

The results of numerical experiments^{196–199} have led Fisher and Hughes²⁰⁵ and Bray and Moore²⁰⁶ to a heuristic model of the Ising spin glass, which can be referred to as the drop model (see also Ref. 207). According to this model, large-scale excitations with characteristic size L, whose energy has been shown to be^{196–199} $E \approx \tilde{J}(L) \sim L^{y}$, y = -1 ⁻¹, can be produced by overturning a large number of spins in "drops" whose surface area is of the order of d_x . The very significant point is that the index d_x , whose meaning is close to fractal dimension, is not related in the trivial manner $d_x = d - 1$ to the dimension of space, as in the case of the Ising ferromagnet. In general, $d_x \ge d - 1$ because, be-

ing a fractal object, the drop can have an anomalously large surface area.

Let us now consider in such a system a perturbation of the exchange integrals δJ_{ij} , with zero mean and variance δJ . The redistribution of spin orientations on the drop boundary then leads to the energy gain $\delta J \sim L^{d_s/2}$. This gain is greater than the excitation energy in the drop $E(L) \sim L^{y}$, i.e., when $\theta = (d_s/2) - \gamma > 0$, the redistribution of spin does indeed take place. In other words, when $\theta > 0$, the perturbation δJ of the exchange integrals leads to a redistribution of the orientations of spins lying at the distance $L > L_0 \approx (J/\delta J)^{1/\theta}$ from the perturbed exchange couplings. Numerical simulation has shown that, in two-dimensional glasses, y = -0.3(Refs. 196 and 197), $d_s = 1.6$ (Ref. 206), and $\theta = 1.1$. In three-dimensional glasses, y = 0.2 and, since $d_s \ge d - 1$, we have $\theta > 0.8$. In the three-dimensional case, when there is a phase transition for $T \neq 0$, the spin state is unstable not only with respect to a perturbation of the exchange integrals, but also with respect to weak temperature perturbations: a temperature change δT leads to a reorientation of spins at distances $L > (\delta T)/T^{-1/\theta}$. The spin-glass phase in the threedimensional case was referred to in Ref. 206 as chaotic; the spins do not freeze in this phase for any $T \neq 0$.

A number of phenomena in the relaxation dynamics of spin glasses is discussed in Ref. 205 on the basis of the drop model.

10.3.2 Heisenberg spin glass. Since the basic experiments were performed with Heisenberg systems, their numerical simulations are of particular interest. All the numerical simulation data obtained for systems with nearest-neighbor interaction indicate that $d_c > 3$ in the Heisenberg spin glass.²⁰⁸⁻²¹² The authors of Refs. 209 and 210 have calculated the energy defect and have concluded that $d_{\rm c} = 4$. The Monte Carlo method was used in Ref. 212 to calculate the spin glass susceptibility and the autocorrelation function for samples with dimensions of 8³, 16³, 32³ and a normal distribution of exchange integrals with unit variance. The mean field theory shows that, in this case, there should be a transition at $T_g^{MF} = \sqrt{6/3} = 0.82$, whereas the Bethe-Peierls gives $T_g^{BP} = 0.52$. Numerical simulation in the range 0.20 < T < 0.80 gives no indication of a phase transition. The susceptibility and the characteristic relaxation time τ increase with decreasing temperature: $\chi_{SG} \sim T^{-\gamma}$, $\gamma = 3.4; \ \tau \sim T^{-zv}$, $zv = 5.9, \ z = 5.1$, so that the transition temperature is most likely to be zero, as in the two-dimensional Ising spin glass. However, in contrast to the two-dimensional Ising glass, in which τ increases exponential- $1y^{213-215}$ as the temperature is reduced, here we see a power-type increase, i.e., transitions between different states are not due to overcoming of barriers. At the same time, q(t) falls logarithmically with time.

All these results are in clear conflict with experiment, which shows that there is serious evidence that the transition to the spin glass occurs at a finite temperature. This is indicated by the rapid increase in χ_{nl} as $T \rightarrow T_g$, and also by dynamic properties, i.e., by the dependence of T_g on ω .

For some spin glasses, in particular for CuMn, this function is known in a very wide range, namely, between times of the order of 10^{-8} s, which are typical for neutron experiments,²¹⁶ and times of the order of 10^4 s, which correspond to statistical experiments. It has been shown²¹⁷ that,



FIG. 20. Fulcher's law (a) and the power-type law (b) for the temperature dependence of the relaxation time of the alloy CuMn (4.6 at.% Mn).

for times in the range $10^{-8} \text{ s} < \tau < 10^4$ s, these data are satisfactorily described by the scaling relationship $\tau = \tau_0 [T/(T-T_0)]^{zv}$, where zv = 5.5 (Fig. 20) and T_0 is close to the temperature at which the static susceptibility exhibits a break. At the same time, Fulcher's law $\tau \sim \exp[E/(T-T_0)]$, and the generalized Arrhenius law²⁰⁸ $\tau \sim \exp(E/T^{zv})$, based on the assumption that infinite barriers arise in the system as $T \rightarrow 0$, do not describe the measured function $T(\tau)$ for reasonable parameter values at all times.

The advantages of the scaling relationship, as compared with other possible descriptions, also follow from other experimental data.^{219–223}

The experimental situation in the case of Heisenberg magnets is closer to the numerical simulations for three-dimensional Ising glasses¹⁹¹ than Heisenberg glasses. This is well illustrated by Fig. 21 which is taken from Ref. 212. Even the measured indices are in reasonable agreement with calculations.¹⁹¹

A resolution of this contradiction was suggested in Refs. 210 and 223 in which the special role of random anisotropy in spin glasses was noted. It can ensure that the transi-



FIG. 21. a—Susceptibility χ_{SG} as a function of temperature for the threedimensional Ising model with nearest-neighbor interaction, based on numerical simulation; T_g^{MF} —transition temperature in the molecular field approximation. b—Experimental data from Ref. 21. c—Numerical simulation of the three-dimensional Heisenberg model with nearest-neighbor interaction.

tion becomes Ising-like, and the transition temperature in spin glasses with short-range interaction is then given by

$$T_{\rm g} \sim J \left(\frac{D}{J}\right)^{1/4}.$$
 (10.10)

Spin glasses with the RKKY interaction belong²²⁴ to another universality class with $d_c = 3$ (see also Refs. 225). Random anisotropy in such glasses brings them closer to the Ising case, so that

$$T_{g} \sim J \left(\ln \frac{J}{D} \right)^{-1/2}, \tag{10.11}$$

which, as in the case of short-range interactions, and realistic values of the parameter D/J, is not very different from J.

However, this picture is also found to give rise to difficulties. First, Monte Carlo calculations for glasses with the RKKY interaction show²¹¹ that $d_c > 3$ in these systems. Second, as noted in Ref. 212, if (10.10) and (10.11) are valid, then glasses with a very small degree of anisotropy should exhibit a transition from the Heisenberg to the Ising type behavior as the temperature is reduced. However, this is never observed experimentally.

11. SPIN-GLASS MODELS IN COMBINATORIAL OPTIMIZATION PROBLEMS AND IN BIOLOGY

The ideas and methods developed in the theory of spin glasses have had a very considerable influence on problems in combinatorial optimization which, at first sight, are very different, but in which frustrations are found to occur in one form or another. $^{14,226-230}$ We refer here to the so-called problems with exponentially large samples i.e., problems for which it is probably impossible to construct an algorithm for a solution in a time that is a power-type function of the number N of objects in the sample.

In combinatorial optimization problems, we are usually interested in a set of N parameters that enable us to obtain an absolute extremum, whereas in statistical physics we investigate the macroscopic properties of a system. However, application of the methods of statistical physics enable us, first, to provide a reasonable analytic estimate of the quantity for which the optimum is desired and, second, by introducing in a natural manner the "cost" function (analog of free energy) and a parameter analogous to temperature, it is possible to carry out a Monte Carlo simulation of the cooling of the system and obtain the distribution of the "microscopic" parameters for a nearly optimal solution.

As an example, consider the problem of partition of a plane graph, whose vertices are randomly joined, into two subsystems with an equal number of vertices, so that the number of links joining the vertices in different subsystems is a minimum. Such problems arise, for example, in the design of complex electronic systems when the number of connections between blocks has to be minimized.²²⁶

Let us assign a variable σ_i to each vertex *i*, which can assume values ± 1 depending on to which subsystem the vertex belongs. Every two vertices will be assumed to be coupled with probability $\alpha = p/N$, and let us associate with these couplings an energy $J_{ij} = J > 0$. It is then readily seen that the Ising Hamiltonian

$$\tilde{\mathscr{D}} = -\sum_{i \in \mathcal{I}} J_{ij} \sigma_i \sigma_j$$
 (11.1)

is simply related to the number of couplings $C(\alpha)$ between

159 Sov. Phys. Usp. 32 (2), February 1989

vertices belonging to different subsystems²²⁷

$$\mathscr{H} = -\frac{J}{2}N(N-1)p + 2C(\alpha), \qquad (11.2)$$

and the combinatorial optimization problem reduces to finding the minimum of \mathscr{H} , subject to the condition that the total moment must be $\Sigma_i \sigma_i = 0$, which ensures that the number of vertices in the subsystems is equal. Frustrations occur as a result of competition between the ferromagnetic interaction J_{ij} and the condition that the resultant moment is zero. We note that this model is identical with the dilute spin glass model.¹⁸⁶

The replica symmetric analysis performed by Fu and Anderson²³⁷ has led to an estimate for $C(\alpha)$, which is better than that provided by previously known optimization methods.

Statistical physics models for the classical traveling salesman problem^{226,228} the plane graph coloring problem,²²⁹ and so on have been formulated in a similar way.

In addition to the replica-symmetric analysis and computer simulations of the "cooling" of the system, attempts have been made to investigate the properties of the phase with broken symmetry.²³⁰

Unexpectedly, spin-glass methods have also been found to be successful in biology, especially as models of associative memory.^{14,231–239} These models are based on the idea of memory as a cooperative property of neural networks. The Ising variable $\sigma_i = \pm 1$ can be assigned to the excited and unexcited states of a neuron *i*. The neuron potential V_i arises as a result of its interaction with other neurons in the network, and is equal to

$$\sum_{i,j} J_{ij} \sigma_j$$

The quantities J_{ij} characterize the strength of the interaction, and can have different signs. Assuming that the neuron excitation threshold U_i is given by

$$U_i = \sum_i J_{ij}$$

and that the interaction is symmetric, we see that the stationary state of the system corresponds to the condition for the minimum of the Hamiltonian (11.1).

To ensure that the system can memorize patterns and recognize them associatively, stable configurations must correlate with patterns to be memorized. This is achieved by considering interactions of the form

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{l} \xi_{i}^{\mu} \xi_{j}^{\mu} \quad (i \neq j),$$

where ξ_i^{μ} are random variables equal to ± 1 . The sets $\{\xi_i^{\mu}\}$ correspond to patterns stored in memory. When the ratio $\alpha = l/N$ does not exceed the critical value $\alpha_0 = 0.138$, the system has *l* stationary states, each of which is close to one of the sets $\{\xi_i^{\mu}\}$. If the initial state $\{\sigma_i\}$ is not very different from some pattern $\{\xi_i^{\mu}\}$, the result of the dynamic reduction of energy will be that the system will undergo a transition to this pattern, which is close to the initial state, i.e., the system will "recognize" the pattern by association.

12. CONCLUSION

Intensive studies of spin glasses are continuing. The lowest critical dimension of the vector glass appears to be three or more, so that the molecular field theory cannot literally describe real spin glasses. This has stimulated attempts to construct a theory of spin glasses that does not start with the molecular field theory as the zero-order approximation. Theories have been advanced and developed in which the spin glass is looked upon as a chaotic phase that is unstable with respect to small changes in the boundary conditions and in the temperature, and attempts have continued to find models in which the replica-symmetric solution is stable down to T = 0.

However, it must not be assumed that the molecular field theory has turned out to be useless. Correlations due to spatial dispersion decay very slowly, so that the order parameter differs from zero for actually attainable times, and the molecular field theory provides a good description of the static properties of spin glasses.

New theoretical searches have run parallel with experiments devoted to comprehensive studies of different properties of spin glasses. Recent papers have reported studies of the temperature dependence of the thermostatic and isothermal susceptibilities²⁴⁰ for $T \ll T_g$, the absorption of ultrasound in dielectric spin glasses,¹²⁴ precision measurements of remanent magnetization and of aging as functions of the magnetic field,²⁴² new studies of the mixed state in disordered ferro-and antiferromagnets,243-246 analyses of the phase transition to the spin glass state,²⁴⁷ and so on.²⁴⁸⁻²⁵³

The drop model of the spin glass has been extended²⁵⁴ and the nonergodic state on the Bethe lattice has been investigated.255

New publications have appeared on the optimization problem, using the replica method. $^{256-261}$ The literature is undoubtedly continuing to expand.

- ¹⁾We are using a system of units in which the Bohr magneton is $\mu_{\rm B} = 1$ and the temperature is expressed in energy units.
- ²⁾Ginzburg's work is close in spirit to these investigations.

³⁾See Ref. 185 on the properties of the Bethe lattice

- ¹K. Fischer, Phys. Status Solidi 116, 357 (1983); 130, 13 (1985).
- ²I. Ya. Korenblit and E. F. Shender, Izv. Vussh. Uchebn. Zaved. Fiz. No. 10, 23 (1984) [Sov. Phys. J. 27, 822 (1984)].
- ³C. Y. Huang, J. Magn. Magn. Mater. 51, 1 (1985).
- ⁴K. Binder and A. P. Young, Rev. Mod. Phys. 58, 801 (1986).
 ⁵B. E. Vugmeister and M. D. Glinchuk, Usp. Fiz. Nauk 146, 493 (1985) [Sov. Phys. Usp. 28, 608 (1985)]
- ⁶M. Takashige, H. Terauchi, Y. Miura, and S. Hoshino, J. Phys. Soc. Jpn. 54, 3250 (1985)
- ⁷J. M. Rowe, J. J. Rush, D. G. Hinks, and S. Susman, Phys. Rev. Lett. 43, 1158 (1979).
- *F. Lüty, Defects in Insulating Crystals, Springer-Verlag, Berlin, Heidelberg, New York, 1981.
- ⁹D. H. Reich, T. F. Rosenbaum, G. Aeppli, and H. J. Guggenheim, Phys. Rev. B 34, 4956 (1986)
- ¹⁰S. John and T. C. Lubensky, *ibid.* 4815.
- ¹¹V. M. Vinokur, L. B. Ioffe, A. I. Larkin, and M. V. Feigel'man, Zh.
- Eksp. Teor. Fiz. 93, 343 (1987) [Sov. Phys. JETP 66, 198 (1987)].
- ¹²S. F. Edwards and P. W. Anderson, J. Phys. F 35, 1792 (1975).
- ¹³V. Cannella and J. A. Mydosh, Phys. Rev. B 6, 4220 (1972).
- 14R. Rammal, G. Toulouse, and M. A. Virasoro, Rev. Mod. Phys. 58, 765 (1986).
- ¹⁵R. V. Chamberlin, M. Hardiman, L. A. Turkevich, and R. Orbach, Phys. Rev. B 25, 6720 (1982).
- ¹⁶M. Suzuki, Prog. Theor. Phys. 58, 1151 (1977).
- ¹⁷J. Chalupa, Solid State Commun. 24, 429 (1977).
- 18 Y. Mikao, S. Chikazawa, T. Soito, and Y. G. Yuochunas, J. Phys. Soc. Jpn. 46, 1951 (1979). ¹⁹B. Barbara, A. P. Malozemoff, and Y. Imry, Phys. Rev. Lett. 47, 1852
- (1981)
- ²⁰P. Monod and H. Boichiat, J. Phys. Lett. (Paris) 43, 45 (1982).
- 160 Sov. Phys. Usp. 32 (2), February 1989

- ²¹R. Omari, J. J. Prejean, and J. Souletie, J. Phys. (Paris) 44, 25 (1983).
- ²²P. Beauvilloin, C. Chappert, and J. R. Renard, J. Phys. Lett. (Paris) 45, L665 (1984)
- ²³L. P. Levy and A. T. Ogielski, Phys. Rev. Lett. 57, 3288 (1986).
- ²⁴A. V. Deryabin, A. V. Tkov, V. K. Kazantsev, Yu. A. Chirkov, and I. V. Zakharov, Phys. Status Solidi A 75, 229 (1986).
- ²⁵A. V. Deryabin, V. K. Kazantsev, I. V. Zakharov, and A. V. T'kov, Zh. Eksp. Teor. Fiz. 91, 607 (1986) [Sov. Phys. JETP 64, 358 (1986)].
- ²⁶F. Benzamka, D. Bertrand, A. R. Fert, and J. P. Redoules, J. Phys. C 19, 4741 (1986)
- ²⁷T. Taniguchi, Y. Miyako, and J. L. Tholence, J. Phys. Soc. Jpn. 54, 220 (1985)
- ²⁸P. Svedlindh, L. Lundgren, P. Nordblad, and H. S. Chen, Europhys. Lett. 2, 805 (1986).
- ²⁹H. Bouchiat, J. Phys. (Paris) 47, 71 (1986).
- ³⁰Y. Yeshurun and H. Sompolinsky, Phys. Rev. Lett. 56, 984 (1986). ³¹N. de Courtenay, H. Bouchiat, H. Hurdequiut, and A. Fert, J. Phys.
- (Paris) 47, 1507 (1986).
- ³²G. Kotliar and H. Sompolinsky, Phys. Rev. Lett. 53, 1751 (1982).
- ³³L. E. Wenger and P. H. Keesom, Phys. Rev. B 13, 4053 (1976).
- ³⁴H. Löhneysen, R. van den Berg, G. V. Lecomte, and W. Zinn, *ibid.* 31, 2920 (1985).
- ³⁵W. E. Fogle, J. D. Boyer, R. A. Fisher, and N. E. Phillips, Phys. Rev. Lett. 50, 1815 (1983)
- ³⁶J. Wosnitza, H. von Löhneysen, W. Zinn, and U. Krey, Phys. Rev. B 33, 3436 (1986).
- ³⁷P. Svenlindh, P. Granberg, P. Nordblad, L. Lundgren, and H. S. Chen, ibid. 35, 268 (1987).
- ³⁸D. Müser, L. E. Wenger, A. J. van Duyneveldt, and J. A. Mydosh, ibid. 27, 3100 (1983).
- ³⁹W. Reim, R. H. Koch, A. P. Malozemoff, and M. B. Ketchen, Phys. Rev. Lett. 57, 905 (1986)
- ⁴⁰M. Ocio, H. Bouchiat, and P. Monod, J. Phys. Lett. (Paris) 46, L647 (1985).
- ⁴¹M. Ocio, H. Bouchiat, and P. Monod, J. Magn. Magn. Mater. 54-57, 185 (1986).
- ⁴²P. Refregier, M. Ocio, and H. Bouchiat, Europhys. Lett. 3, 503 (1987). ⁴³O. L. Lundgren, P. Svedlindh, and O. Beckman, J. Magn. Magn. Mater. 25, 33 (1981).
- ⁴⁴A. J. van Duyneveldt and C. A. M. Mulder, Physica (Utrecht) 82, 1148 (1982).
- ⁴⁵C. C. Paulsen, S. J. Williamson, and H. Maletta, J. Magn. Magn. Mater. 54-57, 209 (1986).
- ⁴⁶E. Pytte and Y. Imry, Phys. Rev. B 35, 1465 (1987).
- ⁴⁷B. de Mayo, H, Phys. Chem. Solids 35, 1525 (1974).
- 48C. N. Guy, J. Phys. F 5, L242 (1975).
- ⁴⁹L. Lundgren, P. Norblad, and P. Svendlindh, Phys. Rev. B 34, 8164 (1986).
- ⁵⁰R. V. Chamberlin, G. Mozurkevich, and R. Orbach, Phys. Rev. Lett. 52, 867 (1984)
- ⁵¹R. V. Chamberlin, J. Appl. Phys. 57, 3377 (1985).
- ⁵²A. K. Jonscher, Nature 267, 673 (1977)
- ⁵³K. L. Ngai, Comm. Solid State Phys. 9, 127 (1979).
- ⁵⁴R. G. Palmer, D. L. Stein, E. Abrahams, and P. W. Anderson, Phys. Rev. Lett. 53, 958 (1984).
- ⁵⁵L. Lundgren, P. Svendlindh, P. Nordblad, and O. Beckman, *ibid.* 51, 911 (1983).
- ⁵⁶P. Nordblad, P. Svendlindh, L. Lundgren, and L. Sandlund, Phys. Rev. B 33, 645 (1986); Phys. Lett. A 120, 475 (1987).
- ⁵⁷P. Nordblad, L. Lundgren, and K. Sandlund, J. Magn. Magn. Mater. 54-57, 185 (1986).
- ⁵⁸M. Alba, M. Ocio, and J. Hamman, Europhys. Lett. 2, 45 (1986).
- ⁵⁹L. Lundgren, P. Nordblad, and L. Sandlund, *ibid.* 1, 529.
- ⁶⁰L. Lundgren, P. Nordblad, P. Svendlindh, and O. Beckman, J. App. Phys. 57, 3371 (1985).
- ⁶¹M. Alba and J. Hammann, J. Magn. Magn. Mater. 54-57, 213 (1986).
- ⁶²H. Maletta and W. Felsch, Z. Phys. B 37, 55 (1980).
- ⁶³H. Maletta, J. Appl. Phys. 53, 2185 (1982).
- ⁶⁴Y. Yeshurun, M. B. Salamon, K. V. Rao, and H. S. Chen, Phys. Rev. Lett. 45, 1366 (1980)
- ⁶⁵A. Berton, J. Chaussy, J. Odin, R. Rammal, J. Souletie, J. L. Tholence, and R. Tournier, J. Appl. Phys. 52, 1763 (1981)
- ⁶⁶R. B. Goldfarb, F. R. Fickett, L. V. Rao, and H. S. Chen, J. Appl. Phys. 53, 7687 (1982).
- ⁶⁷D. Bertrand, A. R. Fert, M. S. Schmidt, F. Bensamka, and S. Legrand, J. Phys. C 15, L883 (1982).
- 68 J. Gelard, F. Bensamka, D. Bertrand, A. R. Fert, J. R. Redouless, and S. Legrand, ibid. 16, K939 (1983); 17, 1725 (1984).
- 69G. V. Lecomte, H. von Löhneysen, W. Baunhofer, and G. Güntherodt, Sold State Commun. 52, 535 (1984).
- ⁷⁰G. V. Lecomte, H. von Löhneysen, A. Schröder, W. Bauhofer, and G.
 - 160 I. Ya. Korenblit and E. F. Shender

Güntherodt, J. Magn. Magn. Mater. 54-57, 59 (1986).

- ⁷¹P. Wong, S. v. Molnar, T. T. M. Palstra, J. A. Mydosh, H. Yoshizawa, S. M. Shapiro, and A. Ito, Phys. Rev. Lett. 55, 2043 (1985).
- ⁷²G. A. Takzei and A. M. Kostyshin, Pis'ma Zh. Eksp. Teor. Fiz. 40, 308 (1984) [JETP Lett. 40, 1099 (1984)].
- ⁷³G. A. Takzei, A. M. Kostyshin, and I. I. Sych, Fiz. Tverd. Tela (Lenin-
- grad) 29, 2434 (1987) [Sov. Phys. Solid State 29, 1401 (1987)]. ⁷⁴S. Geschwindt, G. Delvin, and S. L. McCall, Phys. Rev. Lett. 58, 1595
- (1987).
- ⁷⁵J. J. Hauser, Phys. Rev. B 34, 3453 (1986).
- ⁷⁶W. Abdul-Razzaq and J. S. Kouvel, Phys. Rev. B 35, 1764 (1987).
- ⁷⁷S. Kouvel, W. Abdul-Razzaq, and Kh. Ziq. *ibid.* 1768.
- ⁷⁸A. K. Nigam, Girish Chandra, and S. Ramakrishnan, J. Phys. F 16, 1235 (1986)
- ⁷⁹R. Ranganathan, J. L. Tholence, R. Krishnan, and M. Dancygier, J. Phys. C 18, L1057 (1985).
- ⁸⁰S. Senoussi and Y. Oner, J. Phys. (Paris) 46, 1435 (1985).
- ⁸¹S. Senoussi and D. Elkhatouri, J. Magn. Magn. Mater. 54-57, 153 (1986)
- ⁸²R. B. Goldfarb, K. V. Rao, and H. S. Chen, Solid State Commun. 54, 799 (1985).
- ⁸³A. Z. Men'shikov, G. A. Takzei, and A. E. Teplykh, Fiz. Met. Metalloved. 54, 465 (1982). [Phys. Met. Metallogr. (USSR) 54(3), 41 (1982)
- ⁸⁴G. A. Takzei, I. I. Sych, A. Z. Men'shikov, and A. E. Teplykh, Fiz. Met. Metalloved. 52, 960 (1981) [Phys. Met. Metallogr. (USSR) 52(6), 27 (1981)]
- ⁸⁵A. V. Deryabii, V. K. Kazantsev, A. V. T'kov, and I. V. Zakharov, Zh. Eksp. Teor. Fiz. 92, 1761 (1988) [Sov. Phys. JETP 65, 987 (1988)]
- ⁸⁶H. Kadomatsu, T. Tsuraoka, and H. Fujiwara, J. Phys. Soc. Jpn. 53, 3153 (1984).
- ⁸⁷G. Dublin and Y. Yeshurun, Phys. Rev. B 25, 4899 (1982).
- **M. A. Manheim, S. M. Bhagat, and H. S. Chen, J. Magn. Magn. Mater. 38, 147 (1983)
- ⁸⁹P. V. Erwin, J. W. Lynn, J. J. Phyne, and H. S. Chen, J. Appl. Phys. 57, 3473 (1985).
- ⁹⁰G. Aeppli, S. M. Shapiro, R. J. Birneneau, and H. S. Chen, Phys. Rev. B 29, 2589 (1984).
- ⁹¹H. Maletta, G. Aeppli, and S. M. Shapiro, Phys. Rev. Lett. 48, 1490 (1982)
- ⁹²A. P. Murani, Solid State Commun. 34, 705 (1980).
- ⁹³I. Mirebeau, G. Jehanno, I. A. Campbell, F. Hippert, B. Hennion, and M. Hennion, J. Magn. Magn. Matr. 54-57, 99 (1986). ⁹⁴K. Katsumara, M. Tanimoto, S. Mitsuda, and Y. Eudoh, J. Phys. Soc.
- Jpn. 53, 3315 (1984).
- 95 Y. Miyako, T. Nishioka, T. Sato, Y. Takeda, S. Marimoto, and A. Ito, J. Magn. Magn. Mater. 54-57, 149 (1986).
- ⁹⁶S. B. Roy, A. K. Majumdar, N. C. Mishra, A. R. Raychaudhary, and R. Srinivasan, Phys. Rev. B 31, 7458 (1985)
- ⁴⁷T. Sakakibara, T. Goto, and Y. Miyako, Solid State Commun. 58, 563 (1986).
- 98I. A. Campbell, H. Hurdequint, and F. Hippert, Phys. Rev. B 33, 3540 (1986).
- ⁹⁹D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35, 1792 (1975).
- ¹⁰⁰S. Kirkpatrick and D. Sherrington, Phys. Rev. B 17, 4384 (1978).
- ¹⁰¹J. R. L. de Almeida and D. J. Thouless, J. Phys. A 11, 983 (1978).
 ¹⁰²A. J. Bray and M. A. Moore, Phys. Rev. Lett. 41, 1068 (1978).
- ¹⁰³D. J. Thouless, P. W. Anderson, and R. G. Palmer, Philos. Mag. 35, 593 (1977).
- ¹⁰⁴A. J. Bray and M. A. Moore, J. Phys. C 13, L469 (1980).
- ¹⁰⁵F. Tanaka and S. F. Edwards, *ibid*. F 10, 2769 (1980)
- ¹⁰⁶C. de Dominicis, M. Gabay, T. Garel, and H. Orland, J. Phys. (Paris) 41, 923 (1980).
- ¹⁰⁷G. Parisi, J. Phys. A 13, 1101, 1887 (1980).
- ¹⁰⁸G. Parisi, Phys. Rev. Lett. 50, 1946 (1983).
- ¹⁰⁹B. Duplantier, J. Phys. A 14, 283 (1981)
- ¹¹⁰H. J. Sommers and W. Dupont, *ibid.* C 17, 5785 (1984).
- ¹¹¹G. Parisi, *ibid*. A 13, L115 (1980).
- ¹¹²J. R. L. de Almeida and E. H. S. Lage, *ibid.* C 16, 939 (1983).
- ¹¹³A. V. Goltsev, *ibid.* A 16, L339 (1983).
- ¹¹⁴A. V. Goltsev, *ibid*. 1337.
- ¹¹⁵C. de Dominicis and I. Kondor, Phys. Rev. B 27, 606 (1983).
- ¹¹⁶I. Kondor and C. de Dominicis, J. Phys. A 16, 3931 (1983).
- ¹¹⁷A. J. Bray and M. A. Moore, *ibid.* C 13, 419 (1980)
- ¹¹⁸C. de Dominicis, M. Gabay, and B. Duplantier, *ibid*. A 15, L47 (1982).
- ¹¹⁹H. Sompolinsky, Phys. Rev. Lett. **47**, 935 (1981). ¹²⁰H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. **47**, 359 (1981); Phys. Rev. B 25, 6860 (1982)

AND STREET

 ¹²¹M. Gabay and G. Toulouse, Phys. Rev. Lett. 47, 201 (1981).
 ¹²²G. Toulouse, M. Gabay, T. C. Lubensky, and J. Vannimenus, J. Phys. Lett. (Paris) 43, 109 (1982).

-

161 Sov. Phys. Usp. 32 (2), February 1989 ¹²³M. A. Moore and A. J. Bray, J. Phys. C 15, L301 (1982).

....

- ¹²⁴D. M. Cragg, D. Sherrington, and M. Gabay, Phys. Rev. Lett. 49, 158 (1982).
- ¹²⁵S. L. Ginzburg, Zh. Eksp. Teor. Fiz. 84, 388 (1983) [Sov. Phys. JETP 57, 225 (1983)].
- ¹²⁶M. Gabay, T. Garel, and C. de Dominicis, J. Phys. C 15, 7165 (1982).
- ¹²⁷D. M. Cragg and D. Sherrington, Phys. Rev. Lett. 49, 1190 (1982).
- ¹²⁸S. A. Roberts and A. J. Bray, J. Phys. C 15, L527 (1982).
- ¹²⁹D. Elderfield and D. Sherrington, *ibid.* 16, 4865 (1983).
- ¹³⁰K. Babereschke, P. Pureur, A. Fert, R. Wendler, and S. Senoussi, Phys. Rev. B 29, 4999 (1984)
- ¹³¹S. Murayama, K. Yokosawa, Y. Miyako, and E. F. Wasseeman, Phys. Rev. Lett. 57, 1785 (1986).
- ¹³²N. P. Il'in and I. Ya. Korenblit, Zh. Eksp. Teor. Fiz. 81, 2070 (1981)
 [Sov. Phys. JETP 54, 1100 (1981)].
- ¹³³A. Fert and P. M. Levy, Phys. Rev. Lett. 44, 1538 (1980).
- ¹³⁴P. M. Levy and A. Fert, Phys. Rev. B 23, 4667 (1981).
- ¹³⁵B. Z. Spivak and A. Zyuzin, Pis'ma Zh. Eksp. Teor. Fiz. 43, 185 (1986) [JETP Lett. 43, 234 (1986)].
- ¹³⁶J. J. Prejean, M. J. Joliclerc, and P. Monod, J. Phys. (Paris) 41, 427 (1980).
- ¹³⁷A. Fert and F. Hippert, Phys. Rev. Lett. 49, 1508 (1982).
- ¹³⁸C. G. Morgan-Pond, J. Phys. B 32, 305 (1985).
 ¹³⁹D. Sherrington, Phys. Rev. C 17, L823 (1984).
- ¹⁴⁰A. M. Kostyshin and G. A. Takzei, Zh. Eksp. Teor. Fiz. **91**, 1938 (1986) [Sov. Phys. JETP **64**, 1149 (1986)].
- 141 A. V. Deryabin, V. K. Kazantsev, and I. V. Zakharov, Fiz. Tverd. Tela (Leningrad) 29, 635 (1987) [Sov. Phys. Solid State 29, 366 (1987)].
- 142 A. V. Myagkov, A. A. Minakov, and v. G. Veselago, Zh. Eksp. Teor. Fiz. 92, 1754 (1987) [Sov. Phys. JETP 65, 983 (1987)].
- 143 L. J. P. Ketelsen and M. B. Salamon, Phys. Rev. B 30, 2718 (1984): 33, 3610 (1986).
- ¹⁴⁴A. J. Bray and M. A. Moore, J. Phys. C 13, L655 (1980).
- ¹⁴⁵H. J. Sommers and H. D. Usadel, Z. Phys. B 47, 63 (1982).
- 146 Ya. V. Fedorov and E. F. Shender, Pis'ma Zh. Eksp. Teor. Fiz. 43, 526 (1986) [JETP Lett. 43, 681 (1986)]
- ¹⁴⁷H. D. Usadel, Solid State Commun. 58, 629 (1986).
- ¹⁴⁸I. Ya. Korenblit and E. F. Shender, Zh. Eksp. Teor. Fiz. 89, 1785 (1985) [Sov. Phys. JETP 62, 1030 (1985)]
- ¹⁴⁹I. Ya. Korenblit, Ya. V. Federov, and E. F. Shender, Zh. Eksp. Teor. Fiz. 92, 710 (1987) [Sov. Phys. JETP 65, 400 (1987)].
- ¹⁵⁰I. Ya. Korenblit, Ya. V. Fedorov, and E. F. Shender, Preprint LIYaF-1278 [in Russian], Leningrad, 1987. ¹⁵¹Ya. V. Fyodorov, I. Ya. Korenblit, and E. F. Shender, Europhys. Lett.
- 4,827 (1987)
- ¹⁵²I. Ya. Korenblit and E. F. Shender, Zh. Eksp. Teor. Fiz. 93, 1060 (1987) [Sov. Phys. JETP 66, 597 (1987)].
- ¹⁵³I. Ya. Korenblit and E. F. Shender, Pis'ma Zh. Eksp. Teor. Fiz. 42, 370 (1985) [JETP Lett. 42, 456 (1985)]
- ¹⁵⁴G. A. Takzei, A. M. Kostyshin, and I. I. Sych, *ibid.* 43, 425 (1986) [JETP Lett. 43, 547 (1986)].
- ¹⁵⁵ M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. A. Virasoro, J. Phys. (Paris) 45, 843 (1984).
- ¹⁵⁶M. Mezard and A. Virasoro, *ibid.* 46, 1293 (1985).
- ¹⁵⁷A. J. Bray, M. A. Moore, and A. P. Young, J. Phys. C 17, L155 (1984). ¹⁵⁸M. Mezard, G. Parisi, and M. A. Virasoro, J. Phys. Lett. (Paris) 46,
- L217 (1985)
- ¹⁵⁹M. Mezard, G. Parisi, and M. Virasoro, Europhys. Lett. 1, 77 (1986).
- ¹⁶⁰A. Young, A. J. Bray, and M. A. Moore, J. Phys. C 17, L149 (1984).
- ¹⁶¹N. Parga, G. Parisi, and M. A. Virasoro, J. Phys. Lett. (Paris) 45, L1 (1984)
- ¹⁶²R. N. Bhatt and A. P. Young, J. Magn. Magn. Mater. 54–57, 191 (1986).
- ¹⁶³B. Derrida and G. Toulouse, J. Phys. Lett. (Paris) 46, L223 (1985).
- ¹⁶⁴C. de Dominicis, H. Orland, and F. Lainee, *ibid.* L469.
- ¹⁶⁵A. T. Ogielski and D. L. Stein, Phys. Rev. Lett. 55, 1634 (1985).
- 166G. Paladin, M. Mezard, and C. de Dominicis, J. Phys. (Paris) 46, L985 (1985).

¹⁷⁰A. V. Gol'tsev, Zh. Eksp. Teor. Fiz. 91, 1725 (1986) [Sov. Phys. JETP

¹⁷¹D. Kutasow, A. Aharony, E. Domany, and W. Kinzel, Phys. Rev. Lett. 56, 2229 (1986).

 ¹⁷²S. L. Ginzburg, Use of Nuclear Reactors and Accelerators in Condensed-State Physics [in Russian], Leningrad, 1986, p. 3. ¹⁷³H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. 50, 1297 (1983)

¹⁷⁴C. de Dominicis and I. Kondor, J. Phys. Lett. (Paris) 45, L205 (1984).

I. Ya. Korenblit and E. F. Shender

161

¹⁶⁷S. Grossman, F. Wegner, and K. H. Hoffmann, *ibid*. L575. ¹⁶⁸B. A. Huberman and M. Kerzberg, J. Phys. A 18, L331 (1985).
 ¹⁶⁹C. de Dominicis, J. Magn. Magn. Mater. 54-57, 17 (1986).

¹⁷⁵C. de Dominicis and I. Kondor, *ibid.* 46, L1037 (1985).

¹⁷⁶A. V. Gol'tsev, J. Phys. C 17, L241 (1984).

64, 1021 (1986)].

- ¹⁷⁷A. V. Gol'tsev, *ibid*. **19**, L219 (1986).
- ¹⁷⁸I. Kondor and C. de Dominicis, Europhys. Lett. 2, 617 (1986).
- ¹⁷⁹A. J. Bray and M. A. Moore, J. Phys. C 12, 79 (1979)
- ¹⁸⁰D. S. Fisher and H. Sompolinsky, Phys. Rev. Lett. 54, 1063 (1985)
- ¹⁸¹L. B. Ioffe and M. V. Feigel'man, Zh. Eksp. Teor. Fiz. 89, 654 (1985) [Sov. Phys. JETP 62, 376 (1985)].
- ¹⁸²D. Bowman and K. Levin, Phys. Rev. B 15, 3438 (1982).
- ¹⁸³D. J. Thouless, Phys. Rev. Lett. 56, 1082 (1986)
- ¹⁸⁴Ya. V. Fedorov, Pis'ma Zh. Eksp. Teor. Fiz. 46, 270 (1987) [JETP Lett. 46, 340 (1987)].
- ¹⁸⁵I. Ya. Korenblit and E. F. Shender, Usp. Fiz. Nauk 126, 233 (1978) [Sov. Phys. Usp. 21, 832 (1978)].
- ¹⁸⁶L. Vianna and L. J. Bray, J. Phys. C 18, 3037 (1985).
- ¹⁸⁷I. Kanter and H. Sompolinsky, Phys. Rev. Lett. 58, 164 (1987).
- 188C. de Dominicis and P. Mottishaw, Europhys. Lett. 3, 87 (1987).
- ¹⁸⁹P. Mottishaw and C. de Dominicis, J. Phys. A 20, L375 (1987).
- ¹⁹⁰A. Z. Patashinakii and V. L. Pokrovskii, Fluctuation Theory of Phase Transitions, Pergamon Press, Oxford, 1979 [Russ. original, Nauka, M., 1975 and 1982].
- ¹⁹¹A. T. Ogielski, Phys. Rev. B **32**, 7384 (1985)
- ¹⁹²I. Morgenstern and K. Binder, *ibid.* 22, 288 (1980).
- ¹⁹³P. Reed, A. J. Bray, and M. A. Moore, J. Phys. F 11, L139 (1978).
- ¹⁹⁴A. P. Young, Phys. Rev. Lett. 50, 917 (1983).
- ¹⁹⁵R. Fish and A. B. Harris, *ibid.* 38, 785 (1977)
- ¹⁹⁶A. J. Bray and M. A. Moore, J. Phys. C 17, L463, L613 (1984).
 ¹⁹⁷A. J. Bray and M. A. Moore, Phys. Rev. B 131, 631 (1985).
- ¹⁹⁸W. L. McMillan, *ibid*. **30**, 476 (1984)
- ¹⁹⁹W. L. McMillan, *ibid.* **31**, 340 (1985).
- ²⁰⁰R. R. Singh and S. Chakravarty, Phys. Rev. Lett. 57, 245 (1986).
- ²⁰¹R. N. Bhatt and A. P. Young, Phys. Rev. Lett. 54, 924 (1985).
- ²⁰²A. P. Young and R. N. Bhatt, J. Magn. Magn. Mater. 54-57, 6 (1986).
- ²⁰³N. Sourlas, J. Phys. (Paris) 45, L969 (1984).
- ²⁰⁴N. Sourlas, Europhys. Lett. 1, 189 (1986).
- ²⁰⁵D. S. Fisher and D. A. Huse, Phys. Rev. Lett. 56, 1601 (1986).
- ²⁰⁶A. J. Bray and M. A. Moore, *ibid.* 58, 57 (1987).
- ²⁰⁷J. Villain, Europhys. Lett. 2, 871 (1986).
- ²⁰⁸J. R. Banavar and M. Cieplak, Phys. Rev. Lett. 48, 832 (1982).
- ²⁰⁹W. L. McMillan, Phys. Rev. B **31**, 342 (1985).
- ²¹⁰B. W. Morris, S. G. Colborne, M. A. Moore, A. J. Bray, and J. Conisius, J. Phys. C 19, 1157 (1986).
- ²¹¹A. Chakrabarti and C. Dasgupta, Phys. Rev. Lett. 56, 1404 (1986).
 ²¹²J. A. Olive, A. P. Young, and D. Sherrington, Phys. Rev. B 34, 6341 (1986).
- ²¹³W. L. McMillan, *ibid*. 28, 5216 (1983).
- ²¹⁴A. P. Young, Phys. Rev. Lett. 50, 917 (1983).
- ²¹⁵J. D. Reger and A. Zippelius, Phys. Rev. B 31, 5900 (1985).
- ²¹⁶A. P. Murani, J. Magn. Magn. Mater. 25, 68 (1981).
- ²¹⁷J. Souletie and J. L. Tholence, Phys. Rev. B 32, 516 (1985).
 ²¹⁸K. Binder and A. P. Young, *ibid.* 516.
- ²¹⁹N. Bontemps, J. Rajchenbach, R. V. Chamberlin, and R. Orbach, *ibid*. 30, 6514 (1984); J. Magn. Magn. Mater. 54-57, 1 (1986).
- ²²⁰P. Svendlich, L. Lundgren, P. Norblad, and H. S. Chan, Europhys. Lett. 3, 243 (1987)
- ²²¹M. Saint-Paul, J. L. Tholence, and W. Giriat, Solid State Commun. 60, 621 (1986).
- ²²²C. Meyer, F. Hartman-Boutron, and Y. Gross, J. Phys. (Paris) 47, 1395 (1986).

- ²²³P. Beauvillain, J. P. Renard, M. Malecki, and J. J. Prejean, Europhys. Lett. 2, 23 (1986).
- ²²⁴A. J. Bray, M. A. Moore, and A. P. Young, Phys. Rev. Lett. 56, 2641 (1986).
- ²²⁵I. E. Dzyaloshinsky and G. E. Volovik, J. Phys. (Paris) 39, 693 (1978). ²²⁶S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, Science 220, 671 (1983).
- ²²⁷Y. Fu and P. W. Anderson, J. Phys. A 19, 1605 (1966)
- ²²⁸M. Mezard and G. Parisi, J. Phys. (Paris) 47, 1285 (1986)
- ²²⁹J. P. Bouchaud and P. Le Doussal, Europhys. Lett. 1, 91 (1986).
- ²³⁰S. Kirkpartick and G. Toulouse, J. Phys. (Paris) 46, 1277 (1985).
- ²³¹J. J. Hopfield, Proc. Nat. Acad. Sci. USA 79, 2554 (1982).
- ²³²J. J. Hopfield, D. I. Feinstein, and R. G. Palmer, Nature 304, 18
- (1983). ²³³D. J. Amit, H. Gutfreund, and H. Sompolinsky, Phys. Rev. A 32, 1007 (1985).
- ²³⁴J. L. van Hemmen and R. Kühn, Phys. Rev. Lett. 57, 913 (1986).
- ²³⁵Vik. S. Dotsenko, J. Phys. C 18, L1017 (1985).
 ²³⁶L. B. Ioffe and M. V. Feigel'man, Usp. Fiz. Nauk 160, 323 (1986) [Sov. Phys. Usp. 29, 986 (1986)]
- ²³⁷J. P. Nadal, G. Toulouse, J. P. Changeux, and S. Dehaene, Europhys. Lett. 1, 535 (1986).
- ²³⁸E. Domany, R. Meir, and W. Kinzel, *ibid.* 2, 175.
- ²³⁹A. Crisanti, D. J. Amit, and H. Gutfreund, *ibid.* 3, 113 (1987).
- ²⁴⁰P. Mendels, H. Alloul, and M. Ribault, *ibid.* 3, 113 (1987).
- ²⁴¹P. Doussineau, A. Levelut, M. Matecki, J. P. Renard, and W. Schön, *ibid* 251.
- ²⁴²P. Nordblad, L. Lundgreen, and L. Sandlund, *ibid.* 243.
- ²⁴³H. Yoshizawa, S. Mitsuda, H. Aruga, and A. Ito, Phys. Rev. Lett. 59, 2364 (1987)
- ²⁴⁴G. Aeppli, H. Maletta, S. M. Shapiro, and D. Abernaty, Phys. Rev. B 36, 3956 (1987).
- ²⁴⁵S. G. Barsov, A. L. Getalov, S. L. Ginzburg, et al., Preprint LIYaF-1332 [in Russian], Leningrad, 1987
- ²⁴⁶V. V. Runov, S. L. Ginzburg, B. P. Toperverg, A. D. Tret'yakov, A. I. Okorokov, and E. I. Mal'tsev, Zh. Eksp. Teor. Fiz. 94(1), 325 (1988) [Sov. Phys. JETP 67, 181 (1988)].
- ²⁴⁷M. B. Salamon, J. Appl. Phys. 61, 4228 (1987).
- ²⁴⁸H. Pinkvos and Ch. Schwink, Phys. Status Solidi B 144, 351 (1987).
- ²⁴⁹R. H. Koch, W. Rein, A. P. Malozemoff, and M. B. Ketchen, J. Appl. Phys. 61, 3678 (1987)
- ²⁵⁰M. Alba, J. Hamman, M. Ocio, and Ph. Reffregier, *ibid.* 3683.
- ²⁵¹B. Bontemps, R. Hoogerbeets, Loo Wei-li, and R. Orbach, *ibid.* 4089. ²⁵²T. Datta, A. Barrientos, J. Amirzadeh, and E. R. Jones, Solid State
- Commun. 62, 571 (1987)
- ²⁵³P. Mazumdar and S. M. Bhagat, J. Appl. Phys. 61, 3633 (1987).
- ²⁵⁴D. S. Fisher and D. A. Huse, J. Phys. A 20, L997 (1987); L1005.
- ²⁵⁵P. Mottishow, Europhys. Lett. 4, 333 (1987).
- ²⁵⁶D. Sherrington and K. Y. M. Wong, J. Phys. A 20, L785 (1987).
- ²⁵⁷K. Y. M. Wong and D. Sherrington, *ibid*. L793.
- ²⁵⁸K. Y. M. Wong, D. Sherrington, P. Mottishaw, R. Dewar, and C. de Dominicis, Preprint of Imperial College, London, 1987.
- ²⁵⁹M. Mezard and G. Parisi, J. Phys. (Paris) 48, 1451 (1987).
- ²⁶⁰I. Morgenstern and D. Würtz, Z. Phys. B 67, 397 (1987)
- ²⁶¹C. de Dominicis and P. Mottishaw, J. Phys. A 20, L841 (1987).

Translated by S. Chomet