Autosolitons

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A phenomenon of nonlinear physics is studied in this review: the formation of characteristic, time-independent, solitary states—autosolitons (AS)—in different physical, chemical, and biological systems. The physics of As in some types of systems—systems with "positive" and "negative" thermal diffusion, uniformly generated "combustion material," and local self production of matter—is explained for the example of semiconductor and gas plasma, where AS consist of a region of carriers with high temperature and low (or high) density. The conditions for the formation of As in the form of strongly nonequilibrium regions in systems departing slightly from thermodynamic equilibrium are discussed; examples of systems in which this phenomenon is realized are presented. The systems are classified with respect to the physics and types of AS formed in them. A theory describing AS from a unified viewpoint based on a mathematical analogy is described, and general results are presented, determining the basic parameters and properties of static, pulsating, and traveling AS that form in a wide class of the most diverse systems— chemical and biochemical reactions, nonequilibrium gases and superconductors, photoconductors and magnets, magnetic semiconductors, composite superconductors, active lightguiding transmission lines, and many other systems.

CONTENTS

Introduction. One of the striking phenomena of nonlinear physics is the formation of solitons and autosolitons—localized stationary states in different physical, chemical, and biological systems. Solitons are solitary waves that form in dispersive nonlinear media and whose properties are in many ways reminiscent of particles. There is an extensive literature on solitons (see, for example, Refs. 1–7). Autosolitons (AS) are solitary stationary states of a different type whose properties can differ fundamentally from particles. In a general sense AS differ from solitons in the same way that self-oscillations differ from oscillations[8–11] and autowaves from waves.[12–18]

An autosoliton is a stationary, solitary, characteristic state (autostate) of a system. The parameters of AS (form, amplitude, velocity, frequency of pulsations, etc.) are completely determined by the parameters of the system and do not depend on the kind of the perturbation generating an AS of a given type.

Autosolitons form in stable systems, in which small disturbances are damped. An autosoliton is a stable localized

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state, which at the periphery transforms into the same stable uniform state of the system.[1)] To excite AS a localized disturbance with sufficiently large amplitude and sufficiently long duration must be applied to the system. After this additional disturbance ceases one of the possible types of AS can form spontaneously in the system. In this sense the formation of AS can be regarded as a phenomenon of self-organization.[2)]

We emphasize that AS can form in monostable noequilibrium systems that have a *solitary, uniform state* for any degree of nonequilibrium. The nontriviality of the formation of such AS is associated with the fact that they do not disperse owing to diffusion, but rather, on the contrary, diffusion processes are responsible for their existence (Sec. 1).

Attractors, characterized by a definite region of attraction, are formally an example of both self-oscillations and AS. The simplest attractor—a stable limit cycle in the phase space of dynamic variables—corresponds to periodic self-oscillations.[8,11] An attractor in the configuration space, i.e., the space whose every point corresponds to a definite coordinate distribution of the parameters of the system, corresponds to an AS. In such a configuration space the system can be characterized by several attractors, i.e., AS of different type and form can arise in it. To excite AS, the local short-time disturbance exciting it must transfer the system into a state corresponding to a region of attraction (with respect to the initial conditions) of an attractor corresponding to the given type of AS. In this case after the disturbance is switched off AS will form spontaneously.

The following types of AS can be excited in a system depending on its parameters and the form of the disturbance: static AS, whose velocity equals zero and whose form remains constant in time[25–30] (Secs. 3–5); pulsating AS, whose velocity equals zero but whose form varies periodically in time[29–34] (Sec. 6); *traveling* AS,[3)] which move with a definite finite velocity without damping[1,12–16,27–30,35]; and, AS in the form of other, more complicated, solitary autostates (Sec. 2) and autowaves (see Secs. 7 and 8).

We emphasize once again that the velocity and form of a traveling AS, unlike a soliton, is determined uniquely by the parameters of the system and not by the energy of the initial disturbance.[1,12–16] When two traveling AS collide they can annihilate[12–16] or transform into AS of a different type (for example, into static or pulsating AS), depending on the parameters of the system.[32,35–37]

The nature of autosolitons is extremely diverse (Secs. 1 and 2, and Conclusions). Thus an AS in the form of a traveling, undamped electric pulse can be excited in a nerve fiber[1,12,38,39] or its electronic analog (neuristor).[1,40–42] In a high-frequency gas discharge an AS in the form of a static solitary striation[44] can be excited in addition to periodic striations.[43] In semiconductors and semiconductor structures AS are observed in the form of glowing regions, where the temperature of hot carriers[45,46] or their density[47] is high. In media where autocatalytic reactions of the Belousov-Zhabotinskiĭ type occur different traveling AS and other autowaves of a more complicated type form and are quite sharp. Autosolitons in the form of strongly nonequilibrium regions can form[4)] in gas and semiconductor plasma[50–51] as well as in thermodynamically slightly nonequilibrium neutral gases[52] (Sec. 5).

The properties of traveling AS (pulses) and some other autowaves were analyzed qualitatively back in 1946 by Wiener and Rosenblueth[54] on the basis of an axiomatic discrete model. In 1952 Hodgkin and Huxley proposed and studied a model of the propagation of pulses in a nerve fiber.[55] The form and velocity of traveling AS (pulses) were analyzed in greatest detail in the simplest two-parameter models of a nerve fiber—models of the Fitz-Hugh–Nagumo (FHN) type.[56–60,1,12–16 5)] The theory of static and pulsating AS is developed in Refs. 25–28 and 31.

It later became clear that most static, pulsating, and traveling AS studied are realized in the same (from the viewpoint of the mathematical description) class of active systems with diffusion, whose properties are determined by a system of two nonlinear differential equations of the diffusion type[6)] (Secs. 1 and 2). In one of the limiting cases (Sec. 2.2) these equations correspond to models of the FHN type and admit solutions only in the form of traveling AS and other autowaves (Sec. 7), while in the other limit solutions in the form of the simplest static autostructures are obtained (Sec. 3.1). Traveling AS (pulses) and other atuowaves have been studied very completely in models of the FHN type. There is an extensive literature devoted to them.[12–16,18,39,49,63–66,68,69] For this reason this review is confined primarily to static and pulsating AS; traveling AS are discussed only to the extent necessary in order to present a complete picture of the properties of AS.

In this review the properties of AS arising in a wide class of active distributed systems[7)] are discussed and the basic results of the theory are presented. First, the physics of AS in the basic types of monostable active systems with diffusion (Sec. 1) is studied for concrete examples, after which a classification of such systems is given and the basic properties of the AS realized are discussed (Sec. 2). The theory and properties of static (Secs. 3–5), pulsating (Sec. 6), and traveling (Sec. 7) AS in active systems with diffusion are then presented. In the last section (Sec. 8) the characteristics of AS in trigger systems and active media with long-range couplings are studied.

1. AUTOSOLITONS IN SOME SYSTEMS

1.1. Thermal-diffusion AS in systems with "positive" thermal diffusion. Thermal-diffusion AS are formed as a result of the competition between diffusion and thermal-diffusion fluxes.[25,26,51] In this section AS in systems with "positive" thermal diffusion, i.e., in systems in which the thermal-diffusion particle flux is directed from the hot region into the cold region, are studied.

For definiteness we shall study the formation of AS in an electron-hole plasma (EHP), heated in the process of carrier photogeneration.[25,26] Let the energy of the exciting photons $\hbar\omega$ exceed the gap width of the semiconductor E_g by an amount $2\Delta = \hbar\omega - E_g$. Then, when a photon is absorbed hot carriers are formed and the EHP is heated up as a result of interelectronic collisions. When the effective masses of the electrons and holes are not very different and their density ($n = p$) is high enough, the carriers are heated as a single system up to some effective temperature T. The value of T is determined from the equation describing the local balance of the energy of the carriers:

$$2G\Delta = 2n\,(T - T_0)\,\tau_\varepsilon^{-1}, \qquad (1.1)$$

where G is the rate of generation of carriers, τ_ε is the charac-

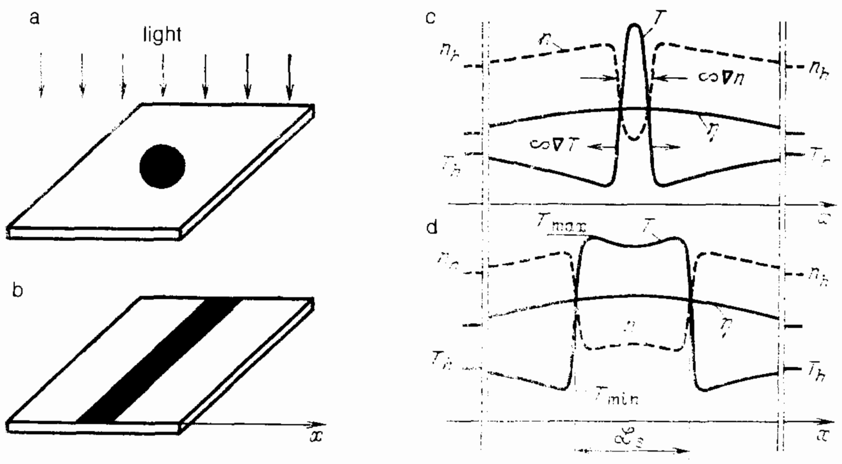

FIG. 1. Autosolitons (AS) in systems with "positive" thermal diffusion: schematic illustration of a radially-symmetric (a) and one-dimensional (b) AS in a semiconductor film; the concentration $n(x)$ and temperature $T(x)$ distributions of a hot electron-hole plasma (EHP) in a narrow spike (c) and wide (d) AS.

teristic relaxation time of the energy of the carriers, and T_l is the lattice temperature of the semiconductor. If the lifetime of the carriers $\tau_r \approx$const, then a single uniform state of the EHP is associated with fixed values of G and Δ: $n = n_h = G\tau_r$ and $T = T_h = T_l + \Delta\cdot\tau_\varepsilon/\tau_r$. In such a stable EHP a hot AS in the form of a self-maintaining region of high temperature and low carrier density can nonetheless be excited (Fig. 1).[25,26]

To excite AS the EHP must be additionally heated for a short time in some region with light absorbed by free carriers. At the end of this light pulse an As whose form is determined solely by the parameters of the EHP and does not depend on the parameters of the pulse exciting the AS, is formed at the point of illumination (Fig. 2).

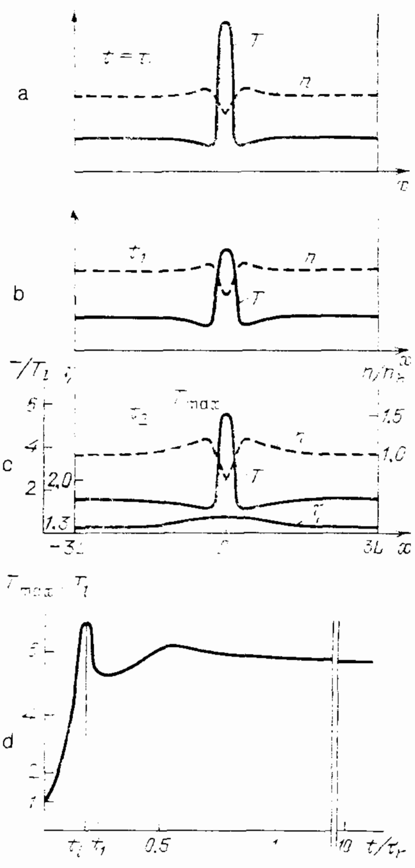

FIG. 2. Kinetics of the formation of a hot AS in a stable, heated EHP[89] with local short-duration ($t_i = \tau_r/6$) action of a radiation pulse (heating the carriers). a, b) distribution of the temperature T and density of the carriers n at intermediate times $t = t_i$ (a) and $t_1 = 2\tau_r/9$ (b). c) steady-state form of the AS ($t_2 = 10\tau_r$). d) time-dependence of the maximum temperature in the AS formed.

The existence of a hot AS is determined by the fact that thermal diffusion causes intense transfer of hot carriers out of the high-temperature region (see Fig. 1c). As a result the carrier density and therefore the power density removed form the system of hot carriers into the lattice $P = 2n(T - T_l)/\tau_\varepsilon$, decrease in the high-temperature region. According to (1.1) this maintains the high value of the carrier temperature at the center of the AS, since the power $W = 2\Delta\cdot G =$ const fed to the carriers maintains a high temperature, i.e., it does not depend on n and T. The region of low carrier density at the center of the AS does not spread out because the diffusion flux of carriers at the center of the AS is practically balanced by the oppositely directed thermal diffusion flux (Fig. 1c).

The carrier density and temperature distributions in a quasineutral EHP ($n = p$) are described by the equations of balance of the number of particles and their average energy $\bar{\varepsilon}$[83-85]:

$$\frac{\partial n}{\partial t} = -\operatorname{div} \mathbf{j}_e + G - R, \tag{1.2}$$

$$2\frac{\partial n\bar{\varepsilon}}{\partial t} = -\operatorname{div} \mathbf{j}_\varepsilon + W - P, \tag{1.3}$$

where $\mathbf{j}_e$ and $\mathbf{j}_\varepsilon$ are the electron flux density and the energy flux density of the carriers; $R = n/\tau_r$ is the carrier recombination rate. In a degenerate EHP[20,83-85] $\bar{\varepsilon} = (3/2)T$, and

$$\mathbf{j}_e = -\nabla(nD(T)) = -D\nabla n - (1+\alpha)DT^{-1}n\nabla T, \tag{1.4}$$

where D is the coefficient of bipolar diffusion and $1 + \alpha = \partial \ln D/\partial \ln T$. It follows from (1.2) and (1.4) that

$$\tau_r\frac{\partial n}{\partial t} = L^2\Delta\eta + 1 - n, \tag{1.5}$$

where the carrier density is measured in units of $n_h = G\tau_r$, $\eta = nD(T)\times(D^0 n_h)^{-1}$; $D^0 = D(T_l)$, $L = (D^0\tau_r)^{1/2}$ is the bipolar diffusion length.

One can see from Eq. (1.5) that in a hot EHP the quantity L characterizes the spatial scale of variation not of the carrier density n, but rather the quantity $\eta \propto nD(T)$, i.e., the flux j_e (1.4). For this reason the larger the value of L the lower is the value of j_e in the AS i.e., the more accurately the diffusion flux is compensated by the thermal diffusion flux (Fig. 1c). It follows from here that the size of the hot region *of the AS $\mathscr{L}_s$ has the upper limit $\sim L$. On the other hand,* in order for the hot region of the AS not to spread out owing to heat conduction by the carriers, $\mathscr{L}_s$ must be greater than the characteristic length of variation of the temperature of the carriers $l \approx (D\tau_\varepsilon)^{1/2}$, whence follows the necessary condition for the existence of AS: $L \gg l$. In an AS of size $l \ll \mathscr{L}_s \ll L$ it can be assumed that the quantity η satisfies $\eta = nD(T)/D^0 n_h$ = const and Eq. (1.1) holds. Substituting $n = n_h \eta D^0/D(T)$ into (1.1) we obtain an equation for the local energy balance in the AS

$$W = \frac{2\eta (T - T_l) D^0 n_h}{D(T)\tau_\varepsilon(T)}, \tag{1.6}$$

which takes into account, with η = const, the compensation of the diffusion and thermal-diffusion fluxes of the carriers, i.e., the fact that j_e (1.4) is small in the AS.

It follows from the theory of AS (see Sec. 3.2) that the roots of Eq. (1.6) with W = const and $\eta = \eta_s$ determine the values of the maximum T_{max} and minimum T_{min} temperatures in a wide AS[26] (Fig. 1d). Equation (1.6) has several roots when in some range of temperatures of the carriers $\alpha + s > 0$, where $s = \partial \ln \tau_\varepsilon / \partial \ln T$. Thus the inequalities $L \gg l$ and $\alpha + s > 0$ are the conditions for the existence of AS in a photogenerated, weakly heated EHP.[25,26]

These conditions can also hold in a strongly asymmetric (with $m_h^* \gg m_e^*$) EHP,[85] in a gas plasma (F-layer of the ionosphere),[86] and in an ideal gas of heated excitons in semiconductors[87]; for this reason, according to Refs. 25 and 26, AS can be excited in these systems also.

The thermal-diffusion AS (Figs. 1 and 2) studied above can be excited in a symmetric EHP, heated with a constant[50,88,89] or high-frequency electric field.[51] When an EHP is heated in a constant field a distinguished direction appears, so that a one-dimensional AS[88,50] in the form of a hot layer (Fig. 1b) perpendicular to the lines of current $j = \sigma E$ forms in it. For such an AS the current is j = const, and the power heating the EHP is $W = j^2/\sigma \propto T/\eta$. Substituting $W \propto T/\eta$ into (1.6), we find[50,88] that the conditions for the existence of such a transverse AS reduce to $L \gg l$ and $\alpha + s > -1$. Depending on the mechanisms responsible for the dissipation of the energy and momentum of the carriers, i.e., the form of the functions $D(T)$ and $\tau_s(T)$, wide AS of size $\mathscr{L}_s \gg l$ (see Fig. 1d) or spike AS of large amplitude[25,89] (Sec. 5) can be excited in the EHP heated by a constant field.

In a "dense" EHP heated by an electric field a longitudinal AS in the form of a pinch or a layer directed along the lines of current[90,91] can be excited. This is associated with the fact that the mobility μ of the carriers in a "dense" EHP is determined by electron-hole scattering, i.e., $\mu \propto T^{3/2} n^{-1}$. In a "dense" EHP the electronic current $j_e \propto \partial P/\partial x$, i.e., $\eta \equiv P = nT$ is the pressure of the electron gas, while the conditions for the existence of AS reduce to[90,91] $L \gg l$ and $s > -3/2$.

A longitudinal thermal-diffusion AS (Fig. 1b) can be excited under the same conditions in a gas plasma heated with a constant or high-frequency field.[90,51] In a gas plasma the electron and ion pressure plays the role of the quantity η, varying smoothly as a function of the bipolar diffusion length L (Fig. 1c): $\eta \equiv P = n(T + T_i)$, where T_i is the ion temperature.

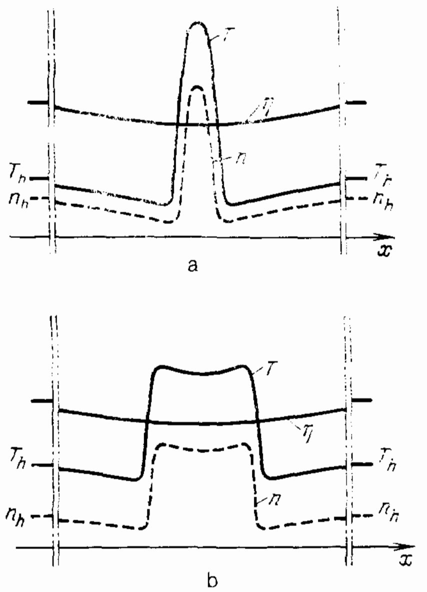

FIG. 3. Autosolitons in systems with "negative" thermal diffusion: distribution of the density $n(x)$ and temperature $T(x)$ in an EHP thermalized with the lattice in a narrow spike (a) and wide (b) AS.

Numerical studies of the kinetics of formation of a thermal-diffusion AS (Fig. 2) have established[89] that to excite the AS the light pulse additionally heating the carriers must have the following parameters: the duration $t_u \gtrsim \tau_r$ and the spot size $(d_0) l \lesssim d_0 \lesssim L$. These conditions follow[89] from the physics of a thermal-diffusion AS.[25,50]

1.2. Thermal-diffusion AS in systems with "negative" thermal diffusion. In many systems, because of the fact that the scattering cross section of the particles increases as the velocity of the particles increases, the thermal diffusion flux of the particles is directed from the cold region into the hot region. Such "negative" thermal diffusion can be observed in a mixture of neutral gases,[92] in chemical reactions,[93] and in semiconductors.[25,51] Thermal-diffusion AS, consisting of a region of high temperature and particle density,[25,52] can arise in such systems (Fig. 3).

We shall illustrate the physics of formation of such an AS using the example of a nonequilibrium EHP, thermalized with the lattice of the semiconductor.[25]

For sufficiently high carrier temperatures and densities there is enough time for the EHP generated in a thin semiconductor film to be thermalized with the lattice. The distribution of the carrier density and temperature are described by Eq. (1.2) and the heat-conduction equation

$$c\rho \frac{\partial T}{\partial t} = \nabla_\perp (\varkappa(T) \nabla_\perp T) + W - (T - T_t)\varkappa(T) l_T^{-2}, \tag{1.7}$$

averaged over the thickness of the film; here c, ρ, and $\varkappa$ are the specific heat capacity, density, and thermal conductivity of the lattice; l_T is the characteristic length of temperature variation; and, T_t is the temperature of the substrate (thermostat). The lattice is heated as the carriers recombine[25] and as they are heated by the radiation,[94] i.e., $W = n(E_g/\tau_r + \sigma_{ph}\Phi)$, where σ_{ph} and Φ are the photon absorption cross section and the photon density.

For carriers thermalized with the lattice[51]

$$\mathbf{j}_e = -D\nabla n + (\delta - \alpha - 1)\, nDT^{-1}\nabla T, \tag{1.8}$$

where $\delta = \partial \ln \tau_d (\varepsilon, T)/\partial \ln T$, and τ_d is the momentum dissipation time of a carrier with energy ε. In some semiconductors $\delta - \alpha - 1 > 0$, i.e., for carriers thermalized with the lattice "negative" thermal diffusion can occur.

The existence of AS is linked with the fact[25] that for $L \gg \mathscr{L}_s \gg l_T$ thermal diffusion leads to a carrier flux directed from the cold peripheral regions into the hot central region of the AS (Fig. 3). Carriers accumulating in the AS increase, as a result of their recombination and absorption of radiation by them, the power W flowing from the carriers into the lattice and therby maintain a high temperature at the cener of the AS.[51]

A thermal-diffusion AS (Fig. 3) can also be excited in a mixture of reacting light and heavy gases, heated by radiation selectively absorbed by the light component of the gas mixture.[52] The existence of an AS in such a mixture is determined by the fact that the highest absorption of electromagnetic radiation occurs, i.e., the greatest heating is realized, in the region where the density n of light gas particles is high (Fig. 3). Diffusion spreading of the light gas is prevented by the thermal diffusion flux of light gas particles away from the periphery to the center of the hot region. In the case under study narrow spike AS with a high temperature are formed (Fig. 3a).[52] *In addition, the smaller the ratio l_T/L the higher are the temperature and density of the light gas at the center of the AS and the lower is the value of W, i.e., the degree of nonequilibrium of the mixture, with which an AS can be excited in it*[52] *(Sec. 5.4).*

1.3. Static, pulsating, and traveling AS in systems with uniformly generated "combustion material." The combustion of a substance with concentration n is described by the equations[72,73]

$$c\rho \frac{\partial T}{\partial t} = \nabla (\varkappa \nabla T) + E\Phi(n, T) - P, \tag{1.9}$$

$$\frac{\partial n}{\partial t} = D\Delta n + G - \Phi(n, T), \tag{1.10}$$

in which G and $P = 0$; Φ and E are the rate and heat of the reaction. The rate of many reactions is given by $\Phi \propto n \exp(-\Delta/T)$, i.e., it is of a thermal-activation character.[72,73] Equations (1.9) and (1.10) with $G = 0$ describe the propagation of the combustion front, i.e., the wave of switching from one stable stationary state into another.[12,15,72,73]

More complicated phenomena can be observed in systems in which uniform generation of combustion material occurs, i.e., $G \neq 0$. In such systems AS can arise in the form of static, pulsating, and traveling regions of combustion, outside which the concentration of combustion material and the temperature are everywhere constant and below the threshold for spontaneous combustion. A good example of a traveling AS is buring of grass in a steppe followed by regeneration of the grass owing to growth.

An example of a static AS is a steadily burning sphere. Such a sphere can arise in a mixture of gases (including the atmosphere), in which H_2O, O_2, or CO_2 molecules dissociate under the action of electromagnetic radiation (or other source). As a result of such dissociation combustion material—hydrogen, ozone, or carbon monoxide—is generated uniformly. Let the steady-state density of the combustion material and the temperature of the mixture be much lower than the threshold for spontaneous combustion of the mixture. If, now, some small region of the mixture is ignited, i.e., heated by an additional source (for example, in the form of a glowing metal filament) above the flash point, then when this source of heat is switched off a steadily buring sphere can appear in the mixture.

The rate of combustion $\Phi \infty \exp(-\Delta/T)$ in such a sphere is high because of the high temperature. Steady combustion is maintained in the sphere by a constant diffusion inflow of uniformly generated combustion material from the peripheral regions of the sphere toward its center.[95] This inflow of combustion material is all the stronger the longer is the diffusion length L compared with the size of the sphere. The latter is determined by the characteristic distance (l) over which the temperature changes. From here it follows that a steadily burning sphere can exist only if $L \gg l$, which is a necessary condition for the existence of static AS.[25-29]

The chemical reactions determining combustion processes are very complicated and, as a rule, have been poorly studied.[72] But there exist simple physical systems, whose properties are described by equations of the type (1.9) and (1.10) with $G \neq 0$. A nonequilibrium gas or semiconductor plasma, in which the rate of recombination of electrons increases rapidly with the temperature, for example, as $\exp(-\Delta/T)$, is such a system with uniformly generated combustion material.[32,96]

For definiteness we shall study a semiconductor in which an EHP is generated uniformly with such high density that the electrons and holes in it are degenerate, while their rate of recombination R is determined by Auger processes.[32] Carriers whose energy is of the order of the gap width of the semiconductor E_g, which are produced in the process of Auger recombination, heat the EHP as a result of electron-electron collisions.[97,98] For this reason, here, carrier recombination can be regarded as an exothermal combination reaction in which heat $E = E_g$ is released.

In a degenerate EHP the thermal current is suppressed compared with the diffusion current,[83] i.e., $j_c = -D\nabla n$. Equations (1.2) and (1.3) then actually reduce to (1.9) and (1.10), in which $\Phi = R(n,T)$, $E = E_g$, $c\rho = n\partial\bar{\varepsilon}/\partial T$; G is the rate of generation of electrons and holes; and, $P = P(n,T)$ is the power transferred from the hot EHP to the semiconductor lattice.[83,99]

If the EHP, weakly heated in the process of Auger recombination, is additionally illuminated in a region of size $d_0 < L$ with a light pulse with duration of the order of τ_r, then an AS in the form of a static (Fig. 4a) or pulsating (Fig. 4b) region of "combustion"—high carrier temperature— can arise in it.[32,100] The appearance of such AS is linked with the fact[32] that in semiconductors, as a rule, the rate of Auger recombination is $R \propto \exp(-\Delta/T)$,[101-103] while $\tau_\varepsilon \ll \tau_r$, i.e., $L \approx (D\tau_r)^{1/2} \gg l \approx (\varkappa \tau_\varepsilon/n)^{1/2}$.

A static AS exists because[32] strong recombination of carriers occurs at its center owing to the fact that the rate of Auger recombination R increases with the temperature T. Because strong diffusion of carriers from peripheral regions into the AS, however, the carrier density decreases significantly less than R increases, i.e., than would happen in a uniform EHP. The carriers flowing into the AS owing to diffusion from the periphery recombine intensively in it, generating in the process of Auger recombination carriers with energy of the order of E_g, which, in their turn, maintain a high temperature in the AS.

The stability of a static AS (Sec. 4) is linked with the

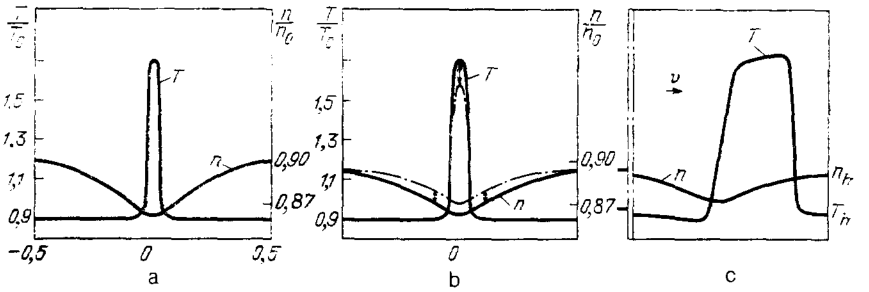

FIG. 4. Static (a), pulsating (b), and traveling (c) AS in a stable degenerate EHP, heated in the process of Auger recombination.[32] The results of the numerical calculation were taken from Ref. 100.

fact that the increase of the temperature in the AS is damped by a corresponding decrease in the carrier density. This damping is realized only in some range of variation of G, at the limits of which the static AS spontaneously transforms into a pulsating AS[32] (Sec. 6). The pulsating AS appears as a result of the growth of temperature fluctuations, which vary with some frequency $\omega = \omega_c$ ($\omega_c \tau_r > 1$, but $\omega_c \tau_\varepsilon \ll 1$). The growth of these fluctuations is associated with the fact that because of the long lifetime of the carriers ($\tau_r \gg \tau_\varepsilon$) the carrier density cannot follow such rapid changes in the temperature, i.e., its damping action is weakened.

The conditions $L \gg l$ and $\tau_r \gg \tau_\varepsilon$, under which an AS traveling in any direction can be excited[32] (Fig. 4c) in addition to static (Fig. 4a) and pulsating AS (Fig. 4b), hold in the EHP under study. In addition, depending on the parameters of the EHP, the AS can have a different form (the solid curves in Fig. 5; $\theta = T/T_h$, $\eta = n/n_h$).

The appearance of a traveling AS can be explained as follows. It follows from the equation describing the local energy balance of the carriers $E_g R(n,T) = P(n,T)$ that two states of the plasma can correspond to the same carrier density in a uniform EHP $n = n_h$: $T = T_h$ and $T = T_{max}$. For $\tau_r \gg \tau_\varepsilon$ local, brief heating of the EHP by radiation absorbed by free carriers can transfer the plasma at the location of the illumination from the state with $T = T_h$ into the state with $T = T_{max}$. Because of heat conduction the hot carriers produced can heat the neighboring regions of the EHP, transferring them successively into the state $T = T_{max}$, i.e., a wave of transfer from the state $T = T_h$ into the state with $T = T_{max}$ arises. Over the time $\tau = \tau_\varepsilon T/F$ (F is the Fermi energy of the electrons) the heat flux of the carriers propagates over a distance $\sim l$, so that the velocity of such a wave $v \sim l/\tau$. Behind this transfer wave (the front wall of the traveling AS, Fig. 4c) the temperature and therefore the rate of Auger recombination are high, so that the carrier density will decrease over a time $\sim \tau_r$, i.e., it will drop off over the drift length $\tilde{L} = v\tau_r$. This burnup of the density has a lower limit $n = n_{min}$ (Fig. 4c), at which the velocity of the back wall equals the velocity of the front wall. Behind the back wall the density is restored to the value $n = n_h$ in a region of size $\sim \tilde{L}$ (Fig. 4c).

The appearance of pulsating AS as well as traveling AS in the form of a combustion wave followed by restoration of the temperature and density of the combustion material is

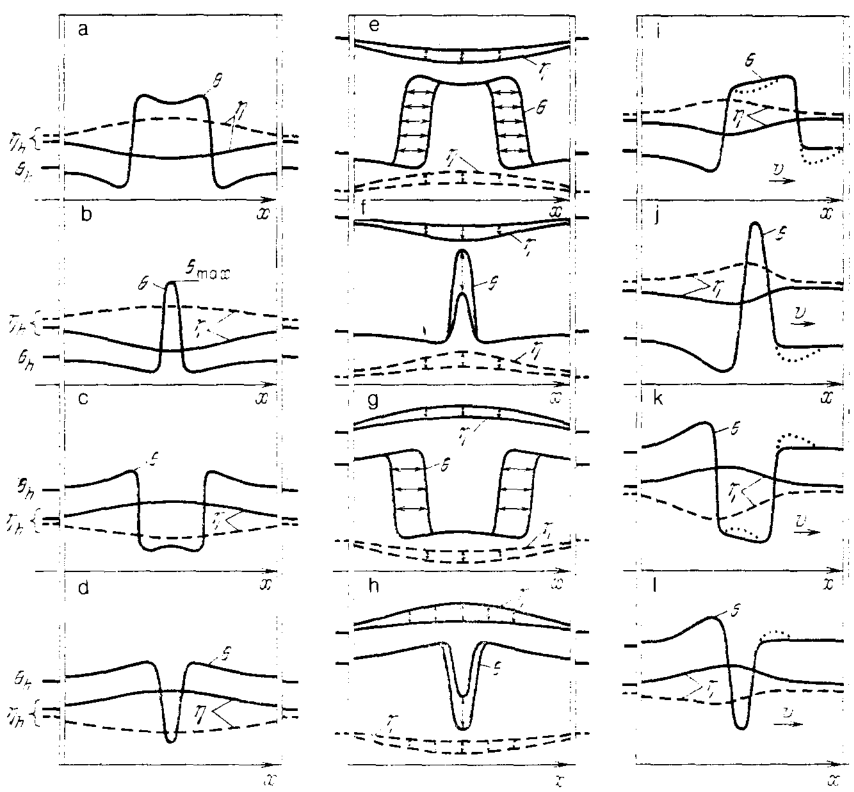

FIG. 5. The basic types of static (a–d), pulsating (e–h), and traveling (i–l) autosolitons. The dotted curves show the distribution $\eta(x)$ in N- and Λ-systems, while the solid curves show the distribution in И- and Λ-systems; the arrows in Figs. 3–h show the process of oscillation of the activator and inhibitor between two extreme positions of the pulsating AS; the dots in Figs. i–l show the possible difference in the distribution of the activator in KΩ-systems and Ω-systems.

characteristic for the class of systems of the combustion type studied here.

1.4. Autosolitons in systems with local self-production of matter. In many chemical and biological reactions self-production of one of the chemical substances occurs as a result of autocatalysis, cross-catalysis, fermentation processes, or replication.[15,16,18,23,104–113] Real reactions of this type are very complicated.[48,104,23,111–113] For this reason, for convenience, we shall study a hypothetical reaction, proposed by Prigogine and his coworkers,[23] whose scheme goes back to the classical work of Türing[110] and has the form

$$A \to X,\ 2X + Y \to 3X,\ B + X \to Y + C,\ X \to E, \tag{1.11}$$

where A and B are the starting, C and E are the final, and X and Y are the intermediate products of the reactions. The second reaction is autocatalytic. It describes the self-production of the substance X, which is controlled by the chemical substance Y, X, and Y are customarily called the activator and the inhibitor, respectively.[16,107,109] The equations of chemical kinetics (the Brusselator model)[23,104] follow from (1.11):

$$\begin{aligned}\tau_\theta \frac{\partial\theta}{\partial t} &= l^2\Delta\theta + B + \theta^2\eta - \theta(1+A),\\ \tau_\eta \frac{\partial\eta}{\partial t} &= L^2\Delta\eta - \theta^2\eta + A\theta,\end{aligned} \tag{1.12}$$

where θ and η are, respectively, the concentrations of the activator X and inhibitor Y; τ_θ, τ_η and l, L are the characteristic times and distances over which θ and η change; A and B are constant coefficients. In such a stable, uniformly flowing, chemical reaction AS in the form of a region of size $\sim l$ with a high concentration of the activator θ and a somewhat lower concentration of the inhibitor η (solid curves in Fig. 5b) can be excited when $L \gg l$.[25,37,114] This is linked with the fact that in a small region of size $\sim l \ll L$ self-production of the activator, which is described by the second of the reactions (1.11), cannot be suppressed by a corresponding local change in the inhibitor concentration η because of the strong diffusion inflow of inhibitor from the peripheral regions into the center of the AS (Fig. 5b).

Traveling AS (pulses) as well as other autowaves have been studied in greatest detail experimentally in investigations of reactions of the Belousov-Zhabotinskiĭ type.[48,62–64,66,68,69,111] Examples of such autocatalytic reactions are the oxidation-reduction reactions for cerium, magnesium, or ion irons in the presence of bromine ions, potassium bromate, cerium sulfate, and organic reducers—such as malonic, bromomalonic, acetoacetic, malic, citric, and other acids.[48,111] These multistage reactions are described by a system of many differential equations of chemical kinetics.

If, however, stages describing activation and inhibition processes can be separated in the reaction, then by using general mathematical methods[115,116] the description of AS and autowaves in them can be reduced to a system of two or three diffusion equations.[15,16] Thus autowaves in the Belousov-Zhabotinskiĭ reactions are described with the help of a three-component Field-Körös-Noyes model ("Oregonator"),[12,15,23,106] as well as two-component models,[15,111] including also models described by equations of the type[48,117,118]

$$\begin{aligned}\tau_\theta \frac{\partial\theta}{\partial t} &= l^2\Delta\theta + \theta\{1 - \eta[2 + (\theta-1)^2]\} + B,\\ \tau_\eta \frac{\partial\eta}{\partial t} &= L^2\Delta\eta - A\eta - \theta(\eta - 1).\end{aligned} \tag{1.13}$$

Numerical studies of Eqs. (1.13) showed that in the model under study for $L \gg l$ and $\tau_\eta \gg \tau_\theta$, in accordance with the general theory,[25,29,30] pulsating AS[33,34] can be excited in addition to traveling and static AS[117,118] (Sec. 6).

The formation of shape (morphogenesis) and other processes[110,16,23,106–109] have been linked with self-production of matter in biochemical reactions.

Thus Gierer and Meinhardt showed[119–122] that the experimental data on the development of hydra[109] can be explained on the basis of the model

$$\begin{aligned}\tau_\theta \frac{\partial\theta}{\partial t} &= l^2\Delta\theta + \frac{A\theta^2}{\eta} + B - \theta,\\ \tau_\eta \frac{\partial\eta}{\partial t} &= L^2\Delta\eta + C\theta^2 - \eta.\end{aligned} \tag{1.14}$$

In (1.14) θ is a short-range activator while η is a long-range inhibitor,[109] i.e., $L \gg l$.[8)] Here the AS (Fig. 5b; broken curve for η)[114] describes the distribution of θ and η in the "head" of the hydra. The growth of such a "head" (excitation of an AS) in a morphologically uniform fragment of the hydra can be provoked by translating the corresponding cells, taken from the body of the adult hydra.[109] The properties of AS (Sec. 5) permit explaining[37,114] some results of experiments with hydra[109] and numerical studies of the model (1.14).[119–122] In particular, the instability of two close-lying AS with respect to the "transfer" effect (Sec. 4.3), established in the theory,[25] explains[114] why it has not been possible to excite two close-lying "heads" in the body of the hydra in the experimental and numerical studies.[109,119–122]

Local self-production of particles can also occur in semiconductors and gases. In a self-maintained gas discharge as well as with interband or impurity breakdown of a semiconductor this process is determined by the increase in the rate of ionization of electrons ν_i with an increase in the electron density. The increase in ν_i with increasing n is associated with the increasing importance of excited centers and electron-electron collisions in the ionization process.[123,124] In such systems an AS is a region of high electron density and somewhat low effective electron temperature[44] (Fig. 5b, where $\theta \equiv n$, $\eta \equiv T$—solid curve). In other words, the electron density plays the role of the activator ($\theta \equiv n$) and the effective electron temperature plays the role of the inhibitor ($\eta \equiv T$).[44,125] For this reason an AS exists when the characteristic distance over which the electron density changes is much shorter than the distance over which the electron temperature changes. In a gas discharge this condition holds owing to the fact that the bipolar diffusion length of electrons and ions is short owing to the large mass of the ions. Under conditions of impurity breakdown of a semiconductor it can hold when the Debye screening length $r_d \ll l_\varepsilon$—the characteristic cooling length of hot electrons.[125]

2. ACTIVE SYSTEMS WITH DIFFUSION

2.1. Definition and examples of systems. In the last few years, primarily owing to the work of Glandsdorf and Prigogine,[104] Nicolis and Prigogine,[23] Haken,[107,24] Ebeling,[106] as well as Vasil'ev, Romanovskiĭ, and Yakhno,[12] it has become obvious that the properties of many physical, chemical, and biological systems (including those studied in Sec. 1) are described by the following system of nonlinear equations of the diffusion type[9)]

$$\tau_i \frac{\partial X_i}{\partial t} = \sum_{j=1}^{N} \nabla (L_{ij}\nabla X_j) - g_i(X_1, \ldots, X_i, \ldots, X_N, A). \tag{2.1}$$

We call attention to the fundamental character of these equations. They are the equations of macroscopic kinetics and describe, in particular, chemical and biological reactions.[12–16,104–113] In the latter case X_i are the concentrations of the intermediate products and A are the constants of the chemical reactions (Sec. 1.4). In physical kinetics Eqs. (2.1) in the hydrodynamic approximation follow from the first moments of Boltzmann's kinetic equation[2] and describe, for example, the properties of hot carriers in semiconductors[20,83,85] and gas plasma.[84,86,124,127] For physical systems X_i denote the temperature, the carrier density, the potential, the current density, etc., while A is the emf of the power supply, the intensity of electromagnetic radiation, etc., (Sec. 1).

Interest in this class of distributed systems with diffusion continues to grow owing to the fact that striking nonlinear phenomena are realized in them. The uniform state of such systems with some (bifurcation) value of the parameter $A = A_c$ can become unstable,[12–16,23,24,106–110] and self-excited oscillations[11,17,23] or autostructures (dissipative structures)[12,16,23,24,29,106–109] can appear spontaneously in them. For $A < A_c$, i.e., in the region of stability of the uniform state, different types of static and pulsating AS[29,30] and autowaves[12–16] can be excited in systems with external, short-time, local perturbation. Thus in the class of systems under study two types of nonlinear phenomena are realized: 1) spontaneous formation of autostructures and their subsequent evolution[10)]; 2) induced formation of different types of AS and complicated autowaves in stable systems.

Systems whose properties are described by Eqs. (2.1) and in which these nonlinear phenomena are realized are now called *active systems with diffusion*. They are said to be active in the sense that positive feedback is realized on at least one parameter $X_1 \equiv \theta$—the *activator*; this leads to self-production of the activator and is responsible for the formation of autostructures. The activator production process is controlled by some other parameter $X_2 \equiv \eta$— the *inhibitor*, which suppresses the activator production process. For this reason, in the simplest but quite general case, active systems with diffusion are described by two equations of the type

$$\tau_\theta \frac{\partial \theta}{\partial t} = l^2 \Delta\theta - q(\theta, \eta, A), \tag{2.2}$$

$$\tau_\eta \frac{\partial \eta}{\partial t} = L^2 \Delta\eta - Q(\theta, \eta, A), \tag{2.3}$$

where A is a bifurcation parameter.

Equations (2.2) and (2.3) are the basic equations of the theory of autowaves and autostructures (dissipative structures) in biological systems.[12,14,16,23,104,106–109] In particular, the Gierer-Meinhardt model (1.14) and the simplified Jacob-Mono model of morphogenesis[12,16,128] as well as models of the FHN type, describing propagation of pulses (traveling AS) in a nerve fiber,[1,12,15,16,39] along the myocardium of the heart,[18,63,66,67] and in electronic neuristor circuits[1,40–42] reduce to these equations. Analysis of some ecological systems as well as population genetics reduces to the study of equations of the type (2.2) and (2.3).[129–131] Analysis of other systems with self-production of matter (Sec. 1.4), including the simplest models of chemical reactions (1.12) and (1.13), also reduces to the analysis of equations of the type (2.2) and (2.3). In particular, they describe AS and striations in high-frequency gas discharge[44,123,124] and accompanying shock ionization in semiconductors[125] (Sec. 1.4).

Equations (1.10) and (1.11), describing processes of the combustion type (Sec. 1.3), are a particular case of Eqs. (2.2) and (2.3). In such systems the temperature plays the role of the activator ($\theta \equiv T$) while the concentration of the "combustion material" plays the role of the inhibitor ($\eta \equiv n$).

In systems with thermal diffusion (Secs. 1.1 and 1.2) the temperature also plays the role of the activator ($\theta \equiv T$) while some function of the carrier density and carrier temperature plays the role of the inhibitor (the specific form of η for some systems is presented in Secs. 1.1 and 1.2). It is easy to verify that by transforming to the variables θ and η in Eqs. (1.2) and (1.3) these equations are reduced to Eqs. (2.2) and (2.3) with a somewhat more complicated left side. The equations describing the stratification of the uniform state of systems with "mutual diffusion" of two components can be transformed in an analogous manner.[132,133] In other words, static AS in systems with thermal diffusion (Secs. 1.1 and 1.2) are described by Eqs. (2.2) and (2.3) for the stationary case.[25,26]

Equations of the type (2.2) and (2.3) also describe many other phenomena,[12,16] including the formation of autostructures in ferroelectrics—photoconductors,[134] in hot plasma of semiconductors,[32,135] in nonequilibrium superconductors,[22,136] in materials with phase transitions,[137,138] and in magnetic photoconductors[139]; the propagation of a reaction wave along the surface of a catalyst,[15,140] of a light pulse in an active optical fiber,[141] of photoinduced autowaves in semiconductors,[142–145] and in magnets[146,147]; stratification of the lattice temperature in semiconductors[25,94,96,148–150] and in semiconductor structures,[151–157] as well as the appearance of regions of local ionization in uniform semiconductors[125,158] and in *p-n* junctions.[47,159]

2.2. Classification of monostable systems and the properties of AS realized in them. In the theory of AS (Secs. 3–7) an important characteristic determining their form is the dependence $\eta(\theta)$, which is given by the equation

$$q(\theta, \eta, A) = 0 \quad \text{for} \quad A = \text{const}. \tag{2.4}$$

This dependence determines the relation between η and θ in the regions of AS, where $\theta(\mathbf{r})$ varies smoothly (Sec. 3), so that we shall call it the *local coupling* (LC). The uniform state of the system $\theta = \theta_h$ and $\eta = \eta_h$, according to (2.2) and (2.3), satisfies the equations

$$q(\theta_h, \eta_h, A) = 0, \; Q(\theta_h, \eta_h, A) = 0, \tag{2.5}$$

i.e., it corresponds to the point of intersection of the LC curve and the equation-of-state (ES) curve. The ES curve corresponds to the dependence $\eta(\theta)$, following from the equation

$$Q(\theta, \eta, A) = 0 \quad \text{for} \quad A = \text{const}. \tag{2.6}$$

The qualitative form of the LC an ES curves (Fig. 6) for

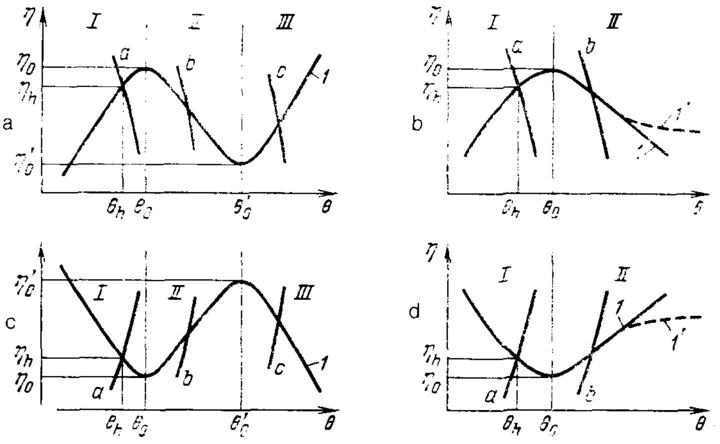

FIG. 6. The basic types of curves of local coupling (LC) (cures 1) and equation of state (ES) (curves a, b, c). The ES curves a, b, and c correspond, respectively, to cold ($A < A_0$, $\theta_h < \theta_0$), heated ($A_0 < A < A_0$, $\theta_0 < \theta_h < \theta'_0$), and hot ($A > A'_0, \theta_h > \theta'_0$) systems.

monostable systems in which AS originate can be established from general considerations.[27,28 11)] The existence of negative feedback on the inhibitor and positive feedback on the activator means that in some range of values of the parameters θ, η, and A we have

$$Q'_\eta \equiv \frac{\partial Q}{\partial \eta} > 0, \text{ a } q'_\theta \equiv \frac{\partial q}{\partial \theta} < 0. \tag{2.7}$$

For all A the monostable systems under study have a unique uniform state, i.e., the dependences $\eta_h(A)$ and $\theta_h(A)$ are single-valued. The latter is valid, according to (2.5), when

$$q'_\theta Q'_\eta - q'_\eta Q'_\theta > 0. \tag{2.8}$$

When the conditions (2.7) hold this inequality is satisfied only for $q'_\eta Q'_\theta < 0$, i.e., when

$$Q'_\theta > 0, \quad q'_\eta < 0 \tag{2.9}$$

or

$$Q'_\theta < 0, \quad q'_\eta > 0. \tag{2.10}$$

Let the inequalities (2.7) hold in the range $\theta_0 < \theta < \theta'_0$. Then, when (2.9) holds the derivative satisfies $d\eta/d\theta < 0$ for both the LC and ES curves (Fig. 6a), since

$$\frac{d\eta}{d\theta} = -\frac{q'_\theta}{q'_\eta} \text{ on the curve ЛС: } q(\theta, \eta) = 0, \tag{2.11}$$

$$\frac{d\eta}{d\theta} = -\frac{Q'_\theta}{Q'_\eta} \text{ on the curve УС: } Q(\theta, \eta) = 0. \tag{2.12}$$

At the points $\theta = \theta_0$ and θ'_0 the derivative is $q'_\theta = 0$ (the point $\theta = \theta'_0$ does note necessarily exist, but it characteristically exists for many real systems). According to (2.11) the derivative $d\eta/d\theta$ at the points $\theta = \theta_0$ and θ'_0 changes sign, i.e., when (2.9) holds the LC curve is N-shaped (Fig. 6a), while in the case when the point θ'_0 does not exist it is Λ-shaped (Fig. 6b). In the general case the sign of Q'_η (or Q'_θ) is not related with the sign of q'_θ, so that for the ES curve the condition $d\eta/d\theta < 0$ can also hold outside the region $\theta_0 \leqslant \theta \leqslant \theta'_0$ (Figs. 6a and b, curves 1-c).

Analogous arguments for the case when the conditions (2.10) hold lead to И shaped or V-shaped LC curves (Figs. 6c and d).

Depending on the form of the LC curve we shall call the system an N-, И-, Λ-, or V-system. In many physical systems the parameter θ characterizes the temperature (Sec. 1) and increases as the degree of nonequilibrium increases (bifurcation parameter A). For this reason we shall call the region I, where $\theta \leqslant \theta_0$, the *cold* region; the region II, where $\theta_0 < \theta < \theta'_0$, the *heated* region; and, the region III where $\theta \geqslant \theta'_0$, the *hot* region (see Fig. 6). By analogy we shall say that a system is cold, heated, or hot depending on the region of values to which its uniform state, i.e., the quantity $\theta = \theta_h$, corresponds (Fig. 6). The uniform state of cold and hot systems is stable and corresponds to values of θ and η for which $q'_\theta \geqslant 0$; the heated system is unstable, and it corresponds to $q'_\theta < 0$.

Autosolitons of a different type (Fig. 5) form in systems[29,30] in which the diffusion (L) or drift ($\tilde{L}$) length of variation of the inhibitor distribution η is much larger than the characteristic diffusion length (l) of variation of the activator distribution θ (Sec. 1), i.e., for $L \gg l$, or $\tilde{L} = v\tau_\eta \gg l$, where v is the velocity of the traveling AS.

In AS (Figs. 1–5 and 7) the activator distribution θ in some small regions (of the order of l in size)—the walls of the AS—changes rapidly from $\theta_{\max} > \theta_h$ to $\theta_{\min} < \theta_h < \theta_0$. Outside these regions $\theta(\mathbf{r})$ varies smoothly with the same characteristic length as $\eta(\mathbf{r})$. The inhibitor distribution $\eta(\mathbf{r})$ everywhere varies smoothly in a static AS (Figs. 1–3, 4a and 5a–d) with characteristic length of the order of L (Secs. 3 and 5), while in a traveling AS it varies over a characteristic length of the order of $\tilde{L}$ (Figs. 4c and 5i–l; Sec. 7). At the periphery of the AS the functions $\theta(\mathbf{r})$ and $\eta(\mathbf{r})$ approach their values for the uniform state $\theta = \theta_h$ and $\eta = \eta_h$ (Figs. 1–5 and 7). In N- and Λ-systems $\eta(\mathbf{r})$ outside the walls of the AS varies in phase with $\theta(\mathbf{r})$ [Fig. 5, solid curves for $\eta(x)$], while in И- and V-systems it varies an antiphase (Fig. 5, broken curves; Secs. 3 and 5).[29,30]

The type of AS is determined primarily by the quantities $\varepsilon = l/L$ and $\alpha = \tau_\theta/\tau_\eta$ (Fig. 8). We shall therefore divide the systems into three qualitatively different classes[29,30]:

K-systems, for which $\varepsilon \ll 1, \alpha > 1$ (Secs. 3–5), (2.13a)

Ω-systems, $\varepsilon \gtrsim 1, \alpha \ll 1$ (Sec. 7), (2.13b)

KΩ-systems, $\varepsilon \ll 1, \alpha \ll 1$ (Sec. 6,7). (2.13c)

According to (2.2), (2.3), and (2.13) in K-systems the inhibitor η is much longer-ranged but more responsive than the activator θ; in Ω-systems it is unresponsive but shorter-

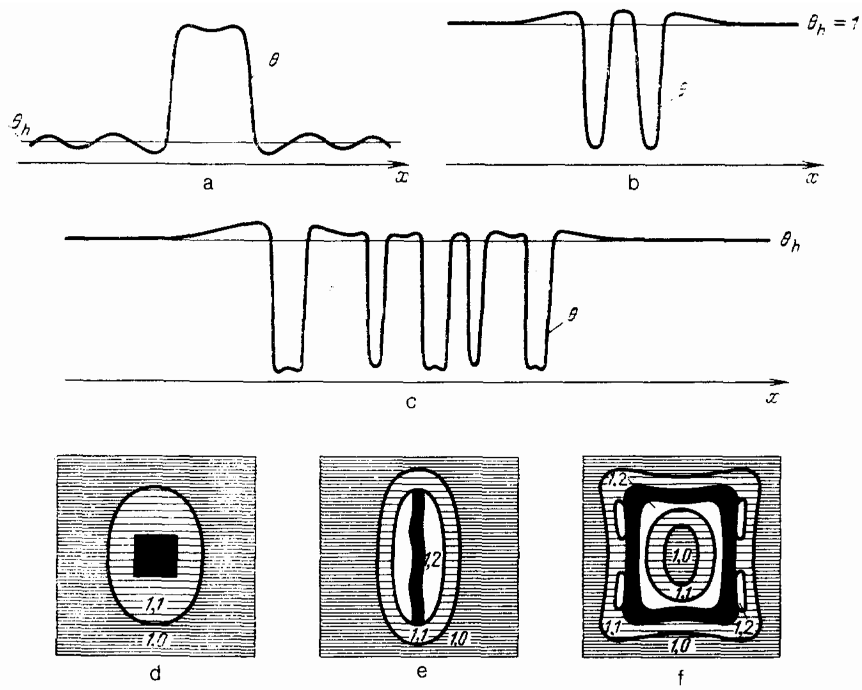

FIG. 7. Distribution of the activator in some static AS of a complicated type: a–c) one-dimensional; d–f) two-dimensional. b, d–f) results of numerical studies of the model (1.13)[117,118] (in Figs. d–f in the dark regions $\theta \approx 0.1$; the uniform state corresponds to $\theta_h = 1.0$).

ranged; and, in KΩ-systems it is both longer-ranged and unresponsive.

The uniform state of heated (Fig. 6, curve c) K-*systems* with some $A = A_c$, when θ_h exceeds θ_0, stratifies,[12)] i.e., it becomes unstable (Türing instability[110)]) wth respect to aperiodic (with frequency $\omega = 0$) growth of fluctuations with a particular wave number $k = k_0$.[12,16,23,106–109] It is convenient to write the value of k_0 for Eqs. (2.2) and (2.3) in the form[28,153]

$$k_0 = (lL)^{-1/2}\,(q'_\theta Q'_\eta - q'_\eta Q'_\theta)^{1/4} \approx (lL)^{-1/2}. \tag{2.14}$$

In K-systems autostructures (dissipative structures) with large amplitude form abruptly as a result of stratification.[25–29] Uniform oscillations and pulsating and traveling autostructures do not form in them. In some regions of stability of the uniform state in K-systems (Fig. 6, curves a and c) static AS[25–30] can be excited (Fig. 5, a–d; Secs. 3–5). The properties and parameters of AS depend significantly on the form of the LC curve (Fig. 6).

Only hot static AS of the spike type[25,30] (Fig. 5b) with amplitude $\theta_{max} \gg \theta_0 > \theta_h$, whose value increases with A and

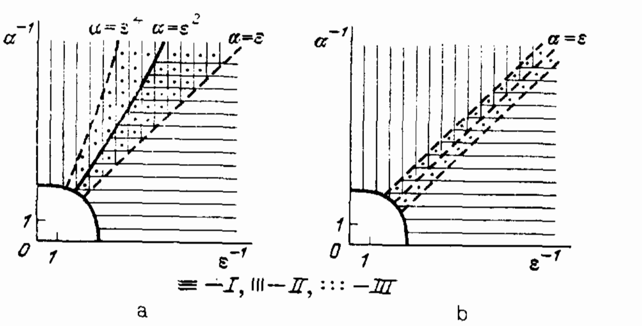

FIG. 8. Regions of existence of static (I), traveling (and other autowaves) (II), and pulsating (III) autosolitons in the $\alpha - \tau_\theta/\tau_\eta$ and $\varepsilon = l/L$ plane in N- and И-systems (a) and Λ- and V-systems (b).

all the more the lower the value of ε, can arise in KΛ- and KV-systems (Sec. 5). Depending on the nonlinearity of the system, narrow ($\sim l$ in size) and wide ($\sim L$ in size) spike AS can form in them. The minimum level of excitation $A = A_b$, for which narrow spike AS still exist, is proportional to ε^n ($n > 0$). In other words AS can be excited for $A_b \ll A_c$, i.e., in systems that depart slightly from the state of thermodynamic equilibrium (Sec. 5.4). In two- and three-dimensional systems the one-dimensional, narrow, spike AS are unstable; radially symmetric spike AS can exist in them.

In KN- and KИ-systems (Secs. 3 and 4) the minimum level of excitation $A = A_b$ for which AS still exist is determined by the nonlinearities of the system and is virtually independent of ε (Sec. 4.3). In cold systems (with $A_b < A < A_c$) hot wide ($\mathcal{L}_s > l$ in size) AS with $\theta_{max} > \theta'_0$ form (Figs. 1d and 5a), while in hot systems (with $A'_c < A < A'_b$) cold wide AS with $\theta_{min} < \theta_0$ form (Fig. 5c). As $A \to A_b\,(A'_b)$ the size of a hot (cold) AS decreases and reaches the value $\mathcal{L}_s \sim l \ln \varepsilon^{-1}$ at the point $A = A_b\,(A'_b)$, where the AS vanishes abruptly (Sec. 4.2). As $A \to A_c\,(A'_c)$ the AS increases in size, and the monotonic dropoff of η and θ at the periphery of the AS can be replaced by an oscillating dropoff (Fig. 7a). In some systems, not reaching the point $A = A_c\,(A'_c)$, the width of the AS reaches the critical size $\sim L$, at which division of the AS occurs (Sec. 3.5).

In two- and three-dimensional systems one-dimensional and radially symmetric AS are stable in a wide range of values of A. At the boundary of these regions AS with fluted (cellular) walls form spontaneously and they can even fragment (Sec. 4.4).

Aside from such AS many other complicated AS, both one-dimensional (Figs. 7b and c) and two(three)-dimensional (Fig. 7d–f), can form in the system. The form of the distributions of η and θ in the section of a complex two-dimensional AS is reminiscent of the distributions of η and θ in one-dimensional AS. As A is varied a complicated AS of one type can transform spontaneously into an AS of a different type as a result of "local breakdown" (Sec. 3.5), stratification of the walls (Sec. 4.4), or "transfer" (Sec. 4.3).

In K-systems with A close to A_c (A'_c) a small local nonuniformity can lead to the abrupt, spontaneous appearance of a static AS; the parameters of the AS formed are determined by the characteristics of the system and are virtually independent of the parameters of the nonuniformity.[13)]

An external field puts static AS in ideally uniform systems into motion. Drifting AS can be pinned on a small nonuniformity (pinning effect). For a sufficiently strong external field the AS generated spontaneously at a small nonuniformity or near the boundary of the sample can become detached from it and lead to the appearance of a periodic or stochastic sequence of moving AS.[51]

The uniform state of heated *Ω-systems* is unstable with respect to uniform (with $k = 0$) fluctuations with the particular frequency $\omega = \omega_0$,[104,23] whose value is conveniently written in the form[28,153]

$$\omega_0 = \alpha^{1/2}\tau_\theta^{-1}\,(q'_\theta Q'_\eta - q'_\eta Q'_\theta)^{1/2} \sim (\tau_\theta\tau_\eta)^{-1/2}. \qquad (2.15)$$

As a result of such an instability relaxational oscillations appear in an abrupt fashion in Ω systems.[104,23] For this reason such systems are sometimes said to be self-oscillatory. Static and pulsating AS do not form in Ω systems (Fig. 8). Traveling AS and other autowaves of a more complicated form can be excited in stable Ω-systems[12–16,54–69,111] (Sec. 7).[14)] In cold ΩN- and ΩИ-systems hot traveling AS form (Figs. 5 i and k), while cold traveling AS form in hot systems (Figs. 5 l and m); they are $\sim\tilde{L} = v\tau_\eta$ in size and their velocity depends on the value of A and varies from $v \sim l/\tau_\theta$ as $A \to A_c$ up to $v \sim \alpha^{1/2} l/\tau_\theta$ as $A \to A_v < A_c$ (for $A < A_v$ traveling AS are not formed). In Ω systems traveling AS are annihilated in collisions; this determines many properties and the interaction of complex autowaves.[12–16,54–69,111]

In KΩ-*systems* all types of AS can be realized[29–32,35]: simple (Fig. 5) and complicated static (Fig. 7), pulsating and traveling AS, as well as different autowaves (Sec. 7). A pulsating AS of large size can be represented, in a simplified manner, in the form of a hot or cold static AS, whose width[33] or radius[34] varies periodically in time (Figs. 5f–g). A pulsating spike AS is reminiscent of a static spike AS with an oscillating amplitude (Fig. 5f; Sec. 6).

The basic properties and parameters of static AS in KΩ-systems are analogous to those described above for K-systems. When the excitation level is changed, however, static AS in KΩ-systems can transform spontaneously into pulsating or traveling AS (Sec. 6.2). The larger is the value of the ratio α/ε, the smaller is the range A of existence of traveling AS and the higher is their minimum velocity (Sec. 7.1).[35]

In KΩ-systems (Sec. 7), unlike Ω-systems, a diffusion precursor, representing a "refractory zone" (points in Figs. 5i–l), propagates in front of the wall (front) of a traveling AS. For this reason, traveling AS may not be annihilated in collisions, but rather they can repel one another or even form an AS of a different type (static or pulsating). Traveling AS also collide inelastically with static or pulsating AS.

Depending on the parameters of the system (primarily α and ε) a small nonuniformity can lead to spontaneous formation of static, pulsating, or traveling AS.

Based on the foregoing classification a gas discharge[44] and a semiconductor under conditions of impact ionization[125] as well as systems with "negative" thermal diffusion (Sec. 1.2) are KN- and KΛ-systems. The "Brusselator" model (1.12) is a KΛ-system for $\tau_\theta > \tau_\eta$ and for a KΩΛ-system $\tau_\theta < \tau_\eta$. Systems with "positive" thermal diffusion (Sec. 1.1) are KИ- and KV-systems; for them the form of the LC curve is determined by Eq. (1.6). The model of Gierer and Meinhardt (1.14) for $\tau_\theta > \tau_\eta$ is a KV-system. Models of the "Oregonator" type (1.13) are KΩИ- or ΩИ-systems. Systems with uniformly generated "combustion material" (Sec. 1.3), including EHP, heated in the process of Auger recombination,[32] are, as a rule, KΩN- or KΩΛ-systems. In models of the FHN type[12–16] (see Introduction) $L = 0$, while $\tau_\theta \ll \tau_\eta$, i.e., they are a limiting case of ΩN- or ΩИ-systems in the limit $\varepsilon \to \infty$ (Sec. 7). In many semiconductor devices L is much greater than the size of the sample $\mathcal{L}$.[154,157] For this reason they can be regarded as a limiting case ($L \to \infty$) of K- or KΩ-systems[28] (Secs. 3.1, 4.1, 5.1, and 6.1).

In real systems the LC curve may not have a distinct N-, И- or Λ-, or V-shaped form. In Λ- and V-systems with a "degenerate" LC curve (broken curves 1′ in Figs. 6b and d) wide AS (Fig. 5a) with large amplitude can form. Conversely, based on their properties N- and И-systems with $\theta'_0 \gg \theta_0$ are closer to Λ- and V-systems, i.e., spike AS with large amplitude (Sec. 5) are realized in them in a large range of values of A. In other words, more accurately, systems for which $\theta'_0 \sim \theta_0$ (Figs. 6a and c) should be classified as N- and И-systems (Secs. 3 and 4).

3. STATIC AUTOSOLITONS (KИ- AND KN-SYSTEMS)

Before presenting the theory of AS, it is convenient, from the methodological viewpoint, to study the simplest structures realized in systems of small size $\mathcal{L} \ll L$.

3.1. Structures in small systems.[25,28,157] In systems of size $\mathcal{L} \ll L$, but $\mathcal{L} \gg l$ the inhibitor density η, unlike the activator density θ, actually does not vary in space. Its value under cyclic or neutral conditions at the boundaries of the system S

$$\mathbf{S}\nabla\eta\,|_S = \mathbf{S}\nabla\theta\,|_S = 0, \qquad (3.1)$$

can be found by averaging Eq. (2.3) over the volume (V) of the system:

$$\tau_\eta\frac{\partial\eta}{\partial t} = -V^{-1}\int_V Q(\theta(\mathbf{r}), \eta, A)\,d\mathbf{r} \equiv -\langle Q(\theta, \eta, A)\rangle. \qquad (3.2)$$

For the stationary, one-dimensional case Eq. (2.2) with $\eta(x) = \text{const}$ can be written as

$$l^2\frac{d^2\theta}{dx^2} + \frac{dU_\theta}{d\theta} = 0, \quad U_\theta = -\int^\theta q(\theta, \eta, A)\,d\theta. \qquad (3.3)$$

Equation (3.3) is formally identical to the equation describing the conservative motion of a "particle" with coordinate θ and time x in a potential U_η. The form of the potential for a fixed value of A is determined by the value of η, which, in its turn, depends on the solution $\theta(x)$. This functional relation for the stationary case, according to (3.2), has the form

$$\langle Q(\theta(x), \eta, A)\rangle = 0. \qquad (3.4)$$

The extrema of the potential U_θ (3.3) with fixed A correspond to the condition $dU_\theta/d\theta = -q(\theta,\eta,A) = 0$, i.e., according to (2.4), they are determined by the points of inter-

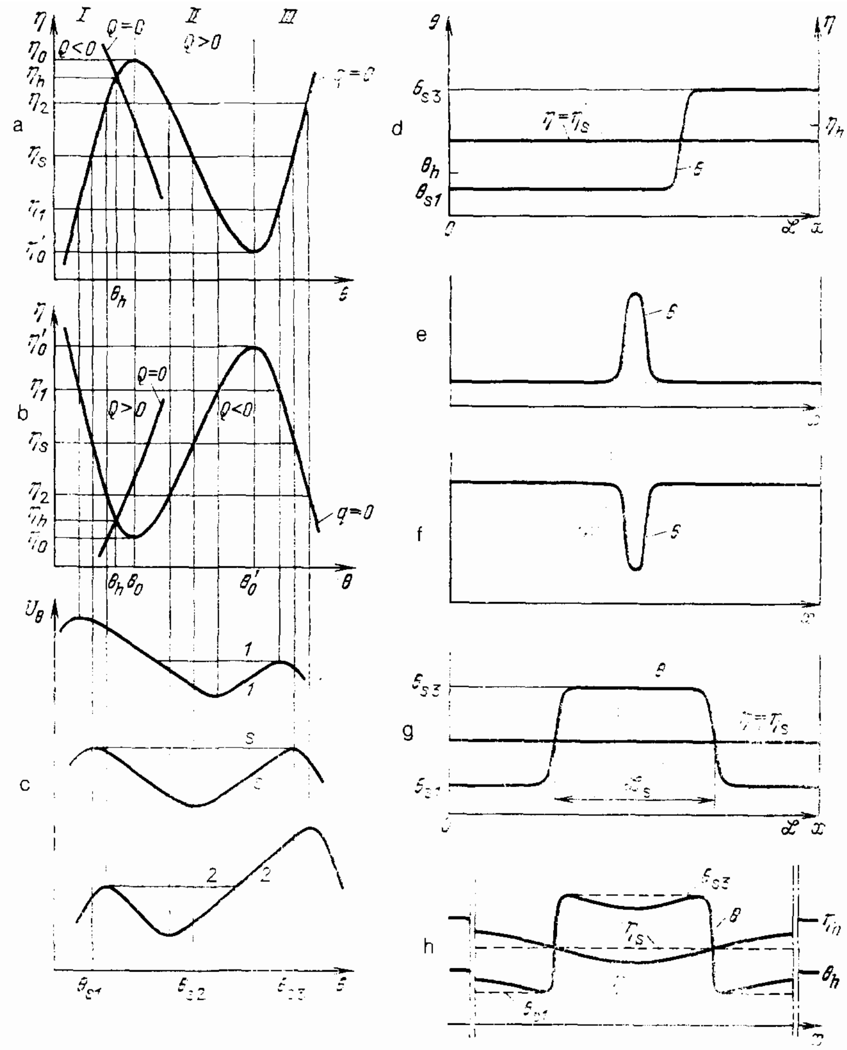

FIG. 9. Illustrating construction of states in small systems. a, b) Form of the LC curve for N- and И-systems. c) form of the potentials U_θ, the highest trajectories of a particle in which s, 2 and 1 correspond to the solutions $\theta(x)$ in Figs. d, e, and f, respectively. g) wide striation; h) form of a wide AS in large systems.

section of the LC curve with the straight line $\eta=\text{const}$ (Figs. 9a and b). This permits reconstructing the form of the potential U_θ from the form of the LC curve for different values of η (Fig. 9c). Indeed, at the extremal points of U_θ, according to (3.3), $\partial^2 U_\theta/\partial\theta^2 = -q'_\theta$. For this reason the point of intersection of the straight line $\eta=\text{const}$ and the branch II of the LC curve ($\theta_0<\theta<\theta'_0$, Figs. 9a and b), where $q'_\theta<0$ (Sec. 2.2), corresponds to the minimum of U_θ, while intersection with the branches I ($\theta<\theta_0$) and III ($\theta>\theta'_0$), where $q'_\theta>0$, corresponds to a maximum of U_θ (Fig. 9c). This implies that for values of η ranging from η'_0 to η_0 [where $\eta_0=\eta(\theta_0)$ and $\eta'_0=\eta(\theta'_0)$ are the extremal points of LC (Figs. 9a and b)], U_θ has the form of a potential well (Fig. 9c). For this reason $\theta(x)$ can have periodic solutions with a characteristic length of the order of l. Such activator distributions are, however, unstable (Sec. 4.1).

The solutions $\theta(x)$ corresponding to trajectories of a "particle" passing through a saddle point of Eq. (3.3), corresponding to a maximum of the potential U_θ (Fig. 9c), have a special form (Fig. 9d–g). This is associated with the fact that as the saddle point is approached the variation of $\theta(x)$ becomes increasingly smoother.[8] When the LC curve is N- (or И-) shaped (Figs. 9a and b), the interval (η_0,η'_0) of values of η contains two saddle points (Fig. 9c). The values of U_θ at these points $\theta=\theta_{s1}$ and θ_{s3} are equal for some $\eta=\eta_s$, satisfying, according to (3.3), the equations

$$\int_{\theta_{s1}}^{\theta_{s3}} q(\theta,\ \eta_s,\ A)\,d\theta=0,\quad q(\theta_{si},\ \eta_s,\ A)=0\quad (i=1,\ 2,\ 3). \tag{3.5}$$

For $\eta=\eta_s$ the particle trajectory (s in Fig. 9c) from one saddle point θ_{s1} to another θ_{s3} describes, with exponential accuracy, the distribution $\theta(x)$ in the form of wide striations of size $\mathscr{L}_s\gg l$ at one of the boundaries of the system (Fig. 9d), while a trajectory close to this saddle trajectory describes a wide striation at the center of the system (Fig. 9g).

One can see from Fig. 9a that in KN-systems with $\eta>\eta_s$ the distribution $\theta(x)$ has the form of a narrow hot striation (Fig. 9e), while for $\eta<\eta_s$ it has the form of a narrow cold striation (Fig. 9f). In KИ-systems (Fig. 9b) narrow hot striations are realized for $\eta<\eta_s$, while narrow cold striations are realized for $\eta>\eta_s$. The value of η and therefore the form of $\theta(x)$, according to (3.4), depend on the size of the system $\mathscr{L}$ and the value of A, i.e., on the values of $\eta=\eta_h$ and $\theta=\theta_h$ for the uniform state of the system (Fig. 9a). Replacing the wide striation with a step, from (3.4) with accuracy up to $l/\mathscr{L}\ll 1$, we obtain an equation determining its size:

$$\mathscr{L}_s Q(\theta_{s3},\ \eta_s,\ A) = -(\mathscr{L}-\mathscr{L}_s)\,Q(\theta_{s1},\ \eta_s,\ A). \tag{3.6}$$

Equation (3.6) is satisfied with different values of A owing to the fact that Q has opposite signs (Figs. 9a and b) to the right and left of the ES curve ($Q = 0$; Sec. 2.2). It follows from (3.6) that as A decreases the width of the striation $\mathscr{L}_s$ (Fig. 9g) decreases. This is linked with the fact that as A, more accurately the value of θ_h, decreases the value of η_h approaches increasingly more closely to η_s, while θ_h approaches θ_{s1}, i.e., the quantity $Q(\theta_{s1},\eta_s,A) \to Q(\theta_h,\eta_h,A) = 0$ (Fig. 9a). For this reason Eq. (3.6) can hold in the limit $\eta_h \to \eta_s$ only if $\mathscr{L}_s \to 0$. It also follows from here that for $\theta_h < \theta_{s1}$ (or $\theta_h > \theta_{s3}$) the condition (3.6) obviously does not hold, since in this case $Q(\theta_{s3}, \eta_s, A)$ and $Q(\theta_{s1}, \eta_s, A)$ have the same sign. In other words, for $A < A_s$ and $A > A_s'$ ($A =$ AS corresponds to $\eta_h = \eta_s$ and $\theta_h = \theta_{s1}$; $A = A_s' - \eta_h = \eta_s$ and $\theta_h = \theta_{s3}$) the solution in the form of striations is not realized, i.e., it exists only when $A_s < A < A_s'$. From here there follows a fundamental difference between the states in the two-parameter systems under study (Figs. 9d–g) and states with analogous form realized in bistable (or trigger) one-parameter systems with two stable states $\theta = \theta_{h1}$ and $\theta = \theta_{h3}$ (for example, in materials with a structural phase transition, in semiconductors with an S- or N-shaped current-voltage characteristic (IVC),[19,20] and in other bistable systems[21,22]). In the latter cases the solution $\theta(x)$ in the form of a step (Fig. 9d) describes a domain wall between two stable states of the system $\theta = \theta_{h1}$ and $\theta = \theta_{h3}$.

The systems studied have, for all values of A, only one uniform state $\theta = \theta_h$ ($\eta = \eta_h$), which, as one can see from the formula (3.6), is not the same as the quantities θ_{s1} and θ_{s3} for any values of A for a wide striation (Fig. 9d). The appearance of virtually uniform states $\theta(x) \approx \theta_{s1}$ and $\theta(x) \approx \theta_{s3}$ outside the wall of the striation (domain) is linked with strong diffusion spreading of the inhibitor through a system of size $\mathscr{L} \ll L$, which is taken into account by Eq. (3.4) or (3.6). Diffusion fluxes of the inhibitor cause the system to deviate strongly from a uniform state and lead to the formation of nonuniform states with $\eta \neq \eta_d$ of the type shown in Figs. 9d–g.

Thus in systems with $\mathscr{L} \ll L$ (more accurately, with $L = \infty$) there exists, in a wide range of values of A, a solution in the form of a wide striation, for which $\eta = \eta_s \neq \eta_h$ (Fig. 9g). Obviously, in systems with $\mathscr{L} \gg L$ at the periphery of the striation the value of η should transform smoothly with a characteristic length of the order of L to the value $\eta = \eta_h$, while θ should transform to θ_h. The qualitative form of a hot wide AS (Fig. 9h) can already be reconstructed easily from these considerations only.

3.2. Method for constructing an antisoliton.[26–28] It is convenient to write Eqs. (2.2) and (2.3) for the one-dimensional stationary case in the form

$$\varepsilon^2 \frac{d^2\theta}{dx^2} + \frac{dU_\theta}{d\theta} = 0, \quad U_\theta = -\int^{\theta} q(\theta, \eta(\theta), A)\, d\theta, \tag{3.7}$$

$$\frac{d^2\eta}{dx^2} + \frac{dU_\eta}{d\eta} = 0, \quad U_\eta = -\int^{\eta} Q(\theta(\eta), \eta, A)\, d\eta, \tag{3.8}$$

where x is measured in units of L. It follows from (3.7) and (3.8) that the distributions $\theta(x)$ and $\eta(x)$ describing a solitary, symmetric (relative to the point $x = 0$) state, must satisfy the integral equations

$$\int_0^{\infty} q(\theta(x), \eta(x))\, dx = \int_{\theta_m}^{\theta_h} q(\theta, \eta(\theta))\, d\theta = 0, \tag{3.9}$$

$$\int_0^{\infty} Q(\theta(x), \eta(x))\, dx = \int_{\eta_m}^{\eta_h} Q(\theta(\eta), \eta)\, d\eta = 0, \tag{3.10}$$

where $\theta_m = \theta(0)$, $\eta_m = \eta(0)$. The presence of the small parameter $\varepsilon = l/L$ in (3.7) permits using, when studying stationary states, the method of qualitative analysis, based on the concepts of slow and fast motions[8,9]—in our case smooth and sharp distributions.

The solutions of Eqs. (3.7) and (3.8) can be studied as phase trajectories in the four-dimensional phase space of the variables X_i ($i = 1,\dots,4$):

$$X_1 \equiv \theta, \quad X_2 \equiv \varepsilon \frac{d\theta}{dx}, \quad X_3 \equiv \eta, \quad X_4 \equiv \frac{d\eta}{dx}, \tag{3.11}$$

satisfying, according to (3.7) and (3.8), the system of equations

$$\varepsilon \frac{dX_j}{dx} = f_j(X_i), \quad j = 1,\ 2,$$
$$\frac{dX_j}{dx} = f_j(X_i) \quad (j = 3,\ 4;\ i = 1,\ \dots,\ 4). \tag{3.12}$$

where

$$f_1 = X_2,\ f_2 = q(X_1,\ X_3,\ A),$$
$$f_3 = X_4,\ f_4 = Q(X_1,\ X_3,\ A). \tag{3.13}$$

According to the qualitative theory of differential equations,[8] for $\varepsilon \ll 1$ all phase trajectories of the system of equations (3.12) pass near trajectories corresponding to smooth or sharp distributions, as well as their combinations (see, however, Sec. 5.3).

Smooth distributions correspond to solutions of the system (3.12) with $\varepsilon = l/L = 0$; more precisely, $l = 0$. The characteristic length of these distributions is L. One can see from (3.7) and (3.8) that for $\varepsilon = 0$ the quantities η and θ are related with one another locally by Eq. (2.4), while the distribution $\eta(x)$ is described by the equation

$$L^2 \frac{d^2\eta}{dx^2} + \frac{dU_\eta}{d\eta} = 0, \quad U_\eta = -\int^{\eta} Q(\theta(\eta),\ \eta,\ A)\, d\eta. \tag{3.14}$$

For this reason the dependence $\eta(\theta)$, given by (2.4), is called the local coupling (LC).

Sharp distributions correspond to solutions of the system of equations (3.7) and (3.8) with $L = \infty$, i.e., they satisfy Eq. (3.3) with $\eta(x) = \text{const}$. Thus the solutions $\theta(x)$ studied in Sec. 3.1 are sharp distributions.

In KN- and KИ- systems the LC curve (Figs. 9a and b) has three sections (I, II, and III) where the dependence $\theta(\eta)$ is single-valued (Fig. 10a). For this reason the potential U_η in (3.14) consists of three independent branches (I, II, and III), which correspond to $\theta \leqslant \theta_0$, $\theta_0 \leqslant \theta \leqslant \theta_0'$ and $\theta \geqslant \theta_0'$ (Fig. 10b), respectively. According to (3.14)

$$\frac{dU_\eta}{d\eta} = -Q(\theta(\eta),\ \eta,\ A). \tag{3.15}$$

It follows from (2.4)–(2.6) and (3.15) that the potential U_η has an extremum at the point $\eta = \eta_h$ and $\theta = \theta_h$, corresponding to a uniform state of the system. The form of the

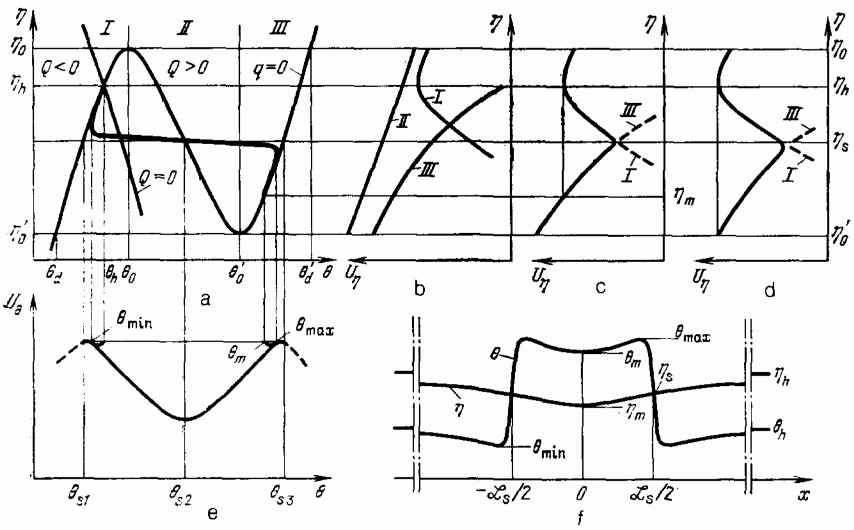

FIG. 10. Illustrating construction of a hot, wide AS (f) in N-systems: form of the LC ($q=0$) and ES ($Q=0$) curves (a), the branches I–III of the potential U_η for smooth distributions (b), the true potential U_η, and the trajectory of a "particle" in it (c), corresponding to the distribution $\eta(x)$ in AS (f), of the true potential U_η with $A=A_d$ (d), of the true potential U_θ (solid curves) (e), the highest particle trajectory in which corresponds to the distribution $\theta(x)$ in the AS (f).

extremum is determined by the sign of the derivative

$$\frac{d^2U_\eta}{d\eta^2}=-\frac{q'_\theta Q'_\eta-q'_\eta Q'_\theta}{q'_\theta}, \tag{3.16}$$

i.e., according to (2.8) the sign of q'_θ. In a cold system $\theta_h<\theta_0$ and the uniform state corresponds to $q'_\theta>0$ (Sec. 2.2). Therefore the branch I of the potential U_η has the form of a potential hump (Fig. 10b). Branches II and III of the potential U_η have the form of potential steps (Fig. 10b), whose slope, according to (3.15), is determined by the sign of Q (Fig. 10a).

It follows from the form of U_η that the motion of a "particle" in each branch of the potential U_η (Fig. 10b) is infinite, i.e., it cannot describe the distribution $\theta(x)$ and $\eta(x)$ in the form of an AS. Thus a solitary state—an AS—cannot be constructed in the class of only smooth or only sharp distributions. The distributions $\theta(x)$ and $\eta(x)$ in the form of an AS consist of a combination of segments of smooth and sharp distributions.

For definiteness we shall study the construction of the form of a hot wide AS (Fig. 9h), realized in a cold KN-system (Fig. 10). The wall of such an AS is described by the solution corresponding to the separatrix of Eq. (3.3) with $\eta=\eta_s$, closed at the two saddle points $\theta=\theta_{s1}$ and θ_{s3}, whose values correspond to sections I and III on the LC curve (Fig. 10a). To construct $\eta(x)$ we shall arrange the branches I and III of the potential U_η (Fig. 10b) so that they intersect at the point $\eta=\eta_s$ (Fig. 10c). Then the potential U_η in (3.8) will assume the form of a potential well (Fig. 10c), in which the highest particle trajectory corresponds to the distribution $\eta(x)$ in AS (Fig. 10f). Indeed, the uniform state $\eta=\eta_h$ ($\theta=\theta_h$) corresponds, as already pointed out, to the point of maximum of the potential U_η, where $dU_\eta/d\eta=0$ (Fig. 10c). This point is the saddle point of Eq. (3.14) and the trajectory, closing at this point, describes the smooth distribution $\eta(x)$, asymptotically transforming at the periphery to the value $\eta=\eta_h$ (Fig. 10f).

In smooth distributions $\theta(x)$ is locally related with $\pi(x)$ by Eq. (2.4). The values of θ outside the walls of AS in KN-systems correspond to the branch I of the LC curve for $\eta(x)>\eta_s$ or the branch III for $\eta(x)<\eta_s$ (Fig. 10). Joining at the points $\eta=\eta_s$, $\theta=\theta_{s1}$ and $\eta=\eta_s$, $\theta=\theta_{s3}$ the sections of the smooth distributions $\theta(x)$ with sharp distributions in the form of walls (Fig. 9d), we obtain a distribution $\theta(x)$ in the form of a wide hot AS (Fig. 10f). Thus to order $\varepsilon\ll1$ the distributions $\theta(x)$ and $\eta(x)$ in a wide AS ($\mathscr{L}_s\gg l$), taking into account their symmetry relative to the point $x=0$ (Fig. 10f), can be written in the form

$$\begin{aligned}\theta(x)&=\theta_{sh}(x)+\theta_{III}-\theta_{s3},\quad 0\leqslant x\leqslant\frac{\mathscr{L}_s}{2},\\&=\theta_{sh}(x)+\theta_I-\theta_{s1},\quad \frac{\mathscr{L}_s}{2}\leqslant x<\infty,\\ \eta(x)&=\eta_{III},\quad 0\leqslant x\leqslant\frac{\mathscr{L}_s}{2},\\&=\eta_I,\quad \frac{\mathscr{L}_s}{2}\leqslant x<\infty;\end{aligned} \tag{3.17}$$

here $\theta_{sh}(x)$ is a sharp distribution describing the wall of the AS; it corresponds at $\eta=\eta_s$ to the separatrix of Eq. (3.3), passing from one saddle point θ_{s3} to another θ_{s1}; $\eta_{I,III}(x)$, $\theta_{I,III}(x)$ are smooth distributions, which describe $\eta(x)$ and $\theta(x)$ outside the walls of the AS. The latter, as follows from (3.14) and (2.4), are solutions of the equations[32]

$$L^2\frac{d^2\eta_j}{dx^2}=Q_j\equiv Q(\theta_j(\eta),\eta,A),$$
$$q(\theta_j,\eta,A)=0,\quad (j=\mathrm{I,III}), \tag{3.18}$$

which satisfy the boundary conditions

$$\eta_I(\infty)=\eta_h,\quad \eta_I\left(\frac{\mathscr{L}_s}{2}\right)=\eta_{III}\left(\frac{\mathscr{L}_s}{2}\right)=\eta_s,$$
$$\left.\frac{d\eta_{III}}{dx}\right|_{x=0}=0. \tag{3.19}$$

The value of the inhibitor distribution at the wall of the wide AS $\eta=\eta_s$ as well as the extremal values of the activator $\theta_{max}=\theta_{s3}$ and $\theta_{min}=\theta_{s1}$ can be found from simple, as a rule, algebraic equations (3.5). The latter also determine the value of θ at the point $x=\mathscr{L}_s/2$: $\theta_{sh}(\mathscr{L}_s/2)=\theta_{s2}$ (Fig. 10f). Integrating Eq. (3.18), taking into account the smoothness of the inhibitor concentration at the point $x=\mathscr{L}_s/2$ (i.e., the conditions $d\eta_I/dx=d\eta_{III}/dx$ at $x=\mathscr{L}_s/2$), we obtain the equations for determining $\mathscr{L}_s$, as well as the values of $\eta(0)=\eta_m$ and $\theta(0)=\theta_m$ at the center of the AS (at the point $x=0$; Fig. 10f)[32,29,30]

$$\int_{\eta_s}^{\eta_h} Q_{\mathrm{I}}\,d\eta+\int_{\eta_m}^{\eta_s} Q_{\mathrm{III}}\,d\eta=0,$$

$$\mathscr{L}_s=\sqrt{2}L\int_{\eta_m}^{\eta_s}\left(\int_{\eta_m}^{\eta} Q_{\mathrm{III}}\,d\eta\right)^{-1/2}d\eta, \tag{3.20}$$

$$q(\theta_m,\ \eta_m,\ A)=0.$$

We note that Eqs. (3.5) and (3.20), determining the main parameters of the AS, satisfy, to order ε, (3.9) and (3.10), respectively.

To simplify the presentation we have omitted many details, which can be established[27] from an analysis of the true dependence $\eta(\theta)$ in an AS (thick curve in Fig. 10a). By studying its behavior one can determine more accurately the form of $\eta(x)$ and $\theta(x)$ and construct, in a self-consistent manner,[27] the potentials U_η in (3.8) and U_θ in (3.7), taking into account (3.9) and (3.10). In particular, it can be shown that the potential U_θ in (3.7) is a potential well, which, to order ε, is close to the potential U_θ in (3.3), shown in Fig. 9c (curve s). Near the extremal points $\theta_{\min}\approx\theta_{s1}$ and $\theta_{\max}\approx\theta_{s3}$ (Fig. 10a) of this potential (Fig. 10e) it branches off with two shallow potential wells (with depth of the order of ε). The regions of smooth distributions correspond to the highest trajectory of the motion in these shallow wells, while the regions of the sharp distribution (walls of AS) correspond to the highest trajectory in a deep well (Fig. 10e).

It follows from the procedure described above for constructing the form of the AS that in KN-systems (Fig. 10a) the functions $\theta(x)$ and $\eta(x)$ outside the walls of the AS change in phase (Fig. 10f), while in KИ-systems (Fig. 9b) they change in antiphase (Fig. 5, broken curves). This is essentially the only qualitative difference between AS in KN- and KИ-systems.

In hot systems with $\theta_h>\theta_0'$, i.e., for $A>A_c'$ (Figs. 6a and c, curves c), there exist cold AS[26-30] (Figs. 5c and d), the procedure for constructing $\theta(x)$ and $\eta(x)$ in which is analogous to that described above for a hot AS. The maximum $\theta_{\max}=\theta_{s3}$ and minimum $\theta_{\min}=\theta_{s1}$ values of θ as well as the value of $\eta=\eta_s$ at the wall of a cold AS are also determined, to order ε, from Eqs. (3.5), while its width, as well as the distributions $\theta(x)$ and $\eta(x)$ are determined from Eqs. (3.3) and (3.20), (3.17)–(3.19), in which the indices I and III as well as 1 and 3 must be interchanged.[29,32]

We note that Eqs. (3.3), (3.5), and (3.17)–(3.20) also determine the parameters and form of the AS for $l\ll L$ in systems whose properties are described by more complicated equations than (2.2) and (2.3), containing, for example, cross terms $\nabla\eta\cdot\nabla\theta$. This and other results can be rigorously proven,[32] using the asymptotic theory of AS presented below.

It also follows from the qualitative procedure described above[25,27] that aside from the simplest static autosolitons (Figs. 5a–d) AS of a complicated form, both mirror-symmetric (Fig. 7b) and unsymmetric (Fig. 7c), can be excited in the system. The latter contain several regions—striations—with high (low) activator density and different width, and located at different distances from one another. Numerical calculations indicate that excitation of complicated AS is possible.[117,118,15)]

3.3. Asymptotic theory. The walls of an AS, in which the activator density changes sharply over a short length $\sim l\ll L$, can be regarded as boundary layers. The existence of such boundary layers makes it possible to use the ideas of the theory of singular perturbations, developed for other problems with boundary layers,[116,160-162] for analyzing AS. Thus it can be verified that to order $\varepsilon=l/L\ll 1$ Eqs. (2.2) and (2.3) for stationary states in accordance with the general theory[8] reduce[59,60,163] to equations for sharp (3.3) and smooth (3.14) distributions. To construct AS from a set of sharp and smooth distributions one must construct a solution that satisfies the integral (3.9) and (3.10) and corresponding boundary conditions.[27]

Using the ideas of the theory of singular perturbations,[116] it can be shown[29,32] that the functions $\theta(x)$ and $\eta(x)$, satisfying (3.17)–(3.20), are the solution describing the form of the AS (Fig. 10f) to order $\varepsilon\ll 1$. Indeed, taking into account the symmetry of the AS relative to the point $x=0$ (Fig. 10f), we study two sections: $m=1$, $0\leqslant x\leqslant x_0\equiv\mathscr{L}_s/2$ and $m=2$, where $x_0\leqslant x<\infty$. The boundary conditions for the functions $X_i^{(m)}(x)$ in Eqs. (3.12) on each of the sections $m=1$ and 2 have the form

$$X_2^{(1)}(0)=X_4^{(1)}(0)=0,\quad X_1^{(2)}(\infty)=\theta_h,\quad X_3^{(2)}(\infty)=\eta_h,\quad X_i^{(1)}(x_0)=X_i^{(2)}(x_0)\quad (i=1,\ \ldots,\ 4). \tag{3.21}$$

Following the theory of singular perturbations[116] we write the solutions of the system (3.12) in the form

$$X_i^{(m)}=\widetilde{X}_i^{(m)}(\zeta,\ \varepsilon)+\overline{X}_i^{(m)}(\xi,\ \varepsilon)\quad (i=1,\ \ldots,\ 4;\ m=1,\ 2), \tag{3.22}$$

where $\zeta=x-x_0$, $\xi=(x-x_0)/\varepsilon$; we shall seek the so-called[116] exterior solution $\widetilde{X}_i^{(m)}(\zeta,\varepsilon)$ and interior (boundary) solution $\overline{X}_i^{(m)}(\xi,\varepsilon)$ in the form of series in ε:

$$\widetilde{X}_i(\zeta,\ \varepsilon)=\widetilde{X}_{i,0}(\zeta)+\varepsilon\widetilde{X}_{i,1}(\zeta)+\ldots+\varepsilon^k\widetilde{X}_{i,k}(\zeta)+\ldots, \tag{3.23}$$

$$\overline{X}_i(\xi,\ \varepsilon)=\overline{X}_{i,0}(\xi)+\varepsilon\overline{X}_{i,1}(\xi)+\ldots+\varepsilon^k\overline{X}_{i,k}(\xi)+\ldots \tag{3.24}$$

We substitute (3.22)–(3.24) into (3.12), (3.13). Then, expanding the functions $f_j(X_i)$ in a series in powers of ε and equating the coefficients with equal powers of ε (depending separately on ζ and ξ[116]), we obtain to zeroth order in ε the system of equations

$$\widetilde{X}_{2,0}^{(m)}=0,\quad f_2(\widetilde{X}_{i,0}^{(m)}(\zeta))=0,\quad \frac{d\widetilde{X}_{3,0}^{(m)}}{d\zeta}=\widetilde{X}_{4,0}^{(m)}(\zeta), \tag{3.25}$$

$$\frac{d\widetilde{X}_{4,0}^{(m)}}{d\zeta}=f_4(\widetilde{X}_{i,0}^{(m)}(\zeta)), \tag{3.26}$$

$$\frac{d\overline{X}_{1,0}^{(m)}}{d\xi}=\overline{X}_{2,0}^{(m)}(\xi),\quad \frac{d\overline{X}_{2,0}^{(m)}}{d\xi}=\overline{f}_{2,0},\quad \frac{d\overline{X}_{3,0}^{(m)}}{d\xi}=0, \tag{3.27}$$

$$\frac{d\overline{X}_{4,0}^{(m)}}{d\xi}=0\quad (i=1,\ \ldots,\ 4;\ m=1,\ 2), \tag{3.28}$$

where

$$\overline{f}_{j,0}=f_j(\widetilde{X}_{i,0}^{(m)}(0)+\overline{X}_{i,0}^{(m)}(\xi))-f_j(\widetilde{X}_{i,0}^{(m)}(0)), \tag{3.29}$$

and the boundary conditions

$$\widetilde{X}_{4,0}^{(1)}(-x_0)=0;\quad \widetilde{X}_{i,0}^{(1)}(0)=\widetilde{X}_{i,0}^{(2)}(0)\quad (i=3,\ 4),\quad \widetilde{X}_{1,0}^{(2)}(\infty)=\theta_h,\quad \widetilde{X}_{3,0}^{(2)}(\infty)=\eta_h, \tag{3.30}$$

$$\widetilde{X}_{1,0}^{(1)}(0)+\overline{X}_{1,0}^{(1)}(0)=\widetilde{X}_{1,0}^{(2)}(0)+\overline{X}_{1,0}^{(2)}(0), \tag{3.31}$$

$$\overline{X}_{2,0}^{(1)}(0)=\overline{X}_{2,0}^{(2)}(0), \tag{3.32}$$

in which the conditions[116]

$$\bar{X}^{(1)}_{i,0}(-\infty)=\bar{X}^{(2)}_{i,0}(\infty)=0 \quad (i=1,\ \ldots,\ 4) \qquad (3.33)$$

were employed. It follows from (3.27), (3.28), and (3.33) that

$$\bar{X}^{(m)}_{3,0}(\xi)=\bar{X}^{(m)}_{4,0}(\xi)=0 \quad (m=1,\ 2). \qquad (3.34)$$

In order for the conditions (3.32)–(3.34) to hold at the same time the solutions $\bar{X}^{(m)}_{1,0}(\xi)$, $\bar{X}^{(m)}_{2,0}(\xi)$ of the first two equations (3.27) must correspond to the separatrix passing from one saddle point to another. This situation is realized only if $\tilde{X}^{(m)}_{3,0}(0)=\eta_s$ $(m=1,2)$, where η_s satisfies (3.5). Assuming that $\bar{X}^{(1)}_{1,0}(0)=\theta_{s2}-\theta_{s3}$, $\bar{X}^{(2)}_{1,0}(0)=\theta_{s2}-\theta_{s3}$, we obtain (3.17)–(3.20) from (3.25)–(3.34).

The form of the static AS in KN- and KИ-systems (Figs. 5a and c), established in the theory described in Refs. 26–28, has been confirmed in detail by numerical and analytical studies of different models[16,33,100,117,118,128,164–167] (see, for example, the figures in Sec. 1).

3.4. Radially symmetric autosolitons.[27,28] Radially symmetric static AS, as follows from (2.2) and (2.3), are described by the following system of equations:

$$\varepsilon^2\frac{d^2\theta}{d\rho^2}+\varepsilon^2 2^s\rho^{-1}\frac{d\theta}{d\rho}+\frac{dU_\theta}{d\theta}=0, \qquad (3.35)$$

$$\frac{d^2\eta}{d\rho^2}+2^s\rho^{-1}\frac{d\eta}{d\rho}+\frac{dU_\eta}{d\eta}=0, \qquad (3.36)$$

where the radius ρ is measured in units of L, while U_θ and U_η are determined in (3.7) and (3.8); $s=0$ (or 1) in the case of cylindrical (or spherical) symmetry. The solutions (3.35) and (3.36) can be formally regarded as trajectories of two interacting "particles," moving with time ρ along the θ and η axes in the potentials U_θ and U_η, but, unlike Eqs. (3.7) and (3.8), in the presence of friction forces that diminish as ρ increases. As ρ increases the "work of the friction forces" decreases, and Eqs. (3.35) and (3.36) become increasingly more like Eqs. (3.7) and (3.8) for a one-dimensional AS. It follows from here that the distributions $\theta(\rho)$ and $\eta(\rho)$ in an AS with large radius $\rho_0\sim L$ (Fig. 11a) are qualitatively identical to $\theta(x)$ and $\eta(x)$ in a one-dimensional, wide AS (Fig. 5a). In addition, in the wall of an AS with radius $\rho_0\gtrsim L$, to order $\varepsilon=l/L\ll 1$, $\eta=\eta_s$ (3.5), while the distribution $\theta(\rho)$ is identical to $\theta(x)$ for a one-dimensional AS (Sec. 3.2). These results based on the theory of singular perturbations are proved in Refs. 29 and 32, where simple equations, describing $\theta(\rho)$ and $\eta(\rho)$ in an AS with a large radius, are also derived.

To describe AS with radius $\rho_0<L$ (Fig. 11b) it is necessary to take into account the fact that because of the work of "friction forces" accompanying the motion the "energies of the particles" decrease, i.e., the "particles" move in potentials U_θ and U_η along downwards sloping trajectories. It follows from here that in the wall of an AS with radius $\rho_0<L$ $\eta=\eta_{sh}\neq\eta_s$, and for a hot AS in KN-systems $\eta_{sh}>\eta_s$ (Fig. 11b), while in KИ-systems $\eta_{sh}<\eta_s$. Aside from the AS studied above, in the form of a hot cluster or a spot (Figs. 11a and b), in cold systems ($A<A_c$, Fig. 6, curves a) there exist AS in the form of a hollow sphere or cylinder (Fig. 11c). Such states with a large interior radius ρ_{01} (Fig. 11c) have in cross-section, the distributions $\theta(\rho)$ and $\eta(\rho)$ that are close to $\theta(x)$ and $\eta(x)$ in a one-dimensional AS (Fig. 5a).

In hot systems (for $A>A'_c$, Figs. 6a and b, curves c) radially symmetric AS in the form of a cold cluster or a spot (Fig. 11d) as well as in the form of a cold spherical or cylindrical layer (Fig. 11e) can be excited.

3.5. "Local breakdown" and division of AS. Analysis of (3.20) or (3.10) shows that as the degree of nonequilibrium of the system (the bifurcation parameter A) increases the size of the hot AS $\mathscr{L}_s$ (Fig. 10f) increases.[26,28] At the same time it follows from the physics of the existence of AS that the size $\mathscr{L}_s$ has the upper limit $\sim L$ (Sec. 1). This result also follows from the procedure for constructing AS (Sec. 3.2).[26,168] Indeed, the distribution $\eta(x)$ in an AS corresponds to the highest trajectory of a "particle" with coordinate η in the potential U_η (Fig. 10c). The coordinate η has the lower limit η'_0. For this reason the highest trajectory of the "particle," passing through the point $\eta=\eta'_0$, in principle limits the width of the AS.

Thus the highest trajectory (Fig. 10d) is realized in the case when for some $A=A_d<A_c$ (the point $A=A_c$ determines the point of stratification of the uniform state of the system, see Sec. 2.2) the condition $U_\eta(\theta'_0,\eta'_0,A_d)=U_\eta(\theta_h,\eta_h A_d)$ holds. This condition, determining the critical value $A=A_d$, can be written, based on the results of Sec. 3.2, to order $\varepsilon\ll 1$ in the form

$$\int_{\eta_s}^{\eta_h} Q(\theta_{\rm I}(\eta),\ \eta,\ A_d)\,d\eta=\int_{\eta_s}^{\eta'_0} Q(\theta_{\rm III},\ \eta,\ A_d)\,d\eta. \qquad (3.37)$$

It follows from (3.20) that the maximum possible size of an

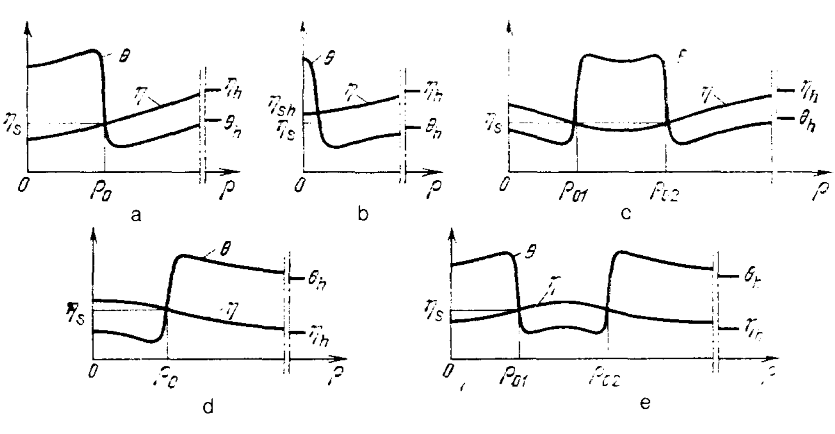

FIG. 11. Radially symmetric AS: in the form of a hot sphere or spot with large (a) and small (b) radii, hollow hot sphere or ring (c), cold sphere or spot (d), and hollow cold sphere or spot (e).

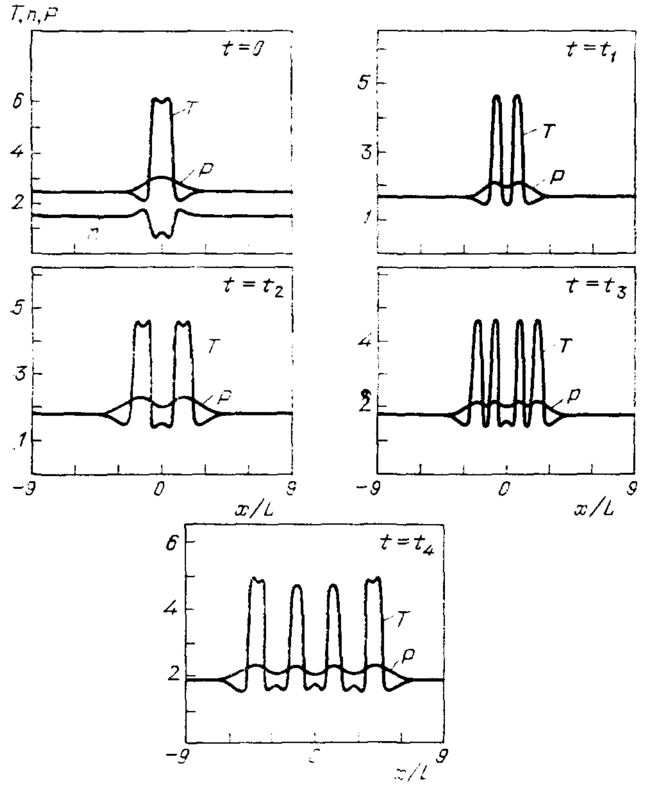

FIG. 12. Kinetics of division of AS in a "dense"

AS is

$$\mathscr{L}_{\max} = \mathscr{L}_s(A_d) = \sqrt{2}L\int_{\eta'_0}^{\eta_s}\left(\int_{\eta'_0}^{\eta} Q(\theta_{III}(\eta),\ \eta,\ A_d)\,d\eta\right)^{-1/2} d\eta. \quad (3.38)$$

Since for $A > A_d$ a solution in the form of a hot AS does not exist, the starting state in the form of an AS must be restructured dynamically. Such restructuring will occur as a result of "local breakdown,"[29,168] i.e., a jump-like drop in the activator density at the center of the AS. The latter is determined by the fact that at $A = A_d$ the value of η at the center of the AS reaches η'_0, so that for $A > A_d$ the activator density, as one can see from Fig. 10a, should drop abruptly from $\theta_m \approx \theta'_0$ to θ_d.

The successive division of an AS owing to "local breakdown" was observed in numerical studies of AS in a "dense" EHP (Fig. 12).[91] The division of AS (in the form of domains), observed in numerical studies of composite superconductors,[169,167,22] is apparently also explained by "local breakdown."

Successive division of AS, owing to "local breakdown," is one of the scenarios of dynamic restructuring of autostructures,[168] as a result of which self-organization occurs without the participation of fluctuations. The "local breakdown" effect can determine not only self-organization, occurring with a quasistationary change of the bifurcation parameter A, but also the kinetics of formation of autostructures. Thus, for example, two striations, which can move away from one another, can form as a result of "local breakdown" accompanying the excitation of the system with a short-duration but sufficiently wide pulse,[170,117] and two AS form in the system with $A < A_d$.[171]

Autosolitons of a complicated form can form spontaneously in a system as a result of "local breakdown." Thus, for example, an AS in the form of a hot sphere (or spot) (Fig. 11a) for $A > A_d$ can transform into an AS in the form of a hollow, hot sphere (or ring) (Fig. 11c).

It follows from the procedure for constructing (Sec. 3.2) a cold wide AS (Fig. 5c) in a hot system [168,29] that for some $A = A_d$ "local breakdown" can appear at the center of such an AS, i.e., the attractor density can increase in an avalanche-like fashion from $\theta_m \approx \theta_0$ up to $\theta = \theta'_d$ (Fig. 10a). The values of A_d and $\mathscr{L}_{\max} = \mathscr{L}_s(A'_d)$ for a cold AS are determined from the conditions

$$\int_{\eta_s}^{\eta_{II}} Q(\theta_{III}(\eta),\ \eta,\ A'_d)\,d\eta = \int_{\eta_s}^{\eta_0} Q(\theta_I(\eta),\ \eta,\ A'_d)\,d\eta,$$
$$\mathscr{L}_{\max} = \sqrt{2}L\int_{\eta_0}^{\eta_s}\left(\int_{\eta_0}^{\eta} Q(\theta_I(\eta),\ \eta,\ A'_d)\,d\eta\right)^{-1/2} d\eta. \quad (3.39)$$

4. STABILITY AND EVOLUTION OF STATIC ASs (KN- and KИ-SYSTEMS).

4.1. Stability and evolution of structures in small systems.[28,31,157] To study the stability of states in one-dimensional systems of size $\mathscr{L} \ll L$ we shall linearize Eqs. (2.2) and (3.2) and the cyclic boundary conditions near the stationary distribution $\theta(x)$, studied above, relative to perturbations of the form

$$\delta\theta(x,\ t) = \delta\theta(x)\,e^{-\gamma t},$$
$$\delta\eta(t) = \delta\eta\,e^{-\gamma t},\quad \delta A(t) = \delta A\,e^{-\gamma t}. \quad (4.1)$$

As a result we obtain

$$(\hat{H}_\theta - \gamma)\,\delta\theta = -q'_\eta\delta\eta - q'_A\delta A,$$
$$\hat{H}_\theta = -\frac{d^2}{dx^2} + V_\theta,\quad V_\theta = q'_\theta(\theta(x),\ \eta,\ A), \quad (4.2)$$
$$\delta\eta = -[\langle Q'_\theta\delta\theta\rangle + \langle Q'_A\rangle\,\delta A]\,(\mu_0 - \gamma\alpha^{-1})^{-1}, \quad (4.3)$$
$$\delta\theta(0) = \delta\theta(\mathscr{L}),\quad \left.\frac{d\delta\theta}{dx}\right|_{x=0} = \left.\frac{d\delta\theta}{dx}\right|_{x=\mathscr{L}}, \quad (4.4)$$

where $\mu_0 = \langle Q'_\eta\rangle$; x and the time are measured in units of l and τ_θ, respectively. It is obvious from (4.1) that the distribution $\theta(x)$ is unstable if $\mathrm{Re}\,\gamma < 0$.

We note that for uniform one-parameter systems[16)] with no inhibitor ($\delta\eta = 0$) it can be shown, based solely on the translational symmetry of the problem, that all nonuniform distributions $\theta(x)$ are unstable. Indeed, it follows from (4.2) that the problem of the stability of the distributions $\theta(x)$ with $\delta\eta = 0$ and fixed value of A reduces to the analysis of the eigenvalues of the self-adjoint problem

$$\hat{H}_\theta\delta\theta_n = \lambda_n\delta\theta_n \quad (4.5)$$

and (4.4) with normalized eigenfunctions $\delta\theta_n$. Following Ref. 71, to find the spectrum λ_n we differentiate Eq. (3.3) with respect to x:

$$\hat{H}_\theta\frac{d\theta}{dx} = 0. \quad (4.6)$$

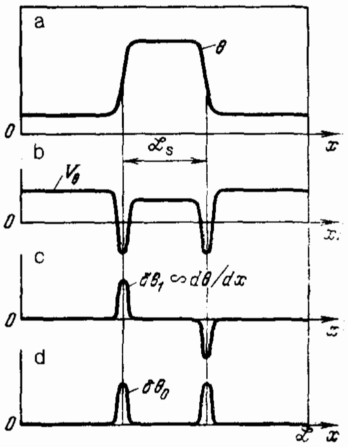

FIG. 13. Illustrating the analysis of the stability of a wide striation (a): form of the "potential" V_θ (b) and corresponding functions of the first "excited" $\delta\theta_1$ (c) and "ground" $\delta\theta_0$ (d) state.

One can see from (4.6) and (4.5) that $\delta\theta_m \propto d\theta/dx$ is the m-th eigenfunction of the operator $\hat{H}_\theta$, corresponding to $\lambda_m = 0$, where m is the number of nodes of the function $d\theta/dx$ in the interval $(0, \mathscr{L})$. For the cyclic boundary conditions (4.4) under study, the function $\theta(x)$ has at least one extremum. As a result $\delta\theta \propto d\theta/dx$ has at least one zero, i.e., according to the oscillation theorem[172,173] it is not the ground-state function of the problem (4.5), (4.4), so that $\gamma = \lambda_0 < 0$.

In small, two-parameter systems distributions $\theta(x)$ with several extrema (Sec. 3.1) are also unstable. To prove this, we expand the function $\delta\theta(x)$ in (4.2)–(4.4) in terms of the eigenfunctions $\delta\theta_n$ of the problem (4.5), (4.4) and substitute them into (4.2) and (4.3). Substituting (4.3) into (4.2) with $\delta A = 0$, after the corresponding transformations, we obtain

$$\left[1 + \sum_{n=0}^{\infty} a_n \mu_0 (\lambda_n - \gamma)^{-1} (\mu_0 - \alpha^{-1}\gamma)^{-1}\right] \prod_{n=0}^{\infty} (\lambda_n - \gamma) = 0, \tag{4.7}$$

where the coefficients are

$$a_n = -\langle q'_\eta \delta\theta_n \rangle \langle Q'_\theta \delta\theta_n \rangle \mu_0^{-1} \geqslant 0 \tag{4.8}$$

in accordance with the conditions (2.7), (2.9) and (2.10). Using the properties of translational symmetry of the solutions $\theta(x)$, the problem (4.2)–(4.4) can be reduced to the analysis of the stability of the distributions $\theta(x)$, symmetric relative to the center of the striation—the point $x = \mathscr{L}/2$ in Fig. 13a. With respect to this point $q'_\eta\{\theta(x)\}$ and $Q'_\theta\{\theta(x)\}$ are even functions; the functions $\delta\theta_n(x)$ are even functions of x for even n (Fig. 13d) and odd functions for odd n (Fig. 13c).[172,173] For this reason, for odd n the coefficients are $a_n = 0$, and according to (4.7) $\gamma = \lambda_n$. From here it follows that if the distribution $\theta(x)$ has more than one extremum, i.e., $n > 1$, then it is unstable,[17)] since for it as least $\gamma = \lambda_1 < 0$.

The distribution $\theta(x)$ in the form of one striation (Figs. 9g and 13), owing to the damping effect of the uniformly varying inhibitor concentration, turns out to be stable in a wide range of values of A. This distribution has one extremum, so that the function $d\theta/dx$ (Fig. 13c) has one node, and according to (4.5) and (4.6) it corresponds to the function $\delta\theta_1$ with $\lambda_1 = 0$. It follows from here that $\gamma = \lambda_n > 0$. for all $n > 1$. We have for even n that only $\lambda_0 < 0$, while all others satisfy $\lambda_n > 0$. According to (4.7), an isolated striation (Fig. 13) is unstable if the function

$$D(\gamma \equiv i\omega) = 1 + \sum_{n=0}^{\infty} \frac{a_n \left[(\lambda_n - \alpha^{-1}\mu_0^{-1}\omega^2) + i\omega(1 + \alpha^{-1}\mu_0^{-1}\lambda_n)\right]}{(\lambda_n^2 + \omega^2)(1 + \alpha^{-2}\mu_0^{-2}\omega^2)} \tag{4.9}$$

has at least one zero in the upper half-plane of the complex frequency ω. In (4.9) n are even numbers. The number of such zeros (N), based on the principle of the argument of Ref. 126, is

$$N = P + (2\pi)^{-1} \Delta \arg D(\omega), \tag{4.10}$$

where P is the number of poles of the function $D(\omega)$ in the upper half-plane of ω. According to (2.7) $\mu_0 > 0$; hence, since for a striation only $\lambda_0 < 0$, it follows from (4.9) that $P = 1$. It is obvious from (4.9) that $\mathrm{Re}D(\omega)$ is an even function of ω, while $\mathrm{Im}D(\omega)$ is an odd function, and $D(\pm\infty) = 1$ (Fig. 14). Thus when $D(0) > 0$ (curve 1 in Fig. 14a), $N = 1$ and therefore the striation is unstable.

When $D(0) < 0$ the quantity $\Delta \arg D(\omega)$ depends on the behavior of the function

$$K(\omega) = \sum_{n=0}^{\infty} a_n (\alpha\mu_0 + \lambda_n)(\lambda_n^2 + \omega^2)^{-1} \propto \omega^{-1} \,\mathrm{Im}\, D(\omega). \tag{4.11}$$

In KN- and KИ-systems, owing to the fact that $\alpha > 1$ (Sec. 2.2), the condition

$$\alpha\mu_0 + \lambda_0 > 0, \tag{4.12}$$

under with $K(\omega)$ is known to be greater than 0 for all values of ω, since all $a_n \geqslant 0$ (4.8), usually holds. In this case $\Delta \arg D(\omega) = -2\pi$ (curve 2 in Fig. 14a), i.e., $N = 0$, and therefore when $D(0) < 0$ the striation is stable. Thus when the degree of nonequilibrium of the system changes at the point $A = A_b$, where $D(0)$ changes sign, the function $D(\omega)$ (4.9) in the upper half-plane acquires a zero with $\mathrm{Im}\gamma = 0$. It follows from here that when the condition (4.12) holds the instability of the striation is of an aperiodic character, while the limit of its stability is determined by the equation

$$D(0) = 1 + \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n} = 0. \tag{4.13}$$

We shall show that the condition (4.13) determines the point $A = A_b$, where $d\eta/dA = \infty$. For this we expand the function $\delta\theta$ in (4.2) and (4.3) in a series in $\delta\theta_n$. Let us premultiply (4.2) on the left by $\delta\theta_n$ and average over the volume of the system. As a result, taking (4.3) into account and setting $\gamma = i\omega$, after appropriate transformations we obtain

$$\frac{d\eta}{dA} = (D(\omega))^{-1} \left[\sum_{n=0}^{\infty} \langle Q'_\theta \delta\theta_n \rangle \langle q'_A \delta\theta_n \rangle (\lambda_n - i\omega)^{-1} - \langle Q'_A \rangle\right] \times (\mu_0 - i\omega\alpha^{-1})^{-1}. \tag{4.14}$$

Comparison of (4.14) and (4.13) shows that in K-systems a striation becomes unstable at points where $d\eta/dA = \infty$ at the frequency $\omega = 0$.

The critical width of a striation at the point where the striation becomes unstable can be determined from the condition (4.13). For this we note that the function $\delta\theta_1 \propto d\theta/dx$ is localized in the regions of the walls of a wide striation (Fig. 13c) and decays exponentially away from it. The latter behavior follows from the fact that the extremal points θ_{max} and θ_{min} of the distribution $\theta(x)$ in the form of a wide striation are exponentially close to the saddle points $\theta = \theta_{s3}$ and

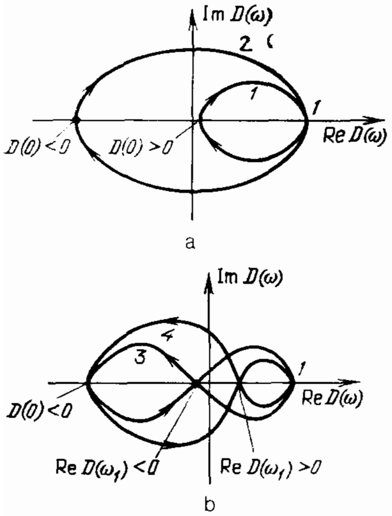

FIG. 14. Qualitative behavior of the complex function $D(\omega)$ over a circuit in the upper half-plane ω for $K(\omega)>0$ and $K(0)<0$ (b).

θ_{s1} of Eq. (3.3). As these points are approached[8]

$$\theta(x)-\theta_{s1,3}\propto e^{-\tilde{x}/l_{1,3}},\quad l_{1,3}=l\,(V_\theta(\theta_{s1,3}))^{-1/2}\approx l, \tag{4.15}$$

where $\tilde{x}$ is the distance from the wall of the striation. The same results can also be obtained from a quantum-mechanical analogy, based on which the form to $\delta\theta_0(x)$ can also be constructed and λ_0 can be determined. Indeed, it follows from the construction (Sec. 3.1) of a wide striation (Fig. 9) that for $\theta=\theta_{\min}=\theta_{s1}$ and $\theta=\theta_{\max}=\theta_{s3}$ the derivative satisfies $q'_\theta>0$. The derivative satisfies $q'_\theta<0$ only for values of θ close to the points of intersection of the branch II of the LC curve (Sec. 2.2) and the straight line $\eta=\eta_s$ (Figs. 9a and b). From here it follows that in the "Hamiltonian" $\hat{H}_\theta$ the "potential" $V_\theta=q'_\theta$ in (4.2) for a wide striation has the form of two narrow (of size $\sim l$) potential wells, localized in the walls of the striation (Fig. 13b) and separated by potential barriers of height $V_\theta=q'_\theta(\theta_{s1,3})\sim 1$. Thus the value $\lambda_1=0$ lies deep in these wells. Therefore the function $\delta\theta_1$ must be an asymmetric combination of the functions $\delta\theta_0^{(0)}$ of the ground state of each of the isolated wells of the potential V_θ (Fig. 13b). Outside these potential wells the localized functions $\delta\theta_0^{(0)}$ decay exponentially, and according to the same law[172] as in (4.15). The ground state function $\delta\theta_0$ (Fig. 13d) of the potential V_θ (Fig. 13b) is a symmeric combination[172] of the functions $\delta\theta_0^{(0)}$ of the ground state of each of the isolated wells. Owing to the fact that the overlapping of the functions $\delta\theta_0^{(0)}$ is exponentially small the function $\delta\theta_0$ corresponds[172] to the eigenvalue $\lambda_0\sim-\exp(-\mathscr{L}_s/l)$, where $\mathscr{L}_s$ is the size of the hot striation (Fig. 13a). Since $|\lambda_0|\ll 1$, the sums in (4.7) and (4.13) can be restricted to the first term only and for $\alpha>1$ and $\gamma\to 0$ we find that

$$\gamma=\lambda_0+a_0,\quad D(0)=1+\frac{a_0}{\lambda_0}=0. \tag{4.16}$$

The expressions (4.16) graphically illustrate the fact, presented above, that the condition (4.13) corresponds to the point where $\gamma=0$. We recall that the critical fluctuation $\delta\theta_0$ with $\delta\eta=0$ has the increment $-\lambda_0$. Its growth is damped by a corresponding change in the inhibitor density, which is described by the coefficient $a_0>0$ in the expression (4.16) for γ. Taking into account the conditions (2.9) and (2.10) as well as the localized character of the normalized function $\delta\theta_0$ (Fig. 13d), from (4.8) we find that

$$a_0\approx\frac{l}{\mathscr{L}}\,|\langle q'_\eta\rangle_{sh}\,\langle Q'_\theta\rangle_{sh}|\,\mu_0^{-1}\sim\frac{l}{\mathscr{L}}, \tag{4.17}$$

where the symbol $\langle\ldots\rangle_{sh}$ denotes averaging of the function over the region of the striation wall, more precisely, the region of localization of the function $\delta\theta_0$ (Fig. 13d). Substituting into (4.16) the estimates for λ_0 and a_0, from the condition $\gamma=0$ we find that the critical size of the striation approximately equals

$$\mathscr{L}_s(A_b)=\mathscr{L}_b\sim l\ln\frac{\mathscr{L}}{l}. \tag{4.18}$$

We note that (4.18) was obtained from the condition for the threshold of stability of the striation (4.13), which determines the point $A=A_b$, where $d\eta/dA=\infty$. As A decreases the width of the striation decreases (Sec. 3.1), so that at the point $A=A_b$ the striation, having the width $\mathscr{L}_s=\mathscr{L}_b$ (4.18), vanishes abruptly. The evolution of a striation with $\mathscr{L}\ll L$ was analyzed in greater detail in Ref. 28, while radially symmetric states were analyzed in Ref. 31.

In studying the stability of a striation in the two- and three-dimensional cases the fluctuations $\delta\theta\propto\delta\theta(x)\exp(i\mathbf{k}_\perp\mathbf{r}_\perp-\gamma t)$ with $k_\perp\neq 0$ ($k_\perp^2=(2\pi l\cdot n_1/\mathscr{L}_y)^2+(2\pi l\cdot n_2/\mathscr{L}_z)^2\,n_{1,2}=1,\ldots$) must be taken into account. For such fluctuations which are nonuniform along the striation walls, in (4.7) all coefficients are $a_n=0$, while λ_n change to $\lambda_n+k_\perp^2$. Therefore it follows from (4.7) that

$$\gamma_{\min}=\lambda_0+\left(\frac{2\pi l}{\overline{\mathscr{L}}}\right)^2,\quad \overline{\mathscr{L}}=\max\{\mathscr{L}_y,\mathscr{L}_z\}. \tag{4.19}$$

Substituting $\lambda_0\sim-\exp(-\mathscr{L}_s/l)$ into (4.19) we obtain the critical width of a striation for which it stratifies in the plane of its walls: $\mathscr{L}_s\sim l\cdot\ln(\overline{\mathscr{L}}/l)^2$.

4.2. Stability of AS in one-dimensional systems.[25-28] Let us linearize Eqs. (2.2) and (2.3) near solutions in the form of a one-dimensional AS with respect to fluctuations of the form

$$\delta\theta(r,t)=\delta\theta(x)\exp(i\mathbf{k}_\perp\mathbf{r}_\perp-\gamma t),\ \delta\eta(r,t)=\delta\eta(x)\exp(i\mathbf{k}_\perp\mathbf{r}_\perp-\gamma t). \tag{4.20}$$

As a result, for the one-dimensional case ($k_\perp=0$) we arrive at the system of equations

$$(\hat{H}_\theta-\gamma)\,\delta\theta=-q'_\eta\delta\eta,\quad \hat{H}_\theta=-\frac{d^2}{dx^2}+V_\theta,\quad V_\theta=q'_\theta(\theta(x),\eta(x),A), \tag{4.21}$$

$$(\hat{H}_\eta-\alpha^{-1}\gamma)\,\delta\eta=-Q'_\theta\delta\theta,\quad \hat{H}_\eta=-\varepsilon^{-2}\frac{d^2}{dx^2}+V_\eta,\quad V_\eta=Q'_\eta(\theta(x),\eta(x),A), \tag{4.22}$$

in which length and time are measured in units of l and τ_θ, respectively.

Differentiating Eqs. (3.7) and (3.8) with respect to x it is easy to verify that $\delta\theta\propto d\theta/dx$ and $\delta\eta\propto d\eta/dx$ are the eigenfunctions of the problem (4.21), (4.22) with cyclic boundary conditions, corresponding to the eigenvalue $\gamma=0$. This result is a consequence of the translational symmetry of the problem. It follows uniquely from it for systems with $\eta(x) = \text{const}$ that any solution $\theta(x)$ for which the function $d\theta/dx$ has more than one node is unstable (Sec. 4.1).

For the complicated problem under study [the fourth-order system of equations (4.21) and (4.22)], unlike the problem (4.5) and (4.4), there is no oscillation theorem. States not only in the form of a wide AS, for which $d\theta/dx$ has five nodes (Fig. 10f), but also more complicated states are stable.[27,166]

It is convenient to analyze the stability and evolution of AS as the degree of nonequilibrium of the system changes (Sec. 4.3) with the help of the bifurcation characteristic of the system—the A-dependence (Fig. 15) of the value $\eta = \eta_s$ in the wall of the AS (Fig. 10f). In K-systems the threshold of stability of AS on this characteristic corresponds to the point $A = A_b$, where $d\eta_s/dA = \infty$ (Fig. 15). This assertion is a generalization of the result presented in Sec. 4.1 for systems with $\mathscr{L} \ll L$, for which $\eta(x) = \eta_s$. It can be established from the following simple considerations.

Small changes $d\theta$ and $d\eta$ in the starting distributions $\theta(x)$ and $\eta(x)$ caused by a small change in the parameter A are determined, according to (3.7) and (3.8), from the equations

$$\hat{H}_\theta\, d\theta = -q'_\eta\, d\eta - q'_A\, dA, \quad \hat{H}_\eta\, d\eta = -Q'_\theta\, d\theta - Q'_A\, dA. \tag{4.23}$$

The existence of a point where $d\eta_s/dA = \infty$ means that at this point with $dA = 0$ the increments $d\eta(x)$ and $d\theta(x)$ do not vanish and do not correspond to the functions $\delta\eta \propto d\eta/dx$ and $\delta\theta \propto d\theta/dx$, describing a small shift of the AS. It is obvious from (4.23) with $dA = 0$ that such increments

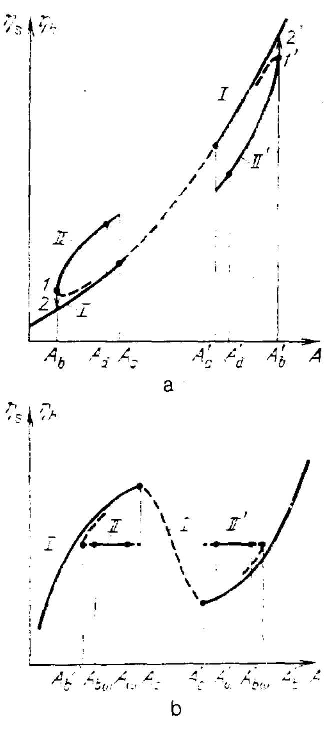

FIG. 15. Form of the bifurcation characteristics for a radiation-heated or electric-field-heated EHP (Sec. 1.1)[26,88] (a), for a degenerate EHP heated in the process of Auger recombination (Sec. 1.3)[32] (b). The curves I, II, and II' correspond, respectively, to the uniform state of an EHP and hot and cold AS; the broken sections of the curves correspond to unstable states.

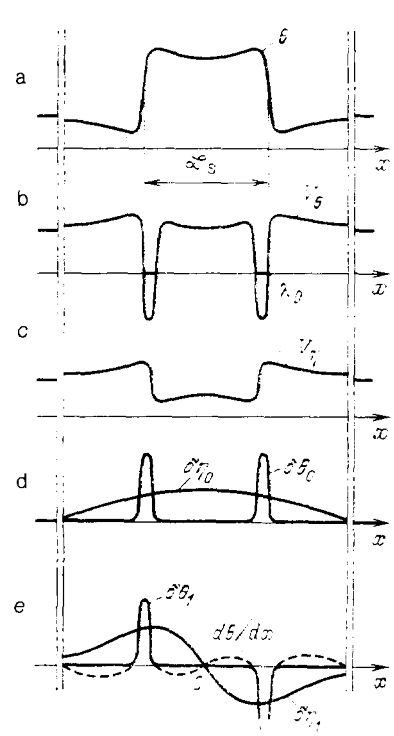

FIG. 16. Illustrating the study of the stability of a static AS. (a) activator distribution in a wide AS. b,c) form of the "potentials" V_θ (b) and V_η (c) for fluctuations. d,e) form of the critical (d) and "shear" (e) fluctuations of the activator $\delta\theta_{0,1}$ and the corresponding disturbances $\delta\eta_{0,1}$ damping them.

$d\eta(x)$ and $d\theta(x)$ correspond to a nontrivial solution of the system of equations (4.21) and (4.22) with $\gamma = 0$. From here it follows that the threshold of stability of AS in K-systems ($\gamma = 0$) is correlated with the point $A = A_b$ (Fig. 15), where the derivatives of the quantities characteristic for AS (η_s, $\mathscr{L}_s$, etc.) with respect to A become infinite.[18] It does not follow uniquely from this assertion that other values $A \neq A_b$ for which Reγ changes sign cannot, in principle, exist. For example, AS of complicated form (Fig. 7b and c) can become unstable without reaching the point $A = A_b$ (see the last paragraph of Sec. 4.3).

The stability of wide AS for $A > A_b$ is associated with the fact that the "dangerous" fluctuations of the activator density are damped by a corresponding nonuniform change in the inhibitor density (Fig. 16). To verify this and to evaluate the critical size of an AS at the point $A = A_b$, we shall study the eigenfunctions and eigenvalues of the problems

$$\hat{H}_\theta \delta\theta_n = \lambda_n \delta\theta_n, \quad \hat{H}_\eta \delta\eta_k = \mu_k \delta\eta_k, \tag{4.24}$$

for which the functions $\delta\theta_n$ and $\delta\eta_k$ are normalized and satisfy cyclic bounary conditions. One can see from (4.21) and (4.22) that the eigenfunctions $\delta\theta_n$ and eigenvalues λ_n describe fluctuations of the activator density with $\delta\eta = 0$, while $\delta\eta_k$ and μ_k describe fluctuations of the inhibitor density with $\delta\theta = 0$.

According to (4.22) and (2.7) in the "Hamiltonian" $\hat{H}_\eta$ the "potential" satisfies $V_\eta = Q'_n > 0$. It follows from here[172] that all $\mu_k > 0$; in addition, according to the oscillation theorem[172,173] the larger the index k the higher is the value of μ_k. This reflects the fact that because $Q'_n > 0$ (Sec. 2.2) all fluctuations of the inhibitor density with $\delta\theta = 0$ are damped.

The "potential" $V_\theta = q'_\theta$ in the "Hamiltonian" $\hat{H}_\theta$ (4.21), as in the case of a small system (Fig. 13b), consists of two narrow (of the order of l in size) potential wells (Fig. 16), localized in the walls of the AS, i.e., located at a distance $\mathscr{L}_s$ from one another. Indeed, in KN- and $K\textit{И}$-systems (Sec. 2.2) the derivative $q'_\theta < 0$ for the branch II of the LC curve and $q'_\theta > 0$ for the branches I and III of the LC curve (Fig. 10a). For this reason, it follows from the procedure for constructing an AS (Sec. 3.2) that $V_\theta = q'_\theta > 0$ in regions where $\theta(x)$ is a smooth function (outside the walls of the AS in Fig. 10) and $V_\theta < 0$ only in regions where $\theta(x)$ changes sharply, i.e., at the walls of the AS. Outside the walls of the AS $V_\theta = q'_\theta \sim 1$, so that the ground state functions $\delta\theta_0^{(0)}$ of each solitary potential well (Fig. 13b) are strongly localized.

To find the eigenvalue $\lambda_0^{(0)}$ corresponding to these functions we differentiate Eq. (3.7) with respect to x, multiply it on the left by $\delta\theta_0^{(0)}$, and average over x. Next, we multiply the first of the equations (4.24) for $n = 0$ on the left by $d\theta/dx$ and average it over x. Subtracting the equations so obtained from one another and taking into account the hermitian character of the operator $\hat{H}_\theta$ we find that

$$\lambda_0^{(0)} = -\left\langle \frac{d\theta}{dx} q'_\eta \left(\frac{d\eta}{d\theta}\right) \delta\theta_0^{(0)} \right\rangle \left\langle \delta\theta_0^{(0)} \frac{d\theta}{dx} \right\rangle^{-1} \sim -\frac{\varepsilon \mathscr{L}_s}{L}. \tag{4.25}$$

In determining $\lambda_0^{(0)}$ in (4.25) we employed the fact that the eigenfunction $\delta\theta_0^{(0)}$ is localized in the region of the well of size $\sim 1(l)$, where θ changes by an amount ~ 1, while η changes by an amount $\sim \varepsilon \mathscr{L}_s/L$. Since the depth and width of the well ~ 1, $\lambda_1^{(0)} - \lambda_0^{(0)} \sim 1$,[172] i.e., the value $\lambda_1^{(0)} \sim 1$ corresponds to the first "excited" state in the well.

The "potential" V_θ consists of two identical wells, separated by a hump of magnitude ~ 1 (Fig. 16b), so that owing to the exponentially weak overlapping of the ground state of each of the "isolated" wells the level $\lambda_0^{(0)}$ will split into two exponentially close levels[172]: $\lambda_1 \sim -\varepsilon \mathscr{L}_s/L$ and

$$\lambda_0 \sim -\varepsilon \frac{\mathscr{L}_s}{L} - e^{-\mathscr{L}_s/l}. \tag{4.26}$$

The eigenfunctions λ_0 and λ_1 of the operator $\hat{H}_\eta$ correspond to the functions $\delta\theta_0$ and $d\theta_1$, which are the binding and antibinding combinations of the ground-state functions $\delta\theta_0^{(0)}$ of each of the "isolated" wells (Figs. 16d and e).[172] Since $\lambda_1^{(0)} \sim 1$ it can be concluded that the spectrum λ_n contains only two negative values. The corresponding functions $\delta\theta_0$ and $\delta\theta_1$ are the only growing (with increment $-\lambda_0$ and $-\lambda_1$) fluctuations of the activator density with $\delta\eta = 0$. The damping action of the inhibitor on the growth of these functions is described by the term $-q'_\eta \delta\eta$ in (4.21). Taking into account the fact that $\delta\theta_0(x)$ is an even function of x while $\delta\theta_1(x)$ is an odd function of x with respect to the center of the AS (Fig. 16) it is easy to verify that, as in small systems (Sec. 4.1), these functions grow independently.

The function $\delta\theta_1$ to order $\varepsilon \ll 1$ is close to the true fluctuation of the activator density $\delta\theta \propto d\theta/dx$ (Fig. 16e), which describes a small translational shift of the AS. Therefore the eigenvalue of the problem (4.21) and (4.22) corresponding to the function $\delta\theta \approx \delta\theta_1$ is $\gamma_1 = 0$. Thus among the fluctuations of the activator density $\delta\theta$ in the problem (4.21) and (4.22) the fluctuation $\delta\theta \approx \delta\theta_0$ is the only "dangerous" one (Fig. 16d). Its growth can be suppressed by a corresponding change in the inhibitor density which, according to (4.22), equals

$$\delta\eta = -\langle \Gamma Q'_\theta \delta\theta_0 \rangle = -\sum_{k=0}^{\infty} \delta\eta_k \langle \delta\eta_k^* Q'_\theta \delta\theta_0 \rangle (\mu_k - \alpha^{-1}\gamma)^{-1}, \tag{4.27}$$

where $\Gamma(x, x', \gamma)$ is the Green's function of the homogeneous problem (4.22); the symbol $\langle ... \rangle$ denotes averaging over the volume of the system, while k are even numbers, since for odd k the integrals in (4.27) vanish because of the different symmetry of the functions $\delta\eta_k$ and $\delta\theta_0$.

We substitute (4.27) into (4.21) with the function $\delta\theta = \delta\theta_0$, premultiply the equation obtained on the left by $\delta\theta_0$, and average it over the volume of the system. As a result we obtain an equation for the critical value of γ:

$$\Phi(\gamma) = \lambda_0 - \gamma + \sum_{k=0}^{\infty} a_k \mu_k (\mu_k - \alpha^{-1}\gamma)^{-1} = 0, \tag{4.28}$$

where all the coefficients are

$$a_k = -\langle \delta\eta_k^* Q'_\theta \delta\theta_0 \rangle \langle \delta\eta_k q'_\eta \delta\theta_0 \rangle \mu_k^{-1} \geqslant 0$$

in accordance with the conditions (2.7), (2.9), and (2.10). It follows from (4.28) that the AS is stable, when the functions $\Phi(\gamma = -i\omega)$ has no zeros in the upper half-plane of the complex frequency ω. Analysis of this function (see Sec. 6.2) shows that in K systems in which the condition (4.12) holds the quantity γ near the threshold of stability of the AS equals

$$\gamma = \lambda_0 + \sum_{k=0}^{\infty} a_k, \tag{4.30}$$

where k are even numbers. As in (4.16), the second term in (4.30) describes the damping effect of the inhibitor on the growth of the critical fluctuation $\delta\theta \approx \delta\theta_0$. The quantity μ_k increases with the index k, and the number of nodes of the function $\delta\eta_k$ increases. The sum (4.30) can therefore be truncated at the first term. Setting in (4.30) $\gamma = 0$ and taking into account (4.26), we obtain the following estimate for the critical width of the AS (at the point $A = A_b$):

$$\mathscr{L}_s(A_b) = \mathscr{L}_b \sim l \ln \frac{L}{l}. \tag{4.31}$$

In deriving (4.31) the fact that $a_0 \sim l/L$ was taken into account. This estimate follows from the formula (4.29), if the conditions of normalization of the functions $\delta\theta_0$ and $\delta\eta_0$ and the fact that $\delta\eta_0$ is localized in walls of the AS of size $\sim l$ and the smoothly varying function $\delta\eta_0$ is localized in regions of size $\sim L$ are taken into account (Fig. 16d).

We note that (4.31) is identical to (4.18), if $\mathscr{L}$ in the latter equation is replaced by L—the characteristic size of the region of localization of the AS (Fig. 10f).

4.3. Evolution of autosolitons.[25,26,29] For $A < A_c$ a hot AS (Fig. 5a), for which $\eta = \eta_s$ at the wall of the AS (Sec. 3.2) differs strongly from $\eta = \eta_h$ (Fig. 15), can be excited in a stable system. As A (the level of excitation of the system) is lowered the AS becomes narrower (Sec. 3.2) and vanishes abruptly at the point $A = A_b$, where its size is $\mathscr{L}_s = \mathscr{L}_b$ (4.31) (the jump $1 \to 2$ in Fig. 15a). The value $A = A_b$ corresponds to the point where $d\eta_s/dA = \infty$ (Sec. 4.2). Since according to (4.31) $\mathscr{L}_b \gg l$ for $L \gg l$, in virtually the entire region of existence of the AS its basic parameters and the dependence $\eta_s(A)$, i.e., the bifurcation characteristic of the system, are determined from the simple equations (3.5) and (3.20). In addition, to order $\varepsilon \ll 1$, $A_b = A_s$. (The quantity $A = A_s$ is determined from Eqs. (3.5) and (2.5), in which $\eta_h = \eta_s$, while $\theta_h = \theta_{s1}$ (Sec. 3.1)).

As A is increased the AS becomes larger, and for $A > A_c$ striations arise in the system as a result of instability of the uniform state outside the region of localization of the AS. A different scenario of restructuring of the AS is, however, also possible. For $A = A_d < A_c$ the width of the AS can reach a critical size, at which "local breakdown," leading to division of the AS (Sec. 3.5), will occur at the center of the AS. As a result of successive division of the AS formed the entire system will be filled with striations (Fig. 12).[91]

When $A > A'_c$ it is possible to excite in a hot stable system a cold AS (Fig. 5c), which as A is increased narrows and vanishes abruptly at the point $A = A_b$, where $d\eta_s/dA = \infty$ (the jump $1' \to 2'$ in Fig. 15). The formula (4.31) holds for the critical size of the AS at $A = A'_b$. Thus the main parameters of the cold AS, like that of the hot AS, are determined in virtually the entire region of its existence, to order $\varepsilon \ll 1$, from the simple equations (3.5) and of the type (3.20) (see Sec. 3.2); in addition, $A'_b = A'_s$. [The quantity $A = A'_s$ is determined from Eqs. (3.5) and (2.5) with $\eta_h = \eta_s$ and $\theta_h = \theta_{s3}$ (Sec. 3.1)]. As A is reduced the width of the cold AS increases and at the point $A = A'_c$ or $A'_d > A'_c$ (Fig. 15a) the entire system is filled with striations as a result of fluctuational or dynamic restructuring (Sec. 3.5).

An AS can also form in a system spontaneously. In uniform systems with $A = A_c$ striations form in a jump-like fashion as a result of instability of the uniform state (Sec. 2.2). As A is lowered the number of striations decreases spontaneously in a jump-like fashion as a result of instability (of the "transfer" type),[19),25,26] and as $A \to A_b$ an AS forms in it.[25,26,29] In a real system an AS can form spontaneously in a jump-like fashion at a small nonuniformity[29] for A close to A_c (or A'_c).

The foregoing picture of the evolution of an AS describes the results of numerical studies of models of composite superconductors[22,167,169] and electron-hole plasma (Fig. 15).[91,100,171]

We call attention to the characteristics of the restructuring of the form of the complex AS as A changes. In a complex AS (Sec. 3.2) "local breakdown" can occur only in the widest striation (the region of high or low activator density). This can bring about not the filling of the entire sample with striations (Sec. 3.5), but rather the appearance of a more complicated AS, containing a larger number of striations. In a complex AS an instability of the "transfer" type, arising in the vicinity of the two closest striations, can lead to the formation of an AS containing a smaller number of striations. Such an instability appears before the points A_b or A'_b, which determine the limits of stability of a simple AS, are reached (Fig. 15a).

4.4. Autosolitons in two- and three-dimensional systems.[27–29] A hot, wide, one-dimensional AS (Fig. 16a) in two- and three-dimensional systems is stable, but in a smaller range of values of A than in one-dimensional systems. At the limits of this range $[A_{b1}, A_{c1}]$ a one-dimensional AS becomes unstable with respect to fluctuations of the form (4.20) with $k_\perp \neq 0$, nonuniform in the region of the walls of the AS, i.e., leading in a two-dimensional system to fluting of the surface of its walls, and in a three-dimensional system to the appearance of a cellular structure on them or to division of the AS into smaller regions. To prove this we take into account in (4.20) $k_\perp \neq 0$ and after appropriate transformations we arrive at Eqs. (4.28) and (4.29), in which λ_0 must be replaced by $\lambda_0 + k_\perp^2$ and μ_k by $\mu_k + \varepsilon^{-2}k_\perp^2$. As a result, instead of (4.30) we obtain

$$\gamma = \lambda_0 + k_\perp^2 + \sum_{k=0}^{\infty} a_k \mu_k (\mu_k + \varepsilon^{-2}k_\perp^2)^{-1}, \tag{4.32}$$

where a_k is given by (4.29). Retaining in the sum (4.32) only the first term and taking into account the estimate for $a_0 \sim \varepsilon$ and the localized character of the functions $\delta\theta_0$ (Fig. 16d) we find that the walls of the AS stratify relative to critical fluctuations $\delta\theta \approx \delta\theta_0(x)\exp(i\mathbf{k}_\perp \mathbf{r}_\perp)$ with $k_\perp \sim (l/L)^{1/4}(lL)^{-1/2}$, while the width of the AS at the points $A = A_{b1}$ and $A = A_{c1}$ equals, correspondingly, to

$$\mathcal{L}_{b1} \equiv \mathcal{L}_s(A_{b1}) \sim l \ln\left(\frac{L}{l}\right)^{3/2}, \quad \mathcal{L}_{c1} \equiv \mathcal{L}_s(A_{c1}) \sim (lL)^{1/2}. \tag{4.33}$$

A more rigorous result, which takes into account the entire sum in (4.32), can be obtained when $Q'_\eta \equiv B = \text{const}$. For such systems the Green's function in (4.27) equals

$$\begin{aligned} \Gamma(x, x', \gamma) &= \frac{\varepsilon}{2w} \exp[-\varepsilon w(x - x')] \quad \text{for} \quad x \geqslant x', \\ &= \frac{\varepsilon}{2w} \exp[\varepsilon w(x - x')] \quad \text{for} \quad x \leqslant x', \end{aligned} \tag{4.34}$$

where $w = (B + \varepsilon^{-2}k_\perp^2 - \gamma\alpha^{-1})^{1/2}$. Using (4.34), we obtain an expression for γ, analysis of which shows that $k_\perp \sim (l/L)^{1/6}(lL)^{-1/2}$, and

$$\mathcal{L}_{b1} \sim l \ln\left(\frac{L}{l}\right)^{4/3}, \quad \mathcal{L}_{c1} \sim (lL)^{1/2}\left(\frac{l}{L}\right)^{-1/6}. \tag{4.35}$$

Comparing (4.33) and (4.35) with (4.31) shows that $\mathcal{L}_b < \mathcal{L}_{b1}$, i.e., as A is lowered the autosoliton stratifies at the point $A_{b1} > A_b$ (Fig. 15). This result can be obtained in a more rigorous fashion by analyzing (4.32) and (4.30).

On the other hand, since $\mathcal{L}_{c1} \ll L$, as A is increased the widening AS can stratify along the walls at $A = A_{c1}$, without reaching the point $A = A_d$ or $A = A_c$ (Sec. 4.3).

Analogous conclusions can also be drawn regarding the stratification of a stable one-dimensional cold AS (Figs. 5c and d) as the degree of nonequilibrium of the hot system is varied. Numerical calculations also show that wide, one-dimensional AS in two-dimensional systems are stable.[118]

A radially symmetric AS (Sec. 3.4) in the form of a cluster (Figs. 11a, b, and d) or a hollow sphere (Figs. 11c and e) can be excited in three-dimensional systems, while an AS in the form of a spot or ring can be excited in two-dimensional systems. The critical fluctuations $\delta\theta$ are localized in a surface layer (of thickness $\sim l$) of the cluster or spot with radius $\rho = \rho_0 \gg l$. The growth increment of such a radially symmetric fluctuation $\delta\theta_0$ with $\delta\eta = 0$ equals approximately[35] $\lambda_0 \sim -\varepsilon\rho_0/L-(l/\rho_0)^2$. Growth of $\delta\theta_0$ is damped by a corresponding change in the inhibitor density right up to the point $A = A_b$, where $d\eta_s/dA = \infty$. At $A = A_b$ the cluster vanishes abruptly (Fig. 15). As A is increased the radius of the cluster (spot) increases, but it can become unstable without reaching the points $A = A_d$ or $A = A_c$ (Fig. 15) with respect to radially unsymmetric fluctuations. An AS in the form of a hollow sphere or ring with an inner radius $\rho_{01} > L$ is stable for values of A for which its thickness $\mathcal{L}_s = \rho_{02} - \rho_{01}$ falls in the range whose limits $\mathcal{L}_{b1}$ and $\mathcal{L}_{c1}$ are determined by (4.33) or (4.35). Outside this range the AS becomes unstable with respect to radially unsymmetric fluctuations.[29]

It follows from here that the picture of the evolution of radially symmetric AS can be very complicated. As A is increased an expanding AS in the form of a cluster (Fig. 11a) can transform as a result of "local breakdown" (Sec. 3.5) into a hollow sphere (Fig. 11c), which can then fragment, as a result of the growth of radially unsymmetric fluctuations, into smaller parts. As a result many AS or one AS, but of a very complicated form, can arise in the system.

5. Static spike autosolitons (KΛ-, KV-systems).

In Secs. 3 and 4 we studied AS in KN- and $KИ$-systems, whose amplitude, according to (3.5) ($\theta_{max} - \theta_{min} = \theta_{s3} - \theta_{s1}$), is determined by the existence of the branch III of the single-valued dependence $\theta(\eta)$ on the N- or $И$-shaped LC curve (Figs. 9a and b). The values $\theta > \theta_0'$, corresponding to this branch, can be regarded, in a certain sense, as the "hot stable phase" of a system with "temperature" $\theta \sim \theta_{s3}$ corresponding to $\eta = \eta_s$ (Fig. 10).

A different situation is realized in $K\Lambda$- and KV-systems (Sec. 2.2), whose LC curve is Λ- or V-shaped (Figs. 17a and b). In such systems for $l \ll L$ spike AS of two types can exist: narrow (Sec. 5.2)[25,28] and wide (Sec. 5.3).[89,174] The width of the spike ($\mathscr{L}_s$) of a narrow AS, irrespective of the smallness of $\varepsilon = l/L \ll 1$, is of the order of $\sim l$, while the width of the spike of the wide AS is of the order of $\sim L$.

5.1. Structures in small systems.[25,28,157] The distribution $\theta(x)$ and the value $\eta(x) = \eta_{sh} = \text{const}$ in systems of size $\mathscr{L} \ll L$ are determined by Eqs. (3.3) and (3.4). It follows (Sec. 3.1) from the LC curve in Λ- and V-systems (Figs. 17a and b) that the potential U_θ in (3.3) has two extrema, one of which $\theta = \theta_1$ corresponds to a maximum of U_θ (Fig. 17c). The highest trajectory of the "particle" in such a potential U_θ corresponds to the separatrix of Eq. (3.3), terminating at the point $\theta = \theta_1$. It describes the only solution $\theta(x)$, in the form of a narrow striation of size $\sim l$ (Fig. 17d), with only one maximum. It was shown in Sec. 4.1 that with cyclic boundary conditions, among all possible distributions $\theta(x)$ (Sec. 3.1) only a solution in the form of one striation (Fig. 17d) is stable right up to point $A = A_b$ on the bifurcation characteristic, where $d\eta_{sh}/dA = \infty$ (Fig. 18).

To illustrate our results we shall study the distribution $\theta(x)$ and the form of the bifurcation characteristic in models that can be solved analytically. For the Brusselator model (1.12) Eqs. (3.3) and (3.4) have the form

$$l^2 \frac{d^2\theta}{dx^2} + B + \theta^2\eta - (1 + A)\,\theta = 0, \quad \eta_{sh} = A\,\langle\theta\rangle\,(\langle\theta^2\rangle)^{-1}. \tag{5.1}$$

It follows from them[25,175] that for $\mathscr{L} \gg l$ (more precisely, for $\eta_{sh} \ll (1 + A)^2/4B$)

$$\eta_{sh} = A\ (1 + A)^2 \left[1 \mp \{1 - 24l\,[\mathscr{L} A^2\,(1 + A)^{1/2}]^{-1}\}^{1/2}\right] (2\theta_h)^{-1}, \tag{5.2}$$

$$\theta(x) = \theta_1 + (\theta_{max} - \theta_1)\,\mathrm{ch}^{-2}\frac{\xi x}{l}, \tag{5.3}$$

where

$$\theta_1 = \theta_h\,(1 + A)^{-1}, \quad \theta_{max} - \theta_1 = \frac{3\,(1 + A)}{2\eta_{sh}}, \quad \xi = (1 + A)^{1/2}/2, \quad \theta_h = B. \tag{5.4}$$

For the model (1.14) with $\mathscr{L} \gg l$ the inhibitor density[176]

$$\eta_{sh} = \frac{A^2\mathscr{L}}{6Cl}\left[1 \pm \left(1 - \frac{A_b^2}{A^2}\right)^{1/2}\right], \quad A_b = \left(\frac{24l}{\mathscr{L}}\right)^{1/2} CB, \tag{5.5}$$

while the activator density $\theta(x)$ is determined by (5.13),[114] in which

$$\theta_1 = \eta_{sh}\,(2A)^{-1} - 2A^{-1}\xi^2\eta_{sh}, \quad \theta_{max} - \theta_1 = 6\eta_{sh}\xi^2 A^{-1}, \quad \xi^2 = (4\eta_{sh})^{-1}\,(\eta_{sh}^2 - 4AB\eta_{sh})^{1/2}. \tag{5.6}$$

The expressions (5.2) and (5.5) determine the bifurcation characteristics of the models studied (Fig. 18). The values of the inhibitor density for the uniform state in the model (1.12) $\eta_h = A/B$, while in (1.14) $\eta_h = C[(A/C) + B]^2$ and, as follows from (5.2) and (5.5), they differ significantly from the values of η_{sh} for striations (Fig. 18). The upper sign in (5.2) and (5.5) determines the value of η_{sh} in a stable striation of high amplitude, for which the expressions (5.4) and (5.6) are valid. It follows from (5.2) and (5.5) that the limiting value of the parameter $A = A_b$, for which the striation vanishes abruptly (Fig. 18), $A_b \propto (l/\mathscr{L})^{1/2}$, i.e., it decreases as the size of the system $\mathscr{L}$ increases.

Substituting (5.2) and (5.3) into the expression for the "Hamiltonian" $\hat{H}_\theta$ in (4.2) we find that the "potential" appearing in it is given by

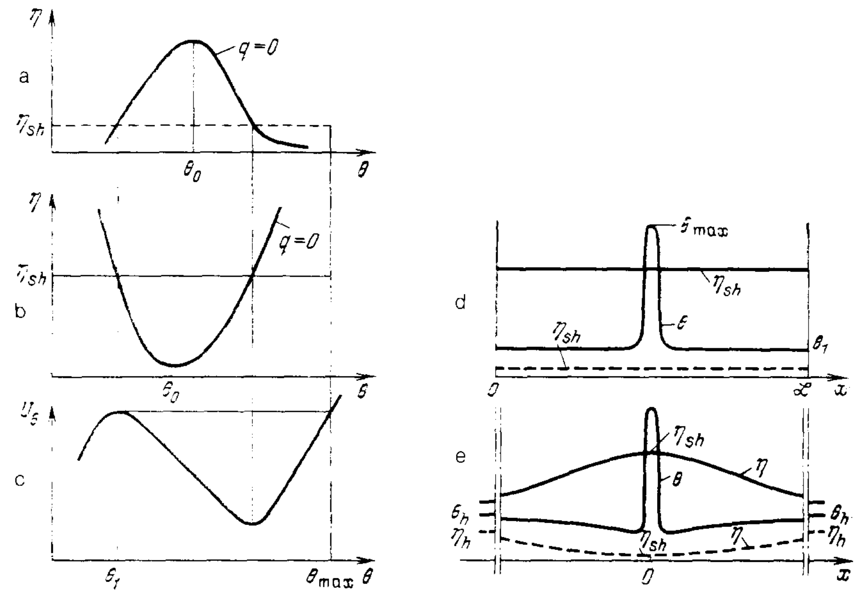

FIG. 17. Illustrating construction of structures in small Λ and V-systems. a,b) local-coupling curves for Λ-(a) and V-systems (b). c) form of the potential U_θ. d,e) form of a narrow spike striation (d) and a narrow spike AS (e). Broken curves—value of the inhibitor in Λ-systems.

$$V_\theta = q_\theta'(\theta(x),\ \eta_{sh}) = (1+A)\left[1 - 3\,\mathrm{ch}^{-2}\frac{(1+A)^{1/2}x}{2l}\right]. \tag{5.7}$$

In this potential [the problems (4.5) and (4.4)] with exponential accuracy $\lambda_1 = 0$, while λ_0 and $\delta\theta_0$ equal[172]

$$\lambda_0 = -5\xi^2 = -\frac{5}{4}(1+A),$$
$$\delta\theta_0 = \left[\frac{15}{32}(1+A)^{1/2}\frac{\mathscr{L}}{l}\right]^{1/2}\mathrm{ch}^{-3}\frac{(1+A)^{1/2}x}{2l}. \tag{5.8}$$

It is obvious from (5.8) that the function $\delta\theta_0$ is localized in the region of striation of size $\sim l$ and it corresponds to $\lambda_0 \sim -1$.

The conclusion that in the spectrum of λ_n of the problem (4.5) and (4.4) for a narrow striation $\lambda_1 = 0$ while $\lambda_0 \sim -1$ is of a general character. Indeed, the strict equality $\lambda_1 = 0$ follows from the translational symmetry of the problem for the striation (Sec. 4.1). The condition $\lambda_1 - \lambda_0 \sim 1$ follows from the fact that the size and width of the "potential" V_θ for the striation is ~ 1. It follows from (4.16) that the condition $a_0 \gtrsim 1$ is a necessary condition for the striation to be stable for $\lambda_1 \sim -1$. For a striation amplitude $\theta_{max} - \theta_1 \lesssim 1$, according to (4.17), we have the coefficient $a_0 \sim l/\mathscr{L} \ll 1$. For this reason the condition $a_0 \gtrsim 1$ cannot hold for striations with small amplitude (Fig. 18, broken curves). It follows from the procedure for constructing a narrow striation (Fig. 17) that the condition (3.4) can be written approximately in the form

$$\mathscr{L}_s\ |\ Q\ (\theta_{max},\ \eta_{sh})\ | = (\mathscr{L} - \mathscr{L}_s)\ |\ Q\ (\theta_1,\ \eta_{sh})\ |. \tag{5.9}$$

For narrow striations with large amplitude $|Q(\theta_1)| \gtrsim 1$ usually holds, i.e., the condition (5.9) with $\mathscr{L}_s \sim l$ is valid, when $|Q(\theta_{max})| \gtrsim \mathscr{L}/l \gg 1$. The latter condition can be satisfied only in systems for which $Q(\theta)$ is an increasing function of θ. In Λ-and V- systems for which this is not realized wide spike striations,[176] which according (5.9) should have a size $\mathscr{L}_s \sim \mathscr{L}/2$, can form.

5.2 Narrow spike autosolitons.[25,27,28] The qualitative form of a narrow spike AS in extended Λ- and V-systems can be established from simple considerations: as $\mathscr{L}$ increases the value $\eta = \eta_{sh} \neq \eta_h$ at the center of a striation (Fig. 17d) should, at the periphery of the striation, transform smoothly over a characteristic distance $\sim L$ to the value $\eta = \eta_h(\theta - \theta_h)$ for the uniform state (Fig. 17e).

To construct a narrow spike AS we shall employ the method described in Sec. 3.2. Unlike N- and И-systems the potential for sharp distributions U_θ in Λ- and V-systems (Fig. 17c) has only one maximum. This difference leads to the fact that only spike ASs can exist in Λ- and V-systems. The construction of narrow spike AS (Fig. 17e) is illustrated in Fig. 19. To construct the true potential U_η (Fig. 19c) in (3.8) the first of the conditions (3.10) is employed; it follows from this condition that

$$\langle Q \rangle_{sh} = -\varepsilon^{-1}\langle Q \rangle_{sm}, \tag{5.10}$$

where the symbols $\langle\ldots\rangle_{sh}$ and $\langle\ldots\rangle_{sm}$ denote averaging of the function $Q(\theta,\eta)$ over the region of sharp and smooth distributions, respectively. Usually $|\langle Q \rangle_{sm}| \gtrsim 1$, i.e, according to (5.10) $|\langle Q \rangle_{sh}| \gtrsim \varepsilon^{-1} \gg 1$. The derivative satisfies $dU_\eta / d_\eta = -Q$, so that the potential U_η in (3.8) near $\eta = \eta_{sh}$ branches sharply upwards away from the branch I of the potential U_η in (3.14) for smooth distributions (Fig. 19b), forming a steep wall, i.e., it acquires the form of a potential well (Fig. 19c). The distribution $\eta(x)$ (Fig. 19d) corresponds to the finite motion of a "particle" in such a well (trajectory 1 in Fig. 19c).

It follows from the procedure for constructing an AS (Sec. 3.2) that the distributions $\theta(x)$ and $\eta(x)$, taking into account the symmetry of the AS with respect to the point $x = 0$, can be written approximately in the form

$$\theta\ (x) = \theta_{sh}\ (x) - \theta_1 + \theta_I\ (x)\ (0 \leqslant x < \infty),$$
$$\eta\ (x) = \eta_I\ (x)\ (0 \leqslant x < \infty), \tag{5.11}$$

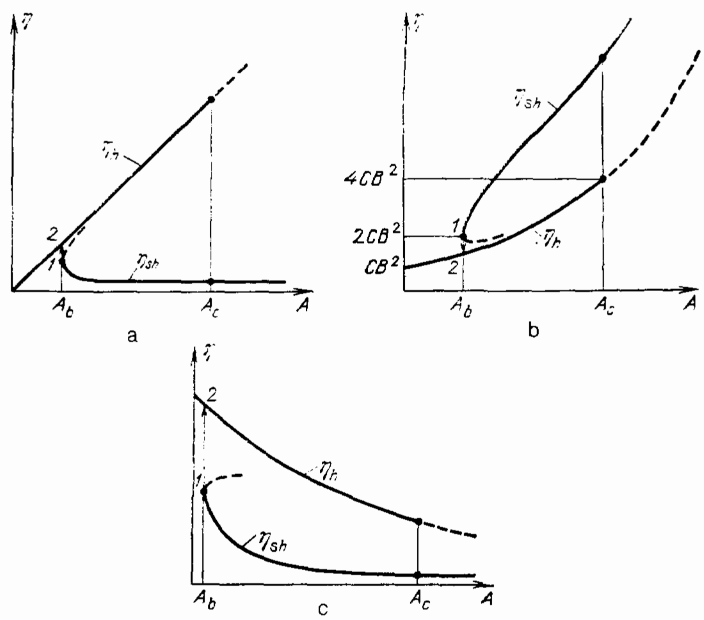

FIG. 18. Form of the bifurcation characteristics. a) For the Brusselator model (1.12)[25,175]; b) for the Gierer-Meinhardt model (1.14)[37,176]; c) for a mixture of neutral gases heated with electromagnetic radiation.[52] The broken sections of the curves correspond to unstable states.

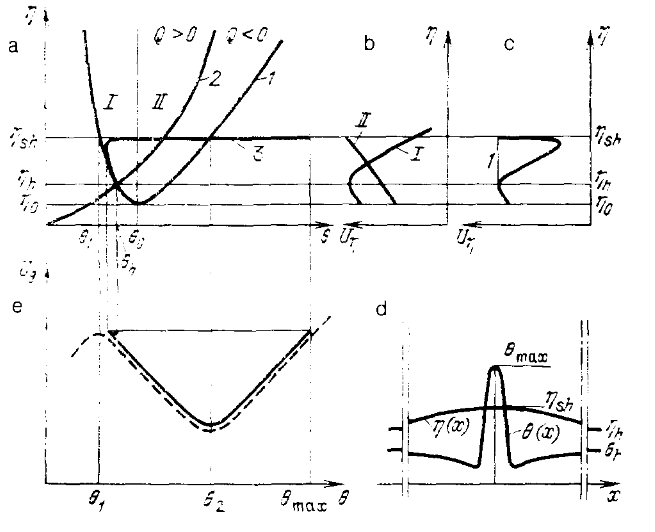

FIG. 19. Illustrating construction of a narrow spike AS. a) Form of local coupling (curve 1), ES curve (2), and the true dependence $\eta(\theta)$ (3). b) form of the branches I and II of the potential U_η for smooth distributions; c) form of the true potential U_η, in which the highest trajectory (1) corresponds to the distribution $\eta(x)$ in the autosoliton (d); e) form of the potential U_θ: the broken curve shows the potential in the approximation of sharp distributions, while the solid curve shows the true potential U_θ, the highest particle trajectory in which corresponds to the distribution $\theta(x)$ in d).

where $\theta_{sh}(x)$ is the sharp distribution, obtained in Sec. 5.1 and corresponding to the separatrix of Eq. (3.3), terminating at the saddle point $\theta=\theta_1$ (Fig. 17d); the functions $\eta_1(x)$ and $\theta_1(x)$ are smooth distributions of the inhibitor and activator densities, which are solutions of Eqs. (3.18) corresponding to the boundary conditions $\eta_1(\infty)=\eta_h$, $\eta_1(0)=\eta_{sh}$; η_{sh} can be evaluated from the equation[29,176]

$$\int_0^\infty Q(\theta_1(x),\ \eta_1(x),\ A)\,dx$$
$$= -\int_0^\infty \left(Q(\theta_{sh}(x),\ \eta_{sh},\ A) - Q(\theta_1,\ \eta_{sh},\ A)\right) dx, \quad (5.12)$$

which essentially follows from (3.10). The form of the narrow spike AS (Fig. 19), found in the general theory,[25,27] is confirmed by the results of analytical and numerical studies of different models.[16,114,120,175–177]

Just as for a striation (Sec. 5.1), for a narrow spike AS (Fig. 2a) the potential V_θ in the "Hamiltonian" $\hat{H}_\theta$ (4.21), determining the growth increment and the form of the critical fluctuations of the activator $\delta\theta_n$ for $\delta\eta=0$ (4.24), has the form of a narrow well (Fig. 20b). In the spectrum of such a well we have only λ_0 and $\lambda_1<0$, and in addition[28] $\lambda_1 \lesssim -\varepsilon$, while $\lambda_0 \sim -1$. The functions $\delta\theta_0$ and $\delta\theta_1$ corresponding to λ_0 and λ_1 are localized in the region of the AS spike (Figs. 20c and d). The function $\delta\theta_1$ is close to $\delta\theta \propto d\theta/dx$ (Fig. 20d), which describes a small shift of the AS, i.e., for it $\gamma_1=0$.

An autosoliton exists in the range from $A=A_b$ up to A_c (Sec. 4.2): for $A<A_b$ it vanishes abruptly (Fig. 18); as A is increased the amplitude of the AS increases, and for $A>A_c$ as a result of the instability of the uniform state (Sec. 2.2) outside the AS a more complicated autostructure appears in the system.[25,29]

The expressions (5.11) for some form of the functions $Q(\theta,\eta)$ and $q(\theta,\eta)$ can be rigorously substantiated[176] based on the theory of singular perturbations.[116] Unlike the procedure for constructing an AS in KN- and KИ-systems (Sec. 3.3), for narrow spike AS in KΛ- and KV-systems we shall seek the exterior $\tilde{X}_i$ and interior $\bar{X}_i$ solutions in the form of series in powers of ε with exponents that can take on negative, including fractional, values:

$$\tilde{X}_i(x,\ \varepsilon)=\sum_{k=0}^{\infty}\varepsilon^{\alpha_i+(k/m)}\tilde{X}_{i,\ \alpha_i+(k/m)}(x),$$
$$\bar{X}_i(\xi,\ \varepsilon)=\sum_{k=0}^{\infty}\varepsilon^{\beta_i+(k/m)}\bar{X}_{i,\ \beta_i+(k/m)}(\xi), \quad (5.13)$$

where $\xi=x/\varepsilon$, $\alpha_i=N_i/m$, $\beta_i=M_i/m$; N_i and M_i are any integers; and m is a positive integer. The coefficients α_i and β_i are determined by the boundary conditions and the specific form of the functions $Q(\theta,\eta)$ and $q(\theta,\eta)$.

Using this procedure for the model (1.14) with $C=B=1$ makes it possible to prove that the expressions (5.11) and (5.12) are valid to order $\varepsilon \ll 1$, and it can be shown that in leading order in ε[176]

$$\theta(x)=\varepsilon^{-1}\frac{A}{2}\,\mathrm{ch}^{-2}\frac{x}{2l}+2\left[1+\left(1-\frac{4A}{\eta(x)}\right)^{1/2}\right]^{-1}, \quad (5.14)$$

$$\eta(x)=\varepsilon^{-1}\frac{A^2}{3}e^{-x/L}+\eta_h,\quad \eta_h=(1+A)^2. \quad (5.15)$$

The next correction for η equals

$$\frac{A^2}{6}\left(2\frac{x}{l}+\mathrm{ch}^{-2}\frac{x}{2l}-4\ln\mathrm{ch}\frac{x}{2l}-4\ln 2\right) \quad (5.16)$$

and makes it possible for the condition $d\eta/dx=0$ to hold at the point $x=0$. The expressions (5.14)–(5.16) are valid for $A^2>\varepsilon$. The quantity $A=A_b$, for which the solution in the form of an AS vanishes (Fig. 18), can be determined from (5.15), by setting $\mathscr{L}=2L$ in it. This follows from the fact that the expressions (5.14) and (5.15) in the neighborhood of an AS spike (the point $x=0$, Fig. 17e) essentially transform into the expressions (5.5) and (5.6), if we set $\mathscr{L}=2L$ in them.

We note that in two-and three-dimensional systems one-dimensional narrow spike ASs are unstable[27] relative to fluctuations with $k_\perp \sim (lL)^{1/2}$ that are nonuniform along the plane of the AS. This follows from (4.32), if we set in it

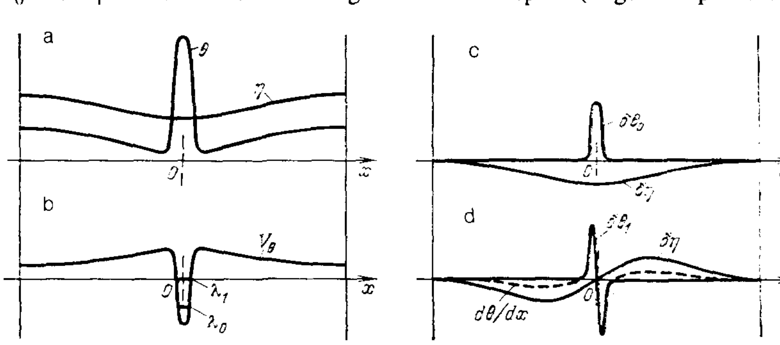

FIG. 20. Illustrating the analysis of the stability of a narrow spike AS (a): form of the "potential" V_θ (b) and corresponding eigenfunctions $\delta\theta_0$ (c) and $\delta\theta_1$ (d) and perturbations $\delta\eta$, damping the growth of the dangerous fluctuation $\delta\theta\approx\delta\theta_0$ and "shear" fluctuation $\delta\theta\propto d\theta/dx$.

$\lambda_0 \sim -1$. In such systems, radially symmetric spike AS with radius $\sim l$, whose θ and η distributions in the cross section are identical to those shown in Fig. 17e, are stable. These conclusions explain the results of numerical studies.[119]

5.3 Wide spike autosolitons.[89,174] In some KΛ- and KV-systems with $l \ll L$ wide spike AS, the width of whose peak is of the order of L, form. This, as already pointed out in Sec. 5.1 in the discussion of the condition (5.9), can be realized in systems in which the quantity $|Q|$ does not grow as θ increases; more precisely, the conditions (5.10) does not hold, even for large values of $\theta_{\max}$. To construct AS in such systems the concepts of sharp and smooth distributions (Secs. 3.2 and 5.2), which form the basis of the method described in Ref. 25, cannot be employed. The expansions (5.13) in the form of an exponential boundary layer also turn out to be unjustified.[176]

Wide spike AS are realized, for example, in a nondegenerate heated EHP, described by Eqs. (1.2)–(1.5).[89,174] It follows from Sec. 1.1 [Eq. (1.5)] that

$$Q = \frac{(n - G\tau_r)}{n_h} = \eta\theta^{-1-\alpha} - 1, \qquad (245c)$$

where $\theta = T/T_l$, $\eta = nD(T)/D^0 n_h = n\theta^{1+\alpha}/n_h$, $n_h = G\tau_r$, $D \propto T^{1+\alpha}$. In this case the condition (3.10) has the form

$$\int_{-\infty}^{\infty} (n - G\tau_r)\, n_h^{-1}\, dx = \int_{-\infty}^{\infty} (\eta\theta^{-1-\alpha} - 1)\, dx = 0, \qquad (5.17)$$

i.e., it describes the overall balance of particles in AS: carriers ejected owing to thermal diffusion (Sec. 1.1) from the AS spike, where the carrier temperature is high (Fig. 21a), accumulate at the periphery of the AS (Fig. 21b). Since at the center of a spike $Q \simeq -1$ (Fig. 21b), the condition (5.17) can hold only when $\mathscr{L}_s \sim L$. Precisely such a wide AS was discovered[89] in a numerical study of a heated nondegenerate EHP with T_l greater than the Debye temperature of the semiconductor (Fig. 21).

Wide spike AS have the amazing property that their amplitude, i.e., $\theta_{\max}$ at the center of the AS, can have a huge value for not very small values of $\varepsilon = l/L$. Thus numerical studies of AS in EHP have established[89] that for $\varepsilon = 1/10$ $\theta_{\max} \approx 100$ (Fig. 21), and $\theta_{\max} \simeq 10^3$ already for $\varepsilon = 1/15$.

5.4. Strongly nonequilibrium regions in weakly nonequilibrium systems.[50–52]

Studies of the models (1.12) and (1.14) show (Secs. 5.1 and 5.2) that the smaller the value of $\varepsilon = l/L$ the larger is $\theta_{\max}$ [see (5.14)] and the smaller is the minimum value of $A = A_b$ for which an AS exists (Fig. 18). This means that for $\varepsilon \ll 1$ AS in the form of strongly nonequilibrium regions can form in physical systems that depart slightly from thermodynamic equilibrium. We shall illustrate this effect for the example of a mixture of light and heavy gases, heated weakly by radiation absorbed by the light component of the gas (Sec. 1.2).[20]

Let the gas be confined in a small tube. The distribution of the temperature T in such a tube is described by Eq. (1.7) for the one-dimensional case, in which $W = n\sigma_{\rm ph}\Phi$, where n is the number density of light particles; $\sigma_{\rm ph}$ and Φ are the photon absorption cross section and the photon flux density. Diffusion of the light gas in the heavy gas with density $N \gg n$ is described by Eq. (1.2), in which $j_c = -T(\partial/\partial x)(nD(T)T^{-1})$ is the flux of particles of the light gas.[2,178] From an analysis of these equations, describing the distribution of n and T along the tube, analogous to that presented in Secs. 5.1 and 5.2 for models (1.12) and (1.14), it follows that the minimum radiation power for which an AS exists is $\Phi = \Phi_b \propto (l/L)^{1/2}$, where L equals $\mathscr{L}$—the size of the tube in the case of nonreacting gases ($G = R = 0$) or the diffusion length of light paticles. For $\Phi = \Phi_b$ uniform heating of the mixture $T = T_h$ can constitute a fraction of a percent of its equilibrium temperature, while the temperature of the gas and the density at the center of the AS (Fig. 3a) can be several orders of magnitude higher than the equilibrium value: $T_{\max} \approx T_h (L/l)^{1/2}$, $n_{\max} \approx (L/l) n_h$.

Regions of high temperature can form in semiconductor films, heated weakly by radiation absorbed by electrons and holes in thermal equilibrium with the lattice, whose density distribution ($n = p$) and temperature are described by Eqs. (1.2), (1.7), and (1.8). Regions of high temperature ("hot spots") have been studied in greatest detail experimentally (see, for example, Refs. 151 and 179) and theoretically[157] in semiconductor structures, since the appearance of such regions determines the quality and reliability of many modern electronics devices (there is a very extensive literature on this question; see the reviews of Refs. 180 and 181).

6. PULSATING AUTOSOLITONS (KΩ-SYSTEMS)

The conditions $\varepsilon \ll 1$ and $\alpha \ll 1$, defining KΩ-systems (Sec. 2.2), hold for many physical and chemical systems (Sec. 1). In most of them $l = (D_\theta\tau_\theta)^{1/2}$ and $L = (D_\eta\tau_\eta)^{1/2}$, where D_θ and D_η are the coefficients of diffusion. The smallness of $\varepsilon = (\alpha D_\theta/D_\eta)^{1/2} \ll 1$, is, as a rule, associated with the smallness of $\alpha = \tau_\theta/\tau_\eta \ll 1$. For this reason, depending on the ratio D_θ/D_η, different ratios of ε and α can be realized.

Since $\varepsilon \ll 1$ static AS in K-systems (Sec. 3.5) and in KΩ-systems have the same form. In KΩ-systems, however, static AS exist in a smaller range of values of A. At the boundaries of this region, because α is small the condition

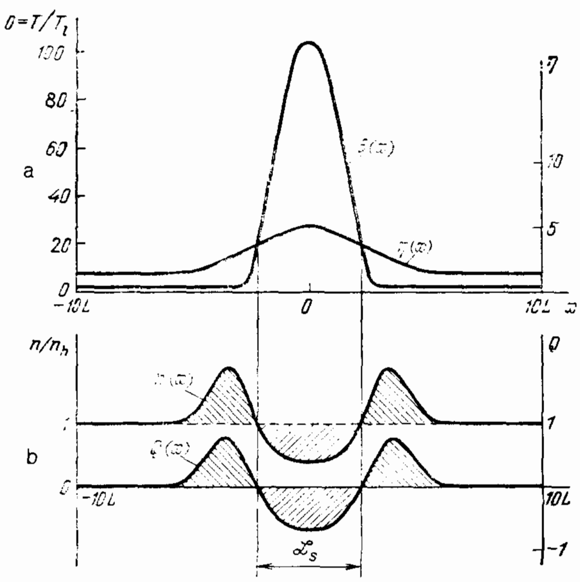

FIG. 21. Form of a wide spike AS in a heated degenerate EHP[89]: activator distribution—temperature $\theta = T/T_l$, inhibitor distribution $\eta = nD(T)/n_h D(T_l)$ (a), carrier density n, and Q (b).

$$\lambda_0 + \alpha\mu_0 < 0, \tag{6.1}$$

can hold[31]; this condition essentially determines the threshold for instability of a static AS relative to pulsations,[21] i.e., growth of fluctuations of the form (4.20) with Re $\gamma<0$ and Im $\gamma=\omega_c\neq 0$. This is linked with the fact that because of the slowness of the change in the inhibitor density ($\tau_\eta \gg \tau_\theta$) there is not enough time for the inhibitor to damp the critical fluctuations of the activator density, varying with frequency $\omega=\omega_c:\tau_\eta^{-1}<\omega_c<\tau_\theta^{-1}$. For this reason pulsating AS exist in KΩ-systems in addition to static AS (Figs. 5e–h).[31–34]

6.1 Pulsating structures in small systems.[31] The criterion for the appearance of pulsations of wide striations (Fig. 13a) for $\alpha \ll 1$ can be found by analyzing the zeros of the function $D(\gamma=i\omega)$ (4.9) in the upper half-plane of the complex frequency ω. In the region of stability of a striation with respect to fluctuations with Im $\gamma=0$ $D(0)<0$ (Sec. 4.1). Since for wide striations $\lambda_0<0$ and $|\lambda_0|\ll 1$ (Sec. 4.1), it follows from the expression (4.11) that when (6.1) holds $K(0)<0$. However, already for some $\omega=\omega_1\ll 1$ the quantity $K(\omega)$ changes sign. Analogously, at some $\omega=\omega_c$ Re $D(\omega_c)=0$, since Re $D(0)=D(0)<0$, and Re $D(\infty)=1$. Analysis of $D(\omega)$ (4.9) shows that for $K(0)<0$ the striation is stable [in (4.10) $N=P-1=0$], if Re $D(\omega_1)<0$ (the curve 3 in Fig. 14b), and unstable ($N=P+1=2$), if Re $D(\omega_1)>0$ (the curve 4 in Fig. 14b). For some $A=A_{b\omega}$ the frequency $\omega_1=\omega_c$, while $D(\omega_c)=0$, i.e., the real frequency ω_c is a zero of the function $D(\omega)$ at the threshold of stability of the striation. Thus according to (4.14) the threshold of stability of a striation relative to pulsations coincides with the point $A=A_{b\omega}$, where the "susceptibility" of the system is $d\eta/dA=\infty$ at the frequency $\omega=\omega_c$.

In order to be able to determine the value of ε_c and the critical width of a striation $\mathscr{L}_{b\omega}=\mathscr{L}_s(A_{b\omega})$ for $\alpha\ll 1$ from the condition $D(\omega_c)=0$, because $|\lambda_0|\ll 1$ the sum in (4.9) can be truncated at the first term. As a result we find that

$$\omega_c \approx (\alpha\mu_0)^{1/2}(a_1+\lambda_0)^{1/2}\tau_\theta^{-1}, \tag{6.2}$$

while the criterion for the appearance of pulsations of a striation reduces to (6.1). Using the fact that $\alpha_0\sim l/\mathscr{L}$ (4.17) and $\lambda_0\sim-\exp(-\mathscr{L}_s/l)$ (Sec. 4.1), it is easy to find from (6.1) and (6.2) that

$$\mathscr{L}_{b\omega}\sim l\ln(\mu_0\alpha)^{-1},\quad \omega_c\sim(\tau_\theta\tau_\eta)^{-1/2}\left(\frac{l}{\mathscr{L}}\right)^{1/2}. \tag{6.3}$$

One can see by comparing (4.18) with (6.3) that $\mathscr{L}_{b\omega}>\mathscr{L}_b$, i.e., $A_{b\omega}>A_b$, in systems with $\alpha\ll l/\mathscr{L}$. Thus as A is reduced the narrowing striation (Sec. 4.1) in systems with $\alpha\ll l/\mathscr{L}$ becomes unstable at the point $A=A_{b\omega}$ relative to pulsations, before reaching the point $A=A_b$ on the bifurcation characteristic, where $d\eta/dA=\infty$ (Fig. 15b).

The formula (6.2) and the condition (6.1) also determine the frequency and criterion for pulsations of radially symmetric structures in two- and three-dimensional systems.[31] In addition, it follows from the estimates that at the point at which pulsations arise the critical radius and frequency are given by

$$\rho_{b\omega}=\rho_0(A_{b\omega})\sim l(\alpha\mu_0)^{-1/2},\quad \omega_c\sim(\tau_\theta\tau_\eta)^{-1/2}\left(\frac{l\rho_0^{1+s}}{R^{2+s}}\right)^{1/2}, \tag{6.4}$$

where R is the radius of the system and $s=0$ (1) for the cylindrically (spherically) symmetric state.

For $A=A_{b\omega}$ the instability is associated with the growth of the fluctuations $\delta\eta=\delta\eta\cos(\omega_c t)$ and $\delta\theta\approx\delta\theta_0(\rho)\cos(\omega_c t)$. From the form of the functions $\delta\eta=$ const and $\delta\theta_0$ (Figs. 13d), corresponding to $\lambda_0<0$ (Sec. 4.1), we can conclude that the pulsations are periodic oscillations of the size and amplitude of the striation (spot or drop), accompanied by uniform oscillations of the inhibitor density.

These results have been confirmed by numerical studies of pulsating states in the model (1.13).[34] These studies also showed that in accordance with (6.3) and (6.4) the frequency of the pulsating structures is lower than the frequency of the critical fluctuation $\omega_0\sim(\tau_\theta\tau_\eta)^{-1/2}$ at the threshold of stability relative to uniform oscillations (Sec. 2.2).

We note that for stable states in Λ- and V-systems $\lambda_0\sim-1$ (Sec. 5.1), so that according to the condition (6.1) spike structures pulsating in amplitude can be observed in these systems even for $\alpha\sim 1$.[157,175]

6.2 Conditions for the appearance of and form of pulsating AS.[31] It was shown in Sec. 6.1 that the threshold for pulsations of a striation with $\mathscr{L}\ll L$ is determined by the point where the "susceptibility" of the system is $d\eta/dA=\infty$ at the frequency $\omega_c\neq 0$. The same criterion also determines the threshold of stability of AS. Indeed the change in the activator $d\theta$ and the inhibitor $d\eta$ density in a static AS, caused by a small change in the bifurcation parameter $dA(t)=dA\exp(-i\omega_c t)$, as follows from (2.2) and (2.3), is determined by the equations

$$\begin{aligned}(\hat H_\theta-i\omega_c)\,d\theta&=-q'_\eta\,d\eta-q'_A\,dA,\\ (\hat H_\eta-i\omega_c\alpha^{-1})\,d\eta&=-Q'_\theta\,d\theta-Q'_A\,dA,\end{aligned} \tag{6.5}$$

where, as in (4.21) and (4.22), the time is measured in units of τ_θ. The existence of points where $d\eta_s/dA=\infty$ for $\omega_c\neq 0$ means that at these points with $dA=0$ the increments are $d\eta(x)$ and $d\theta(x)\neq 0$. One can see from (6.5) with $dA=0$ that such increments $d\eta(x)$ and $d\theta(x)$ correspond to a nontrivial solution of the system of equations (4.21) and (4.22) with Re $\gamma=0$ and Im $\gamma=\omega_c$. Thus the threshold of stability of AS relative to pulsations is correlated with the existence of points where the derivative of the characteristic quantities of an AS (η_s, $\mathscr{L}_s$, etc.) with respect to A at the particular frequency $\omega=\omega_c$ becomes infinite.

The frequency of the pulsations and the critical size of an AS can be evaluated by analyzing the zero of the function $\Phi(\gamma=i\omega)$ (4.28) in the upper half-plane of the complex frequency ω. Since the function $\Phi(\omega)$ has no poles in this half-plane, the number of its zeros according to the principle of the argument in Ref. 126 equals $N=(2\pi)^{-1}\times\Delta\arg\Phi(\omega)$. The change in the argument of the function $\Phi(\omega)$ along a circuit in the upper half-plane can be established with the help of the properties of the function $\Phi(\omega)$. It is obvious from (4.28) that Re $\Phi(\omega)$ is an even while Im $\Phi(\omega)$ is an odd function of ω; for Re $\omega=\pm\infty$ and Im $\omega=0$ the values are Re $\Phi=\lambda_0<0$, Im $\Phi=-\omega=\mp\infty$, while for Im $\omega=\infty$ the values are Re Φ and Im $\Phi=\infty$. It follows from here that for $\Phi(0)<0$ $N=1$ (curve 1 in Fig. 22), and therefore the AS is unstable. For $\Phi(0)>0$ $\Delta\arg\Phi(\omega)$ depends on the sign of the function $K(\omega)=\omega^{-1}\mathrm{Im}\,\Phi(\omega)$. When $K(0)<0$ the quantity $K(\omega)$ is

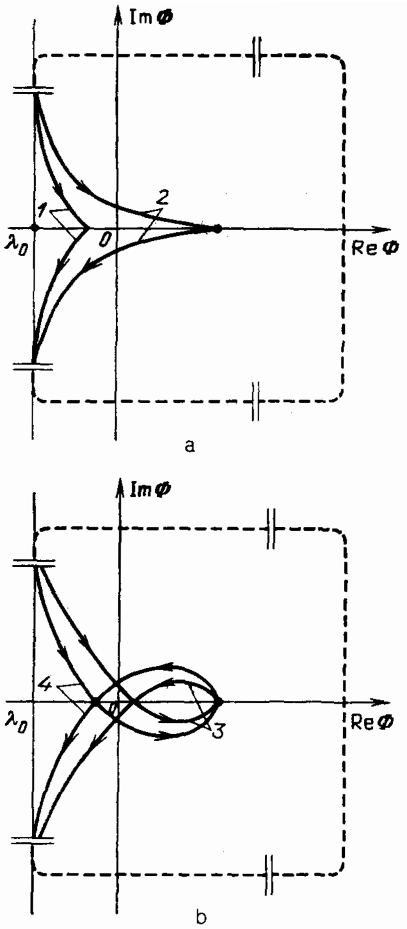

FIG. 22. Qualitative behavior of the complex function $\phi(\omega)$ over a circuit in the upper half-plane of ω for $K(0)<0$ (a) and $K(0)>0$ (b).

obviously less than zero for all real ω, and $N=0$ (the curve 2 in Fig. 22), i.e., the AS is stable.

The conditon $K(0)<0$ holds in K-systems because $\alpha>1$, so that in these systems an AS becomes unstable at the point where $\Phi(0)$ becomes negative. The sign of $\Phi(0)$ is correlated with the sign of γ in the expression (4.30), following from (4.28) with $\alpha>1$ and $|\gamma|\ll 1$.

In KΩ-systems a static AS is stable in a smaller range of values of A than in K-systems. It becomes unstable when $\Phi(0)>0$, i.e., in the region of stability of static AS in K-systems ($\alpha>1$), owing to the fact that $K(0)>0$ when $\alpha\ll 1$. In addition, $K(\omega)$ changes sign already for some real frequency $\omega=\omega_1\ll 1$. Analogously $\operatorname{Re}\Phi(\omega_c)=0$ for some real frequency $\omega=\omega_c$, since $\operatorname{Re}\Phi(0)=\Phi(0)>0$, while $\operatorname{Re}\Phi(\omega)=\lambda_0<0$. One can see from Fig. 22 that when $K(0)>0$ the AS is stable ($N=0$) if $\operatorname{Re}\Phi(\omega_1)>0$ (curve 3 in Fig. 22) and unstable ($N=2$) if $\operatorname{Re}\Phi(\omega_1)<0$ (curve 4 in Fig. 22).

Thus the limit of stability of the AS corresponds to the condition $\omega_1=\omega_c$, under which $\Phi(\omega_c)=0$. As in the analysis of (4.30), only the first term need be retained in the sum (4.28) and from the condition $\Phi(\omega_c)=0$ we find that the frequency of and the condition for pulsations of AS are determined by (6.2) and (6.1), respectively.

Substituting into (6.1) the estimate of λ_0 from (4.26) we obtain

$$\alpha\mu_0-\varepsilon\frac{\mathscr{L}_s}{L}-e^{-\mathscr{L}_s/l}<0. \tag{6.6}$$

This condition for $\alpha\ll\varepsilon^2\ll 1$ holds for any values of $\mathscr{L}_s$, i.e., static AS are not realized in such KΩ-systems (Fig. 8). According to (6.6) for $\varepsilon^2\ll\alpha\lesssim\varepsilon\ll 1$ a static AS in one-dimensional KΩ-systems is stable in the range of A where the width $\mathscr{L}_s$ of the AS falls within the limits

$$\mathscr{L}_{b\omega}(A_{b\omega})<\mathscr{L}_s<\mathscr{L}_\omega(A_\omega),\ \mathscr{L}_{b\omega}\sim l\ln(\alpha\mu_0)^{-1},$$
$$\mathscr{L}_\omega\sim L\mu_0(\alpha/\varepsilon). \tag{6.7}$$

Thus under conditions of both cooling and heating a static AS can become a pulsating AS when $A=A_{b\omega}>A_b$ and $A_\omega<A_c, A_d$ (Fig. 15b), i.e., before the point limiting the existence of a static AS in K-systems (Sec. 4.3) is reached. Since for AS $a_0\sim\varepsilon$ (Sec. 4.2) it follows from (6.2) that the frequencies of the pulsations of a one-dimensional AS with $A=A_{b\omega}$ and A_ω are equal in order of magnitude and are given by

$$\omega_c\sim\left(\frac{l}{L}\right)^{1/2}(\tau_\theta\tau_\eta)^{-1/2}. \tag{6.8}$$

It can be shown analogously that the radius of a stable radially symmetric AS in the form of a spot (cylinder) or a drop (sphere) ranges from $\rho_{b\omega}\sim l(\alpha\mu_0)^{-1/2}$ to $\rho_\omega\sim L\mu_0(\alpha/\varepsilon)$. The frequency of the pulsations of an AS with radius $\rho_0=\rho_\omega$ equals[29]

$$\omega_c\sim\alpha\left(\frac{\alpha}{\varepsilon}\right)^{(n-2)/2}\tau_\theta^{-1}, \tag{6.9}$$

where $n=1,2$ or 3 is the dimension of the AS.

It follows from (6.9) that when $\alpha<\varepsilon$ the frequencies of the pulsations at the threshold of stability of a static AS in the form of a wide layer (Fig. 5a), cylinder, or sphere with a large radius (Fig. 11a) decrease as the dimension of the AS increases. Numerical studies of the model (1.13) have shown that this result is also true for frequencies of steadily pulsating AS in the form of a layer, cylinder, or sphere.[34]

Somewhat different estimates for the quantites $\mathscr{L}_{b\omega}$, $\mathscr{L}_\omega$ and ω_c can be obtained for one-dimensional systems, in which the "potential" $V_\eta=Q'_\eta=B=\text{const}$, in the "Hamiltonian" $\hat{H}_\eta$, i.e., the corresponding $\delta\eta_0$ (4.24) is not localized. In this case the relation between $\delta\eta$ and $\delta\theta_0$ is determined by the formula (4.27), in which Γ is given by (4.34). We substitute $\delta\eta$ in (4.21) with $\delta\theta=\delta\theta_0$, multiply this expression on the left by $\delta\theta_0$, and average over the volume of thc system. Analysis of the equation so obtained from γ shows that[32]

$$\mathscr{L}_{b\omega}\sim l\ln(\varepsilon^2\alpha)^{-1/3},\ \mathscr{L}_\omega\sim L(\alpha/\varepsilon)^{1/3},$$
$$\omega_c\sim\varepsilon^{1/2}(\alpha/\varepsilon)^{-1/6}(\tau_\theta\tau_\eta)^{-1/2}. \tag{6.10}$$

Strictly speaking, conclusions about the frequency of pulsating AS in the case when they appear in a soft excitation regime can be drawn from the formulas (6.9) and (6.10). At the same time numerical studies have shown[33,34] that for $\alpha\ll 1$ hard excitation of pulsating AS can be realized. There then can exist an interval of A in which a static, pulsating, and traveling AS can be excited[34,100] (Sec. 7). To excite each AS it is necessary to choose a disturbance that would transfer the system into the region of attraction of a given type of AS.

The form of pulsating AS can be predicted from the fact that at the point at which pulsations appear the solution

$\theta(x)$ and $\eta(x)$ in the form of a static AS branches with the solution $\theta(x) \pm \delta\theta_0(x)\cos(\omega_c t)$ and $\eta(x) \pm \delta\eta_0(x)\cos(\omega_c t)$ in the form of a pulsating AS. The form of the functions $\delta\theta_0(x)$, $\delta\eta_0(x)$ in a wide AS is shown in Fig. 16d. It follows from the figure that a pulsating AS forms at $A = A_\omega$. Its size varies periodically in time, more precisely, in it the activator describes antiphase auto-oscillations of the walls of the AS, while the inhibitor varies periodically in the entire region of localization of the AS (Fig. 5e).

When И- or N-systems are cooled a wide AS narrows and becomes increasingly more like a spike AS (Fig. 5b) with an amplitude ~ 1. Since in such an AS the functions $\delta\theta_0(x)$ are localized at the center of the AS, a pulsating AS with an oscillating amplitude (Fig. 5f) can arise at the point $A = A_{b\omega}$ (Fig. 15b).

Such large-amplitude, pulsating, spike ASs can arise in Λ- and V-systems, and even for $\alpha \sim 1$. The latter result follows from the condition (6.1), if the fact that for spike AS $\lambda_0 \sim -1$ (Sec. 5.2) is taken into account in it.

From an analysis of the growth of fluctuations $\delta\theta$, close to the functions $\delta\theta_n$ of the problem (4.24), corresponding to $\lambda_n < 0$ with $n \neq 0$, it may be concluded that pulsating AS of a complicated form can exist.[31] Thus in the analysis of the stability of a wide AS it was shown (Sec. 4.2) that the spectrum of eigenvalues λ_n contains, aside from an eigenvalue $\lambda_0 < 0$, an eigenvalue $\lambda_1 < 0$ whose value is exponentially close to λ_0. Fluctuations close to $\delta\theta_0$ and $\delta\theta_1$ grow independently. It follows from the form of the fluctuation $\delta\theta \approx \delta\theta_1$ (Fig. 16e) that as a result of the growth of such fluctuations for A close to A_ω rocking (with walls oscillating in phase) or traveling AS can arise in KΩN- and KΩИ-systems with $\varepsilon^2 \ll \alpha \lesssim \varepsilon \ll 1$.[35] Numerical studies, in which rocking striations in a finite sample were observed, also indicate that rocking AS can arise.[100]

Study of the growth of the fluctuation $\delta\theta \approx \delta\theta_0(x) \exp(i\mathbf{k}_\perp \mathbf{r}_\perp - i\omega_c t)$ with $\lambda_0 + k_\perp^2 < 0$ leas to the conclusion that wide AS with wavy walls can be excited in two- and three-dimensional systems.[31]

Thus radially symmetric AS with periodically varying amplitude can be excited in two- and three-dimensional KΩΛ- and KΩV-systems, while in KΩN- and KΩИ-systems, in addition to such AS, AS excited in which primarily their radius oscillates, as well as can be, AS in the form of drops or spots, undergoing radially unsymmetric oscillations, and also pulsating autostructures of a complicated type.[31]

The qualitative results presented above are also valid for cold pulsating AS (Figs. 5, g and h), which can be excited in hot stable KΩN- and KΩИ-systems. The existence of pulsating, cold ASs was established in numerical studies of the model (1.13).[33,34]

7. TRAVELING AUTOSOLITONS (KΩ- AND Ω-SYSTEMS).

7.1 Form and velocity of traveling AS. To constuct the form of one-dimensional AS traveling with a constant velocity v we transform in Eqs. (2.2) and (2.3) to the self-similar variable $x \Rightarrow x - vt$:

$$\frac{d^2\theta}{dx^2} + v\frac{d\theta}{dx} + \frac{dU_\theta}{d\theta} = 0,$$
$$\varepsilon^{-2}\frac{d^2\eta}{dx^2} + \alpha^{-1}v\frac{d\eta}{dx} + \frac{dU_\eta}{d\eta} = 0. \tag{7.1}$$

Here and below the velocity is measured in units of l/τ_θ, length is measured in units of l, and time is measured in units of τ_θ; the potentials U_θ and U_η are defined in (3.7) and (3.8). The equations (7.1) with cyclic boundary conditions comprise an eigenvalue problem, whose spectrum determines the possible velocities, while the corresponding eigenfunctions $\theta(x)$ and $\eta(x)$ are the forms of different traveling AS.

Using $\theta(x)$ and $\eta(x)$ for static AS (Sec. 3) as the zeroth-order approximation, it is easy to verify that this problem does not admit solutions with $v \ll \alpha, \varepsilon^2$, i.e., traveling AS have only a finite velocity.[35] It follows from analysis of (7.1)[35] that in N- and И-systems the minimum velocity is $v_{\min} \sim \alpha^{1/2}$, and small AS ($\mathscr{L}_s \sim l \ln \alpha^{-1}$) have this velocity and only in systems with $\alpha < \varepsilon^4$.

The solutions of Eqs. (7.1) with $\alpha, \varepsilon \ll 1$ according to the general theory[8] are close to the solutions corresponding to combinations of smooth and sharp distributions (Sec. 3.2), which in this case satisfy the equations

$$\frac{d^2\theta}{dx^2} + v\frac{d\theta}{dx} + \frac{dU_\theta}{d\theta} = 0, \quad \eta = \text{const}, \tag{7.2}$$

$$\varepsilon^{-2}\frac{d^2\eta}{dx^2} + \alpha^{-1}v\frac{d\eta}{dx} + \frac{dU_\eta}{d\eta} = 0, \quad q(\theta, \eta, A) = 0. \tag{7.3}$$

The solution of the system (7.1) can be regarded as the trajectory of two "particles," moving with time x along the η and θ axes in the potentials U_η and U_θ, but unlike the case of static AS (Sec. 3.2), in the presence of friction forces that are proportional to the velocity v and have a constant sign.[27,28,35] The forms of the potentials U_θ and U_η in (7.2) and in (7.3) were studied in Sec. 3.2. As the "particles" move their energies decrease owing to the work of the friction forces, and the higher v the more does their energy decrease, i.e., the more does the form of the traveling AS (Fig. 23) differ from a statis AS (Fig. 10).

Thus the "particles" in the potentials U_θ and U_η move along downward sloping trajectories (Fig. 23). Premultiplying (7.2) by $d\theta/dx$ and (7.3) by $d\eta/dx$ and integrating over x, respectively, along the ith and jth elementary sections of $\theta(x)$ and $\eta(x)$, at whole limits $d\theta/dx$ or $d\eta/dx$, respectively, vanish (the extrema of $\theta(x)$ and $\eta(x)$ are located at different points x), we obtain

$$v = \Delta U_{\theta_i}\left[\int_i \left(\frac{d\theta}{dx}\right)^2 dx\right]^{-1} = \alpha\Delta U_{\eta_j}\left[\int_j \left(\frac{d\eta}{dx}\right)^2 dx\right]^{-1}, \tag{7.4}$$

where ΔU_{θ_i} and ΔU_{η_j} is the decrease in the potential energy of the particles owing to the "work of the friction forces" on the corresponding elementary sections (Fig. 23).

According to the general procedure of constructing AS (Sec. 3.2) in И- and N-systems (Fig. 23d) a sharp distribution $\theta(x)$, describing the walls of the AS (Fig. 3b), can transform into a smooth distribution, describing $\theta(x)$ and $\eta(x)$ outside the walls and between them, only near the peaks of the potential U_θ (Fig. 23a), i.e., the saddle points of Eq. (7.2). In the case of a static AS $\Delta U_\theta = 0$ and a half-

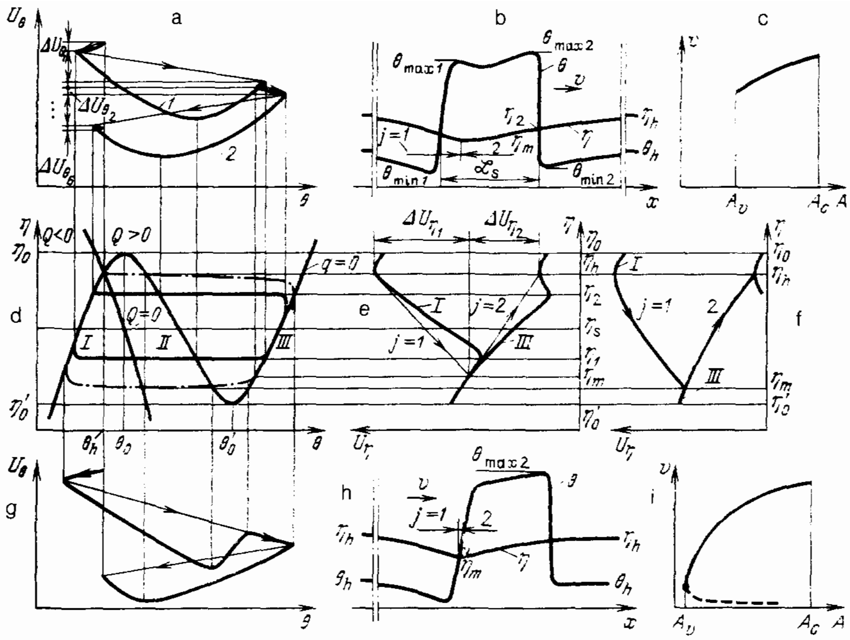

FIG. 23. Illustrating construction of the form of traveling AS: the distributions $\theta(x)$ and $\eta(x)$, which are shown in (b) for the case $\varepsilon^2 \ll \alpha < \varepsilon \ll 1$ and (h) for $\alpha \ll \varepsilon^2$ (see text for explanation).

oscillation of a sharp distribution, describing the wall of an AS, corresponds to the value $\eta = \eta_s$ (Sec. 3.2), which is determined from (3.5). Near $\eta = \eta_s$ the branches I and III of the potential U_η, the highest trajectory in which describes $\eta(x)$ in a static AS (Fig. 10), are joined. For a traveling AS $\Delta U_\theta > 0$, so that with the transfer from the branch I to the branch III of U_η (Fig. 23e, the trajectory of the particle with $j = 1$) the potential U_θ should have the form of the curve 1 in Fig. 23(a). Such a poetntial, as one can see from Fig. 9, is realized for $\eta_1 < \eta_s$ ($\eta_1 > \eta_s$ for И-systems). In other words, the joining of the branches I and III of the potential U_η in (7.3) occurs for $\eta_1 < \eta_s$ (Fig. 23e). One can see from Fig. 9 that the value $\eta_1 < \eta_s$ corresponds to $\theta_{\min 1}$ and $\theta_{\max 1}$ on the trailing edge (wall) of a traveling AS (Fig. 23b), lower than $\theta_{\min} = \theta_{s1}$ and $\theta_{\max} = \theta_{s3}$ for a static AS (Sec. 3.2).

The turning point of the trajectory of a particle in the potential U_η corresponds to the values $\eta = \eta_m$ (Fig. 23e). At this point a transition occurs from the trajectory $j = 1$ to the trajectory $j = 2$ in the potential U_η in (7.1), whose minimum lies at $\eta_2 < \eta_s$ (Fig. 23e). Indeed, the transition from the branch III to the branch I of the potential U_η in (7.3) can occur for smooth distributions when the potential U_θ has the form of the curve 2 in Fig. 9c, which is possible only when $\eta_2 > \eta_s$. It follows from here that $\theta_{\max 2}$ and $\theta_{\min 2}$ in the leading edge (wall) of the traveling AS (Fig. 23b) are larger than the corresponding values for a static AS. Successive joining of the segments of smooth and sharp distributions leads to a self-consistent adjustment of the potentials U_θ and U_η and the distributions $\theta(x)$ and $\eta(x)$ (Figs. 23a, b, e).[35,28]

It follows from the above-described procedure for constructing a traveling AS that v has an upper limit of ~ 1 (l/τ_θ). Indeed for $v \gg 1$ it is obvious from (7.2) that the work of positive friction forces is much larger than the kinetic energy of a "particle" moving in the potential U_θ, i.e., it will be stuck in its minimum (Fig. 23a), without reaching the second maximum, corresponding to the point of joining with the corresponding smooth distribution (Fig. 23). According to (7.4) the velocity $v \sim l/\tau_\theta$ is realized when η_1 and η_2 differ from η_s by an amount ~ 1. It follows from the construction of the potential U_η (Fig. 23e) that this is possible when the kinetic energy of a "particle," moving in the potential U_η is less than, or of the order of, the work of friction forces. The latter, according to (7.3) is realized when the diffusion length satisfies $L \lesssim v\tau_\eta \equiv \tilde{L}$—the drift length. Since $v \lesssim l/\tau_\theta$, it follows from the last equality that a traveling, wide AS can be excited in N- and И-systems only when $\alpha \lesssim \varepsilon \ll 1$ (Fig. 8).

As A is decreased a wide AS becomes narrower and transforms into a narrow AS. The smaller the size $\mathscr{L}_s$ of a traveling AS the smaller is the change in $\eta(x)$ between its walls, i.e., the smaller is the difference between η_1 and η_2. In its turn, $\eta_1 < \eta_s$, while $\eta_2 > \eta_s$ (Fig. 23), and their difference from η_s determines v. It follows from here that as a wide AS becomes narrower its velocity decreases and for some size $\mathscr{L}_s(A_v) = \mathscr{L}_v$ a solution in the form of an AS moving with velocity $v_{\min} \gg \alpha^{1/2}$ vanishes (Fig. 23c).

This assertion is valid for $\varepsilon^4 \ll \alpha \lesssim \varepsilon \ll 1$, since wide AS can transform on narrowing (as A is decreased) into slow AS, moving with $v \sim \alpha^{1/2}$ only in systems with $\alpha < \varepsilon^4$. It also follows from here that the smaller the ratio α/ε the smaller are the values of $v_{\min}$ and A_v (Fig. 23c). We note that the results presented here also hold for a periodic sequence of traveling striations (pulses).[35,28]

The form of traveling AS, shown in Fig. 23b, is realized in KΩN- (and KΩИ-)-systems for which α is not too small compared with $\varepsilon \ll 1$. When $\alpha \ll \varepsilon$ the distributions $\theta(x)$ and $\eta(x)$ at the top and between the walls of the AS become monotonic (Fig. 23h). Indeed for $\alpha \ll \varepsilon$, more precisely, $\tilde{L} \gg L$, the "kinetic energy of the particle," i.e., the term $\varepsilon^{-2} d^2\eta/dx^2$ in (7.1), can be neglected and it can be assumed that the change in the "potential energy" U_η is entirely expended on the work of "friction forces." This means that the trajectory of a particle corresponds to slipping along the branches of a multivalued potential U_η, i.e., for $\tilde{L} \gg L$ the trajectories $j = 1$ and 2 in Fig. 23e degenerate into $j = 1$ and 2 in Fig. 23f. In the process, the minimum in the distribution $\eta(x)$ $\eta = \eta_1 = \eta_m$ lies on the trailing edge of the AS (Fig. 23, h); on the leading edge $\eta = \eta_h$ (Fig. 23f), and $\theta_{\min} = \theta_h$

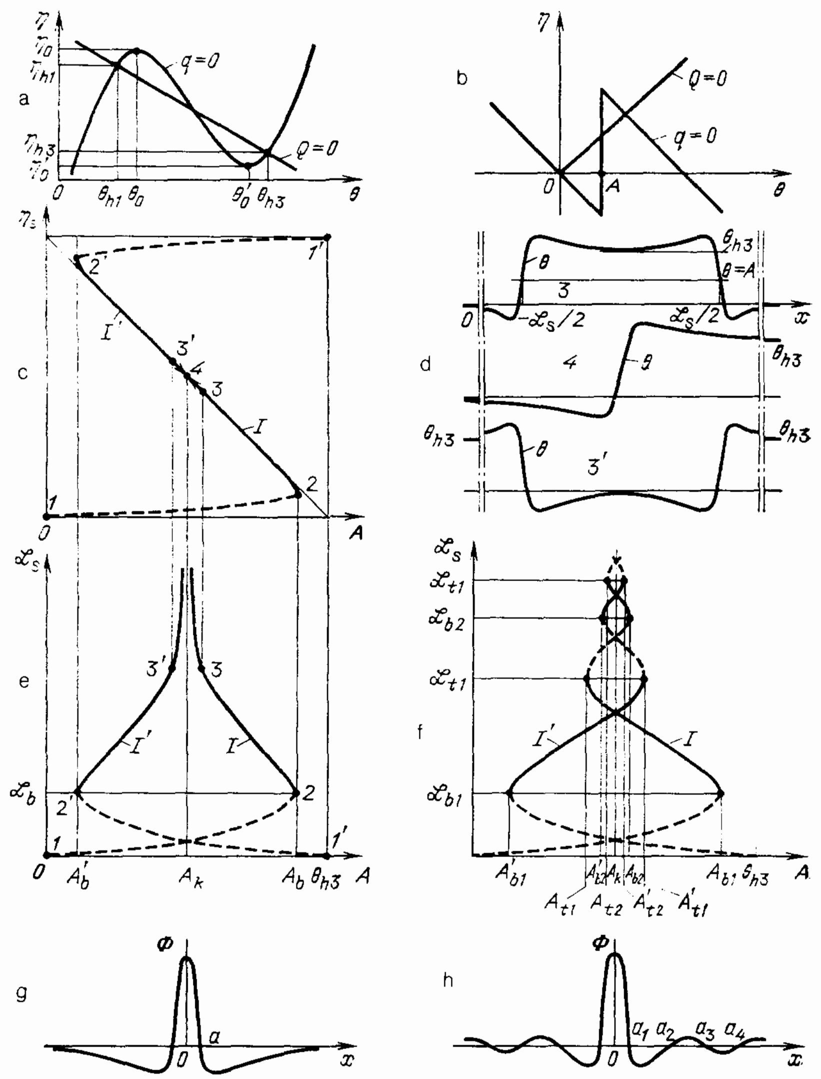

FIG. 24. Elucidation of the properties of AS in trigger systems. a,b) LC and ES curves. d) Form of wide AS (3 and 3′) and of a complicated domain wall (4). c,e,f) bifurcation characteristics. g,h) form of the coupling functions. The solid sections in c), e), and f) correspond to stable AS, while the broken sections correspond to unstable states. The numbers on the curves 3, 3′, and 4 in (d) correspond to points on the bifurcation characteristics in c) and e).

and $\theta_{\max_2}$ are the minimum and maximum roots of the equation $q(\theta,\eta_h,A) = 0$.

When $\alpha < \varepsilon^2$, for all possible velocities $\tilde{L} > L$, i.e., the distribution $\eta(x)$ is characterized not by the diffusion length L but rather by the drift length $\tilde{L}$. Thus for $\alpha < \varepsilon^2 (\tilde{L} \gg L)$ the condition for using the approximation of sharp and smooth distributions reduces to $\tilde{L} \gg l$, i.e., the small parameter of the problem is not ε, but rather $\alpha \ll 1$. In addition, there are no upper limits on ε.

It follows from here that in ΩN (ΩИ)-systems ($\varepsilon > 1$, $\alpha \ll 1$; Sec. 2.2), in which the static and pulsating AS are not realized (Fig. 8; see Sec. 3.6), an AS, whose form is identical to that shown in Fig. 23h and which travels with velocity $v \gtrsim \alpha^{1/2}$ can be excited. These results were essentially established from studies of models of the FHN type,[1,12,58–60] which are limiting cases of Ω-systems as $\varepsilon \to \infty$, i.e., they are described by Eqs. (2.2) and (2.3) with $L = 0$. Studies of these models have also revealed[1,59] that the dependence of v on the bifurcation parameter A has the form shown in Fig. 23i: wide stable AS correspond to high velocities, while narrow, unstable AS (broken section of the curve in Fig. 23i) correspond to low velocities. The minimum velocity is $v_{\min} \sim \alpha^{1/2}(l/\tau_\theta)$, while the maximum velocity is $v \sim l/\tau_\theta$ and is reached for A close to A_c (Fig. 23i).[59] For $A > A_c$ the uniform state of the system is unstable (Sec. 2.2), i.e., a traveling AS cannot be excited.

It follows from the construction of a traveling AS in Ω-systems (Fig. 23h), which was proven rigorously[59,60] with the help of the theory of singular perturbations,[116] as well as the physics of its formation (Sec. 1.4), that its leading edge represents a wave of transfer of activator from the state $\theta = \theta_h$ to $\theta = \theta_{\max 2}$ at $\eta \approx \eta_h = \text{const}$. For this reason, two AS traveling towards one another annihilate when they collide.[12,16]

As we have already pointed out, a traveling AS is realized in KΩN- and KΩИ-systems for $\alpha < \varepsilon \ll 1$ (Fig. 8).[35,28] The larger the ratio α/ε the larger are the values of $v_{\min}$ and A_v (Fig. 23c), i.e., the smaller is the range of $A (A_v < A < A_c)$ for which a traveling AS can be excited in the system. For A close to A_c or A_d (see Sec. 3.5) in KΩN (KΩИ-)-systems with $\varepsilon^2 \ll \alpha \lesssim \varepsilon \ll 1$, static and pulsating AS can be excited together with traveling AS (Sec. 3 and 6).[35] It follows from here that when two traveling AS collide in such systems a static or pulsating AS can form. This is attributable to the fact that in KΩ-systems, unlike Ω-systems, a "diffusion precursor"—a region where the inhibitor distribution changes smoothly from η_2 to η_h (Fig. 23b)—travels in front of the leading edge of the traveling AS.

We note that all results presented above also hold for cold traveling AS (Figs. 5l and m), which can be excited in hot Ω- and KΩ-systems with an N- or И-shaped LC curve.

In Ω- and KΩ-systems, in which the LC curve has a Λ or V shape (Figs. 6b and d), traveling spike AS (Fig. 5j) with a large amplitude can be excited. Such one-dimensional traveling AS are unstable in two- and three-dimensional systems with respect to division into smaller regions for the same reason as static, spike AS (Sec. 5.2), so that autowaves of a complicated type (spiral, radially diverging, etc.) cannot be excited in them.

7.2 More complicated autowaves. In one-dimensional systems a periodic[1,12–16] and possibly also a stochastic[16,27,182] sequence of pulses, whose shape is close to one of those shown in Figs. 5i–l, can be excited in addition to traveling AS (pulses).

Based on analysis of the stability, analogous to that performed in Secs. 4.2 and 4.4, it can be concluded[29,32] that for $\alpha \ll \varepsilon$ one-dimensional, wide, traveling AS are stable with respect to fluting of their walls in two- and three-dimensional systems. Numerical studies of two-dimensional autowaves have established that in models of the FHN type traveling AS in Ω-systems are stable (see, for example, Refs. 18, 61–69). A one-dimensional traveling AS in two-dimensional systems can excite, on encountering a nonuniformity, a spiral autowave—a reverberator[54,13,15,183–185]; this was first observed and studied in greatest detail in chemical reactions of the Belousov-Zhabotinskiĭ type (Sec. 1.4).[48,49] In three-dimensional systems complicated autowaves in the form of loops, rings, and other vortices can be excited.[49,66–69,111]

In KΩ-systems a pulsating AS (Sec. 6.2) and other autostructures can be excited in addition to the autowaves enumerated above.[31,29] A pulsating AS with a quite large amplitude in the form of a cluster (spot or layer) can lead to the excitation of autowaves that diverge away from it spherically (cylindrically or one-dimensionally) without damping and which are, far from the center, in the cross section, close to a traveling AS (Fig. 23b). In other words, a pulsating AS can be manifested as a guiding center (a source of radially diverging autowaves), observed experimentally in chemical reactions.[48,15] A stationary guiding center in a two-dimensional KΩ-system was observed in numerical studies of the model (1.13).[34] The formation of a guiding center in Ω-systems could be associated with a nonuniformity, which everywhere transforms a stable system in a local region into a self-oscillatory state.[12,15] In KΩ-systems a small nonuniformity can lead to spontaneous appearance of a guiding center with $A_d < A < A_c$: a small nonuniformity will provoke the formation of AS (Sec. 4.3), and in the course of its formation "local breakdown" will occur at the center of the AS (Sec. 4.4), as a result of which radially diverging autowaves can be generated. The conditions for excitation of a guiding center are most easily met in three-component[12,16] and in more complicated active systems with diffusion.[186]

8. CHARACTERISTICS OF AUTOSOLITONS IN BISTABLE "TRIGGER" SYSTEMS

8.1. Systems with diffusion. N- and И-systems, in which LC and ES curves (Sec. 2.2) intersect at three points are customarily called trigger systems (Fig. 24a). This situation can be realized, for example, in semiconductors[149,150,32] and in gas plasma,[90] as well as in chemical reactions.[48,111] Of three uniform states in trigger systems with fixed A two are stable—the states corresponding to the cold (θ_{h1}, η_{h1}) and hot (θ_{h3}, η_{h3}) state of the system (Fig. 24a). The theory of AS in monostable systems, presented in Secs. 3, 4, 6, and 7, also holds for AS in trigger (bistable) systems, for which the conditions (2.7), (2.8), or (2.10) are satisfied.[29,32] It follows from it[29,32] that in such bistable K-systems, unlike monostable systems (Secs. 3 and 4), the regions of existence of hot and cold AS join at some level of excitation $A = A_k$: a hot AS (Figs. 5a and b) can be excited in a cold system ($\theta_{h1} < \theta_0$; Sec. 2.2) right up to $A = A_b$ (Sec. 4.2), while a cold AS (Figs. 5c and d) can be excited in a hot system ($\theta_{h3} > \theta_0'$) right up to $A = A_b'$ (Sec. 4.2).[32]

This is associated with the fact that bistable systems have two stable uniform states, corresponding to the branches I and III of the LC curve (Fig. 24a). These states, according to Sec. 3.2, determine the points of the maximum of the potential U_η in the equation for smooth distributions (3.12), i.e., the branches I and III of the potential U_η have the form of potential humps.[28] To construct the true potential in (3.8) these branches must be joined at the point $\eta = \eta_s$ (Sec. 3.2). According to (3.14) the values of U_η at the maxima are equal when the condition[29,32]

$$\int_{\eta_{h1}}^{\eta_s} Q(\theta_{\rm I}(\eta), \eta, A_k)\, d\eta = \int_{\eta_{h3}}^{\eta_s} Q(\theta_{\rm III}(\eta), \eta, A_k)\, d\eta,$$
$$q(\theta_{\rm I,\,III}, \eta, A_k) = 0, \tag{8.1}$$

holds; this condition determines to order $\varepsilon \ll 1$ the value of $A = A_k$. The highest trajectory of a "particle" in such a potential describes the transition from one stable state into another. In a cold system the maximum of U_η corresponding to branch I is less than for branch III, while the highest trajectory of a "particle" in such a potential corresponds to a hot AS (Fig. 5a). In a hot system the situation is reversed, i.e., only a cold AS is realized in them (Fig. 5c).

As $A \to A_k$ the width of the AS approaches infinity, and a state in the form of a complicated static domain structure, describing the transition from the state $\eta = \eta_{h1}$, $\theta = \theta_{h1}$ into the state $\eta = \eta_{h3}$, $\theta = \theta_{h3}$ (Fig. 24d, curve 4), is realized in the system.[32,39]

Waves of transfer of a complicated structure, transferring the system from a cold state into a hot state or vice versa, can be excited in the K-systems studied. These waves are realized in different ranges of A, joining at the point $A = A_k$, at which the velocity of the waves is $v = 0$.

One-dimensional AS in two-and three-dimensional K-systems are stable with respect to stratification in the plane of their walls (Sec. 4.4) in the range of values of A whose limits are determined by the values of $\mathscr{L}_{b1}$ and $\mathscr{L}_{c1}$, given by (4.33) or (4.35).

In one-dimensional KΩ-systems with $\varepsilon^2 \ll \alpha < \varepsilon \ll 1$ AS become unstable with respect to pulsations (Sec. 6.2) at the boundaries of the region determined by the conditions (6.7) or (6.10).[32,39]

In Ω-systems switching autowaves, transferring the system from a cold state ($\eta = \eta_{h1}$, $\theta = \theta_{h1}$) into a hot state (η_{h3}, θ_{h3}; Fig. 24a) and vice versa—from a hot into a cold state[60,29]—can be excited in addition to traveling AS and other autowaves (Sec. 7). The existence of two such different switching waves, moving in the same direction, in Ω-systems (unlike K-systems) for the same value of A is one of the characteristic effects that distinguish the two-parameter systems under study from one-parameter systems with diffusion, for example, from semiconductors with an S-shaped IVC. In the latter systems only one wave that transforms the system into one of two stable states can be excited.[19] The possibility of switching the system sequentially from a cold state into a hot state and vice versa permits exciting in it very arbitrary sequences of autowaves of different width.

The characteristic properties of AS in trigger systems, following from the general theory, have been confirmed[164–166] by analytical studies of an axiomatic piecewise-linear model (Fig. 24b), described by Eqs. (2.2) and (2.3) in which

$$q(\theta, \eta, A) = \theta + \eta - H(\theta - A), \quad Q(\theta, \eta) = \eta - B\theta, \tag{8.2}$$

where $H = 1$ for $\theta \geqslant A$ and $H = 0$ for $\theta < A$, i.e., A determines the excitation threshold of the medium. The model (2.2) and (2.3) with $L = 0$, $\tau_\eta \gg \tau_\theta (\alpha \ll 1)$, and with q given by (8.2) is widely employed for analyzing autowaves[15,187,188] in Ω-systems (Sec. 7).

It has been established[188] that a traveling AS (pulse) has the form shown in Fig. 23g. A fixed value of A corresponds to two solutions—in the form of a stable, fast, wide pulse and an unstable, slow, narrow pulse. The pulse vanishes abruptly before reaching the point $A = A_v$, where $dv/dA = \infty$ (Fig. 23i) while $v_{\min} = \sqrt{3}\alpha^{1/2}$.[15]

It can be verified that these results are also valid for $\varepsilon \ll 1$, but $\alpha < \varepsilon^4$. As the ratio α/ε increases, in accordance with the general assertions (Sec. 7.1), the minimum value of the velocity $v_{\min}$ increases at which a traveling AS vanishes abruptly (Fig. 23c). For $3^{-1/2}\varepsilon < \alpha \ll 1$ a solution in the form of traveling AS is not realized.

Analytic formulas describing the distributions $\theta(x)$ nd $\eta(x)$ in static AS,[164,166] whose forms are identical to those shown in Figs. 5a–d, can be found for this model for $\varepsilon \ll 1$. The form of the potentials U_θ and U_η in (3.7) and (3.8) can be reconstructed based on these formulas and it can be verified[166] that their form is identical to that following from the general theory (Sec. 3.2). It also follows from the analytical study of Ref. 166 that for the model under study $\theta_{h1} = 0$, $\theta_{h3} = (1 + B)^{-1}$, $A_k = 0.5\theta_{h3}$, $A_b = 0.5 \{1 - \varepsilon B(1 + B)^{-1/2}(1 + \ln[(\varepsilon B)^{-1}(1 + B)^{1/2}])\}$, $A'_b = 2A_k - A_b$, while the bifurcation characteristics, i.e., the dependences of η_s or $\mathscr{L}_s$ on A, have the form shown in Figs. 24c and e. The branch I corresponds to a hot AS (Figs. 5a and b) while I' corresponds to a cold AS (Figs. 5c and d). For $\alpha > 1$ stable AS correspond to the solid sections on the bifurcation characteristics (Figs. 24c and e). As $A \to A_k$ the size of the AS increases without bound (Fig. 24d, curves 3 and 3'), while for $A = A_k$ the size becomes infinite, more precisely, a structure in the form of a complicated domain wall is realized (Fig. 24d, curve 4). As $A \to A_b$ (or A'_b) the size of the AS decreases, and becomes equal to $\mathscr{L}_b = l \ln [\varepsilon^{-1}(1 + B)^{1/2}B^{-1}]$ at the critical points 2 and 2' (Figs. 24c and e), and this agrees with the estimate (4.31). The formulas (4.35) and (6.10) are also confirmed for this model.

8.2 Systems with long-range couplings. Above, we studied the properties of AS in two-parameter systems, in which the long-range action of the inhibitor is determined by diffusion processes. Autosolitons in active (excitable) systems, in which a nonuniform distribution of the activator gives rise to long-range "forces," leading to the suppression (inhibition) of the activation process, have analogous properties. Such a situation is realized, for example, in distributed neuron networks, including, apparently, in the cortex of the brain[189,190] the so-called TV analog,[36] and active distributed optoelectronic media.[191] Long-range forces in physical systems can have a different nature (electric, strain, magnetic). In optoelectronic media long-range "forces" are determined by optical and electric couplings.[191]

In neuristor networks and simple media short-range exciting and long-range damping couplings are realized.[189–192,36] The activation process is realized thanks to the former couplings, while long-range inhibition is realized owing to the latter couplings. In the simplest one-dimensional case the processes in neuristor networks are described by an integral equation of the form[189]

$$\frac{\partial\theta}{\partial t} = -\theta + \int_{-\infty}^{\infty} \Phi(x' - x) H(\theta(x', t) - A)\, dx' \tag{8.3}$$

or

$$\frac{\partial\tilde\theta}{\partial t} = -\tilde\theta + H\left[\int_{-\infty}^{\infty} \Phi(x' - x)\tilde\theta(x', t)\, dx' - A\right], \tag{8.4}$$

where the coupling function $\Phi(x)$ describes short-range activation and long-range inhibition (Fig. 24g).

The solution of Eq. (8.4) in the stationary case describes the local state of the elements of the excitable medium (0 or 1) and has the form of a rectangle: $\tilde\theta(x) = H(x + \mathscr{L}_s/2) - H(x - \mathscr{L}_s/2)$, where $\mathscr{L}_s$ is the width of the AS and is determined from the equation[192]

$$\int_0^{\mathscr{L}_s} \Phi(x)\, dx = A. \tag{8.5}$$

The spatial distribution of the threshold in such a medium has a more complicated form, identical to the activator distribution $\theta(x)$ in the model (8.3). Indeed, the simple transformation[189,192]

$$\theta(x) = \int_{-\infty}^{\infty} \Phi(x' - x)\tilde\theta(x')\, dx' \tag{8.6}$$

reduces Eq. (8.4) to (8.3).

In spite of the fact that Eqs. (8.3) or (8.4) differ qualitatively from the system (2.2), (2.3), and (8.2), the bifurcation characteristic $\mathscr{L}_s(A)$ of the models (8.3) and (8.4),[192] determined by (8.5), is qualitatively the same as that shown in Fig. 24e for the model (8.2)[166] (Sec. 8.1). For the model (8.3) $\theta_{h1} = 0, \theta_{h3} = 2A_k$ (the points 1 and 1', respectively, in Fig. 24e)[192]

$$A_k = \int_0^{\infty} \Phi(x)\, dx, \quad A_b = \int_0^{a} \Phi(x)\, dx,$$
$$A'_b = 2A_k - A_b, \quad \mathscr{L}_b = a, \tag{8.7}$$

where a is the positive root of the function $\Phi(x)$ (Fig. 24g). The branch I of the bifurcation characteristic (Fig. 24e) describes the evolution of the simplest hot AS, for which[192]

$$\theta(x)=\int_{x-(\mathscr{L}_s/2)}^{x+(\mathscr{L}_s/2)}\Phi(x')\,dx'\quad\left(\theta\left(\frac{\mathscr{L}_s}{2}\right)=A\right). \tag{8.8}$$

The branch I′ corresponds to a cold AS, in which the activator distribution equals $2A_k-\theta(x)$, where $\theta(x)$ is given by (8.8).

Based on the form of the coupling function $\Phi(x)$ (Fig. 24g), it can be easily concluded from (8.8) that the distributions $\theta(x)$ in the form of a hot and cold AS are identical to those shown in Figs. 5(a–d) and 24d. In addition, the evolution and basic properties of AS in the models (8.3) and (8.4) under study and in the one-dimensional model of an active medium with diffusion (2.2), (2.3), (8.2) (Sec. 8.1) are completely identical.

When the coupling function $\Phi(x)$ has an oscillating character (Fig. 24g), for A close to A_k (8.7), many AS with different size $\mathscr{L}_s$, whose number increases without bound as $A\to A_k$ (Fig. 24f), can be realized.[192] The distribution of the activator in the nth hot AS is given by the formula (8.8), in which $\mathscr{L}_s$ must be replaced by $\mathscr{L}_{sn}$.[192] It differs from that shown in Figs. 5a and b and 24e (curve 3) only in that $\theta(x)$ does not drop off monotonically at the periphery of the AS, but rather it has an oscillating character (Fig. 7a).

According to the theory of stability of AS (Sec. 4.2) stable and unstable sections of existence of AS on a bifurcation characteristic (Fig. 24f) are separated by the points $A=A_{bn}$ and A_{tn}, where $d\mathscr{L}_s/dA=\infty$. The values of A_{bn} and A_{tn} and the corresponding widths of hot AS equal[192]

$$A_{bn}=\int_0^{\mathscr{L}_{bn}}\Phi(x)\,dx,\quad A_{tn}=\int_0^{\mathscr{L}_{tn}}\Phi(x)\,dx,$$

$$\mathscr{L}_{bn}=a_{2n-1},\quad \mathscr{L}_{tn}=a_{2n}\quad(n=1,\ \ldots), \tag{8.9}$$

where a_n is the positive nth root of the function $\Phi(x)$ (Fig. 24h).

We emphasize that the models (8.3) and (8.4) describe media in which the process of retardation (inhibition) is long-ranged, but inertialess, i.e., they belong to K-systems (Sec. 2.2).

The properties of media in which the retardation process is not only long-ranged, but also slower than the activation (excitation) process, are analogous to those of KΩ-systems (Sec. 2.2); more precisely, pulsating and traveling AS can be realized in them in addition to static AS. A "refractory zone" (here it is associated with long-range damping rather than diffusion (Secs. 2.2 and 7.1)) travels in front of the leading wall of a traveling AS in such systems. This explains why in numerical studies of one of the models of such a medium[36] traveling AS are not annihilated, but rather repelled or form a static AS, i.e., they manifest properties that are also characteristic of other KΩ-systems (Sec. 2.2).

CONCLUSIONS

The properties of many real physical, and especially chemical[49,111] and biological[16] systems are described by several differential equations. If, however, activation and inhibition processes are separated in them, then the description of AS can, as a rule, be reduced to the analysis of two equations of the diffusion type (Sec. 2.1). For this reason the results of the theory of AS presented in this review are of a very general character.

In real systems activation and inhibition processes can be of a completely different nature and can be described not only by equations of the diffusion type, but also by other types of equations, including integral (Sec. 8.2). It follows from the theory of AS presented in this review that the types of AS is determined primarily by the characteristic spatial and temporal scales of variation of the activator and inhibitor densities. In generalizing the results of Sec. 2.2, systems where the inhibition (damping) process is faster than the activation (excitation) process and has a longer range must be classified as K systems, in which static AS form; systems where inhibition is slower, but short-range must be classified as Ω-systems, in which traveling AS form; systems where inhibition is slower and long-range must be classified as KΩ-systems, *in which static, pulsating, and traveling AS are realized*. It is natural to expect that the types and basic properties of AS in each of these classes of systems, irrespective of the physical nature of the activation and inhibition processes, will be analogous to those studied for active systems with diffusion. This conclusion is confirmed by the results presented in Sec. 8.2 for AS realized in active media described in integral equations, as well as by numerical studies of AS in active media described by several differential equations,[112,186] including also those of a complicated type.[193,194]

The results of the theory of AS presented in this review are employed to explain diverse phenomena in chemistry and biology.[12–17,37,109,114] The formation of striations in a gas discharge,[44] clusters of hot carriers in gas and semiconductor plasma,[45,46,51] luminous points under conditions of avalanche breakdown of p-n junctions,[47] and local regions of melting on the surface of a crystal accompanying uniform, pulsed, laser excitation[195] can be explained on the basis of the theory of AS.

In semiconductors and semiconductor structures AS can consist of strongly nonequilibrium regions, in which the electron or lattice temperature is high.[25] For this reason, many degradation effects in microelectronic devices could be associated with spontaneous formation of AS at small nonuniformities.

The diversity of the properties of AS and the important role that AS play in biology[12,16] are stimulating intereste[36,117,118,191] in the development of diverse, microelectronic devices for recording, storing, and processing information based on active distributed media with diffusion or long-range couplings.

As pointed out in Sec. 2.2, a static or pulsating AS can drift under the action of external forces with a velocity proportional to the flux generated by the forces. Autosolitons can also appear in systems whose degree of departure from equilibrium is determined by the magnitude of the flux itself. Autosolitons exist in such systems owing to the fact that dissipative losses in them are compensated by "pumping" of energy from the flux (flow). For example, two-dimensional moving AS can arise in a viscous liquid flowing down along the vertical surface of a pipe[196] or plane[76]; AS in the form of vortices can form in a rotating liquid or in planetary atmospheres.[80–82]

Autosolitons can also be excited in the region of instability of uniform state of a system in which a stable periodic autostructure is formed. The autosoliton here could be one of the types of "defects"[22)] of the periodic structure.[194] Autosolitons can also be excited in a turbulent medium. An example of such an AS is the formation of typhoons in a turbulent atmosphere.[197,198]

Since the conditions for the formation of AS are of an extremely general character, AS can be observed in the most diverse systems. It has not been excluded that many of the puzzling phenomena in nature, such as ball lightning,[199] the formation of high-temperature accumulations of gas in space,[200] as well as the aurora borealis and many other observed nonuniformities in the ionosphere,[201] can be explained based on the ideas regarding AS presented in this review.

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1) In this sense nonuniform states in the form of domain walls, arising in bistable nonequilibrium systems,[19–22] are not AS.

2) The term self-organization is usually employed for spontaneous formation and subsequent evolution of structures in nonequilibrium systems[23,24]—"to ordering through fluctuations"[23] (see Sec. 2.1).

3) The well-known[1,12–16] autowave in the form of a traveling pulse is the simplest one-dimensional, traveling AS. After it passes the system returns to the starting, stable, uniform state.

4) The uniform state of a system that deviates from thermodynamic equilibrium not beyond the limits of validity of Onsager's relations is stable.[53] For this reason the assertion that AS exist means that the uniform state is not the only stable state of even such weakly nonequilibrium systems.

5) Other, more complicated autowaves,[12,15,16,18,39,49,61–69] including spiral (reverberators) waves and guiding centers (Sec. 7) as well as different auto-wave vortices[64–66] have also been studied in models of the FHN type.

6) Autosolitons are not formed in one-parameter, uniform systems, whose properties are described by one diffusion equation. A single stationary process in the form of an autowave of transfer from one stable uniform state into another is realized in such bistable systems.[70,12,15] Such an autowave has been studied in detail in application to problems in the theory of combustion.[71–73] Nonstationary localized regions with the value of the parameter of the system growing in an unbounded fashion in time[73–75]—the temperature in problems of combustion and explosion[72,73]—can arise in one-parameter systems with one stable, uniform state. The theory of such a nonstationary process (regime with peaking) is presented in Ref. 75.

7) Autosolitons can also form in other systems, for example, in hydrodynamic systems in the presence of flows, i.e., convective flows, caused by external disturbances[76–82] (see Conclusions).

8) It is assumed that the action of neuronal networks, for example, in the cortex of the brain, is also controlled by the interaction of short-range activation and long-range inhibition (Sec. 8.2).

9) The system (2.1) is formally a system of autonomous, quasilinear equations of the parabolic type[126]: the "sources" g_i in them are strongly nonlinear functions of X_i and A, but do not depend explicitly on the spatial coordinates and time.

10) It is known that depending on the parameters of the system a soft or hard regime of excitation of autostructures can be realized in them. In the case of the soft regime autostructures with a low amplitude, which equals zero for $A = A_c$ and increases with the supercriticality (the value of $A-A_c$), are formed in the case of the soft regime. The formation and evolution of gradually excited auto-structures were analyzed in detail in monographs by Nicolis and Prigogine[23] and Haken.[24,107] In the case of hard excitation autostructures with large amplitude appear abruptly. The properties and evolution of such autostructures are analyzed in Refs. 25–29. In this review these questions, which refer to the problem of self-organization, are not studied.

11) The LC and ES curves, defined by (2.4) and (2.6), are formally the zero isoclines,[8] corresponding to the point equations (2.2) and (2.3).

12) The uniform state of KN- and K-И-systems stratifies in some range $A_c \leqslant A \leqslant A'_c$. To order $\sim\varepsilon \ll 1 A_c = A_0$ and $A'_C = A'_0$, where $A = A_0$ and $A = A'_0$ correspond to $\theta_h = \theta_0$ and $\theta'_h = \theta'_0$, respectively.

13) This effect is reminiscent of the formation of nuclei of the new phase in a first-order phase transition.

14) Systems in which autowaves can be excited are also called excitable media.[39,111]

15) The existence of different complicated AS in the system means that a set of doubly asymptotic trajectories, terminating in the limit $x \to \pm\infty$ at the singular point $\theta = \theta_h$, $\eta = \eta_h$ is realized in the four-dimensional phase space of the variables θ, $d\theta/dx$, η, and $d\eta/dx$. The latter point, according to the theory of dynamic systems,[9] is a saddle point of the type $O^{2,2}$, and it corresponds to a set of different trajectories terminating at it.

16) The effect of a nonuniformity in one-parameter systems is discussed in Ref. 21.

17) This result was essentially taken from the theory of stability of domains in bistable semiconductors.[19,20] It also holds for neutral boundary conditions (3.1).[28]

18) This assertion is valid in the case when at the threshold of stability $\mathrm{Im}\gamma = 0$. A different situation is realized in $K\Omega$-systems (Sec. 6.2).

19) The instability of two close-lying striations is linked with the fact that because of strong diffusion spreading of the inhibitor the changes in the inhibitor distribution cannot follow locally the growth of the antisymmetric fluctuations of the activator.[25,29] Such "pumping" of the activator between striations causes the amplitude of the width of the striations to grow as a result of "consumption" of a neighboring striation. The "pumping" effect determines the minimum possible distance $\mathscr{L}_{\min}$ between striations. The value of $\mathscr{L}_{\min}$ depends on A.[31] Because of the "pumping" effect two AS at a distance of $\mathscr{L} < \mathscr{L}_{\min}$ cannot be excited; this is also confirmed by numerical studies.[117]

20) Bistability, oscillations, and stratification of such a two-component, strongly heated mixture are studied in Refs. 93, 92, and 178.

21) The condition for the appearance of pulsations was derived previously in an analysis of the stability of "hot spots" in semiconductor structures.[157] It transforms into (6.1), if $\tau_\eta \equiv C_s T/eI_c(1+\alpha)$, $\tau_\theta \equiv \tau_T$, $\mu_0 = 1$ are substituted in it; S_e is the emitter capacitance; I_c and α are the collector current and the current transfer factor; τ_T is the characteristic time of variation of the temperature (T) of the structure.

22) M. I. Rabinovich called our attention to this phenomenon.

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