On the microscopic theory of superfluid liquids

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This review presents a microscopic construction of the thermodynamics and hydrodynamics of superfluid bosons and fermions with singlet pairing, based on the concept of quasiaverages and the hypothesis of reduced description. Here we do not assume that the Hamiltonian possesses any dynamical symmetry. This has permitted obtaining results pertaining to both Galilean-invariant and relativistic systems. Account is taken of dissipative processes. The kinetic coefficients are presented in terms of the correlation functions of the flux operators. The approach is extended to solutions of quantum liquids. The influence of an external ac field on superfluid systems is studied, and the low-frequency asymptotic behavior of the Green's function is found in the hydrodynamic approximation. The symmetry properties of the equilibrium state are formulated, and the thermodynamics is constructed for superfluid Fermi systems with triplet pairing (the superfluid phases ^{3}He-B and ^{3}He-A). For the latter the flux densities of the additive integrals of motion are found in a state of equilibrium and the equations of "ideal" hydrodynamics are derived.

1. INTRODUCTION

At present the theoretical foundation for describing both equilibrium and nonequilibrium states of systems with spontaneously broker symmetry in statistical mechanics is the concept of quasiaverages of N. N. Bogolyubov[1] and the method of reduced description[2] developed by N. N. Bogolyubov to study the dynamics of physical systems based on the Liouville equation. This approach enables one to obtain both the thermodynamics and the equations of motion for such macroscopic systems.

One of the examples of systems with spontaneously broken symmetry is a superfluid liquid. The publications of N. N. Bogolyubov[3,4] established the connection of the phenomenon of superfluidity with that of Bose condensation, and presented a microscopic derivation of the equations of ideal hydrodynamics of a Galilean-invariant superfluid liquid.

We should note that most of the physical results for systems with spontaneously broken symmetry were obtained on the basis of a phenomenological approach. This primarily pertains to the phenomenon of superfluidity of He II, for whose description the two-fluid Tisza-Landau model proved effective. It is precisely within the framework of this model that L. D. Landau[5] constructed the equations of ideal hydrodynamics of superfluid He II, while dissipation processes were taken into account in the studies of I. M. Khalatnikov (see Ref. 6). Subsequently the phenomenological approach was applied to other systems close to superfluid in their properties. In particular, such systems include superconductive systems and quantum crystals (see Refs. 7–9).

In recent years intensive studies have been conducted on superfluid ^{3}He (see, e.g., Refs. 10–17). In contrast to He II, in which the invariance of the equilibrium state with respect to phase transitions is broken, in the case of ^{3}He the symmetry breaking is more complex: in addition to breaking of symmetry with respect to phase transitions, symmetry breaking also occurs with respect to three-dimensional rotations, both in coordinate and in spin space. Consequences of this are a more complex structure of the order parameter, a vortical character of superfluid flow, and the appearance of varied magnetic properties.

This review presents a microscopic construction of the thermodynamics and hydrodynamics of superfluid Bose and Fermi systems with singlet pairing on the basis of the method of quasiaverages and the method of reduced description. Here we have not assumed that the Hamiltonian of the system possess Galilean invariance. In particular, this has enabled both obtaining results pertaining to Galilean-invariant systems and treating relativistic systems. Moreover, the influence is studied of external fields and the low-frequency asymptotic behavior is found of the general Green's functions $G^{\pm}_{ab}$ (for arbitrary quasilocal operators $\hat{a}$ and $\hat{b}$) in the hydrodynamic approximation. Finally, the symmetry properties of the equilibrium state are formulated on the basis of the concept of quasiaverages, the thermodynamics is constructed, and the fluxes of the additive integrals of motion are found for superfluid Fermi systems with triplet pairing (the superfluid phases ^{3}He-A and ^{3}He-B).

This review is based principally on the results of Refs. 18–26.

2. STATE OF STATISTICAL EQUILIBRIUM

2.1. Conservation laws

In describing the state of statistical equilibrium and the nonequilibrium dynamics of condensed media, the conservation laws play a substantial role. In the Schrödinger representation the density operators $\hat{\zeta}_\alpha(\mathbf{x}) \equiv \{\hat{\varepsilon}(\mathbf{x}), \hat{\pi}_k(\mathbf{x}), \hat{\zeta}_a(\mathbf{x})\}$ corresponding to the additive integrals of motion $\gamma_\alpha = \int d^3x \hat{\zeta}_\alpha(\mathbf{x}) (\alpha = 0,1,2,3,a)$ satisfy the differential conservation laws

$$i[\hat{\mathscr{H}}, \hat{\zeta}_\alpha(\mathbf{x})] = i[\hat{\mathscr{P}}_k, \hat{\zeta}_{\alpha k}(\mathbf{x})] = -\frac{\partial \hat{\zeta}_{\alpha k}(\mathbf{x})}{\partial x_k}. \quad (2.1)$$

Here the $\hat{\zeta}_{\alpha k}(\mathbf{x}) \equiv \{\hat{q}_k(\mathbf{x}), \hat{t}_{ik}(\mathbf{x}), \hat{j}_{ak}(\mathbf{x})\}$ are the current-density operators of the additive integrals of motion, $\hat{\mathscr{H}} \equiv \hat{\gamma}_0$ is the Hamiltonian, $\hat{\mathscr{P}}_k \equiv \hat{\gamma}_k$ is the momentum ($k = 1,2,3$), and $\hat{\gamma}_a$ are the integrals of motion associated with the inner

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symmetries of the system. Usually the operators γ are the operators for the number of particles $\hat{N}$ and the spin $\hat{\mathbf{S}}$.

Upon using the operator identity

$$i[\hat{A}, \hat{b}(\mathbf{x})] = -i[\hat{B}, \hat{a}(\mathbf{x})] - \frac{\partial \hat{b}_k(\mathbf{x})}{\partial x_k},$$

$$\hat{b}_k(\mathbf{x}) = i\int d^3x' x'_k \int_0^1 d\lambda\, [\hat{a}(\mathbf{x}-(1-\lambda)\mathbf{x}'),\ \hat{b}(\mathbf{x}+\lambda\mathbf{x}')],$$

which is valid for arbitrary quasilocal operators $\hat{a}(\mathbf{x})$, $\hat{b}(\mathbf{x})$ (here $\hat{A} \equiv \int d^3x \hat{a}(\mathbf{x})$ and $\hat{B} \equiv \int d^3x \hat{b}(\mathbf{x})$), we find the expressions for the operators of the flux densities $\hat{\zeta}_{\alpha k}(\mathbf{x})$ in terms of the operators of the densities of the additive integrals of motion $\hat{\zeta}_\alpha(\mathbf{x})$ (cf. Refs. 18 and 27):

$$\hat{q}_k(\mathbf{x}) = \frac{i}{2}\int d^3x' x'_k \int_0^1 d\lambda\, [\hat{\varepsilon}(\mathbf{x}-(1-\lambda)\mathbf{x}'),\ \hat{\varepsilon}(\mathbf{x}+\lambda\mathbf{x}')],$$

$$\hat{t}_{ik}(\mathbf{x}) = -\hat{\varepsilon}(\mathbf{x})\delta_{ik} + i\int d^3x' x'_k \int_0^1 d\lambda\, [\hat{\varepsilon}(\mathbf{x}-(1-\lambda)\mathbf{x}'),$$

$$\hat{\pi}_i(\mathbf{x}+\lambda\mathbf{x}')], \tag{2.2}$$

$$\hat{\zeta}_{ak}(\mathbf{x}) = i\int d^3x' x'_k \int_0^1 d\lambda\, [\hat{\varepsilon}(\mathbf{x}-(1-\lambda)\mathbf{x}'),\ \hat{\zeta}_a(\mathbf{x}+\lambda\mathbf{x}')].$$

In deriving these formulas we have taken account of the following symmetry properties of the energy density $\hat{\varepsilon}(\mathbf{x})$:

$$i[\hat{\mathcal{P}}_k,\ \hat{\varepsilon}(\mathbf{x})] = -\frac{\partial \hat{\varepsilon}(\mathbf{x})}{\partial x_k}, \quad [\hat{\gamma}_a,\ \hat{\varepsilon}(\mathbf{x})] = 0.$$

In studying the phenomenon of superfluidity one usually assumes invariance of the physical system with respect to Galilean or Lorentzian transformations, which leads to certain transformation properties of the Hamiltonian. A feature of the present treatment is that only the requirements of translational and phase invariance are imposed on the Hamiltonian. This approach allows one to describe a broader class of superfluid systems (we shall term them everywhere below generalized; an example of such a generalized system is the electron gas of metals), while also it permits one to treat as special cases in unitary fashion systems having Galilean or Lorentzian dynamical symmetry.

2.2. Introduction of quasiaverages

For systems having spontaneously broken symmetry, the state of statistical equilibrium has a lower symmetry than the symmetry of the Hamiltonian. A convenient concept that enables describing such systems is the concept of quasiaverages.

According to N. N. Bogolyubov[1] (cf. also the monographs[27-29]) the averages in a state of statistical equilibrium (with broken symmetry) are defined by the formula

$$\langle \ldots \rangle = \lim_{\nu\to 0}\lim_{V\to\infty} \mathrm{Sp}\, w_\nu \ldots, \quad w_\nu = \exp(\Omega_\nu - Y_\alpha \hat{\gamma}_\alpha - \nu Y_0 \hat{f}); \tag{2.3}$$

Here V is the volume of the system, the Y_α are the thermodynamic forces conjugate to the additive integral of motion, and Ω_ν is the thermodynamic potential, which is defined by the condition $\mathrm{Sp}\, w_\nu = 1$. The operator $\hat{f}$ possesses the symmetry of the phase under study and removes the degeneracy of the state of statistical equilibrium. The limit

$$\omega = \lim_{\nu\to 0}\lim_{V\to\infty} \frac{\Omega_\nu}{V}$$

defines the density of the thermodynamic potential. For quasiaverages (in contrast to ordinary averages) the principle of spatial attenuation of correlation always holds. That is, we have

$$\langle \hat{a}(\mathbf{x})\, \hat{b}(\mathbf{y})\rangle \xrightarrow[|\mathbf{x}-\mathbf{y}|\to\infty]{} \langle \hat{a}(\mathbf{x})\rangle \langle \hat{b}(\mathbf{y})\rangle,$$

where $\hat{a}(\mathbf{x})$ and $\hat{b}(\mathbf{y})$ are arbitrary quasilocal operators [the correlation of the operators $\hat{a}(\mathbf{x})$ and $\hat{b}(\mathbf{y})$ can decline by a power law].

It is known from the phenomenological theory that an adequate description of the thermodynamics and kinetics of degenerate systems requires introducing into the theory new thermodynamic parameters q, which are not associated with the conservation laws, but arise from the symmetry of the equilibrium state being studied. Within the framework of the microscopic approach this implies that the operator $\hat{f}$ amounts to a certain linear functional of the order-parameter operator $\hat{\Delta}(\mathbf{x})$,

$$\hat{f} = \int_V d^3x\, (g(\mathbf{x}, t; q)\, \hat{\Delta}(\mathbf{x}) + \text{H.c.}). \tag{2.4}$$

Here $g(\mathbf{x},t;q)$ is a c-number function of the coordinates and the time, which determines the equilibrium values of the order parameter $\Delta(\mathbf{x},t) = \langle \hat{\Delta}(\mathbf{x})\rangle$, and which depends on the thermodynamic parameters q that fix the symmetry properties of the physical phase being studied.

The dependence of $g(\mathbf{x},t;q)$ on the coordinates $\mathbf{x}$ and the time t arises from the fact that the introduction of the term $\nu Y_0 \hat{f}$ into (2.3) can break the invariance of the equilibrium statistical operator with respect to translations in space and time. That is, we have $[w(t), \mathcal{H}] \neq 0$, $[w(t), \hat{\mathcal{P}}_k] \neq 0$, where $w(t)$ is the equilibrium statistical operator

$$w(t) = \lim_{\nu\to 0}\lim_{V\to\infty} w_\nu(Y, q; t), \tag{2.5}$$

which depends on the thermodynamic forces Y_α and also on the parameters q that determine the symmetry of the phase being studied. (We should understand the taking of the limit in (2.5) in the sense of averages (cf. (2.3)).)

To concretize the further presentation, let us examine first a very simple superfluid system—a one-component, spin-free Bose liquid with the order-parameter operator $\hat{\Delta}(\mathbf{x}) = \psi(\mathbf{x})$ [$\psi(\mathbf{x})$ is the operator for annihilation of a particle at the point $\mathbf{x}$]. In line with the idea of Bose condensation, the state of statistical equilibrium of the superfluid Bose liquid is determined by the condition

$$[w, \hat{\mathcal{P}}_k - p_k\hat{N}] = 0, \quad [\hat{f}, \hat{\mathcal{P}}_k - p_k\hat{N}] = 0 \tag{2.6}$$

(the symmetry condition), where p_k is a thermodynamic parameter that has the meaning of the superfluid momentum (momentum of condensate particles). Since we have

$$e^{-i\hat{\vec{\mathcal{P}}}\mathbf{x}}\psi(\mathbf{x}')\, e^{i\hat{\vec{\mathcal{P}}}\mathbf{x}} = \psi(\mathbf{x}+\mathbf{x}'), \quad e^{-i\varphi\hat{N}}\psi(\mathbf{x})\, e^{i\varphi\hat{N}} = e^{i\varphi}\psi(\mathbf{x}), \tag{2.7}$$

we can easily see that $g(\mathbf{x},t) = \exp[i(\mathbf{p}\mathbf{x} + \varphi(t))]$, where $\varphi(t)$ is a certain function of the time.

We note further that

$$[w_\nu(t),\ Y_0\mathcal{H} + \mathbf{Y}\hat{\vec{\mathcal{P}}} + Y_4\hat{N} + \nu Y_0\hat{f}] = 0.$$

Since, owing to the canonical commutation relations the operator $[\hat{f},\hat{a}(\mathbf{x})]$ is also quasilocal, while the average of the quasilocal operator is assumed to be finite, we have

$$\lim_{\nu\to 0}\lim_{V\to\infty} \nu\,\mathrm{Sp}\,w_\nu(\hat{t})\,[\hat{f},\ \hat{a}(\mathbf{x})] = .$$

Therefore we have

$$[w(t),\ Y_0\mathcal{H} + \mathbf{Y}\hat{\vec{\mathcal{P}}} + Y_4\hat{N}] = 0.$$

Thus, upon taking account of (2.6), we have

$$[w(t),\ \mathcal{H} + p_0\hat{N}] = 0,\ p^0 = \frac{Y_4 + \mathbf{Y}\mathbf{p}}{Y_0}. \tag{2.8}$$

We shall call this relationship the stationarity condition. The statistical operator $w(t)$ must satisfy the von Neumann equation. Therefore, using (2.8) we obtain

$$w(t+\tau) = e^{ip_0\hat{N}\tau}w(t)\,e^{-ip_0\hat{N}\tau} \tag{2.9}$$

and hence, according to (2.4) and (2.9) we find

$$g(\mathbf{x},\ t) = \exp(i\varphi(\mathbf{x},\ t)),\quad \varphi(\mathbf{x},\ t) = \mathbf{p}\mathbf{x} + p_0 t + \chi. \tag{2.10}$$

In summarizing we can say that, in the superfluid systems being studied, the state of thermodynamic equilibrium is characterized by the thermodynamic forces Y_α associated with the additive integrals of motion ($Y_0 \equiv T^{-1}$ is the reciprocal temperature, $Y_k/Y_0 \equiv v_{nk}$ is the velocity of a normal component, and $Y_4/Y_0 \equiv \mu$ is the chemical potential), and also with the superfluid momentum $\mathbf{p}$ and the phase $\chi \equiv \varphi(0,0)$, the existence of which arises from the breaking of the symmetry of the state of statistical equilibrium. We stress that the dependence on the thermodynamic variables $\mathbf{p}$ and χ is introduced by means of infinitely small sources, and that in a state with broken symmetry this dependence is maintained as $\nu \to 0$.

Let us introduce the unitary operator

$$U_\varphi(t) = \exp\left(-i\int d^3x\varphi(\mathbf{x},\ t)\,\hat{n}(\mathbf{x})\right). \tag{2.11}$$

Upon choosing the function of (2.10) as $\varphi(\mathbf{x},t)$, we easily see that

$$\begin{aligned} U_\varphi(t)\,w_\nu(t)\,U_\varphi^+(t) &\equiv w_\nu' \\ &= \exp\Big[\Omega_\nu - Y_0\mathcal{H}_\mathbf{p} - Y_k(\hat{\mathcal{P}}_k + p_k\hat{N}) - Y_4\hat{N} - \nu Y_0 \\ &\quad \times \int d^3x\,(\psi(\mathbf{x}) + \psi^+(\mathbf{x}))\Big], \end{aligned} \tag{2.12}$$

where

$$\mathcal{H}_\mathbf{p} = U_\mathbf{p}\mathcal{H}U_\mathbf{p}^+,\quad U_\mathbf{p} = \exp\left(-i\,\mathbf{p}\int d^3x\ \mathbf{x}\hat{n}(\mathbf{x})\right). \tag{2.13}$$

Since $[\mathcal{H}_\mathbf{p},\mathcal{P}] = 0$, then, according to (2.11) and (2.12), we have

$$\mathrm{Sp}\,w_\nu(t)\,\psi(\mathbf{x}) = e^{i\varphi(\mathbf{x},t)}\,\mathrm{Sp}\,w_\nu'\psi(0) = e^{i\varphi(\mathbf{x},t)}\eta. \tag{2.14}$$

Thus the quantity $\varphi(\mathbf{x},t)$ amounts to the phase of the equilibrium order parameter $\mathrm{Sp}\,w_\nu'(t)\psi(\mathbf{x})$ (one can say that $\mathrm{Sp}\,w_\nu\psi(\mathbf{x}) = \mathrm{Sp}\,w_\nu'\psi^+(\mathbf{x})$).

2.3. Thermodynamics

In line with the definition (2.3) and (2.10), we can easily see that the density of the thermodynamic potential ω does not depend on the phase χ and is a function of the thermodynamic parameters Y_0, Y_4, $\mathbf{Y}^2$, $\mathbf{p}^2$, and $\mathbf{y}\mathbf{p}$. Upon differentiating ω with respect to the thermodynamic forces Y_α and the superfluid momentum $\mathbf{p}$, we obtain

$$\begin{aligned} &\frac{\partial\omega}{\partial Y_0} = \mathrm{Sp}\,w'\hat{\varepsilon}_\mathbf{p}(0) = \mathrm{Sp}\,w\hat{\varepsilon}(0) \equiv \varepsilon, \\ &\frac{\partial\omega}{\partial Y_l} = \mathrm{Sp}\,w'(\hat{\pi}_l(0) + p_l\hat{n}(0)) = \mathrm{Sp}\,w\hat{\pi}_l(0) \equiv \pi_l, \\ &\frac{\partial\omega}{\partial Y_4} = \mathrm{Sp}\,w\hat{n}(0) \equiv n, \\ &\frac{\partial\omega}{\partial p_l} = \lim_{\nu\to 0}\lim_{V\to\infty}\left(\frac{Y_0}{V}\,\mathrm{Sp}\,w_\nu'\frac{\partial\mathcal{H}_\mathbf{p}}{\partial p_l} + \frac{Y_l}{V}\,\mathrm{Sp}\,w_\nu'\hat{N}\right). \end{aligned} \tag{2.15}$$

Upon taking account of the formulas of (2.13), we find

$$\lim_{\nu\to 0}\lim_{V\to\infty}\frac{1}{V}\,\mathrm{Sp}\,w_\nu'\frac{\partial\mathcal{H}_\mathbf{p}}{\partial p_l} = -i\int d^3x x_l\,\mathrm{Sp}\,w'\,[\hat{\varepsilon}_\mathbf{p}(\mathbf{x}),\ \hat{n}(0)],$$

$$\hat{\varepsilon}_\mathbf{p} \equiv U_\mathbf{p}\hat{\varepsilon}U_\mathbf{p}^+.$$

On the other hand, upon averaging the expression (2.2) for the operator of the flux density of the number of particles $\hat{j}_l(\mathbf{x}) \equiv \hat{\zeta}_{4l}(\mathbf{x})$ with the statistical operator w, in agreement with (2.12) we have

$$j_l = \mathrm{Sp}\,w\hat{j}_l(\mathbf{x}) = -i\int d^3x\,x_l\,\mathrm{Sp}w'[\hat{\varepsilon}_\mathbf{p}(\mathbf{x}),\ \hat{n}(0)]. \tag{2.16}$$

(We have taken account of the fact that $[w',\ \hat{\vec{\mathcal{P}}}] = 0$). Therefore we have

$$j_l = \frac{1}{Y_0}\frac{\partial\omega}{\partial p_l} - \frac{Y_l}{Y_0}\frac{\partial\omega}{\partial Y_4}. \tag{2.17}$$

Thus we have found an expression for the flux density of the number of particles in terms of the thermodynamic potential ω. Upon using Eqs. (2.15) and (2.17), we find the following fundamental thermodynamic equation:

$$d\omega = \varepsilon\,dY_0 + \pi_k dY_k + n\,dY_4 + (Y_0 j_l + Y_l n)\,dp_l, \tag{2.18}$$

which has the meaning of the second law of thermodynamics for reversible processes in a superfluid liquid.

In constructing the hydrodynamics of an ideal superfluid liquid, and also in studying the low-frequency asymptotic behavior of the Green's function, we shall need expressions for the averages in the state w of the flux-density operators of the momentum $\hat{t}_{ik}$ and the energy $\hat{q}_k$ in terms of the thermodynamic potential ω. In proceeding to find these quantities, we note that, according to (2.2) and (2.12)

$$t_{ik} = \mathrm{Sp}w\,\hat{t}_{ik}(0) = -\langle\hat{\varepsilon}_\mathbf{p}(0)\rangle_0\delta_{ik} - i\,\langle[\hat{\Gamma}_{ki},\ \hat{\varepsilon}_\mathbf{p}(0)]\rangle_0 + p_i j_k, \tag{2.19}$$

where the average is taken over the state w_ν', $\langle\ldots\rangle_0 = \mathrm{Sp}\ w_\nu'\ldots$ in this formula $\hat{\Gamma}_{ki} = \int d^3x x_k\hat{\pi}_i(\mathbf{x})$ is the generator of the group of arbitrary linear transformations $x_i \to x_i' = a_{ik}x_k$. The latter statement stems from the fact that the operators $\psi(\mathbf{x})$ and $\psi'(\mathbf{x}) = \psi(a\mathbf{x})|\det a|^{1/2}$ satisfy identical commutation relationships, and hence are interrelated by the unitary transformation U_a:

$$U_a\psi(\mathbf{x})\,U_a^+ = \psi'(\mathbf{x}) = |\det a|^{1/2}\,\psi(a\mathbf{x}).$$

Upon treating the infinitely small transformations $a_{ik} = \delta_{ik} + \xi_{ik}$, $|\xi| \ll 1$, we easily find that $U_a = 1 - i\xi_{kl}\hat{\Gamma}_{lk}$. The generator $\hat{\Gamma}_{ki}$ satisfies the following commutation relationships:

$$i[\hat{\Gamma}_{kl}, \hat{n}(0)] = -\delta_{ik}\hat{n}(0),$$
$$i[\hat{\Gamma}_{kl}, \hat{\pi}_l(0)] = -\delta_{ik}\hat{\pi}_l(0) - \delta_{ik}\hat{\pi}_i(0). \qquad (2.20)$$

Upon using the first of these relationships, we find that

$$t_{ik} = -\langle\hat{\varepsilon}'_{\mathbf{p}}(0)\rangle_0\delta_{ik} - i\langle[\hat{\Gamma}_{kl}, \hat{\varepsilon}'_{\mathbf{p}}(0)]\rangle_0 + p_i j_k, \qquad (2.21)$$

where $\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x}) \equiv \hat{\varepsilon}_{\mathbf{p}}(\mathbf{x}) + p_0\hat{n}(\mathbf{x}) + \nu(\psi(\mathbf{x}) + \psi^+(\mathbf{x}))$.

From the condition $\mathrm{Sp}\, w' = 1$ we obtain (cf. (2.12))

$$e^{-\Omega} = \mathrm{Sp}\exp\left(-\int_V d^3x\hat{h}(\mathbf{x})\right), \quad \hat{h}(\mathbf{x}) = Y_0\hat{\varepsilon}'_{\mathbf{p}}(\mathbf{x}) + Y_l\hat{\pi}_l(\mathbf{x}'). \qquad (2.22)$$

Upon defining the operator $\hat{h}_a(\mathbf{x})$ by the formula

$$U_a\hat{h}(\mathbf{x})U_a^+ \equiv \hat{h}_a(a\mathbf{x})\,|\det a|, \qquad (2.23)$$

we find that

$$e^{-\Omega} = \mathrm{Sp}\, U_a\exp\left(-\int_V d^3x\hat{h}(\mathbf{x})\right)U_a^+ = \mathrm{Sp}\exp\left(-\int_{V_a} d^3x\hat{h}_a(\mathbf{x})\right),$$

($V_a \equiv V|\det a|$). Since the potential Ω is proportional to V, we have

$$\exp\left(-\frac{\Omega}{|\det a|}\right) = \mathrm{Sp}\exp\left(-\int_V d^3x\hat{h}_a(\mathbf{x})\right).$$

Hence, upon taking account of the fact that $\det a = 1 + \delta_{kl}\xi_{kl}$ for infinitely small transformations, we obtain

$$\mathrm{Sp}\, w\left(\frac{\partial\hat{h}_a(0)}{\partial\xi_{kl}}\right)_{\xi=0} = -\overset{\vee}{\omega}\delta_{kl}, \quad \overset{\vee}{\omega} = \lim_{V\to\infty}\frac{\Omega_{\vee}}{V}.$$

On the other hand, Eq. (2.23) implies that

$$-i[\hat{\Gamma}_{lk}, \hat{h}(0)] = \left(\frac{\partial\hat{h}_a(0)}{\partial\xi_{kl}}\right)_{\xi=0} + \delta_{kl}\hat{h}(0),$$

and hence

$$i\langle[\hat{\Gamma}_{ik}, \hat{h}(0)]\rangle_0 + \delta_{kl}\langle\hat{h}(0)\rangle_0 = \overset{\vee}{\omega}\delta_{kl}.$$

Upon substituting into this formula the expression for $\hat{h}(\mathbf{x})$ and taking account of (2.20), we obtain from (2.21) an expression for the momentum flux density in a state of equilibrium

$$t_{ik} = \frac{p_i}{Y_0}\frac{\partial\omega}{\partial p_k} - \frac{\partial}{\partial Y_i}\frac{\omega Y_k}{Y_0}. \qquad (2.24)$$

To find the average in a state of equilibrium of the energy flux density we shall employ the following theorem[26]:

Theorem. For an equilibrium statistical operator

$$w = \exp\left(\Omega - \int_V d^3x\hat{h}(\mathbf{x})\right),$$

that satisfies the condition of spatial homogeneity (2.6), *the following equation holds*

$$Q_k \equiv \int d^3x x_k\,\mathrm{Sp}\, w[\hat{h}(\mathbf{x}), \hat{h}(0)] = 0. \qquad (2.25)$$

To prove this relationship, let us calculate the trace in the expression for Q_k in the system of eigenvectors of the momentum operator $\hat{\mathbf{P}} \equiv \hat{\mathscr{P}} - \mathbf{p}\hat{N}$, $\hat{P}|\mathbf{P},a\rangle = P_k|\mathbf{P},a\rangle$ (the index a numbers the remaining quantum numbers). Consequently we can represent the quantity Q_k in the form

$$Q_k = -\frac{1}{(2\pi)^3}\int d^3P\frac{\partial}{\partial P_k}\mathrm{tr}\exp(\Omega - \hat{h}_{\mathbf{P}})(1 + \hat{h}_{\mathbf{P}})2.$$

Here the quantity $\hat{h}_P$ is an operator in the subspace of eigenvectors $|\mathbf{P},a\rangle$ of the operator $\hat{P}_k$ belonging to fixed eigenvalues of P:

$$\left\langle\mathbf{P}, a\left|\int d^3x\hat{h}(\mathbf{x})\right|\mathbf{P}, b\right\rangle = \langle a|\hat{h}_{\mathbf{P}}|b\rangle;$$

Here tr denotes the operation of taking the trace in the subspace of the vectors $|a\rangle$. Since we have

$$\lim_{P\to\pm\infty}\hat{h}_{\mathbf{P}} \to \infty,$$

as is necessary for the following average to be finite:

$$\mathrm{Sp}\, w\,\hat{a}(\mathbf{x}) = (2\pi)^{-3}\int d^3P\,\mathrm{tr}\exp(\Omega - \hat{h}_{\mathbf{P}})\langle\mathbf{P}|\hat{a}(0)|\mathbf{P}\rangle,$$

we have $Q_k = 0$. Upon using the explicit form of the operator $\hat{h}(\mathbf{x})$ and taking account of the formulas (2.2) and (2.25), we find

$$Y_\alpha(Y_k\zeta_\alpha + Y_0\zeta_{\alpha k}) = 0. \qquad (2.26)$$

Here we have $\zeta_\alpha = \mathrm{Sp}\, w\hat{\zeta}_\alpha$, $\zeta_{\alpha k} = \mathrm{Sp}\, w\hat{\zeta}_{\alpha k}$. The term $\nu Y_0\hat{f}(\mathbf{x})$ does not contribute to the commutator in the integral in (2.25), since the operators $\hat{\zeta}_\alpha(\mathbf{x})$ and $\hat{f}(\mathbf{x})$ are quasilocal, while it is assumed that the mean of quasilocal operators exists as $\nu\to 0$. Starting from the formulas (2.17) and (2.24) and the relationship (2.26), we arrive at the following expression for the equilibrium average energy flux density:

$$q_k = -\frac{\partial}{\partial Y_0}\frac{\omega Y_k}{Y_0} - \frac{p_0}{Y_0}\frac{\partial\omega}{\partial p_k}. \qquad (2.27)$$

Evidently, we can write the formulas (2.17), (2.24), and (2.27) for the flux densities in the following form:

$$\zeta_{\alpha k} = -\frac{\partial}{\partial Y_\alpha}\frac{\omega Y_k}{Y_0} + \frac{\partial\omega}{\partial p_k}\frac{\partial p_0}{\partial Y_\alpha}. \qquad (2.28)$$

Now let us represent the expressions for the flux densities in a form corresponding to two-liquid hydrodynamics. The thermodynamic potential ω is a function of Y_0, $\mathbf{Y}^2$, Y_4, $\mathbf{p}^2$, and $\mathbf{yp}$. Let us introduce the quantities ρ_n, ρ_s, and m, which are functions of these thermodynamic variables:

$$\rho_n \equiv -2Y_0\frac{\partial\omega}{\partial\mathbf{Y}^2}, \quad \rho_s \equiv \frac{2}{Y_0}\frac{\partial\omega}{\partial\mathbf{p}^2}\overset{*}{m}{}^2, \quad \frac{\rho_n}{\overset{*}{m}} \equiv n - \frac{\partial\omega}{\partial(\mathbf{Yp})}. \qquad (2.29)$$

Then, upon taking account of (2.29), the fluxes j_k, t_{ik} and q_k acquire the form

$$j_k = \frac{\rho_n}{\overset{*}{m}}v_{nk} + \frac{\rho_s p_k}{\overset{*}{m}{}^2}, \quad t_{ik} = -\frac{\omega}{Y_0}\delta_{ik} + \rho_n v_{ni}v_{nk} + \rho_s\frac{p_i p_k}{\overset{*}{m}{}^2},$$
$$q_k = v_{nk}\left[-\frac{\omega}{Y_0} + \varepsilon + \left(n - \frac{\rho_n}{\overset{*}{m}}\right)p_0\right] - \frac{\rho_s p_k}{\overset{*}{m}{}^2}p_0. \qquad (2.30)$$

Hence we see that $\mathbf{p}_n$ has the meaning of the "mass" density of the normal component, while ρ_s is the "mass" density of the superfluid component. If we interpret the quantity m as the effective "mass of a particle", then we must interpret $\mathbf{p}/\overset{*}{m}$ as the superfluid velocity. We note that generally the total density $\rho \equiv \overset{*}{m}n = \overset{*}{m}\partial\omega/\partial Y_4$ does not coincide with the sum of the normal ρ_n and superfluid ρ_s densities: $\rho \neq \rho_n + \rho_s$.

In this section the superfluid liquid was characterized by the order-parameter operator—the field Bose operator ψ. We note that actually the structure of the order-parameter operator involved with the statistics of the particles (Bose or Fermi) is inessential. It suffices to consider that the order-parameter operator $\hat{\Delta}(\mathbf{x})$ is translationally invariant ($\hat{\Delta}(\mathbf{x}) = \exp(-i\hat{\mathscr{P}}\mathbf{x})\hat{\Delta}(0)\cdot\exp(i\hat{\mathscr{P}}\mathbf{x})$) and satisfies the equation

$$[\hat{n}(\mathbf{x}), \hat{\Delta}(\mathbf{x}')] = -\tilde{g}\hat{\Delta}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}'), \tag{2.31}$$

Here the quantity $\tilde{g}$ characterizes the system and is not associated with its state. In particular, for bosons we have $\hat{\Delta}(\mathbf{x}) = \psi(\mathbf{x})$, $\tilde{g} = 1$. For fermions with Cooper pairing in the s-state we have $\hat{\Delta}(\mathbf{x}) = \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{x})$, and Eq. (2.31) is satisfied when $\tilde{g} = 2$.

For relativistic superfluid Fermi systems we must take as the order-parameter operator the relativistically invariant operator $\bar{\psi}(x)\psi^c(x)$, where $\bar{\psi}(x)$ and $\psi^c(x)$ are bispinors conjugate and charge-conjugate with respect to $\psi(x)$.

According to the formulas (2.15) and (2.28) the densities ζ_α and the flux densities $\zeta_{\alpha k}$ in a state of thermodynamic equilibrium were represented in terms of the thermodynamic potential ω, which was a function of the variables Y_α and $\mathbf{p}$. This notation of the formulas is usually employed for Galilean-invariant systems. Moreover we note that we can rewrite Eqs. (2.15) and (2.28) in an equivalent form if we choose as the independent variables the quantities $Y_\mu = (Y_0, Y_k)$, $p_\mu = (p_0, p_k)$ and transform from the potential ω to the usually employed Gibbs potential $\omega' = \omega/Y_0$:

$$j^\mu = \frac{\partial\omega'}{\partial p_\mu}, \quad t^{\mu\nu} = -\frac{\partial\omega' Y^\nu}{\partial Y_\mu} + p^\mu\frac{\partial\omega'}{\partial p_\nu}. \tag{2.32}$$

Here we have introduced the relativistic notation $j^\mu \equiv (n, j_k)$, $t^{00} \equiv \varepsilon$, $t^{0k} \equiv q_k$, $t^{ik} \equiv t_{ik}$, $t^{k0} \equiv \pi_k$, and have assumed that the raising and lowering of indices is performed by using the metric tensor $g_{\mu\nu}$ ($g_{00} = -1$, $g_{ik} = \delta_{ik}$, $g_{0k} = 0$). We shall use these formulas in the following sections in treating relativistic systems.

3. HYDRODYNAMICS OF A SUPERFLUID LIQUID

3.1. The method of reduced description

To study the evolution of spatially inhomogeneous states of a superfluid liquid on the hydrodynamic level, we shall use the hypothesis of reduced description, according to which the nonequilibrium statistical operator $\rho(t)$ at times $t \gg \tau_r$ (τ_r is the relaxation time) depends on the time and on the initial statistical operator $\rho \equiv \rho(0)$ via a certain set of parameters. For a superfluid liquid such parameters are the density of the additive integrals of motion $\zeta_\alpha(\mathbf{x},t)$ and the phase $\varphi(\mathbf{x},t)$ of the order parameter:

$$\rho(t) \xrightarrow[t\gg\tau_r]{} \sigma\{\zeta(\mathbf{x},t,\rho), \varphi(\mathbf{x},t,\rho)\}, \tag{3.1}$$

$$\zeta_\alpha(\mathbf{x}) = \mathrm{Sp}\,\sigma(\zeta,\varphi)\hat{\zeta}_\alpha(\mathbf{x}), \quad \varphi(\mathbf{x}) = \mathrm{Im}\ln\mathrm{Sp}\,\sigma(\zeta,\varphi)\psi(\mathbf{x}).$$

Since $[\mathscr{H},\hat{\mathscr{P}}] = [\mathscr{H},\hat{N}] = 0$, then the equations of (3.1) yield the formulas

$$e^{-i\mathscr{H}\tau}\sigma\{\zeta_\alpha(\mathbf{x}',t), \varphi(\mathbf{x}',t)\}e^{i\mathscr{H}\tau} = \sigma\{\zeta_\alpha(\mathbf{x}',t+\tau), \varphi(\mathbf{x}',t+\tau)\},$$

$$e^{i\hat{\mathscr{P}}\mathbf{x}}\sigma\{\zeta_\alpha(\mathbf{x}',t), \varphi(\mathbf{x}',t)\}e^{-i\hat{\mathscr{P}}\mathbf{x}} = \sigma\{\zeta_\alpha(\mathbf{x}+\mathbf{x}',t), \varphi(\mathbf{x}+\mathbf{x}',t)\} \tag{3.2}$$

$$e^{i\hat{N}\varphi'}\sigma\{\zeta_\alpha(\mathbf{x}',t), \varphi(\mathbf{x}',t)\}e^{-i\hat{N}\varphi'} = \sigma\{\zeta_\alpha(\mathbf{x}',t), \varphi(\mathbf{x}',t)+\varphi'\}.$$

Upon differentiating the first relationship of (3.2) with respect to τ and assuming that $\tau = 0$, we obtain a functional equation for $\sigma(\zeta,\varphi)$:

$$-i[\mathscr{H}, \sigma(\zeta,\varphi)] = \int d^3x\left(\frac{\delta\sigma(\zeta,\varphi)}{\delta\zeta_\alpha(\mathbf{x})}L_\alpha(\mathbf{x}) + \frac{\delta\sigma(\zeta,\varphi)}{\delta\varphi(\mathbf{x})}L_\varphi(\mathbf{x})\right). \tag{3.3}$$

The parameters of reduced description $\zeta_\alpha(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)$ satisfy the equations of motion

$$\dot{\zeta}_\alpha(\mathbf{x},t) = -\frac{\partial}{\partial x_k}\mathrm{Sp}\,\sigma(\zeta,\varphi)\hat{\zeta}_{\alpha k}(\mathbf{x}) \equiv L_\alpha(\mathbf{x}) \equiv L_\alpha(\mathbf{x};\zeta(t),\varphi(t)),$$

$$\dot{\varphi}(\mathbf{x},t) = \mathrm{Re}\frac{\mathrm{Sp}\,\sigma(\zeta,\varphi)[\mathscr{H},\psi(\mathbf{x})]}{\mathrm{Sp}\,\sigma(\zeta,\varphi)\psi(\mathbf{x})} \equiv L_\varphi(\mathbf{x}) \equiv L_\varphi(\mathbf{x};\zeta(t),\varphi(t)). \tag{3.4}$$

In the spatially inhomogeneous case the superfluid momentum is associated with the phase by the formula $\mathbf{p}(\mathbf{x}) = \partial\varphi(\mathbf{x})/\partial\mathbf{x}$.

To find an unambiguous solution of Eqs. (3.3) and (3.4), we need a certain boundary condition that has the meaning of the ergodic relationship. The ergodic relationship expresses mathematically the fact of transition in the region of large t of an arbitrary nonequilibrium state to a state of statistical equilibrium. Let ρ be the initial nonequilibrium statistical operator that satisfies the condition of spatial homogeneity $[\rho, \hat{\mathscr{P}}_k - p_k\hat{N}] = 0$. Then in the process of evolution a transition occurs rapidly, in the time τ_r, to a state of statistical equilibrium. This means that the relationship is fulfilled that

$$\rho(t) \equiv e^{-i\mathscr{H}t}\rho e^{i\mathscr{H}t} \xrightarrow[t\gg\tau_r]{} w(t), \tag{3.5}$$

which should be understood in the sense of averages.

Let us find the dependence of the thermodynamic parameters $Y_\alpha, \varphi(\mathbf{x},t)$ that enter into $w(t)$ (cf. (2.3)) on the initial statistical operator ρ. Since $[\hat{N},\hat{\zeta}_{\alpha k}(\mathbf{x})] = 0$ and the statistical operator $\rho(t)$ in (3.5) satisfies the condition $[\rho(t), \hat{\mathscr{P}}_k - p_k\hat{N}] = 0$, $\mathrm{Sp}\,\rho(t)\hat{\zeta}_\alpha(\mathbf{x})$ does not depend on $\mathbf{x}$ and t. Therefore we have

$$\mathrm{Sp}\,w(t)\hat{\zeta}_\alpha(0) = \mathrm{Sp}\,\rho(0)\hat{\zeta}_\alpha(0). \tag{3.6}$$

This equation determines the dependence of the thermodynamic parameters Y_α on the initial statistical operator ρ.

Upon introducing the phase of the quantity $\psi(\mathbf{x})$ in the state $\rho(t)$: $\bar{\psi}(t) = \mathrm{Im}\ln\mathrm{Sp}\,\rho(t)\psi(0)$ (we have taken account of the fact that $[\rho(t), \hat{\mathscr{P}}_k - p_k\hat{N}] = 0$), we can easily show by using (2.14) that the quantity χ in (2.10) is determined by the formula

$$\chi = \bar{\varphi}(0) + \int_0^\infty d\tau\,(\dot{\bar{\varphi}}(\tau) - p_0) \tag{3.7}$$

[since $\bar{\varphi}(\tau)_{\tau\to\infty} \to p_0\tau + \chi$, the integral converges as $\tau \to \infty$].

This formula solves the problem of finding χ as a functional of ρ. We shall rewrite formula (3.5) in the form

$$\rho(t) \xrightarrow[t\gg\tau_r]{} w(Y_\alpha, \mathbf{p}, p_0 t + \chi) = \lim_{\nu\to 0}\lim_{V\to\infty} w_\nu (Y_\alpha, \mathbf{p}, p_0 t + \chi), \tag{3.8}$$

where

$$w_\nu(Y_\alpha, \mathbf{p}, \chi) = \exp\left\{\Omega_\nu - Y_\alpha\hat{\gamma}_\alpha - \nu Y_0 \int d^3x\,(\psi(x) \times \exp[-i(\mathbf{px}+\chi)] + \text{э. c.})\right\}; \tag{3.9}$$

Together with Eqs. (2.10), (3.6), and (3.7), it defines the ergodic relationship for superfluid Bose systems.

Let us study the average $a(\mathbf{x}) = \mathrm{Sp}\,\sigma(\zeta(\mathbf{x}'), \varphi(\mathbf{x}))\hat{a}(\mathbf{x})$. Owing to the principle of attenuation of spatial correlations, the fundamental contribution to this average *will come from those values of the parameters* $\zeta_\alpha(\mathbf{x}')$ *and* $\varphi(\mathbf{x})$ whose value of the argument $\mathbf{x}'$ is close to $\mathbf{x}$. Accordingly, let us expand $\sigma(\zeta,\varphi)$ in a series in the gradients of the parameters ζ and φ. Here we must bear in mind the fact that the quantity $\nabla\varphi$ is not small; however, the second derivative $\nabla\nabla\varphi$ is of the order of $\nabla\zeta$. In line with what we have said, we have

$$\overset{(0)}{\sigma}(\mathbf{x}) = \sigma\left\{\zeta(\mathbf{x}),\ \varphi(\mathbf{x}) + (x_k' - x_k)\frac{\partial\varphi(\mathbf{x})}{\partial x_k}\right\}.$$

That is, the statistical operator $\overset{(1)}{\sigma}$ is a function (rather than a functional) of the arguments $\zeta(\mathbf{x})$, $\psi(\mathbf{x})$, and $\partial\psi(\mathbf{x})/\partial x_k$. A consequence of Eq. (3.2) is the spatial homogeneity of this state, i.e., $[\overset{(1)}{\sigma}(\mathbf{x}),]\ \hat{\mathscr{P}} - \hat{N}\nabla\psi(\mathbf{x})] = 0$. Upon comparing (3.1) and (3.8), we obtain

$$\overset{(0)}{\sigma}\left\{\zeta(\mathbf{x}),\ \varphi(\mathbf{x}) + \frac{x'\partial\varphi(\mathbf{x})}{\partial\mathbf{x}}\right\} = w\left\{Y(\mathbf{x}),\ \frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{x}},\ \varphi(\mathbf{x})\right\}, \tag{3.10}$$

Here the parameters $Y_\alpha(\mathbf{x})$ as functions of $\zeta(\mathbf{x})$ and $\mathbf{p}(\mathbf{x})$ are determined by the equation $\mathrm{Sp}\,w(Y,\mathbf{p},\varphi)\,\hat{\zeta}_\alpha(\mathbf{x}) = \zeta_\alpha(\mathbf{x})$. Upon substituting the operator $\overset{(1)}{\sigma}$ that we have found into Eq. (3.4) and taking account of (3.10) and the symmetry properties (2.5) and (2.8), we arrive at the following equation of superfluid hydrodynamics in the principal approximation in terms of the spatial gradients:

$$\dot{\zeta}_\alpha = -\nabla_k\overset{(0)}{\zeta}_{\alpha k}, \quad \dot{\varphi} = p_0 \equiv \frac{Y_4 + \mathbf{Y}\nabla\varphi}{Y_0}, \tag{3.11}$$

Here ζ_α and $\overset{(0)}{\zeta}_{\alpha k}$ are associated with the thermodynamic potential by the formulas (2.15) and (2.28). Since the phase ψ enters into the right-hand sides of these equations only via $\nabla\psi$, but not explicitly, the latter equation (3.11) is usually written in the form

$$\dot{\mathbf{p}} = \nabla p_0, \quad \mathrm{rot}\,\mathbf{p} = 0. \tag{3.12}$$

Let us introduce into the treatment the entropy density

$$s = -\lim_{\nu\to 0}\lim_{V\to\infty}\frac{1}{V}\mathrm{Sp}\,w_\nu \ln w_\nu = -\omega + Y_\alpha\zeta_\alpha.$$

Upon using Eq. (3.11), we find that

$$\dot{s} = \nabla_k\frac{sY_k}{Y_0}.$$

This implies that entropy transport is effected by the normal component of the liquid, which moves with the velocity $v_n = -\mathbf{Y}Y_0^{-1}$.

We note that the ergodic relationship (3.8) is necessary also in taking account of the next—the dissipative—approximation.

3.2. Galilean- and relativistically invariant systems

Let the equations of quantum mechanics be invariant with respect to the Galilean transformation, which corresponds to the unitary operator

$$U_\mathbf{v} = \exp\left(-im\,\mathbf{v}\int d^3x\,\mathbf{x}\,\hat{n}(\mathbf{x})\right), \tag{3.13}$$

where m is the mass of the particle. In the unitary transformation (3.13) the operators $\hat{\zeta}_\alpha(\mathbf{x})$ are transformed according to the formulas

$$U_\mathbf{v}\hat{n}(\mathbf{x})U_\mathbf{v}^+ = \hat{n}(\mathbf{x}), \quad U_\mathbf{v}\hat{\pi}(\mathbf{x})U_\mathbf{v}^+ = \hat{\pi}(\mathbf{x}) + m\mathbf{v}\,\hat{n}(\mathbf{x}),$$
$$U_\mathbf{v}\hat{\varepsilon}(\mathbf{x})U_\mathbf{v}^+ = \hat{\varepsilon}(\mathbf{x}) + \mathbf{v}\hat{\pi}(\mathbf{x}) + \frac{1}{2}mv^2\hat{n}(\mathbf{x}). \tag{3.14}$$

Thus the transformed operators $U_\mathbf{v}\hat{\zeta}_\alpha(\mathbf{x})\,U_\mathbf{v}^+$ are a linear combination of the original operators $\hat{\zeta}_\alpha(\mathbf{x})$ with coefficients that depend on the transformation parameters $\mathbf{v}$. We note that Eqs. (3.14) and (2.2) imply the coincidence of the expressions for the mass flux density operator with the momentum density operator, i.e., $m\hat{j}_k(\mathbf{x}) = \hat{\pi}_k(\mathbf{x})$. A consequence of Eqs. (2.2) and (3.14) is the transformation laws for the flux-density operators $\hat{t}_{ik}(\mathbf{x})$ and $\hat{q}_k(\mathbf{x})$:

$$U_\mathbf{v}\hat{t}_{ik}(\mathbf{x})U_\mathbf{v}^+ = \hat{t}_{ik}(\mathbf{x}) + v_i\hat{\pi}_k(\mathbf{x}) + v_k\hat{\pi}_i(\mathbf{x}) + mv_iv_k\hat{n}(\mathbf{x}),$$
$$U_\mathbf{v}\hat{q}_k(\mathbf{x})U_\mathbf{v}^+ = \hat{q}_k(\mathbf{x}) + v_i\hat{t}_{ik}(\mathbf{x}) + v_kv_i\hat{\pi}_i(\mathbf{x}) + v_k\hat{\varepsilon}(\mathbf{x}) + \frac{v^2}{2}(\hat{\pi}_k(\mathbf{x}) + mv_k\hat{n}(\mathbf{x})).$$

Upon using these formulas we easily see that the following relationship holds for the thermodynamic potential of the superfluid liquid:

$$\omega(Y_\alpha, \mathbf{p}) = \omega(Y_\alpha', 0) \equiv \omega(Y_\alpha'),$$
$$Y_0' = Y_0, \quad Y_k' = Y_k + Y_0v_k, \quad Y_4' = Y_4 + mv_kY_k + Y_0\frac{mv^2}{2}. \tag{3.15}$$

Here we have $v_k = p_k/m$. The thermodynamic potential ω of Galilean-invariant systems with account taken of rotational invariance is a function of the three independent variables Y_0', $\mathbf{Y}'^2$, and Y_4'. The transformation $w\to w' = U_\mathbf{v}wU_\mathbf{v}^+$ (cf. (2.12)) corresponds to a transition to a reference system where the condensate is at rest, while the parameter $\mathbf{v}\equiv\mathbf{p}m\equiv\mathbf{v}_s$ has the meaning of the superfluid velocity. In view of (2.18) and (3.15), the second law of thermodynamics for Galilean-invariant superfluid systems has the form

$$d\omega = \zeta_\alpha'\,dY_\alpha',$$
$$\varepsilon' = \varepsilon - \mathbf{v}_s\pi + \frac{mv_s^2}{2}n, \quad \pi' = \pi - m\mathbf{v}_sn, \quad n' = n.$$

A consequence of Eqs. (2.29) and (3.15) is the equations

$$m = \overset{*}{m}, \quad \rho_s = \rho - \rho_n, \tag{3.16}$$

In the special case that we are discussing they lead to coincidence of the equations (3.11) that we have derived with the equations of two-liquid hydrodynamics of L. D. Landau.

Now let us study the case in which the system is invariant with respect to the Lorentz transformations $x^\mu \to x'^\mu = a^\mu_\nu x^\nu$ ($x^0 \equiv t, x^k \equiv x_k$). In this case the 4-vector $\hat{\mathcal{P}}^\mu \equiv (\mathcal{H}, \hat{\mathcal{P}}_k)$ and the charge operator $\hat{Q}$ have the transformation properties

$$\hat{\mathcal{P}}'^\mu = U_a \hat{\mathcal{P}}^\mu U_a^+ = a^\mu_\nu \hat{\mathcal{P}}^\nu, \hat{Q}' = U_a \hat{Q} U_a^+ = \hat{Q} \ (\mu, \nu = 0, 1, 2, 3). \tag{3.17}$$

Here U_a is a unitary transformation in Hilbert space corresponding to the Lorentz transformation, whose explicit form we have not written out. The equilibrium statistical operator of a relativistic superfluid liquid has the form

$$w(Y_\mu, p_\mu, \varphi) = \exp\Big[V\omega - Y_\mu \hat{\mathcal{P}}^\mu - Y_4 \hat{Q} - \nu Y_\mu \int_\sigma d\sigma^\mu \{\psi(x) \exp[-i(p_\nu x^\nu + \varphi)] + \text{H.c.}\}\Big]. \tag{3.18}$$

Here we have $Y_\mu \equiv (Y_0, Y_k)$, $p_\mu \equiv (p_0, p_k)$, $p_0 \equiv (Y_4 + \mathbf{Yp})/Y_0$ (for simplicity we assume that the complex scalar field ψ corresponds to the particles). Equations (3.17) and (3.18) imply that

$$U_a w(Y_\mu, p_\mu, \varphi) U_a^+ = w(Y'_\mu, p'_\mu, \varphi'),$$
$$Y'_\mu = Y_\nu a^\nu_\mu, \quad p'_\mu = p_\nu a^\nu_\mu, \quad \varphi' = \varphi.$$

Thus the quantities Y_μ and p_μ form 4-vectors, while the quantity $Y_4 = -Y_\mu p^\mu$ amounts to an invariant. The condition of spatial homogeneity (2.6) and the condition of stationarity (2.8) are combined into a single relativistically invariant relationship

$$[w, \hat{\mathcal{P}}^\mu - p^\mu \hat{Q}] = 0.$$

Since the volume V is not a relativistic invariant, then, instead of the density of the thermodynamic potential ω, it is expedient to transform to the relativistically invariant potential $\omega' = \omega/Y_0$ (the Gibbs potential), which has the physical meaning of a pressure. The potential ω' is a function of the invariants Y^2, p^2, and $Y_\mu p^\mu$:

$$\omega' = \omega'(Y^2, p^2, Y_\mu p^\mu). \tag{3.19}$$

Therefore the quantity j^μ amounts to the 4-vector of the flux, while $t^{\mu\nu}$ is the energy-momentum tensor [see (2.32)]. The equations of ideal hydrodynamics of a relativistic superfluid liquid, in agreement with (3.11) and (2.32), have the form

$$\frac{\partial t^{\mu\nu}}{\partial x^\nu} = 0, \ \frac{\partial j^\mu}{\partial x^\mu} = 0, \ \frac{\partial p^\mu}{\partial x_\nu} - \frac{\partial p^\nu}{\partial x_\mu} = 0. \tag{3.20}$$

Here the latter equation was derived by combining the equation of motion for the superfluid momentum with the condition of potentiality of flow (3.12), while the 4-momentum p_ν is connected to the phase φ by the relationship $p_\nu = \partial\varphi/\partial x^\nu$. The entropy density $s = -\omega + Y_\alpha \zeta_\alpha$, according to (2.32), is equal to

$$s \equiv \overset{(0)}{s} = Y_0 Y_\mu \frac{\partial \omega'}{\partial Y_\mu}.$$

It is combined with the entropy flux density $s^k = -\overset{(0)}{s} Y_k / Y_0$ into the 4-vector

$$\overset{(0)}{s^\mu} = -Y^\mu Y_\nu \frac{\partial \omega'}{\partial Y_\nu},$$

which amounts to the 4-flux of entropy. A consequence of Eq. (3.20) is the condition of adiabaticity of flow of the superfluid liquid

$$\frac{\partial \overset{(0)}{s^\mu}}{\partial x^\mu} = 0.$$

The system of equations of ideal hydrodynamics (3.20) derived in the microscopic approach is fully equivalent to the equations of Ref. 30 on which the phenomenological approach is based.

3.3. Relaxation processes

To take account of the dissipative terms in the equations of hydrodynamics (3.11), we must find the statistical operator $\sigma(\zeta, \psi)$ in the linear approximation in the gradients ∇Y_α and ∇p_k. This problem was solved in Ref. 19. Without stopping on the method of deriving this operator, we shall present the expressions for the dissipative fluxes $\overset{(1)}{\zeta}_{\alpha k}$, $\overset{(1)}{L}_\varphi$ and the kinetic coefficients.

We can write the equations of hydrodynamics of a superfluid liquid with account taken of dissipation processes in the form

$$\dot{\zeta}_\alpha = -\nabla_k(\overset{(0)}{\zeta}_{\alpha k} + \overset{(1)}{\zeta}_{\alpha k}), \quad \dot{\varphi} = p_0 + \overset{(1)}{L}_\varphi.$$

Here the dissipative fluxes $\overset{(1)}{\zeta}_{\alpha k}$ and $\overset{(1)}{L}_\varphi$ have the form

$$\begin{aligned} \overset{(1)}{\zeta}_{\alpha k} &= \nabla_l Y_\beta I(\hat{\zeta}'_{\beta l}, \hat{\zeta}'_{\alpha k}) + \nabla_i \frac{\partial\omega}{\partial p_i} I(\hat{h}', \hat{\zeta}'_{\alpha k}), \\ \overset{(1)}{L}_\varphi &= \nabla_l Y_\beta I(\hat{\zeta}'_{\beta l}, \hat{h}') + \nabla_i \frac{\partial\omega}{\partial p_i} I(\hat{h}', \hat{h}') \end{aligned} \tag{3.21}$$

and the operators $\hat{\zeta}'_{\alpha k}$, $\hat{h}$, are defined by the formulas

$$\begin{aligned} \hat{\zeta}'_{\alpha k}(\mathbf{x}) &\equiv \hat{\zeta}_{\alpha k}(\mathbf{x}) - \frac{\partial \langle \hat{\zeta}_{\alpha k} \rangle}{\partial \zeta_\beta} \hat{\zeta}_\beta(\mathbf{x}), \\ \hat{h}'(\mathbf{x}) &\equiv \frac{1}{2\operatorname{Re}\langle\psi\rangle}[H, \psi(\mathbf{x}) e^{-i\mathbf{px}} - \psi^+(\mathbf{x}) e^{i\mathbf{px}}] - \frac{\partial p_0}{\partial \zeta_\beta} \hat{\zeta}_\beta(\mathbf{x}). \end{aligned} \tag{3.22}$$

The quantities

$$I(\hat{a}, \hat{b}) \equiv \frac{1}{2} \int_{-\infty}^{\infty} d\tau \int d^3x \langle e^{iH\tau}(\hat{a}(\mathbf{x}) - \langle \hat{a} \rangle) e^{-iH\tau} \hat{b}(0) \rangle \tag{3.23}$$

are the generalized kinetic coefficients and they satisfy the Onsager principle

$$I(\hat{a}, \hat{b}) = I(\hat{b}, \hat{a}).$$

Moreover, Eqs. (3.22) and (3.23) imply the properties

$$I(\hat{a}, \hat{a}) \geqslant 0, \quad I^2(\hat{a}, \hat{b}) \leqslant I(\hat{a}, \hat{a}) I(\hat{b}, \hat{b}),$$
$$I(\hat{\zeta}'_{\alpha k}, \hat{\zeta}_\beta) = 0, \quad I(\hat{h}', \hat{\zeta}_\beta) = 0.$$

The equation of motion for the entropy density in the approximation being studied has the form

$$\dot{s} + \nabla_k (s v_{nk} + \overset{(1)}{s}_k) = \overset{(2)}{I}, \quad \overset{(1)}{s}_k = Y_\alpha \overset{(1)}{\zeta}_{\alpha k} + \frac{\partial \omega}{\partial p_k} \overset{(1)}{L}_\varphi,$$

$$\overset{(2)}{I} = \nabla_k Y_\alpha \overset{(1)}{\zeta}_{\alpha k} + \nabla_k \frac{\partial \omega}{\partial p_k} \overset{(1)}{L}_\varphi$$

$$= I\left(\nabla_k Y_\alpha \cdot \hat{\zeta}'_{\alpha k} + \nabla_k \frac{\partial \omega}{\partial p_k} \cdot \hat{h}', \ \nabla_l Y_\beta \cdot \hat{\zeta}'_{\beta l} + \nabla_l \frac{\partial \omega}{\partial p_l} \cdot \hat{h}'\right) \geqslant 0,$$

Here $\overset{(1)}{s}_k$ and $\overset{(2)}{I}$ are respectively the dissipative flux density and the entropy production.

Since the statistical operator w is a function of two vectors—the normal velocity and the superfluid momentum—the kinetic coefficients have a rather complicated tensor structure. If we neglect the anisotropy, having assumed $\mathbf{p} = 0, \mathbf{v}_n = 0$ in (3.3), then the superfluid liquid being studied is characterized by the following kinetic coefficients:

$$\varkappa = \frac{1}{3T^2} I(\hat{\mathbf{q}}' - \mu \hat{\mathbf{j}}', \ \hat{\mathbf{q}}' - \mu \hat{\mathbf{j}}') \geqslant 0,$$

$$D = \frac{1}{3T^2} I(\hat{\mathbf{q}}' - \mu \hat{\mathbf{j}}', \ \hat{\mathbf{j}}') \left(\mu \equiv -\frac{Y_4}{Y_0}\right),$$

$$\sigma = \frac{1}{3T} I(\hat{\mathbf{j}}', \hat{\mathbf{j}}') \geqslant 0, \quad \eta = \frac{1}{10T} I(\overset{0}{\hat{t}}{}'_{ik}, \overset{0}{\hat{t}}{}'_{ik}) \geqslant 0,$$

$$\overset{0}{\hat{t}}{}'_{ik} \equiv \hat{t}'_{ik} - \frac{1}{3}\delta_{ik} \hat{t}'_{ll}, \quad \zeta_1 = \frac{1}{3T} I(\hat{t}'_{ii}, \hat{h}'),$$

$$\zeta_2 = \frac{1}{9T} I(\hat{t}'_{ii}, \hat{t}'_{kk}) \geqslant 0, \quad \zeta_3 = \frac{1}{T} I(\hat{h}', \hat{h}') \geqslant 0;$$

Here $\varkappa$ is the heat conductivity coefficient, D and σ are the diffusion coefficients, and η, ζ_1, ζ_2, and ζ_3 are the viscosities. In this case the fluxes $\overset{(1)}{s}_k$, $\overset{(1)}{\zeta}_{\alpha k}$, and $\overset{(1)}{L}_\varphi$ have the form

$$\overset{(1)}{s}_k = -\frac{\varkappa}{T}\nabla_k T - D\nabla_k \mu, \quad \overset{(1)}{j}_{4k} = -D\nabla_k T - \sigma \nabla_k \mu,$$

$$\overset{(1)}{t}_{ik} = -\eta u_{ik} - \delta_{ik}\left[\zeta_1 \nabla_l \left(\frac{1}{Y_0}\frac{\partial \omega}{\partial p_l}\right) + \zeta_2 \nabla_l v_{nl}\right],$$

$$\overset{(1)}{L}_\varphi = \zeta_1 \nabla_l v_{nl} + \zeta_3 \nabla_l \left(\frac{1}{Y_0}\frac{\partial \omega}{\partial p_l}\right).$$

Here we have $u_{ik} = \nabla_k v_{ni} + \nabla_i v_{nk} - (2/3)\delta_{ik}\nabla_l v_{nl}$. Thus, if we neglect anisotropy effects, the superfluid liquid being studied is characterized by seven kinetic coefficients.

Let us examine the simplifications to which Galilean invariance leads in the structure of the dissipative fluxes. In this case the kinetic coefficients $I(\hat{a},\hat{b})$ are functions of the difference between the normal and superfluid velocities. Moreover, upon taking account of (3.22), we see that $\hat{\zeta}'_{4k} \equiv 0$. Hence the kinetic coefficients $I(\hat{\zeta}_{\alpha k}, \hat{\zeta}'_{\beta l})$ vanish if one of the indices α or β equals 4. Therefore the dissipative fluxes of a Galilean-invariant superfluid liquid with $\mathbf{v}_n - \mathbf{v}_s \neq 0$ are characterized by 14 independent kinetic coefficients.[19] When $\mathbf{v}_n - \mathbf{v}_s = 0$, the number of kinetic coefficients is reduced to five and the structure of the dissipative fluxes coincides with the results of the phenomenological theory.[6]

Now let the liquid be invariant with respect to Lorentz transformations. The equations of hydrodynamics of a relativistic superfluid liquid with account taken of dissipation processes have the form

$$\frac{\partial(\overset{(0)}{\zeta}{}^{\alpha\mu} + \overset{(1)}{\zeta}{}^{\alpha\mu})}{\partial x^\mu} = 0, \quad \frac{\partial p^\mu}{\partial x_\nu} - \frac{\partial p^\nu}{\partial x_\mu} = \frac{\partial \overset{(1)}{L}{}^\nu}{\partial x_\mu} - \frac{\partial \overset{(1)}{L}{}^\mu}{\partial x_\nu}.$$

The dissipative fluxes $\overset{(1)}{\zeta}{}^{\alpha\mu} \equiv (\overset{(1)}{t}{}^{\nu\mu}, \overset{(1)}{j}{}^\mu)$ $(\alpha = \nu, 4)$ and $\overset{(1)}{L}{}^\mu$ are determined by the formulas

$$\begin{aligned} \overset{(1)}{\zeta}{}^{\alpha\mu} &= \frac{\partial Y_\beta}{\partial x^\nu} I(\hat{\zeta}'^{\beta\nu}, \hat{\zeta}'^{\alpha\mu}) + \frac{\partial K^\lambda_\nu}{\partial x^\lambda} I(\hat{h}'^\nu, \hat{\zeta}'^{\alpha\mu}), \\ \overset{(1)}{L}{}^\mu &= \frac{\partial Y_\beta}{\partial x^\nu} I(\hat{\zeta}'^{\beta\nu}, \hat{h}'^\mu) + \frac{\partial K^\lambda_\nu}{\partial x^\lambda} I(\hat{h}'^\nu, \hat{h}'^\mu), \\ K^{\mu\nu} &= \frac{\partial \omega'}{\partial p_\lambda}\frac{\partial Y^\mu Y^\nu}{\partial Y^\lambda} = K^{\nu\mu}, \end{aligned} \tag{3.24}$$

and the operators $\hat{\zeta}'^{\alpha\mu}$ and $\hat{h}'^\mu$ have the form

$$\hat{\zeta}'^{\alpha\mu}(x) = \hat{\zeta}^{\alpha\mu}(x) - \frac{\partial\langle\hat{\zeta}^{\alpha\mu}\rangle}{\partial(\langle\hat{\zeta}^{\beta\lambda}\rangle u_\lambda)}\hat{\zeta}^{\beta\mu'}(x)u_{\mu'}, \quad u_\mu \equiv \frac{Y_\mu}{(-Y^2)^{1/2}},$$

$$\hat{h}'^\mu(x) = u^\mu\left\{[H, \hat{\varphi}(x)] + \frac{\partial p^\nu u_\nu}{\partial(\langle\hat{\zeta}^{\alpha\nu'}\rangle u_{\nu'})}\hat{\zeta}^{\alpha\lambda}(x)u_\lambda\right\}.$$

[Here $H \equiv u_\mu(\hat{\mathscr{P}}^\mu - p^\mu \hat{Q})$ and $\langle \ldots \rangle \equiv \mathrm{Sp}\, w\ldots$, where w has the form (3.18)]. We can also represent the kinetic coefficients I in the form of averages of the type of (3.23).

We note that the presented expressions (3.24) of the dissipative fluxes are more general as compared with the phenomenological approach.[30] The number of invariants that determine the kinetic coefficients in our treatment is 35: 1 for $I(\hat{h}'^\lambda, \hat{h}'^\nu)$; 2 for $I(\hat{h}'^\mu, \hat{j}'^\nu)$; 4 each for $I(\hat{h}'^\lambda, \hat{t}'^{\mu\nu}) I(\hat{j}'^\mu, \hat{j}'^\nu)$; 10 for $I(\hat{t}'^{\mu\nu} \hat{j}'^\lambda)$, and 14 for $I(\hat{t}'^{\mu\nu}, \hat{t}'^{\lambda\rho})$. Upon comparing these coefficients with the results of Ref. 30, we see that the latter lack the coefficients of the type $I(\hat{h}'^\nu, \hat{\zeta}'^{\alpha\mu})$.

In taking the nonrelativistic limit we must take account of the fact that the relativistic expressions for the operators $\hat{t}_{\mu\nu}$ and $\hat{j}_\mu$ are associated as $c \to \infty$ with the nonrelativistic operators for the density and the flux density by the relationships

$$\hat{j}_0 \to \hat{n}, \quad \hat{j}_k \to \frac{1}{m}\hat{\pi}_k,$$

$$\hat{t}_{00} \to mc^2 \hat{n} + \hat{\varepsilon}, \quad \hat{t}_{0k} \to c\hat{\pi}_k + \hat{q}_k, \quad \hat{t}_{ik} \to \hat{t}_{ik}.$$

(Here we have taken account of the symmetric character of the relativistic energy-momentum tensor.) Upon using these formulas and taking account of the fact that $I(\ldots,\zeta_\beta) = 0$, we can show that the relativistic equations of superfluid hydrodynamics transform into the equations of hydrodynamics of Galilean-invariant theory.

3.4. Solutions of quantum liquids

Let the solution of quantum liquids being studied contain $a = 1,2,\ldots,a \equiv \{a\}$ different components. For the sake of definiteness we shall assume that the components of the liquid $1, \ldots, s \equiv \{s\}$ exist in the superfluid state, while the remaining $s+1,\ldots,a \equiv \{n\}$ exist in the normal (nondegenerate) state. The mixture of liquids contains s order parameters $\langle \psi_s \rangle = \eta_s \exp(i\varphi_s)$. For simplicity of presentation we assume that all the particles of the liquid are bosons. The equilibrium statistical operator of the solution has the form

$$w(t) = \exp\left[\Omega_\nu - Y_\alpha \hat{\gamma}_\alpha - \nu Y_0 \sum_s \int d^3x \,(\psi_s(x) e^{-i\varphi_s(x,t)} + \text{H.c.})\right]$$

and has the following spatial, temporal, and phase symmetry

$$\left[w, \hat{\mathcal{P}} - \sum_s{}' \mathbf{p}_s \hat{N}_s\right] = 0, \quad [w, \hat{N}_n] = 0,$$
$$\left[w, \mathcal{H} + \sum_s \overset{0}{p}_s \hat{N}_s\right] = 0, \quad \overset{0}{p}_s = (Y_s + \mathbf{Y}\mathbf{p}_s) Y_0^{-1}. \tag{3.25}$$

Equation (3.25) implies the structure of the asymptotic phase $\varphi_s(\mathbf{x},t) = \mathbf{p}_s\mathbf{x} + \overset{0}{p}_s t + \varphi_s$. Thus the mixture of superfluid and normal liquids being studied is characterized by the thermodynamic forces Y_α ($\alpha = 0, k, a$), the superfluid momenta $\mathbf{p}_s$ and phases φ_s.

Upon employing the method of Sec. 2.3, we can easily show that the equilibrium averages of the densities of the additive integrals of motion and the flux densities corresponding to them can also be expressed in terms of thermodynamic potential ω:

$$\zeta_\alpha = \frac{\partial \omega}{\partial Y_\alpha}, \quad \zeta_{\alpha k} = -\frac{\partial}{\partial Y_\alpha} \frac{\omega Y_k}{Y_0} + \sum_s \frac{\partial \omega}{\partial p_{sk}} \frac{\partial \overset{0}{p}_s}{\partial Y_\alpha}. \tag{3.26}$$

Hence we obtain the thermodynamic equation

$$d\omega = \zeta_\alpha d Y_\alpha + \sum_s (Y_0 j_{sk} + Y_k n_s) d p_{sk} \tag{3.27}$$

($j_{sk} \equiv \zeta_{sk}$), which has the meaning of the second law of thermodynamics for a solution of quantum liquids. The thermodynamic potential ω contains $2 + a + s + [s(s+1)/2]$ variables

$$\omega(Y_\alpha, \mathbf{p}_s) = \omega(Y_0, \mathbf{Y}^2, Y_a, \mathbf{Y}\mathbf{p}_s, \mathbf{p}_s\mathbf{p}_{s'}).$$

To trace the interrelation of the formulas (3.26) with the expressions for the fluxes of the physical quantities adopted in the "two-liquid" formulation, let us introduce the quantities

$$\rho_n \equiv -2Y_0 \frac{\partial \omega}{\partial \mathbf{Y}^2}, \quad \rho_{ss'} \equiv \frac{\overset{*}{m}_s \overset{*}{m}_{s'}}{Y_0} \frac{\partial \omega}{\partial (\mathbf{p}_s \mathbf{p}_{s'})},$$

where the thermodynamic "mass" m_s is defined by the equation

$$m_s \frac{\partial \omega}{\partial (\mathbf{Y}\mathbf{p}_s)} = \sum_{s'} \rho_{ss'}.$$

With account taken of these definitions, the fluxes $\zeta_{\alpha k}$ acquire the form

$$j_{ak} \equiv \zeta_{ak} = -\frac{Y_k}{Y_0}\left(n_a - \sum_{s,s'} \delta_{as} \frac{\rho_{ss'}}{\overset{*}{m}_s}\right) + \sum_{s,s'} \delta_{as}\rho_{ss'} \frac{p_{s'k}}{\overset{*}{m}_s \overset{*}{m}_{s'}},$$
$$t_{ik} = -\frac{\omega}{Y_0}\delta_{ik} + \rho_n \frac{Y_k Y_i}{Y_0^2} + \sum_{s,s'} \rho_{ss'} \frac{p_{si} p_{s'k}}{\overset{*}{m}_s \overset{*}{m}_{s'}},$$
$$q_k = -\frac{Y_k}{Y_0}\left(-\frac{\omega}{Y_0} + \varepsilon + \sum_{s,s'} \frac{\overset{0}{p}_s \rho_{ss'}}{\overset{*}{m}_s}\right) - \sum_{s,s'} \frac{\rho_{ss'}}{\overset{*}{m}_s \overset{*}{m}_{s'}} \overset{0}{p}_s p_{s'k}.$$

We see that ρ_n has the meaning of the "mass" density of the normal component of the liquid, the matrix ρ_{ss}, has the meaning of the superfluid density associated with the interaction of the particles of the components s and s'. We can interpret the quantity m_s as the effective "mass" of a particle of the s component. We see from the presented formulas that the normal liquids in solution with superfluid ones participate only in the normal motion under any change in the concentration that does not lead to a phase transition. Moreover, we see within the framework of the approach that we have developed that the existence of nondiagonal elements $\rho_{ss'} \neq 0$ ($s \neq s'$) leads to the effect of entraining each of the superfluid motions of the other superfluid components of the solution, as was first noted in Ref. 31.

If the studied solution of quantum liquids possesses Galilean invariance, we can easily show that the thermodynamic potential ω is a function of the following variables:

$$\omega(Y_0, \mathbf{Y}^2, Y_a, \mathbf{Y}\mathbf{p}_s, \mathbf{p}_s\mathbf{p}_{s'}) = \omega\left(Y_0, 0, Y_a - \frac{m_a \mathbf{Y}^2}{2Y_0}, 0, \left(\mathbf{p}_s + \frac{\mathbf{Y} m_s}{Y_0}\right)\left(\mathbf{p}_{s'} + \frac{\mathbf{Y} m_{s'}}{Y_0}\right)\right).$$

Here m_a is the mass of a particle of the a component. This relationship leads to $s + 1$ relations among the introduced quantities ρ_n, $\rho_{ss'}$, and $\overset{*}{m}_s$:

$$m_s = \overset{*}{m}_s, \quad \sum_a n_a m_a = \rho_n + \sum_{s,s'} \rho_{ss'}. \tag{3.28}$$

Now let us study systems invariant with respect to Lorentz transformations. In this case the relativistic equilibrium statistical operator w has the form

$$w = \exp\left[V Y_0 \omega' - Y_\mu \hat{\mathcal{P}}^\mu - \sum_n Y_n \hat{Q}_n - \sum_s Y_s \hat{Q}_s - \nu Y_\mu \int_\sigma d\sigma^\mu \sum_s \{\psi_s(x) \exp[-i(p_{s\nu}x^\nu + \varphi_s)] + \text{э. c.}\}\right].$$

Here $Y_\mu \equiv (Y_0, Y_k)$, $p_{s\mu}$ $\overset{(0)}{p}_s, p_{sk}$) are 4-vectors, with $Y_s = -Y^\mu p_{s\mu}$. In terms of the relativistically invariant thermodynamic potential $\omega' = \omega'(Y^2, p^2, Y_\mu p_s^\mu, Y_n)$, which depends on the independent variables $Y_\mu, p_{s\mu}$, and Y_n, the energy-momentum tensor $t^{\mu\nu}$ and the 4-flux j_a^μ, we can write the a component in the form

$$t^{\mu\nu} = -\frac{\partial Y^\nu \omega'}{\partial Y_\mu} + \sum_s p_s^\mu \frac{\partial \omega'}{\partial p_{s\nu}},$$
$$j_a^\mu = \sum_s \delta_{as} \frac{\partial \omega'}{\partial p_{s\mu}} - \sum_n \delta_{an} Y^\mu \frac{\partial \omega'}{\partial Y_n}. \tag{3.29}$$

Let us introduce the 4-flux of the entropy

$$s^\mu = -Y^\mu \left(Y_\nu \frac{\partial \omega'}{\partial Y_\nu} + \sum_n Y_n \frac{\partial \omega'}{\partial Y_n}\right). \tag{3.30}$$

The thermodynamic relationship (3.27) with account taken of the formulas (3.29) and (3.30) for a relativistically invariant solution of quantum liquids is reduced to the form

$$d s^\mu = \sum_a Y_a d j_a^\mu + Y_\nu d t^{\nu\mu} - \sum_s (Y^\nu j_s^\mu - Y^\mu j_s^\nu) d p_{s\nu}.$$

The method of reduced description developed in Sec. 3 can be easily generalized for deriving the equations of hydrodynamics of the studied mixtures of quantum liquids. The parameters of the reduced description are the densities of the additive integrals of motion ζ_α and φ_s—the phase of the order parameter of the s component of the solution. The equations of hydrodynamics of such systems have the form

$$\dot{\zeta}_\alpha = -\nabla_k \overset{(0)}{\zeta}_{\alpha k}, \quad \dot{\mathbf{p}}_s = \nabla \dot{\varphi}_s = \nabla \overset{0}{p}_s. \tag{3.31}$$

The fluxes $\overset{(0)}{\zeta}_{\alpha k}$ and $\overset{0}{p}_s$ are determined by Eqs. (3.25) and (3.26) and describe an approximation corresponding to the ideal hydrodynamics of the solution of liquids. The dissipative approximation, which we shall not treat here, was studied in a microscopic approach in Ref. 20. In the case in which the solution consists of superfluid and normal liquids, the equations of hydrodynamics (3.31) with account taken of (3.28) coincide with the phenomenological equations,[6] while in the case of a solution of two superfluid liquids they coincide with the equations of Refs. 31 and 32.

4. THE GREEN'S FUNCTION

4.1. Definition and properties of the Green's function

For superfluid systems the Gibbs operator w of (3.8) commutes with the operators $\hat{H}$ and $\vec{\mathbf{P}}$ (but does not commute with the operators $\mathscr{H}$ and $\hat{\vec{\mathscr{P}}}$). Therefore we can naturally give the following definition of the translationally invariant (with respect to the coordinates and the time) retarded (+) and advanced (−) Green's functions:

$$G^{\pm}_{ab}(\mathrm{x},\ t) = \mp i\theta(\pm t)\,\mathrm{Sp}\,w\,[\hat{a}(\mathrm{x},\ t),\ \hat{b}(0)]. \tag{4.1}$$

Here we have

$$\hat{a}(\mathrm{x},\ t) = \exp[i(Ht - \hat{P}_k x_k)]\,\hat{a}(0)\exp[-i(Ht - \hat{P}_i x_i)]. \tag{4.2}$$

The Green's functions that we have introduced determine the linear response of the system to an external perturbation. Actually, let the system exist as $t\to -\infty$ in a state of statistical equilibrium that is described by the statistical operator w of (3.8). At some instant t_0 of time an external field is turned on, so that the Hamiltonian of the system becomes the operator $\mathscr{H}(t) = \mathscr{H} + V(t)$, and correspondingly, the von Neumann equation acquires the form

$$i\frac{\partial\rho(t)}{\partial t} = [\mathscr{H} + V(t),\ \rho(t)]. \tag{4.3}$$

Upon introducing the following operator instead of $\rho(t)$:

$$\tilde{\rho}(t) = e^{-ip_0\hat{N}t}\rho(t)\,e^{ip_0\hat{N}t}, \tag{4.4}$$

we obtain for it the equation

$$i\frac{\partial\tilde{\rho}(t)}{\partial t} = [H + \tilde{V}(t),\ \tilde{\rho}(t)], \tag{4.5}$$

where we have

$$\tilde{V}(t) = e^{-ip_0\hat{N}t}V(t)\,e^{ip_0\hat{N}t} = \int d^3x\,\xi(\mathrm{x},\ t)\,\hat{b}(\mathrm{x}) \tag{4.6}$$

($\xi(\mathbf{x},t)$ − c is the numerical external field and $\hat{b}(\mathbf{x})$ is a quasilocal operator that determines the interaction of the particles with the external field). Since the external field was absent as $t\to -\infty$ and the system existed in equilibrium, we have

$$\begin{aligned}\tilde{\rho}(-\infty) &= e^{-ip_0\hat{N}t}w(t)\,e^{ip_0\hat{N}t}\big|_{t\to-\infty}\\ &= \exp\Big[\Omega - Y_\alpha\hat{\gamma}_\alpha - vY_0\int d^3x(\psi(\mathrm{x})\,e^{-i(\mathbf{p}\mathrm{x}+\chi)} + \mathrm{H.c.})\Big] = w.\end{aligned} \tag{4.7}$$

(We stress that in the representation ~, in contrast to the original representation, the equilibrium statistical operator w does not depend on the time.)

If we consider the interaction of the system with the external field to be weak, we can expand $\tilde{\rho}(t)$ in a power series in $\tilde{V}(t)$: $\tilde{\rho}(t) = w + \rho'(t) + \ldots$. We shall define the mean value of the operator $\hat{a}(\mathbf{x}) \equiv \exp(-i\hat{\mathbf{P}}\mathbf{x})\hat{a}(0)\exp(i\hat{\mathbf{P}}\mathbf{x})$, apart from terms linear in the interaction $\tilde{V}(t)$ by the formula

$$a(\mathrm{x},\ t) = \mathrm{Sp}\,\tilde{\rho}(t)\,\hat{a}(\mathrm{x}) = \mathrm{Sp}\,w\hat{a}(\mathrm{x}) + a_\xi(\mathrm{x},\ t) + \ldots, \tag{4.8}$$

where we have

$$a_\xi(\mathrm{x},\ t) = \int_{-\infty}^{\infty} dt' \int d^3x'\,\xi(\mathrm{x}',\ t')\,G^{+}_{ab}(\mathrm{x}-\mathrm{x}',\ t-t') \tag{4.9}$$

and the Green's function $G^{+}_{ab}(\mathbf{x}-\mathbf{x}', t-t')$ is determined by Eq. (4.1) (the mean value $a(\mathbf{x},t)$ is easily related to the ordinary mean $\mathrm{Sp}\,\rho(t)\exp(-i\hat{\vec{\mathscr{P}}}\mathbf{x})\hat{a}(0)\times\exp(i\hat{\vec{\mathscr{P}}}\mathbf{x})$).

The invariance of the equations of quantum mechanics with respect to continuous transformations leads to certain restrictions on the structure of the Green's function.

Let us study as an example the case in which the equations of quantum mechanics are invariant with respect to Galilean transformations. In this case, according to (3.8), we have

$$U_{\mathbf{p}}w(Y_\alpha,\ \mathbf{p})\,U^{+}_{\mathbf{p}} = w(Y'_\alpha,\ 0).$$

Here the thermodynamic forces Y'_α are related to the thermodynamic forces Y_α by the formulas of (3.15). Further, upon taking account of (3.14) we obtain

$$G^{\pm}_{ab}(\mathrm{x},\ t;\ Y_\alpha,\ \mathbf{p}) = G^{\pm}_{\tilde{a}\tilde{b}}\Big(\mathbf{x} - \frac{\mathbf{p}}{m}t,\ t;\ Y'_\alpha,\ 0\Big),$$

where $\tilde{a}(0)\equiv U_{\mathbf{p}}\hat{a}(0)U^{+}_{\mathbf{p}}, \tilde{b}(0)\equiv U_{\mathbf{p}}\hat{b}(0)U^{+}_{\mathbf{p}}$.

Now let us study the case in which the system possesses the property of relativistic invariance. As before, the Green's function is determined by Eq. (4.1):

$$G^{\pm}_{ab}(x) = \mp i\theta(\pm x^0)\,\mathrm{Sp}\,w(Y_\mu,\ p_\mu,\ \varphi)\,[\hat{a}(x),\ \hat{b}(0)], \tag{4.10}$$

where we have

$$\hat{a}(x) = \exp[-i(\hat{\mathscr{P}}^\mu - p^\mu\hat{Q})\,x_\mu]\,\hat{a}(0)\exp[i(\hat{\mathscr{P}}^\mu - p^\mu\hat{Q})\,x_\mu]$$

and the statistical operator $w(Y_\mu,p_\mu,\varphi)$ is determined by Eq. (3.18). Since, according to (3.18), ψ is a scalar field, we have

$$U_a\psi(0)\,U^{+}_a = \psi(0).$$

Therefore, upon taking account of (3.17), we find

$$G^{\pm}_{ab}(x_\mu,\ Y_\mu,\ p_\mu) = G^{\pm}_{\tilde{a}\tilde{b}}(x'_\mu,\ Y'_\mu,\ p'_\mu).$$

Here $\tilde{a}(0)\equiv u_a\hat{a}(0)U^{+}_a, \tilde{b}(0) = U_{ab}(0)U^{+}_a$, and the primed quantities are related to the unprimed quantities by the formulas $x'_\mu = a^\nu_\mu x_\nu, Y'_\mu = Y_\nu a^\nu_\mu$, and $p''_\mu = p_\nu a^\nu_\mu$.

4.2. The hydrodynamics of a superfluid liquid in external fields

In this section we shall study the influence of quite arbitrary weak, slowly varying external fields on the evolution of the system. To solve this problem we shall turn to the equation of motion (4.3) for the statistical operator $\rho(t)$. Just as in Sec. 4.1, we shall assume that the statistical operator $\tilde{\rho}(t)$ (cf. (4.4)) satisfies the asymptotic relationship (4.7). (For simplicity we shall study states of equilibrium such that $\chi = 0$). In the presence of a weak external field, provided only that its frequency is small in comparison with τ_r^{-1}, the statistical operator $\tilde{\rho}(t)$ for $t \gg \tau_r$ will depend on the time not only via $\zeta_\alpha(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)$ (see Sec. 3.1), but also via the external field $\xi(\mathbf{x},t)$ and all its time derivatives $\dot{\xi}(\mathbf{x},t)$,

$\ddot{\xi}(\mathbf{x},t),...$

$$\tilde{\rho}(t) \xrightarrow[t\gg\tau_r]{} \tilde{\rho}(\zeta_\alpha(t),\varphi(t);\xi(t),\dot{\xi}(t),\ldots)\equiv\tilde{\rho}(\zeta_\alpha(t),\varphi(t);t). \quad (4.11)$$

Here we have

$$\operatorname{Sp}\tilde{\rho}(\zeta_\beta(t),\varphi(t);t)\hat{\zeta}_\alpha(\mathbf{x})=\zeta_\alpha(\mathbf{x},t),$$
$$\operatorname{Im}\ln\operatorname{Sp}\tilde{\rho}(\zeta_\beta(t),\varphi(t);t)\psi(\mathbf{x})=\varphi(\mathbf{x},t). \quad (4.12)$$

(Here $\varphi(\mathbf{x},t)$ is the phase of $\psi(\mathbf{x})$ in the state $\tilde{\rho}(\zeta,\varphi;t)$). We stress that in this formula the functional arguments $\zeta_\beta, \varphi, \xi, \dot{\xi},...$, considered as functions of $\mathbf{x}$, must be assumed independent.

Upon comparing Eq. (4.11) with (3.1), with $\xi=\dot{\xi}=...=0$, we have

$$\tilde{\rho}(\zeta_\alpha(t),\varphi(t);0,0,\ldots)=\sigma(\zeta_\alpha(t),\varphi(t)). \quad (4.13)$$

Let us linearize the asymptotic relationship (4.11) near the state of (4.7), while assuming that $\tilde{\rho}(t)=w+\tilde{\rho}'(t)$. Upon taking account of (4.13), we obtain

$$\tilde{\rho}(t) \xrightarrow[t\gg\tau_r]{} \sigma'(\zeta'_\alpha(t),\varphi'(t))+\rho(\xi(t))+\ldots, \quad (4.14)$$

where

$$\sigma'(\zeta'(t),\varphi'(t))=\int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\zeta'_\alpha(\mathbf{x},t)+\hat{\sigma}_\varphi(\mathbf{x})\varphi'(\mathbf{x},t)\}, \quad (4.15)$$

$$\hat{\sigma}_\alpha(\mathbf{x})\equiv\frac{\delta\sigma(\zeta,\varphi)}{\delta\zeta_\alpha(\mathbf{x})}\bigg|_{\zeta=\bar{\zeta},\varphi=\mathbf{px}'},\quad \hat{\sigma}_\varphi(\mathbf{x})=\frac{\delta\sigma(\zeta,\varphi)}{\delta\varphi(\mathbf{x})}\bigg|_{\zeta=\bar{\zeta},\varphi=\mathbf{px}'}.$$

Further $\zeta'_\alpha(\mathbf{x},t), \varphi'(\mathbf{x},t)$ are the deviations of the parameters $\zeta_\alpha(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)$ from the equilibrium values, i.e., from $\bar{\zeta}_\alpha=\operatorname{Sp} w\hat{\zeta}_\alpha$ and $\bar{\varphi}=\operatorname{Im}\ln\operatorname{Sp} w\psi(\mathbf{x})=\mathbf{px}$, respectively $(\zeta'_\alpha(\mathbf{x},t)=\zeta_\alpha(\mathbf{x},t)-\bar{\zeta}_\alpha, \varphi'(\mathbf{x},t)=\varphi(\mathbf{x},t)-\bar{\varphi})$. In Eq. (4.14) $\rho(\xi(t))\equiv\rho(\xi(t),\dot{\xi}(t),...)$ is the deviation linear in ξ of the statistical operator $\tilde{\rho}(\zeta_\alpha,\varphi;t)$ from w associated with the explicit dependence of $\tilde{\rho}(\zeta_\alpha,\varphi;t)$ on the field $\xi(\mathbf{x},t)$. If the only reason for nonequilibrium is the external field, then the quantities $\zeta'_\alpha(\mathbf{x},t)$ and $\varphi'(\mathbf{x},t)$ amount to linear functionals of $\xi(\mathbf{x},t), \dot{\xi}(\mathbf{x},t),...$.

Upon taking account of (3.2) we have

$$e^{i\hat{\mathbf{P}}\mathbf{y}}\hat{\sigma}_\alpha(\mathbf{x})e^{-i\hat{\mathbf{P}}\mathbf{y}}=\hat{\sigma}_\alpha(\mathbf{x}-\mathbf{y}),\quad e^{i\hat{\mathbf{P}}\mathbf{y}}\hat{\sigma}_\varphi(\mathbf{x})e^{-i\hat{\mathbf{P}}\mathbf{y}}=\hat{\sigma}_\varphi(\mathbf{x}-\mathbf{y}),$$
$$e^{i\hat{N}\varphi}\hat{\sigma}_\alpha(\mathbf{x})e^{-i\hat{N}\varphi}=\hat{\sigma}_\alpha(\mathbf{x}),\quad e^{i\hat{N}\varphi'}\hat{\sigma}_\varphi(\mathbf{x})e^{-i\hat{N}\varphi'}=e^{i\varphi'}\hat{\sigma}_\varphi(\mathbf{x}).$$

These formulas state that, when $\mathbf{p}\neq 0$, the operator $\hat{\mathbf{P}}$ (rather than the operator $\vec{\mathscr{P}}$) can be expediently interpreted as the translation operator.

Since the phase $\bar{\varphi}$ in the state w equals $\mathbf{px}$, we can represent the phase in the state $w+\sigma'(\zeta'(t),\varphi'(t))$ in the form

$$\varphi(\mathbf{x},t)=\operatorname{Sp}\sigma'(\zeta'(t),\varphi'(t))\hat{\varphi}(\mathbf{x})+\mathbf{px}, \quad (4.16)$$

where

$$\hat{\varphi}(\mathbf{x})=\frac{i}{2\eta}(\psi^+(\mathbf{x})e^{i\mathbf{px}}-\psi(\mathbf{x})e^{-i\mathbf{px}})\equiv e^{-i\hat{\mathbf{P}}\mathbf{x}}\hat{\varphi}(0)e^{i\hat{\mathbf{P}}\mathbf{x}}. \quad (4.17)$$

(Here $\eta=|\operatorname{Sp} w\psi|$; see (2.14)). We shall call the operator $\hat{\varphi}(\mathbf{x})$ the phase operator, upon noting that, according to the definition of $\sigma'(\zeta',\varphi')$, we have

$$\zeta'_\alpha(\mathbf{x},t)=\operatorname{Sp}\sigma'(\zeta'(t),\varphi'(t))\hat{\zeta}_\alpha(\mathbf{x}),$$
$$\varphi'(\mathbf{x},t)=\operatorname{Sp}\sigma'(\zeta'(t),\varphi'(t))\hat{\varphi}(\mathbf{x}) \quad (4.18)$$

and using Eq. (4.5), we find the linearized equations of the hydrodynamics of a superfluid liquid in the presence of external fields

$$\dot{\zeta}'_\alpha(\mathbf{x},t)-L_\alpha(\mathbf{x};t)=\eta_\alpha(\mathbf{x},t),$$
$$\dot{\varphi}'(\mathbf{x},t)-L_\varphi(\mathbf{x},t)=\eta_\varphi(\mathbf{x};t), \quad (4.19)$$

where we have

$$L_\alpha(\mathbf{x};t)=i\operatorname{Sp}\sigma'(\zeta'(t),\varphi'(t))[H,\hat{\zeta}_\alpha(\mathbf{x})],$$
$$L_\varphi(\mathbf{x};t)=i\operatorname{Sp}\sigma'(\zeta'(t),\varphi(t))[H,\hat{\varphi}(\mathbf{x})] \quad (4.20)$$

and the "sources" $\eta_\alpha(\mathbf{x};t)$ and $\eta_{\rm ph}(\mathbf{x};t)$ are determined by the formulas

$$\eta_\alpha(\mathbf{x};t)=i\operatorname{Sp} w[\tilde{V}(\xi(t)),\hat{\zeta}_\alpha(\mathbf{x})]+i\operatorname{Sp}\rho(\xi(t))[H,\hat{\zeta}_\alpha(\mathbf{x})],$$
$$\eta_\varphi(\mathbf{x};t)=i\operatorname{Sp} w[\tilde{V}(\xi(t)),\hat{\varphi}(\mathbf{x})]+i\operatorname{Sp}\rho(\xi(t))[H,\hat{\varphi}(\mathbf{x})]. \quad (4.21)$$

(Here $\eta_\alpha(\mathbf{x};t)\equiv\eta_\alpha(\mathbf{x};\xi(t),\dot{\xi}(t),...)$, $\eta_{\rm ph}(\mathbf{x};t)\equiv\eta_{\rm ph}(\mathbf{x};\xi(t),\dot{\xi}(t),...)$ are unknown linear functionals of $\xi(t),\dot{\xi}(t),....$)

Upon taking account of (4.12–4.14) and (4.18), we have

$$\operatorname{Sp}\rho(\xi(t))\hat{\zeta}_\alpha(\mathbf{x})=0,\quad \operatorname{Sp}\rho(\xi(t))\hat{\varphi}(\mathbf{x})=0 \quad (4.22)$$

We obtain the following from Eq. (4.5) with account taken of (4.19) and the expansion (4.14):

$$i\int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})(L_\alpha(\mathbf{x};t)+\eta_\alpha(\mathbf{x};t))+\hat{\sigma}_\varphi(\mathbf{x})(L_\varphi(\mathbf{x};t)+\eta_\varphi(\mathbf{x};t))\}$$
$$+i\frac{\partial\rho(\xi(t))}{\partial t}=[H,\sigma'(\zeta'(t),\varphi'(t))]+[H,\rho(\xi(t))]+[\tilde{V}(\xi(t)),w]. \quad (4.23)$$

We can rewrite this equation by using Eq. (3.3) in the form

$$i\int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\eta_\alpha(\mathbf{x};t)+\hat{\sigma}_\varphi(\mathbf{x})\eta_\varphi(\mathbf{x};t)\}+i\frac{\partial\rho(\xi(t))}{\partial t}=[H,\rho(\xi(t))]+[\tilde{V}(\xi(t)),w]. \quad (4.24)$$

As $\tau\to\infty$, according to (3.1), the following formula holds

$$e^{-iH\tau}\rho(\xi(t))e^{iH\tau}\xrightarrow[\tau\to\infty]{}\sigma'(\underset{\sim}{\zeta}(\tau;t),\underset{\sim}{\varphi}(\tau;t))=e^{-iH\tau}\sigma'(\underset{\sim}{\zeta}(0;t),\underset{\sim}{\varphi}(0;t))e^{iH\tau},$$
$$\sigma'(\underset{\sim}{\zeta}(0;t),\underset{\sim}{\varphi}(0;t))=\int d^3x\,\{\hat{\sigma}_\alpha(\mathbf{x})\underset{\sim}{\zeta}_\alpha(\mathbf{x},0;t)+\hat{\sigma}_\varphi(\mathbf{x})\underset{\sim}{\varphi}(\mathbf{x},0;t)\}.$$

This is because the evolution in τ occurs with the Hamiltonian H, which does not contain the external field. [The parameters $\underset{\sim}{\zeta}_\alpha(\mathbf{x},\tau;t)$, $\underset{\sim}{\varphi}(\mathbf{x},\tau;t)$ satisfy the equations in the variables $\mathbf{x}$ and τ of linearized hydrodynamics with the initial conditions $\underset{\sim}{\zeta}_\alpha(\mathbf{x},0;t)\equiv\underset{\sim}{\zeta}_\alpha(\mathbf{x};t)$, $\underset{\sim}{\varphi}(\mathbf{x},0;t)=\underset{\sim}{\varphi}(\mathbf{x};t)$. The latter are linear functionals of $\xi(\mathbf{x},t),\dot{\xi}(\mathbf{x},t),\ddot{\xi}(\mathbf{x},t),....$ That is, we have $\underset{\sim}{\zeta}_\alpha(\mathbf{x};t)=\underset{\sim}{\zeta}_\alpha(\mathbf{x};\xi(t),\dot{\xi}(t),...)$, $\underset{\sim}{\varphi}(\mathbf{x};t)=\underset{\sim}{\varphi}(\mathbf{x};\xi(t),\dot{\xi}(t),...$, while t is a parameter.] Upon taking account of this limit relationship we obtain from (4.24)

$$\rho(\xi(t)) = \sigma'(\underset{\sim}{\zeta}(\mathbf{x}';t), \underset{\sim}{\varphi}(\mathbf{x}';t)) - \int_0^\infty d\tau e^{-iH\tau}\{i[H, \sigma'(\underset{\sim}{\zeta}(\mathbf{x}';t), \underset{\sim}{\varphi}(\mathbf{x}';t))] + i[\tilde{V}(\xi(t)), w] + \dot{\rho}(\xi(t)) + \int d^3x(\hat{\sigma}_\alpha(\mathbf{x})\eta_\alpha(\mathbf{x};t) + \hat{\sigma}_\varphi(\mathbf{x})\eta_\varphi(\mathbf{x};t))\} e^{iH\tau}. \tag{4.25}$$

The parameters $\zeta_\alpha(\mathbf{x};t)$ and $\varphi(\mathbf{x};t)$ are determined from Eq. (4.22). One can apply to Eq. (4.25) the standard iteration procedure in the spatial and time derivatives of the field $\xi(\mathbf{x},t)$.

To find the "sources" in perturbation theory in the gradients of the external field $\xi(\mathbf{x},t)$, we must find the statistical operator $\rho(\xi(t))$ also in perturbation theory in the inhomogeneities of the external field. According to Eq. (4.25), to do this we must know the expansion of the statistical operator $\sigma'(\zeta(\mathbf{x};t), \varphi(\mathbf{x};t))$ in a series in the gradients of the parameters of the reduced description. In the zero-order approximation, according to (4.15), we have

$$\overset{(0)}{\sigma}{}'(\underset{\sim}{\zeta}(t), \underset{\sim}{\varphi}(t)) = \overset{(0)}{\underset{\sim}{\zeta}}_\alpha(\mathbf{x},t)\int d^3x'\hat{\sigma}_\alpha(\mathbf{x}') + \overset{(0)}{\underset{\sim}{\varphi}}(\mathbf{x},t)\int d^3x'\hat{\sigma}_\varphi(\mathbf{x}'), \tag{4.26}$$

where we have $\overset{(0)}{\zeta}_\alpha(\mathbf{x};t) \sim \xi(\mathbf{x},t), \overset{(0)}{\varphi}(\mathbf{x},t) \sim \xi(\mathbf{x},t)$.

To determine the quantities $\int d^3x'\hat{\sigma}_\alpha(x')$, $\int d^3x'\hat{\sigma}_\varphi(\mathbf{x}')$ we must apply the ergodic relationship (3.8). In this relationship let the statistical operator ρ (such that $[\rho, \hat{\mathscr{P}}_k - (p_k + p_k')\hat{N}] = 0$ differ little from the equilibrium statistical operator w, $\rho = w + \rho'$. Then since $[w, \hat{P}_k] = 0$ (the statistical operator w corresponds to the superfluid momentum $\mathbf{p}$), we have

$$[\rho', \hat{P}_k] = p_k'[w, \hat{N}]. \tag{4.27}$$

In the zero-order approximation we evidently have $\chi = 0$ (the quantity $\mathbf{p}'$ is of the first order of smallness in the deviation from a state of equilibrium). Hence Eq. (3.8) in the linear approximation acquires the form

$$e^{-iHt}\rho' e^{iHt} \xrightarrow[t \gg \tau_r]{} \frac{\partial w}{\partial \zeta_\alpha}\zeta_\alpha' + \frac{\partial w}{\partial p_k}p_k' + \frac{\partial w}{\partial \varphi}(\chi' + p_0' t), \tag{4.28}$$

where

$$\zeta_\alpha' = \mathrm{Sp}\,\rho'\hat{\zeta}_\alpha, \quad p_0' = \zeta_\alpha'\frac{\partial p_0}{\partial \zeta_\alpha} + \mathbf{p}'\frac{\partial p_0}{\partial \mathbf{p}}, \quad \chi' = \mathrm{Sp}\,\rho'\hat{\varphi}(0) + \int_0^\infty d\tau(\mathrm{Sp}\,\dot{\rho}'(\tau)\hat{\varphi}(0) - p_0'). \tag{4.29}$$

In the variation of the relationship (3.8) we chose as the independent variables the quantities $\zeta_\alpha \approx \mathrm{Sp}\,w\hat{\zeta}_\alpha \equiv \zeta_\alpha(Y_\beta, \mathbf{p})$, rather than the quantities Y_α, $\mathbf{p}$, and χ, which are functionals of ρ, since we have $\mathrm{Sp}\,\rho\hat{\zeta}_\alpha = \mathrm{Sp}\,w\hat{\zeta}_\alpha$, owing to the fact that ζ_α is the density of the additive integrals of motion. Moreover, we have taken account of the fact that $p_0 = (Y_4 + Y\mathbf{p})/Y_0$ and have used Eq. (3.7) for the phase χ. We stress that the relationship (4.28) is valid for the initial statistical operators, which satisfy the condition (4.27).

Now let us turn to formula (4.14) for $\xi(\mathbf{x},t) = 0$. Upon choosing the initial statistical operator $\tilde{\rho}'(0)$, which coincides with the statistical operator ρ' in (4.28) and noting that in this case $\zeta_\alpha'(\mathbf{x},t) = \zeta_\alpha'$, owing to the conservation laws, does not depend on $\mathbf{x}$ and t, while the phase $\varphi'(\mathbf{x},t) = \mathbf{p}'\mathbf{x} + p_0't + \chi'$, we obtain

$$e^{-iHt}\rho' e^{iHt} \xrightarrow[t \gg \tau_r]{} \int d^3x\,[\hat{\sigma}_\alpha(\mathbf{x})\zeta_\alpha' + \hat{\sigma}_\varphi(\mathbf{x})(\mathbf{p}'\mathbf{x} + p_0't + \chi')].$$

Upon comparing this expression with (4.28) we find that

$$\frac{\partial w}{\partial \zeta_\alpha} = \int d^3x\,\sigma_\alpha(\mathbf{x}), \quad \frac{\partial w}{\partial \varphi} = \int d^3x\,\hat{\sigma}_\varphi(\mathbf{x}), \quad \frac{\partial w}{\partial p_k} = \int d^3x\,x_k\hat{\sigma}_\varphi(\mathbf{x}). \tag{4.30}$$

Now let ρ' be a rather arbitrary statistical operator. Then, upon neglecting the gradients $\zeta_\alpha'(\mathbf{x},t)$, but taking account of the phase gradients $\varphi'(\mathbf{x},t)$, we can represent the average $\mathrm{Sp}\exp(-iHt)\rho'\exp(iHt)\hat{a}(\mathbf{x})$ for $t \gg \tau_r$, with account taken of (4.30), in the form

$$\mathrm{Sp}\,e^{-iHt}\rho' e^{iHt}\hat{a}(\mathbf{x}) \xrightarrow[t \gg \tau_r]{} \zeta_\alpha'(\mathbf{x},t)\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha} + \varphi'(\mathbf{x},t)\frac{\partial\langle\hat{a}\rangle}{\partial\varphi} + \frac{\partial\varphi'(\mathbf{x},t)}{\partial x_k}\frac{\partial\langle\hat{a}\rangle}{\partial p_k}. \tag{4.31}$$

(Here $\hat{a}(\mathbf{x}) \equiv \exp(-i\hat{\mathbf{P}}\mathbf{x})\hat{a}(0)\exp(i\hat{\mathbf{P}}\mathbf{x})$. We shall use this relationship below in finding the low-frequency asymptotic behavior of the Green's function.

Upon taking account of (4.3), Eq. (4.26) implies that

$$\overset{(0)}{\sigma}{}'(\underset{\sim}{\zeta}(t), \underset{\sim}{\varphi}(t)) = \overset{(0)}{\underset{\sim}{\zeta}}_\alpha(\mathbf{x},t)\frac{\partial w}{\partial \zeta_\alpha} + \overset{(0)}{\underset{\sim}{\varphi}}(\mathbf{x},t)\frac{\partial w}{\partial \varphi}. \tag{4.32}$$

Thus, upon taking account of (4.32), the statistical operator $\rho^{(0)}(\xi(t))$ in the zero-order approximation in the gradients of the external field has the following structure:

$$\overset{(0)}{\rho}(\xi(t)) = \overset{(0)}{\underset{\sim}{\zeta}}_\alpha(\mathbf{x},t)\frac{\partial w}{\partial \zeta_\alpha} + \overset{(0)}{\underset{\sim}{\varphi}}(\mathbf{x},t)\frac{\partial w}{\partial \varphi} - \int_0^\infty d\tau e^{-iH\tau} \times \Big(i[\overset{(0)}{\tilde{V}}(\xi(t)), w] + i\Big[H, \overset{(0)}{\underset{\sim}{\zeta}}_\alpha(\mathbf{x},t)\frac{\partial w}{\partial \zeta_\alpha} + \overset{(0)}{\underset{\sim}{\varphi}}(\mathbf{x},t)\frac{\partial w}{\partial \varphi}\Big] + \frac{\partial w}{\partial \zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x};t) + \frac{\partial w}{\partial \varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x};t)\Big)e^{iH\tau},$$

Here we have $\overset{(0)}{V}(\xi(t)) = \xi(\mathbf{x},t)\int d^3x\hat{b}(\mathbf{x})$. This implies that $[\overset{(2)}{\rho}(\xi(t)), \hat{P}_k] = 0$. Then Eq. (4.21), which determines $\overset{(0)}{\eta}_\alpha(\mathbf{x};t)$ and $\overset{(0)}{\eta}_\varphi(\mathbf{x};t)$ in the zero-order approximation in the field inhomogeneities

$$\overset{(0)}{\eta}_\alpha(\mathbf{x};t) = i\,\mathrm{Sp}\,w[\overset{(0)}{\tilde{V}}(\xi(t)), \hat{\zeta}_\alpha(\mathbf{x})] + i\,\mathrm{Sp}\,\overset{(0)}{\rho}(\xi(t))[H, \hat{\zeta}_\alpha(\mathbf{x})],$$
$$\overset{(0)}{\eta}_\varphi(\mathbf{x};t) = i\,\mathrm{Sp}\,w[\overset{(0)}{\tilde{V}}(\xi(t)), \hat{\varphi}(\mathbf{x})] + i\,\mathrm{Sp}\,\overset{(0)}{\rho}(\xi(t))[H, \hat{\varphi}(\mathbf{x})], \tag{4.33}$$

and the fact that $[H, \hat{\zeta}_\alpha(\mathbf{x})] = [\hat{P}_k, \hat{\zeta}_{\alpha k}(\mathbf{x})]$ together imply that

$$\overset{(0)}{\eta}_\alpha(\mathbf{x};t) = i\,\mathrm{Sp}\,w[\overset{(0)}{\tilde{V}}(\xi(t)), \hat{\zeta}_\alpha(x)]. \tag{4.34}$$

Since the $\hat{\zeta}_\alpha(\mathbf{x})$ are the density operators of the additive integrals of motion and $\mathrm{Sp}\,\rho^{(0)}(\xi(t))\hat{\zeta}_\alpha(\mathbf{x}) = 0$, then, upon using (4.34), we obtain $\overset{(0)}{\zeta}_\alpha(\mathbf{x},t = \mathrm{Sp}\,\sigma'^{(0)}(\xi(t),\varphi(t))\hat{\zeta}_\alpha(\mathbf{x})$

$= 0$. Therefore $\overset{(2)}{\rho}(\xi(t))$ has the form

$$\overset{(0)}{\rho}(\xi(t)) = \overset{(0)}{\underset{\sim}{\varphi}}(\mathbf{x}, t)\frac{\partial w}{\partial \varphi} - \int_0^\infty d\tau e^{-iH\tau}\left(i[\overset{(0)}{V}(\xi(t)), w] + \frac{\partial w}{\partial \zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x}; t) + \frac{\partial w}{\partial \varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x}; t)\right)e^{iH\tau}. \quad (4.35)$$

(We have taken account of the fact that $[H,\partial w/\partial\varphi] = 0$.) The parameter $\overset{(0)}{\varphi}(\mathbf{x},t)$ is determined from the condition $\overset{(2)}{\rho}(\xi(t))\hat{\varphi})(\mathbf{x}) = 0$. Upon taking account of the fact that $[w,\hat{P}_k] = [w,H] = 0$, we find, according to (4.34) and (4.6):

$$\overset{(0)}{\eta}_\alpha(\mathbf{x}; t) = -i\xi(\mathbf{x}, t)\,\mathrm{Sp}\,w[\hat{N}, \hat{b}(0)]Y_0\frac{\partial p_0}{\partial Y_\alpha}. \quad (4.36)$$

To determine $\overset{(0)}{\eta}_\varphi(\mathbf{x};t)$ we must find $[\overset{(0)}{\rho}(\xi(t)),H]$. We obtain from (4.35)

$$[\overset{(0)}{\rho}(\xi(t)), H] = i\lim_{\tau\to\infty} e^{-iH\tau}\left(i[\overset{(0)}{V}(\xi(t)), w] + \frac{\partial w}{\partial\zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x}; t) + \frac{\partial w}{\partial\varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x}; t)\right)e^{iH\tau} - i\left\{i[\overset{(0)}{V}(\xi(t)), w] + \frac{\partial w}{\partial\zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x}; t) + \frac{\partial w}{\partial\varphi}\overset{(0)}{\eta}_\varphi(\mathbf{x}; t)\right\}. \quad (4.37)$$

Since $H = \mathscr{H} + p_0\hat{N}$ depends on ζ_α, $\mathbf{p}$, and $[w,H] = 0$, we have

$$e^{-iH\tau}\frac{\partial w}{\partial\zeta_\alpha}e^{iH\tau} = \frac{\partial w}{\partial\zeta_\alpha} + \tau\frac{\partial p_0}{\partial\zeta_\alpha}\frac{\partial w}{\partial\varphi}.$$

Therefore, noting that $[H,\partial w,\partial\varphi] = 0$, we shall rewrite Eq. (4.37) in the form

$$[\overset{(0)}{\rho}(\xi(t)), H] = -\lim_{\tau\to\infty}\left(e^{-iH\tau}[\overset{(0)}{V}(\xi(t)), w]e^{iH\tau} - i\tau\overset{(0)}{\eta}_\alpha(\mathbf{x}; t)\frac{\partial p_0}{\partial\zeta_\alpha}\frac{\partial w}{\partial\varphi}\right) + [\overset{(0)}{V}(\xi(t)), w].$$

This implies that the second of the formulas of (4.33) acquires the form

$$\overset{(0)}{\eta}_\varphi(\mathbf{x}; t) = -i\lim_{\tau\to\infty}\left(\mathrm{Sp}\,e^{-iH\tau}[\overset{(0)}{V}(\xi(t)), w]e^{iH\tau}\hat{\varphi}(\mathbf{x}) - i\tau\overset{(0)}{\eta}_\alpha(\mathbf{x}; t)\frac{\partial p_0}{\partial\zeta_\alpha}\right). \quad (4.38)$$

Upon using the ergodic relationship (4.28), we easily see that

$$\lim_{\tau\to\infty}\left\{ie^{-iH\tau}[\overset{(0)}{V}(\xi(t)), w]e^{iH\tau} + \tau\frac{\partial w}{\partial\varphi}\overset{(0)}{\eta}_\alpha(\mathbf{x}; t)\frac{\partial p_0}{\partial\zeta_\alpha}\right\} = -\frac{\partial w}{\partial\zeta_\alpha}\overset{(0)}{\eta}_\alpha(\mathbf{x}; t) - \frac{\partial w}{\partial\varphi}\chi'.$$

Here χ' is determined by Eq. (4.29), in which we have chosen the operator $i[w,\overset{(0)}{V}(\xi(t))]$ as the initial operator ρ'. Thus we have

$$\overset{(0)}{\eta}_\varphi(\mathbf{x}; t) = \chi'(\mathbf{x}, t). \quad (4.39)$$

If $[\hat{N},\hat{b}] = 0$, then we have $\overset{(0)}{\eta}_\alpha(\mathbf{x};t) = 0$ (cf. (4.36)). In this case we can simplify the expression (4.38) for $\overset{(0)}{\eta}_\varphi(\mathbf{x};t)$. To do this we again use the ergodic relationship (4.28), in which we have chosen as the initial statistical operator ρ' the quantity $[w, \int d^3x(\psi^+(\mathbf{x})e^{i\mathbf{px}} - \psi(\mathbf{x})e^{-i\mathbf{px}}$. Since this operator commutes with $\hat{P}_k$, then according to (4.38), (4.29), and (4.28), we obtain

$$\overset{(0)}{\eta}_\varphi(\mathbf{x}; t) = \frac{1}{2\eta}\xi(\mathbf{x}, t)\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_\alpha} \times \int d^3x'\,\mathrm{Sp}[w, \psi^+(\mathbf{x}')e^{i\mathbf{px}'} - \psi(\mathbf{x}')e^{-i\mathbf{px}'}]\hat{\zeta}_\alpha(0).$$

Upon taking account of the fact that $[w,H] = 0$ and $[w,\hat{P}_k] = 0$, we can easily transform $\overset{(0)}{\eta}_\varphi(\mathbf{x};t)$ to the form

$$\overset{(0)}{\eta}_\varphi(\mathbf{x}; t) = -\xi(\mathbf{x}, t)\left(\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_\mu}p^\mu + \frac{\partial\langle\hat{b}\rangle}{\partial\zeta_4}\right), \quad [\hat{N}, \hat{b}] = 0. \quad (4.40)$$

[Here we have used the relativistic notation, cf. (2.32).]

Since in the case being studied ($[\hat{N},\hat{b}] = 0$) the principal approximation for $\eta_\alpha(\mathbf{x};t)$ vanishes, we must find the quantities $\eta_\alpha(\mathbf{x};t)$ in the first approximation in the gradients of the external field. Upon using the formulas (4.6), (4.15), (4.21), (4.25), and (4.30), as well as the ergodic relationship (4.28), we obtain as a result the following expression for the "sources" $\overset{(1)}{\eta}_\alpha(\mathbf{x};t)$ (cf. Ref. 24):

$$\overset{(1)}{\eta}_\alpha(\mathbf{x}; t) = -\frac{\partial\xi(\mathbf{x}, t)}{\partial x_l}\left(\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_\beta}K_{l;\alpha\beta} + Y_0\frac{\partial p_0}{\partial Y_\alpha}\frac{\partial\langle\hat{b}\rangle}{\partial p_l}\bigg|_\zeta\right). \quad (4.41)$$

Here we have

$$K_{l;\alpha\beta} \equiv -i\int d^3x\,x_l\,\mathrm{Sp}\,w[\hat{\zeta}_\alpha(\mathbf{x}), \hat{\zeta}_\beta(0)] = K_{l;\beta\alpha}.$$

We can easily show that, if we take account of (2.2) and (2.28), we can represent the elements of the matrix $K_{l;\alpha\beta}$ in the following compact form:

$$K_{l;\mu\beta} = -Y_0\frac{\partial\zeta_{\mu l}}{\partial Y_\beta} - Y_l\frac{\partial\zeta_\mu}{\partial Y_\beta} + p^\mu\frac{\partial\zeta_\beta}{\partial p_l} \quad (\mu = 0, 1, 2, 3),$$
$$K_{l;4\beta} = \delta_{\beta 0}\zeta_{4l} + \delta_{\beta l}\zeta_4. \quad (4.42)$$

We call attention to the fact that the terms in the sources $\eta_\alpha(\mathbf{x};t)$ proportional to the first derivatives of the external field with respect to time $\partial\xi(\mathbf{x},t)/\partial t$ vanish.

Thus the formulas (4.36) and (4.39)–(4.41) yield in the principal approximation expressions for the "sources" $\eta_\alpha(\mathbf{x};t)$ and $\eta_\varphi(\mathbf{x};t)$ in the equations of hydrodynamics of a superfluid liquid in the case of an external field $\xi(\mathbf{x},t)$ that varies slowly in space and time.

4.3. Low-frequency asymptotic behavior of the Green's function

In this section we shall find the concrete structure of the Green's function $G^\pm_{ab}(\mathbf{k},\omega)$ in the region of small wave vectors $\mathbf{k}$ ($kl \ll 1$; l is the free-flight path) and frequencies ω ($\omega\tau_r \ll 1$, τ_r is the relaxation time). For this purpose we shall use the equations of hydrodynamics of the superfluid liquid in the form (3.20). In the presence of the "sources" $\eta_\alpha(\mathbf{x};t)$ and $\eta_\varphi(\mathbf{x};t)$, these equations have the form[24]

$$\frac{\partial t^{\mu\nu}(\mathbf{x},t)}{\partial x^{\nu}}=\eta^{\mu}(\mathbf{x};t),\quad \frac{\partial j^{\nu}(\mathbf{x},t)}{\partial x^{\nu}}=\eta^{4}(\mathbf{x};t),$$

$$\frac{\partial p^{\mu}(\mathbf{x},t)}{\partial x_{\nu}}-\frac{\partial p^{\nu}(\mathbf{x},t)}{\partial x_{\mu}}=\eta^{\mu\nu}(\mathbf{x};t). \tag{4.43}$$

The latter equation in (4.43) is a consequence of the equations

$$\dot{\varphi}(\mathbf{x},t)=p_0(\mathbf{x},t)+\eta_{\varphi}(\mathbf{x};t),\quad p_k(\mathbf{x},t)=\frac{\partial\varphi(\mathbf{x},t)}{\partial x_k}.$$

Hence we see that

$$\eta_{\mu\nu}(\mathbf{x};t)\equiv\frac{\partial\eta_{\varphi}(\mathbf{x};t)}{\partial x_{\lambda}}(g_{\mu 0}g_{\nu\lambda}-g_{\mu\lambda}g_{\nu 0})\ (\mu,\nu,\lambda=0,1,2,3). \tag{4.44}$$

Considering the "sources" to be small, we shall linearize Eq. (4.43) about the equilibrium state, choosing as the parameters describing the deviation from the equilibrium state the quantities $\delta Y_{\lambda}(\mathbf{x},t)=Y_{\lambda}(\mathbf{x},t)-\overline{Y}_{\lambda}$ [the deviation of the thermodynamic forces $Y_{\lambda}(\mathbf{x},t)$ from the equilibrium values $\overline{Y}_{\lambda}$], the quantity $\delta p_0(\mathbf{x},t)=p_0(\mathbf{x},t)-p_0$, and the phase $\delta\varphi(\mathbf{x},t)$ (the phase φ in the equilibrium state is assumed equal to zero). Then the equations of hydrodynamics (4.43) for the Fourier components of the corresponding quantities, when linearized about the equilibrium state with $\varphi=0$, have the form

$$ik_{\nu}\left(\frac{\partial t^{\mu\nu}}{\partial Y_{\lambda}}\delta Y_{\lambda}(k)+\frac{\partial t^{\mu\nu}}{\partial p_{\lambda}}\delta p_{\lambda}(k)\right)=\eta^{\mu}(k),$$

$$ik_{\nu}\left(\frac{\partial j^{\nu}}{\partial Y_{\lambda}}\delta Y_{\lambda}(k)+\frac{\partial j^{\nu}}{\partial p_{\lambda}}\delta p_{\lambda}(k)\right)=\eta^{4}(k), \tag{4.45}$$

$$k_{\nu}\delta p_{\mu}(k)-k_{\mu}\delta p_{\nu}(k)=\eta_{\varphi}(k)(k_{\nu}g_{\mu 0}-k_{\mu}g_{\nu 0}).$$

We shall write the solution of the last equation in the form

$$\delta p_{\nu}(k)=ik_{\nu}\delta\varphi(k)+g_{\nu 0}\eta_{\varphi}(k). \tag{4.46}$$

Upon taking account of (4.46), we find from (4.45) that

$$\delta Y_{\nu}(k)=iD^{-1}_{\nu\mu}[\overline{\eta}^{\mu}(k)-p^{\mu}\overline{\eta}^{4}(k)-(kY)a^{\mu}\delta\varphi(k)],$$

$$\delta\varphi(k)=-\frac{1}{\Delta}[\overline{\eta}^{4}(k)(1-a^{\lambda}D^{-1}_{\lambda\mu}p^{\mu})+a^{\lambda}D^{-1}_{\lambda\mu}\overline{\eta}^{\mu}(k)]. \tag{4.47}$$

Here we have

$$\overline{\eta}^{\mu}(k)\equiv\eta^{\mu}(k)+ik_{\nu}\frac{\partial t^{\mu\nu}}{\partial p_0}\eta_{\varphi}(k),$$

$$\overline{\eta}^{4}(k)\equiv\eta^{4}(k)+ik_{\nu}\frac{\partial j^{\nu}}{\partial p_0}\eta_{\varphi}(k), \tag{4.48}$$

$$\Delta(k)\equiv b-(kY)a^{\lambda}D^{-1}_{\lambda\mu}a^{\mu}, \tag{4.49}$$

$$D^{\mu\nu}\equiv\frac{\partial^2(kY)\omega'}{\partial Y_{\mu}\partial Y_{\nu}},\quad a^{\lambda}\equiv k_{\nu}\frac{\partial^2\omega'}{\partial p_{\nu}\partial Y_{\lambda}},\quad b\equiv k_{\nu}k_{\mu}\frac{\partial^2\omega'}{\partial p_{\nu}\partial p_{\mu}}. \tag{4.50}$$

Now let us study the problem of the concrete structure of the "sources" η^{μ}, η^{4}, and $\eta^{\mu\nu}$. As we saw in Sec. 4.2, their explicit form substantially depends on whether the operator $\hat{b}(\mathbf{x})$ entering into Eq. (4.6) commutes with the operator for the number of particles $\hat{N}$ or not. First let us examine the case in which $[\hat{N},\hat{b}]\neq 0$. According to (4.36) the Fourier components of the sources $\overset{(0)}{\eta}{}^{\mu}(k)$ and $\overset{(0)}{\eta}{}^{4}(k)$ are determined by the formulas

$$\overset{(0)}{\eta}{}^{\mu}(k)=\xi(k)\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}p^{\mu},\quad \overset{(0)}{\eta}{}^{4}(k)=\xi(k)\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}. \tag{4.51}$$

Since in the "sources" $\overline{\eta}^{\mu}(k)$ and $\overline{\eta}^{4}(k)$ the quantity $\eta_{\phi}(k)$ enters in combination with the coefficient k_{ν}, then in the principal approximation in k we have

$$\overset{(0)}{\overline{\eta}}{}^{\mu}(k)=\overset{(0)}{\eta}{}^{\mu}(k),\quad \overset{(0)}{\overline{\eta}}{}^{4}(k)=\overset{(0)}{\eta}{}^{4}(k). \tag{4.52}$$

If $[\hat{N},\hat{b}]=0$, then the "sources" η^{μ} and η^{4} vanish in the principal approximation in k and therefore we must find them in the next approximation in k. According to the formulas (4.40) and (4.41) the "sources" $\overset{(1)}{\eta}\,\eta^{\mu}(k)$ $\overset{(0)}{\eta}{}^{4}(k)$, and $\overset{(0)}{\eta}_{\varphi}(k)$ are determined by the formulas

$$\overset{(1)}{\eta}{}^{\mu}(k)=-ik_l\xi(k)\left(\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_{\beta}}K_{l;\mu\beta}+p^{\mu}\frac{\partial\langle\hat{b}\rangle}{\partial p_l}\right),$$

$$\overset{(1)}{\eta}{}^{4}(k)=-ik_l\xi(k)\left(\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_{\beta}}K_{l;4\beta}+\frac{\partial\langle\hat{b}\rangle}{\partial p_l}\right), \tag{4.53}$$

$$\overset{(0)}{\eta}_{\varphi}(k)=-\xi(k)\left(\frac{\partial\langle\hat{b}\rangle}{\partial\zeta^{\nu}}p^{\nu}+\frac{\partial\langle\hat{b}\rangle}{\partial\zeta^{4}}\right)=-\xi(k)\frac{\partial\langle\hat{b}\rangle}{\partial\zeta^{\alpha}}p^{\alpha}.$$

Here we have $p^{\alpha}\equiv(p^{\mu},1)$, $(\alpha=\mu,4)$. Therefore, upon using (4.48), (4.53), (4.42), and transforming from the derivatives $\partial\langle\hat{b}\rangle\partial\zeta_{\beta}$, $\partial\langle\hat{b}\rangle/\partial\mathbf{p}$ to the derivatives $\partial\langle\hat{b}\rangle/\partial Y_{\mu}$, $\partial\langle\hat{b}\rangle/\partial p_{\mu}$, we obtain the following expressions for the "sources" $\overset{(1)}{\eta}{}^{\mu}(k)$ and $\overset{(1)}{\eta}{}^{4}(k)$:

$$\overset{(1)}{\eta}{}^{\mu}(k)=i\xi(k)\left(\frac{\partial\langle\hat{b}\rangle}{\partial Y_{\nu}}A^{\mu}_{\nu}+\frac{\partial\langle\hat{b}\rangle}{\partial p_{\nu}}B^{\mu}_{\nu}\right),$$

$$\overset{(1)}{\eta}{}^{4}(k)=i\xi(k)\left(\frac{\partial\langle\hat{b}\rangle}{\partial Y_{\nu}}A_{\nu}+\frac{\partial\langle\hat{b}\rangle}{\partial p_{\nu}}B_{\nu}\right), \tag{4.54}$$

$$\overset{(0)}{\eta}_{\varphi}(k)=-\xi(k)p^{\beta}\left(\frac{\partial\langle\hat{b}\rangle}{\partial Y_{\nu}}\frac{\partial Y_{\nu}}{\partial\zeta^{\beta}}\bigg|_{\mathbf{p}}+\delta_{\nu 0}\frac{\partial\langle\hat{b}\rangle}{\partial p_{\nu}}\frac{\partial Y_{\alpha}}{\partial\zeta^{\beta}}\bigg|_{\mathbf{p}}\frac{p^{\alpha}}{Y_0}\right).$$

Here we have

$$A_{\mu}=Y_0a^{\lambda}\frac{\partial Y_{\lambda}}{\partial\zeta^{\mu}}\bigg|_{\mathbf{p}},\ B_{\mu}=-k_{\mu}+\delta_{\mu 0}a^{\lambda}\frac{\partial Y_{\lambda}}{\partial\zeta^{\alpha}}\bigg|_{\mathbf{p}}p^{\alpha}\quad(\alpha=\nu,4),$$

$$A^{\mu}_{\nu}=(kY)g^{\mu}_{\nu}+Y_0\frac{\partial Y_{\lambda}}{\partial\zeta^{\nu}}\bigg|_{\mathbf{p}}(p^{\mu}a^{\lambda}-D^{\mu\lambda})\quad(\mu,\nu,\lambda=0,1,2,3),$$

$$B^{\mu}_{\nu}=-k_{\nu}p^{\mu}+\delta_{\nu 0}\frac{\partial Y_{\lambda}}{\partial\zeta^{\alpha}}\bigg|_{\mathbf{p}}p^{\alpha}(p^{\mu}a^{\lambda}-D^{\mu\lambda}).$$

Let us describe the scheme for finding the low-frequency asymptotic behavior of the Green's function $G^{+}_{ab}(k)$. According to Eq. (4.9) the Fourier component of the quantity $a_{\xi}(\mathbf{x},t)$ is related to the Green's function by the relationship

$$a_{\xi}(\mathbf{k},\omega)=\xi(\mathbf{k},\omega)G^{+}_{ab}(\mathbf{k},\omega). \tag{4.55}$$

On the other hand, upon taking account of (4.8) and (4.14), we can represent the quantity $a_{\xi}(\mathbf{x},t)$ in the region of large t $(t\gg\tau_r)$ in the form

$$a_{\xi}(\mathbf{x},t)=\mathrm{Sp}\,\sigma'(\zeta'(t),\varphi'(t))\,\hat{a}(\mathbf{x})+\mathrm{Sp}\,\rho(\xi(t))\,\hat{a}(\mathbf{x}). \tag{4.56}$$

Here the operator $\sigma'(\zeta'(t),\varphi'(t))$ is determined by Eq. (4.15). Since $\mathrm{Sp}\,\hat{\sigma}_{\alpha}(\mathbf{x}')\hat{a}(\mathbf{x})$ and $\mathrm{Sp}\,\sigma_{\varphi}(\mathbf{x})'\hat{a}(\mathbf{x})$ depend on the difference $\mathbf{x}-\mathbf{x}'$, the Fourier component $a_{\xi}(\mathbf{k},\omega)$ of the quantity $a_\xi(\mathbf{x},t)$ contains terms that arise from the first term in (4.56) and are proportional to $\zeta'_\alpha(\mathbf{k},\omega)\equiv\delta\zeta_\alpha(\mathbf{K},\omega)$, $\varphi'(\mathbf{k},\omega)\equiv\delta\varphi(\mathbf{k},\omega)$. The equations of superfluid hydrodynamics with the "sources" $\eta(\mathbf{k},\omega)$ (which are proportional to $\xi(\mathbf{k},\omega)$) imply that, in the region of small $\mathbf{k}$ and $\omega(k\ll l^{-1}$, $\omega\ll\tau_r^{-1})$, the quantities $\delta\zeta_\alpha(\mathbf{k},\omega)$ and $\delta\varphi(\mathbf{k},\omega)$ are singular (cf. (4.47)), while the Fourier component of the second term in Eq. (4.56), according to the preceding Sec. 4.2, is regular. Upon taking account of what we have said, the Fourier component of the quantity $a_\xi(\mathbf{x},t)$ in the region of small $\mathbf{k}$ and ω can be represented according to (4.31) in the principal approximation in the form

$$a_\xi(\mathbf{k},\omega)=\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha}\delta\zeta_\alpha(\mathbf{k},\omega)+\frac{\partial\langle\hat{a}\rangle}{\partial p_l}ik_l\delta\varphi(\mathbf{k},\omega)+\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}\delta\varphi(\mathbf{k},\omega) \tag{4.57}$$

or

$$a_\xi(k)=\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}\delta Y_\mu(k)+\frac{\partial\langle\hat{a}\rangle}{\partial p_\mu}\delta p_\mu(k)+\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}\delta\varphi(k). \tag{4.58}$$

Upon using the formulas (4.46)–(4.47), which were derived from the equations of linearized hydrodynamics, as well as the expressions for the "sources" (4.51) and (4.52) (with $[\hat{N},\hat{b}]\neq 0$) and (4.54) (with $[\hat{N},\hat{b}]=0$), we find the quantities $\delta Y_\mu(k)$, $\delta p_\mu(k)$, and $\delta\varphi(k)$ (they will be proportional to $\xi(k)$). Then, upon comparing Eq. (4.58) and (4.55), we obtain[24] the final expression for the Fourier components of the Green's functions $G^+_{ab}(k)$ in the region of small k (in the case of arbitrary quasilocal operators $\hat{a}$ and $\hat{b}$):

$$G^+_{ab}(k)=\frac{1}{\Delta}\left\{\left[-\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}-\frac{\partial\langle\hat{a}\rangle}{\partial p_\mu}ik_\mu+i\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}(kY)a^\lambda D^{-1}_{\lambda\mu}\right]\times\left[\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}-\frac{\partial\langle\hat{b}\rangle}{\partial p_\nu}ik_\nu+i\frac{\partial\langle\hat{b}\rangle}{\partial Y_\nu}(kY)a^{\lambda'}D^{-1}_{\lambda'\nu}\right]-\frac{\partial\langle\hat{a}\rangle}{\partial Y_\mu}\frac{\partial\langle\hat{b}\rangle}{\partial Y_\nu}\Delta(kY)D^{-1}_{\mu\nu}\right\}. \tag{4.59}$$

Here we have neglected the contribution of the non-pole term in $G^+_{ab}(k)$, since we have already neglected terms of this type in dropping terms of the type $\mathrm{Sp}\,\rho(\xi)\hat{a}$ in $a_\xi(k)$.

If the potential ω corresponds to a relativistically invariant system (cf. (3.19)), then p_ν and Y_ν amount to 4-vectors, while $D_{\mu\nu}$ is a 4-tensor. In this case Eq. (4.59) yields a relativistically covariant representation of the low-frequency asymptotic behavior of the Green's function (cf. (4.10)).

In certain cases it is convenient to represent the low-frequency asymptotic behavior of the Green's functions in the form of a bilinear combination of the derivatives $\partial\langle\hat{a}\rangle/\partial\zeta_\alpha$, $\partial\langle\hat{a}\rangle/\partial p_l$, and $\partial\langle\hat{a}\rangle/\partial\varphi$ and the derivatives $\partial\langle\hat{b}\rangle/\partial\zeta_\alpha$, $\partial\langle\hat{b}\rangle/\partial p_l$, and $\partial\langle\hat{b}\rangle/\partial\varphi$. This representation is convenient for nonrelativistic generalized superfluid systems. Upon starting with Eq. (4.59), we shall represent the pole component of the low-frequency asymptotic behavior of the Green's function $G^+_{ab}(\mathbf{k},\omega)$ in the form[21]

$$\begin{aligned}G^+_{ab}(\mathbf{k},\omega)&=\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha}\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_\beta}G^+_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\omega)\\&-\frac{i}{\eta}\frac{\partial\langle\hat{a}\rangle}{\partial\zeta_\alpha}\left(\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}-ik_l\frac{\partial\langle\hat{b}\rangle}{\partial p_l}\right)G^+_{\zeta_\alpha\psi}(\mathbf{k},\omega)\\&-\frac{i}{\eta}\frac{\partial\langle\hat{b}\rangle}{\partial\zeta_\beta}\left(\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}+ik_l\frac{\partial\langle\hat{a}\rangle}{\partial p_l}\right)G^+_{\psi\zeta_\beta}(\mathbf{k},\omega)\\&-\frac{1}{\eta^2}\left(\frac{\partial\langle\hat{a}\rangle}{\partial\varphi}+ik_l\frac{\partial\langle\hat{a}\rangle}{\partial p_l}\right)\left(\frac{\partial\langle\hat{b}\rangle}{\partial\varphi}-ik_l\frac{\partial\langle\hat{b}\rangle}{\partial p_l}\right)G^+_{\psi\psi}(\mathbf{k},\omega).\end{aligned} \tag{4.60}$$

The Green's functions that figure here, $G^+_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\omega)$, $G^+_{\zeta_\alpha\psi}(\mathbf{k},\omega)$, $G^+_{\psi\zeta_\alpha}(\mathbf{k},\omega)$, and $G^+_{\psi\psi}(\mathbf{k},\omega)$, are determined by the formulas

$$G^+_{\zeta_\alpha\zeta_\beta}(\mathbf{k},\omega)=-\frac{Z_\alpha Z_\beta}{\Delta}-(kY)\frac{\partial\zeta_\alpha}{\partial Y_\mu}D^{-1}_{\mu\nu}\frac{\partial\zeta_\beta}{\partial Y_\nu},$$

$$G^+_{\zeta_\alpha\psi}(\mathbf{k},\omega)=\frac{\eta}{\Delta}Z_\alpha,\quad G^+_{\psi\zeta_\alpha}(\mathbf{k},\omega)=-\frac{\eta}{\Delta}Z_\alpha,$$

$$G^+_{\psi\psi}(\mathbf{k},\omega)=\frac{\eta^2}{\Delta},\quad \eta=|\mathrm{Sp}\,w\psi|,$$

where we have

$$Z_\alpha=\frac{\partial\zeta_\alpha}{\partial p_\lambda}k_\lambda-\frac{\partial\zeta_\alpha}{\partial Y_\lambda}(kY)a^\mu D^{-1}_{\mu\lambda}.$$

The quantity Δ (cf. (4.49) determines the poles of the Green's function (branches of vibrations) of the superfluid liquid. We stress that there are no poles associated with $\det D=0$. This involves the fact that, near a singularity $\det D=0$, the quantity $1/\Delta$ behaves like $\det D$, $1/\Delta\sim\det D$. Therefore the structure of the asymptotic behavior of the Green's function (4.59) has no poles involving singularities of the matrix D. One can show that the singularities involving the vanishing of $\det D$ cancel.

One can easily show that the dispersion equation $\Delta(\mathbf{k},\omega)=0$ for $\mathbf{Y}=0$, $\mathbf{p}=0$ with account taken of the definitions (2.28) and (2.29) acquires the following form:

$$\Delta(\mathbf{k},\omega)=\omega^4-\omega^2k^2(B+\rho_cC)-k^4\frac{s\rho_s}{Y_0\rho_n}\frac{A}{\mathring{m}^2}=0, \tag{4.61}$$

Here we have

$$\begin{aligned}A&\equiv\frac{\partial P}{\partial\zeta_0}\frac{\partial}{\partial\zeta_4}\frac{Y_4}{Y_0}-\frac{\partial P}{\partial\zeta_4}\frac{\partial}{\partial\zeta_0}\frac{Y_4}{Y_0},\\B&\equiv\frac{1}{\mathring{m}}\left(\frac{\partial P}{\partial\zeta_4}-\frac{Y_4}{Y_0}\frac{\partial P}{\partial\zeta_0}\right)+\frac{s}{Y_0\rho_n}\left(\frac{\partial P}{\partial\zeta_0}+\frac{\rho_s}{\mathring{m}}\frac{\partial}{\partial\zeta_0}\frac{Y_4}{Y_0}\right),\\C&\equiv\frac{1}{\mathring{m}^2}\left(\frac{\partial}{\partial\zeta_4}\frac{Y_4}{Y_0}-\frac{Y_4}{Y_0}\frac{\partial}{\partial\zeta_0}\frac{Y_4}{Y_0}+\mathring{m}\frac{s}{Y_0\rho_n}\frac{\partial}{\partial\zeta_0}\frac{Y_4}{Y_0}\right).\end{aligned} \tag{4.62}$$

(Here $\rho_c\equiv\rho-\rho_n-\rho_s$).

For Galilean-invariant systems, according to (3.16), we have $\rho_c=0$ $\mathring{m}=m$, while the quantities A and B acquire the form

$$A=-m^2\frac{\sigma}{\rho c_V}\left(\frac{\partial P}{\partial\rho}\right)_T,\quad B=\left(\frac{\partial P}{\partial\rho}\right)_\sigma+\frac{\rho_sT\sigma^2}{\rho_nc_V}. \tag{4.63}$$

(Here $P=-\omega/Y_0\equiv-\omega'$ is the pressure; $\sigma=s/\rho=Y_0[(P+\varepsilon+(\rho Y_4/Y_0))]/\rho$ is the entropy per unit mass; c_V is the heat capacity per unit mass at constant volume; and $Y_0^{-1}\equiv T$ is the temperature). In this case the dispersion equation (4.61) leads to the well known expressions for the velocities $u_{1,2}$ of first and second sound:

$$u_{1,2}^2 = \frac{1}{2}\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{T\sigma^2\rho_s}{c_V\rho_n}\right] \pm\left\{\frac{1}{4}\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma + \frac{T\sigma^2\rho_s}{c_V\rho_n}\right]^2 - \left(\frac{\partial P}{\partial \rho}\right)_T \frac{T\sigma^2\rho_s}{c_V\rho_n}\right\}^{1/2}.$$

(The index 1 corresponds to the sign " + ", and the index 2 to the sign " − ".)

In closing this section, let us write the asymptotics of the Green's function $G^{\pm}_{\bar\psi\psi}(\mathbf{k},\omega)$ for $\omega\tau_r \ll 1$, $kl \ll 1$ (the asymptotics of the Green's function $G^{\pm}_{\xi_\alpha\psi}(\mathbf{k},\omega)$ and $G^{\pm}_{\xi_\alpha\xi_\beta}(\mathbf{k},\omega)$ are given in Ref. 24 in the case in which $\mathbf{Y}=0$, $\mathbf{p}=0$ (we shall not assume here that the superfluid system has the property of Galilean or relativistic invariance):

$$G^{\pm}_{\bar\psi\psi}(\mathbf{k},\omega) = -\frac{\eta^2}{\Delta(\mathbf{k},\omega)}\left[\omega^2\left(\frac{Y_4}{Y_0}\frac{\partial}{\partial\zeta_0}\frac{Y_4}{Y_0} - \frac{\partial}{\partial\zeta_4}\frac{Y_4}{Y_0}\right) + k^2\frac{s}{Y_0\rho_n}A\right].$$

(The quantities A, $\Delta(\mathbf{k},\omega)$ are determined by Eqs. (4.62) and (4.61).)

If the superfluid system has the property of Galilean invariance, then the asymptotic behavior that we have found of the Green's functions $G^{\pm}_{\bar\psi\psi}(\mathbf{k},\omega)$ and $G^{\pm}_{\xi_\alpha\psi}(\mathbf{k},\omega)$ with account taken of (4.63) transform at $\mathbf{Y}=0$, $\mathbf{p}=0$ respectively into the results of N. N. Bogolyubov[4] and of Galasevich,[33] while $G^{\pm}_{\xi_\alpha\xi_\beta}(\mathbf{k},\omega)$ transforms into the results of Hohenberg and Martin.[34]

5. THERMODYNAMICS AND HYDRODYNAMICS OF THE SUPERFLUID PHASES OF ^{3}He

In this section we shall briefly treat superfluid systems in which, in addition to breakdown of phase invariance, also rotational symmetry in coordinate and spin spaces is broken (examples of such systems are the A and B phases of superfluid ^{3}He). In the case of Fermi systems the operator $\hat f$ in the Gibbs exponential has the structure [see (2.4)]

$$\hat f = \int d^3x\,(\hat\Delta_{\alpha k}(\mathbf{x})\,g_{\alpha k}(\mathbf{x},t) + \text{H.c.})\quad(\alpha, k = 1,2,3),$$
$$\hat\Delta_{\alpha k}(\mathbf{x}) = \psi(\mathbf{x})\,\sigma_2\sigma_\alpha\frac{\partial\psi(\mathbf{x})}{\partial x_k} - \frac{\partial\psi(\mathbf{x})}{\partial x_k}\,\sigma_2\sigma_\alpha\psi(\mathbf{x}); \qquad (5.1)$$

Here $\hat\Delta_{\alpha k}$ is the operator of the order parameter (we shall denote the spin index of the operator $\hat\Delta$ with Greek letters, and the orbital index with Roman letters), σ_α are the Pauli matrices, $g_{\alpha k}(\mathbf{x},t)$ is a certain function of the coordinates and time that characterizes the equilibrium state of the system. The commutation relationships of the order-parameter operator with the integrals of motion $\hat N$, $\hat{\mathscr P}_i$, as well as with the spin operator $\hat S_\beta$ and the orbital angular momentum of the system $\hat{\mathscr L}_i = \int d^3x\,\varepsilon_{ikl}x_k\hat\pi_l(\mathbf{x})$ have the form

$$[\hat N, \hat\Delta_{\alpha k}(\mathbf{x})] = -2\hat\Delta_{\alpha k}(\mathbf{x}),\ [\hat{\mathscr P}_i, \hat\Delta_{\alpha k}(\mathbf{x})] = i\frac{\partial\hat\Delta_{\alpha k}(\mathbf{x})}{\partial x_i},$$
$$[\hat S_\beta, \hat\Delta_{\alpha k}(\mathbf{x})] = i\varepsilon_{\beta\alpha\gamma}\hat\Delta_{\gamma k}(\mathbf{x}), \qquad (5.2)$$
$$[\hat{\mathscr L}_i, \hat\Delta_{\alpha k}(\mathbf{x})] = i\varepsilon_{ijl}x_j\frac{\partial\hat\Delta_{\alpha k}(\mathbf{x})}{\partial x_l} + i\varepsilon_{ikj}\hat\Delta_{\alpha j}(\mathbf{x}).$$

To establish in explicit form the dependences of the function $g_{\alpha k}(\mathbf{x},t)$ on $\mathbf{x}$, and t, we assume as before that the state of statistical equilibrium is spatially homogeneous in the sense of (2.6). In view of the existence of weak relativistic interactions of the particles of the studied system (which we neglect in the Gibbs exponential), the spin operator is not an integral of motion, and hence the Gibbs distribution of (2.3) lacks a thermodynamic force conjugate with this operator. Therefore the condition of stationarity in the case being studied, as before, has the form (2.8). The relationships (2.6), (2.8), and (5.1) imply that

$$g_{\alpha k}(\mathbf{x},t) = e^{-i2\varphi(\mathbf{x},t)}g_{\alpha k}, \qquad (5.3)$$

where $\varphi(\mathbf{x},t) = \mathbf{px} + p_0 t$, $p_0 = (Y + \mathbf{Yp})Y_0^{-1}$, $g_{\alpha k} \equiv$ const.

The further concretization of the superfluid state (finding the structure of the quantities $g_{\alpha k}$) involves the formulation of the symmetry properties of the operator $\hat f$ with respect to rotations in coordinate and spin spaces. Since the operator $\hat f$ explicitly contains the superfluid momentum $\mathbf{p}$, then, as we have seen, it is convenient to treat the operator $\hat{\mathbf{P}} = \hat{\boldsymbol{\mathscr P}} - \mathbf{p}\hat N$ as the translation operator. By analogous considerations, it is convenient to treat as the rotation operator, instead of the angular momentum operator $\hat{\mathscr L}_i$, the operator $\hat L_i \equiv \hat{\mathscr L}_i - i\varepsilon_{ikl}p_k\partial/\partial p_l$, which acts both in Hilbert space and in the space of functions of the superfluid momentum $\mathbf{p}$. The operators $\hat P_i$ and $\hat L_i$ satisfy the commutation relationships characteristic of the generators of the translation and three-dimensional rotation groups of coordinate space:

$$[\hat P_i, \hat P_k] = 0,\quad [\hat L_i, \hat P_k] = i\varepsilon_{ikl}\hat P_l,\ [\hat L_i, \hat L_k] = i\varepsilon_{ikl}\hat L_l. \quad (5.4)$$

In the case of the A phase of superfluid ^{3}He the symmetry conditions of the operator $\hat f$ (and hence, those of the Gibbs distribution) are determined by commutation relationships of the type

$$\left[\hat f, \mathbf{l}\hat{\mathbf{L}} - \frac{1}{2}m_l\hat N\right] = 0,\ \left[\hat f, \mathbf{d}\hat{\mathbf{S}} - \frac{1}{2}m_s\hat N\right] = 0. \qquad (5.5)$$

Here $\mathbf{l}$ and $\mathbf{d}$ are unit vectors that characterize the equilibrium state (for ^{3}He-A: $m_l = 1$, $m_s = 0$). Let us elucidate the physical meaning of these symmetry conditions. For this purpose we shall introduce the "wave function" of a Cooper pair of particles of the system:

$$\Psi_{\alpha_1\alpha_2}(\mathbf{x}_1, \mathbf{x}_2) = \mathrm{Sp}\,w\psi_{\alpha_1}(\mathbf{x}_1)\,\psi_{\alpha_2}(\mathbf{x}_2).$$

Let us assume that $\mathbf{p}=0$, $\mathbf{Y}=0$. Then, upon noting that in this case $[w, \mathbf{l}\hat{\boldsymbol{\mathscr L}} - (\hat N m_l/2)] = [w, \mathbf{d}\hat{\mathbf{S}} - \hat N m_s/2] = 0$, we have

$$\mathrm{Sp}\left[w, \mathbf{l}\hat{\boldsymbol{\mathscr L}} - \frac{m_l}{2}\hat N\right]\psi_{\alpha_1}(\mathbf{x}_1)\,\psi_{\alpha_2}(\mathbf{x}_2) = 0,$$
$$\mathrm{Sp}\left[w, \mathbf{d}\hat{\mathbf{S}} - \frac{m_s}{2}\hat N\right]\psi_{\alpha_1}(\mathbf{x}_1)\,\psi_{\alpha_2}(\mathbf{x}_2) = 0.$$

Since

$$[\hat{\mathscr L}_i, \psi_\alpha(\mathbf{x})] = -l_i\psi_\alpha(\mathbf{x}),\quad [\hat S_i, \psi_\alpha(\mathbf{x})] = -\frac{1}{2}(\sigma_i)_{\alpha\beta}\psi_\beta(\mathbf{x}), \qquad (5.6)$$

then we have

$$\mathbf{l}\,(\hat{\mathbf{l}}^{(1)} + \hat{\mathbf{l}}^{(2)})\,\Psi_{\alpha_1\alpha_2}(\mathbf{x}_1, \mathbf{x}_2) = m_l\Psi_{\alpha_1\alpha_2}(\mathbf{x}_1, \mathbf{x}_2),$$
$$\mathbf{d}\,(\hat{\mathbf{s}}^{(1)} + \hat{\mathbf{s}}^{(2)})\,\Psi(\mathbf{x}_1, \mathbf{x}_2) = m_s\Psi(\mathbf{x}_1, \mathbf{x}_2),$$

where the $\hat l_i^{(a)} = -i\varepsilon_{ikl}x_k^{(a)}\nabla_l^{(a)}, \hat s_i^{(a)}$ $(a = 1,2)$ are the operators for the moment of momentum and the spin, which act respectively on the first and second arguments of the "wave function". Therefore we shall say that the state of statistical equilibrium for which the relationship (5.5) is satisfied corresponds to a state in which the projection on the direction of $\mathbf{l}$ of the moment of momentum of the Cooper pair equals m_l, while the projection of the spin of the Cooper pair on the direction of $\mathbf{d}$ equals m_s. The choice of the order-parameter operator in the form of a vector in the spin and orbital indices (cf. (5.1)) corresponds to the idea that the spin and the moment of momentum of the Cooper pair are assumed equal to 1.

The relationships (5.5) and (5.1) imply that

$$g_{\alpha k} = d_\alpha (m_s)\, \xi_k (m_l),$$

where

$$d_\alpha (m_s) = d_\alpha^+, \quad m_s = -1, \qquad \xi_k (m_l) = \xi_k^+, \quad m_l = -1,$$
$$= d_\alpha^-, \quad m_s = 1, \qquad = \xi_k^-, \quad m_l = 1,$$
$$= d_\alpha, \quad m_s = 0, \qquad = l_k, \quad m_l = 0;$$

Here we have

$$\mathbf{d}^\pm = \frac{1}{\sqrt{2}} (\Delta_1^s \pm i\, \Delta_2^s), \quad \xi^\pm = \frac{1}{\sqrt{2}} (\Delta_1 \pm i\Delta_2)$$

and Δ_1, Δ_2 (Δ_1^s, Δ_2^s) are real, mutually orthogonal unit vectors orthogonal to the vector $\mathbf{l}$ (or $\mathbf{d}$). In the case of ^{3}He-A we have: $g_{\alpha k} = d_\alpha \xi_k^-$.

The state of equilibrium of ^{3}He-A is described by a statistical operator $w(t)$ of the form of (2.3), in which we should understand as $\hat{f}$ the following expression:

$$\hat{f} = \int d^3 x \hat{\Delta}_{\alpha k} (\mathbf{x}) \exp [- i2 (\mathbf{px} + p_0 t)]\, d_\alpha \xi_k^- + \text{H.c.}$$
$$\equiv \int d^3 x \hat{\Delta}_{\alpha k} (\mathbf{x})\, d_\alpha \xi_k^- (\mathbf{x}, t) + \text{H.c.}, \qquad (5.7)$$

Here we have $\xi_k^- (\mathbf{x},t) = \xi_k^- \exp[-i2(\mathbf{px} + p_0 t)]$. The explicit form of the operator $w(t)$ and the commutation relationships (5.2) imply that

$$e^{-i\hat{N}\varphi} w (Y_\alpha, \mathbf{p}, \xi^-, \xi^+, \mathbf{d})\, e^{i\hat{N}\varphi} = w (Y_\alpha, \mathbf{p}, \xi^+ e^{-i2\varphi}, \xi^- e^{i2\varphi}, \mathbf{d}),$$
$$e^{-i\omega \hat{S}} w (Y_\alpha, \mathbf{p}, \xi^+, \xi^-, \mathbf{d})\, e^{i\omega \hat{S}} = w (Y_\alpha, \mathbf{p}, \xi^+, \xi^-, \mathbf{d} (\omega)).$$

(Here we have $\mathbf{d}(\omega) = \mathbf{d} \exp \hat{\omega}$, and $\omega_{\alpha\beta} \approx \varepsilon_{\alpha\beta\gamma}\omega_\gamma$.) Therefore the potential ω and the averages of the operators $\hat{Q}$, which are invariant with respect to spin rotations and phase transformations ($[\hat{Q}, \hat{S}_i] = 0$, $[\hat{Q}, \hat{N}] = 0$), do not depend on φ and $\mathbf{d}$. Hence they are functions of Y_α, $\mathbf{p}$ and l.

The symmetry of ^{3}He-B is determined by the commutation relationship

$$[\hat{f}, \hat{L}_i + R_{i\alpha} \hat{S}_\alpha] = 0,$$

where $R_{i\alpha}$ is a rotation matrix ($R\tilde{R} = 1$) defined by three parameters that characterize the state of statistical equilibrium of ^{3}He-B. Therefore, upon noting that

$$\left[\hat{\mathbf{L}} + \hat{\mathbf{S}}, \int d^3 x\, (\hat{\Delta}_{ii} (\mathbf{x})\, e^{-i2(\mathbf{px}+p_0 t)} + \text{H.c.})\right] = 0,$$

and performing on this relationship a unitary transformation corresponding to the rotation R in spin space, we easily find that $g_{\alpha k} = R_{\alpha k} \exp(i\varphi)$, $\overset{*}{\varphi} = \varphi$). Thus the equilibrium state of ^{3}He–B is described by the statistical operator $w(t)$ of (2.3), in which the operator $\hat{f}$ has the form

$$\hat{f} = \int d^3 x (\hat{\Delta}_{\alpha k} (\mathbf{x})\, e^{-i2\varphi(\mathbf{x}, t)} R_{\alpha k} + \text{H.c.}), \quad \varphi (\mathbf{x}, t) = \mathbf{px} + p_0 t + \varphi$$

and the potential ω obviously does not depend on φ.

Now let us take up the finding of the equilibrium mean operators of the flux densities $\langle \zeta_{\alpha k} \rangle$ in ^{3}He-A. Proceeding just as in Sec. 2.3, we can easily see that the flux density $\zeta_{4k} \equiv j_k$ has the form

$$j_k = \frac{1}{Y_0} \frac{\partial \omega}{\partial p_k} - \frac{Y_k}{Y_0} \frac{\partial \omega}{\partial Y_4}. \qquad (5.8)$$

We saw in Sec. 2.3 that, in finding the expression for the momentum flux density t_{ik}, an important role is played by the unitary transformations U_a that correspond to the group of arbitrary affine transformations $x_i \to x_i' - a_{ik} x_k$. The reason why one could express t_{ik} in the case of superfluid ^{4}He in terms of the thermodynamic potential consisted in the fact that the operator $\hat{f}(V,\mathbf{p})$ in the unitary transformations U_a transformed in terms of itself, namely, $U_a \hat{f}(V,\mathbf{p}) U_a^+ = \hat{f}(V|\det a|, pa^{-1}) \cdot |\det a|^{-1/2}$. In the case of ^{3}He-A the situation changes, and the operator $\hat{f}$ in the Gibbs exponential in the unitary transformations U_a transforms in terms of itself only under certain restrictions on the matrix a_{ik} of the affine transformation. Let us elucidate these restrictions. According to (5.7) we have

$$U_a \hat{f} (V, \mathbf{p}, \xi^-) U_a^+$$
$$= \int_{V|\det a|} d^3 x\, (\hat{\Delta}_{\beta k} (\mathbf{x}) \exp [- i2 (p_l a_{li}^{-1} x_i + p_0 t)]\, a_{ki} \xi_i^- d_\beta + \text{H.c.}).$$

We can express the right-hand side of this equation in terms of $\hat{f}$ only in the case in which

$$a_{ki} \xi_i^- = \alpha c_{ki} \xi_i^-, \qquad (5.9)$$

where α is an arbitrary scalar parameter and c is an arbitrary matrix of spatial rotations ($c\tilde{c} = 1$). In this case we have

$$U_a \hat{f} (V, \mathbf{p}, \xi^-) U_a^+ = \alpha \hat{f} (V\, |\det a|, \tilde{a}^{-1}\mathbf{p}, c\xi^-). \qquad (5.10)$$

We find from Eq. (5.9) that the most general structure of the matrix a_{ki} has the form

$$a_{kl} = c_{kl} b_{li},$$

where $b_{li} \xi_i^- = \alpha \xi_l^-$. Hence we obtain that $b_{ik} = \alpha \delta_{ik} + \beta l_i l_k$ (α and β are arbitrary parameters). Thus, in contrast to the complete group of homogeneous affine transformations, which is characterized by nine real parameters, the group of transformations that leave invariant the structure of the operator $\hat{f}(V,\mathbf{p},\xi^-)$ in the sense of (5.10) is characterized by five arbitrary parameters. Therefore, in contrast to the case of superfluid ^{4}He, the information obtainable on the momentum flux density t_{ik} starting with the symmetry properties of the operator $\hat{f}$ will be more meager in the case of ^{3}He-A. Upon repeating the calculations of Sec. 2.3, we obtain the following expression for the tension tensor t_{ik}:

$$t_{ik} = \frac{p_i}{Y_0} \frac{\partial \omega}{\partial p_k} - \frac{\partial}{\partial Y_i} \frac{\omega Y_k}{Y_0} + \frac{1}{2Y_0} \left(\eta_i \frac{\partial \omega}{\partial \eta_k} - \eta_k \frac{\partial \omega}{\partial \eta_i} \right) + t'_{ik}. \qquad (5.11)$$

(The potential ω depends only on the ratio $\eta_i/|\eta| \equiv l_i$; the vector η_i has been introduced so that the derivatives $\partial \omega / \partial \eta_i$ may be understood in the ordinary sense.) The term t'_{ik} satisfies only the following relationships:

$$t'_{ik} = t'_{ki}, \quad t'_{ll} = 0, \quad l_i l_k t'_{ik} = 0. \qquad (5.12)$$

(In deriving (5.11) we took account of the fact that the operator $\hat{f}$ is defined essentially only up to a transformation $\hat{f} \to a\hat{f}$, since, according to the method of quasiaverages, $\nu \to 0$).

We note that the thermodynamic potential ω is a function of the invariants Y_0, Y_4, Y^2, $\mathbf{Y}^2$, $\mathbf{Yp}$, $(\mathbf{Yl})^2$, $(\mathbf{pl})^2$, $(\mathbf{Yl})(\mathbf{pl})$, and $\mathbf{p}^2$. Hence, according to (5.11), this implies that the tension tensor $t_{ik} = t_{ki}$ is symmetric.

Upon applying Eq. (2.26) and Eqs. (5.8), (5.11), we can easily find an expression for the energy flux density. Consequently we obtain a formula that combines all the fluxes in the state of statistical equilibrium of ^{3}He-A,

$$\zeta_{\alpha k} = -\frac{\partial}{\partial Y_\alpha}\frac{\omega Y_k}{Y_0} + \frac{\partial \omega}{\partial p_k}\frac{\partial p_0}{\partial Y_\alpha} + \tilde{t}_{ik} Y_0 \frac{\partial}{\partial Y_\alpha}\frac{Y_i}{Y_0},$$
$$\tilde{t}_{ik} = -\frac{1}{2Y_0}\left(\eta_k \frac{\partial \omega}{\partial \eta_i} - \eta_i \frac{\partial \omega}{\partial \eta_k}\right) + t'_{ik}. \qquad (5.13)$$

Here t'_{ik} satisfies the relationships of (5.12). In the phenomenological theory it is assumed that the tensor t'_{ik} vanishes if the potential ω does not depend on the vector $\mathbf{l}$ (in this case eq. (5.13) transforms into (2.28)). Accordingly the tensor t'_{ik} can be represented in the form

$$t'_{ik} \equiv t'_{ik,l}\frac{\partial \omega}{\partial \eta_l}|\eta|, \quad t'_{ik,l} = t'_{ki,l}, \quad l_i l_k t'_{ik,l} = 0. \qquad (5.14)$$

Now let us take account of the invariance of the system being treated with respect to Galilean transformations. In this case the potential ω is a function of the invariants $Y'_0, Y'_4, \mathbf{Y}'^2$, and $(\mathbf{Y'l})^2 (Y'_0 = Y_0,\ Y'_k = Y_k + (Y_0 p_k/m)$, and $Y'_4 = Y_4 + \mathbf{Yp} + (Y_0 \mathbf{p}^2/2m))$, we took account of the pseudovector character of the quantity $\mathbf{l}$). Let us introduce into the treatment the superfluid and normal densities $\rho_s, \rho_n, \rho_n + \rho_s = \rho$ (cf. (2.29)), as well as the quantity ρ_0:

$$\rho_s \equiv \frac{2}{Y_0}\frac{\partial \omega}{\partial \mathbf{p}^2}m^2, \quad \rho_n \equiv -2Y_0\frac{\partial \omega}{\partial \mathbf{Y}'^2}, \quad \rho_0 = -2Y_0\frac{\partial \omega}{\partial (\mathbf{Y'l})^2}.$$

Then we shall represent the flux density of the number of particles j_i and the tension tensor t_{ik} in the form

$$j_i = (\rho_n)_{ik} v_{nk} + (\rho_s)_{ik} v_{sk},$$
$$(\rho_n)_{ik} \equiv \rho_n \delta_{ik} + \rho_0 l_i l_k, \quad (\rho_s)_{ik} = \rho_s \delta_{ik} - \rho_0 l_i l_k,$$
$$t_{ik} = P\delta_{ik} + \rho_n v_{ni} v_{nk} + \rho_s v_{si} v_{sk}$$
$$+ \frac{1}{2}\rho_0 (\mathbf{v}_n + \mathbf{v}_s)\, \mathbf{l}\,[l_i (\mathbf{v}_n + \mathbf{v}_s)_k + l_k (\mathbf{v}_n + \mathbf{v}_s)_i] + t'_{ik};$$

Here we have $\mathbf{v}_n = -\mathbf{Y}/Y_0$, $\mathbf{v}_s = \mathbf{p}/m$.

The formulas of transformation of the tension tensor t_{ik} in Galilean transformations implies that the tensor t'_{ik} is a function of the vectors Y' and l. Therefore the tensor t'_{ik} is constructed from a combination of δ_{ik}, $l_i l_k$, $Y'_i Y'_k$, $[\mathbf{Y'l}]_i l_k + [\mathbf{Y'l}]_k l_i$, $[\mathbf{Y'l}]_i Y'_k + [Y'l]_k Y'_i$, and $l_i Y'_k + l_k Y'_i$. In view of (5.12) it is characterized in the general case by four scalar functions. Upon taking account of the pseudovector character of the quantity $\mathbf{l}$ and assuming that the structure of $t'_{ik}(Y'_l)$ does not contain singularities as $\mathbf{Y}' \to 0$, and also, upon using (5.14), we obtain in the principal approximation in $\mathbf{Y}'$:

$$t'_{ik} = \frac{\alpha}{2Y_0}\left(\eta_i \frac{\partial \omega}{\partial \eta_k} + \eta_k \frac{\partial \omega}{\partial \eta_i}\right).$$

Here α is a certain function of the thermodynamic parameters (the coefficients of the other three scalar functions contain higher powers of $\mathbf{Y}'$).

If we have expressions for the equilibrium average flux densities $\zeta_{\alpha k}$ of (5.13), we can write the equations of balance for ^{3}He-A in an approximation linear in the spatial gradients (ideal hydrodynamics):

$$\dot{\zeta}_\alpha = -\nabla_k \zeta_{\alpha k}. \qquad (5.15)$$

In this approximation in the expressions for the flux densities $\zeta_{\alpha k}$, we can neglect the terms usually written out,[15] which involve the spatial gradients of both the vector $\mathbf{l}$ and the other thermodynamic parameters; the terms containing the gradients of these quantities appear in the next approximation, which leads to both dissipative terms and to modification of the reactive terms in the equations of hydrodynamics of ^{3}He-A.

The equations (5.15) are not closed. Analogously to the way in which it was done in treating superfluid ^{4}He, one must supplement them with the equations of motion for $\mathbf{p}$ and $\mathbf{l}$. We can easily show that the equation of motion for the superfluid momentum $\mathbf{p}$ has the same form in the approximation being treated as in the case of ^{4}He (cf. (3.12)):

$$\dot{p}_k = \nabla_k p_0, \quad p_0 = (Y_4 + \mathbf{Yp})\, Y_0^{-1}. \qquad (5.16)$$

A problem of the microscopic theory is also to derive the equations of motion for the vector $\mathbf{l}$. We shall not present the derivation of this equation, which has the following form in the approximation linear in the gradients (see the review of Ref. 15):

$$\dot{l}_i = -(\mathbf{v}_n \nabla)\, l_i + (\delta_{ip} - l_i l_p)\, t'_{jk,p} \nabla_k v_{nj}$$
$$- \frac{1}{2}[\mathbf{l}\ \mathrm{rot}\ \mathbf{v}_n]_i - \frac{\eta}{Y_0}\left[\mathbf{l}\ \frac{\partial \omega}{\partial \mathbf{l}}\right]_i.$$

Here $t'_{jk,p}$ is determined by the formulas of (5.14) and η is a certain function of the thermodynamic parameters. One can easily derive this equation from Eqs. (5.15) and (5.16) by requiring the adiabaticity of flow of the superfluid liquid. That is, we have

$$\dot{s} = -\nabla(s\mathbf{v}_n),$$

Here the entropy density is determined by the formula $s = -\omega + Y_\alpha \zeta_\alpha$.

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