From the Editorial Board:

On December 25 it will be two years since the death of Vadim Genrikhovich Knizhnik at the age of only twenty five (February 20, 1962–December 25, 1987). Vadim Knizhnik tackled many problems in theoretical physics, but he became widely known through his work in the new field of string theory. The Belavin-Knizhnik theorem established the connection between the string-theoretic approach of A. M. Polyakov and complex geometry, thus achieving a union between modern field theory and contemporary mathematical ideas.

This review is essentially a presentation of this theorem and of its immediate consequences which have provided the basis for many of subsequent researches. The review is based on lectures given by Vadim Knizhnik in the spring of 1987 at the first Republican School of Young Scientists, held in Kiev at the Institute of Theoretical Physics of the Academy of Sciences of the Ukrainian SSR. The results reported in these lectures have since become classical and have frequently been rederived and presented again by other methods. However, the original version remains one of the clearest and most accessible for those embarking on a study of string theory. For specialists, Sec. 12 and IV will be of particular interest: they describe constructs that were left unfinished by the author and should lead to interesting developments.

Multiloop amplitudes in the theory of quantum strings and complex geometry

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The evaluation of multiloop amplitudes in the theory of closed oriented bosonic strings is reduced to the problem of finding the measure on the moduli space of Riemann surfaces. It is shown that the measure is equal to the product of the square of the modulus of a holomorphic function and the determinant of the imaginary part of the period matrix, raised to the power 13. A consequence of this theorem is that the measure can be expressed in terms of theta-functions. A variant of the holomorphy theorem, in the form of Quillen's theorem, is used to evaluate the dependence of the determinants of the Laplace operator on a Riemann surface on the boundary conditions. When the Riemann surface is represented by a branched covering of a plane, the measure is expressed in terms of the coordinates of the branch points, and to each branch point there corresponds a vertex operator. The measure is the correlation function of these operators, and this can be used to represent the sum over all the higher loops as the partition function of a certain two-dimensional conformal field theory.

1. INTRODUCTION

The theory of quantum strings has attracted considerable attention in recent years. This has been due to the remarkable results of Green and Schwarz[1] who showed that superstring theory provides a basis for a self-consistent theory of quantum gravity with acceptable phenomenological properties. The most popular candidate for a theory that unifies all interactions is the heterotic string model[2] with the symmetry group $E_8 \times E_8$ in 10 dimensions, six of which are compactified on the Calabi-Yao manifold.[3]

To be sure that superstring theory is self-consistent, we must verify that there are no divergences in any of the perturbation theory orders. One-loop calculations[1,2] and certain qualitative arguments[4] support this proposition, first put forward in Ref. 1, but a complete proof is lacking for multiloop diagrams (it is still lacking[65]-*Ed.*). It may well be that a more complete understanding of the structure of multiloop corrections would lead to progress in the solution of other problems, as well. As we shall see, hopes in this area rest largely on the fact that the multiloop amplitudes are exceptionally beautiful objects, and their theory makes use of a wide range of physical and mathematical ideas. With increasing energy, the unification of interactions is in one sense accompanied by the unification of ideas.

This review describes many of the results obtained in this area for the simplest model of closed oriented bosonic strings (ESVM).

In the geometric approach, the p-loop scattering amplitudes of closed oriented bosonic strings are sums over closed oriented surfaces of genus p (with p handles). It will be shown in Sec. 2 that, for the critical dimension $D = 26$, the summation reduces to integration over moduli space M_p of genus p Riemann surfaces, and our task is to describe the analytic properties of multiloop amplitudes as functions of coordinates on M_p. It is precisely these properties that determine the structure of divergences in the theory.

We shall see later that these analytic properties are very simple. The amplitudes are constructed with the help of meromorphic and even rational functions on M_p. The formulation of the problem and the result can be stated more precisely as follows. In Polyakov's covariant geometric approach,[5] the sum over surfaces is the sum over topologies (of genus p), internal metrics $g_{ab}(\xi)$, and imbeddings $X_\mu(\xi)$ of the surface with coordinates $\xi^{1,2}$ into D-dimensional flat space-time. When $D = 26$, the conformal anomaly is found to cancel,[5] and the full quantum symmetry group becomes the product of the Weyl group Conf (S) (conformal transformations $g_{ab}(\xi) \to \lambda(\xi) g_{ab}(\xi)$ and the group of general coordinate transformations Diff (S) of the surface S. Thus, for each p, we have to integrate over the orbits of the group $H = \mathrm{Conf}(S) \times \mathrm{Diff}(S)$ in the space $G(S)$ of all metrics on S, i.e., over the factor-space $G/H = M_p$. This space is called the moduli space of genus-p Riemann surfaces, and, as was shown by Riemann, its dimension is finite and equal to 0 for $p = 0$, to 2 for $p = 1$, and to $6p - 6$ for $p \geqslant 2$. Teichmuller *et al.*[6] have shown that M_p has a natural complex structure. More than that, M_p is an algebraic manifold.[7]

Let $y_1, \dots, y_{3p-3}$ be some complex coordinates on M_p. For $D = 26$, the sum over genus-p surfaces then has the following form after the volume of the gauge group has been extracted (cf. Sec. 2):

$$Z_p = \int_{M_p} d\Omega \exp \widetilde{W}(y_i, \bar{y}_i),$$
$$d\Omega = \left(\frac{i}{2}\right)^{3p-3} dv \wedge d\bar{v}, \tag{1.1}$$
$$dv = dy_1 \wedge \dots \wedge dy_{3p-3},$$

where $\widetilde{W}$ is a function of the coordinates $y_i, \bar{y}_i$. The natural question is: how does the complex structure on M_p manifest itself in the analytic properties of $\widetilde{W}(y_i, \bar{y}_i)$? We recall that the one-loop calculation ($p = 1$) gives[8]

$$Z_1 = \int_{M_1} d^2y\, |y\Delta(y)|^{-2} (\ln|y|)^{-14},$$
$$\Delta(y) = y \prod_{k=1}^{\infty} (1 - y^n)^{24}, \tag{1.2}$$

where $y = \exp(2\pi i \tau)$ and τ (the ratio of the periods of a torus) runs over the fundamental region of the modular group $SL(2,\mathbb{Z})$. Formula (1.2) suggests that these properties may turn out to be relatively simple for $p > 1$, as well. The measure in (1.2) is, to within a power of the logarithm, the square of the modulus of the analytic function of y that has no zeros anywhere and has a second-order pole at $y = 0$ at which the torus degenerates. We find that, for $p > 1$, the measure has almost the same properties[9]:

A) $\exp \widetilde{W}(y_i, \bar{y}_i) = |F(y_1, \dots, y_{3p-3})|^2 (\det \mathrm{Im}\, \tau)^{-13}$, (1.3)

where $F(y)\, dv$ is a holomorphic $(3p-3,0)$-form with no zeros on M_p and τ is the period matrix of the Riemann surface with coordinates $y_1, \bar{y}_1, \dots, y_{3p-3}, \bar{y}_{3p-3}$ on M_p.

B) *the form $F(y)\, dv$ has a second-order pole at infinity D of the space M_p at which the surfaces degenerate*[1)].

This pole leads to the divergence in (1.1), and its presence is closely related to the fact that the ground state of the bosonic string is a tachyon. These results are reviewed in detail in Section I, and were originally obtained by A. A. Belavin and the present author in Ref. 9.

It is readily shown that conditions A) and B) define the form $F(y)dv$ uniquely to within a constant factor. In particular, this enables us to express $F(y)$ for $p = 2$, 3, and 4 in terms of the Riemann theta-functions. These results will be presented in Sec. 6 together with the necessary information from the theory of automorphic Siegel forms. Moreover, Beilinson and Manin succeeded in using properties A) and B) of the measure to express it in terms of theta functions and Abelian differentials for arbitrary p (Ref. 11). An alternative derivation of this result is given in Sec. 7 and is based on the theory of analytic fields on Riemann surfaces, constructed in Ref. 12.

In Sec. 4 we determine the order of a pole by means of a direct estimate of functional integrals, but A. Beilinson and V. Drinfeld have told us that Mumford's theorem[7] and the results reported by Wolpert[13] on cohomologies $\overline{M}_p$ can be used together to determine the order of a pole on the assumption that $F(y)$ is meromorphic on $\overline{M}_p$ and has no zeros or poles on M_p. Further details can be found in Sec. 5 which gives a precise mathematical formulation of the results.

In classical mathematics, Riemann surfaces are regarded as branched coverings of the complex plane. The parameters y_i are then the complex coordinates of some of the branch points. The corresponding calculations, based on the representation of a branch point by a vertex operator,[14] are given in Sec. 9–12.

The appearance of the complex-analytic structure in string theory is closely related to the conformal invariance and the cancelation of the gravitational anomaly separately in right and left moving sectors of string excitations.[15] The product $F(y)\ \overline{(F(y))}$ is then the contribution of right (left) excitations to the measure. Three anomalies, namely, conformal, gravitational, and analytic, cancel simultaneously.

I. A THEOREM ON HOLOMORPHY

2. From the sum over surfaces to integration over moduli space

According to Ref. 5, the sum over surfaces is defined by

$$\sum_{\text{over the surface}} e^{-\text{area}} \stackrel{\text{def}}{=} \sum_{p=0}^{\infty} \int Dg_{ab}(\xi)\, DX_\mu(\xi) \exp(-S[X_\mu, g_{ab}]), \tag{2.1}$$

where $g_{ab}(\xi)$ is the internal metric on the surface with coordinates ξ_1, ξ_2, and $X_\mu(\xi)$ determines the imbedding of the surface into D-dimensional space-time; S is the Nambu-Goto action

$$S = \int g^{1/2} (g^{ab} \partial_a X^\mu \partial_b X_\mu + \mu_0^2) d^2\xi. \tag{2.2}$$

We shall put $D = 26$ in the discussion given below. The integration measure in (2.1) is defined in terms of intervals in the fundamental space

$$\|\delta g\|^2 = \int g^{aa_1} g^{bb_1} \delta g_{ab} \delta g_{a_1 b_1} g^{1/2} d^2\xi,$$
$$\|\delta X_\mu\|^2 = \int (\delta X_\mu)^2 g^{1/2} d^2\xi. \tag{2.3}$$

Each metric determines the volume form $dv = g^{1/2} d\xi^1 \wedge d\xi^2$ and the complex structure

$$J_a^b = g^{1/2} \varepsilon_{ac} g^{cb}, \tag{2.4}$$

compatible with it, where $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$. The harmonic coordinates $z, \bar{z}$ are connected with this complex structure, where z is determined from the solution of the Beltrami equation

$$J_a^b \frac{\partial z}{\partial \xi^b} = i \frac{\partial z}{\partial \xi^a}. \tag{2.5}$$

In terms of these coordinates, the metric assumes the conformal form

$$\hat{g} \equiv g_{ab} d\xi^a d\xi^b = \rho(z, \bar{z})\, dz\, d\bar{z}, \quad \rho \equiv e^{\varphi}. \tag{2.6}$$

For infinitesimal conformal $\varphi \to \varphi + \delta\varphi$ and general coordinate $z \to z + \varepsilon(z,\bar{z})$ transformations, the variation of the metric is

$$\begin{aligned} \delta\hat{g} &= \rho\delta\tilde{\varphi}\, dz\, d\bar{z} + \rho\bar{\partial}\varepsilon(d\bar{z})^2 + \rho\partial\bar{\varepsilon}(dz)^2, \\ \rho\delta\tilde{\varphi} &= \rho\delta\varphi + \partial(\rho\varepsilon) + \bar{\partial}(\rho\bar{\varepsilon}), \\ \partial &\equiv \frac{\partial}{\partial z}, \quad \bar{\partial} \equiv \frac{\partial}{\partial\bar{z}}, \end{aligned} \tag{2.7}$$

and its length is the sense of (2.3) is

$$\|\delta\hat{g}\|^2 = \int [\rho(\delta\varphi)^2 + \rho(\partial\bar{\varepsilon})\bar{\partial}\varepsilon]\, d^2\xi \tag{2.8}$$

where $d^2\xi$ is understood (here and henceforth) as being $i/2 dz \wedge d\bar{z}$.

To extract from (2.1) the volume of the gauge group of general-coordinate and conformal transformations, we must[5] pass from integration over Dg_{ab} to integration over $D\varphi$ and $D\varepsilon$. Since φ is a scalar and ε a vector, it follows that for these fields the intervals in function space are

$$\|\delta\varphi\|^2 = \int \rho(\delta\varphi)^2 d^2\xi, \quad \|\varepsilon\|^2 = \int \varepsilon\bar{\varepsilon}\rho^2 d^2\xi. \tag{2.9}$$

From (2.8) and (2.9) we find that[2)]

$$Dg_{ab} = \det(-\rho^{-2}\partial\rho\bar{\partial})\, D\varphi D\varepsilon\, Dg^{\perp}_{ab} \tag{2.10}$$

where $Dg^{\perp}_{ab}$ represents integration over the directions in the function space of metrics that are orthogonal to the variations (2.7). To show that such directions exist, consider an infinitesimal variation of the metric $\delta\hat{g}^*$ [not to be confused with $\delta\hat{g}$ from (2.7)]

$$\delta\hat{g}^* = \delta\varphi^*\rho\, dz\, d\bar{z} + f(dz)^2 + \bar{f}(d\bar{z})^2. \tag{2.11}$$

From the orthogonality condition

$$\langle\delta\hat{g}, \delta\hat{g}^*\rangle = \int \delta\tilde{\varphi}\delta\varphi^*\rho\, d^2\xi + \int f\bar{\partial}\varepsilon\, d^2\xi + \int \bar{f}\partial\bar{\varepsilon}\, d^2\xi = 0$$

we then find that

$$\delta\varphi^* = \bar{\partial}f = \partial\bar{f} = 0. \tag{2.12}$$

Variations of the metric that are orthogonal to the gauge group, i.e., satisfy (2.12), are called holomorphic quadratic differentials. It is known that on a surface of genus $p \geqslant 2$, the complex dimension of a linear space V of such differentials is $3p - 3$ (1 for $p = 1$ and 0 for $p = 0$). Thus, integration over $Dg^{\perp}_{ab}$ is integration over the finite-dimensional space M_p of complex structures of Riemann spaces of genus p (moduli space) that is related to variations of the metric of the form of (2.12).

The complex-analytic coordinates in M_p are introduced as follows.[6] We take the basis $f_i(z)(dz)^2 (i = 1,...,3p - 3)$ in V and its dual basis

$$\eta^j(z, \bar{z})\frac{d\bar{z}}{dz} \quad (j = 1, \ldots, 3p-3)$$

in the space of the Beltrami differentials[3)]

$$\int \eta^k f_i\, d^2\xi = \delta^k_i. \tag{2.13}$$

Any complex structure J close to $J^{(0)}$ that is compatible with the metric $\rho dz d\bar{z}$ can then be parametrized by the complex parameters $y_1,...,y_{3p-3}$ and is compatible with the metric

$$\hat{g}(y) = \rho\,|dz + y_i\eta^i d\bar{z}|^2 = \tilde{\rho}\, du\, d\bar{u}, \tag{2.14}$$

where the coordinate u is determined from the Beltrami solution

$$\frac{\partial u}{\partial\bar{z}} = y_i\eta^i\frac{\partial u}{\partial z} \tag{2.15}$$

and is a *holomorphic* function of y_i (Ref. 6).

The conditions given by (2.13) define η^k to within the total derivative

$$\eta^k \to \tilde{\eta}^k = \eta^k + \bar{\partial}\varepsilon^k, \tag{2.16}$$

but the complex structures that correspond to $\delta y_k\eta^k$ and $\delta y_k\tilde{\eta}^k$ and are infinitesimally close to $J^{(0)}$ are found to coincide. The arbitrariness in (2.16) is fixed by the orthogonality of the metric (2.14) to the variations (2.7), which leads to the choice

$$\eta^k = \rho^{-1}(N_2^{-1})^{ki}\bar{f}_i, \quad (N_2)_{ik} \overset{\text{def}}{=} \int \bar{f}_i f_k \rho^{-1} d^2\xi, \tag{2.17}$$

so that

$$\begin{aligned} \|\delta\hat{g}^{\perp}(y)\|^2 &= \delta y_i\delta\bar{y}_k(N_2^{-1})^{ik}, \\ Dg^{\perp}_{ab} &= (\det N_2)^{-1} d\Omega. \end{aligned} \tag{2.18}$$

We now substitute (2.18) in (2.10), perform the Gaussian integration over $DX_\mu(\xi)$ in (2.1) (taking into account the zero mode $X^{(0)}_\mu(\xi) = \text{const}$), and extract the infinite volume of the group of general coordinate and conformal transformations of $\int D\varepsilon D\varphi$. The problem then reduces to the evaluation of the following integral over the moduli space M_p:

$$\begin{aligned} Z_p &= \int d\Omega \exp W(y_i, \bar{y}_i)(\det N_1)^{-13}, \\ \exp W(y_i, \bar{y}_i) &= \frac{\det\Delta_{-1}}{\det N_2}\left(\frac{\det N_0 \cdot \det N_1}{\det'\Delta_0}\right)^{13} \end{aligned} \tag{2.19}$$

where $\Delta_j = -\rho^{j-1}\partial\rho^{-j}\bar{\partial}$ is the Laplace operator acting in the space of the j-differentials, i.e., tensors $\phi_{+\ldots+}(z,\bar{z})$, that transform like $(dz)^{-j}$

$$(N_j)_{\alpha\beta} = \int \rho^{1-j}\bar{\phi}^{(j)}_\alpha\phi^{(j)}_\beta\, d^2\xi, \tag{2.20}$$

and is a matrix of the scalar products of the zero modes $\phi^{(j)}_\alpha$ of the operator Δ_j. We note that $\det N_1$ does not depend on ρ, and $\det N_{-1}$ is absent from (2.19), since Δ_{-1} does not have zero modes for $p \geqslant 2$. Because of the cancellation of the conformal anomaly,[5] the product of the remaining terms in (2.19) is also independent of ρ and is therefore a function of $y_i, \bar{y}_i$ alone. This can be verified by means of the formula[5,16]

$$\begin{aligned} \delta_\rho \ln\frac{\det'\Delta_j}{\det N_j \cdot \det N_{1-j}} &= \frac{C_j}{6\pi}\int \delta\rho\rho^{-1}\partial\bar{\partial}\ln\rho\, d^2\xi, \\ C_j &= 6j^2 - 6j + 1, \end{aligned} \tag{2.21}$$

where $N_k \equiv 1$ if Δ_k has no zero modes. The expression given by (2.21) contains 6π instead of the usual 24π, since $\partial\bar{\partial} = (1/4)(\partial_1^2 + \partial_2^2)$.

3. Holomorphy of *F(y)* (Ref. 9)

We shall now show that $\exp W(y_i,\bar{y}_i)$ in (2.19) is the square of the modulus of a holomorphic function of y_i. This will involve the evaluation of the variation of $W(y_i,\bar{y}_i)$ for an infinitesimal variation of the complex structure generated by a variation of the metric of the form

$$\delta\hat{g} = \rho[\eta(y)(d\bar{z})^2 + \overline{\eta(y)}(dz)^2],\quad \eta(y) = \sum_{i=1}^{3p-3} \delta y_i \eta^i. \quad (3.1)$$

The function $\exp W(y_i,\bar{y}_i)$ is the square of the modulus of a holomorphic function if and only if the second variation of W does not contain the terms $\overline{\eta}\eta$:

$$\delta_\eta \delta_{\bar{\eta}} W = 0. \quad (3.2)$$

A tedious quasiclassical calculation[9] shows that

$$\delta_\eta \delta_{\bar{\eta}} \ln \frac{\det' \Delta_j}{\det N_j \cdot \det N_{1-j}} = -\frac{C_j}{6\pi} \int \rho^{-2} (\bar{\partial} f \partial \bar{f} + f\bar{f} \partial\bar{\partial} \ln \rho)\, d^2\xi,$$
$$f \equiv \rho\bar{\eta}, \quad (3.3)$$

if

$$\delta_\eta \phi_\beta^{(j)} = \delta_{\bar{\eta}} \phi_\beta^{(j)} = 0,$$

so that, if we recall (2.19), we find (3.2). The analytic anomaly (3.3) cancels and

$$\exp W(y_i, \bar{y}_i) = |F(y_i)|^2. \quad (3.4)$$

Moreover, it follows from (3.3) that any expression of the form

$$\prod_j \left(\frac{\det' \Delta_j}{\det N_j \cdot \det N_{1-j}}\right)^{n_j} \quad (3.5)$$

will be the square of the modulus of a holomorphic function on M_p, provided that

$$\sum_j C_j n_j = 0. \quad (3.6)$$

We therefore conclude that, to within $(\det N_1)^{-13}$, the measure (2.19) is indeed the square of the modulus of an analytic function, provided the basis $\phi_\alpha^{(1)}$ in the space of the holomorphic 1-differentials is chosen to be a holomorphic function of y_i. The latter can be achieved as follows. Let us choose on a genus-p surface S a symplectic basis of $2p$ closed orientable uncontractible paths $a_i, b_i, i = 1,...,p$, so that

$$a_i \circ a_j = b_i \circ b_j = 0 \;(i \neq j,\; a_i \circ b_j = \delta_{ij}), \quad (3.7)$$

where $a \circ b$ is the algebraic number of intersections (intersections are taken into account with natural signs). We know that the space of holomorphic 1-differentials (Abelian differentials of the first kind) has the complex dimension p, and we can take in this space a basis $\omega_i = \phi_i^{(1)}(z)dz$ of normalized differentials such that

$$\oint_{a_i} \omega_j = \delta_{ij}. \quad (3.8)$$

The matrix

$$\tau_{ij} = \oint_{b_i} \omega_j \quad (3.9)$$

is then called the period matrix of the surface S. In this basis,

$$(N_1)_{kj} = \frac{i}{2} \int_S \omega_k \wedge \bar{\omega}_j = \operatorname{Im} \tau_{kj}. \quad (3.10)$$

Substituting this in (2.19), and recalling (3.4), we obtain (1.3). The fact that $F(y)$ is holomorphic and has no zeros then follows from the fact that the regularized determinants in (2.19) should not vanish on nondegenerate surfaces (since the number of modes for each is constant: one in Δ_0 and none in Δ_{-1}), or become infinite. In principle, (3.2) does not preclude the possibility that $F(y_1)$ in (3.4) acquires a nonzero phase on a path γ in $\overline{M}_p$ and is thus not a function on $\overline{M}_p$, but on a covering of it. However, if $F(y)dV$ is a meromorphic form [which we shall prove by proving B)], then γ should be uncontractible. However, we know that there are no such paths: $H_1(\overline{M}_p,\mathbb{Z}) = 0 (p \geqslant 3)$, so that this possibility can be excluded. This proves property A) formulated in the Introduction.

We note that, soon after the publication of our preprint,[9] a more detailed discussion of the connection between proposition A) and the index theorem for a family of operators and the critical dimension 26 was given in Refs. 10 and 17. In addition, it was noted in Ref. 10 that (3.3) was a special case of the local variant of the index theorem obtained in Ref. 18.

Let us now briefly consider the connection between the holomorphy of the measure and conformal invariance. The second variation $\delta\bar{\delta}\widetilde{W}$ of the effective action of ghosts and fields X_μ can be expressed in terms of the correlation functions for the energy-momentum tensor operator $T_{ab} = T_{ab}^{X\mu} + T_{ab}^{gh}$:

$$\delta\bar{\delta}\widetilde{W} = \int d^2\xi \eta(\xi) \int d^2\xi' \overline{\eta(\xi')} \langle T_{++}(\xi) T_{--}(\xi')\rangle. \quad (3.11)$$

The naive conservation law yields $\partial_- T_{++} = \partial_+ T_{--} = 0$, so that, to within zero modes, $\langle T_{++}(\xi) T_{--}(\xi')\rangle = 0$. Because of conformal anomaly, this is not valid separately for ghosts and fields X_μ, since $\langle T_{+-}\rangle \neq 0$, $\partial_- T_{++} = -\partial_+ T_{+-}$. However, for $D = 26$, the anomaly cancels, so that $\langle T_{++}(\xi) \times T_{--}(\xi')\rangle = 0$ to within the zero modes. When the latter are included, we again obtain (3.3).

We now turn to an analysis of the behavior of the measure at infinity D of the space $\overline{M}_p$, at which the surfaces degenerate, and prove proposition B).

4. Singularities at infinity of moduli space and divergences[9]

In this Section, we shall be dealing in detail not with determinants, but with functional integrals. We shall examine the divergence of the integral

$$Z_p = \int \frac{d\Omega}{\det(f_i, f_j)} \frac{\left[\int D\varphi \exp\left(-\int \partial\bar{\varphi}\, \bar{\partial}\varphi\, d^2\xi\right)\right]^{13}}{\int D\varepsilon \exp\left(-\int \rho \partial\bar{\varepsilon} \bar{\partial}\varepsilon d^2\xi\right)}, \quad (4.1)$$

where φ is a complex scalar field, ε is the complex vector field of ghosts, f_j is a basis in the space of holomorphic quadratic differentials, related to the deformations of the complex structure

$$dz \to dz + \delta y_i \eta^i d\bar{z} \quad (4.2)$$

by

$$\int \eta^i f_j\, d^2\xi = \delta_j^i; \quad (4.3)$$

and y_i are complex coordinates in the moduli space M_p,

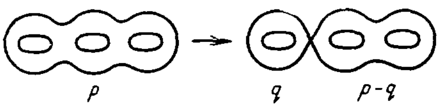

FIG. 1.

specified in the neighborhood of the given complex structure dz by expression (4.2). The scalar product is

$$(f_i, f_j) \overset{\text{def}}{=} \int f_i \bar{f}_j \rho^{-1} d^2\xi. \tag{4.4}$$

Formula (4.1) is also valid for $p = 1$. The measure in Z_p diverges in two cases.

Case I. A surface of genus p degenerates into two surfaces of genus q and $q - p$, respectively, with punctures at which the two surfaces are glued together. The manifold of such surfaces in moduli space $\bar{M}_p$ will be denoted by $D_q, q = 1,2,..., [p/2]$.

Case II. A surface of genus p degenerates into a genus $p - 1$ surface with two glued points that are the residues of a degenerate handle (Fig. 2). In $\bar{M}_p$, such surfaces lie on a manifold that we shall denote by D_0. Let us find the codimensions, i.e., $\dim\bar{M}_p - \dim D_\alpha$, of the subspace D_q and D_0 in $\bar{M}_p$. We shall use the fact that the complex dimension of the moduli space of a surface of genus $p \geqslant 1$ with n marked points is $3p - 3 + n$ [$2n$ coordinates of points on a polygon on the Lobachevskiĭ plane, $+ (6p - 6)$ parameters of the polygon], so that the dimensions D_q and D_0 are given by

$$\dim D_q = 3q - 3 + 1 + 3(p - q) - 3 + 1 = 3p - 4, \tag{4.5}$$
$$\dim D_0 = 3(p - 1) - 3 + 2 = 3p - 4.$$

Hence all the D_α have a complex codimension of 1 in $\bar{M}_p$, and actually augment the moduli space of the nonsingular spaces M_p to the compact space $\bar{M}_p$. To analyze the behavior of the measure in Z_p in a neighborhood $D = D_0 \cup D_1 \cup ... \cup D_{[p/2]}$, let us choose a coordinate y_1 in this neighborhood across D, and $y_2,...,y_{3p-3}$ along D, so that D is specified locally by

$$y_1(D) = 0. \tag{4.6}$$

We shall examine the measure as a function of y_1 for fixed $y_2,...,y_{3p-3}$. It can be shown that, in the neighborhood of y_1, a conformal transformation of the metric can be used to convert a degenerate neck into a very long cylinder (Fig. 3). To show this, let us take the coordinate $\tau, 0 \leqslant \tau \leqslant T$ along a cylinder of length T. When $T \gg 1$, the multiplication of the flat metric of the cylinder by the conformal factor $\lambda = \exp(-2\tau) + \exp(2\tau - 2T)$ converts it into two disks that have unit radius and are joined at the centers by a neck of radius $e^{-T} \equiv |y_1| \ll 1$. More precisely, the complex structure of the degenerate surface near the narrow neck is the same as the neck of the hyperbola $uv = y_1$ in $\mathbb{C}^2$ that degenerates into two planes at y_1: $u = 0$ and $v = 0$, which intersect

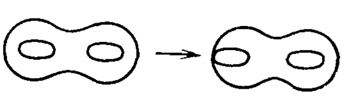

FIG. 2.

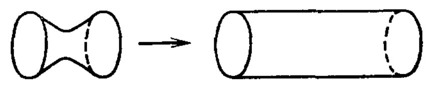

FIG. 3.

transversally at the point $u = v = 0$. The metric $\hat{g} = |du/u|^2$ transforms the neck of the hyperbola into a cylinder of length $T \sim \ln(1/|y_1|)$. Actually, $u = 0$ and $v = 0$ are the equations of the surfaces into which the original surface degenerates. This representation is convenient for the analysis of asymptotic behavior. We recall that the measure in Z_p does not depend on the choice of the conformal metric.

In both cases, the surface S_p will be considered as being the result of gluing together the cylinder and one (case II) or two (case I) disks (Fig. 4). The coordinates (τ,σ) on the cylinder K will be chosen as shown in Fig. 5. The surface from D corresponds to $T \to \infty$, and it is precisely this limit that will be of interest to us here. It will be clear from the ensuing discussion that the "natural" coordinate is

$$y_1 = \exp[-(T + i\delta)], \tag{4.7}$$

where δ is the angle of rotation of the right-hand edge of the cylinder K relative to the left-hand edge when the gluing to the disk is carried out.

We must first evaluate the functional integrals. We start by imposing the boundary conditions on the contours Γ_1, Γ_2 and Γ_3, Γ_4, then evaluate the integrals for the given boundary conditions, and finally multiply them together and integrate over the boundary conditions. In case I, we have for scalars

$$I_0 = \int D\varphi \exp\Big(-\int_{S_p} \partial\bar{\varphi}\,\bar{\partial}\varphi\, d^2\xi\Big) = \int D\varphi(0,\sigma) D\varphi(T,\sigma)$$
$$\times \exp(-v_1[\varphi(0,\sigma)] - v_2[\varphi(T,\sigma)])\, G[\varphi(0,\sigma), \varphi(T,\sigma)], \tag{4.8}$$

where

$$\exp(-v_1[\varphi(0,\sigma)]) = \int_{\text{condition } \varphi(0,\sigma)} D\varphi \exp\Big(-\int_{V_1} \partial\bar{\varphi}\,\bar{\partial}\varphi\, d^2\xi\Big),$$
$$G = \int_{\varphi(0,\sigma),\ \varphi(T,\sigma)} D\varphi \exp\Big(-\int_K \partial\bar{\varphi}\,\bar{\partial}\varphi\, d\sigma\, d\tau\Big). \tag{4.9}$$

Similarly, we determine $\exp(-v_2[\varphi(T,\sigma)])$ and $\exp(-v_3[\varphi(0,\sigma)\varphi(T,\sigma)])$ in case II. The same applies to ghosts, except that the action is $\int \rho\partial\bar{\varepsilon}\bar{\partial}\xi\, d^2\xi$ and not $\int \partial\bar{\varphi}\bar{\partial}\varphi\, d^2\xi$.

Since $\rho = 1$ on the cylinder K, we find that G is the same

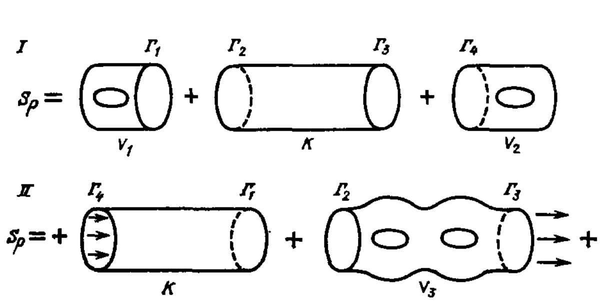

FIG. 4.

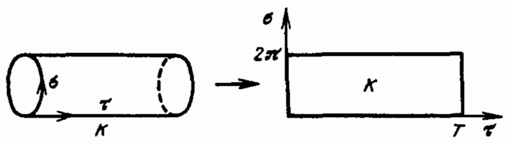

FIG. 5

for both scalars and ghosts:

$$G[\varphi(0,\sigma),\varphi(T,\sigma)]=\exp(-S_{\rm cl}[\varphi(0,\sigma),\varphi(T,\sigma)])\times(\det{}_{\rm Dir}\Delta_0)^{-1}, \quad (4.10)$$

where $S_{\rm cl}$ acts on the solution of the Laplace equation $\partial\bar\partial\varphi=0$ with boundary conditions

$$\varphi(0,\sigma)=\sum_{n=-\infty}^{+\infty}\varphi_n(0)e^{in\sigma},\quad \varphi(T,\sigma)=\sum_{n=-\infty}^{+\infty}\varphi_n(T)e^{in\sigma}, \quad (4.11)$$

and $\det_{\rm Dir}\Delta_0$ is the determinant of the Laplace operator on the cylinder K with Dirichlet conditions at the edges. Simple algebra then leads to

$$G=T^{-1}e^{T/6}\prod_{n=1}^{\infty}(1-e^{-2nT})^{-2}\times\exp\left(-\sum_{n=-\infty}^{+\infty}\frac{n}{\operatorname{sh}nT}e^{nT}|\varphi_n(0)-e^{-nT}\varphi_n(T)|^2\right). \quad (4.12)$$

Thus, in case I, we should find the asymptotic behavior for $T\to\infty$ of the following expression:

$$I_0=T^{-1}e^{T/6}\prod_{n=1}^{\infty}(1-e^{-2nT})^{-2}\int\prod_{n=-\infty}^{+\infty}D^2\varphi_n(0)D^2\varphi_n(T)\times\exp(-v_1[\varphi_1(0)]-S_{\rm cl}[\varphi_n(0),\varphi_n(T)]-v_2[\varphi_n(T)]). \quad (4.13)$$

As $T\to\infty$, we can neglect in $S_{\rm cl}$ the terms that tend to zero, provided only that this does not give rise to an additional degeneracy of the quadratic form $v_1+S_{\rm cl}+v_2$, i.e., to independence of $\varphi_n(0)$ or $\varphi_n(T)$.

It is clear from (4.12) that

$$S_{\rm cl}|_{T\to\infty}\sim T^{-1}|\varphi_0(0)-\varphi_0(T)|^2+\sum_{n>0}2n|\varphi_n(0)|^2+\sum_{n<0}2|n||\varphi_n(T)|^2\quad\ldots, \quad (4.14)$$

where the ellipsis represents exponentially small terms. We have retained $T^{-1}|\varphi_0(0)-\varphi_0(T)|^2$ because there are scalar zero modes on V_1 and V_2, and v_1 and v_2 do not depend on $\varphi_0(0)$ and $\varphi_0(T)$. The exponentially small terms can be neglected if v_1 does not become degenerate on any vector of the form

$$\sum_{n<0}a_ne^{in\sigma},$$

and v_2 on a vector of the form

$$\sum_{n>0}b_ne^{in\sigma}$$

We shall show that this is actually the case. Let us suppose that the opposite is true. In that case, there is a solution φ^* of the equation $\bar\partial\varphi=0$ that assumes the form

$$\sum_{n<0}a_ne^{in\sigma}$$

on Γ_1. Let us now imagine that the cylinder K extends to infinity on the right ($\tau\geqslant0$), and the metric on it is $e^{-2\tau}=\rho^*$ and not unity. In terms of the coordinates

$$u=\exp[-(\tau+i\sigma)]\equiv e^{-z}$$

the cylinder $0\leqslant\sigma<2\pi$, $0\leqslant\tau$ is the circle $|u|\leqslant1$ with constant unit metric

$$\rho^*dz\,d\bar z=|u|^2d\ln u\,d\ln\bar u=du\,d\bar u.$$

The surface V_1 is sealed by this circle and becomes the compact surface V_1^* of genus $q\geqslant1$.

The solution φ^* can be extended to the cylinder while maintaining its holomorphy:

$$\varphi^*(\tau+i\sigma)=\sum_{n<0}a_n\exp[n(\tau+i\sigma)].$$

Consequently, a holomorphic function φ^* that is not a constant exists on the surface V_1^*. However, it is known that such functions do not in fact exist, and this contradiction proves the original proposition.

The conclusion therefore is that the form v_1 does not have zero vectors such as

$$\sum_{n<0}a_ne^{in\sigma}.$$

Similarly, v_2 does not have zero vectors such as

$$\sum_{n>0}b_ne^{in\sigma}$$

and the exponentially small terms can indeed be neglected in $S_{\rm cl}$. Consequently,

$$I_0|_{T\to\infty}\sim T^{-1}e^{T/6}\int d^2\varphi_0(0)d^2\varphi_0(T)\times\exp(-|\varphi_0(0)-\varphi_0(T)|^2T^{-1})\sim e^{T/6}\int d^2(\varphi_0(0)+\varphi_0(T)). \quad (4.15)$$

As expected, we are left with the integral over the zero mode, i.e., the volume of the "Universe". Next, let us suppose that the integral is equal to unity. We are thus left with

$$I_0(T)|_{T\to\infty}\backsim e^{T/6}\quad(\text{на }D_q;\ q\neq0). \quad (4.16)$$

Let us now consider the integral over the ghosts:

$$I_1(T)=\int D\varepsilon\exp\left(-\int\rho\,\partial\bar\varepsilon\,\bar\partial\varepsilon\,d^2\xi\right). \quad (4.17)$$

The only difference as compared with the scalars is that we need not retain $|\varepsilon_0(0)-\varepsilon_0(T)|^2/T$ in (4.10) in $S_{\rm cl}$, since v_1^{gh} does not have zero vectors of the form

$$\sum_{n\leqslant0}a_ne^{in\sigma}$$

($n\leqslant0$ and not $n<0$ as in the case of scalars!), whereas v_2^{gh} does not have zero vectors of the form

$$\sum_{n\geqslant0}b_ne^{in\sigma}.$$

To show this, let us again assume the opposite, as in the scalar case. A solution ε^* of the equation $\bar\partial\varepsilon=0$ then exists on V_1 and can be extended to the cylinder K by means of the holomorphic function

$$\varepsilon^*(\tau+i\sigma)=\sum_{n\leqslant 0} a_n e^{n(\tau+i\sigma)}.$$

However, ε is now a vector and not a scalar. In terms of the coordinates u the solution

$$\varepsilon^*(u)=\varepsilon^*(z)\frac{du}{dz}=-\varepsilon^*(z)u=-\sum_{n\leqslant C} a_n u^{1-n}$$

vanishes for $u=0$. Consequently, the *nonconstant* holomorphic vector field $\varepsilon^*(z)$ exists on the sealed surface V_1^*. However, we know that such fields do not exist if the genus q of the surface V_1^* is greater than or equal to unity. Moreover, when $q>1$, no holomorphic vector fields exist, and for $q=1$ there is one holomorphic vector field that does not vanish *anywhere. Hence we conclude that v_1^{gh} has no zero vectors of* the form

$$\sum_{n\leqslant 0} a_n e^{in\sigma}.$$

Since, in this case, we need not retain the term $|\varepsilon_0(0)-\varepsilon_0(T)|^2/T$ in S_{c1} in (3.10), the entire leading part of the dependence on T is determined by the cofactor $\det_{\rm Dir}\Delta_0(T)$ in G:

$$I_{-1}(T)|_{T\to\infty}\propto T^{-1}e^{T/6}\quad(\text{on } D_q,\ q\neq 0). \tag{4.18}$$

On the other hand, the ratio of the functional integrals in the measure Z_p exhibits the following behavior:[4]

$$I_{(I)}=\frac{I_0^{13}}{I_{-1}}\bigg|_{T\to\infty}\propto Te^{2T}\ (\text{on } D_q,\ q\neq 0). \tag{4.19}$$

Case II can be treated in exactly the same way, except that V_3 is now sealed by two discs, one on the right and the other on the left. The only difference is that now, and in the case of the scalar fields, we can neglect the term $|\varphi_0(0)-\varphi_0(T)|^2/T$ in S_{c1}. This is so because, in v_3, it is only the sum $\varphi_0(0)+\varphi_0(T)$ and not $\varphi_0(0)$ and $\varphi_0(T)$ separately that is the zero mode (as was the case for v_1+v_2 in case I).

We therefore conclude that, in case II, we have $I_0\sim I_{-1}$, and the ratio of the functional integrals is

$$I_{(II)}=\frac{I_0^{13}}{I_{-1}}\bigg|_{T\to\infty}\sim T^{-12}e^{2T}\ (\text{on } D_0). \tag{4.20}$$

All that remains is to evaluate the form of the volume

$$\frac{d\Omega}{\det(f_i,f_j)}=\prod_{l=1}^{3p-3} dy_l\wedge d\bar{y}_l \det{}^{-1}(f_i,f_j). \tag{4.21}$$

We shall do it by exploiting the fact that the variations of T is related to a Beltrami differential that is constant on K. Actually, a variation of the complex structure generated by this differential $dz\to dz+ad\bar{z}$ may be looked upon as the transformation $z\to\tilde{z}=z+a\bar{z}$ that converts the rectangle $0\leqslant\sigma\leqslant 2\pi$, $0\leqslant\tau\leqslant T(z=\tau+i\sigma)$ into a parallelogram. The new value of the coordinate $(T+i\delta)/2\pi i$ in the space $\overline{M}_p$ is the complex ratio of the periods of this parallelogram

$$d\,\frac{T+i\delta}{2\pi i}=\frac{\tilde{z}(T)}{\tilde{z}(2\pi i)}-\frac{T}{2\pi i}=\frac{aT}{\pi i},$$

and hence $a=D(T+i\delta)/2T$. Consequently, the coordinate $\tilde{y}_1=T+i\delta$ in moduli space $\overline{M}_p$ corresponds to the Beltrami differential $\eta^1=1/2T$, since the shift by $d\tilde{y}_1$ in $\overline{M}_p$ corresponds, as we have shown, to the following variation of the complex structure:

$$dz\to dz+d\tilde{y}_1\eta^1 d\bar{z}.$$

From (4.3) we find that the quadratic differential[5] f_1 on K is equal to unity. The coordinate $\tilde{y}_1$ is directed across D. The remaining quadratic differentials can be chosen so that $(f_i,f_1)\sim 1$ and the corresponding coordinates are directed along D. In that case,

$$\det(f_i,f_j)\propto(\tilde{f}_1,\tilde{f}_1)\propto T,$$

$$\begin{aligned}\frac{d\Omega}{\det(f_i,f_j)}&\propto\frac{d\tilde{y}_1\wedge d\bar{\tilde{y}}_1}{T}\wedge d\Omega_{\parallel}\\&=\frac{dT\,d\delta}{T}d\Omega_{\parallel}=\frac{e^{2T}}{T}dy_1\wedge d\bar{y}_1\wedge d\Omega_{\parallel}\\&=\frac{dy_1\wedge d\bar{y}_1}{|y_1|^2\ln(1/|y_1|)}\wedge d\Omega_{\parallel},\end{aligned} \tag{4.22}$$

where

$$y_1=e^{-\tilde{y}_1}=\exp[-(T+i\delta)].$$

The surface D is specified by the equation $y_1(D)=0$. This coordinate is acceptable since, as we can see from (4.12), the ratio of the determinants can be expanded into a series in $y_1,\bar{y}_1$, and the divergences that appear to be exponential in terms of the coordinates $y_1,\ldots$ are in fact terms raised to a power.

Collecting (4.19)–(4.22) together, we find that the asymptotic behavior of the measure in the neighborhood of $D_q, q\neq 0$ is

$$\frac{d\Omega}{\det(f_i,f_j)}\frac{I_0^{13}}{I_{-1}}\bigg|_{|y_1|\to 0}\sim\frac{d^2y_1}{|y_1|^4}\sim dTe^{2T} \tag{4.23}$$

and in the neighborhood of D_0

$$\frac{d\Omega}{\det(f_i,f_j)}\frac{I_0^{13}}{I_{-1}}\bigg|_{|y_1|\to 0}\sim\frac{d^2y_1}{|y_1|^4[\ln(1/|y_1|)]^{13}}\sim\frac{dT}{T^{13}}e^{2T}. \tag{4.24}$$

To find the order of the pole of the form $F(y)dv$ from (1.3) and (3.4), we need only estimate the period matrix (3.25) in the neighborhood of the surface D. In case I, in which the surface splits into the two surfaces S_q and S_{p-q} of genus q and $p-q$, respectively, we find that the p holomorphic 1-differentials ω_i transform into the holomorphic 1-differentials $\omega'_\alpha, \alpha=1,\ldots,q$ on S_q and $\omega''_\beta,\beta=1,\ldots,p-q$ on S_{p-q}. The period matrix $\hat{\tau}$ then assumes a block form, and $\det\operatorname{Im}\hat{\tau}(y)$ has a finite limit as $y_1\to 0$. Therefore, in the neighborhood of $D_q, q\neq 0$, we have

$$\det\operatorname{Im}\hat{\tau}(y)|_{y_1\to 0}\propto\det\operatorname{Im}\hat{\tau}'\cdot\det\operatorname{Im}\hat{\tau}'', \tag{4.25}$$

where $\hat{\tau}'(\hat{\tau}'')$ is the period matrix of $S_q\,(S_{p-q})$. Hence, to estimate the period matrix in case II, i.e., for a degenerate handle, we take the basis of cycles (3.23) so that the cycle a_p runs across the cylinder K, as in Fig. 4 (case II), and the cycle b_p runs along the cylinder. We now choose the coordinate $z=\tau+i\sigma$ on K (see Fig. 5). From the relation

$$\oint_{a_l}\omega_j=\delta_{lj}$$

and the holomorphic condition $\bar\partial\omega_i = 0$ it then follows that, when $T \gg 1$, all the differentials other than ω_p decay exponentially on the cylinder

$$\omega_i \lesssim e^{-\tau} + e^{-(T-\tau)} \quad (i = 1, \ldots, p-1),$$

and $\omega_p = 1/2\pi i$ for $\tau \gg 1, T - \tau \gg 1$. Hence, the only divergent element of the period matrix for $T \to \infty$ is

$$\tau_{pp} = \oint_{b_p} \omega_p \approx \frac{T}{2\pi i}$$

and, in the neighborhood of D_0, we have[6)]

$$\det \operatorname{Im} \hat\tau |_{y_1 \to 0} \propto T = \ln \frac{1}{|y_1|}. \qquad (4.26)$$

Substituting (4.25) and (4.26) in (1.2), and comparing with (4.23) and (4.24), respectively, we find that the form $F(y)\mathrm{d}v$ has a second-order pole on D:

$$F(y)\,\mathrm{d}v|_{y_1 \to 0} \propto y_1^{-2} \mathrm{d}y_1 \wedge \mathrm{d}v_{\parallel}. \qquad (4.27)$$

We have therefore proved property B) of the measure, formulated in the Introduction.

We shall now show that the condition for the absence of zeros in M_p and the asymptotic behavior (4.27) define the form $F(y)\mathrm{d}v$ uniquely, except for a constant factor. Actually, the ratio of any two forms F' and F'' satisfying these conditions is a meromorphic function on $\bar M_p$ that does not vanish or become infinite anywhere except, possibly, for intersections of the components D_α of the surface D [i.e., points at which the coefficient in front of y_i^{-2} in (4.27) can have a singularity]. Hence, it follows that, either $F'/F'' = \text{const}$, or the manifold of zeros and poles of the function F'/F'' has the complex codimension 2 in $\bar M_p$. However, we know that the manifold of zeros and poles of a nonconstant meromorphic function on a compact algebraic manifold has the complex codimension 1. This means that $F'/F'' = \text{const}$.

The asymptotic behavior of (4.23) and (4.24) has a very simply interpretation. If we look upon string theory as a theory of an infinite number of interacting particles, then K in Fig. 5 can be interpreted as the propagator in the representation of the proper time T, and the integral of the measure over T can be written in the following form in case II:

$$Z_p^{\text{div.II}} = \int \mathrm{d}^D p_\mu \int_0^\infty \mathrm{d}T \sum_r V_3(r, p_\mu) \exp[-(p_\mu^2 + m_r^2)T], \qquad (4.28)$$

where the sum is evaluated over all particles corresponding to different excited states of the string (p_μ is the momentum in the loop). For large T, small p_μ^2 are significant in the integral with respect to momenta in (4.28), and the measure in (4.28) has the asymptotic form

$$\mathrm{d}T T^{-D/2} \sum_r \exp(-m_r^2 T) V_3(r, 0), \qquad (4.29)$$

from which it follows that the main contribution to the integral in (4.28) for large T is provided by tachyons and massless states

$$Z_p^{\text{div.II}} \sim \int^\infty \mathrm{d}T T^{-D/2} \sum_{m_r^2 \leqslant 0} e^{-m_r^2 T} V_3(r, 0). \qquad (4.30)$$

We now recall that, in a closed bosonic string, the ground state is a tachyon[20] with $m_0^2 = -2$ (in our normalization) and the excited states have $m_r^2 \geqslant 0$ (multiplet of massless excitations containing the graviton $g^{\mu\nu}$, the tensor $A^{[\mu\nu]}$, and the dilaton ϕ). Since $D = 26$, we then find from (4.30) that

$$Z_p^{\text{div.II}} \sim \int^\infty \mathrm{d}T T^{-13} e^{2T},$$

which is identical with (4.24). We note that, when $D > 2$, the massless states do not contribute to the divergence of the integral (4.30).

In case I, the particles propagate between vertices V_1 and V_2 (see Figs. 4 and 1) with zero momentum, and instead of (4.30) we have

$$Z_p^{\text{div.I}} \sim \int^\infty \mathrm{d}T \sum_{m_r^2 \leqslant 0} e^{-m_r^2 T} V_1(r, 0) V_2(r, 0), \qquad (4.31)$$

so that the main contribution to the divergence in (4.31) is again associated with a tachyon and has the form

$$\int^\infty \mathrm{d}T e^{2T},$$

which is identical with (4.23). However, it is clear that, in this case, there is also a divergence associated with the massless dilaton ($g^{\mu\nu}$ and $A^{[\mu\nu]}$ are not created from vacuum), which evidently leads to the renormalization of the slope of the Regge trajectory.

The order γ of the pole of $F(y)\mathrm{d}v$ is therefore

$$\gamma = 1 - \frac{m_0^2}{2}. \qquad (4.32)$$

To conclude this Section, we note that a similar method of estimating functional integrals and of analyzing divergences was used in Refs. 21 and 22 in which it was based on the Selberg trace formula.

5. Measure in a bosonic string and Mumford's theorem

In this Section, we examine the mathematically rigorous formulation of the above results, and consider their relation to Mumford's theorem.[7] It follows from (2.19) and (3.4) that $F(y)$ is not a function but a section of a line bundle E above M_p (i.e., the complex dimension of a fibre is equal to unity). More precisely, $F(y)$ is the contribution to the measure due to left excitations of the string

$$F(y) = \det \bar\partial_{-1} \cdot (\det \bar\partial_0)^{-13}, \qquad (5.1)$$

where $\bar\partial_j$ acts in the space of j-differentials and $\det \bar\partial_j$ is a section of a line bundle[23] with a fiber generated by the vector

$$\Phi_1^{(j)} \wedge \ldots \wedge \Phi_{\alpha_j}^{(j)} \wedge \Phi_1^{(1-j)} \wedge \ldots \wedge \Phi_{\alpha_{1-j}}^{(1-j)},$$

in which $\Phi_\alpha^{(j)}$ is a basis in the space $\operatorname{Ker}\bar\partial_j$ of holomorphic j-differentials and $\Phi_\alpha^{(1-j)}$ is a basis in $(\operatorname{Coker}\bar\partial_j)^* \approx \operatorname{Ker}\partial_{1-j}$. Thus, E is the tensor product of two line bundles over M_p

$$E = K \otimes \lambda^{-13}, \qquad (5.2)$$

where K is a bundle of $3p-3{,}0$ forms with fibers generated by the vector $\mathrm{d}v = \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_{3p-3}$, and λ is a bundle of modular forms with fibers generated by $\omega_i \wedge \ldots \wedge \omega_p$ where $\{\omega_i\}$ is a basis in the space of holomorphic 1-forms. We have chosen it just as at the end of Sec. 3. The bundle λ is nontrivial because the basis of cycles can change as we circulate on a closed curve g in $\overline{M}_p$. The section of $F(y)$ is well defined only when the gravitational anomalies cancel[24] in (5.1). This does actually occur,[15] and the condition for the cancelation of the gravitational anomaly is in fact equivalent[25] to the condition for the cancelation of the conformal anomaly in the ratio $\det\Delta_{-1}/\det'\Delta_0$.

The theorem[7)] proved by Mumford[7] by evaluating the characteristic class $c_1(E)$ of a bundle E states that this bundle is trivial on M_p (in particular,this reflects the absence of topological obstacles to the cancelation of anomalies). Moreover, it also follows from the evaluation of $c_1(E)$ that a section of F that is holomorphic and does not vanish on M_p exists in E and has a second-order pole at the infinity D. Moreover, the Wolpert theorem[13] on the independence of components $D_0,...,D_{[p/2]}$ of infinity D in the group $H_{6p-6}(\overline{M}_p,Q)$ of homologies of $\overline{M}_p$ enables us to conclude that any holomorphic section E that does not vanish on M_p differs from F by a constant factor. As noted by Beilinson and Drinfield, the square of the modulus of the section of F can be used to determine the measure on $\overline{M}_p$:

$$d\mu = d v \wedge d\bar{v}\,|F(y)|^2(\det(\omega_i,\ \omega_j))^{-13}, \qquad (5.3)$$

where

$$(\omega_i,\ \omega_j)\overset{\text{def}}{=}\frac{i}{2}\int \omega_i \wedge \bar{\omega}_j,$$

and $\det(\omega_i,\omega_j)$ is a natural Hermitian matrix on λ. In the basis chosen at the end of Sec. 3, it is identical with $\det \operatorname{Im} \hat{\tau}$. Comparison of (5.3) with (2.19), (3.4), and (4.27) shows that it is precisely this measure that arises in the theory of bosonic strings. We have thus proved the following theorem:

Theorem.[9] *The integration measure in the theory of closed oriented bosonic strings is the square of the modulus of the global, holomorphic, and nonzero on M_p section of the bundle $K \otimes \lambda^{-13}$, divided by the thirteenth power of the natural metric on λ.*

Since the holomorphic structure on the moduli space arises from an algebraic structure, any holomorphic object upon it, e..g., the section of F that arises in string theory, is an algebraic object (in accordance with the GAGA principle[27]).

Our results are naturally generalized by the following conjecture.

Conjecture.[9] Multiloop amplitudes (and not only vacuum amplitudes) in any conformally invariant string theory (such as the bosonic string in $D = 26$ or the superstring in $D = 10$) can be expressed in terms of algebraic objects (functions or sections of holomorphic bundles) on the moduli space of Riemann surfaces.

Quantum geometry is therefore the complex geometry of the space $\overline{M}_p$.

II. EXPLICIT FORMULAS FOR THE MEASURE IN TERMS OF THETA-FUNCTIONS

We showed above that summation over closed oriented surfaces of genus $p \geqslant 2$ (which determines p-loop vacuum amplitudes in the theory of bosonic strings) reduces for the critical dimension $D = 26$ to integration over the space M_p of complex structures of Riemann surfaces of genus p. We have investigated the analytic properties of the integration measure as a function of complex coordinates on M_p. The measure multiplied by $(\det \operatorname{Im}\hat{\tau})^{-13}$ ($\hat{\tau}$ is the period matrix) is the square of the modulus of a function that is holomorphic on M_p and does not vanish anywhere. This function has a second-order pole at the infinity $D = \overline{M}_p/M_p$ of the compactified moduli space M_p. These properties define the measure uniquely to within an arbitrary constant, and this enables us to construct explicit formulas for genus $p = 2$, 3, and 4 in terms of theta-functions.

6. The measure for $p = 2, 3, 4$

In this Section, we reproduce the formulas for the measure with $p = 2$ and $p = 3$ that were reported in Ref. 28, and will formulate a conjecture about the form of the measure for $p = 4$ (Refs. 9 and 29). The direct evaluation of the measure for $p = 2$ is given in Sec. 11. Simple formulas for $p = 2$ and $p = 3$ can be written down because, in these cases, we have an explicit parametrization of the space M_p by the period matrices, which we shall now describe.

On an arbitrary Riemann surface S_p of genus p in a symplectic basis of cycles (closed paths) $a_i, b_i, i = 1,...,p$,

$$a_i \circ a_j = b_i \circ b_j = 0 \quad (i \neq j,\ a_i \circ b_j = \delta_{ij}), \qquad (6.1)$$

which was introduced at the end of Sec. 3, and the related basis of holomorphic 1-differentials ω_i such that

$$\int_{a_i} \omega_j = \delta_{ij}, \qquad (6.2)$$

we can construct the period matrix

$$\tau_{ik} = \oint_{b_i} \omega_k, \qquad (6.3)$$

satisfying the Riemann relations[30]

$$\tau_{ik} = \tau_{ki},\ \operatorname{Im} \tau > 0. \qquad (6.4)$$

These relations ensue from

$$\int_{S_p} \omega \wedge \bar{\omega}' = \sum_{i=1}^{p} \left(\int_{a_i} \omega \overline{\int_{b_i} \omega'} - \int_{b_i} \omega \overline{\int_{a_i} \omega'} \right),$$

where ω and ω' are arbitrary holomorphic 1-differentials, and from the fact that the norm of the nonzero differentials ω is positive:

$$\|\omega\|^2 = \frac{i}{2} \int_{S_p} \omega \wedge \bar{\omega} > 0.$$

The Torelli theorem states that a complex structure is uniquely determined by the period matrix to within a diffeomorphism. Thus, complex structures can be parametrized by the matrices τ. However, an infinite number of matrices τ can correspond to a given surface. Actually, the basis $\{a_i, b_i\}$ is not uniquely determined by (6.1). We can find another basis

$$b_i' = A_{ik} b_k + B_{ik} a_k, \quad a_i' = C_{ik} b_k + D_{ik} a_k, \qquad (6.5)$$

that will satisfy (6.1) if the integer matrices A, B, C, and D satisfy the conditions

$$AB^T - BA^T = CD^T - DC^T = 0,\ AD^T - BC^T = 1, \quad (6.6)$$

i.e.,

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in S_p(p,\mathbb{Z})\equiv\Gamma_p.$$

The group Γ_p is the Siegel modular group of degree p. Under the transformations (6.5), the basis of differentials (6.2) becomes

$$\omega_i'=\omega_k(C\tau+D)^{-1}_{ki}, \qquad (6.7)$$

from which we find that the period matrix in the basis (6.5) takes the form

$$\tau'=(A\tau+B)(C\tau+D)^{-1}. \qquad (6.8)$$

Thus, to avoid including the same surface more than once, we must confine our attention to the factor space

$$\mathfrak{S}_p=\frac{\mathscr{H}_p}{\Gamma_p},$$

where $\mathscr{H}_p$ denotes the space of all the symmetric $p\times p$ matrices with positive-definite imaginary parts and is called Siegel's upper half-plane. The group Γ_p acts upon it by the transformations (1.8). The manifold $\mathfrak{S}_p$ has the complex dimension $p(p+1)/2$ that is equal to the dimension of the space M_p for $p=1, 2, 3$. Actually, $\mathfrak{S}_p$ and M_p coincide in these cases. Thus, finally, for $p=1,2,3$, the space M_p can be parametrized by matrices covering the fundamental region $\mathfrak{S}_p$ of the group Γ_p in Siegel's upper half-space $\mathscr{H}_p$.

It follows from (1.13) for $p=2$ and 3 that the measure should take the form[28]

$$Z_p=\int\limits_{\mathfrak{S}_p}\prod_{k\leq i}\frac{i}{2}\,d\tau_{kj}\wedge d\bar{\tau}_{kj}\,|\chi_{12-p}(\tau)|^{-2}(\det\operatorname{Im}\tau)^{-13}. \qquad (6.9)$$

It can be shown that the natural modular-invariant measure on $\mathfrak{S}_p$ is

$$d\mu_p=\prod_{k\leq i}\frac{i}{2}\,d\tau_{kj}\wedge d\bar{\tau}_{kj}\,(\det\operatorname{Im}\tau)^{-p-1}.$$

In addition, it follows from (6.6) that

$$\det\operatorname{Im}\tau'=|\det(C\tau+D)|^{-2}\det\operatorname{Im}\tau. \qquad (6.10)$$

Hence, the condition that the measure in (6.9) is the measure on $\mathfrak{S}_p$, i.e., it is modular-invariant, takes the form

$$\chi_k(\tau')=[\det(C\tau+D)]^k\chi_k(\tau),\quad k=12-p \qquad (6.11)$$

[for $p=3$ this formula must be made more precise; see below]. Next, the form $\Pi_{i<j}d\tau_{ij}$ has a first-order pole on the component D_0 of infinity ($\operatorname{Im}\tau_{11}\to\infty$), and a zero of order $p-2$ on the component D_1 for which τ assumes a block form (for $p=2$, 3 there are no other components). It therefore follows from property B) of Section 1 and (6.11) that $\chi_{k(\tau)}$ is a parabolic modular form of weight $k=12-p$ on $\mathfrak{S}_p$, which has a zero of order p on D_1. The function τ that transforms in accordance with (6.11) and is holomorphic on $\mathscr{H}_p$ is called a modular form (of Siegel) of weight k on $\mathfrak{S}_p$. The weight k must be even for odd p. The modular form that vanishes on D_0 is called parabolic. If, on the other hand, p and k are odd, the form must be determined by further multiplication by the character of the group Γ_p, since τ' does not change and the right-hand side of (6.11) changes sign when the signs of A, B, C, and D are simultaneously reversed.

The space of modular forms on $\mathfrak{S}_p$ has been well studied for $p=1, 2$. For $p=1$, all the forms are linear combinations of forms of weight 4 and 6, and their number is given by

$$p_1(t)=\sum_{k=0}^{\infty}d_{2k}(1)t^{2k}=(1-t^4)^{-1}(1-t^6)^{-1}, \qquad (6.12)$$

where $d_{2k}(p)$ represents the number of linearly independent modular forms of weight $2k$ on $\mathfrak{S}_p$. The situation is similar for $p=2$, but is somewhat more complicated.[31] If we confine our attention to forms of even weight, we find that there are four main forms of weight 4, 6, 10, and 12:

$$p_2(t)=(1-t^4)^{-1}(1-t^6)^{-1}(1-t^{10})^{-1}(1-t^{12})^{-1}. \qquad (6.13)$$

The expressions for the main forms are given in Ref. 31 in terms of the Eisenstein series and theta-constants. It is also shown in Ref. 31 that there is a unique parabolic form of weight 10. It must therefore be identical with χ_{10} in (6.9) and have a second-order zero on D_1. This can be readily verified with the aid of the following formula:[31]

$$\chi_{10}(\tau)=\prod_m\theta_m^2(\tau), \qquad (6.14)$$

where the theta-constants are defined by

$$\theta_m(z;\tau)=\sum_{\mathbf{n}\in Z^p}\exp\left[\pi i\left(\mathbf{n}+\frac{\mathbf{m}'}{2}\right)^t\tau\left(\mathbf{n}+\frac{\mathbf{m}'}{2}\right)+2\pi i\left(\mathbf{n}+\frac{\mathbf{m}'}{2}\right)^t\left(\mathbf{z}+\frac{\mathbf{m}''}{2}\right)\right], \qquad (6.15)$$

$$\theta_m(\tau)\equiv\theta_m(0;\tau),\quad \theta_{m,k_1\ldots k_s}=\left.\frac{\partial^s\theta_m(z;\tau)}{\partial z_{k_1}\ldots\partial z_{k_s}}\right|_{z=0},\quad m\equiv(\mathbf{m}',\mathbf{m}''),$$

and the components of the vectors $\mathbf{m}'$, $\mathbf{m}''$ assume the values 0, 1. The quantity

$$e(m)=\mathbf{m}'\cdot\mathbf{m}''\ (\mathrm{mod}\ 2) \qquad (6.16)$$

is called the parity of the characteristic m, and the product in (6.14) is evaluated only over even characteristics. For genus p, there are $2^{p-1}(2^p+1)$ even and $2^{p-1}(2^p-1)$ odd characteristics. If $e(m)=1$, then $\theta_m(0,\tau)=0$. It follows from (6.14) and (6.15) that, as $\tau_{12}\to0$,

$$\chi_{10}\sim-2^{14}\exp[2\pi i(\tau_{11}+\tau_{22})](\pi\tau_{12})^2, \qquad (6.17)$$

as required.

We shall now show, following Ref. 28 (see also Ref. 32), that the form $\chi_{10}(\tau)$ has no zeros inside $\mathfrak{S}_2$. We shall do thus using the following formula for the fermionic determinant[9,33] proved in Sec. 7:

$$\det{}_m\bar{\partial}_{1/2}\cdot(\det\bar{\partial}_0)^{1/2}=\theta_m(\tau) \qquad (6.18)$$

where $\det_m\bar{\partial}_{1/2}$ is the determinant of the Dirac operator acting on the space of left-handed Weyl fermions "living" on S_p and satisfying the periodic (antiperiodic) boundary conditions on complete cycles a_i for $\mathbf{m}_i'=1(0)$ and cycles b_i for $\mathbf{m}_i''=1(0)$:

$$|\det\bar{\partial}_0|^2\overset{\mathrm{def}}{=}\frac{\det'\Delta_0}{(\det\operatorname{Im}\tau)\left(\int\rho d^2\xi\right)}. \qquad (6.19)$$

A more precise definition is discussed in Sec. 7.

It follows from (6.18) and (6.19) that $\theta_m(\tau)$ vanishes on surfaces S_2 on which $\bar{\partial}_{1/2}$ acquires zero fermionic modes with boundary conditions m, i.e., when holomorphic 1/2-

differentials with the characteristic m exist on S_2. According to Riemann's theorem on singularities, the parity of the number of such $\frac{1}{2}$-differentials is the same as the parity $e(m)$ of the characteristic m. Moreover, for a general surface, their number is $e(m)$. Thus, the vanishing of $\theta_m(\tau)$ for $e(m)=0$ on the surface S_2^* signifies that at least two holomorphic 1/2-differentials $\psi_1(z)(dz)^{1/2}$ and $\psi_2(z)(dz)^{1/2}$ with the characteristic m exist on S_2^*. Moreover, we know that the holomorphic j-differential on a genus-p surface has $2j(p-1)$ zeros. Hence, the meromorphic function $f(z)=\psi_1(z)/\psi_2(z)$ exists on the surface S_2^* and has one zero and one pole. It follows that S_2^* has genus 0. This contradiction demonstrates that $\theta_m(\tau)$ has no zeros for $e(m)=0$ inside $\mathfrak{S}_2$. The absence of 1/2-differentials with even characteristics also follows from the fact that any genus-2 surface is a hyperelliptic curve

$$y^2=(z-a_1)\cdot\ldots\cdot(z-a_6) \qquad (6.20)$$

in $\mathbb{C}^2=(y,z)$, and it is readily verified that there are exactly six holomorphic $\frac{1}{2}$-differentials

$$\psi_i=\left(\frac{z-a_i}{y}\right)^{1/2}(dz)^{1/2} \qquad (6.21)$$

(one for each of the six odd characteristics).

Let us now turn to $p=3$. The measure

$$\prod_{i\leq j} d\tau_{ij}$$

now has a pole of order 1 on D_0 and a zero of order 1 on D_1. Moreover, as noted above, the form χ_9 of weight 9 can be determined only with values in the character of Γ_3. On the other hand, the usual complex-valued form of weight 18 is

$$\chi_9^2=\chi_{18}. \qquad (6.22)$$

It must have a zero of order 2 on D_0 and of order 6 on D_1. This form exists and is given by[34]

$$\chi_{18}=\prod_m \theta_m(\tau), \qquad (6.23)$$

where the product is evaluated over all the 36 even characteristics. However, χ_{18} vanishes not only on D_0 and D_1, but also on the manifold D_* of hyperelliptic surfaces inside $\mathfrak{S}_3$ (Ref. 8). Actually, on the genus-3 hyperelliptic surface

$$y^2=(z-a_1)\cdot\ldots\cdot(z-a_8), \qquad (6.24)$$

there are in addition to the 28 fermionic zero modes

$$\psi^{(ij)}=\left[\frac{(z-a_i)(z-a_j)\,dz}{y}\right]^{1/2},\quad i<j, \qquad (6.25)$$

(one for each of the 28 odd characteristics), two modes

$$\psi^{(0)}=y^{-1/2}(dz)^{1/2},\quad \psi^{(1)}=z\psi^{(0)} \qquad (6.26)$$

with the same even characteristic, the particular value of which depends on the choice of the basis of cycles (6.1) on the curve (6.24). Hence, χ_{18} does indeed vanish on D_* and, in terms of the coordinates y_i of Sec. 2, this is a second-order zero. Conversely, if the form χ_{18} vanishes on the surface S_3^*, then there are on S_3^* at least two holomorphic 1/2-differentials $\psi^{(0)}$ and $\psi^{(1)}$ with the same even characteristic, and their ratio $f(z)=\psi^{(0)}/\psi^{(1)}$ is a meromorphic function on S_3^* with two zeros and two poles, i.e., S_3^* is a two-sheet covering $\mathbb{CP}^1$. Consequently, S_3^* is a hyperelliptic surface.

We conclude that χ_{18} vanishes in the interior of $\mathfrak{S}_3=\mathbf{M}_3$ on hyperelliptic surfaces and only on such surfaces. In terms of the coordinates y_i of Sec. 2, this is a second-order pole, and the root $\chi_9=\chi_{18}^{1/2}$ can be extracted. We still have to demonstrate that, in terms of the coordinates y_i, the measure $\Pi_{i\leq j}d\tau_{ij}$ has a zero of order 1. To show this, let us take the following basis of holomorphic quadratic differentials on the surface (6.24):

$$f_k=z^{k-1}y^{-2}(dz)^2\quad(k=1,\ldots,5),\ f_6=y^{-1}(dz)^2 \qquad (6.27)$$

and the metric $\rho dzd\bar z$ that is symmetric under the transformation γ: $(y,z)\to(-y,z)$. We can then take η^6 to be odd in γ:

$$\eta^6=\text{const}\cdot\frac{\bar f_6}{\rho\, d z\, d\bar z}. \qquad (6.28)$$

Moreover, the Abelian holomorphic differentials $\omega_i=\phi_i(z)dz$ are linear combinations of

$$\frac{dz}{y},\ \frac{z\,dz}{y},\ \frac{z^2dz}{y}.$$

It then follows from (6.28) and from the formula

$$\frac{\partial\tau_{ab}}{\partial y_i}=-\int_{S_p}(\eta^i\omega_a)\wedge\omega_b, \qquad (6.29)$$

which is valid for any genus, that

$$\frac{\partial\tau_{ab}}{\partial y_6}=0, \qquad (6.30)$$

so that, having taken $y_6(D_*)=0$, we find that

$$\prod_{i\leq j}d\tau_{ij}\underset{y_6\to0}{\sim}y_6dy_1\ldots dy_6.$$

Consequently, the measure

$$\prod_{i\leq j}d\tau_{ij}\chi_{18}^{-1/2}\propto dy_1\ldots dy_6 \qquad (6.31)$$

is holomorphic in y_i in the neighborhood of D_*, and does not vanish, as expected from the main theorem of Sec. 5. The integral root-type singularity that appears for $p=3$ in the measure (6.9), (6.22), and (6.23), naively contradicts the holomorphy proved in Secs. 1–5 and is due to the fact that M_3 does not cover $\mathfrak{S}_3$ smoothly in the neighborhood of the manifold of hyperelliptic surfaces (of symmetry γ). We have thus demonstrated the validity of (6.9), (6.22), and (6.23) for the measure in the case of genus $p=3$. We note also that the analytic properties of the forms χ_{10} and χ_{18} were investigated by Igusa[31,34] by other methods.

The explicit formulas given by (6.9) and (6.19), (6.23) are usefully augmented by the formulas for the tachyon scattering amplitudes.[9,29] This involves the evaluation of the Gaussian integral[5]

$$\int DX_\mu\exp\left(-\frac{1}{\pi}\int\partial X^\mu\bar\partial X_\mu d^2\xi\right)\prod_{k=1}^N\int\exp(ip_k^\mu X_\mu(\xi_k))\rho(\xi_k)d^2\xi_k$$
$$=\left(\frac{\det N_0}{\det'\Delta_0}\right)^3\exp\left(-\frac{1}{\pi}\int\partial X_{\rm cl}^\mu\bar\partial X_{{\rm cl},\mu}d^2\xi\right)$$
$$\times\prod_{k=1}^N\int\exp(ip_k^\mu X_{\mu,{\rm cl}}(\xi_k))\rho(\xi_k)d^2\xi_k$$
$$\overset{\rm def}{=}\left(\frac{\det N_0}{\det'\Delta_0}\right)^{13}K(\mathbf{p}_1,\ldots,\mathbf{p}_N;\tau), \qquad (6.32)$$

where $X^\mu_{cl}(\xi)$ is a solution of

$$-2\partial\bar{\partial}X^\mu_{cl}(\xi)=\sum_{k=1}^{N} i\pi p^\mu_k\delta(\xi-\xi_k). \qquad (6.33)$$

Because of the conservation of momentum

$$\sum_{k=1}^{N} p^\mu_k=0, \qquad (6.34)$$

the solution (6.33) can be readily expressed in terms of the theta-functions (6.15):

$$X^\mu_{cl}(\xi)=-\sum_{k=1}^{N} ip^\mu_k[\ln|\theta_m(z(\xi)-z(\xi_k);\ \tau)|^2 + 4\pi\,\mathrm{Im}\,(z(\xi))^t(\mathrm{Im}\,\tau)^{-1}\mathrm{Im}\,z(\xi_k)], \qquad (6.35)$$

where m is any odd characteristic. The argument $\mathbf{z}(\xi)$ in (6.35) is equal to the integral of the vector $\boldsymbol{\omega}=(\omega_1,...,\omega_p)$ from among holomorphic Abelian differentials, evaluated over a path joining the point ξ to the fixed point ξ_0

$$z(\xi)=\int_{\xi_0}^{\xi}\omega. \qquad (6.36)$$

By regularizing the function (6.35) at $\xi=\xi_k$ as in Ref. 5, we find that the dependence on ρ in (6.32) cancels out on the mass surface $\mathbf{p}^2=2$, and after some simple algebra the factor $K(\mathbf{p}_i;\tau)$ reduces to the form

$$K(p_i;\ \tau)=\int\prod_{k=1}^{N}\frac{i}{2}v^2_m(\xi_k)\wedge\overline{v^2_m(\xi_k)}\prod_{j<i}|\chi_{ij}|^{2p_ip_0}, \qquad (6.37)$$

$$\chi_{ij}\equiv\theta_m(z_{ij};\ \tau)\exp[-\pi\,\mathrm{Im}\,z^t_{ij}(\mathrm{Im}\,\tau)^{-1}\mathrm{Im}\,z_{ij}],$$

where $\mathbf{z}_{ij}=\mathbf{z}(\xi_i)-\mathbf{z}(\xi_j)$ and the 1-differential $v^2_m(\xi)$ has the form

$$v^2_m(\xi)=\omega_i(\xi)\left.\frac{\partial\theta_m(z;\ \tau)}{\partial z_i}\right|_{z=0}. \qquad (6.38)$$

The expression given by (6.37) does not depend on the choice of the odd characteristic m, and can be substituted into the measure so as to obtain the amplitudes

$$A_p(p_1\ldots p_N)=\int_{M_p} d\Omega\,|F(y)|^2(\det\mathrm{Im}\,\tau)^{-13}K(p_1\ldots p_N;\ \tau). \qquad (6.39)$$

For $p=1$, we can use (6.37) and (1.2) to reproduce a well-known result.[8]

We now turn to the case $p=4$. The complex dimension of $\mathfrak{S}_4$ is greater by one than the dimension of M_4, so that a single relation is available for the matrix τ. It is called the Schottky relation[35] and is the condition for the vanishing of a certain parabolic form J_8 of weight 8:

$$J_8(\tau)=0. \qquad (6.40)$$

Strictly speaking, Schottky showed that any matrix τ of a genus-4 Riemann surface satisfies (6.29); the reverse proposition was proved only relatively recently in Ref. 8 which gives the formula for J_8 in terms of $\theta_m(\tau)$. The results of Ref. 36 enable us to formulate the following conjecture:[9,29]

Conjecture 1. The measure for $p=4$ is

$$Z_4=\int_{\mathfrak{S}_4} d\mu_4\,|\delta(J_8)|^2(\det\mathrm{Im}\,\tau)^{-8}$$

$$\stackrel{\mathrm{def}}{=}\int_{M_4\subset\mathfrak{S}_4}\mathrm{res}\,(d\tau J_8^{-1}(\tau))\wedge\overline{\mathrm{res}\,(d\tau J_8^{-1}(\tau))}(\det\mathrm{Im}\,\tau)^{-13}. \qquad (6.41)$$

7. Analytic fields on Riemann surfaces and the Beilinson-Manin formula

We showed in Secs. I.A and I.B that the measure in string theory can be expressed in terms of the holomorphic objects $\det\bar{\partial}_j$, i.e., sections of determinant bundles over M_p, and that the evaluation of the measure is thus reduced to the problem of constructing such sections. This occurs because right and left excitations of two-dimensional quantum fields do not interact with one another. This enables us to extract in a consistent manner the chiral (analytic) sectors of these excitations, the structure of which we shall examine in detail in this section, following Ref. 12. To formulate the problem more precisely, let us consider a two-dimensional surface S with coordinates ξ^1,ξ^2 and metric $g_{ab}(\xi)$. Let us introduce on this surface certain analytic coordinates $z,\bar{z}$, in terms of which the metric assumes the conformal form

$$g_{ab}(\xi)\,d\xi^a d\xi^b=\rho(z,\ \bar{z})\,dz\,d\bar{z},$$

and let us examine this set $\{\phi^{(j)}\}$ of fields of spin j oı j-differentials which transform as follows under analytic replacements of the two-dimensional coordinate z on the surface S:

$$\phi^{(j)}(z,\ \bar{z})=\left(\frac{df(z)}{dz}\right)^j\tilde{\phi}^{(j)}(f,\ \bar{f}). \qquad (7.1)$$

The two anticommuting fields $\phi^{(j)}$ and $\phi^{(1-j)}$ can be used to construct the action[25]

$$S_j=\int_S\phi^{(1-j)}\bar{\partial}\phi^{(j)}\,dz\wedge d\bar{z},\quad \bar{\partial}\equiv\frac{\partial}{\partial\bar{z}}, \qquad (7.2)$$

where, in general, the integral is evaluated over a surface S of genus p. In the discussion that follows, we shall always consider that $p\geqslant 2$.

Formally, the free energy of this system is given by

$$F_j=-\ln\det\bar{\partial}_j \qquad (7.3)$$

where the subscript j on $\bar{\partial}$ signifies that $\bar{\partial}$ acts on the j-differentials, and our main problem is to find the explicit formula for a_j. In actual fact, for each j we construct a section $\det\bar{\partial}_j$ of the corresponding determinant bundle, such that any conformally invariant product of these sections has neither zeros nor poles in the interior of M_p. Because of this, the formulas obtained below can be used to evaluate the conformally-invariant products (3.5) and (3.6) and, in particular, to evaluate the measure in the theory of bosonic strings

$$Z_p=\int_{M_p}\prod_{i=1}^{3p-3}dy_i\wedge d\bar{y}_i\,|F(y)|^2(\det\mathrm{Im}\,\tau)^{-13}\quad(p\geqslant 2), \qquad (7.4)$$

$$F=\det\bar{\partial}_{-1}(\det\bar{\partial}_0)^{-13}.$$

Actually, there is as yet no satisfactory quantum-field definition of $\det\bar{\partial}_j$ that could serve as the starting point for an evaluation, so that the formulas given below must be treated more as definitions than equations. Nevertheless, they seem to us to be very instructive, and can be used to evaluate such quantities as $F(y)$ in (7.4); they provide the connection between the quantum field theory of j-differentials (7.1), (7.2) and the geometric approach of Quillen[23] and Faltings.[37] We shall see later that the formulas obtained by Beilinson and Manin arise naturally in our approach. Our strategy will be to start with representations of quantum field theory and elucidate the properties that any reasonable expression for $\det\bar\partial_j$ should have. We shall then construct the simplest form with such properties and verify it in certain special cases.

We begin with $j<1$ and try to introduce the formal definition

$$\det\bar\partial_j = \int \mathrm{D}\varphi\,\mathrm{D}f \exp\left(\int \varphi\,\bar\partial f\,\mathrm{d}z\wedge\mathrm{d}\bar z\right), \tag{7.5}$$

where f and φ are, respectively, the j- and $(1-j)$-differentials. Of course, this formula cannot be correct because $\bar\partial_j$ has $n_j=(2j-1)(p-1)$ zero modes, i.e., holomorphic j-differentials $f_1(z)(\mathrm{d}z)^j,\ldots,f_{n_j}(z)(\mathrm{d}z)^j$. We must extract their contribution from (7.5) and write

$$\det\bar\partial_j = \frac{\langle f(z_1)\ldots f(z_{n_j})\rangle}{\det\|f_i(z_k)\|}, \tag{7.6}$$

where the matrix $\|f_i(z_k)\|$ contains the element $f_i(z_k)$ on the intersection of the i-th row and k-th column, and

$$\langle f(z_1)\ldots f(z_{n_j})\rangle = \int \mathrm{D}\phi\,\mathrm{D}f\,f(z_1)\ldots f(z_{n_j}) \times \exp\int_S \phi\,\bar\partial f\,\mathrm{d}z\wedge\mathrm{d}\bar z. \tag{7.7}$$

To ensure that (7.6) does not depend on the choice of z_i, the correlation function (7.7) must be antisymmetric in all the z_i; it must also be a holomorphic j-differential for each z_i. However, we know that, because of the gravitational anomaly,[24] it is impossible to construct the correlation function (7.7) so that it does not depend either on the choice of the coordinates at other points on the surface or on the choice of the conformal metric ρ. This follows from the fact that, according to Ref. 24, the variation of F_j for a small general coordinate transformation has the form

$$\delta_\varepsilon F_j = \frac{C_j}{24\pi}\int \rho^{-1}\partial(\rho\varepsilon)(\Delta\ln\rho)\,i\,\mathrm{d}z\wedge\mathrm{d}\bar z, \tag{7.8}$$
$$C_j = 6j^2-6j+1.$$

Thus, F_j depends on the complex structure x of the surface S, the conformal metric ρ upon it, and the choice of the analytic coordinate z: $F_j=F_j(X,\rho,z)$. In the discussion that follows, we shall not consider the general case of arbitrary X,ρ,z, and will confine our attention to the dependence of F_j on X and z for a certain fixed conformal metric $\rho=\rho_*$. The most convenient choice is

$$\rho_* = |\nu_*(z)|^4, \tag{7.9}$$

where $\upsilon_*(z)$ is a holomorphic 1/2-differential (fermionic zero mode) and the asterisk represents its boundary conditions or characteristic (determined below). The important point for us is that, in general, υ_* has exactly $p-1$ zeros of order 1, which we shall denote by $R_1\ldots R_{p-1}: \upsilon_*(R_i)=0$.

For the matrix (7.9), the variation of F_j given by (7.8) has the form

$$\delta_\varepsilon F_j = \frac{2}{3}C_j\sum_{i=1}^{p-1}\left(\varepsilon'(R_i)+2\frac{\nu_*'}{\nu_*}\varepsilon(R_i)\right) = \frac{2}{3}C_j\delta_\varepsilon\ln\prod_i \nu_*^2(R_i), \tag{7.10}$$

$$\delta_\varepsilon\nu_*(z) = \varepsilon\partial\nu_* + \frac{1}{2}(\partial\varepsilon)\nu_*(z), \tag{7.11}$$

so that

$$\det\bar\partial_j[X,\rho_*,z] = \prod_{i=1}^{p-1}\left(\frac{\mathrm{d}f}{\mathrm{d}z}(R_i)\right)^{-2C_j/3}\det\bar\partial_j|X,\rho_*,f|. \tag{7.12}$$

Thus, $\det\bar\partial_j$ transforms as the product $(-2C_j/3)$ of differentials at the points R_i.

The last condition for $\det\bar\partial_j$ can be stated by saying that it should not depend on the coordinates $\bar y_i$ on the moduli space M_p.

An important property of (7.6) is that it depends on the choice of the basis $\{f_i\}$ in the space of holomorphic j-differentials, in agreement with Quillen's definition of $\det\bar\partial_j$ as the section of a determinant bundle on M_p (Ref. 23: see Sec. 5). For arbitrary j, there is no special basis $\{f_i\}$ with the exception of the case $j=1$ for which there is a normalized basis of 1-differentials $\{\omega_i, i=1,\ldots,p\}$ introduced in Sec. 3.

It is precisely this basis that is convenient for the definition of $\det\bar\partial_0$:

$$\det\bar\partial_0 = \frac{\int \mathrm{D}\omega\,\mathrm{D}\varphi\,\omega(z_1)\ldots\omega(z_p)\,\varphi(z)\exp\int\omega\bar\partial\varphi\,\mathrm{d}z\wedge\mathrm{d}\bar z}{\det\|\omega_i(z_j)\|}, \tag{7.13}$$

where the p 1-differentials $\omega(z_i)$ and the scalar $\varphi(z)$ appear because we have three zero modes ω and one zero mode φ: $\varphi(\xi)=\text{const}$. Moreover, it is assumed that (7.13) is independent of z. We note that, in relation to the transformation (6.5)–(6.7), the expression given by (7.13) behaves as a modular form of weight 1.

The simplest formulas are obtained not for the $\det\bar\partial_j$ themselves, but for the combinations

$$\lambda_j = \det\bar\partial_j\cdot(\det\bar\partial_0)^{1/2}. \tag{7.14}$$

It follows from the foregoing discussion that λ_j should be a $-(2j-1)^2$-differential with respect to conformal transformations at the points R_i, and a modular form of weight 1/2 with respect to Γ_p or, more precisely, with respect to the subgroup Γ_p that conserves the characteristic $\upsilon_*(z)$ in (7.9). Moreover, for half-integral j, there are 2^{2p} different boundary conditions, and λ_j must depend upon them.

All that remains is to explain how we can parametrize these boundary conditions and introduce the final concepts and notations that are necessary for constructing the formula for λ_j. We shall do this by considering, for a given $j\in\mathbb{Z}+(1/2)$, a meromorphic j-differential f and its formal sum (f)

$$(f) = \sum_i n_i Q_i - \sum_k m_k P_k, \tag{7.15}$$

where Q_i (P_k) are the zeros (poles) of $f(z)$ of order n_i (m_k). Let us also consider a point Q on the surface S and define the map

$$\xi\to\xi = \int_Q^\xi \omega \tag{7.16}$$

of the surface S into the complex torus $J=\mathbb{C}^p/\mathbb{Z}^p\oplus\tau\mathbb{Z}^p$, where the period matrix τ is defined by (6.3).

An important property of (7.15) is that

$$(\mathbf{f})=\sum_i n_i\mathbf{Q}_i-\sum_k m_k\mathbf{P}_k \qquad (7.17)$$

does not depend on the choice of f, and $(f)=0$ for $j=0$ (Abel's theorem). Hence it follows that

$$2(\mathbf{f})=2j\mathbf{k}\ (\text{in J}), \qquad (7.18)$$

where the Riemann constant is $\mathbf{k}=(\omega)$ for an arbitrary meromorphic 1-differential ω and, consequently,

$$(\mathbf{f})=j\mathbf{k}+\frac{1}{2}\tau\mathbf{m}'+\frac{1}{2}\mathbf{m}'', \qquad (7.19)$$

where the components of the p-vectors $\mathbf{m}',\mathbf{m}''$ are either 0 or 1. The set of 2^{2p} different pairs $(\mathbf{m}',\mathbf{m}'')\equiv\mathbf{m}$, called the characteristics of f, parametrizes invariantly all the possible boundary conditions on f. The number $e(m)=\mathbf{m}'\cdot\mathbf{m}''$ (mod2) is called the parity of the characteristic m. If $j>1$, it follows from the Riemann-Roch theorem that the number of holomorphic j-differentials does not depend on m and is equal to $n_j=(2j-1)(p-1)$, but for $j=1/2$ the situation is rather more involved. In the latter case, we know from the Riemann theorem on singularities[38] that the parity of the number of holomorphic 1/2-differentials, where $n_{1/2}(m)=e(m)$ for a surface of general position. We recall that, for odd characteristics m, the holomorphic 1/2-differentials, $v_m(z)$ can be constructed explicitly:

$$\mathbf{v}_m^2(z)=\theta_{m,i}\omega_i(z), \qquad (7.20)$$

where the Riemann θ-function was defined in Sec. 6 and

$$\theta_{m,i}\equiv\left.\frac{\partial\theta_m(\mathbf{z})}{\partial z_i}\right|_{\mathbf{z}=0}. \qquad (7.21)$$

Everything is now ready for λ_j. For $j\in\mathbb{Z}+(1/2)$, we propose the expression

$$\lambda_j=\frac{\mathbf{v}_*^{2j-1}(z_1)\ldots\mathbf{v}_*^{2j-1}(z_{n_j})\langle V_1(z_1)\ldots V_1(z_{n_j})V_{1-2j}(R_1)\ldots V_{1-2j}(R_{p-1})\rangle_*}{\det\|f_i(z_k)\|(\mathbf{v}_*'(R_1)\ldots\mathbf{v}_*'(R_{p-1}))^{(2j-1)^2}}\times\theta_m\left(\sum_i\mathbf{z}_i-(2j-1)\sum_\alpha\mathbf{R}_\alpha\right), \qquad (7.22)$$

where the asterisk represents an arbitrary odd characteristic, $(v_*)=R_1+\ldots+R_{p-1}$, m is the characteristic of f_j, and

$$\left\langle\prod_i V_{q_i}(z_i)\right\rangle\overset{\text{def}}{=}\prod_i(\theta_{*,k}\omega_k(z_i))^{q_i^2/2}\prod_{i<k}(\theta_*(\mathbf{z}_i-\mathbf{z}_k))^{q_iq_k}, \qquad (7.23)$$
$$\sum_i q_i=0.$$

By using known analytic properties of the θ-functions,[38] we can readily verify that (7.22) satisfies all the conditions enumerated above.

For $j\in\mathbb{Z}$, the expression is the same, except that m becomes $*$.

Formula (7.23) can be directly generalized to the case of correlation functions. This can be done by associating each additional pair $f(z)$, $\Phi(z')$ in (7.7) with operators $v^{2j-1}(z)V_1(z), v_*^{1/2j}(z')V_1(z')$ in (7.22), and adding $\mathbf{z}-\mathbf{z}'$ to the argument of the θ-function.

We note that (7.23) is the chiral part of the correlation function of the exponents of the free scalar φ on the surface. This follows from the corresponding formula of Sec. 6:

$$\left\langle\prod_k e^{iq_k\varphi(z_k,\bar z_k)}\right\rangle=\left|\left\langle\prod_i V_{q_i}(z_i)\right\rangle_*\right|^2\times\exp\left[-2\pi\sum_{i<j}q_iq_j\,\mathrm{Im}\,(\mathbf{z}_i-\mathbf{z}_j)^t(\mathrm{Im}\,\tau)^{-1}\times\mathrm{Im}\,(\mathbf{z}_i-\mathbf{z}_j)\right],\quad\sum_i q_i=0. \qquad (7.24)$$

We now turn to the examination of the different special cases. First, consider $j=0$. In this case, (7.13) and (7.23) assume the form

$$\lambda_0=\frac{\mathbf{v}_*^{-1}(z)\mathbf{v}_*(z_1)\ldots\mathbf{v}_*(z_p)\langle V_1(z)V_{-1}(z_1)\ldots V_{-1}(z_p)V_1(R_1)\ldots V_1(R_{p-1})\rangle_*}{\det\|\omega_i(z_k)\|\mathbf{v}_*'(R_1)\ldots\mathbf{v}_*'(R_{p-1})}\times\theta_*(\mathbf{z}-\mathbf{z}_1-\ldots-\mathbf{z}_p+\mathbf{R}_1+\ldots+\mathbf{R}_{p-1}). \qquad (7.25)$$

We now use the fact that (7.25) is independent of z, z_i and substitute $z_p=z$, $z_i=R_i, i=1,\ldots,p-1$. Hence

$$(\det\bar\partial_0)^{3/2}=\frac{\theta_{*,i}\omega_i(z)}{\det\|\omega(z)\,\omega(R_1)\ldots\omega(R_{p-1})\|}. \qquad (7.26)$$

This expression does not depend on z and looks like a direct generalization of θ_1' to $p=1$. As explained above, $\omega(R_i)$ appears in (7.26) as a consequence of the gravitational anomaly which does not appear for $p=1$ because, according to (7.9), $\rho^*=$ const and the right-hand side of (7.8) vanishes. By combining (7.22) and (7.26), we can also obtain formulas for the $\det\bar\partial_j$ themselves.

Another interesting and important case is $j=1/2$. For even characteristics, we have

$$\det{}_m\bar\partial_{1/2}(\det\bar\partial_0)^{1/2}=\theta_m(0),\quad e(m)=0 \qquad (7.27)$$

and for odd m

$$\det{}_m\bar\partial_{1/2}(\det\bar\partial_0)^{1/2}=\frac{\langle V_1(z_1)V_1(z_2)\rangle_m}{\mathbf{v}_m(z_1)\mathbf{v}_m(z_2)}\theta_m(\mathbf{z}_1-\mathbf{z}_2)=\mathbf{v}_m^{-2}(z)\,\theta_{m,i}\omega_i(z),\quad e(m)=1, \qquad (7.28)$$

in which the two zero modes in (2.28) appear because the action $S_{1/2}=\int\tilde\psi\bar\partial\psi dz\wedge d\bar z$ contains two different fermionic fields of spin 1/2, namely, $\tilde\psi$ and ψ, and each of them has the zero mode $v_m(z)$. The following expression for the correlation functions of the fermions ψ and $\tilde\psi$ can also be readily obtained. For any m,

$$\left\langle\prod_{i=1}^N\psi(z_i)\tilde\psi(z_i')\right\rangle_m\det{}^{1/2}\bar\partial_0=\prod_{i=1}^N(\theta_{*,l}\omega_l(z_i)\,\theta_{*,k}\omega_k(z_i'))^{1/2}\times\prod_{i<j}\theta_*(\mathbf{z}_i-\mathbf{z}_j)\,\theta_*(\mathbf{z}_i'-\mathbf{z}_j')\times\prod_{i,j}\theta_*^{-1}(\mathbf{z}_i-\mathbf{z}_j')\,\theta_m\left(\sum_{i=1}^N(\mathbf{z}_i-\mathbf{z}_i')\right). \qquad (7.29)$$

The determinant of the Dirac operator, given by (7.28), is in agreement with the expression in Refs. 9 and 33 and can be derived as follows.[12,39,40]

Starting with the results of Secs. 2 and 3, we write for $\lambda_{1/2}$

$$|\lambda_{1/2}|^2 = \frac{\left|\int D\psi D\tilde{\psi} \exp\left[(1/2\pi i)\int \tilde{\psi}\bar{\partial}\psi dz d\bar{z}\right]\right|^2}{(\det \operatorname{Im}\tau)^{1/2}\int D\varphi \exp\left(\frac{1}{4\pi i}\int \partial\varphi\bar{\partial}\varphi dz \wedge d\bar{z}\right)},$$

where φ is a real scalar field. For a small deformation of the complex structure

$$dz \to d\tilde{z} = dz + \eta d\bar{z}$$

the variation of $\lambda_{1/2}$ is

$$\delta_\eta \ln \lambda_{1/2} = \frac{1}{2\pi i}\left\langle \int \eta T dz \wedge d\bar{z}\right\rangle + \frac{1}{4i}\operatorname{tr}[(\operatorname{Im}\tau)^{-1}\delta_\eta\tau], \tag{7.30}$$

where $T \equiv T_{++}$ is the left component of the energy-momentum tensor of the fields $\psi, \tilde{\psi}$, and φ:

$$\langle T\rangle = \langle T^{(1/2)}\rangle - \langle T^{(0)}\rangle. \tag{7.31}$$

The first term, $T^{(1/2)} = (1/2)((\partial\psi)\tilde{\psi} - \psi\partial\tilde{\psi})$, is the energy-momentum tensor of the fermions $\psi, \tilde{\psi}$, and the second term, $T^{(0)} = -(1/2-(\partial\varphi)^2$, is the energy-momentum tensor of the scalar field φ. Their vacuum expectation values can be found by substituting the corresponding operator expansions into the two-point correlation functions

$$\langle \psi(z)\tilde{\psi}(z')\rangle_m = \frac{(\theta_{*,i}\omega_i(z)\,\theta_{*,k}\omega_k(z'))^{1/2}}{\theta_*(z-z')}\frac{\theta_m(z-z')}{\theta_m(0)}$$
$$\approx (z-z')^{-1} + (z-z')\langle T^{(1/2)}(z')\rangle + o((z-z')^2), \tag{7.32}$$

$$\langle e^{i\varphi(z,\bar{z}')}e^{-i\varphi(z',\bar{z}')}\rangle = |z-z'|^2[1+(z-z')^2\langle T^{(0)}(z')\rangle + \ldots]$$
$$= \left|\frac{(\theta_{*,i}\omega_i(z)\,\theta_{*,k}\omega_k(z'))^{1/2}}{\theta_*(z-z')}\right|^2$$
$$\times \exp[2\pi \operatorname{Im}(z-z')^t(\operatorname{Im}\tau)^{-1}\operatorname{Im}(z-z')]. \tag{7.33}$$

The last expression in (7.32) is the unique antisymmetric $(1/2,1/2)$ differential in (z,z') with characteristic m and a pole of order 1 at $z=z'$; it is holomorphic in z, z' if $z \neq z'$.

Strictly speaking, (7.32) and (7.33) are valid only when the metric ρ is a constant in a certain neighborhood containing z and z'. However, the combination given by (7.31), which we must find, does not depend on ρ because of the cancelation of all the anomalies. Comparison of (7.31) and (7.32) shows that

$$T(z) = \frac{1}{2}\frac{\theta_{m,ij}\omega_i(z)\,\omega_j(z)}{\theta_m(0)} + \frac{\pi}{2}(\operatorname{Im}\tau)^{-1}_{ik}\,\omega_i(z)\,\omega_k(z)$$
$$= 2\pi i\frac{\partial \ln\theta_m(0)}{\partial\tau_{kl}}\omega_k(z)\,\omega_l(z) + \frac{\pi}{2}(\operatorname{Im}\tau)^{-1}_{ik}\,\omega_i(z)\,\omega_k(z). \tag{7.34}$$

Substituting this expression in (7.39), and using the formula

$$\delta_\eta\tau_{ij} = \int \eta\omega_i\omega_j dz \wedge d\bar{z}, \tag{7.35}$$

we find that

$$\delta_\eta \ln\lambda_{1/2} = \delta_\eta \ln\theta_m(0),$$

from which (7.27) follows.

It is interesting that the expression for $T^{(0)}$ obtained from (7.33) contains the so-called projective connectivity Γ (Ref. 41)

$$T^{(0)} = \Gamma - \frac{\pi}{2}(\operatorname{Im}\tau)^{-1}_{ik}\,\omega_i\omega_k,$$
$$\Gamma = \frac{1}{12}\left[\frac{\theta_{*,i}\omega_i''}{\theta_{*,i}\omega_i} - \frac{3}{2}\left(\frac{\theta_{*,i}\omega_i'}{\theta_{*,i}\omega_i}\right)^2\right] - \frac{1}{3}\frac{\theta_{*,ijk}\omega_i\omega_j\omega_k}{\theta_{*,k}\omega_k}. \tag{7.36}$$

In contrast to T in (7.34), $T^{(0)}$ is not a quadratic differential and we see from (7.36) that it transforms under the replacement $z \to f(z)$ so that it is expressed in terms of the Schwarz derivative

$$T^{(0)}(z) = \frac{1}{12}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right] + \left(\frac{df(z)}{dz}\right)^2\tilde{T}^{(0)}(f(z)) \tag{7.37}$$

in accordance with the general rules of conformal field theory.[42] The expression given by (7.36) transforms into a 2-differential if we restore the dependence on the conformal metric $\phi = \ln\rho$:

$$T^{(0)}[\phi] = \Gamma - \frac{\pi}{2}\omega^t(\operatorname{Im}\tau)^{-1}\omega + \frac{1}{12}\left[\frac{1}{2}(\partial\phi)^2 - \partial^2\phi\right]. \tag{7.38}$$

This formula can also be deduced from the operator expansion for the products of the currents ω and $\partial\varphi$

$$\partial_{z'}G_\xi(z,z') = \langle\langle \omega(z)\,\partial_{z'}\varphi(z')\rangle\rangle = (z-z')^{-2} - \langle T^{(0)}_{\text{hol}}(z')\rangle$$
$$\ldots = \partial_{z'}\frac{\left\langle \oint_{a_1}\omega(z_1)\,dz_1\ldots\oint_{a_p}\omega(z_p)\,dz_p\varphi(\xi)\,\omega(z)\,\varphi(z')\right\rangle}{\left\langle \oint_{a_1}\omega(z_1)\,dz_1\ldots\oint_{a_p}\omega(z_p)\,dz_p\varphi(\xi)\right\rangle}. \tag{7.39}$$

It follows from (7.39) that $G_\xi(z,z')$ is a $(1,0)$-differential in (z,z') with poles at $z=z', z=\xi$, which satisfies

$$\oint_{a_i} G_\xi(z,z')\,dz = 0,\quad \oint_{\gamma_{z'}} G_\xi(z,z')\,dz = -\oint_{\gamma_\xi} G_\xi(z,z')\,dz = 2\pi i.$$

These conditions define $G_\xi(z,z')$ uniquely:

$$G_\xi(z,z') = \partial_z \ln\frac{\theta_*(z-z')}{\theta_*(z-\xi)},$$

and we find that

$$T^{(0)}_{\text{hol}} = \Gamma. \tag{7.40}$$

It is interesting that all the possible classical identities involving the 1-differentials can be derived by substituting the Ward identities from the conformal theory of the fields ω, φ. For example, (7.35) can be obtained by taking the following identity as the starting point:

$$\tau_{ik} = \frac{1}{2\pi i}\left\langle\!\left\langle \oint_{b_i}\omega dz \oint_{b_k}\partial\varphi\, dz'\right\rangle\!\right\rangle.$$

The two remaining examples that we need to consider are $j=3/2$ and $j=2$. Once again, we shall try to place all the points z_i from (7.22) in the corresponding R_i, and this will yield

$$\lambda_{3/2}(m) = \theta_m(0)\det{}^{-1}\|\zeta(R_1)\,\zeta'(R_1)\ldots\zeta(R_{p-1})\,\zeta'(R_{p-1})\|,$$
$$e(m) = 0, \tag{7.41}$$

$$\lambda_{3/2}(m) = \theta_{m,i}\omega_i(R_1)\det{}^{-1}\|\zeta'(R_1)\,\zeta'''(R_1)\,\zeta(R_2)\,\zeta'(R_2)$$
$$\ldots\zeta(R_{p-1})\,\zeta'(R_{p-1})\|,\quad e(m) = 1, \tag{7.42}$$

where ζ is a column of $2p-2$ holomorphic 3/2-differentials with the characteristic m, and

$$\lambda_2 = \theta_{*,i}\omega_i''(R_1)$$
$$\times\det{}^{-1}\|\mathbf{f}(R_1)\mathbf{f}''(R_1)\mathbf{f}^{\mathrm{IV}}(R_1)\mathbf{f}(R_2)\ldots\mathbf{f}'(R_{p-1})\mathbf{f}''(R_{p-1})\|, \quad (7.43)$$

where $\mathbf{f}$ is a column of holomorphic 2-differentials. Expression (7.41) is particularly simple. It describes, for example, the dependence of the ghost determinant in the heterotic string on the boundary conditions that fermions must satisfy on the world sheet. The form of (7.41) suggests[43,44] that the superstring renormalization theorems[4] are a consequence of the Riemann identity.[38] (The present state of the problem is discussed in Ref. 65-*Ed.*).

The formulas for $\det\bar\partial_j$ can be used to evaluate the conformally-invariant products of different Laplace operators $\Delta_j = -\rho^{j-1}\partial\rho^{-j}\bar\partial$ acting on the j-differentials. According to Secs. 2 and 3, we have

$$\prod_j\left(\frac{\det'\Delta_j}{\det N_j\cdot\det N_{1-j}}\right)^{n_j} = \left|\prod_j(\det\bar\partial_j)^{n_j}\right|^2, \quad \sum_j C_j n_j = 0,$$
$$(N_j)_{ik} = \int f_i^{(j)}\bar f_k^{(j)}\rho^{1-j}dz\wedge d\bar z. \quad (7.44)$$

It follows from (7.9) and (7.22) that the product $\Pi_j(\det\bar\partial_j)^{n_j}$ does not depend on the choice of coordinates at the points R_i because

$$\sum_j C_j n_j = 0.$$

This shows that there is a close connection between the cancelation of the analytic anomaly in (7.44) and the cancelation of the gravitational anomaly in $\Pi_j(\det\bar\partial_j)^{n_j}$. We shall now show how the connection arises with the formula of Beilinson and Manin[11] for the measure in the theory of bosonic strings. In the ESVM model, the measure is a special case of (7.44), and we find them from (7.4), (7.26), and (7.43) that

$$F = \frac{\lambda_2}{\lambda_0^9}$$
$$= \frac{\det^9\|\omega(R^1)\,\omega''(R_1)\,\omega(R_2)\ldots\omega(R_{p-1})\|}{(\theta_{*,i}\omega_i''(R_1))^8\det\|\mathbf{f}(R_1)\,\mathbf{f}'(R_1)\,\mathbf{f}^{\mathrm{IV}}(R_1)\mathbf{f}(R_2)\ldots\mathbf{f}'(R_{p-1})\|}, \quad (7.45)$$

where the holomorphic quadratic differentials $f_i(z)$ in the column $\mathbf{f} = (f_1,\ldots,f_{3p-3})'$ define the coordinates y_i on M_p that are used in (7.4) and in Sec. 2. It is readily verified that (7.45) does not depend on the choice of the conformal coordinates at the points R_i which can be defined by the conditions

$$v_*'(R_i) = 1 \quad (i = 1,\ldots,\; p-1). \quad (7.46)$$

Next, we shall try to modify the basis $\{\omega_i\},\{f_i\}$ so that the determinants (7.45) become as simple as possible. This is achieved by the following choice:

$$\omega_\alpha(R_i) = \delta_{\alpha i}\;\; (\alpha = 1,\ldots,\; p-1),\;\; \omega_p = v_*^2,$$
$$\zeta_k = \omega_k v_*(k = 1,\ldots,\; p), \zeta_{p+l}(R_{l+1}) = \delta_{i,l}(l = 1,\ldots,\; p-2), \quad (7.47)$$
$$f_k = v_*\zeta_k\;(k = 1,\ldots,\; 2p-2),\; f_{2p-2+l}(R_i)$$
$$= \delta_{il}\;\;(l = 1,\ldots,\; p-1).$$

The normalization conditions in (7.47) are written in terms of the coordinates (7.46) and (7.47), and define $\omega_\alpha,\zeta_k,\ldots$ uniquely. Substituting (7.47) in (7.45), we obtain the Beilinson-Manin formula

$$F = (\theta_{*,i}\omega_i''(R_1))^{-8} = \left(\frac{v_*^2(z)}{\theta_{*,i}\omega_i(z)}\right)^8, \quad (7.48)$$

where we have used the fact that the last ratio does not depend on z. If, in addition, we determine v^* as in (7.20), we find that $F = 1$, which is in accord with the Mumford theorem[7] on the triviality of the corresponding bundle on ∂M_p. We also note that, for $p = 2$, it can be shown that (7.45) is actually independent of the choice of the characteristic, and coincides with (6.9), (6.14).

We have thus constructed a conformal field theory for analytic j-differentials on Riemann surfaces of genus p, and the explicit formula (7.45) for the measure in string theory. We have used the special metric $\rho_* = |v_*|^4$, which led us to the following fermionization rules [cf. (7.23), (7.24), and (7.30)]:

$$f^{(j)} \longleftrightarrow v_*^{2j-1}\,\psi,\quad \varphi^{(1-j)} \longleftrightarrow v_*^{1-2j}\tilde\psi,$$
$$\text{vacuum}\rangle$$
$$\mapsto(\det\bar\partial_0)^{-1/2}\prod_{i=1}^{p-1}[\tilde\psi(R_i)\tilde\psi(R_i)\ldots\tilde\psi^{(2j-2)}(R_i)(v_*'(R_i))^{-(2j-1)^2}], \quad (7.49)$$

where the fields ψ and $\tilde\psi$ are 1/2-differentials on which we impose the boundary conditions * for $j\in\mathbb{Z}$ and the same boundary conditions as are imposed on $f^{(j)}$ for $j\in\mathbb{Z}+(1/2)$. We thus see that the generalization of the bosonization rules for genus $p\geqslant 2$ is not trivial. It would be interesting to transform the above formulas to an arbitrary metric ρ.

We note in conclusion that some of the formulas given in this Section (mostly for $j = 1/2$) were recently obtained by several authors in connection with multiloop superstring evaluations[44,45] and with generalizations of the bosonization formulas.[46] Formula (7.27) appeared in Refs. 9 and 33 (in Ref. 33, the dependence of the fermionic determinant on the boundary conditions was found using a striking analogy with the holomorphy theorem for the moduli space of line bundles over a Riemann surface). These results were described in the last Section. An interesting alternative approach to the evaluation of det Δ_j by means of bosonization is given in Ref. 47.

8. Quillen's theorem and the dependence of determinants on the boundary conditions

In this Section, we examine the dependence on the boundary conditions for the determinant of the Laplace operator Δ_j acting in the space of the j-differentials $\phi(z,\bar z)$ on a Riemann surface X of genus $p>0$. These conditions can be parametrized by the factors $\exp(2\pi i x_k)$ and $\exp(2\pi i y_k)$ that are acquired by the field ϕ in completed basis cycles a_k and b_k. The space $\mathcal{T}$ of all the possible boundary conditions is thus at $2p$-dimensional torus with coordinates $x\leqslant x_k,\; y_k<1$. We shall evaluate $\det\Delta_j = \det\Delta_j(\mathbf{x},\mathbf{y})$ as a function on $\mathcal{T}$.

We know that there is a natural complex structure on $\mathcal{T}$. This enables us to proceed in the spirit of Sec. 3 and evaluate det Δ_j as a function of the complex coordinates on $\mathcal{T}$, using an appropriate analog of propositions A) and B) of the introduction concerning Quillen's theorem.[23] This method was used in Ref. 33 to find det $\Delta_j(\mathbf{z},\mathbf{y})$ for $j = 1/2$.

It will be useful to reformulate the problem in order to introduce the complex structure on $\mathcal{T}$ correctly. In particular, we shall use the replacement

$$\phi(z, \bar{z}) = \varphi_0(z, \bar{z}) \times\exp\left\{2\pi i\left[x_k \operatorname{Re}\int_{z_0}^{z}\omega_k dz + (y_k - x_m \operatorname{Re}\tau_{mk})(\operatorname{Im}\tau)^{-2}_{kl}\operatorname{Im}\int_{z_0}^{z}\omega_l dz\right]\right\}, \quad (8.1)$$

where z_0 is an arbitrary point on X and ω_k is the normalized basis of holomorphic 1-differentials. We thus reduce the problem to the evaluation of the determinant of the operator

$$\Delta_j(A) = -\rho^{j-1}(\partial - A)\rho^{-j}(\bar\partial + \bar A), \quad (8.2)$$

which appears in the action for the j-differentials $\phi_0(z,\bar z)$ with charge 1 in the Abelian gauge field

$$A = \pi i x_k\omega_k + \pi(y_l - x_m \operatorname{Re}\tau_{ml})(\operatorname{Im}\tau)^{-1}_{lk}\omega_k \quad (8.3)$$

on X with zero strength

$$F = \partial\bar A - \bar\partial A = 0. \quad (8.4)$$

We now consider the space $\mathcal{A}$ of all fields A with zero strength. The gauge group $\mathcal{G}$ of transformations

$$A \to A - \partial f, \quad \varphi_0 \to e^{f}\varphi_0, \quad (8.5)$$

then acts in this space and is conveniently represented by the product of the group $\mathcal{G}_0$ (with single-valued functions f on X) and the group Γ, by replacing (8.1) with integral x_k and y_k. By definition, to each orbit of the group $\mathcal{G}$ in $\mathcal{A}$ there corresponds a holomorphic line bundle, and the factor space

$$J_X = \mathcal{A}/\mathcal{G}$$

is called the moduli space of holomorphic line bundles over X [of degree $2j(p-1)$] or the Jacobian of X. It is readily shown that in each orbit of the group $\mathcal{G}$ there is a unique representative (8.3) that corresponds to the gauge

$$\bar\partial A = 0, \quad 0 \leqslant x_k, \ y_k < 1. \quad (8.6)$$

Hence the space $\mathcal{T}$ of boundary conditions coincides with the Jacobian

$$\mathcal{T} = J_X.$$

As in Sec. 2, the complex structure on J_X is determined with the aid of $\mathcal{G}_0$-invariant complex coordinates $\mathbf{z}$

$$\mathbf{z}(\bar A) = \int \omega_k \bar A \frac{d\bar z \wedge dz}{2\pi} = \mathbf{x} + \tau\mathbf{y}, \quad (8.7)$$

that uniquely map $\mathcal{A}/\mathcal{G}_0$ into $\mathbb{C}^p$. The group Γ obviously acts upon it by shifts of the vectors in the lattice $\mathbb{Z}^p \oplus \tau\mathbb{Z}^p$. It then follows that

$$J_X = \mathbb{C}^p/\mathbb{Z}^p \oplus \tau\mathbb{Z}^p. \quad (8.8)$$

Finally, $\mathcal{T} = J_X$ is a complex torus and we must naturally try to elucidate the analytic properties of det Δ_j as a function of the complex coordinates (8.7) upon it. The answer to this question is provided by the following Quillen's theorem.

Quillen's theorem:

$$\delta\bar\delta \ln \frac{\det{}' \Delta_j(A)}{\det N_j(A)\cdot\det N_{1-j}(A)} = \frac{1}{2\pi}\int \delta A\delta\bar A\, dz \wedge d\bar z, \quad (8.9)$$

where $N_j(A)$ is the matrix of the scalar products of the zero modes of the operator $\bar\partial_j + \bar A$. This theorem can readily be proved by a quasiclassical procedure. Using (8.7), we find from (8.9) that

$$\frac{\det{}' \Delta_j}{\det N_j \circ \det N_{1-j}}(\mathbf{z}, \bar{\mathbf{z}}) = \varkappa_j \exp\left[-\frac{1}{2\pi}\operatorname{Im}\mathbf{z}(\operatorname{Im}\tau)^{-1}\operatorname{Im}\mathbf{z}\right]|f_j(\mathbf{z})|^2, \quad (8.10)$$

where the constant $\varkappa_j$ depends, of course, on τ, and the function $f_j(\mathbf{z})$

A) *is a holomorphic section of the bundle* $\det(\bar\partial_j + \bar A)$ *over the Jacobian* J, which is determined by analogy with the bundle det $\bar\partial_j$ over M_p, introduced in Sec. 5. Moreover, for $p \geqslant 2$ and $j \neq 1/2$, the number of zero modes of the operator $\bar\partial_j + \overline{A(\mathbf{z})}$ does not depend on [8)] $\mathbf{z}$, so that

B_1) *$f_j(z)$ vanishes nowhere if $p \geqslant 2, j \neq 1/2$.* For $j = 1/2$ and general X and $\mathbf{z}$, the operator $\bar\partial_{1/2} + \overline{A(z)}$ does not have zero modes, so that it is clear that the function $f_{1/2}(\mathbf{z})$ must vanish at points of J_X at which the zero modes appear, and the multiplicity of the zeros must be equal to their number. We know that if the characteristic of the 1/2-differential φ_0 on which the operator $\bar\partial_{1/2}$ acts is m (see Sec. 7), then the set of these bundles is specified in J_X by the equation $\theta_m(\mathbf{z}) = 0$, so that

B_2) *the multiplicity and the position of the zeros of $f_{1/2,m}(\mathbf{z})$ and $\theta_m(\mathbf{z})$ are found to coincide.*

It is clear that, for all j, the function $f_j(\mathbf{z})$ is defined by A) and B) up to a factor that does not depend on $\mathbf{z}$. In point of fact, if this were not so, the ratio of any two such functions, other than a constant, would not have zeros or poles on $\mathbb{C}^p$, and would have a periodic modulus, which would be in conflict with the maximum principle.

Explicit formulas for $f_j(\mathbf{z})$ can be readily constructed with the aid of θ-functions by starting with a generalization of (7.6) for det $(\bar\partial_j + \bar A)$. In the notation of Sec. 7, we have

$$f_j(\mathbf{z}) = \lambda_j(\det\bar\partial_0)^{-1/2}\frac{\theta_*(\mathbf{z}_1 + \ldots + \mathbf{z}_{n_j} - \mathbf{z}'_1 - \ldots - \mathbf{z}'_{n_{1-j}} - (2j-1)(\mathbf{R}_1 + \ldots + \mathbf{R}_{p-1}) - \mathbf{z})}{\theta_*(\mathbf{z}_1 + \ldots + \mathbf{z}_{n_j} - \mathbf{z}'_1 - \ldots - \mathbf{z}'_{n_{1-j}} - (2j-1)(\mathbf{R}_1 + \ldots + \mathbf{R}_{p-1}))} \quad (8.11a)$$

for $j \in \mathbb{Z}$ and

$$f_j(\mathbf{z}) = \lambda_{j,m}(\det\bar\partial_0)^{-1/2}\frac{\theta_m(\mathbf{z}_1 + \ldots + \mathbf{z}_{n_j} - \mathbf{z}'_1 - \ldots - \mathbf{z}'_{n_{1-j}} - (2j-1)(\mathbf{R}_1 + \ldots + \mathbf{R}_{p-1}) - \mathbf{z})}{\theta_m(\mathbf{z}_1 + \ldots + \mathbf{z}_{n_j} - \mathbf{z}'_1 - \ldots - \mathbf{z}'_{n_{1-j}} - (2j-1)(\mathbf{R}_1 + \ldots + \mathbf{R}_{p-1}))} \quad (8.11b)$$

for $j \in \mathbb{Z} + (1/2)$, where in all the formulas for λ_j in Sec. 7, the basis of holomorphic j-differentials must be replaced with the basis of zero modes of the operator $\bar\partial_j + A(\mathbf{z})$. The common factor in (8.11) is chosen so that the constant $\varkappa = \Pi_j \varkappa_j$ in the conformally-invariant product of determinants (8.10) does not depend on τ.

We emphasize that we do not need the discussion of Sec. 7 to verify the validity of (8.11) as functions of $\mathbf{z}$ for fixed τ. It is sufficient to verify that (8.11) does not depend on the choice of the points z_k, z'_l on X, and that they satisfy properties $B_{1,2}$). Both facts can be readily established with the aid of the known analytic properties of theta-functions.

Formula (8.11b) assumes a particularly simple form for $j = 1/2$:

$$f_{1/2,m}(\mathbf{z}) = \theta_m(\mathbf{z}) (\det \bar\partial_0)^{-1/2}. \qquad (8.12)$$

For each $\mathbf{z} = \mathbf{x} + \tau\mathbf{y}$, it is convenient to introduce the characteristic $m(\mathbf{z}) = 1/2(\mathbf{y},\mathbf{x})$ and then use it to transform (8.12) to the following form[33]:

$$\det{}_m \Delta_{1/2}(\mathbf{z}) \left(\frac{\det' \Delta_0(0)}{\int \rho d^2\xi \cdot \det \operatorname{Im} \tau} \right)^{1/2} = |\theta_{m+m(\mathbf{z})}(\tau)|^2. \qquad (8.13)$$

This is in agreement with (7.27) because fermions with integral characteristic m_1 are the same as fermions with integral characteristic m_2 in the gauge field A_{12}:

$$m(\mathbf{z}(A_{12})) = m_1 - m_2,$$

i.e.,[9]

$$\det{}_{m_2} \Delta_{1/2}(A_{12}) = \det{}_{m_1} \Delta_{1/2}(0). \qquad (8.14)$$

Similarly, we can show for other half-integral values of j that (7.22) provide the correct description of the dependence of $\det\bar\partial_j$ on the characteristic.

III. RIEMANN SURFACES AS BRANCHED COVERINGS

In this Section, we shall use the classical idea of the Riemann surface as a branched covering of a plane, and will evaluate the multiloop measure of string theory as a function of the complex coordinates of the branch points. The results established above enable us to reduce the problem to the study of the behavior of analytic fields on branched coverings. It will be shown in Sec. 9 that a branch point in the theory of analytic fields plays the part of a vertex operator with simple conformal properties. This enables us to derive explicit formulas for $\det\Delta_j$ and $\det\bar\partial_j$ in the case of coverings with the Abelian monodromy group (see Sec. 10). Unfortunately, in this approach, the measure can only be evaluated completely for $p = 2$ (see Sec. 11), although a number of interesting propositions can be obtained also in the general case (see Sec. 12; Ref. 48). The presentation given in Sec. 9–11 will largely follow Ref. 15. We note that most of the results of Sections 10–11 were obtained by other methods in Ref. 49, and the quantity $\det'\Delta_0$ on hyperelliptic surfaces was first found in Ref. 50.

9. Branch points as primary conformal fields

Let us examine the behavior of analytic fields on an arbitrary Riemann surface X in a small neighborhood U of a branch point of order n. The analytic coordinate y on X that is single-valued in U will be chosen so that the covering map $X \to CP^1$ takes the following form in U:

$$z(y) = a + y^n. \qquad (9.1)$$

The metric on X will then be considered to be z-flat

$$g_{zz} = g_{\bar z\bar z} = 0, \quad g_{z\bar z} = 1. \qquad (9.2)$$

Let us label the n sheets of the Riemann surface X of the inverse map

$$y(z) = (z - a)^{1/n} \qquad (9.3)$$

with the numbers $f^{(l)}$ and $\varphi^{(l)}$, so that, as we circulate around the point a in the z plane, we pass from sheet l to sheet $l + 1$ $((n-1) + 1 \equiv 0)$. We shall use the symbol $\hat\pi_a$ to represent a circuit around the point a. On each sheet l we consider a pair of analytic anticommuting fields $f^{(l)}$ and $\varphi^{(l)}$, with spins j and $1 - j$, respectively, and action

$$S^{(l)} = \int f^{(l)} \bar\partial \varphi^{(l)} dz \wedge d\bar z \qquad (9.4)$$

where $f^{(l)}(z,\bar z)$ represents the field f at the point $y(z)$ on the l-th sheet of X.

We recall that the energy-momentum tensor of the fields $f^{(l)}, \varphi^{(l)}$ takes the form[25]

$$T^{(l)} = -jf^{(l)}\partial\varphi^{(l)} + (j-1)\varphi^{(l)}\partial f^{(l)}, \qquad (9.5)$$

and under the conformal transformations

$$z \to \alpha(z), \quad \bar z \to \bar\alpha(\bar z) \qquad (9.6)$$

these fields transform so that

$$\tilde f^{(l)}(\alpha, \bar\alpha) \left(\frac{d\alpha}{dz}\right)^j = f^{(l)}(z - \bar z),$$
$$\tilde\varphi^{(l)}(\alpha, \bar\alpha) \left(\frac{d\alpha}{dz}\right)^{1-j} = \varphi^{(l)}(z, \bar z). \qquad (9.7)$$

The following normalization is assumed in (9.7):

$$f^{(l)}(z')\varphi^{(l)}(z) \sim I(z' - z)^{-1} + \text{Regge terms}, \qquad (9.8)$$

where I is the unit operator.

A circuit around a branch point takes us from one sheet to another. This means that we must have the following boundary conditions:

$$\hat\pi_a f^{(l)}(z) = f^{(l+1)}(z),$$
$$\hat\pi_a f^{(l)}(z) = \varphi^{(l+1)}(z). \qquad (9.9)$$

(We shall usually omit the argument $\bar z$ of fields f, φ because, by virtue of the equations of motion $\bar\partial f = \bar\partial\varphi = 0$, their correlation functions do not depend on $\bar z$.) In order to establish what happens to the fields $f^{(l)}, \varphi^{(l)}$, $l = 0,...,n-1$ in the neighborhood of a branch point, it is convenient to transform to a basis in which the operator $\hat\pi_a$ is diagonal:

$$f_k = \frac{1}{n^{1/2}} \sum_{l=0}^{n-1} \exp\left\{-\frac{2\pi i}{n}[k + j(1-n)]l\right\} f^{(l)},$$
$$\varphi_k = \frac{1}{n^{1/2}} \sum_{l=0}^{n-1} \exp\left\{\frac{2\pi i}{n}[k + j(1-n)]l\right\} \varphi^{(l)} \qquad (9.10)$$

where we have shifted k by $j(1-n)$ for the sake of convenience, as explained below. It follows from (9.10) that

$$\hat{\pi}_a f_k = \exp\left\{\frac{2\pi i}{n}\,[k + j(1-n)]\, f_k\right\},$$
$$\hat{\pi}_a \varphi_k = \exp\left\{-\frac{2\pi i}{n}\,[k + j(1-n)]\, \varphi_k\right\}. \tag{9.11}$$

An important consequence of (9.11) from our point of view is that the conserved currents

$$J_k = :f_k\varphi_k:, \quad \bar{\partial} J_k = 0 \tag{9.12}$$

are single-valued functions of z in the neighborhood of a.

It is found that, for each of the currents J_k, the branch point has the charge

$$q_k = \frac{k + j(1-n)}{n}, \tag{9.13}$$

i.e., in the neighborhood of the point a, the current J_k has a pole of order 1:

$$J_k(z) = \frac{q_k}{z-a} + \text{ Regge terms}. \tag{9.14}$$

(We recall that this and similar relationships must be understood as identities for different correlation functions.) To elucidate the significance of these relationships, it is convenient to express the operators f_k, φ_k $(k = 0,...,n-1)$ in terms of n analytic bosonic scalar fields $\phi_k(z)$ $(k = 0,1,...,n-1)$ normalized so that

$$\langle \phi_k(z)\,\phi_0(z')\rangle = -\ln(z - z') \tag{9.15}$$

by means of the following bosonization rules:

$$f_k = :e^{i\phi_k}:, \quad \varphi_k = :e^{-i\phi_k}:, \quad J_k = i\partial\phi_k,$$
$$T_k = -jf_k\partial\varphi_k - (1-j)\,\varphi_k\partial f_k = -\frac{1}{2}(\partial\phi_k)^2 + \left(\frac{1}{2} - j\right) i\partial^2\phi_k. \tag{9.16}$$

In terms of the fields ϕ_k, (9.13) and (9.14) show that the following operator corresponds to a branch point:

$$V_{\mathbf{q}}(a) = :e^{i\mathbf{q}\vec{\phi}(a)}:, \quad \mathbf{q}\vec{\phi} = \sum_{k=0}^{n-1} q_k\phi_k. \tag{9.17}$$

Using (9.13) and (9.16), we find that the conformal dimension Δ_n of the operator $V_q(a)$ is

$$\Delta_n = \sum_{k=0}^{n-1}\left(\frac{1}{2}q_k^2 + (j - 1/2)\,q_k\right) = \frac{nC_j^F}{24}\left(1 - \frac{1}{n^2}\right),$$
$$C_j^F = (-2)(6j^2 - 6j + 1), \tag{9.18}$$

where $C = nC_j^F$ is the central charge of the Virasoro algebra constructed from the Lourant components of the total energy-momentum tensor[42]

$$T = \sum_{k=0}^{n-1} T_k$$

of the set of fields $f_k, \varphi_k, k = 0,...,n-1$.

We note that the equation

$$\Delta_n(C) = \frac{C}{24}\left(1 - \frac{1}{n^2}\right) \tag{9.19}$$

is a consequence of the general law for the transformation of the energy-momentum tensor T under analytic changes of coordinates in any conformal field theory with central charge C (Ref. 42)

$$T(y) = \left(\frac{dz}{dy}\right)^2 T(z) + \frac{C}{12}\left[\frac{\partial^3 z/\partial y^3}{\partial z/\partial y} - \frac{3}{2}\left(\frac{\partial^2 z/\partial y^2}{\partial z/\partial y}\right)^2\right]. \tag{9.20}$$

Since in terms of the coordinates y (9.3), the function $T(y)$ is regular as $y \to 0$, it follows from (9.20) that, in terms of the coordinates z, T has an additional singularity at $z = a$:

$$T(z) = \frac{C}{24}(z-a)^{-2}(1 - n^{-2}),$$

from which (9.19) follows.

We now turn to the proof of (9.13) and (9.14). We begin by considering the operator expansion of $f^{(l)}(z')\varphi^{(m)}(z)$ around a branch point:

$$f^{(l)}(z')\,\varphi^{(m)}(z) = \left(\frac{dy'}{dz'}\right)^j f(y')\left(\frac{dy}{dz}\right)^{1-j}\varphi(y)$$
$$= n^{-1}(y')^{j(1-n)}(y)^{(1-j)(1-n)}(y' - y)^{-1} + \ldots$$
$$= n^{-1}(z' - z)^{-1}\sum_{l=0}^{n-1}\left(\frac{y'}{y}\right)^{l + j(1-n)} \tag{9.21}$$

where we have used (9.7) in a conformal transformation to the coordinates y, and also the fact that the expansion is trivial in terms of these coordinates:

$$f(y')\,\varphi(y) = (y' - y)^{-1} + \ldots$$

(It is implied in (9.21) that the value of $y'(z')(y(z))$ is taken on the sheet $l(m)$.) It follows from (9.21) and (9.11) that

$$f_k(z')\,\varphi_m(z) = \delta_{k,m}(z' - z)^{-1}\left(\frac{y'}{y}\right)^{k + j(1-n)} + \ldots$$
$$= \delta_{k,m}\left((z' - z)^{-1} + \frac{k + j(1-n)}{n}\times(z-a)^{-1} + O(z' - z)\right) + \ldots \tag{9.22}$$

Comparison with

$$f_k(z')\,\varphi_m(z) = \delta_{k,m}(z' - z)^{-1} + :f_k\varphi_m(z): + O(z' - z),$$

for $k = m$ leads to (9.13) and (9.14). We note that, according to (9.17), the product $f_k(z')\varphi_k(z)$ provides the following contribution to the correlation function at a branch point:

$$f_k(z')\,\varphi_k(z)\,V_{\mathbf{q}}(a)$$
$$= (z' - z)^{-1}[(z' - a)(z-a)^{-1}]^{q_k}:$$
$$\times \exp(i\phi_k(z') - i\phi_k(z) + i\mathbf{q}\phi(a)):, \tag{9.23}$$

which is also in agreement with (9.22).

Formulas (9.13) and (9.17) are the fundamental results of this section.

10. Interaction between branch points in the case of the Abelian monodromy group

We shall now apply the results of Sec. 9 to the simple but interesting surfaces

$$y^n = (z - a_1)\ldots(z - a_M), \quad M = mn, \tag{10.1}$$

for which the basis (9.10) diagonalizes all the operators $\hat{\pi}_{a_i}$ simultaneously, and formulas (9.16) and (9.17) are globally valid, i.e., they are valid on the entire surface X. The number of points M in (10.1) is chosen to be a multiple of n in order to ensure that $z=\infty$ is not a branch point.

As shown in Sec. 7, the main quantity that plays the part of the partition function for analytic fields is the determinant of the operator $\bar{\partial}_j$, found with the aid of the vacuum expectation value (7.7) in accordance with (7.6).

It follows from the results of Sec. 2 that the quantity $\det\bar{\partial}_j$ can be used to evaluate the determinant of the Laplace operator acting on the space of j-differentials:

$$\det{}'\Delta_j=|\det\bar{\partial}_j|^2\det N_j\cdot\det N_{1-j}\exp(C_jS_L), \qquad (10.2)$$
$$S_L=\frac{1}{2\pi}\int(\partial\varphi\bar{\partial}\varphi+\mu^2e^{\varphi})\,d^2\xi,\quad \varphi=\ln\rho,$$

where $N_j^{\alpha\beta}=\int f_\alpha\bar{f}_\beta d^2\xi$ is the matrix of the scalar products of holomorphic j-differentials f_α in the metric (9.2), and S_L is the Liouville action[5] evaluated in this metric. Since the latter does not depend on a_i, we can readily transform (10.2) to

$$\det{}'\Delta_j=\int d^2z_1\ldots d^2z'_{n_{1-j}}|\langle f(z_1)\ldots f(z_{n_j})\varphi(z'_1)\ldots\varphi(z'_{n_{1-j}})\rangle|^2, \qquad (10.3)$$

where we have omitted a numerical (possibly infinite) constant that does not depend on the position of the points a_i, and have used the fact that (7.6) does not depend on z_k, z'_l. The operators $\varphi(z'_k)$ are introduced in (2.3) to the extent necessary for the absorption of all the zero modes of the field φ. Accordingly, n_{1-j} is the number of holomorphic $(1-j)$-differentials. We also recall that

$$\det{}'\Delta_j=\det{}'\Delta_{1-j}. \qquad (10.4)$$

We now turn to the evaluation of the averages in (10.3) in accordance with the rules expressed by (9.13) and (9.17). According to (9.16), each field ϕ_k has a charge $2j-1$ at infinity, so that the only correlators that do not vanish are those for which the total charge of all the operators for each field ϕ_k is $1-2j$, i.e.,

$$d_j^{(k)}\equiv N(f_k)-N(\varphi_k)=1-2j-mnq_k\ (k=0,1,\ldots,n-1) \qquad (10.5)$$

where $N(f_k)$ [correspondingly, $N(\varphi_k)$] represents the number of operators f_k (φ_k) in the average under consideration. We note that (10.5) takes into account the fact that the operators f_k and φ_k have charges $\delta_{k,m}$ and $-\delta_{k,m}$ in the field ϕ_m, and the branch point has the charge q_m. We note that, by summing (10.5) over all k, we obtain the Riemann-Roch theorem

$$\mathrm{ind}(\bar{\partial}_j)\equiv n_j-n_{1-j}=(2j-1)(p-1), \qquad (10.6)$$

where

$$p=1-n+\frac{1}{2}mn(n-1)$$

is the genus of the surface (10.1). The latter follows from the general Riemann-Hurwitz formula which states that the genus of a Riemann surface X that is an n-sheet covering of $\mathbb{CP}^1$ with branch points a_i, $i=1\ldots,N$ of orders n_i is

$$p=1-n+\sum_{i=1}^{N}\frac{n_i-1}{2}. \qquad (10.7)$$

Formulas (9.13) and (9.17) together with the rules given in Ref. 5 and 25 provide a complete description of all the correlation functions of analytic fields on the surfaces (10.1). The determinants of the Laplace operators Δ_j in the metric (9.2) are then found to have the simple integral representation (10.3) of the Coulomb gas type, which is analogous to the Feynman-Fuchs integral representation of correlation functions in minimal models of conformal quantum field theory.[51]

In the important special case $j=1$ $d_1^{(k)}=m(n-k-1)-1$, and if we take (10.4) into account, we find the following representation for the determinant of the Laplace operator Δ_0 (see Ref. 12 and also Ref. 50):

$$\det{}'\Delta_0=\int\prod_{k=0}^{n-2}\prod_{i=1}^{d_1^{(k)}}d^2z_{i,k}\left|\left\langle\prod_{i,k}f_k(z_{i,k})\varphi_{n-1}(z)\right\rangle\right|^2$$
$$=\prod_{k=0}^{n-2}\int\prod_{i=1}^{d_1^{(k)}}d^2z_{i,k}\left|\prod_{i<j}^{d_1^{(k)}}(z_{i,k}-z_{j,k})\times\prod_{i=1}^{d_1^{(k)}}y(z_i)^{k+1-n}\prod_{\alpha<\beta}^{m-n}(a_\alpha-a_\beta)^{\left(\frac{k+1-n}{n}\right)^2}\right|^2 \qquad (10.8)$$

where the infinite constant $\int d^2z$, has been omitted from this formula since $\varphi_{n-1}(z)$ absorbs the scalar zero mode and the average in (10.8) is independent of z (this does not actually occur because $q_{m-1}=0$ for $j=1$).

Let us now consider separately the case $n-2$ of hyperelliptic surfaces defined by

$$y^2=(z-a_1)\ldots(z-a_{2p+2}) \qquad (10.9)$$

in $\mathbb{C}^2=(y,z)$. The evaluation of the determinant of the Laplace operator on this surface is of particular interest not only for two-loop calculations, but also in connection with the correlation functions of spin operators in the Ashkin-Teller model,[50] i.e., the correlation functions of twist fields that arise when a string extends over a Z_2-orbifold.[52]

To evaluate $\det'\Delta_0$, we use (3.9) which, for $n=2$, assumes the particularly simple form

$$\det{}'\Delta_0=\int\prod_{k=1}^{p}d^2z_k|\langle f(z_1)\ldots f(z_p)\varphi(z)\rangle|^2$$
$$=\int\prod_{k=1}^{p}d^2z_k\left|\prod_{i<j}^{p}(z_i-z_j)\prod_{l=1}^{p}y^{-1}(z_l)\prod_{\alpha<\beta}^{2p+2}(a_\alpha-a_\beta)^{1/4}\right|^2. \qquad (10.10)$$

It is useful to transform this expression to the form obtained in Ref. 50 in which this determinant was first evaluated. It will be convenient to return to (7.6) which, for $j=1$ and the surface (10.9), can be written in the form

$$\langle f(z_1)\ldots f(z_p)\varphi(z)\rangle=\det\bar{\partial}_1\cdot\det\|\omega_i(z_j)\| \qquad (10.11)$$

where we have set the scalar zero mode equal to 1 and the basis ω_i, $i=1,\ldots,p$ in the space of holomorphic 1-differentials has been chosen as in Sec. 6. The matrix N_1^{ij} of the scalar products of the differentials ω_i and $\bar{\omega}_j$ in (10.2) is then identical with the imaginary part of the period matrix, so that if we apply the operator

$$\prod_{i=1}^{p}\oint_{a_i}dz_i,$$

to both sides of (10.11) we obtain, using (10.10) and (10.2),

$$\det \bar{\partial}_1 = \det \bar{\partial}_0 = \prod_{i<j} (a_i - a_j)^{1/4} \det K, \qquad (10.12a)$$

$$\det' \Delta_0 = |\det K|^2 \prod_{i<j} |a_i - a_j|^{1/2} \det \operatorname{Im} \tau, \qquad (10.12b)$$

where the $p \times p$ matrix K is given by

$$K_{ji} = \oint_{a_i} z^{j-1} y^{-1}(z)\, dz \quad (i,\ j = 1, \ldots, p). \qquad (10.13)$$

It is precisely in this form that $\det' \Delta_0$ was obtained in Ref. 50.

Let us now consider the fields on hyperelliptic surfaces (10.9) with half integral j. We then have the possibility of imposing different boundary conditions on the fields f and φ: they can be periodic or aperiodic as we cover the basis cycles of the surface (10.9). By analyzing the order of the singularities of the fields f and φ at the branch points for different boundary conditions, we can readily establish that the arbitrariness of their choice has the consequence that not only the operators

$$V_+(a) = \exp\left[i \left(-\frac{j}{2} \phi_0 + \frac{1-j}{2} \phi_1 \right) \right], \qquad (10.14a)$$

but also

$$V_-(a) = \exp\left[i \left(\frac{1-j}{2} \phi_0 - \frac{j}{2} \phi_1 \right) \right] \qquad (10.14b)$$

can correspond to the branch points. (We recall that $f_l = \exp(i\phi_l), \varphi_l = \exp(-\phi_l), l = 0,1$). The total charge of all the operators $V_\pm(a_i)$ in each of the fields ϕ_0, ϕ_1 should then be a number and, up to the replacement $\phi_0 \leftrightarrow \phi_1$, we have exactly 2^{2p} variants for a genus-p surface, which is in accord with the number of different boundary conditions.

We shall now confine our attention to $j = 1/2$, which will enable us to obtain a number of identities for the theta-functions on the surfaces (10.9). This possibility arises because of the general formula (7.27) for the fermionic determinant

$$\det_m \bar{\partial}_{1/2} \cdot (\det \bar{\partial}_0)^{1/2} = \theta_m. \qquad (7.27')$$

We recall that this formula implies that there are no fermionic zero modes (holomorphic 1/2-differentials) on the surface. However, if such modes are present, then $\det_{m\bar{\partial}_{1/2}}$ must be determined with the aid of (7.28), which in general has the form

$$\langle \psi \tilde{\psi}(z_1) \ldots \psi \tilde{\psi}(z_{n_{1/2}}) \rangle (\det \bar{\partial}_0)^{1/2}$$

$$= \theta_{m,k_1 \ldots k_{n_{1/2}}} \omega_{k_1}(z_1) \ldots \omega_{k_{n_{1/2}}}(z_{n_{1/2}}), \qquad (10.15)$$

where the fields f and φ have been renamed as ψ and $\tilde{\psi}$. It is clear that the number of $n_{1/2}$ zero modes of the fields ψ and $\tilde{\psi}$ is the same. Different useful expressions for the theta-constants and their derivatives can be obtained by evaluating the left hand sides of (7.27) and (10.15) by the above methods.

We begin with the case in which there are no zero modes. Since, for $j = 1/2$, the fields ϕ_0 and ϕ_1 have no charge at infinity, only the averages

$$\left\langle \prod_{i', i''} V_+(a_{i'}) V_-(a_{i''}) \right\rangle$$

with equal numbers of operators V_+ and V_- do not vanish. Using (10.14) and $\{i'|i''\}$ to denote the corresponding characteristic, we obtain the following expression:

$$\det_{\{i'|i''\}} \bar{\partial}_{1/2}$$
$$= \prod_{i'<j'} (a_{i'} - a_{j'})^{1/8} \prod_{i''<j''} (a_{i''} - a_{j''})^{1/8} \prod_{i', i''} (a_{i'} - a_{i''})^{-1/8}. \qquad (10.16)$$

Substituting this in (7.27), and using (10.12a), we obtain the Thomie formula[53]

$$\theta_{\{i'|i''\}} = \prod_{i'<j'} (a_{i'} - a_{j'})^{1/4} \prod_{i''<j''} (a_{i''} - a_{j''})^{1/4} \det^{1/2} K. \qquad (10.17)$$

We emphasize once again that the characteristics $\{i'|i''\}$ exhaust all the boundary conditions for which there are no fermionic zero modes.[53]

In order to transform to (10.15), we must consider averages with different numbers of operators V_+ and V_-. The condition that the total charge must be integral then signifies that the difference between the numbers of operators V_+ and V_- is divided by 4. Thus, for genus $p = 2$ we have six such characteristics $m_k, k = 1,\ldots,6$ (cf. Sec. 6). The fields ψ and $\tilde{\psi}$ thus acquire one zero mode each. The corresponding averages are

$$\left\langle \tilde{\psi}_0(z) \psi_0(z') V_+(a_k) \prod_{i \neq k} V_-(a_i) \right\rangle$$
$$= \left(\frac{z - a_k}{y(z)} \right)^{1/2} \left(\frac{z' - a_k}{y(z')} \right)^{1/2} \prod_{i \neq k} (a_i - a_k)^{-1/8} \prod_{\substack{i,j \neq k \\ i<j}} (a_i - a_j)^{1/8} \qquad (10.18)$$

and the identity analogous to (10.17) is

$$\frac{z - a_k}{y(z)} \prod_{\substack{i,j \neq k \\ i<j}} (a_i - a_j)^{1/4} \det^{1/2} K = \theta_{m_k, i}\, \omega_i(z). \qquad (10.19)$$

We can now use these identities to show, in the next Section, that the Beilinson-Manin formula (7.48) for the measure in the theory of bosonic strings with $p = 2$ does not depend on the choice of the odd reference characteristic and actually reduces to the formulas obtained in Sec. 6.

11. Two-loop measure in a bosonic string

In this Section, we apply the above methods to the evaluation of the two-loop measure in the model of closed oriented bosonic strings with critical dimension $D = 26$ (ESVM), and show that the derived expression is identical with the formulas given in Sec. 6 and 7 in terms of the theta-functions. The starting point for all the calculations will be the general relation (7.4) for a p-loop measure in which $\det \bar{\partial}_1$ and $\det \bar{\partial}_0$ must be determined with the aid of (7.6). When $p = 2$, all the surfaces are hyperelliptic and can be described by the equation

$$y^2 = (z - a_1) \cdot \ldots \cdot (z - a_6) \qquad (11.1)$$

in $\mathbb{C}^2 = (y, z)$. The moduli space M_2 can be parametrized by the coordinates of any three branch points, for example, $\mathbb{C}^3$, for fixed positions of the three remaining points. In this way, it will be covered by the hyperplane $\mathbb{C}^3 = (a_1, a_2, a_3)$ a total of $6! = 720$ times and, to find the partition function z_2, we can integrate over $\mathbb{C}^3$, having divided the result by 720. In the metric given by (9.2), we have actually already found all the correlators that appear in the determinants in (7.4), and all that remains to be done is to determine the basis of the holomorphic quadratic differentials $f_i(z)$ in (7.6) that corresponds to the coordinates $a_i i = 1,2,3,$ and M_2. It is readily shown that this basis is

$$f_i(z) = (z - a_i)^{-1} \prod_{k=4}^{6} \frac{a_i - a_k}{z - a_k} \quad (i = 1, 2, 3), \tag{11.2}$$

and the determinant in the denominator in (7.6) is

$$\prod_{i=1}^{3} da_i \det{}^{-1} \| f_k(z_j) \|$$
$$= \frac{da}{dv_{pr}} (a_{45}a_{56}a_{64})^{-1} \det{}^{-1} \|(z_j - a_i)^{-1}\| \prod_{j=1}^{3} \prod_{k=4}^{6} \frac{z_j - a_k}{a_j - a_k}$$
$$= \frac{da}{dv_{pr}} \prod_{i<j}^{3} z_{ij}^{-1} \prod_{k<l}^{6} (a_{kl})^{-1} \prod_{i=1}^{3} y^2(z_i), \tag{11.3}$$

where

$$da \equiv da_1 \ldots da_6, \quad z_{ij} = z_i - z_j, \quad a_{kl} = a_k - a_l,$$
$$dv_{pr} \equiv da_4 da_5 da_6 (a_{45}a_{56}a_{64})^{-1}.$$

The correlator in the numerator of (7.6) is readily evaluated with the aid of the general rules defined by (9.13) and (9.17). For $j = 2$, $n = 2$, and using (10.5), we find that

$$\left\langle \prod_{i=1}^{3} f(z_i) \right\rangle = \left\langle \prod_{i=1}^{3} f_0(z_i) \prod_{k=1}^{6} V_{(-1,-1/2)}(a_k) \right\rangle$$
$$= \prod_{i<j}^{3} z_{ij} \prod_{i=1}^{3} y^{-2}(z_i) \prod_{k<l}^{6} (a_k - a_l)^{5/4}. \tag{11.4}$$

The remaining factors in (7.4) were evaluated previously in Sec. 10 [cf. (10.12)]. Collecting all this together, we find that

$$Z_2 = \int \frac{d^2a}{dV_{pr}} \left| \prod_{k<l}^{6} a_{kl}^{-3} \det{}^{-13} K \right|^2 \det{}^{-13} \operatorname{Im} \tau$$
$$= \int \frac{d^2a}{dV_{pr}} \left| \prod_{k<l}^{6} a_{kl}^{-3} \right|^2 \left(\int d^2z_1 d^2z_2 \, | z_{12} y^{-1}(z_1) y^{-1}(z_2) |^2 \right)^{-13}, \tag{11.5}$$

where $d^2a \equiv da \wedge d\bar{a}$, and $dV_{pr} = dv_{pr} \wedge d\bar{v}_{pr}$ represents an element of the volume of the projective group. Since the complex structure of the surface under consideration remains unaffected by a permutation of the points a_i, the single-valued coordinate in M_2 that runs across the submanifold $a_i = a_j$ of surfaces with a degenerate handle is

$$y_{ij} = (a_i - a_j)^2.$$

In the coordinates $a_i, a_k, y_{ij}, k \neq j$ near $y_{ij} = 0$, the measure in (11.5) obviously has a pole of order 2, in accord with the general theorem of Sec. 4. The integral in parenthesis in (11.5) gives the required power of $\ln|y_{ij}|$. It is also readily verified that the measure in (11.5) has a pole of order 2 for a partition into two toruses.

We must now establish the correspondence between (11.5) and (6.5), (6.14). This will be done by transforming from the coordinates a_i to τ_{ij}, i.e., by replacing the basis (11.2) in the expression for $\det\bar{\partial}_{-1}$ given by (7.6) with the basis related to variations of τ_{ij}. It follows from (7.35) that the elements of this basis are the products of the corresponding holomorphic 1-differentials

$$d\tau_{ik} \to \tilde{f} = \omega_i \omega_k, \tag{11.6}$$

so that, denoting the elements of the new basis by $\tilde{f}_i$, where

$$\tilde{f}_1 = \omega_1^2, \quad \tilde{f}_2 = \omega_2^2, \quad \tilde{f}_3 = \omega_1\omega_2, \tag{11.7}$$
$$d\tau_{11} d\tau_{22} d\tau_{12} \quad \text{by} \quad d\tau,$$

we find that

$$\frac{da}{dv_{pr}} \det{}^{-1} \| f_i(z_j) \|$$
$$= d\tau \det{}^{-1} \| \tilde{f}_i(z_j) \|$$
$$= d\tau \left(\det \| \omega_\alpha(z_\beta) \| \det \| \omega_\alpha(z_\gamma) \| \det \| \omega_\alpha(z_\delta) \| \right)^{-1}$$
$$= d\tau \det{}^3 K \prod_{i<j}^{3} z_{ij}^{-1} \prod_{i=1}^{3} y^2(z_i) \quad (\beta = 1, 2, \ \gamma = 2, 3, \ \delta = 3, 1), \tag{11.8}$$

where in the last equation we have transformed, with the help of (10.13), from the basis ω_i to the basis of 1-differentials

$$\omega_i = z^{i-1} y^{-1}(z) \quad (i = 1, 2). \tag{11.9}$$

Recalling (11.3), we find that

$$\frac{da}{dv_{pr}} = d\tau \det{}^3 K \cdot \prod_{k<l}^{6} (a_k - a_l), \tag{11.10}$$

and (11.5) assumes the form

$$Z_2 = \int_{M_2} d^2\tau \left| \prod_{k<l}^{6} a_{kl}^{-2} \det{}^{-10} K \right|^2 \det{}^{-13} \operatorname{Im} \tau. \tag{11.11}$$

Finally, multiplying together the identities (10.17) for all the ten even characteristics, we obtain

$$\chi_{10}(\tau) = \prod_{\substack{m \\ e(m)=0}} \theta_m^2 = \det{}^{10} K \cdot \prod_{k<l}^{6} a_{kl}^2, \tag{11.12}$$

which leads to the result of Sec. 6, namely,

$$Z_2 = \int_{M_2} d^2\tau \, | \chi_{10}(\tau) |^{-2} \det{}^{-13} \operatorname{Im} \tau. \tag{11.13}$$

We now turn to the verification of the validity of the Beilinson-Manin formula (7.48) in the form given by (7.45). As already noted for $p = 2$, there are six holomorphic 1/2-differentials

$$\psi_\alpha(z) = \left(\frac{z - a_\alpha}{y(z)} \right)^{1/2} \quad (\alpha = 1, \ldots, 6) \tag{11.14}$$

with different odd characteristics m_α, each of which has a simple zero of order 1 at a_α. For $* = m_\alpha$, (7.45) assumes the form

$$F(y) = \frac{\det{}^9 \| \omega(a_\alpha) \, \omega''(a_\alpha) \|}{(\theta_{m_\alpha, i} \omega_i'(a_\alpha))^8 \det \| \mathfrak{f}(a_\alpha) \, \mathfrak{f}'(a_\alpha) \, \mathfrak{f}^{IV}(a_\alpha) \|}. \tag{11.15}$$

We shall show that this reduces to (3.11). Thus, substituting

$$\mathfrak{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)^t, \quad \omega = K^{-1} \mathbf{w}, \tag{11.16}$$

in (11.15), where $\tilde{f}_i$ and w_i are given by (11.7) and (11.9), and we have used the identity given by (10.19). Replacing a_α with z in the numerator and the denominator in (11.5), and extracting the contribution that is the most singular for $z \to a_\alpha$, we find that

$$\det\|\omega(z)\,\omega''(z)\| \sim (z-a_\alpha)^{-2}\det{}^{-1}K\cdot\prod_\beta{}'(a_\alpha-a_\beta)^{-1},$$

$$\det\|f(z)\,f''(z)\,f^{IV}(z)\| \underset{(11.8)}{\propto} (z-a_\alpha)^{-6}\det{}^{-3}K\cdot\prod_\beta{}'(a_\alpha-a_\beta),$$

$$\theta_{m_\alpha,i}\,\omega_i''(z) \propto (z-a_\alpha)^{-3/2}$$

$$\times\det{}^{1/2}K\cdot\prod_\beta(a_\alpha-a_\beta)^{-1/2}\prod_{\substack{\beta<\beta'\\ \beta,\beta'\neq\alpha}}(a_\beta-a_{\beta'})^{1/4}. \qquad (11.17)$$

(The singularities appear at $z=a_\alpha$ because we are evaluating all the quantities in terms of the coordinates $z,\bar z$, which are singular near the branch points.) Hence it follows that the resultant power of $z-a_\alpha$ in (7.45) is zero and, as $z\to a_\alpha$, we have the finite limit

$$F=\det{}^{-10}K\prod_{k<l}^{6}(a_k-a_l)^{-2}, \qquad (11.18)$$

This limit does not depend on the choice of m_α (!) and, after substitution in (7.4), gives (11.11) which in turn reduces to (11.13). This is an additional argument in favor of the validity of (7.45) for arbitrary genus.

12. The sum of all the higher loops as a conformal field theory

Let us now consider an arbitrary N-sheet covering X of the plane $\mathbb{CP}^1$ with branch points a_i of order n_i. For each branch point a_i, we then have, as before, a vertex operator $V(a_i,\bar a_i)$ that is the product of the ghost part $V^{gh}(a_i)$ and the boson spin operator $\mathfrak{S}(a_i,\bar a_i)$ of the theory of fields X_μ:

$$V(a_i,\bar a_i)=V^{gh}(a_i)\,\bar V^{gh}(\bar a_i)\,\mathfrak{S}(a_i,\bar a_i), \qquad (12.1)$$

but the fields ϕ_k introduced in Sec. 9 cannot be confined globally because the operators $\hat\pi_{a_i}$ cannot be simultaneously reduced to the diagonal form.

In this Section, we shall try to represent the sum of all the multiloop diagrams

$$Z_*=\sum_{p\geqslant 2}Z_p \qquad (12.2)$$

as the partition function in a two-dimensional conformally invariant field theory with Lagrangian containing the operators (12.1). In view of this, it is convenient not to transform to the first-order formalism in the theory of the scalar field x_μ. This will enable us to avoid vacuum averages in the denominator.

We begin with the transformation of the individual terms in (12.2). It follows from (9.19) that the dimension of the operator (12.1) is zero because of the cancelation of the conformal anomaly in the critical dimension. However, it is readily verified that, for example, the conformal dimension of the integrand in (11.5) under the group of projective transformations is 1 (in a_i and $\bar a_i$). This increase in the dimension is due to the factor (11.3), and it may be shown that, in general, its role is reduced to the fact that the operator $V^{gh}(a_i)$ in (12.1) must be replaced (for $j=2$ in the notation of Sec. 9) with the operator

$$U^{gh}(a_i)=:f_{n-2}V^{gh}(a_i):\;=:\exp\left(i\sum_{\substack{k=0,\\ k\neq n-2}}^{n-1}\frac{k+2-2n}{n}\phi_k(a_i)\right): \qquad (12.3)$$

whose dimension is greater by 1. We can now deal with $\bar V^{gh}(\bar a_i)$ in (12.1) in a similar way, and to place at infinity the operator

$$A^{gh}=\varepsilon_{N-2}\partial\varepsilon_{N-2}\partial^2\varepsilon_{N-2}\bar\varepsilon_{N-2}\bar\partial\bar\varepsilon_{N-2}\bar\partial^2\bar\varepsilon_{N-2} \qquad (12.4)$$

with the ghost charge $(-3,-\bar 3)$ and zero dimension. The field $\varepsilon(z)$ in (12.4) is a (-1)-differential that is the conjugate of the quadratic differential[10] f in (12.3). The replacement of the operators V^{gh} of charge -1 in the ghost field ε_{n-2} with the operators U^{gh} of charge 0 in the vacuum (12.4) is as a whole analogous to the corresponding transformation in the tree level amplitudes.[54]

Thus, in general, the p-loop measure can be written in the form

$$Z_p=\int\prod_{i=1}^{l}\frac{d^2a_i}{dV_{pr}}\left\langle A^{gh}(\infty)\prod_{i=1}^{l}U^{gh}(a_i)\,\bar U^{gh}(\bar a_i)\,\mathfrak{S}(a_i,\bar a_i)\right\rangle, \qquad (12.5)$$

where it is assumed that the covering $X\overset{z}{\to}\mathbb{CP}^1$ with branch points $a_i, i=1,\ldots,l$ is rigid, i.e.,

α) $l=3p$;

β) the coordinates of any $l-3$ branch points locally parametrize in M_p the neighborhood of zero codimension.

Moreover, it would be desirable for the integral in (12.5) to be evaluated over $\mathbb{C}^{l-3}$, i.e., so that, as in the case of $p=2$,

γ) the map $\mathbb{C}^{l-3}=(a_1,\ldots,a_{l-3})\to M_p$ is a finite-sheet covering.

It is clear that conditions α) and β) are not satisfied[11] for an arbitrary covering X. However, it turns out that the ghost charge is not conserved in this case and the average in (12.5) vanishes! To show this, consider the current

$$J=\sum_{k=0}^{N-1}:f^{(k)}\varepsilon^{(k)}:\,. \qquad (12.6)$$

In general, this is the only current that does not vary as we circulate around the branch points:

$$\pi_{a_i}J=J,\quad i=1,\ldots,l.$$

Using (12.3) and the results of Sec. 9, we can readily show that the charges of the operators $A^{gh}(\infty)$ and $U^{gh}(a_i)$ for this current are -3 and $1-(3/2)(n_i-1)$, respectively. The total charge Q should be equal to $-3N$, from which we find that

$$l=3\left(1-N+\sum_{i=1}^{l}\frac{n_i-1}{2}\right)=3p. \qquad (12.7)$$

This was derived using the Riemann-Hurwitz formula (10.7).

Thus, only the averages (12.5) corresponding to rigid coverings do not vanish. This suggests that the sum of multiloop contributions can be represented by the partition function of the conformal field theory with action

$$S_N=S_N^{(0)}+\int\sum_{\alpha=1}^{d(N)}\lambda_\alpha\Phi_\alpha(a,\bar a)\,d^2a-\ln A_N^{gh}(\infty), \qquad (12.8)$$

in which operators

$$\Phi_\alpha(a,\bar a)=U^{gh}(a)\,\bar U^{gh}(\bar a)\,\mathfrak{S}(a,\bar a) \qquad (12.9)$$

with different α correspond to different types of branch points on the N-sheet surface. It is readily shown that the number $d(N)$ of these types is

$$d(N)=\sum_{k=0}^{N-2}\frac{N!}{k!\,(N-k)},$$

since substitutions corresponding to the operators $\hat{\pi}_a$ that take us once round the point a should contain exactly one cycle. The term $S_N^{(0)}$ describes the free action of ghost fields and X_μ on each of the N sheets in the metric (9.2), and the term $\ln A_N^{gh}(\infty)$ describes boundary conditions imposed on the ghosts: the total charge Q in the current (12.6) must be $-3N$. The adjustable parameters λ_n play the part of coupling constants. Their values will be discussed below.

By expanding $\langle\exp(-S_N)\rangle$ in λ_n, we obtain a sum of terms of the form given by (12.5) but, in addition to the rigid simply-connected coverings, terms will now appear corresponding to multiply-connected coverings.[12] Moreover, the genus of the simply-connected coverings is limited: $p\leqslant 2N-2$. Both defects can readily be removed. All that needs to be done is to replace at ∞ A^{gh} with V_N, which corresponds to a branch point of order N (this enables us to avoid multiply-connected coverings) and confine our attention to the sector with the ghost charge

$$Q=2-3N \qquad (12.10)$$

in the current (12.6), and then proceed to the limit as $N\to\infty$. Clearly, the operators $\Phi_\alpha^{(n)}$ corresponding to branchings of the same order n should have equal coefficients λ_n. We can therefore seek Z_* in the form

$$Z_*=\lim_{N\to\infty}Z(N),$$

$$Z(N)=\int DX_\mu\prod_{k=0}^{N-1}Df^{(k)}D\varepsilon^{(k)}\bigg|_{Q=2-3N}V_N(\infty)$$
$$\times\exp\left(-S^{(0)}-\int\sum_{n=2}^{N}\lambda_n(N)\,\Phi^{(n)}(a,\bar{a})\,d^2a\right),\qquad(12.11)$$
$$\Phi^{(n)}(a,\bar{a})=\sum_\alpha\Phi_\alpha^{(n)}(a,\bar{a}),$$

where in the last line the sum is evaluated over all the operators corresponding to branchings of order n. This expression readily transforms to

$$Z_*=\int DX_\mu\prod_{k=0}^{N-1}Df^{(k)}D\varepsilon^{(k)}\bigg|_{Q=-3N}A^{gh}(\infty)\int\Phi_N(a,\bar{a})\,d^2a$$
$$\times\exp\left(-S_N^{(0)}-\int\sum_{n=2}^{N}\lambda_n(N)\,\Phi^{(n)}(a,\bar{a})\,d^2a\right),\qquad(12.12)$$

from which it follows that all possible terms (12.5) are found to arise when (12.11) is expanded in $\lambda_n(N)$, and only those that correspond to rigid coverings do not vanish (!) and are automatically simply connected.

It is readily shown that the expansion has a finite number of nonzero terms for each N. They do not contain any operators $\Phi^{(n)}$ with $n>N$, and all coverings have the genus $2\leqslant p\leqslant N$. Hence, the equation

$$Z(N)=\sum_{p=2}^{N}Z_p \qquad (12.13)$$

contains $N-1$ conditions on the $N-1$ parameters $\lambda_n(N)$ with $2\leqslant n\leqslant N$. It is natural to suppose that these parameters are uniquely determined by these conditions, and that (12.11) must therefore exist.

We have thus obtained the following result.

The sum of all the higher loops in the theory of closed oriented bosonic strings is equal to the limit as $N\to\infty$ of the partition function of the two-dimensional conformal field theory with the action

$$S_N=S_N^{(0)}+\int\sum_{n=2}^{N}\lambda_n(N)\,\Phi^{(n)}(a,\bar{a})\,d^2a-\ln V_N(\infty)\qquad(12.14)$$

and boundary conditions (12.10) *when the parameters $\lambda_n(N)$ are specially chosen.*

It would be interesting to investigate the properties of the theory (12.14) for $N\to\infty$. It is quite possible that, in this limit, the model contains nonperturbative phenomena, e.g., operators Φ_{cusp} that correspond to branch points of order ∞ on the Riemann surface of the function $y=\ln z$. Logarithmic divergences, discussed in Ref. 4, ensure that the model (12.14) has a nontrivial renormalization group that leads, apart from the renormalization of the constants $\lambda_n(N)$, to the appearance of new operators in the action that correspond to double points, and so on. Moreover, it may well be that a new type of divergence is associated with the appearance of the operators Φ_{cusp} as $N\to\infty$. If an analogous phenomenon were to be discovered in a superstring, this could signify the instability of flat 10-dimensional vacuum.

IV. CONCLUDING REMARKS

I should now like to say a few words about results that have not been covered by this review. The general trend of our discussion has been that, having derived the general theorem on holomorphy in Chapter I, we used this in Chapter II to consider analytic fields and beyond. Having fixed the coordinates on M_p, we calculated the measure. If we parametrize M_p by the matrices τ, the measure is expressed in terms of theta-functions, and the problem of finding Z_p reduces to the problem of characterization of period matrices of Riemann surfaces (Schottky's problem; see Sec. 6). This problem was recently solved by Mulase and Shiota[56] who proved the validity of the Novikov conjecture that

$\tau\in\mathscr{H}_p$ is the period matrix of the nondegenerate Riemann surface of genus p if, and only if, there exist vectors $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\in\mathbb{C}^p$, $\mathbf{a}_1\neq 0$ and a quadratic form

$$Q(t)=\sum_{i,j=1}^{3}Q_{ij}t_it_j,\quad Q_{ij}\in\mathbb{C}$$

such that, for any $\xi\in\mathbb{C}^p$, the function

$$\tau(t)=\exp[Q(t)]\,\theta_0(t_1\mathbf{a}_1+t_2\mathbf{a}_2+t_3\mathbf{a}_3+\xi)\qquad(\text{IV}.1)$$

is the τ-function of the Kadomtsev-Petviashvili (KP) equation [*$\tau(t)$ must not be confused with the period τ-matrix!*], *i.e., it satisfies the equation*

$$(D_1^4+3D_2^2-D_1D_3)\,\tau\cdot\tau=0,$$

where

$$D_i\tau\cdot\tau=\frac{\partial}{\partial y_i}\tau(t_i-y_i)\cdot\tau(t_i+y_i)\bigg|_{y_i=0}.$$

The τ-function and the connection of the KP hierarchy with string theory are discussed in greater detail in Ref. 57.

We note that there is another method of expressing the measure in terms of theta-functions.[58] It relies on the results of Ref. 37. An interesting interpretation of these formulas was proposed in Ref. 47 in terms of the non-Abelian bosonization on a Riemann surface of arbitrary genus.

The holomorphic coordinates on M_p were introduced in a different way in the present paper. Complex structures were parametrized by the coordinates of branch points. It has been found[48] that the evaluation of the measure in this case necessarily involves the methods of the theory of holonomic quantum fields.[59] These methods were developed in Ref. 59 for the Riemann problem on the construction of the matrix coefficients A_i of the linear equation

$$\frac{\partial Y}{\partial z} = \sum_{i=1}^{l} \frac{A_i}{z - a_i} Y \quad \text{(IV.2)}$$

with given monodromy matrices M_i

$$\hat{\pi}_{a_i} Y(z) = Y(z) M_i, \quad \text{(IV.3)}$$

where $Y(z)$ represents the fundamental matrix of the solutions of (2).

This connection arises as follows. Consider the Green's function for analytic fields f and φ with spins j and $1-j$ on a surface X specified in the form of a covering of the z-plane with branch points a_i, $i = 1,...,l$:

$$Y^{km}(z, z_0) = (z_0 - z)\left\langle \varphi^{(k)}(z_0) f^{(m)}(z) \prod_i V_{\mathbf{q}_i}(a)\right\rangle \times \left\langle \prod_i V_{\mathbf{q}_i}(a_i)\right\rangle^{-1} \quad \text{(IV.4)}$$

$(k, m = 0, \ldots, N-1)$

where the upper index on the fields φ and f represents the number of the sheet and the operators $V_{\mathbf{q}_i}(a_i)$ correspond to branch points as in Sec. 9 and 10. We assume, for the sake of simplicity, that the charges $\mathbf{q}_i$ are chosen so that

$$\tau(a_1 \ldots a_l) \stackrel{\text{def}}{=} \left\langle \prod_{i=1}^{l} V_{\mathbf{q}}(a_i)\right\rangle \neq 0 \quad \text{(IV.5)}$$

[not to be confused with the τ-function in (IV.1)!].

It is clear that, for the function (IV.4), the matrices M_i in (IV.3) are permutation matrices and do not depend on z_0 and a_i:

$$\frac{\partial M_i}{\partial z_0} = \frac{\partial M_i}{\partial a_j} = 0 \quad (i, j = 1, \ldots, l). \quad \text{(IV.6)}$$

The function $\partial_z Y \cdot Y^{-1}$ is an analytic function of z with poles of order 1 at the points a_i and a zero at infinity, i.e., it satisfies (IV.2) with certain $A_i = A_i(z_0, a_i \ldots a_N)$. Condition (IV.6) and the normalization condition

$$Y^{km}(z_0, z_0) = \delta^{km} \quad \text{(IV.7)}$$

lead to the relations

$$\frac{\partial Y}{\partial a_i} Y^{-1} = \frac{A_i}{z_0 - a_i} - \frac{A_i}{z - a_i},$$

$$\frac{\partial Y}{\partial z_0} Y^{-1} = -\sum_{i=1}^{l} \frac{A_i}{z_0 - a_i}. \quad \text{(IV.8)}$$

The conditions for the consistency of (IV.8) and (IV.2) lead to the Schlesinger deformation equations (see Ref. 59)

$$dA_m = \sum_k{}' [A_k, A_m]\, d \ln \frac{a_m - a_k}{z_0 - a_k}, \quad \text{(IV.9)}$$

where

$$d = dz_0 \frac{\partial}{\partial z_0} + \sum_{i=1}^{l} da_i \frac{\partial}{\partial a_i}.$$

On the other hand, by substituting the operator expansion

$$\sum_k f^{(k)}(z) \varphi^{(k)}(z_0) = N(z - z_0)^{-1} + J(z_0) + (z - z_0)(j\partial_{z_0} J(z_0) + T(z_0)) + O((z - z_0)^2) \quad \text{(IV.10)}$$

in (IV.4), we can express in terms of the coefficients A_i the average energy-momentum tensor $\langle T(z_0)\rangle$ whose residues at $z_0 = a_i$ are equal to[42]

$$\frac{\partial \ln \tau(a_1 \ldots a_l)}{\partial a_i}.$$

On integrating the latter, we obtain the remarkable results of Ref. 59:

$$d \ln \tau(a_1 \ldots a_l) = \sum_{i<k} \operatorname{tr}(A_i A_k)\, d \ln (a_i - a_k). \quad \text{(IV.11)}$$

The τ-functions (IV.5) and, in particular, the p-loop measure in (12.5), are thus expressed in terms of the solutions of the deformation equations (IV.9). The complete description of the boundary conditions for these equations is still lacking, as is the elucidation of the connection with the results of Section II. At any rate, it is clear that, for monodromy generated by permutation, the τ-functions (IV.5) are theta-functions, by analogy with (IV.1).

Finally, there is one other method of parametrizing M_p, which specifies the Riemann surface in the form of the fundamental polygon F of the Fuchs group $\Gamma \subset SL(2,\mathbb{R})$ with $2p$ generators $\hat{a}_i, \hat{b}_i \in SL(2,\mathbb{R})$ and the relation

$$\prod_{i=1}^{p} \hat{a}_i \hat{b}_i \hat{a}_i^{-1} \hat{b}_i^{-1} = 1. \quad \text{(IV.12)}$$

The coordinates in M_p are the $6p - 6$ *real* parameters of the group Γ, and the determinants and the measure are expressed in terms of these parameters with the help of Selberg's zeta-function. The basic formulas used in this approach are:

$$\det{}' \Delta_j = \det{}' \Delta_{1-j} = \zeta(j) = \prod_\gamma \prod_{n=0}^{\infty} \{1 - \exp[-(j+n) l_\gamma]\} \quad (j = 2, 3, \ldots),$$

$$\left(\int_F \frac{dx\,dy}{y^2}\right)^{-1} \det{}' \Delta_0 = \zeta'(1), \quad \text{(IV.13)}$$

$$Z_p = \int_{M_p} [d\tau]\, \zeta(2) (\zeta'(1))^{-13}, \quad [d\tau] = \prod_{i=1}^{3p-3} l_i\, dl_i\, d\nu_i,$$

where in the first formula the product is evaluated over all the primitive oriented geodesics γ of length l_γ (the corresponding elements $\hat{\gamma} \in \Gamma$ are not powers of any other elements from Γ), and the metric in Δ_j has a constant negative curvature in the upper Labachevskiĭ half-plane on which $SL(2,\mathbb{R})$ acts. Geodesics of length l_i in the measure $[d\tau]$, taken from Ref. 60, do not cross one another, but are otherwise arbitrary.

The angle v_i parametrizes the surfaces obtained by cutting the original surface along the geodesics l_i and regluing after rotation through the angle v_i.

The approach leading to (IV.13) has been used by numerous authors; details can be found in Refs. 10 and 61. In our view, the only defect of the beautiful formulas given by (IV.13) is that they express the measure in terms of the real coordinates on M_p (or, more precisely, on the Teichmuller space $\mathcal{T}$ that is its covering). The simple complex-analytic structure of the theory is then hidden,which is a serious impediment to its study when superstrings are considered. We have not discussed multiloop amplitudes in the theory of supersymmetric and heterotic strings because existing results (at the beginning of 1987-*Ed.*[65]) seem to us to be merely preliminary.

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ADDENDUM FROM THE EDITORIAL BOARD OF USP. FIZ. NAUK

This review was not prepared by its author specifically for publication in Uspekhi Fiz. Nauk. This is why its format is unusual: it does not contain the traditional introductory discussion of physical applications of string theory and of its position in modern theoretical physics. The most popular application of string theory is still the quantum theory of gravity, i.e., the realistic unification of all fundamental interactions. The reason why the theory of strings is capable of solving this problem is discussed in the popular articles of Ref. 62.

A thorough account of the classical period of the development of string theory (1968–1985) is presented in the monographs of Ref. 63 (see also Ref. 64).

1) After D is added to M_p, it becomes a compact algebraic manifold[10] $\overline{M}_p$.

2) In our case of genus $(p \geqslant 2)$, the operator $\rho^{-2}\partial\rho\bar{\partial}$ does not have zero modes and the usual prime on the determinant can be discarded.

3) The Beltrami differential is defined as the quantity $\eta(z,\bar{z})$ whose connection to the complex structure $J(\eta)$ is given by the following expression for the metric matched to η: $\hat{g}(\eta) = |dz + \eta d\bar{z}|^2$.

4) The case of genus zero V_1^* (sealing with a disk) has been examined by Polyakov to whom the present author is indebted for an explanation. The quantity $I_{(1)}$ is then independent of T.

5) We recall that (4.2) and (4.3) define the directions of η^i and of the corresponding coordinates δy_i in $\overline{M}_p$ in terms of the basis f_j.

6) In this case, the degenerate surface can be imagined as being of genus $p-1$ with two punctures R and Q. The cycle a_p runs around one of them and the cycle b_p joins them. The differential ω_p is a renormalized Abelian differential of genus 3 with poles at R and Q. A detailed discussion can be found in Ref. 19.

7) Manin[26] has noted the puzzling coincidence between the number 13 in Mumford's theorem and 26/2 in string theory. I am indebted to him for drawing Ref. 7 to my attention.

8) For $j \in 1/2\mathbb{Z}$ it is 0 for $j<0$, 1 for $j=0$, p for $j=1$, and $(2j-1)(p-1)$ for $j>1$.

9) We note that with the definitions we have adopted, we have $\det_{m_2}(\bar{\partial}_{1/2} + \bar{A}_{12}) \neq \det_{m_1}\bar{\partial}_{1/2}$.

10) We recall that, in terms of the notation of Sec. 9,

$$\varepsilon_{N-2} = \frac{1}{N^{1/2}} \sum_{k=0}^{N-1} \varepsilon^{(k)},$$

where $\varepsilon^{(k)}$ denotes the ghost field on the k-th sheet of X. We also note that the definition of operators (2.13) involves the fields $f^{(k)}$, $\varepsilon^{(k)}$ only on those sheets k that are glued at the branch point a_i.

11) As far as condition γ) is concerned, it is definitely not satisfied for $p \geqslant 24$ because M_p is then irrational.[55] Fortunately, condition γ) is not essential. The situation described below, in (12.14), presupposes the existence of a finite-sheet covering $M_p \to \mathbb{C}^{3p-3}$. This is not forbidden by the general theorems.

12) It was assumed in the derivation of (12.7) that the covering was singly-connected, i.e., that X was one surface and not two.

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