Description of collective processes and fluctuations in classical and quantum plasmas

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The purpose of this review is to show how the development of ideas regarding classical fluctuations in a system of charged particles initially led to an understanding of the nature of collective collisions of particles, transition scattering, and transition bremsstrahlung and also to an understanding of why the cross sections for emission and scattering in a system of many particles are fundamentally different from those for individual charged particles. It is then shown how the development of these concepts and their generalization to the relativistic and quantum-mechanical cases necessarily lead to the appearance of collective radiative-resonant interactions.

I. INTRODUCTION

A fully ionized plasma is the simplest system of charged particles which are interacting with each other and with fields in accordance with the known laws of electrodynamics, which may be regarded as classical in a certain approximation but which in a more rigorous treatment would generally have to be regarded as a quantum electrodynamics. Accordingly, there are both classical and quantum fluctuations in such a system. Taking an average over fluctuations leads to a determination of macroscopically observable quantities. The development of modern plasma physics has actually taken the path of simplifying the averaging methods and working toward a deeper understanding of the basic physics of collective processes.

In speaking of collective processes one usually has in mind both the changes which occur in the cross sections for various processes due to the presence of the ensemble of particles, on the one hand, and the excitation of nonequilibrium collective fields and other perturbations, on the other. The former are related to the "dressing" of particles and to the appearance of elementary excitations for which dressed particles serve as an image. A well-known example is the Debye screening of charges which are inserted into a plasma from the exterior. If we are dealing with the plasma particles themselves, in a situation which is far from equilibrium, we are faced with the question of how each particle participates in the Debye screening of the other particles while at the same time is one of the charges whose field is screened by the other charges. This question can be answered on the basis of a more detailed analysis of fluctuations. In a plasma which is homogeneous on the average (i.e., after an average is taken over fluctuations), there is actually no field of the particles; such a field may be "manifested" only through fluctuations. The collective processes in a highly nonequilibrium system may be both regular and fluctuating (the description of collective fluctuation fields will be discussed below), and the changes caused in cross sections by collective effects arise both for interactions of the particles with external fields and for interactions of the particles with collective perturbations.

If, for example, we are discussing scattering of such collective fields by particles we must bear in mind that the particles participate to some extent in the production of the fields and also in the Debye screening of the particles, while at the same time they are the particles which do the scattering. The only principle for drawing a distinction in this case is based not on particles but on motions: fluctuating and average. It is this distinction which makes it possible to introduce the concept of dressed particles or excitations as entities which can be described by distributions averaged over fluctuations. One might think that a dressing of particles would be characteristic only of condensed media. However, it can be seen in the example of an extremely low-density plasma that this naive expectation is wrong. For example, the cross sections for the scattering of waves whose lengths exceed the Debye screening length are always fundamentally different from the cross sections for scattering in vacuum (by a "bare" charge). The screening radius increases in a simple way as the plasma becomes more rarefied (i.e., as its density decreases), and this assertion becomes valid for longer waves as the density decreases.

Taking an average over fluctuations leads to certain equations for average quantities. In general, this circumstance does not pose any difficulties to reaching an understanding of the situation. However, in addition to the formal side of the question, the result itself and its physical interpretation are also important. This interpretation points to a fundamental change in the cross sections for average quantities.

Bogolyubov's method of a chain of correlation functions[1] actually reduces to the derivation of an equation for these averages. However, in the voluminous literature on this problem, which includes numerous monographs, an exceedingly cumbersome mathematical procedure is used. Today that procedure can be simplified to an extreme degree. The derivation of the Landau-Balescu collision integral[2–4] takes only a few lines (see the discussion below and also Refs. 18 and 24). Below we will use a generalization of Refs. 18 and 24 to the relativistic quantum-mechanical case).

It was because of the cumbersome mathematical method which has previously been used that it was not possible to go into a more detailed analysis of the problems of collective effects in plasmas. This was true not only for processes involving the collisions of particles but also for processes in

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which collective fields participate. Some examples are transition scattering[5,6] and transition bremsstrahlung,[6–9] in which the picture of dressed particles is verified most clearly; in particular, one can clearly see the physical reason for the radical change in the scattering and bremsstrahlung cross sections for particles in a plasma. An important point is that the very concept of elementary excitations (or dressed particles) arises automatically as a result of a rigorous averaging over fluctuations through the use of a perturbation theory in the fluctuating fields. The effects of quantum fluctuations in nearly classical systems, which stem from virtual transverse fields in a system of many particles, [10] play a special role. The latter may transfer energy from certain particles to others. Among the interactions of this sort are the radiative-resonant interactions which result in an additional exchange of energy and momentum between particles in the presence of very nonequilibrium collective fields,[11] along with the (naturally small) radiative corrections to the interaction of particles with these collective fields. The exchange of energy and momentum between "different" particles may become the predominant process,[12] instead of amounting to corrections (although it still retains the small factor $e^2/\hbar c$), if the density of the particles of one species involved in this exchange is predominant. (See the discussion below; crudely speaking, the density ratio n_0/n_f must "make up for" the small value of $e^2/\hbar c$.)

In discussing these quantum fluctuations we need to bear in mind that the particle distribution must be averaged not only over the fluctuations of the particles but also over the "zero-point" fluctuations, as in quantum electrodynamics (QED). We are of course thinking of noninteracting particles or, more precisely, a description of the process in the interaction picture. When the interactions are taken into account, the quantum fluctuations in a system of particles are naturally different from the zero-point fluctuations and depend on the density of particles. This part of the quantum fluctuations turns out to be responsible for the additional exchange of energy and momentum between particles as they interact with collective fields.

In an interaction of this sort, collective quantum effects (an exchange of energy and momentum which depends on the density of particles) thus have an effect on the interaction of the particles with the collective fields. Since the general methods of fluctuation theory (including quantum fluctuations) have been cumbersome, it was not possible for a long time to carry out calculations on such interactions, which can apparently be manifested in many laboratory experiments and in astrophysical observations.[13–15]

Our purposes in this review are partly methodological, specifically, to present an extremely simple approach to the construction of kinetic equations, averaged over fluctuations, for the dressed particles of a plasma. The relations which we use here reproduce the known procedures which have been used by Klimontovich and, to some extent, by Silin[4] for collision integrals[16] and by Sitenko[18] in a description of fluctuations in situations with nonlinear interactions. Using some general and simple arguments, we will derive the existing results in a few lines. The simplification of the derivation makes it possible also to discuss some questions which are not reflected in Refs. 16–18 and, furthermore, to offer a simple interpretation of the results, including such results as the place of transition scattering and transition bremsstrahlung in modern plasma physics. It was pointed out already in Refs. 19 and 20 (see Ref. 21) that the total cross sections for the scattering by dressed plasma particles must include both ordinary Thomson scattering (on the one hand) and transition scattering and the interference of the two (on the other). Particular emphasis was placed in Refs. 19 and 21 on induced scattering, for which a description can be derived through a statistical averaging of the nonlinear equations, without any need to resort to the theory of linear and nonlinear fluctuations. The scattering probabilities found from induced processes have also been used in calculations on spontaneous scattering (i.e., on the scattering which is usually discussed in the case of low intensities). An independent method for calculating transition scattering processes, based on nonlinear plasma currents, has also been offered.[21,22]

The scattering in a homogeneous medium, however, is a scattering by fluctuations, as we know quite well.[23] To illustrate the use of the simple methods for calculations on fluctuations, we will show below that the spontaneous scattering by fluctuations is described by the square of the sum of the amplitudes for transition and Thomson scattering. In other words, transition scattering can be derived rigorously in the theory of fluctuations. It is also possible to offer a different interpretation of this result in terms of dressed particles: The scattering by the fluctuations of a unit volume of a plasma occurs in the way that it would if the dressed electrons and ions were scattering independently (here we mean the electrons and ions whose fields are screened by the Debye sphere). The number of dressed electrons and ions in a unit volume is equal to the number of (real) electrons and ions in a unit volume of the plasma. From the scattering standpoint, the plasma volume is filled in a sense by classical quasineutral atoms whose electron clouds (the Debye screening spheres) are produced by dynamically free plasma electrons. It is clear that the Debye spheres themselves are produced by the same electrons (and, in general, ions) of the plasma, so that a real electron plays two roles: a scattering center and an element of a Debye sphere. It is in this manner that the picture of dressed particles actually emerges. The dressed particles themselves are described by a distribution function averaged over fluctuations.

An important point is that the same picture appears for processes involving the collisions of particles and bremsstrahlung processes, i.e., the emission which occurs during the collisions of particles. The bremsstrahlung of a unit plasma volume which arises because of fluctuations is equal to the sum (interference is taken into account here) of ordinary and transition bremsstrahlung.[7] The latter is none other than the transition scattering of a virtual photon of the colliding particles into a real photon. This statement means that the picture of dressed particles is completely "workable" again in this case. After discussing these examples we will discuss quantum-fluctuation effects and the transfer of energy and momentum between particles as a result of the effect of thess fluctuations on the interaction of particles with collective fields. We will generalize the simple equations of classical fluctuations, discuss methods for taking averages over quantum fluctuations, and report the results of a calculation of the quantum fluctuations in a system of many particles which are interacting with collective fields.

We need to point out that in this discussion we are using a rather profound idea which was pointed out by Kadomtsev in a 1964 review[24]: It is possible to take classical "zero-point" fluctuations into account correctly while taking the average of a one-particle distribution function which obeys a collisionless kinetic equation. This classical limit of the fluctuations in a quantum-mechanical treatment actually follows from the "zero-point" fluctuations of particles, but the result, being purely classical, has a very simple interpretation in terms of the classical fluctuations of independent events (in this case, of noninteracting particles).

2. FLUCTUATIONS OF INDEPENDENT FREE PARTICLES

We first consider the classical limit. We denote by $f_{\mathbf{p},\mathbf{k},\omega}$ a one-particle distribution function which describes the distribution of particles with respect to the momentum **p** and with respect to the 4-momentum transfer $k = \{\mathbf{k},\omega\}$: $f_\mathbf{p}(\mathbf{r},t) = \int f_{\mathbf{p},\mathbf{k},\omega} \exp[i\mathbf{kr} - i\omega t] d\mathbf{k} d\omega$ ($f_{\mathbf{p},\mathbf{k},\omega}$ also has the meaning of Fourier coordinate and time components). Each of the particles is moving freely, and the function $f_\mathbf{p}$ is uniform, on the average, over space and time. Microscopic motion lead to a dependence of $f_\mathbf{p}$ on **k** and ω, and they give rise to fluctuations. We set

$$f_\mathbf{p}(\mathbf{r}, t) = \Phi_\mathbf{p} + \delta f_\mathbf{p}(\mathbf{r}, t), \quad \Phi_\mathbf{p} = \langle f_\mathbf{p}\rangle, \tag{2.1}$$

$$\frac{\partial}{\partial t}\delta f_\mathbf{p}(\mathbf{r}, t) + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\delta f_\mathbf{p}(\mathbf{r}, t) = 0, \tag{2.2}$$

where $\Phi_\mathbf{p}$ is independent of **r** and t. Equation (2.2) is a consequence of the general Liouville theorem in the absence of fields. The presence of a field causes interactions. In a first approximation, these interactions can be ignored. For simplicity, we assume that there are no external fields.

The one-particle distribution function $f_\mathbf{p}(\mathbf{r},t)$ is found by integrating the distribution function of the variables of all the particles over all the variables except those of the given particle $(\mathbf{p},\mathbf{r},t)$. Equation (2.2) then naturally arises. In Fourier spatial components, Eqs. (2.1) and (2.2) take the form

$$\frac{\partial}{\partial t}\delta f_{\mathbf{p},\mathbf{k}}(t) + i\mathbf{kv}\delta f_{\mathbf{p},\mathbf{k}}(t) = 0, \quad \langle f_{\mathbf{p},\mathbf{k}}\rangle = \Phi_\mathbf{p}\delta(\mathbf{k}). \tag{2.3}$$

According to general considerations, the fluctuation correlation function $\langle \delta f_{\mathbf{p},\mathbf{k},\omega}\delta f_{\mathbf{p}',\mathbf{k}',\omega'}\rangle$ in a steady-state, homogeneous medium should be proportional to $\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')$. Furthermore, the distribution functions for different values of **p** should be totally uncorrelated (there is no interaction). In other words, the correlation function should contain $\delta(\mathbf{p}-\mathbf{p}')$. Finally, by virtue of (2.3), it should contain $\delta(\omega - \mathbf{kv})$:

$$\langle \delta f_{\mathbf{p},\mathbf{k},\omega}\delta f_{\mathbf{p}',\mathbf{k}',\omega}\rangle = |\delta f|^2_\mathbf{p}\delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\delta(\omega-\mathbf{kv}). \tag{2.4}$$

In order to determine $|\delta f|^2_\mathbf{p}$ it is necessary to use the theorem[25] from statistical physics which states that the average fluctuation of the square of the number of particles, $\langle \sigma N^2\rangle$, in a volume V is equal to the average number of particles, $\langle N\rangle$, in V. As these particles we adopt particles with a momentum **p**. We take the volume to be a cube of side L; we expand all functions in Fourier series, and we take the limit $L\to\infty$. Under the normalization condition

$$\left\langle \int f_\mathbf{p}\frac{d\mathbf{p}}{(2\pi)^3}\right\rangle = \int \frac{\Phi_\mathbf{p}\,d\mathbf{p}}{(2\pi)^3} = n, \tag{2.5}$$

where n is the density of particles, we then find $|\delta f|^2_\mathbf{p} = \Phi_\mathbf{p}$, i.e.,

$$\langle \delta f_{\mathbf{p},k}\delta f_{\mathbf{p}',k'}\rangle = \Phi_\mathbf{p}\delta(\mathbf{p}-\mathbf{p}')\delta(\omega-\mathbf{kv})\delta(k+k'), \quad k = \{\mathbf{k},\omega\}. \tag{2.6}$$

This extremely simple expression actually makes it possible to describe the entire rich set of collective processes and fluctuations in a system of classical charged particles which form a plasma.

It is possible to find a quantum-mechanical generalization of (2.6) and to find (2.6) from it in the classical limit. In the second-quantization representation of $\Phi_\mathbf{p}$, this is the average of the occupation number of the particles of momentum **p** over a statistical ensemble. The occupation numbers themselves are known to be defined as the vacuum expectation values $\langle 0|...|0\rangle$ of the operator $\hat{a}^+_\mathbf{p}\hat{a}_\mathbf{p}$, where $|0\rangle$ is the vacuum state, and the operator $\hat{a}^+_\mathbf{p}$ creates a particle ($\hat{a}_\mathbf{p}$ is the operator which annihilates the particle) with momentum **p**:

$$\Phi_\mathbf{p} = \int \langle\langle 0|\hat{a}^+_{\mathbf{p}'}\hat{a}_\mathbf{p}|0\rangle\rangle d\mathbf{p}'. \tag{2.7}$$

The first (outer) angle brackets here correspond to a statistical average, while $\langle 0|$ and $|0\rangle$ correspond to the final and initial vacuum states. In other words, we are taking an average over both the statistical ensemble and the vacuum fluctuations in (2.7). In the discussion below we assume that specifically this type of averaging is carried out for the quantized operator quantities. We will omit the double angle brackets from the expressions, and we will also omit the 0 for the vacuum state. In other words, we rewrite (2.7) as

$$\Phi_\mathbf{p} = \int \langle \hat{a}^+_{\mathbf{p}'}\hat{a}_\mathbf{p}\rangle d\mathbf{p}', \quad \langle a^+_{\mathbf{p}'}a_\mathbf{p}\rangle = \Phi_\mathbf{p}\delta(\mathbf{p}-\mathbf{p}'), \tag{2.8}$$

where the $\Phi_\mathbf{p}$ are simply the corresponding occupation numbers or, more precisely, their averages; over the statistical ensemble.

Clearly, both (2.6) and (2.8) hold for any distributions. In other words, we are not restricting the discussion to equilibrium (e.g., Maxwellian) distributions.

We can introduce the particle operator in the momentum representation, $\hat{\Psi}$. For free particles ($\hbar = c = 1$) we can write

$$\hat{\Psi}_\mathbf{p}(t) = \hat{a}_\mathbf{p}e^{-i\varepsilon_\mathbf{p}t}, \tag{2.9}$$

where $\varepsilon_\mathbf{p}$ is the energy of the particles [in the general relativistic case we would have $\varepsilon_\mathbf{p} = (\mathbf{p}^2 + m^2)^{1/2}$, but in the nonrelativistic theory here we have $\varepsilon_\mathbf{p} = m + (\mathbf{p}^2/2m)$, or $\varepsilon_\mathbf{p} = \mathbf{p}^2/2m$ if we discard the constant phase factor $\exp(-imt)$ from (2.9)]. By virtue of $\mathbf{p} = \mathbf{p}'$ we have $\varepsilon_{\mathbf{p}'} = \varepsilon_\mathbf{p}$ and

$$\Phi_\mathbf{p} = \int \langle \hat{\Psi}^+_{\mathbf{p}'}\hat{\Psi}_\mathbf{p}\rangle d\mathbf{p}'. \tag{2.10}$$

It is convenient to introduce the Wigner operator $\hat{f}_{\mathbf{p},\mathbf{k}}$ ($\hbar = c = 1$):

$$\hat{f}_{\mathbf{p},\mathbf{k}} = \hat{\Psi}^+_{\mathbf{p}-\frac{\mathbf{k}}{2}}(t)\hat{\Psi}_{\mathbf{p}+\frac{\mathbf{k}}{2}}(t). \tag{2.11}$$

We can then write (2.8) as

$$\langle \hat{f}_{\mathbf{p},\mathbf{k}}\rangle = \Phi_\mathbf{p}\delta(\mathbf{k}). \tag{2.12}$$

The only distinction between (2.12) and the classical expression, (2.3), is the operator sign on the left side of (2.12).

We introduce the fluctuation operator

$$\delta\hat{f}_{\mathbf{p},\mathbf{k}}(t)=\hat{f}_{\mathbf{p},\mathbf{k}}(t)-\langle\hat{f}_{\mathbf{p},\mathbf{k}}(t)\rangle$$
$$=\hat{\Psi}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}(t)\,\hat{\Psi}_{\mathbf{p}+\frac{\mathbf{k}}{2}}(t)-\langle\hat{\Psi}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}(t)\,\hat{\Psi}_{\mathbf{p}+\frac{\mathbf{k}}{2}}(t)\rangle. \quad (2.13)$$

It is easy to see that this operator satisfies the following equation by virtue of (2.9):

$$\frac{\partial}{\partial t}\delta\hat{f}_{\mathbf{p},\mathbf{k}}(t)+i\left(\varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}}-\varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right)\delta\hat{f}_{\mathbf{p},\mathbf{k}}(t)=0. \quad (2.14)$$

Under the condition $k\ll p$ we have

$$\varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}}-\varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}\approx\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\varepsilon_{\mathbf{p}}=\mathbf{kv},$$

and Eq. (2.14) becomes the same as (2.3). A solution of (2.14) is

$$\delta\hat{f}_{\mathbf{p},\mathbf{k}}(t)=\delta\hat{f}_{\mathbf{p},\mathbf{k}}(0)\exp\left[-i\left(\varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}}-\varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right)t\right], \quad (2.15)$$

$$\delta\hat{f}_{\mathbf{p},\mathbf{k}}(0)=\hat{a}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{a}_{\mathbf{p}+\frac{\mathbf{k}}{2}}-\langle\hat{a}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{a}_{\mathbf{p}+\frac{\mathbf{k}}{2}}\rangle, \quad (2.16)$$

$$\delta\hat{f}_{\mathbf{p},\mathbf{k},\omega}=\delta\hat{f}_{\mathbf{p},\mathbf{k}}(0)\,\delta\left(\omega-\varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right). \quad (2.17)$$

We can now find the expectation value which we are seeking:

$$\langle\delta\hat{f}_{\mathbf{p},\mathbf{k},\omega}\delta\hat{f}_{\mathbf{p}',\mathbf{k}',\omega'}\rangle$$
$$=\delta\left(\omega-\varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right)\delta\left(\omega'-\varepsilon_{\mathbf{p}'+\frac{\mathbf{k}'}{2}}+\varepsilon_{\mathbf{p}'-\frac{\mathbf{k}'}{2}}\right)$$
$$\times\left(\langle\hat{a}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{a}_{\mathbf{p}+\frac{\mathbf{k}}{2}}\hat{a}^{+}_{\mathbf{p}'-\frac{\mathbf{k}'}{2}}\hat{a}_{\mathbf{p}'+\frac{\mathbf{k}'}{2}}\rangle-\langle\hat{a}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{a}_{\mathbf{p}+\frac{\mathbf{k}}{2}}\rangle\langle\hat{a}^{+}_{\mathbf{p}'-\frac{\mathbf{k}'}{2}}\hat{a}_{\mathbf{p}'+\frac{\mathbf{k}'}{2}}\rangle\right). \quad (2.18)$$

Since the only nonzero expectation values are those corresponding to the case in which there is a creation operator on the right and then an annihilation operator (if we are working from right to left), by breaking up the expectation value of the four operators into products of paired expectation values (this approach naturally presupposes that the free particles are statistically independent, as is assumed in the classical derivation) and by noting that we have $\mathbf{k}\neq 0$ and that the operator $a_{\mathbf{p}+\mathbf{k}/2}$ commutes with $\hat{a}^{+}_{\mathbf{p}-\mathbf{k}/2}$ and $\hat{a}_{\mathbf{p}-\mathbf{k}/2}$, we find

$$\langle\hat{a}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{a}_{\mathbf{p}+\frac{\mathbf{k}}{2}}\hat{a}^{+}_{\mathbf{p}'-\frac{\mathbf{k}'}{2}}\hat{a}_{\mathbf{p}'+\frac{\mathbf{k}'}{2}}\rangle-\langle\hat{a}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{a}_{\mathbf{p}+\frac{\mathbf{k}}{2}}\rangle\langle\hat{a}^{+}_{\mathbf{p}'-\frac{\mathbf{k}'}{2}}\hat{a}_{\mathbf{p}'+\frac{\mathbf{k}'}{2}}\rangle$$
$$=\langle\hat{a}^{+}_{\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{a}_{\mathbf{p}'+\frac{\mathbf{k}'}{2}}\rangle\langle\hat{a}_{\mathbf{p}+\frac{\mathbf{k}}{2}}\hat{a}^{+}_{\mathbf{p}'-\frac{\mathbf{k}'}{2}}\rangle$$
$$=\Phi_{\mathbf{p}-\frac{\mathbf{k}}{2}}\left(1-\Phi_{\mathbf{p}+\frac{\mathbf{k}}{2}}\right)\delta(\mathbf{p}-\mathbf{p}')\,\delta(\mathbf{k}+\mathbf{k}'). \quad (2.19)$$

Here we have used (2.8) and $\hat{a}^{+}_{\mathbf{p}'}\hat{a}_{\mathbf{p}}+\hat{a}_{\mathbf{p}}\hat{a}^{+}_{\mathbf{p}'}=\delta(\mathbf{p}-\mathbf{p}')$. We now assume $\Phi_{\mathbf{p}}\ll 1$, and we ignore $\Phi_{\mathbf{p}}$ in comparison with unity. By virtue of the δ-function in (2.19), we easily see that one of the δ-functions in (2.18) becomes $\delta(\omega'+\varepsilon_{\mathbf{p}+\mathbf{k}/2}-\varepsilon_{\mathbf{p}-\mathbf{k}/2})=\delta(\omega+\omega')$.

We thus find[17]

$$\langle\delta\hat{f}_{\mathbf{p},\mathbf{k},\omega}(\delta\hat{f}_{\mathbf{p}',\mathbf{k}',\omega'}\rangle$$
$$=\Phi_{\mathbf{p}-\frac{\mathbf{k}}{2}}\delta(\mathbf{p}-\mathbf{p}')\,\delta(\mathbf{k}+\mathbf{k}')\,\delta\left(\omega-\varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}\right), \quad (2.20)$$

which serves as a generalization of classical relation (2.6) [under the condition $k\ll p$, (2.6) follows from (2.20)].

It might appear at first glance that here we have not made use of the relation $\langle\delta N^2\rangle=\langle N\rangle$, which is used in the classical derivation. However, this relation itself is actually derived by a method which is analogous to our breakup of the complex expectation values into paired expectation values in the quantum derivation.

We have one more comment. It concerns the meaning of the quantum-mechanical operator $\delta\hat{f}$ in comparison with the classical operator δf. This meaning is not particularly transparent. It turns out, however, that we do not even need such an interpretation, since we are actually interested in only observable quantities, and for a system of particles for which the fluctuations correspond in a first approximation to fluctuations of free particles (and this is the case for weakly interacting particles which make up a plasma under the condition $nd_i^3\gg 1$), the observable quantities which are of interest are expressed in terms of expectation values of quadratic combinations of $\delta\hat{f}$, and they are given in terms of an observable quantity, the expectation value of the probability for observing particles with a momentum $\Phi_{\mathbf{p}}$, by relation (2.20).

Actually, the subject of this analysis will be not perfectly free particles but particles which are interacting weakly with each other. When the interaction is taken into account, $\hat{\Psi}_{\mathbf{p}}(t)$ is a more complicated function of the time than (2.9) is. This function in (2.10) will yield a quantity $\Phi_{\mathbf{p}}$ which depends on t. An important point is that the meaning of (2.10) as the expectation value of a probability is preserved when we take interactions into account, if the operator $\hat{\Psi}^{+}_{\mathbf{p}}$ is a particle creation operator (with the interactions being taken into account), while $\hat{\Psi}_{\mathbf{p}}$ is an annihilation operator. The quantity $\Phi_{\mathbf{p}}$ has a completely specific and observable meaning. For this reason, the observational meaning of the operator $\delta\hat{f}$ may even not be of much interest to us, if the final relations contain $\Phi_{\mathbf{p}}$, provided that we can write dynamic equations for $\delta\hat{f}$ or $\hat{f}$ (taking the interactions into account) or provided that we can express $\delta\hat{f}$ (with the interactions being taken into account) in terms of $\delta\hat{f}^{(0)}$ for free particles and use (2.20) in taking the expectation value of the expressions for $\delta\hat{f}^{(0)}$. Either approach can be taken: We can either write an equation for $\delta\hat{f}$ on the basis of Schrödinger equations, or we can use an $\hat{S}$ matrix and the interaction picture.

That digression was necessary in order to illustrate the fundamental role played by relations (2.20) for the entire procedure used below to calculate the average characteristics of the plasma and to illustrate the point that the quantum-mechanical description of the fluctuations actually requires knowledge of the expectation values of the operators $\delta\hat{f}$, which actually play the same role as the fluctuation part of the distribution function for a classical description.

If we wish to describe the system of particles in a relativistically invariant way, we must introduce some positive and negative energies. For spin-1/2 particles, the operators $\hat{\Psi}_{\alpha}$ have spinor indices α. It is thus convenient to introduce the Wigner operator not as in (2.11) but in the form

$$\hat{f}_{\alpha\beta,\mathbf{p},\mathbf{k}}(t)=\frac{1}{2}\left(\hat{\Psi}^{+}_{\beta,\mathbf{p}-\frac{\mathbf{k}}{2}}(t)\,\hat{\Psi}_{\alpha,\mathbf{p}+\frac{\mathbf{k}}{2}}(t)-\hat{\Psi}_{\alpha,\mathbf{p}+\frac{\mathbf{k}}{2}}(t)\,\hat{\Psi}^{+}_{\beta,\mathbf{p}-\frac{\mathbf{k}}{2}}(t)\right). \quad (2.21)$$

This expression differs from (2.11) in that we have assigned spinor indices α and β to the operators $\hat{\Psi}$ and $\hat{\Psi}^+$, and we have also taken half the sum of the expressions which are found by interchanging $\hat{\Psi}$ and $\hat{\Psi}^+$. For $\mathbf{k}\neq 0$, Ψ and Ψ^+ anticommute. In the case $\mathbf{k} = 0$, definition (2.21) is convenient in the sense that in the absence of particles the vacuum expectation value of (2.21) vanishes.

The 4-current density is expressed in terms of $\hat{f}$ in the following way (for the metric $g_{ii} = 1$, $g_{44} = i$, $\gamma_\mu = \{\gamma,\beta\}$, where γ and β are Dirac matrices):

$$\hat{j}_{\mu,\mathbf{k}}(t) = e\int \mathrm{Sp}\, i\beta\gamma_\mu \hat{f}_{\mathbf{p},\mathbf{k}}(t)\frac{d\mathbf{p}}{(2\pi)^3}\,. \tag{2.22}$$

For free particles we then find the charge density

$$\rho_{\mathbf{k}} = \langle\hat{\rho}_{\mathbf{k}}\rangle = e\int \Phi_{\mathbf{p}}\frac{d\mathbf{p}}{(2\pi)^3}\,\delta(\mathbf{k}),\qquad \mathrm{Sp}\,\langle\hat{f}_{\mathbf{p},\mathbf{k}}\rangle = \Phi_{\mathbf{p}}\delta(\mathbf{k}). \tag{2.23}$$

The quantity $\Phi_{\mathbf{p}}$ thus describes the difference between the numbers of particles and antiparticles, although this point is of minor importance, since we can introduce occupation numbers for both positive and negative energies.

The operators $\hat{\Psi}$ can be classified on the basis of the sign of the energy, $\lambda = \pm 1$, by writing $\hat{\Psi}^\lambda$. The quantity $\hat{f}$ will then have the two indices λ and λ':

$$\hat{f}^{\lambda,\lambda'}_{\alpha\beta,\mathbf{p},\mathbf{k}} = \frac{1}{2}\left(\hat{\Psi}^{+\lambda}_{\beta,\mathbf{p}-\frac{\mathbf{k}}{2}}\hat{\Psi}^{\lambda'}_{\alpha,\mathbf{p}+\frac{\mathbf{k}}{2}} - \hat{\Psi}^{\lambda'}_{\alpha,\mathbf{p}+\frac{\mathbf{k}}{2}}\hat{\Psi}^{+\lambda}_{\beta,\mathbf{p}-\frac{\mathbf{k}}{2}}\right), \tag{2.24}$$

$$\hat{\Psi}^{\lambda}_{\alpha,\mathbf{p}+\frac{\mathbf{k}}{2}} = \sum_\mu u^{\lambda,\mu}_{\alpha,\mathbf{p}}\hat{a}^{\lambda,\mu}_{\mathbf{p}}e^{-i\lambda\varepsilon_{\mathbf{p}}t},$$

where $\mu = \pm 1$ is the projection of the spin onto $\mathbf{p}$, and $u^{\lambda,\mu}_{\alpha,\mathbf{p}}$ is a bispinor. If the particles are unpolarized, then we have $\Phi_{\mathbf{p}} = \Sigma_\mu \Phi^\mu_{\mathbf{p}}, \Phi_{\mathbf{p}}^{+1} = \Phi_{\mathbf{p}}^{-1}$,

$$\Phi^\mu_{\mathbf{p}} = \frac{1}{2}\Phi_{\mathbf{p}},$$

$$\langle\hat{a}^{\mu'+}_{\mathbf{p}'}\hat{a}^\mu_{\mathbf{p}}\rangle = \delta_{\mu,\mu'}\Phi^\mu_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}') = \frac{1}{2}\Phi_{\mathbf{p}}\delta_{\mu\mu'}\delta(\mathbf{p}-\mathbf{p}'),$$

$$\hat{a}^{+1,\mu}_{\mathbf{p}} \equiv \hat{a}^\mu_{\mathbf{p}},\ \hat{a}^{-1,\mu}_{\mathbf{p}} = \hat{b}^{\mu+}_{-\mathbf{p}}. \tag{2.25}$$

The operators which project onto positive and negative energies, $\Lambda^\pm_{\mathbf{p}} = \Lambda^\lambda_{\mathbf{p}}$, are introduced by

$$\Lambda^\lambda_{\mathbf{p}} = \frac{m - i\gamma\mathbf{p} + \lambda\varepsilon_{\mathbf{p}}\beta}{2\varepsilon_{\mathbf{p}}},\qquad \sum_\mu u^{\lambda,\mu}_{\alpha,\mathbf{p}}u^{\lambda,\mu}_{\beta,\mathbf{p}} = \lambda(\Lambda^\lambda_{\mathbf{p}}\beta)_{\alpha\beta}. \tag{2.26}$$

If we assume that there are no positrons, we find from (2.24)

$$\langle\hat{f}^{+,+}_{\alpha\beta,\mathbf{p},\mathbf{k}}\rangle + \langle\hat{f}^{-,-}_{\alpha\beta,\mathbf{p},\mathbf{k}}\rangle = (\Lambda^+_{\mathbf{p}}\beta)_{\alpha\beta}\,\delta(\mathbf{k})\frac{1}{2}(\Phi^+_{\mathbf{p}} - 1)$$

$$+\frac{1}{2}(\Lambda^-_{\mathbf{p}}\beta)_{\alpha\beta}(\Phi^-_{\mathbf{p}} - 1)\,\delta(\mathbf{k}),\ \langle\hat{f}^{+,-}_{\alpha\beta,\mathbf{p},\mathbf{k}}\rangle = \langle\hat{f}^{-,+}_{\alpha\beta,\mathbf{p},\mathbf{k}}\rangle = 0, \tag{2.27}$$

Alternatively, assuming $\Lambda_{\mathbf{p}}^+\beta - \Lambda_{\mathbf{p}}^-\beta = 1, \mathrm{Sp}\Lambda_{\mathbf{p}}^+\beta = 2, \mathrm{Sp}\Lambda_{\mathbf{p}}^-\beta = -2$, we find

$$\mathrm{Sp}\,\Lambda^+_{\mathbf{p}}\beta\hat{f} = (\Phi^+_{\mathbf{p}} - 1)\,\delta(\mathbf{k}),\qquad \mathrm{Sp}\,\Lambda^-_{\mathbf{p}}\beta\hat{f} = (\Phi^-_{\mathbf{p}} - 1)\,\delta(\mathbf{k}), \tag{2.28}$$

$$\mathrm{Sp}\,\hat{f} = \mathrm{Sp}(\Lambda^+_{\mathbf{p}}\beta - \Lambda^-_{\mathbf{p}}\beta)\,\hat{f} = (\Phi^+_{\mathbf{p}} - \Phi^-_{\mathbf{p}})\,\delta(\mathbf{k}). \tag{2.29}$$

We will assume below that there are no positrons ($\Phi_{\mathbf{p}}^- = 0$). The constant terms in (2.28), i.e., the -1 on the right side of (2.28), reflect the unobservable vacuum parts. These terms cancel out in the observable final quantities, such as $\mathrm{Sp}\langle f\rangle$, the permittivity, and the equation for Φ. For this reason, we will retain in the expressions written below only the terms which are proportional to $\Phi_{\mathbf{p}}$; in certain cases we will also be discarding terms which are quadratic in $\Phi_{\mathbf{p}}$ (we will thereby be ignoring degeneracy).

Actually, by repeating the derivation of (2.20), but with a spinor operator in (2.21), and by using (2.24) and (2.26), we find

$$\langle\delta\hat{f}^{+,+}_{\alpha\beta,\mathbf{p},\mathbf{k}}\delta\hat{f}^{+,+}_{\alpha'\beta',\mathbf{p}',\mathbf{k}'}\rangle = \frac{1}{2}\Phi_{\mathbf{p}-\frac{\mathbf{k}}{2}}(\Lambda^+_{\mathbf{p}+\frac{\mathbf{k}}{2}}\beta)_{\alpha\beta'}$$

$$\times(\Lambda^+_{\mathbf{p}-\frac{\mathbf{k}}{2}}\beta)_{\alpha'\beta}\,\delta(\mathbf{p}-\mathbf{p}')\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega - \varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}} + \varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}); \tag{2.30}$$

$$\langle\delta\hat{f}^{+,-}_{\alpha\beta,\mathbf{p},\mathbf{k}}\delta\hat{f}^{-,+}_{\alpha'\beta',\mathbf{p}',\mathbf{k}'}\rangle = \frac{1}{2}\Phi_{\mathbf{p}+\frac{\mathbf{k}}{2}}(\Lambda^+_{\mathbf{p}+\frac{\mathbf{k}}{2}}\beta)_{\alpha\beta'}$$

$$\times(\Lambda^-_{\mathbf{p}-\frac{\mathbf{k}}{2}}\beta)_{\alpha'\beta}\,\delta(\mathbf{p}-\mathbf{p}')\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega - \varepsilon_{\mathbf{p}+\frac{\mathbf{k}}{2}} - \varepsilon_{\mathbf{p}-\frac{\mathbf{k}}{2}}). \tag{2.31}$$

Other combinations contribute nothing if there are no positrons ($\Phi_{\mathbf{p}}^- = 0$). Fluctuations also occur at frequencies far from the actual frequencies of collective modes or the frequencies of photons; i.e., (2.31) describes fluctuations not with a real pair production but (in many cases) a virtual pair production. With regard to the projection operators in (2.30), we note that they are completely understandable and reflect the same properties as in (2.27). An extremely important point is that the spinor indices on these operators are intermingled, and for this reason the virtual processes are coupled. A relativistically invariant analysis of fluctuations is also possible for particles of other spins. We will not write the corresponding relations here, in order to avoid a further complication of this discussion (for spin 0 see, for example, Refs. 18 and 26).

3. SEPARATING OUT THE COLLECTIVE FIELDS

To explain the fundamental points, it is convenient to use the example of purely electrostatic fields. When fields are present, the classical equation for the distribution function of the particles of species α is written in the form

$$\frac{\partial f^\alpha_{\mathbf{p}}}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)f^\alpha_{\mathbf{p}} + e_\alpha\left(\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)f^\alpha_{\mathbf{p}} = 0, \tag{3.1}$$

$$\mathrm{div}\,\mathbf{E} = 4\pi\sum_\alpha e_\alpha\int\frac{f^\alpha_{\mathbf{p}}d\mathbf{p}}{(2\pi)^3}\,. \tag{3.2}$$

These equations are found from the Liouville equation for the exact many-particle problem through an integration over all the variables except the variables of the given particle α, which are designated $\mathbf{p}$ and $\mathbf{r}$. The fluctuations of the particles are thus taken into account.

We assume for simplicity that there are no regular fields; i.e.,

$$\mathbf{E} = \delta\mathbf{E},\ \langle\mathbf{E}\rangle = 0. \tag{3.3}$$

The fluctuations of the free particles discussed above occur as they would if these particles had no charge and produced no fields. Actually, this is not the case, and the particles cease to be completely independent by virtue of the fields. In a collisionless system (or, more precisely, in a collisionless zeroth approximation), however, the correlations through the fluctuating fields are weak, and the relations of the preceding section of this paper hold in a first approximation. It follows from the Poisson equation, (3.2) (and, in general, from Maxwell's equations), that fluctuations in the number of particles [on the right side of (3.2)] automatically cause fluctuations in the electric fields also. This is in a sense a trivial point if the particles have charges. However, the fluctuating fields also change the fluctuations of the particles according to (3.1). It turns out that these additional fluctuations are indeed relatively small and that the condition under which they are small in the absence of collective fields is the condition that collisions be of minor importance.

First, however, we need to introduce collective fields. We take the average of Eqs. (3.1) and (3.2), and we calculate average equations from (3.1):

$$\frac{\partial \Phi_{\mathbf{p}}^{\alpha}}{\partial t}+\mathbf{v}\frac{\partial \Phi_{\mathbf{p}}^{\alpha}}{\partial \mathbf{r}}=-e_{\alpha}\left\langle \delta \mathbf{E}\frac{\partial}{\partial \mathbf{p}}\delta f_{\mathbf{p}}^{\alpha}\right\rangle, \quad (3.4)$$

$$\operatorname{div}\delta\mathbf{E}=4\pi\sum_{\alpha}e_{\alpha}\int\frac{\delta f_{\mathbf{p}}^{\alpha}d\,\mathbf{p}}{(2\pi)^{3}}, \quad (3.5)$$

$$\frac{\partial}{\partial t}\delta f_{\mathbf{p}}^{\alpha}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\delta f_{\mathbf{p}}^{\alpha}=-e_{\alpha}\delta\mathbf{E}\frac{\partial}{\partial \mathbf{p}}\Phi_{\mathbf{p}}^{\alpha}-e_{\alpha}\left(\delta\mathbf{E}\frac{\partial}{\partial \mathbf{p}}\delta f_{\mathbf{p}}^{\alpha}-\left\langle\delta\mathbf{E}\frac{\partial}{\partial \mathbf{p}}\delta f_{\mathbf{p}}^{\alpha}\right\rangle\right). \quad (3.6)$$

The collective fields may be regular, but they may also be random. The simplest case is that of linear collective fields. However, nonlinear collective fields, both regular and irregular, are also possible. At first we restrict the discussion to linear random collective fields.

We introduce

$$\delta\tilde{f}_{\mathbf{p},k}^{\alpha}=\delta f_{\mathbf{p},k}^{\alpha}-\frac{e_{\alpha}\delta\mathbf{E}_{k}}{i(\omega-\mathbf{kv})}\frac{\partial\Phi_{\mathbf{p}}^{\alpha}}{\partial\mathbf{p}}\equiv\delta f_{\mathbf{p},k}^{\alpha}-\delta f_{\mathbf{p},k}^{\alpha(L)}, \quad (3.7)$$

$$\frac{\partial}{\partial t}\delta f_{\mathbf{p}}^{\alpha}+\left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\delta f_{\mathbf{p}}^{\alpha}+e_{\alpha}\delta\mathbf{E}\frac{\partial\Phi_{\mathbf{p}}^{\alpha}}{\partial\mathbf{p}}=\frac{\partial}{\partial t}\delta\tilde{f}_{\mathbf{p}}^{\alpha}+\left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\delta\tilde{f}_{\mathbf{p}}^{\alpha}. \quad (3.8)$$

We then have $\delta\mathbf{E}_k=\delta E_k(\mathbf{k}/k)$,

$$ik\varepsilon_{k}^{l}\delta E_{k}=4\pi\sum_{\alpha}e_{\alpha}\int\frac{\delta\tilde{f}_{\mathbf{p},k}^{\alpha}d\mathbf{p}}{(2\pi)^{3}}, \quad (3.9)$$

$$\frac{\partial}{\partial t}\delta\tilde{f}_{\mathbf{p}}^{\alpha}+\left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\delta\tilde{f}_{\mathbf{p}}^{\alpha}=-e_{\alpha}\left\{\left(\delta\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)\delta\tilde{f}_{\mathbf{p}}^{\alpha}+\left(\delta\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)\delta f_{\mathbf{p}}^{\alpha(L)}-\left\langle\left[\left(\delta\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)\delta\tilde{f}_{\mathbf{p}}^{\alpha}+\left(\delta\mathbf{E}\frac{\partial}{\partial\mathbf{p}}\right)\delta f_{\mathbf{p}}^{\alpha(L)}\right]\right\rangle\right\}, \quad (3.10)$$

where ε_k^l is the linear longitudinal dielectric constant, which is determined from the regular distribution function

$$\varepsilon_{k}^{l}=1+\frac{4\pi}{k^{2}}\sum_{\alpha}e_{\alpha}^{2}\int\frac{1}{\omega-\mathbf{kv}+i0}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\Phi_{\mathbf{p}}^{\alpha}\frac{d\mathbf{p}}{(2\pi)^{3}}. \quad (3.11)$$

Transformation (3.7) thus makes it possible to derive an equation for the fluctuations in which the right side contains only terms which are nonlinear in $\delta\mathbf{E}$ [since $\delta f\mathbf{E}^{\alpha(L)}$ is proportional to $\delta\mathbf{E}$, and it is div **D** which appears in the Poisson equation, not div **E** as in (3.5)]. We define the linear collective field as the partial solution of the homogeneous version of Eq. (3.9):

$$\varepsilon_{k}^{l}\delta E_{k}^{\mathrm{w}}=0,\ \ \varepsilon_{k}^{l}=0,\qquad \delta\mathbf{E}_{k}^{\mathrm{w}}=\frac{\mathbf{k}}{k}\delta E_{k}^{\mathrm{w}}. \quad (3.12)$$

We denote this field by δE^{w}. An important point is that this field is, like $\delta f^{(0)}$, a solution of the homogeneous equation. Since the field amplitude is arbitrary, and the field is random, the correlation function on its right side is arbitrary:

$$\langle\delta E_{k,i}^{\mathrm{w}}\delta E_{k',j}^{\mathrm{w}}\rangle=\frac{k_{i}k_{j}}{k^{2}}|E^{\mathrm{w}}|_{k}^{2}\,\delta(k+k')\,\delta(\varepsilon_{k}^{l})\,\frac{1}{2\pi^{3}}. \quad (3.13)$$

This relation serves as a definition of $|E^{\mathrm{w}}|_k^2$. It should be compared with (2.6), where $\delta(\omega-\mathbf{kv})$, like $\delta(\varepsilon_k^l)$ in (3.13), is a solution of the homogeneous equation, and $\delta(k+k')$ in both relations is a consequence of the spatial homogeneity and the steady-state nature (over time). Finally, $|E^{\mathrm{w}}|_k^2$ in (3.13) and $\Phi_{\mathbf{p}}$ in (2.6) are arbitrary distribution functions of the particles and waves. Furthermore, for greater clarity we can also introduce in (3.13) the number of quanta of the waves of random field (3.13).

We write the field energy:

$$W^{\mathrm{w}}=\int\frac{\omega\partial\varepsilon_{k}^{l}/\partial\omega}{8\pi}e^{i(k+k')x}\langle E_{k,i}E_{k',j}\rangle\,dk\,dk'=\int\frac{\omega_{\mathbf{k}}|E^{\mathrm{w}}|_{\mathbf{k},\omega=\omega_{\mathbf{k}}}^{2}d\,\mathbf{k}}{(2\pi)^{3}}. \quad (3.14)$$

Here we are using

$$\delta(\varepsilon_{k}^{l})=(\delta(\omega-\omega_{\mathbf{k}})+\delta(\omega+\omega_{\mathbf{k}}))\left|\frac{\partial\varepsilon_{k}^{l}}{\partial\omega}\right|_{\omega=\omega_{\mathbf{k}}}^{-1}. \quad (3.15)$$

This energy can also be written in terms of $N_{\mathbf{k}}$, the number of photons (occupation numbers), of longitudinal waves ($\hbar=1$):

$$W^{\mathrm{w}}=\int\frac{\omega_{\mathbf{k}}N_{\mathbf{k}}\,d\mathbf{k}}{(2\pi)^{3}}. \quad (3.16)$$

We thus have

$$|E^{\mathrm{w}}|_{\mathbf{k},\omega_{\mathbf{k}}}^{2}=N_{\mathbf{k}},\qquad |E^{\mathrm{w}}|_{\mathbf{k},-\omega_{\mathbf{k}}}^{2}=N_{-\mathbf{k}}, \quad (3.17)$$

and relation (3.13) can be written in the form

$$\langle\delta E_{k,i}^{\mathrm{w}}\delta E_{k',j}^{\mathrm{w}}\rangle=\frac{k_{i}k_{j}}{2\pi^{2}\mathbf{k}^{2}}\frac{N_{\mathbf{k}}\delta(\omega-\omega_{\mathbf{k}})+N_{-\mathbf{k}}\delta(\omega+\omega_{-\mathbf{k}})}{|\partial\varepsilon_{k}^{l}/\partial\omega|_{\omega=\omega_{\mathbf{k}}}}\delta(k+k'). \quad (3.18)$$

Again, we wish to stress that singling out the linear collective fields is not the only possibility here. If the collective fields are strong, their nonlinear interaction must also be taken into account, generally speaking. In addition to the collective field (which are assumed here to be random), however, there are also fields due to the fluctuations of the particles, $\delta f^{(0)}$, as we have already pointed out. We denote them by $\delta E^{(0)}$. In the simplest case, in which $\delta f^{(0)}$ appears on the right side, we can write

$$\delta E_{k}^{(0)}=\frac{4\pi}{ik\varepsilon_{k}^{l}}\sum_{\alpha}e_{\alpha}\int\frac{\delta f_{\mathbf{p},k}^{(0)\alpha}d\mathbf{p}}{(2\pi)^{3}}. \quad (3.19)$$

According to linear equation (3.9), the solution consists of the solution of the homogeneous equation, δE^{w}, and that of inhomogeneous equation (3.19): $\delta E=\delta E^{\mathrm{w}}+\delta E^{(0)}$. We wish to stress this point specifically in connection with the assumed linearity of the collective fields.

For nonlinear fields the result might be different. We can illustrate this point at a qualitative level by singling out on the left side a term which is nonlinear in the fields:

$$ik(\varepsilon_{k}^{l}+\hat{\varepsilon}_{k}^{N}\{\delta E\})\delta E=4\pi\sum_{\alpha}e_{\alpha}\int\frac{\delta f_{\mathbf{p}}^{(0)\alpha}}{(2\pi)^{3}}d\mathbf{p}+\ldots. \quad (3.20)$$

Here we have already separated out a nonlinear term which is functionally dependent on δE from the right side of (3.9). We have moved it to the left side in the form of the operator $\hat{\varepsilon}_k^N$. On the right side we have written a first term which is linear in $\delta f^{(0)}$ and which does not depend on δE; however, there are of course also some terms which are linear in $\delta f^{(0)}$ but which contain certain terms which are linear and nonlinear in δE, as can be seen from (3.10).

The collective field might be defined as the nonlinear field which satisfies the equation

$$\delta E^w [\varepsilon_k^l + \hat{\varepsilon}_k^N \{\delta E^w\}] = 0. \tag{3.21}$$

Linearizing (3.20) with respect to $\delta E^{(0)}$, we then find

$$\hat{L}\delta E^{(0)} = (\varepsilon_k^l + \hat{\varepsilon}_k^N \{\delta E^w\}) \delta E^{(0)} + \left(\frac{\delta \hat{\varepsilon}_k^N \{\delta E\}}{\delta (\delta E)} \delta E^w \right) \delta E^{(0)}$$
$$= 4\pi \sum_\alpha e_\alpha \int \frac{f_p^{(0)\alpha} dp}{(2\pi)^3}. \tag{3.22}$$

This equation is of course different from (3.19). It shows that the fluctuating fields $\delta E^{(0)}$ which stem from fluctuations of particles in the presence of strong, nonequilibrium collective fields generally change in form [terms which are nonlinear in δE^w and which are of the same order of magnitude as those which are incorporated on the left side of (3.22) must of course be taken into account on the right side]. In several cases, however, despite the fact that the inverse operators on the left side of (3.22) and on the left side of (3.11) are different from each other, integrals of $\delta E^{(0)}$ over $\mathbf{k}$ appear in the final results, and the terms which are nonlinear in $\hat{L}^{-1}$ can be treated by perturbation theory.

This is the situation, for example, in the case of a weak turbulence of collective nonlinear fields δE^w, in which, despite their (weak) nonlinearity, we can use perturbation theory for the fluctuating fields $\delta E^{(0)}$. In this case, however, we can work from (3.19) as an approximation and deal with all the nonlinear terms in (3.9) and (3.10) by perturbation theory. In general, however, it is correct to state that the field fluctuations caused by the particle fluctuations may be altered substantially by collective fields.

Below we will take a special look at this case in connection with the so-called quasilinear description of collective fields. At this point, we instead recall the method for deriving a very simple quasilinear equation from (3.11). We substitute $\delta E = \delta E^w$ into (3.4), and as δf we use $\delta f^{(L)\alpha}$ with $\delta E = \delta E^w$:

$$\delta f_{\mathbf{p},k}^{(L)\alpha} = - \frac{e_\alpha \delta \mathbf{E}_k^w}{i(\omega - \mathbf{k}\mathbf{v})} \frac{\partial \Phi_\mathbf{p}^\alpha}{\partial \mathbf{p}}. \tag{3.23}$$

Here we have

$$\frac{d\Phi_\mathbf{p}^\alpha}{dt} = \frac{\partial \Phi_\mathbf{p}^\alpha}{\partial t} + \mathbf{v} \frac{\partial \Phi_\mathbf{p}^\alpha}{\partial \mathbf{r}}$$
$$= e_\alpha^2 \pi \int |E^w|_k^2 \left(\mathbf{k} \frac{\partial}{\partial \mathbf{p}}\right) \delta(\omega - \mathbf{k}\mathbf{v}) \left(\mathbf{k} \frac{\partial \Phi_\mathbf{p}^\alpha}{\partial \mathbf{p}}\right) dk. \tag{3.24}$$

This equation shows how $\Phi_\mathbf{p}^\alpha$ varies under the influence of δE^w. This circumstance, like the nonlinearities discussed above, should alter the fluctuating fields of the particles. The effect does not reduce to simply a change in $\Phi_\mathbf{p}^\alpha$ in (2.6) with $\Phi_\mathbf{p}^\alpha = \Phi_\mathbf{p}^\alpha(t)$; there are also some additional electromagnetic fluctuations, generally of a quantum nature. With this circumstance in mind, we carried out a corresponding generalization of the equations for the fluctuations of the particles to the quantum case in the preceding section of this review. The analogy between the nonlinear variation in the fluctuations of the type described by (3.22), on the one hand, and the appearance of additional fluctuations due to the quasilinear time dependence, on the other, is of course rather superficial. An important point is that both depend on the amplitude of the collective fields, and the change in the fluctuations due to the presence of collective fields is a common feature.

It is also a simple matter to single out the collective fields in a quantum system. It is sufficient to write an equation for the operator $\hat{f}$ and Maxwell's equation for the field potentials $\hat{A}_\mu$ with a current $\hat{j}_\mu$:

$$\hat{j}_\mu = \sum_\alpha \frac{e_\alpha}{(2\pi)^3} \int \mathrm{Sp}\, i\beta\gamma_\mu \hat{f}_\mathbf{p}^\alpha d\mathbf{p}. \tag{3.25}$$

An equation for $\hat{f}$ for the spin 1/2 case is found from the Dirac equations for the operations for the operators $\hat{\Psi}$ and from definition (2.21):

$$\frac{\partial \hat{f}_{\mathbf{p},\mathbf{k}}^\alpha(t)}{\partial t} + i\beta\left(i\gamma\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) + m\right) \hat{f}_{\mathbf{p},\mathbf{k}}^\alpha(t)$$
$$- \hat{f}_{\mathbf{p},\mathbf{k}}^\alpha(t)\left(-i\gamma\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) + m\right) i\beta$$
$$= \int d\mathbf{k}_1 d\mathbf{k}_2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) (\hat{f}_{\mathbf{p}+\frac{\mathbf{k}_2}{2},\mathbf{k}_1}^\alpha(t) \beta \hat{A}_{\mathbf{k}_2}(t)$$
$$- \beta \hat{A}_{\mathbf{k}_2}(t) \hat{f}_{\mathbf{p}-\frac{\mathbf{k}_2}{2},\mathbf{k}_1}^\alpha(t)); \tag{3.26}$$
$$\hat{A}_\mathbf{k}(t) = \gamma_\mu \hat{A}_{\mathbf{k},\mu}(t).$$

Once these equations have been written, we actually do not need to repeat the procedure which we have described, since it is the same as that for (3.1). Everywhere below we will assume that the collective field δE^w is classical, but we will also take quantum-mechanical effects into account in the fluctuations. In the simplest case of collisions of particles, quantum effects may influence the expression within the Coulomb logarithm, but when there is a quasilinear interaction the fluctuations of higher order in e^2 diverge in the classical limit, while in the quantum limit they introduce finite observable effects.

4. LANDAU-BALESCU COLLISION INTEGRAL

We begin this analysis by completely ignoring the collective fields: $\delta E^w = 0$. We are then left with the fluctuations in the particles themselves. On the right side of (3.4), which is an equation describing the average distribution $\Phi_\mathbf{p}^\alpha$, we need retain only the terms which are quadratic in $\delta f^{(0)}$. It can be seen from (3.9) that even in the first approximation δE is linear in $\delta f^{(0)}$. We can thus replace δE in (3.4) by $\delta E^{(0)}$:

$$\delta E_k = \delta E_k^{(0)} = \frac{4\pi}{ik\varepsilon_k^l} \sum_{\alpha'} e_{\alpha'} \int \frac{\delta f_{\mathbf{p}',k}^{(0)\alpha'} d\mathbf{p}'}{(2\pi)^3}, \tag{4.1}$$
$$\delta \mathbf{E}_k = \frac{\mathbf{k}}{k} \delta E_k.$$

For δf in (3.4), the term which is linear in $\delta f^{(0)}$ is, according to (3.7),

$$\delta f_{\mathbf{p},k}^\alpha = \delta f_{\mathbf{p},k}^{(0)\alpha} + \delta f_{\mathbf{p},k}^{(0)\alpha(L)} = \delta f_{\mathbf{p},k}^{(0)\alpha} + \frac{e_\alpha \delta \mathbf{E}_k^{(0)}}{i(\omega - \mathbf{k}\mathbf{v})} \frac{\partial \Phi_\mathbf{p}^\alpha}{\partial \mathbf{p}}. \tag{4.2}$$

Substituting these expressions into (3.4), we find[18,24]

$$
\begin{aligned}
&\frac{\partial \Phi^\alpha_{\mathbf p}}{\partial t} + \mathbf v\,\frac{\partial \Phi^\alpha_{\mathbf p}}{\partial \mathbf r} = -e_\alpha \int d\mathbf k\, d\mathbf k'\, e^{i(\mathbf k+\mathbf k')\mathbf r - i(\omega+\omega')t} \left\langle \left(\delta \mathbf E^{(0)}_{\mathbf k}\,\frac{\partial}{\partial \mathbf p}\right)\right.\\
&\times\left.\left(\delta f^{(0)\alpha}_{\mathbf p,\mathbf k'} + \frac{e_\alpha \delta \mathbf E^{(0)}_{\mathbf k'}}{i(\omega - \mathbf k\mathbf v)}\,\frac{\partial \Phi^\alpha_{\mathbf p}}{\partial \mathbf p}\right)\right\rangle = -\sum_{\alpha'} e_\alpha e_{\alpha'} \int \frac{d\mathbf k\, d\mathbf k'}{2\pi^2 i k^2 \varepsilon^l_k}\left(\mathbf k\,\frac{\partial}{\partial \mathbf p}\right)\\
&\times\left(\langle \delta f^{(0)\alpha}_{\mathbf p',\mathbf k}\,\delta f^{(0)\alpha'}_{\mathbf p,\mathbf k'}\rangle - \frac{e_\alpha}{2\pi^2 \varepsilon^l_{k'} k'^2}\sum_{\alpha''} e_{\alpha''}\int d\mathbf p''\,\langle \delta f^{(0)\alpha'}_{\mathbf p',\mathbf k}\,\delta f^{(0)\alpha''}_{\mathbf p'',\mathbf k'}\rangle\right.\\
&\times\left.\frac{1}{\omega - \mathbf k\mathbf v}\left(\mathbf k\,\frac{\partial \Phi^\alpha_{\mathbf p}}{\partial \mathbf p}\right)\right) = -e_\alpha^2 \int \frac{d\mathbf k}{2\pi^2 i k^2 \varepsilon^l_k}\left(\mathbf k\,\frac{\partial}{\partial \mathbf p}\right)\delta(\omega - \mathbf k\mathbf v)\,\Phi^\alpha_{\mathbf p}\\
&-\sum_{\alpha'} e_\alpha^2 e_{\alpha'}^2 \int \frac{d\mathbf k\, d\mathbf p'}{i4\pi^2 k^4 |\varepsilon^l_k|^2}\left(\mathbf k\,\frac{\partial}{\partial \mathbf p}\right)\\
&\times \Phi^{\alpha'}_{\mathbf p'}\,\delta(\omega - \mathbf k\mathbf v')\,\frac{1}{\omega - \mathbf k\mathbf v}\left(\mathbf k\,\frac{\partial \Phi^\alpha_{\mathbf p}}{\partial \mathbf p}\right)\\
&= \sum_{\alpha'} 2 e_\alpha^2 e_{\alpha'}^2 \int \frac{d\mathbf p'}{(2\pi)^3}\,\frac{d\mathbf k}{\mathbf k^4}\left(\mathbf k\,\frac{\partial}{\partial \mathbf p}\right)\frac{\delta(\mathbf k(\mathbf v - \mathbf v'))}{|\varepsilon^l_{\mathbf k,\mathbf k\mathbf v}|^2}\\
&\times\left[\Phi^{\alpha'}_{\mathbf p'}\left(\mathbf k\,\frac{\partial \Phi^\alpha_{\mathbf p}}{\partial \mathbf p}\right) - \Phi^\alpha_{\mathbf p}\left(\mathbf k\,\frac{\partial \Phi^{\alpha'}_{\mathbf p'}}{\partial \mathbf p'}\right)\right]. \qquad (4.3)
\end{aligned}
$$

In the latter equation we have used the relations

$$
\begin{aligned}
&\operatorname{Im}\frac{1}{\omega - \mathbf k\mathbf v} = -\pi\delta(\omega - \mathbf k\mathbf v),\\
&\operatorname{Im}\frac{1}{\varepsilon^l_k} = -\frac{\operatorname{Im}\varepsilon^l_k}{|\varepsilon^l_k|^2} \qquad (4.4)\\
&= \frac{4\pi^2}{k^2|\varepsilon^l_k|^2}\sum_{\alpha'} e_{\alpha'}^2 \int \delta(\omega - \mathbf k\mathbf v')\left(\mathbf k\,\frac{\partial \Phi^{\alpha'}_{\mathbf p'}}{\partial \mathbf p'}\right)\frac{d\mathbf p'}{(2\pi)^3},
\end{aligned}
$$

which give us respectively the first and second terms of the last equation in (4.3), which is none other than the well-known Landau-Balescu collision integral[3] [if we ignore $(\varepsilon^l - 1)$ in (4.3), then (4.3) becomes the Landau collision integral[2]].

The collision integral is thus derived on a single line. This entire derivation is simply the quintessence of numerous studies. Equation (4.3) contains only average distribution functions, which appear not only in biquadratic combinations but also in the screening factor $1/|\varepsilon|^2$. This factor describes dynamic screening during collisions. The phrase "dynamic screening" applies here because the frequency ω in ε is not zero (a zero value would correspond to static Debye screeening). It instead has the value

$$\omega = \mathbf k\mathbf v \qquad (4.5)$$

or, in a quantum description, $\omega = \varepsilon_{\mathbf p} - \varepsilon_{\mathbf p - \mathbf k}$. This expression describes energy conservation at a vertex (Figs. 1 and 2). At the other vertex (Fig. 2) we have $\omega = \varepsilon_{\mathbf p' + \mathbf k} - \varepsilon_{\mathbf p'} \approx \mathbf k\mathbf v'$. Together, these two conservation laws give us

$$\mathbf k\mathbf v - \mathbf k\mathbf v' = 0. \qquad (4.6)$$

Relation (4.5) does not express the condition for (Vavilov-) Cherenkov radiation; it instead gives the frequency of a virtual photon. The interaction in (4.3) is described as a collision through a virtual longitudinal wave. Its Green's function $1/k\varepsilon^l_k$ also appears in (4.3).

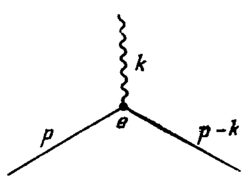

FIG. 1.

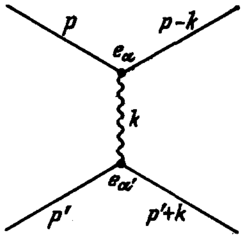

FIG. 2.

The physical content of (4.3), on the other hand, is very important. It is that taking an averge over fluctuations leads to the concept of dressed particles, which are now described by the average distribution function $\Phi^\alpha_{\mathbf p}$.

Since $\mathbf k/k$ is a unit vector (a direction vector), and since the component of $\mathbf k$ parallel to the velocity, $k_\parallel = \mathbf k(\mathbf v - \mathbf v')/|\mathbf v - \mathbf v'|$, vanishes according to (4.3), we see that the integration in (4.3) reduces to an integral over the momentum transfer which is transverse with respect to $\mathbf v - \mathbf v'$, i.e., $\mathbf k_\perp$:

$$\int \frac{d\mathbf k_\perp}{k_\perp^2} = \int \frac{\pi\, dk_\perp^2}{k_\perp^2}. \qquad (4.7)$$

This integral diverges logarithmically; this is the so-called Coulomb logarithm. The divergence at the lower limit is eliminated by screening, i.e., by the quantity $|\varepsilon^l_{\mathbf k,\mathbf k\mathbf v}|^2$ in (4.3), while that at the upper limit is determined by quantum effects (provided only that the condition $\hbar/mv \gg e^2/mv^2$ or $e^2/\hbar v \ll$ holds). At large values of the transverse momentum transfer, the quantum nature of the fluctuations comes into play.

For nonrelativistic quantum-mechanical particles we can use the Schrödinger equation in the p representation ($\hbar = 1$),

$$i\,\frac{\partial \hat\Phi^\alpha_{\mathbf p}(t)}{\partial t} - \varepsilon_{\mathbf p}\hat\Psi^\alpha_{\mathbf p}(t) - e_\alpha \int \hat\varphi_{\mathbf k}\hat\Psi^\alpha_{\mathbf p-\mathbf k}(t)\,d\mathbf k = 0, \quad \varepsilon_{\mathbf p} = \frac{\mathbf p^2}{2m}, \qquad (4.8)$$

and the equation for the operator $\hat f^\alpha_{\mathbf p,\mathbf k}$ given by (2.11),

$$
\begin{aligned}
&i\,\frac{\partial \hat f^\alpha_{\mathbf p,\mathbf k}}{\partial t} - \left(\varepsilon_{\mathbf p+\frac{\mathbf k}{2}} - \varepsilon_{\mathbf p-\frac{\mathbf k}{2}}\right)\hat f^\alpha_{\mathbf p,\mathbf k}\\
&-e_\alpha \int d\mathbf k'\left(\hat\varphi_{\mathbf k'}(t)\,\hat f^\alpha_{\mathbf p-\frac{\mathbf k'}{2},\mathbf k-\mathbf k'}(t) - \hat f^\alpha_{\mathbf p+\frac{\mathbf k'}{2},\mathbf k-\mathbf k'}(t)\,\hat\varphi_{\mathbf k'}(t)\right) = 0. \qquad (4.9)
\end{aligned}
$$

We then find

$$\frac{d\Phi^\alpha_{\mathbf p}}{dt} = \frac{e_\alpha}{i}\int d\mathbf k\, d\mathbf k'\left\langle \hat\varphi_{\mathbf k'}\delta\hat f^\alpha_{\mathbf p-\frac{\mathbf k'}{2},\mathbf k-\mathbf k'} - \delta\hat f^\alpha_{\mathbf p+\frac{\mathbf k'}{2},\mathbf k-\mathbf k'}\hat\varphi_{\mathbf k'}\right\rangle; \qquad (4.10)$$

$$\delta\hat f^\alpha_{\mathbf p,\mathbf k'} = \delta\hat f^{(0)\alpha}_{\mathbf p,\mathbf k'} + \frac{e_\alpha\hat\varphi_{k'}\left(\Phi^\alpha_{\mathbf p-\frac{\mathbf k'}{2}} - \Phi^\alpha_{\mathbf p+\frac{\mathbf k'}{2}}\right)}{\omega' - \varepsilon_{\mathbf p+\frac{\mathbf k'}{2}} + \varepsilon_{\mathbf p-\frac{\mathbf k'}{2}}}, \qquad (4.11)$$

$$
\begin{aligned}
&k^2\varepsilon^l_k\hat\varphi^{(0)}_k = 4\pi\sum_{\alpha'} e_{\alpha'}\int \frac{\delta f^{\alpha'(0)}_{\mathbf p,k}\,d\mathbf p}{(2\pi)^3},\\
&\varepsilon^l_{\mathbf k,\omega} = 1 + \sum_\alpha \frac{4\pi e_\alpha^2}{k^2}\int \frac{\Phi^\alpha_{\mathbf p+\frac{\mathbf k}{2}} - \Phi^\alpha_{\mathbf p-\frac{\mathbf k}{2}}}{\omega - \varepsilon_{\mathbf p+\frac{\mathbf k}{2}} + \varepsilon_{\mathbf p-\frac{\mathbf k}{2}}}\,\frac{d\mathbf p}{(2\pi)^3}. \qquad (4.12)
\end{aligned}
$$

Going through the same procedure as in the derivation of (4.3), we find

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}=\sum_{\alpha'}4e_{\alpha}^{2}e_{\alpha'}^{2}\int\frac{d\mathbf{p}'\,d\mathbf{k}}{(2\pi)^{3}\mathbf{k}^{4}}\frac{(\Phi_{\mathbf{p}'}^{\alpha'}\Phi_{\mathbf{p}-\mathbf{k}}^{\alpha}-\Phi_{\mathbf{p}}^{\alpha}\Phi_{\mathbf{p}'-\mathbf{k}}^{\alpha'})}{|\varepsilon_{\mathbf{k},\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}}^{l}|^{2}}\times\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}-\varepsilon_{\mathbf{p}'}+\varepsilon_{\mathbf{p}'-\mathbf{k}}). \quad (4.13)$$

After an expansion in the momentum transfer $\mathbf{k}$, this expression gives us (4.3).

In taking the classical approach, we can restrict the discussion to the interaction through a virtual longitudinal wave only in the case of nonrelativistic particles. For relativistic particles we have to allow for the circumstance that the Lorentz force appears in the complete equation for $\delta f^{(0)}$ and that for the fluctuating fields we must write Maxwell's equations with the current determined by $\delta f^{(0)}$. Going through this procedure, we find the Belyaev-Budker collision integral[27]

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}=\left(\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}\right)^{l}+\left(\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}\right)^{t}, \quad (4.14)$$

where $(d\Phi_{\mathbf{p}}^{\alpha}/dt)^{l}$ is given by (4.3) and describes a process which occurs through a longitudinal virtual wave. The quantity

$$\left(\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}\right)^{t}=\sum_{\alpha'}e_{\alpha}^{2}e_{\alpha'}^{2}\int\frac{d\mathbf{p}'\,d\mathbf{k}}{(2\pi)^{3}\mathbf{k}^{4}}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\frac{[\mathbf{k}\mathbf{v}]^{4}\,\delta(\mathbf{k}(\mathbf{v}-\mathbf{v}'))}{|k^{2}-(\mathbf{k}\mathbf{v})^{2}\varepsilon_{\mathbf{k},\mathbf{k}\mathbf{v}}^{t}|^{2}}\times\left[\Phi_{\mathbf{p}'}^{\alpha'}\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}^{\alpha}}{\partial\mathbf{p}}\right)-\Phi_{\mathbf{p}}^{\alpha}\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}'}^{\alpha'}}{\partial\mathbf{p}'}\right)\right] \quad (4.15)$$

describes a process which goes through a transverse virtual wave.

In the general—relativistic and quantum-mechanical—case we can find a generalization of (4.13) and (4.15) from (3.26), (3.25), (2.30), (2.31), and (2.29), taking account of processes which go through the longitudinal and transverse virtual waves. For the discussion below it is sufficient to write Green's functions for the corresponding waves. For the transverse virtual wave these functions are

$$\frac{1}{k^{2}-(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}})^{2}\varepsilon_{\mathbf{k},\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}}^{t}},\qquad \frac{1}{k^{2}-(\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}+\mathbf{k}})^{2}\varepsilon_{\mathbf{k},\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}+\mathbf{k}}}^{t}}. \quad (4.16)$$

The second of these functions corresponds to a virtual pair production; the first can be written in the following form in the limit of a large momentum transfer $\mathbf{k}$, with $\varepsilon_{k,\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}}^{t}\approx 1$

$$\frac{1}{2|\mathbf{k}|}\left(\frac{1}{\varepsilon_{\mathbf{p}-\mathbf{k}}-\varepsilon_{\mathbf{p}}+|\mathbf{k}|}-\frac{1}{\varepsilon_{\mathbf{p}-\mathbf{k}}-\varepsilon_{\mathbf{p}}+|\mathbf{k}|}\right). \quad (4.17)$$

We have written these relations in order to show the structure of the incoming Green's functions and of the transverse fields, which will appear below for other processes. Another motivation has been to show that processes with virtual pair production appear in the collision integrals. An important point is that it is technically a straightforward matter to derive even expressions (4.15) by the procedure described above, whose usefulness is thereby demonstrated. On the other hand, incorporating quantum effects is not, as it might appear, a matter of dealing with certain subtleties: These effects play an important role here. For example, one could use (4.15) to find the absorption of electromagnetic waves due to collisions, by calculating the perturbations $\delta\Phi$ due to collisions and the wave field. However, the direct process for absorption is bremsstrahlung, whose maximum frequencies are determined by exclusively quantum effects in the case of fast particles. The quantum effects must therefore be taken into account in a system of charged particles.

We will be interested below in the changes which occur in the fluctuations because of the collective fields. First, however, we will conclude the discussion of the overall picture of the classical fluctuations, which lead to a clear physical model of the dressed particles, which is manifested in the writing of a collision integral. It turns out that this picture is completely valid for other processes also.

5. FLUCTUATIONS AND TRANSITION SCATTERING

Let us examine scattering effects. We assume that we have a collective longitudinal field δE_{k}^{w}. We assume that the scattered field is also longitudinal, δE_{k}^{w}. In fluctuation theory, we can single out scattering processes which are proportional to $|E^{w}|_{k}^{2}$ if we collect all the terms which contain the δ-function characteristic of scattering. By virtue of the conservation laws in the elementary scattering event ($\hbar=1$),

$$\varepsilon_{\mathbf{p}}+\omega_{\mathbf{k}}=\varepsilon_{\mathbf{p}'}+\omega_{\mathbf{k}'}, \quad (5.1)$$

$$\mathbf{p}+\mathbf{k}=\mathbf{p}'+\mathbf{k}'. \quad (5.2)$$

We have the δ-function

$$\delta(\varepsilon_{\mathbf{p}}+\omega_{\mathbf{k}}-\varepsilon_{\mathbf{p}+\mathbf{k}-\mathbf{k}'}-\omega_{\mathbf{k}'}). \quad (5.3)$$

Along with the terms which are proportional to the field intensity $|E^{w}|_{k}^{2}$, the collision integral has terms which are proportional to the intensity of the scattered field, $|E^{w}|_{k}^{2}$. These terms describe induced scattering. All these terms can be derived, but the important points can be demonstrated by simply discussing spontaneous scattering, which is proportional to the intensity of the initial scattered field, $|E^{w}|_{k}^{2}$.

We begin with the classical description, in which case (5.3) becomes

$$\delta(\omega_{\mathbf{k}}-\mathbf{k}\mathbf{v}-\omega_{\mathbf{k}'}+\mathbf{k}'\mathbf{v}), \quad (5.4)$$

and our starting point is the system of equations (3.9), (3.10).

We assume that the Cherenkov condition does not hold for the scattered wave $\{\omega_{\mathbf{k}},\mathbf{k}\}$: $\omega_{\mathbf{k}}\neq\mathbf{k}\mathbf{v}$. For the scattered wave, this condition again should not hold; i.e., we should have $\omega_{\mathbf{k}}\neq\mathbf{k}'\mathbf{v}$. Otherwise we would have $\omega_{\mathbf{k}}=\mathbf{k}\mathbf{v}$ according to (5.4), in contradiction of our assumption. We rewrite our initial equation, (3.4), in the form

$$\frac{\partial\Phi_{\mathbf{p}}^{\alpha}}{\partial t}+\mathbf{v}\frac{\partial\Phi_{\mathbf{p}}^{\alpha}}{\partial\mathbf{r}}=-e_{\alpha}\frac{\partial}{\partial\mathbf{p}}\int dk\,dk'\left\langle\frac{\mathbf{k}}{k}\delta E_{k}\,\delta f_{\mathbf{p},k'}^{\alpha}\right\rangle. \quad (5.5)$$

We will not go into the procedure for singling out the terms which leave only (a) a spontaneous scattering which is linear in $|E^{w}|_{k}^{2}$ and (b) a transition scattering, without taking into account ordinary Thomson scattering. We will simply give the recipe, which shows what we need to take into account in order to derive the result we are seeking. To test the validity of this recipe we must ensure that none of the other terms make an additional contribution to the process of interest. However, this circumstance would mean deriving a general result, which would be overly complex. We will leave that to the reader. The general result contains both ordinary and transition scattering and the interference of the two; furthermore, it contains induced scattering as well as spontaneous scattering.

Here, then, is the recipe for deriving the terms which contain only the spontaneous transition scattering:

1) In place of $\delta f^{\alpha}_{\mathbf{p},k'}$ in (5.5) we must use the zeroth approximation $\delta f^{(0)\alpha}_{\mathbf{p},k}$.

2) On the right side of (3.10), we retain only the term with $\delta f^{\alpha(L)}_{\mathbf{p}}$; i.e., we write (3.9) in the form

$$\delta E_k = \frac{4\pi}{i\omega\varepsilon^l_k}\int S_{k,k_1}(\delta E_{k_1}\delta E_{k-k_1} - \langle\delta E_{k_1}\delta E_{k-k_1}\rangle)\,dk_1, \quad (5.6)$$

where

$$S_{k,k_1} = -\frac{\omega}{2kk_1|\mathbf{k}-\mathbf{k}_1|}\sum_\beta e^3_\beta\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\omega-\mathbf{kv}} \times\left\{\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v}}\left((\mathbf{k}-\mathbf{k}_1)\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right) + \left((\mathbf{k}-\mathbf{k}_1)\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega_1-\mathbf{k}_1\mathbf{v}}\left(\mathbf{k}_1\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\right\}. \quad (5.7)$$

3) We are left with only quadratic combinations of $\delta f^{(0)}$ and δE^{w}_k. This procedure is carried out in the following way. Relation (5.5) already contains $\delta f^{(0)}$, so in δE we need to take into account only the term which is linear in $\delta f^{(0)}$. The quantity $\delta f^{(0)}$ itself can appear only through δE [see (3.19)]. However, after substutiting one of the fields, $\delta E^{(0)}$, and the other field of the scattered wave, δE^{w}, into (5.6), we find zero in (5.5), since we have $\langle\delta E^{\mathrm{w}}\rangle = 0$. It is necessary to integrate (5.6) with a virtual field (which we denote by δE^{v}):

$$\delta E_k = \frac{8\pi}{i\omega\varepsilon^l_k}\int S_{k,k_1}(\delta E^{\mathrm{w}}_{k_1}\delta E^{\mathrm{v}}_{k-k_1} - \langle\delta E^{\mathrm{w}}_{k_1}\delta E^{\mathrm{v}}_{k-k_1}\rangle)\,dk_1, \quad (5.8)$$

where

$$\delta E^{\mathrm{v}}_{k-k_1} = \frac{8\pi}{i(\omega-\omega_1)\varepsilon^l_{k-k_1}} \times\int S_{k-k_1,k_2}(\delta E^{\mathrm{w}}_{k_2}\delta E^{(0)}_{k-k_1-k_2} - \langle\delta E^{\mathrm{w}}_{k_2}\delta E^{(0)}_{k-k_1-k_2}\rangle)\,dk_2. \quad (5.9)$$

4) After substituting (5.9) and (5.8) into (5.5), we need to use relations (2.6) and (3.18), derived above, for the expectation values $(\delta f^{(0)})^2$ and $(\delta E^{\mathrm{w}})^2$.

5) Finally, we need to make use of the condition that there are no Cherenkov resonances in the nonlinear responses S_{k,k_1}. We will make use of that condition in the following way. By virtue of (3.18) we have $k_2 = -k_1$; i.e., the result will contain S_{k,k_1} and $S_{k-k_1,-k_1}$. According to (5.7), the latter resolution has the form

$$S_{k-k_1,-k_1} = \frac{\omega-\omega_1}{2kk_1|\mathbf{k}-\mathbf{k}_1|}\sum_\beta e^3_\beta\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{(\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v})} \times\left[\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega-\mathbf{kv}}\left(\mathbf{k}\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right) - \left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{\omega_1-\mathbf{k}_1\mathbf{v}}\left(\mathbf{k}_1\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\right]. \quad (5.10)$$

In deriving (5.10) from (5.7), in making the substitution $k_1 \to -k_1$, $(k_1 = \{\mathbf{k}_2,\omega_1\})$, we replaced the factor $1/(\omega_1 - \mathbf{k}_1\mathbf{v})$ by $-1/(\omega_1 - \mathbf{k}_1\mathbf{v})$. This replacement is valid only in the absence of a resonance, $\omega_1 \neq \mathbf{k}_1\mathbf{v}$, since in the presence of a resonance the quantity $1/(\omega_1 - \mathbf{k}_1\mathbf{v} + i0)$ becomes $-1/(\omega_1 - \mathbf{k}_1\mathbf{v}_1 - i0)$; i.e., the sign of the imaginary part changes. Taking this circumstance into account, and also taking into account the circumstance that a resonance in the scattering is not forbidden in either (5.10) or (5.7) {i.e., the denominator $1/[\omega - \omega - (\mathbf{k} - \mathbf{k}_1)\mathbf{v}]$ may vanish}, we find the rigorous relation

$$S_{k-k_1,-k_1} = \frac{\omega-\omega_1}{\omega}S_{k,k_1}. \quad (5.11)$$

This relation is derived in the following way. The derivatives $(\mathbf{k}_1\partial/\partial\mathbf{p})$ in the first term in (5.7) and $(\mathbf{k}-\mathbf{k}_1)\partial/\partial\mathbf{p})$ in the second are taken by parts (on the left); i.e., (5.7) is written in the form

$$\frac{S_{k,k_1}}{\omega} = \frac{1}{2kk_1|\mathbf{k}-\mathbf{k}_1|}\sum_\beta e^3_\beta\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\frac{1}{(\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v})} \times\frac{(\mathbf{kk}_1)}{(\omega-\mathbf{kv})^2}\left((\mathbf{k}-\mathbf{k}_1)\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right) - (\mathbf{k}(\mathbf{k}-\mathbf{k}_1))\frac{1}{(\omega-\mathbf{kv})^2(\omega_1-\mathbf{k}_1\mathbf{v})}\left(\mathbf{k}_1\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\right]\frac{1}{m}. \quad (5.12)$$

The derivatives $(\mathbf{k}_1\partial/\partial\mathbf{p})$ in the first term in (5.10) and $(\mathbf{k}\partial/\partial\mathbf{p})$ in the second are taken on the right, and the result is written

$$\frac{S_{k-k_1,-k_1}}{\omega-\omega_1} = \frac{1}{2kk_1|\mathbf{k}-\mathbf{k}_1|}\sum_\beta e^3_\beta\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\frac{1}{\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v}} \times\frac{(\mathbf{kk}_1)}{(\omega-\mathbf{kv})^2}\left((\mathbf{k}-\mathbf{k}_1)\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right) - \frac{(\mathbf{kk}_1)(\omega_1-\mathbf{k}_1\mathbf{v}+\omega-\mathbf{kv})}{(\omega-\mathbf{kv})^2(\omega_1-\mathbf{k}_1\mathbf{v})^2}\left(\mathbf{k}_1\frac{\partial\Phi^\beta_{\mathbf{p}}}{\partial\mathbf{p}}\right) - \frac{m}{(\omega-\mathbf{kv})(\omega_1-\mathbf{k}_1\mathbf{v})}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\Phi^\beta_{\mathbf{p}}\right]\frac{1}{m}. \quad (5.13)$$

In the first term in (5.13) we have subtracted a term with $(\mathbf{k}_1\partial\Phi^\beta_{\mathbf{p}}/\partial\mathbf{p})$, and we have added the same term to the other terms; the denominator $\omega - \omega_1 - (\mathbf{k} - \mathbf{k}_1)\mathbf{v}$ then cancels out. Already from (5.13) and (5.12) we see that these expressions are identical if we take $(\mathbf{k}\partial/\partial\mathbf{p})$ in the last term by parts (on the left).

To derive a final result, which describes the principal scattering process, our only remaining task is to collect all the expressions, using (5.11) in order to express S_{k,k_1} in terms of $S_{k-k_1,-k_1}$. In the result we need to make the substitutions $k \to k - k_1, k_1 \to -k_1$.

As a result, (5.5) acquires a relation

$$\frac{S^2_{k,k_1}}{\varepsilon^2_{k-k_1}}\delta(\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v})\,\Phi^\beta_{\mathbf{p}}\frac{|E^{\mathrm{w}}|^2_{k_1}}{l\varepsilon^l_k} \quad (5.14)$$

(this relation is written out with all the factors a bit further on). It is important to emphasize here that only $\mathrm{Im}(1/\varepsilon^l_k)$, which contains two δ-functions, contributes to (5.14):

$$\mathrm{Im}\frac{1}{\varepsilon^l_k} = -\pi\frac{\omega}{|\omega|}\delta(\varepsilon^l_k) = -\frac{\pi(\delta(\omega-\omega_{\mathbf{k}})-\delta(\omega+\omega_{\mathbf{k}}))}{|\partial\varepsilon^l_k/\partial\omega|_{\omega=\omega_{\mathbf{k}}}}. \quad (5.15)$$

Precisely the same two δ-functions are in the expression for $|E^{\mathrm{w}}|^2_{k_1}$, which is proportional to

$$N_{\mathbf{k}_1}\delta(\omega-\omega_{\mathbf{k}_1}) + N_{-\mathbf{k}_1}\delta(\omega+\omega_{\mathbf{k}_1}). \quad (5.16)$$

Scattering corresponds to combinations of frequencies $\omega = \omega_{\mathbf{k}}, \omega_1 = \omega_{\mathbf{k}_1}$ and $\omega = -\omega_k, \omega_1 = -\omega_k$. In the second case we can make the substitution $\mathbf{k}_1 \to -\mathbf{k}_1$ and $\mathbf{k} \to -\mathbf{k}$. According to (5.14) $\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - (\mathbf{k}-\mathbf{k}_1)\mathbf{v}$ will appear in the result; this is precisely the δ-function describing scattering which we have been seeking. The result also contains

$$\frac{S^2_{k,k_1}}{\varepsilon^{l2}_{k-k_1}} + \frac{S^2_{-k,-k_1}}{\varepsilon^{l2}_{k_1-k}} = \frac{S^2_{k,k_1}}{\varepsilon^{l2}_{k-k_1}} + \frac{S^{*2}_{k,k_1}}{(\varepsilon^{l*}_{k-k_1})^2}.$$

The latter expression can be rewritten in the form

$$\frac{S^2_{k,k_1}}{\varepsilon^{l2}_{k-k_1}} + \frac{S^{*2}_{k,k_1}}{\varepsilon^{l*2}_{k-k_1}} = 2\left|\frac{S_{k,k_1}}{\varepsilon^l_{k-k_1}}\right|^2 - 4\left(\operatorname{Im}\frac{S_{k,k_1}}{\varepsilon^l_{k-k_1}}\right)^2. \quad (5.17)$$

The imaginary parts S_{k,k_1} and $\varepsilon^l_{k-k_1}$ in the last term in (5.17) contain the integral

$$\int d\mathbf{p}'\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - (\mathbf{k}-\mathbf{k}_1)\mathbf{v}').$$

Because of the δ-function in (5.14), they are proportional to $\delta((\mathbf{k}-\mathbf{k}_1)(\mathbf{v}-\mathbf{v}'))$, and when we make the substitution $\mathbf{k}-\mathbf{k}_1 \to \mathbf{k}$ they are proportional to $\delta(\mathbf{k}(\mathbf{v}-\mathbf{v}'))$. This δ-function is characteristic of collisions. Terms of this type thus make contributions of the order of $\langle \delta E^{w2}\rangle/8\pi nmv^2$ to the collision integral.

If we collect all the terms of this type, we can put them in the collision integral. It turns out (curiously) that all these terms combine into an integral of the previous type (Sec. 4) with $\Phi_{\mathbf{p}}$, in which corrections $\sim |E^w|^2$ are taken into account. For this reason, we need consider only the first term in (5.17) below.

Note that in this derivation we have also discarded terms with induced transition radiation (more on this below), which are also proportional to the first power of $N_{\mathbf{k}_1}$ [but which contain derivatives $\partial\Phi_{\mathbf{p}}/\partial\mathbf{p}$, not $\Phi_{\mathbf{p}}$, as in result (5.18) just below]. This final result is

$$\frac{\partial\Phi^\alpha_{\mathbf{p}}}{\partial t} + \mathbf{v}\frac{\partial\Phi^\alpha_{\mathbf{p}}}{\partial\mathbf{r}} = -\frac{\partial}{\partial\mathbf{p}}\int \frac{(\mathbf{k}-\mathbf{k}_1)N_{\mathbf{k}_1}32e^2_\alpha|S_{k,k_1}|^2\delta(\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v})}{|\mathbf{k}-\mathbf{k}_1|^2\pi\omega^2_{\mathbf{k}}|\varepsilon^l_{k-k_1}|^2(\partial\varepsilon^l_k/\partial\omega)\,\partial\varepsilon^l_{k_1}/\partial\omega_1}\Phi^\alpha_{\mathbf{p}}\Bigg|_{\substack{\omega=\omega_{\mathbf{k}},\\ \omega_1=\omega_{\mathbf{k}_1}}}. \quad (5.18)$$

This result is surprising in two regards. First, the square moduli of the complex quantities which appear in (5.17) are interpreted as the square moduli of the probability amplitudes for the process, and $(\mathbf{k}-\mathbf{k}_1)$ in (5.18) is interpreted as a quantity which is proportional to the momentum which is transferred to the particles in the course of the scattering. So far, we have not used quantum-mechanical ideas anywhere.

If we introduce the probability for transition scattering $\omega^\alpha_{\mathbf{p}}(\mathbf{k},\mathbf{k}_1)/\hbar$, and note that the momentum of a particle changes by $\hbar(\mathbf{k}-\mathbf{k}_1)$ in the course of the scattering, according to the quantum relations, we can easily find an expression which describes the change in the particle distribution due to spontaneous scattering:

$$\begin{aligned}\frac{\partial\Phi^\alpha_{\mathbf{p}}}{\partial t} &+ \mathbf{v}\frac{\partial\Phi^\alpha_{\mathbf{p}}}{\partial\mathbf{r}}\\ &= -\frac{1}{\hbar}\int[w^\alpha_{\mathbf{p}}(\mathbf{k},\mathbf{k}_1)\Phi^\alpha_{\mathbf{p}}\\ &\quad - w^\alpha_{\mathbf{p}+\hbar\mathbf{k}-\hbar\mathbf{k}_1}(\mathbf{k},\mathbf{k}_1)\Phi^\alpha_{\mathbf{p}+\hbar\mathbf{k}-\hbar\mathbf{k}_1}]N_{\mathbf{k}_1}d\mathbf{k}\frac{1}{(2\pi)^3}\\ &\approx -\frac{\partial}{\partial\mathbf{p}}\int(\mathbf{k}-\mathbf{k}_1)w^\alpha_{\mathbf{p}}(\mathbf{k},\mathbf{k}_1)N_{\mathbf{k}_1}\frac{d\mathbf{k}_1}{(2\pi)^3}\Phi^\alpha_{\mathbf{p}}. \quad (5.19)\end{aligned}$$

Approximate inequality (5.19) corresponds to a small value of the momentum transfer, with $\hbar$ dropping out of the result. The probability is thus

$$w^\alpha_{\mathbf{p}}(\mathbf{k},\mathbf{k}_1) = \frac{32e^2_\alpha|S_{k,k_1}|^2\delta(\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v})}{\pi\omega^2|\mathbf{k}-\mathbf{k}_1|^2|\varepsilon^l_{k-k_1}|^2(\partial\varepsilon^l_k/\partial\omega)\,\partial\varepsilon^l_{k_1}/\partial\omega_1}\Bigg|_{\substack{\omega=\omega_{\mathbf{k}},\\ \omega_1=\omega_{\mathbf{k}_1}}} \quad (5.20)$$

This expression is exactly the same as that which was originally derived in Ref. 6. The quantity $1/\hbar$ has of course has been introduced in the definition of the probability. It is important to note that the final result, (5.18), is purely classical, as is (5.20).

Second, there is the extremely curious point that the nonlinear vertex S_{k,k_1} and $\varepsilon^l_{k-k_1}$ have imaginary parts, which arise from $\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - (\mathbf{k}-\mathbf{k}_1)\mathbf{v})$ and $\Phi^\alpha_{\mathbf{p}}$, i.e., which correspond to scattering processes. We have of course put the result in a form which contains only the moduli of S_{k,k_1} and $\varepsilon^1_{k-k_1}$, but even without this step the corresponding expressions would contain $\operatorname{Im} S_{k,k_1}$ and $\operatorname{Im}\varepsilon_{k-k_1}$, which are proportional to $\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - (\mathbf{k}-\mathbf{k}_1)\mathbf{v})$. The problem is completely self-consistent here: The particles are scattered, and in the course of this scattering they generate a self-interaction which is responsible for scattering (part of the reason for the generation of this self-interaction is that the scattering is determined by not only the imaginary but also the real parts of S_{k,k_1} and $\varepsilon^l_{k-k_1}$). We wish to stress that $\Phi^\alpha_{\mathbf{p}}$ in $\varepsilon^l_{k-k_1}$ and S'_{k,k_1} may describe the same particles as are described by $\Phi^\alpha_{\mathbf{p}}$ in (5.19).

Here we are seeing a new feature, which goes beyond what has been known previously for transition scattering proper. Transition scattering was treated in Ref. 6 as a scattering by a test particle in a medium with a given permittivity. Now, the distribution of the particles which are scattered, $\Phi^\alpha_{\mathbf{p}}$, appears both in (5.19) and in characteristics of the medium, S_{k,k_1} and $\varepsilon^l_{k-k_1}$. In this sense, the result in (5.20) is more general than that which was derived in Ref. 6. Corrections to the particle collision integral of course also arise. Incorporating more of the terms which have been omitted from the original equation leads to (in addition to transition scattering) ordinary Thomson scattering and an interference of the two scattering processes. The final result contains the square modulus of the total scattering amplitude.

In order to carry out these calculations it is sufficient to use in (5.5) an expression for the field which refines (5.6). For this purpose we need to incorporate on the right side of

(3.10) a term with $\delta\tilde{f}$, by making the substitution $\delta\tilde{f} = \delta f^{(0)} + \delta f^{(1)}$, where $\delta f^{(1)}$ is given by (3.10), in which $\delta\tilde{f}$ is replaced by $\delta f^{(0)}$. In place of (5.6) we find

$$\delta E_k = \delta E_k^{\mathrm{w}} + \frac{4\pi}{i\omega\varepsilon_k^l}\int S_{k,k_1}\delta E_{k_1}\delta E_{k-k_1}\,dk_1 + \frac{4\pi}{ik\varepsilon_k^l}\sum_\alpha e_\alpha^2\int\frac{\delta E_{k_1}\,d\mathbf{p}\,dk_1}{(2\pi)^3(\omega-\mathbf{kv})\,k_1}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\delta f^{(0)}_{\mathbf{p},\mathbf{k}-\mathbf{k}_1}$$
$$+\frac{2\pi}{ik\varepsilon_k^l}\sum_\alpha e_\alpha^3\int\frac{\delta E_{k_1}\delta E_{k_2}\,dk_1\,dk_2\,d\mathbf{p}}{(2\pi)^3(\omega-\mathbf{kv})\,k_1k_2}\times\left(\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{i(\omega-\omega_1-(\mathbf{k}-\mathbf{k}_1)\mathbf{v})}\left(\mathbf{k}_2\frac{\partial}{\partial\mathbf{p}}\right)+\left(\mathbf{k}_2\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{i(\omega-\omega_2-(\mathbf{k}-\mathbf{k}_2)\mathbf{v})}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\right)\delta f^{(0)}_{\mathbf{p},k-k_1-k_2}. \quad (5.21)$$

We can set δE_{k_1} and δE_{k_2} equal to the δE_k^{w} in the last term. We need to substitute the next-to-last term into the second term; only then do we set δE_k equal to δE_k^{w}. The last term in (5.21) describes Thomson scattering, and the next-to-last term describes an interference between Thomson scattering and transition scattering.

Incorporating induced processes leads to terms proportional to $N_{\mathbf{k}}$. It is of course simpler to immediately write the final expressions, which contain the total scattering probability $w_{\mathbf{p}}^{\mathrm{tot}}$, on the basis of quantum considerations than to derive this equation through the averaging procedure described above ($\hbar = 1$):

$$\frac{\partial\Phi_{\mathbf{p}}^\alpha}{\partial t} + \mathbf{v}\frac{\partial\Phi_{\mathbf{p}}^\alpha}{\partial\mathbf{r}} = -\int\{w_{\mathbf{p}}^{\mathrm{tot}}(\mathbf{k},\mathbf{k}_1)[(N_{\mathbf{k}}+1)N_{\mathbf{k}_1}\Phi_{\mathbf{p}}^\alpha - (N_{\mathbf{k}_1}+1)N_{\mathbf{k}}\Phi^\alpha_{\mathbf{p}+\mathbf{k}_1-\mathbf{k}}] + w^{\mathrm{tot}}_{\mathbf{p}+\mathbf{k}-\mathbf{k}_1}(\mathbf{k},\mathbf{k}_1)[(N_{\mathbf{k}_1}+1)N_{\mathbf{k}}\Phi_{\mathbf{p}}^\alpha - N_{\mathbf{k}_1}(N_{\mathbf{k}}+1)\Phi^\alpha_{\mathbf{p}+\mathbf{k}-\mathbf{k}_1}]\}\,d\mathbf{k}_1\,d\mathbf{k}\frac{1}{(2\pi)^6}$$
$$\approx\frac{\partial}{\partial p_i}\int N_{\mathbf{k}}d\mathbf{k}N_{\mathbf{k}_1}d\mathbf{k}_1 w_{\mathbf{p}}^{\mathrm{tot}}(\mathbf{k},\mathbf{k}_1)(k_i-k_{1,i})(k_j-k_{1,j})\times\frac{1}{(2\pi)^6}\frac{\partial\Phi_{\mathbf{p}}^\alpha}{\partial p_j} + \frac{\partial}{\partial p_i}\int N_{\mathbf{k}_1}(k_{1,i}-k_i)\,w_{\mathbf{p}}^{\mathrm{tot}}(\mathbf{k},\mathbf{k}_1)\frac{d\mathbf{k}\,d\mathbf{k}_1}{(2\pi)^6}\Phi_{\mathbf{p}}^\alpha + \frac{\partial}{\partial p_i}\int N_{\mathbf{k}}(k_i-k_{1,i})\,w_{\mathbf{p}}^{\mathrm{tot}}(\mathbf{k},\mathbf{k}_1)\frac{d\mathbf{k}\,d\mathbf{k}_1}{(2\pi)^6}\Phi_{\mathbf{p}}^\alpha. \quad (5.22)$$

With $N_{\mathbf{k}} = 0$ this equation becomes (5.19).

We turn now to the quantum-mechanical generalization of the results derived above. Curiously, a quantum calculation based on Eqs. (4.10) and (4.9) and relation (2.20) immediately gives us the first of relations (5.19) for spontaneous processes; when induced processed are taken into account, it gives us the first of relations (5.22). The probability for the transition scattering [see (5.20)] includes the quantities $\varepsilon^l_{k-k_1}$, which are given by quantum-mechanical expression (4.12), and S_{k,k_1}, which is determined by the corresponding quantum generalization of (5.7), which can be found by perturbation theory from (4.9). [The fields φ_k are assumed to be classical here; this assumption is legitimate if their 4-momenta $\mathbf{k}_1$ are small ($k_1 \ll p$), but the frequency are momentum $k = \{\mathbf{k},\omega\}$ of the scattered wave are arbitrary.] Finally, the δ-function in (5.20) is replaced by the corresponding "quantum-mechanical generalization" $\delta(\omega-\omega_1-\hbar^{-1}(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}}-\hbar\mathbf{k}+\hbar\mathbf{k}_1))$. Admittedly, under ordinary conditions the probability for the emission of frequencies ω of the order of $\epsilon_{\mathbf{p}}/\hbar$ (in which case quantum effects must be taken into account) is extremely small for longitudinal waves, if such a process is possible at all (if $\omega_{\mathbf{k}} \sim \omega_{\mathrm{pe}}$, the plasma density must be very high; only in this case would ω_{pe} be of the order of even $mc^2/\hbar$).

This situation is possible for electromagnetic (transverse) waves. This approach was taken in Ref. 28 to derive a quantum theory for transition scattering into electromagnetic waves and in Ref. 29 to derive a quantum theory for transition radiation (through a summation of the transition-scattering processses at those harmonics into which the "step" in the permittivity decomposes).

Finally, we need to say a word about a term corresponding to $\omega_1 = -\omega_{\mathbf{k}_1}$, $\omega = \omega_{\mathbf{k}}$ and $\omega = \omega_{\mathbf{k}}$, $\omega_1 = \omega_{\mathbf{k}}$ which we discarded in the derivation of (5.18). That term will contain

$$\delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} - (\mathbf{k}+\mathbf{k}_1)\mathbf{v}). \quad (5.23)$$

In general, that term should not be discarded; it describes the simultaneous emission of two photons and their absorption. In several cases, these processes are important, especially if one of the photons has a low frequency. In contrast with (5.22), the quantum equation is

$$\frac{\partial\Phi_{\mathbf{p}}^\alpha}{\partial t} + \mathbf{v}\frac{\partial\Phi_{\mathbf{p}}^\alpha}{\partial\mathbf{r}} = -\int\{w_{\mathbf{p}}^{2\mathrm{quant}}(\mathbf{k},\mathbf{k}_1)[(N_{\mathbf{k}}+1)(N_{\mathbf{k}_1}+1)\Phi_{\mathbf{p}}^\alpha - N_{\mathbf{k}}N_{\mathbf{k}_1}\Phi^\alpha_{\mathbf{p}-\hbar\mathbf{k}_1-\hbar\mathbf{k}}] + w^{2\,\mathrm{quant}}_{\mathbf{p}+\hbar\mathbf{k}+\hbar\mathbf{k}_1}(\mathbf{k},\mathbf{k}_1)[N_{\mathbf{k}_1}N_{\mathbf{k}}\Phi_{\mathbf{p}}^\alpha - (N_{\mathbf{k}}+1)(N_{\mathbf{k}_1}+1)\Phi^\alpha_{\mathbf{p}+\hbar\mathbf{k}+\hbar\mathbf{k}_1}]\}\,d\mathbf{k}\,d\mathbf{k}_1(2\pi)^{-6}. \quad (5.24)$$

In the classical limit the induced processes are also described by a diffusion equation [except that $(k_i - k_{1,i})(k_j - k_{1,j})$ is replaced by $(k_i + k_{1,i})(k_j + k_{1,j})$; $\hbar^2$ cancels out because both $N_{\mathbf{k}_1}$ and $N_{\mathbf{k}_2}$ are proportional to $1/\hbar$]. For spontaneous processes, two terms with $N_{\mathbf{k}_1}$ and $N_{\mathbf{k}_2}$, which contain $k_i + k_{1,i}$ in place of $k_i - k_{1,i}$ (or $k_{1,i} - k_i$), also arise; here again, one $\hbar$ from $k_{1,i} + k_i$ is canceled by one N. Interestingly, yet another spontaneous term, independent of $N_{\mathbf{k}}$, arises. This term will be of a purely quantum nature even in the limit $k \ll p$. From (5.24) we find

$$\frac{\partial\Phi_{\mathbf{p}}^{\alpha,\mathrm{spont}}}{\partial t} + \mathbf{v}\frac{\partial\Phi_{\mathbf{p}}^{\alpha,\mathrm{spont}}}{\partial\mathbf{r}} = \hbar\frac{\partial}{\partial p_i}\int(k_i + k_{1,i})\,w_{\mathbf{p}}^{2\mathrm{quant}}(\mathbf{k},\mathbf{k}_1)\Phi_{\mathbf{p}}^\alpha\frac{d\mathbf{k}\,d\mathbf{k}_1}{(2\pi)^6}. \quad (5.25)$$

The intensity of the spontaneous emission from an individual particle also contains $\hbar$:

$$Q = \int\hbar(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_1})\,w_{\mathbf{p}}^{2\mathrm{quant}}(\mathbf{k},\mathbf{k}_1)\frac{d\mathbf{k}\,d\mathbf{k}_1}{(2\pi)^6}. \quad (5.26)$$

This circumstance was pointed out in Ref. 27, where a calculation was carried out in the classical limit for $w_{\mathbf{p}}^{2\,\mathrm{quant}}$, the probability for a two-photon Cherenkov emission, and where this effect was analyzed in detail.

It is worthwhile to recall that the calculations on both the scattering and the two-photon emission started from the assumption that a single-photon Cherenkov process is forbidden by conservation laws. If it is instead allowed, the qua-

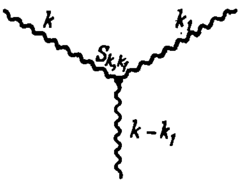

FIG. 3.

silinear interaction will be the predominant process, and the scattering will always amount to a small correction. This circumstance is intuitively clear from the situation that the oscillations of a particle in the field of the scattered wave are usually small, and if there are many scattered waves (a random field) then in going from one resonance to another particle will ultimately not spend much time in the field of a given harmonic. If, on the other hand, the field of the scattered wave is monochromtic, then the changes in the velocity of the particle due to the wave drive the particle away from resonance in a finite time. An exact theory for scattering in the case of a quasilinear Cherenkov interaction is being derived with allowance for turbulent renormalizations; i.e, a Cherenkov field is turned on in the zeroth approximation.[24,31–33]

These renormalization processes are important if the scattering is intense, in which case the linear permittivity ε_k^l is replaced by the nonlinear response ε_k^N when the collective field is singled out in (3.9). This nonlinear response stems from transition scattering processes, i.e., is proportional to $|E^w|_k^2$. In other words, we are dealing with a discrimination of nonlinear collective fields.[34]

It is not by chance that the virtual field δE_k^v arises in calculations on transition scattering. The quantity S_{k,k_1} describes the vertex representing an interaction of three fields (Fig. 3). It contains the two variables k and k_1, since the third is determined unambiguously by the conservation laws. Its 4-momentum is $k-k_1$. This is a virtual field, and the diagram of the transition scattering is as shown in Fig. 4. The square of the Green's function $1/|\mathbf{k}-\mathbf{k}_1|\varepsilon_{k-k_1}^l$ naturally appears in (5.20). For transverse virtual waves, their Green's function, $[(\mathbf{k}-\mathbf{k}_1)^2-(\omega-\omega_1)^2\varepsilon_{k-k_1}^t]^{-1}$, correspondingly arises.

6. FLUCTUATIONS AND TRANSITION BREMSSTRAHLUNG

We can show that in emission involving fluctuations there is an effect which stems from the transition scattering of virtual waves into real waves (along with the well-known ordinary bremsstrahlung, which is the Thomson scattering of virtual waves into real waves). Figure 5 shows the corre-

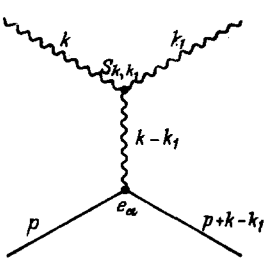

FIG. 4.

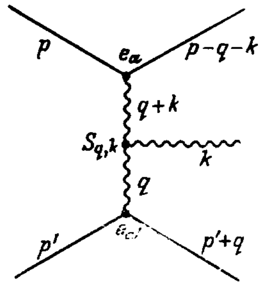

FIG. 5.

sponding diagram of transition bremsstrahlung. All the wavy lines here may correspond to both longitudinal and transverse waves, in any combinations (i.e., to both real and virtual waves).

To illustrate the fundamental points it is most convenient to consider the case in which all the waves are longitudinal. In this case we can work from (5.5) [i.e., from (3.4) and (3.9), (3.10)]. We now assume that there is no collective field δE_k^w. We accordingly write relation (5.8) for $\delta E^{(0)}$, and we write

$$\delta E_k=\frac{8\pi}{i\omega\varepsilon_k^l}\int S_{k,k_1}(\delta E_{k_1}^{(0)}\delta E_{k-k_1}^v-\langle\delta E_{k_1}^{(0)}\delta E_{k-k_1}^v\rangle)\,dk_1. \qquad (6.1)$$

This component was previously discarded, since our purpose was to derive the response which was linear in $|E^w|_k^2$, but in that case δE_k^v would have been quadratic in δE_k^w, and the result would have vanished by virtue of the independence of the averaging over the fluctuations of the fields, δE_k^w, and the fluctuations of the particles. Actually, the argument was simply the principle of selecting effects in such a way that we would be left with a δ-function associated with the scattering. When we move away from that case, we should take $\delta E^{(0)}$ into account in (6.1). If both fields in (6.1) are equal to $\delta E^{(0)}$, the result is zero, since the expectation value of the three fields is zero. For $\delta E^v_{-k_1}$ we now write, in place of (5.9),

$$\delta E_{k-k_1}^{\mathrm{v}}=\frac{8\pi}{i\,(\omega-\omega_1)\,\varepsilon_{k-k_1}^l}\int S_{k-k_1,k_2}(\delta E_{k_2}^{(0)}\delta E_{k-k_1-k_2}^{(0)}$$
$$-\langle\delta E_{k_2}^{(0)}\delta E_{k-k_1-k_2}^{(0)}\rangle)\,d\mathbf{k}_2. \qquad (6.2)$$

From this point on the calculations are the same as those for transition scattering, except that the expectation value $(\delta E^{(0)})^2$ here must be evaluated through the sole use of (2.6) for the averaging over particle fluctuations. We find

$$\frac{\partial\Phi_{\mathbf{p}}^\alpha}{\partial t}+\mathbf{v}\frac{\partial\Phi_{\mathbf{p}}^\alpha}{\partial\mathbf{r}}$$
$$=\frac{\partial}{\partial\mathbf{p}}\sum_{\alpha'}\int d\mathbf{q}\,d\mathbf{k}\frac{2e_\alpha^2e_{\alpha'}^2\mathbf{q}}{\omega_{\mathbf{k}}^2\mathbf{q}^2(\mathbf{k}-\mathbf{q})^2}\delta(\omega_{\mathbf{k}}-\mathbf{q}\mathbf{v}'-(\mathbf{k}-\mathbf{q})\mathbf{v})$$
$$\times\frac{|S_{\mathbf{k},\omega_{\mathbf{k}};\mathbf{q},\mathbf{q}\mathbf{v}'}|^2}{|\varepsilon_{\mathbf{k}-\mathbf{q},(\mathbf{k}-\mathbf{q})\mathbf{v}}^l|^2}\frac{\Phi_{\mathbf{p}}^\alpha}{|\varepsilon_{\mathbf{q},\mathbf{q}\mathbf{v}'}^l|^2\,\partial\varepsilon_k^l/\partial\omega\,|_{\omega=\omega_{\mathbf{k}}}}\frac{\Phi_{\mathbf{p}'}^{\alpha'}}{(2\pi)^3}d\mathbf{p}'. \qquad (6.3)$$

The meaning of (6.3) is obvious: This is the friction force created by the transition bremsstrahlung of the wave $\mathbf{k},\omega$ in a collision of particles α and α'; $1/|\mathbf{k}-\mathbf{q}|\varepsilon_{\mathbf{k}-\mathbf{q},(\mathbf{k}-\mathbf{q})\mathbf{v}}^l$ and $l/q\varepsilon_{\mathbf{q},\mathbf{q}\mathbf{v}'}^l$ correspond to the Green's functions of the two longitudinal virtual waves in Fig. 5; and $\delta(\omega_k-\mathbf{q}\mathbf{v}'-(\mathbf{k}-\mathbf{q})\mathbf{v})$ describes energy conservation in

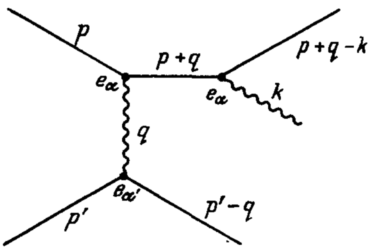

FIG. 6.

the elementary event of the transition bremsstrahlung. The arguments at the vertex S are also clear from the diagram in Fig. 5. The two particles α and α' are of course equivalent, as can be seen by changing the notation: $\mathbf{q}\to\mathbf{k}-\mathbf{q}$. We can write relation (6.3) in the form

$$\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial t}+\mathbf{v}\,\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial\mathbf{r}}=\frac{\partial}{\partial\mathbf{p}}\sum_{\alpha'}\int d\mathbf{q}\,d\mathbf{k}\,d\mathbf{p}'\,w^{\alpha,\alpha'}_{\mathbf{p},\mathbf{p}'}(\mathbf{k},\mathbf{q})\,\Phi^{\alpha}_{\mathbf{p}}\Phi^{\alpha'}_{\mathbf{p}'}\,\frac{1}{(2\pi)^3},\tag{6.4}$$

where $w^{\alpha,\alpha'}_{\mathbf{p},\mathbf{p}'}$ is the probability for the bremsstrahlung, which is precisely the same as the expression derived in Refs. 6 and 7 by a different method.

Incorporating the other terms with $\delta f^{(0)}$ [see (5.21)] leads to ordinary bremsstrahlung in the classical limit (Fig. 6) and to an interference of the ordinary bremsstrahlung with the transition bremsstrahlung. Incorporating transverse waves leads to the entire spectrum of possible bremsstrahlung processes, while incorporating $|E^{w}|^2_k$ does the same for induced bremsstrahlung processes. The transition bremsstrahlung is thus manifested as a necessary component of the volume emission from a plasma due to fluctuations.

7. ZERO-POINT FLUCTUATIONS IN THE QUASILINEAR INTERACTION OF RESONANT FIELDS WITH PARTICLES

As has already been mentioned, the literature reveals a great deal of interest in processes associated with scattering in the presence of resonant fields or, more precisely, nonlinear interactions of such a type that the resonance condition holds approximately for all waves (if this condition holds for the scattered waves). An expansion in the wave amplitudes is inappropriate in the resonance region in this case, and all effects amount to corrections to the quasilinear interaction which are not analytic in $\langle(\delta E^{w})^2\rangle/4\pi nT$. Theories of this type are called "turbulent broadening of a resonance."[22] The basic idea of these theories is that collective modes near the resonance which are already nonlinear are adopted as the initial modes. This approach makes it possible to construct an analytic theory without an expansion in $\langle(\delta E^{w})^2\rangle/4\pi nT$ (admittedly, only within the resonance[32]). An analysis of that sort completely ignores the zero-point fluctuations of the particles, $\delta f^{(0)}$ (Sec. 2). When the method described above, involving the sequential expansion of a collision integral in δE and δf, is used, it is completely reasonable to ask about the effect of the particle fluctuations $\delta f^{(0)}$ on the quasilinear interaction.

We need to begin with the simplest problem, in which (as is frequently the case experimentally) the number of resonant particles, $\Phi^{\alpha}_{\mathbf{p}}$, is small, and we seek only the result which is linear in the number of particles, $\Phi^{\alpha}_{\mathbf{p}}$. In this case it is of course necessary to solve equations for the field; the Green's functions of the longitudinal fields and, in general, of the transverse fields are involved. In an analysis of longitudinal fields it is necessary to use relation (5.5). In order to derive an expression which is linear in $\Phi^{\alpha}_{\mathbf{p}}$, it is necessary to incorporate in δf^{α} (a) terms which are proportional to $\delta f^{(2)}\sim\delta f^{(0)}(\delta E^{w})^2$ along with $\delta E^{(0)}$, (b) terms which are proportional to $\delta f^{(1)}\sim\delta f^{(0)}\delta E^{w}$ and $(\delta f^{(0)})^2\delta E^{w}$ along with $\delta E^{(1)}\sim\delta E^{w}$ and $\delta E^{w}\delta f^{(0)}$ (products of the first and second terms of these expansions appear), and (c) $\delta f^{(0)}$ along with $\delta E^{(2)}\sim\delta f^{(0)}(\delta E^{w})^2$.

Calculations show that, according to the equation for longitudinal waves in (5.5), all these corrections cancel out exactly, and their net contribution is zero. It is apparently for this reason that the effect of the "zero-point" fluctuations in the quasilinear interaction did not attract attention for a long time. It turned out that transverse corrections had to be taken into account only for relativistic particles, and that case did not attract much interest (just as the collision integral with an interaction through a virtual transverse wave is not used much, since it is important for relativistic particles, and for such particles collisions of particles with each other are not as important).

As it turns out, the zero-point fluctuations actually affect the quasilinear interaction only when relativistic effects and transverse virtual waves are taken into account (the result depends explicitly on the velocity of light). In several cases these fluctuations play a very important role.

In discussing transverse fields it is necessary to solve not the Poisson equation but the complete Maxwell's equation, and in a collision integral of the type in (5.5) it is necessary to consider, along with the electric field, the term with the magnetic field (the Lorentz force). In this case the result is not zero; furthermore, the result diverges in the wave numbers of the virtual transverse fields. They are determined by the Green's function

$$\frac{1}{q^2-[(\mathbf{qv})^2/c^2]\,\varepsilon^{t}_{\mathbf{q},\mathbf{qv}}}.\tag{7.1}$$

Since large values of $\mathbf{q}$ are dominant here (the integral over $\mathbf{q}$ diverges), we will ignore polarization effects, i.e., set $\varepsilon^{t}=1$, for simplicity. We are thus solving Maxwell's equations in vacuum. There is no need here to single out transverse collective modes (as in Sec. 3), since we are assuming that there are no transverse waves. The resonant fields δE^{w}_{k}, which contribute a quasilinear interaction in the first approximation, are assumed to be longitudinal in the corrections to the quasilinear interaction which we consider. With $\varepsilon^{t}=1$, the Green's function (7.1) can be written in the form ($c=1$)

$$\frac{1}{2q}\left(\frac{1}{q-\mathbf{qv}}+\frac{1}{q-\mathbf{qv}}\right).\tag{7.2}$$

Since the integration runs over all virtual $\mathbf{q}$, by using the replacement $\mathbf{q}\to-\mathbf{q}$ one can arrange events such that the result contains

$$\frac{1}{q\,(q-\mathbf{qv})}.\tag{7.3}$$

A direct, purely classical calculation yields[10,11,35,36]

$$\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial t}+\mathbf{v}\,\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial\mathbf{r}}=\frac{\partial}{\partial p_i}(D^{ql}_{ij}+D^{fl}_{ij})\,\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial p_j}+\frac{\partial}{\partial p_i}F^{fl}_{i}\Phi^{\alpha}_{\mathbf{p}},\tag{7.4}$$

where D^{ql}_{ij} is the ordinary quasilinear diffusion coefficient [see (3.24)], given by

$$D_{ij}^{ql}=e_\alpha^2\int\frac{k_ik_j}{k^2}|E^w|_k^2\,\delta(\omega-\mathbf{kv})\,d\mathbf{k}, \quad (7.5)$$

and the "corrections" for the particle fluctuations are

$$D_{ij}^{fl}=-\frac{e_\alpha^4}{4\pi}\int\frac{|E^w|_k^2\,dk\,d\mathbf{q}}{k^2(q-\mathbf{qv})}\delta(\omega-\mathbf{kv})$$
$$\times\left[k_i\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\left(v_j-\frac{q_j(\mathbf{qv})}{q^2}\right)\right.$$
$$\left.+k_j\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\left(v_i-\frac{q_i(\mathbf{qv})}{q^2}\right)\right]\frac{1}{q-\mathbf{qv}}-\frac{e_\alpha^4}{2\pi\varepsilon_\mathbf{p}}\int|E^w|_k^2\,dk\,d\mathbf{q}$$
$$\times\frac{\delta(\omega-\mathbf{kv})}{k^2q(q-\mathbf{qv})^3}\{(q-\mathbf{qv})(k_i[\mathbf{v}[\mathbf{qk}]]_j+k_j[\mathbf{v}[\mathbf{qk}]]_i)$$
$$+((\mathbf{kq})-(\mathbf{qv})(\mathbf{kv}))(k_i[\mathbf{v}[\mathbf{qv}]]_j+k_j[\mathbf{v}[\mathbf{qv}]]_i)\}, \quad (7.6)$$

$$E_i^{fl}=\frac{e_\alpha^4}{2\pi\varepsilon_\mathbf{p}^2}\int\frac{|E^w|_k^2k_i\,dk\,d\mathbf{q}}{k^2q^2v^2}\delta(\omega-\mathbf{kv})\left(\frac{1}{2v}\ln\frac{1+v}{1-v}-1\right). \quad (7.7)$$

The quantity F_i can also be put in a form which contains the Green's function (7.3). However, since F^{fl} contains an integration over all $\mathbf{q}$, we have taken an average over the angles $\mathbf{q}$ and $\mathbf{v}$ in (7.7).

Note that the corrections for fluctuations contain, in addition to a diffusion, a friction force [as Landau collision integral (4.3) does].

The divergence of the type

$$e_\alpha^2\int\frac{d\mathbf{q}}{q^2}$$

in (7.6) and (7.7) suggests a contribution from a renormalization of the mass (the part of it associated with the transverse virtual field). However, a calculation shows that (7.6) and (7.7) contain effects beyond these. Specifically, the result (7.6) contains nonzero terms with $\delta E^{(1)}\sim\delta E^w\delta f^{(0)}$, i.e., with the field generated by both the resonant fields and the zero-point fluctuations. They describe a change in the resonant fields due to the fluctuations of the particles. There are also terms which describe changes in the fluctuations of the particles due to the presence of resonant fields.

To some extent, this mutual effect is similar to the radiation corrections in the case in which a particle which has emitted a virtual transverse photon interacts with a resonant field and then absorbs the virtual photon which has been emitted. In the case at hand, however, all the fluctuating currents contain integrals over all particles, and it is not clear whether another particle can absorb a photon. This effect is then distinct from the simply radiative corrections to the interaction of individual particles with external fields. It is necessary to carry out a quantum-mechanical calculation with renormalizations; this can be done correctly only in relativistic quantum theory.

The renormalization procedure is the standard procedure.[37,38] Relations which we have already written, (2.30) and (2.31), which describe zero-point fluctuations in a system of relativistic, quantum, spin 1/2 particles, can serve as a starting point. As the equation for $\delta\hat{f}_{\mathbf{p},\mathbf{k}}$ it is necessary to use a generalization of Eq. (4.9) found from the Dirac equations for $\hat{\Psi}_\mathbf{p}$:

$$\frac{\partial\hat{f}_{\mathbf{p},\mathbf{k}}(t)}{\partial t}+i\beta\left(i\gamma\left(\mathbf{p}+\frac{\mathbf{k}}{2}\right)+m\right)\hat{f}_{\mathbf{p},\mathbf{k}}(t)$$
$$-\hat{f}_{\mathbf{p},\mathbf{k}}(t)\left(-\gamma\left(\mathbf{p}-\frac{\mathbf{k}}{2}\right)+m\right)i\beta$$
$$=e\int d\mathbf{k}_1\,d\mathbf{k}_2\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$$
$$\cdot\times\left(\hat{f}_{\mathbf{p}+\frac{\mathbf{k}_2}{2},\mathbf{k}_1}(t)\beta\hat{A}_{\mathbf{k}_2}(t)-\beta\hat{A}_{\mathbf{k}_2}(t)\hat{f}_{\mathbf{p}-\frac{\mathbf{k}_2}{2},\mathbf{k}_1}(t)\right),$$
$$\hat{A}_\mathbf{k}(t)=\gamma_\mu\hat{A}_{\mathbf{k},\mu}(t); \quad (7.8)$$

where $\gamma_\mu=\{\gamma,i\beta\}$ are the Dirac matrices.

We expand the final result, which contains $|E^w|^2(\delta f^{(0)})^2$, in $k\ll p$, assuming that the resonant fields are classical and that only a Cherenkov resonance $\varepsilon_\mathbf{p}-\varepsilon_{\mathbf{p}-\mathbf{k}}-\omega=0$ is possible for them—not a resonance involving pair production, $\varepsilon_\mathbf{p}+\varepsilon_{\mathbf{p}+\mathbf{k}}-\omega\neq0$. The final result is[11,10,35,36,39-41]

$$\frac{\partial\Phi_\mathbf{p}^\alpha}{\partial t}+\mathbf{v}\frac{\partial\Phi_\mathbf{p}^\alpha}{\partial\mathbf{r}}=\frac{d\Phi_\mathbf{p}^\alpha}{dt}=\hat{I}_\mathbf{p}^{ql}\Phi_\mathbf{p}^\alpha+\pi^2e_\alpha^4\int\frac{|E^w|_k^2}{k^2}\frac{dk\,d\mathbf{q}}{(2\pi)^3}$$
$$\times\left\{R_{\mathbf{p}+\mathbf{q},\mathbf{p}}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta\left(\omega-\mathbf{k}\frac{\partial\varepsilon_{\mathbf{p}+\mathbf{q}}}{\partial\mathbf{p}}\right)\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\Phi_{\mathbf{p}+\mathbf{q}}^\alpha\right.$$
$$-R_{\mathbf{p},\mathbf{p}+\mathbf{q}}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta\left(\omega-\mathbf{k}\frac{\partial\varepsilon_\mathbf{p}}{\partial\mathbf{p}}\right)$$
$$\times\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\Phi_\mathbf{p}^\alpha-\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta\left(\omega-\mathbf{k}\frac{\partial\varepsilon_\mathbf{p}}{\partial\mathbf{p}}\right)$$
$$\left.\times\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)(R_{\mathbf{p},\mathbf{p}+\mathbf{q}}\Phi_\mathbf{p}^\alpha-R_{\mathbf{p}+\mathbf{q},\mathbf{p}}\Phi_{\mathbf{p}+\mathbf{q}}^\alpha)\right\}, \quad (7.9)$$

where

$$R_{\mathbf{p},\mathbf{p}+\mathbf{q}}=-\frac{1}{|\mathbf{q}|}\left[\frac{\mathrm{Sp}\left(\gamma-\frac{\mathbf{q}(\mathbf{q}\gamma)}{q^2}\right)\Lambda_{\mathbf{p}+\mathbf{q}}^+\left(\gamma-\frac{\mathbf{q}(\mathbf{q}\gamma)}{q^2}\right)\Lambda_\mathbf{p}^+\beta}{(\varepsilon_{\mathbf{p}+\mathbf{q}}-\varepsilon_\mathbf{p}+|\mathbf{q}|)^2}\right.$$
$$\left.-\frac{\mathrm{Sp}\left(\gamma-\frac{\mathbf{q}(\mathbf{q}\gamma)}{q^2}\right)\Lambda_{\mathbf{p}+\mathbf{q}}^-\left(\gamma-\frac{\mathbf{q}(\mathbf{q}\gamma)}{q^2}\right)\Lambda_\mathbf{p}^+\beta}{(\varepsilon_{\mathbf{p}+\mathbf{q}}+\varepsilon_\mathbf{p}+|\mathbf{q}|)^2}\right], \quad (7.10)$$

$$I_\mathbf{p}^{ql}=\pi e_\alpha^2\int\frac{|E^w|_k^2}{k^2}dk\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta\left(\omega-\mathbf{k}\frac{\partial\varepsilon_\mathbf{p}}{\partial\mathbf{p}}\right)\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right), \quad (7.11)$$

$I_\mathbf{p}^{ql}$ is a quasilinear operator, and $\Lambda_\mathbf{p}^\pm$ are the projection operators, introduced above, which project onto positive (negative) energies.

Note that (7.9) contains, in addition to terms proportional to $\Phi_\mathbf{p}^\alpha$, terms with $\Phi_{\mathbf{p}+\mathbf{q}}^\alpha$. They of course do not require any renormalization. A renormalization procedure has already been carried out in (7.9); it has resulted in changes in specifically the terms containing $\Phi_\mathbf{p}^\alpha$. It is easy to show that these terms no longer have divergences as $q\to\infty$, as was the case in (7.6) (Ref. 12). We should point out that a convergence occurs only if, among all parts of the Green's function, only those combinations which are written in (7.10) contribute to the result. In speaking of "parts" of a Green's function we mean that (7.2) contains $1/(q-\mathbf{qv})$ and $1/(q+\mathbf{qv})$, which are reduced to (7.3) by the replacement $\mathbf{q}\to-\mathbf{q}$ only in the nonquantum limit. In the quantum case, combinations of the type

$$\frac{1}{\varepsilon_{\mathbf{p}+\mathbf{q}}-\varepsilon_\mathbf{p}+|\mathbf{q}|},\quad\frac{1}{\varepsilon_{\mathbf{p}+\mathbf{q}}-\varepsilon_\mathbf{p}-|\mathbf{q}|}$$

could arise; the second "part" of the Green's function is no longer reduced to the first part by the replacement $\mathbf{q}\to-\mathbf{q}$.

Furthermore, the expression $1/(\varepsilon_{\mathbf{p}+\mathbf{q}} + \varepsilon_{\mathbf{p}} + |\mathbf{q}|)$ might appear along with $1/(\varepsilon_{\mathbf{p}+\mathbf{q}} + \varepsilon_{\mathbf{p}} - |\mathbf{q}|)$. These terms with the "other" parts of the Green's function do indeed appear in all the intermediate (and rather complicated) results, but the coefficients of those "parts" which are not given by (7.10) are strictly zero in the final form. Also strictly zero are the coefficients of the set of other terms of the type $1/2\varepsilon_{\mathbf{p}}(\varepsilon_{\mathbf{p}+\mathbf{q}} \pm \varepsilon_{\mathbf{p}} \pm |\mathbf{q}|)$, etc. The final result is expressed in terms of the one function $R_{\mathbf{p},\mathbf{p}+\mathbf{q}}$, given by expression (7.10).

The circumstance that it is possible to write the result in the compact form in (7.9) and (7.10) might constitute an "esthetic" criterion for the validity of the result.

The terms with $\Phi^{\alpha}_{\mathbf{p}+\mathbf{q}}$ of course do not diverge as $\mathbf{q}\to\infty$, since we have $\Phi^{\alpha}_{\mathbf{p}+\mathbf{q}}\to 0$ in this case. Such terms are explicitly related to the momentum transfer (and thus the energy transfer) from certain particles to others. This circumstance answers the question which was raised above about the absorption of a virtual photon. This photon is indeed absorbed in this system of particles, but not necessarily by the same particle which emitted it. Incidentally, the indistinguishability of particles is a necessary consequence of the quantum-mechanical treatment. The transfer of momentum from certain particles to others, however, is a completely observable process; in particular, a large number of relatively low-energy particles might transfer energy to a small number (roughly speaking, several) high-energy particles.

An expansion in $\mathbf{q}$ in (7.9) is impossible in principle, since the integral converges only for large values of $\mathbf{q}$. We can discuss only the form of the integrand in (7.9) at small values of $\mathbf{q}$. Only in this limit do $\Phi^{\alpha}_{\mathbf{p}}$ and its derivatives appear [cf. (7.6)].

The result of the expansion is

$$\frac{d\Phi^{\alpha\,\mathrm{fl}}_{\mathbf{p}}}{dt} = -\frac{e_\alpha^4}{8\pi}\int \frac{|E^{\mathrm{w}}|^2_k}{k^2 q}\,dk\,d\mathbf{q}\left\{\left(\mathbf{q}\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega-\mathbf{kv})\right.$$
$$\times\left(\mathbf{k}\frac{\partial G}{\partial\mathbf{p}}\right)\left(\mathbf{k}\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial\mathbf{p}}\right)$$
$$+\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega-\mathbf{kv})\left(\mathbf{k}\frac{\partial G}{\partial\mathbf{p}}\right)\left(\mathbf{q}\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial\mathbf{p}}\right)$$
$$+\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega-\mathbf{kv})\,\Phi^{\alpha}_{\mathbf{p}}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\left(\mathbf{q}\frac{\partial}{\partial\mathbf{p}}\right)G$$
$$\left.+\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)G\left(\mathbf{k}\frac{\partial\Phi^{\alpha}_{\mathbf{p}}}{\partial\mathbf{p}}\right)\left(\mathbf{q}\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega-\mathbf{kv})\right\}, \quad (7.12)$$
$$G=\frac{2[\mathbf{qv}]^2}{q^2(|\mathbf{q}|-\mathbf{qv})^2}.$$

This result is very similar to (7.6), but not exactly the same, since a renormalization has been carried out in (7.12), but not in (7.6) [and since in the classical expression (7.6) it is not clear how we would carry out such a renormalization at all]. More precisely, in the classical expression (7.6) the renormalization is not an unambiguous procedure. Only in the framework of quantum electrodynamics[37] does this procedure become unambiguous (it is also unambiguous within the framework of the quantum kinetics with which we are concerned here). From (7.12) and (7.6) we can find the correct form of the classical limit of the radiative-resonant interactions, with renormalizations. Accordingly, (7.12) is the correct expression for the integrand in the effects which stem from the influence of the zero-point fluctuations on the quasilinear interaction. Incidentally, (7.9) incorporates, in addition to the $\delta f^{(0)}$ effects, zero-point fluctuations of the electromagnetic field. Specifically, we know from quantum electrodynamics that these effects are of the same order of magnitude (see Ref. 37, for example, for a proof).

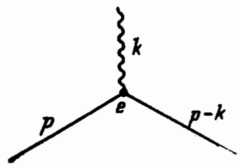

FIG. 7.

Note that (7.10) contains specifically the Green's function of transverse virtual fields incorporting virtual pair production [the second term in (7.10)]: that Green's function whose square appears in the relativistic quantum-mechanical collision integral.

This result is not surprising, since the collision integral also describes the transfer of a virtual photon but contains the square of a matrix element, while (7.9) should correspond crudely to the product of the Cherenkov-radiation matrix element (Fig. 7) and the matrix element associated with the emission of a resonant photon and a single exchange of virtual momentum in the system of particles [Figs. 8(a) and 8(b)].

The arbitrariness of assertions regarding any products of matrix elements can be seen from the circumstance that the fluctuations are described only by operators; i.e., we are dealing with only operator vertices, which appear in Lagrangians, not their combinations which have been constructed by a graphical method in accordance with the S-matrix rules [as in Figs. 8(a) and 8(b)]. The quantity $\Phi^{\alpha}_{\mathbf{p}}$ is defined as the vacuum expectation value of the corresponding combinations of operators.

We recall that although actual pair production does not occur in this process the general relativistic definition of $\Phi^{\alpha}_{\mathbf{p}}$ contains $\mathrm{Sp}\,f = \mathrm{Sp}(\Lambda_{\mathbf{p}}^{+}\beta f - \Lambda_{\mathbf{p}}^{-}\beta f)$, i.e., gives us the dif-

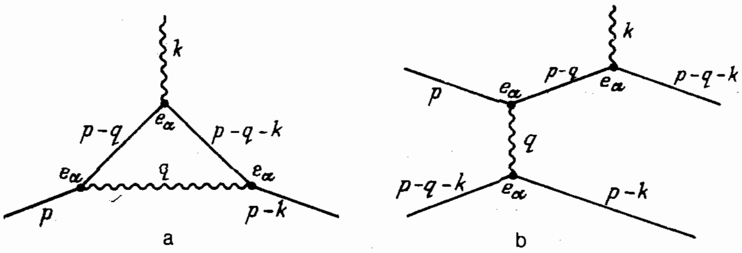

FIG. 8.

ference between the numbers of particles and antiparticles, which is of course conserved. The number of particles is also conserved in (7.9).

Introducing $\mathbf{p}' = \mathbf{p} + \mathbf{q}$, we rewrite (7.9) as

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt} = \hat{I}_{\mathbf{p}}^{\mathrm{ql}}\Phi_{\mathbf{p}}^{\alpha} + \pi e_{\alpha}^{2}\int\frac{d\mathbf{p}'}{(2\pi)^{3}}\{R_{\mathbf{p}',\mathbf{p}}\hat{I}_{\mathbf{p}'}^{\mathrm{ql}}\Phi_{\mathbf{p}'}^{\alpha} - R_{\mathbf{p},\mathbf{p}'}\hat{I}_{\mathbf{p}}^{\mathrm{ql}}\Phi_{\mathbf{p}}^{\alpha} + \hat{I}_{\mathbf{p}}^{\mathrm{ql}}(R_{\mathbf{p},\mathbf{p}'}\Phi_{\mathbf{p}}^{\alpha} - R_{\mathbf{p}',\mathbf{p}}\Phi_{\mathbf{p}'}^{\alpha})\}, \quad (7.13)$$

from which we find the conservation of $n_{\alpha} = \int\Phi_{\mathbf{p}}^{\alpha}d\mathbf{p}(2\pi)^{-3}$:

$$\frac{dn_{\alpha}}{dt} = 0. \quad (7.14)$$

Collective effects thus also play a role in processes which correspond to the known radiation corrections for individual particles.

8. *S*-MATRIX METHOD

In a discussion of the role played by fluctuations in the quasilinear interaction, the virtual fields $\mathbf{q}$ actually appear as vacuum fields (the difference between the permittivity and unity and the Green's function has been ignored), and the longitudinal resonant collective fields are assumed to be simply given fields (external fields). In this case, however, we can treat these problems by the standard *S*-matrix procedure and thus shed some light on the role played by collective effects in (7.9) (i.e., on the role played by the terms with a momentum transfer from certain particles to others).

In addition to the quantized fields we introduce a classical random longitudinal field, described by the potential $\varphi_{k}^{\mathrm{w}}(\delta\mathbf{E}_{k}^{\mathrm{w}} = -i\mathbf{k}\varphi_{k}^{\mathrm{w}})$. This field (with $k \ll p$) may satisfy the Cherenkov condition

$$\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega = 0, \quad (8.1)$$

so the matrix element of the *S* matrix of first order in the field (in contrast with the electromagnetic fields in vacuum) is not zero:

$$M_{\mathbf{k}}^{(1)} = \langle\mathbf{p}|S^{(1)}(t)|\mathbf{p}-\mathbf{k}\rangle = u_{\alpha,\mathbf{p}}^{+1,\mu'+}u_{\alpha,\mathbf{p}-\mathbf{k}}^{+1,\mu}\int\varphi_{\mathbf{k},\omega}d\omega\int_{0}^{t}e^{-i(\omega-\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}-\mathbf{k}})t'}\,dt'. \quad (8.2)$$

The field φ_{k} is turned on adiabatically at $t = 0$.

The probability for Cherenkov radiation is[42]

$$w_{\mathbf{p}}^{\alpha} = \frac{1}{2}\sum_{\mu,\mu'}\frac{d}{dt}\left\langle\left|\int_{t\to\infty}M_{\mathbf{k}}^{(1)}dk\right|^{2}\right\rangle = \frac{1}{2}\sum_{\mu,\mu'}\int\frac{|E^{\mathrm{w}}|_{k}^{2}}{k^{2}}dk\cdot|u_{\alpha,\mathbf{p}}^{+1,\mu'+}u_{\alpha,\mathbf{p}-\mathbf{k}}^{+1,\mu}|^{2}$$

$$\times\left\{\exp[-i(\omega-\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}-\mathbf{k}})t]\int_{0}^{t}\exp[i(\omega-\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}-\mathbf{k}})t']\,dt' + \exp[i(\omega-\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}-\mathbf{k}})t]\int_{0}^{t}\exp[-it'(\omega-\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}})]\,dt'\right\}$$

$$= \int\frac{|E^{\mathrm{w}}|_{k}^{2}}{k^{2}}\,dk\,\frac{\sin(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}-\omega)t}{(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}-\omega)}\sum_{\mu,\mu'}|u_{\alpha,\mathbf{p}}^{+1,\mu'+}u_{\alpha,\mathbf{p}-\mathbf{k}}^{+1,\mu}|^{2}, \quad (8.3)$$

$dk = d\mathbf{k}\,d\omega$.

Here we have used the relation

$$\langle\varphi_{k}^{\mathrm{w}}\varphi_{k'}^{\mathrm{w}}\rangle = \frac{|E^{\mathrm{w}}|_{k}^{2}}{k^{2}}\delta(k+k'). \quad (8.4)$$

Using

$$\frac{\sin xt}{x}\underset{t\to\infty}{\to}\pi\delta(x) \quad (8.5)$$

and

$$\frac{1}{2}\sum_{\mu,\mu'}|u_{\alpha,\mathbf{p}}^{+1,\mu'+}u_{\alpha,\mathbf{p}-\mathbf{k}}^{\mu}|^{2} = \frac{1}{2\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}-\mathbf{k}}}(\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}-\mathbf{k}} + m^{2} + (\mathbf{p}(\mathbf{p}-\mathbf{k}))) \equiv w_{\mathbf{p},\mathbf{p}-\mathbf{k}}^{(0)}, \quad (8.6)$$

we find

$$w_{\mathbf{p}}^{\alpha} = 2\pi e_{\alpha}^{2}\int\frac{|E^{\mathrm{w}}|_{k}^{2}}{k^{2}}\,dk\,w_{\mathbf{p},\mathbf{p}-\mathbf{k}}^{(0)}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}-\omega). \quad (8.7)$$

We cannot go any further in the calculations with the *S* matrix; i.e., we can either derive an equation for $\Phi_{\mathbf{p}}^{\alpha}$ directly from its definition or postulate a relationship between $\Phi_{\mathbf{p}}^{\alpha}$ and $w_{\mathbf{p}}$ on the basis of physical considerations. Incidentally, these two paths lead to the same result for the quasilinear equation. We write (postulate) a balance between the direct and inverse processes as follows:

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt} = \int w_{\mathbf{p}}^{\alpha}(\mathbf{k})(\Phi_{\mathbf{p}-\mathbf{k}}^{\alpha} - \Phi_{\mathbf{p}}^{\alpha})\frac{d\mathbf{k}}{(2\pi)^{3}}, \quad (8.8)$$

where

$$w_{\mathbf{p}}^{\alpha} = \int w_{\mathbf{p}}^{\alpha}(\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^{3}}. \quad (8.9)$$

An expression for $w_{\mathbf{p}}^{\alpha}(\mathbf{k})$ can be written easily on the basis of (8.7).

Using (8.8), and taking half the sum of the expressions with the replacement $k \to -k$ (and using $|E^{\mathrm{w}}|_{-k}^{2} = |E^{\mathrm{w}}|_{k}^{2}$), we find

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt} = \pi e_{\alpha}^{2}\int\frac{|E^{\mathrm{w}}|_{k}^{2}}{k^{2}}\,dk\,[\delta(\varepsilon_{\mathbf{p}+\mathbf{k}}-\varepsilon_{\mathbf{p}}-\omega)(\Phi_{\mathbf{p}+\mathbf{k}}^{\alpha}-\Phi_{\mathbf{p}}^{\alpha})w_{\mathbf{p}+\mathbf{k},\mathbf{p}}^{(0)} - \delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}-\mathbf{k}}-\omega)(\Phi_{\mathbf{p}}^{\alpha}-\Phi_{\mathbf{p}-\mathbf{k}}^{\alpha})w_{\mathbf{p},\mathbf{p}-\mathbf{k}}^{(0)}]. \quad (8.10)$$

Equation (8.10) is a relativistic quantum-mechanical generalization of quasilinear equation (3.24). It contains only the Cherenkov δ-function, because in (8.2) we have retained only the states with positive energy (the states $\mathbf{p}$ and $\mathbf{p}-\mathbf{k}$ have been assumed to be positive-energy states).

It is not difficult to derive an equation to describe pair production in the general case in which the following resonance occurs:

$$\varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{p}+\mathbf{k}} - \omega = 0. \quad (8.11)$$

A requirement here is that the field $\delta E_{k}^{\mathrm{w}}$ can satisfy condition (8.11). We will not write out the corresponding relations, under the assumption that condition (8.11) does not hold.

From (8.10) we can derive (3.24), by means of an expansion in $k \ll p$. The terms which are linear in $\mathbf{k}$ vanish by virtue of the relation $|E^{\mathrm{w}}|_{k}^{2} = |E^{\mathrm{w}}|_{-k}^{2}$. The first term in brackets in (8.10) differs from the second in that $\mathbf{p}$ has been replaced by $\mathbf{p}+\mathbf{k}$; i.e., in the first approximation in the expansion in $\mathbf{k}$, the result reduces to the application of an operator $(\mathbf{k}\partial/\partial\mathbf{p})$ to the second term, which is in turn proportional to the following quantity in the first approximation:

$$(\Phi_{\mathbf{p}}^{\alpha}-\Phi_{\mathbf{p}-\mathbf{k}}^{\alpha})\approx\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}^{\alpha}}{\partial\mathbf{p}}\right).$$

In a first approximation for $w_{\mathbf{p},\mathbf{p}-\mathbf{k}}^{(0)}$ we can then ignore the **k** dependence: $w_{\mathbf{p},\mathbf{p}-\mathbf{k}}^{(0)}\approx w_{\mathbf{p},\mathbf{p}}^{(0)}=1$. We then immediately find Eq. (3.24).

The S' matrix method has the advantage that a standard renormalization procedure can be used. It is thus possible to find corrections to $w_{\mathbf{p}}$ for the radiation corrections [Fig. 8(a)].

To find now an equation for $\Phi_{\mathbf{p}}^{\alpha}$ we are obliged to postulate a balance equation. We have no ground for making such a postulate, since in that approach, when the ensemble of particles is taken into account, only the ordinary radiation corrections are considered; collective effects involving a transfer of momentum from certain particles to others are lost [Fig. 8(b)]. The result is curious if only because it lets us see just what is lost in the process. We have[42]

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}=\hat{I}_{\mathbf{p}}\Phi_{\mathbf{p}}^{\alpha}+2\pi^{2}e_{\alpha}^{4}\int\frac{|E^{w}|_{k}^{2}}{k^{2}}\,dk\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega-\mathbf{kv})\,\Phi_{\mathbf{p}}^{\alpha}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)R_{\mathbf{p},\mathbf{p}+\mathbf{q}}\,d\mathbf{q}. \tag{8.12}$$

It turns out that (8.12) contains $R_{\mathbf{p},\mathbf{p}+\mathbf{q}}$, which is the same quantity as in the correct expression, (7.9); the order of magnitude in (8.12) is the same as in (7.9). There has simply been a certain "redistribution" of terms, and in addition (and quite naturally) only the function $\Phi_{\mathbf{p}}^{\alpha}$ appears (not $\Phi_{\mathbf{p}+\mathbf{q}}^{\alpha}$, with a transfer of momentum to other particles). A redistribution of "this" type also occurred for transition scattering in the case in which the scattering by the cloud of the ion was of the order of the Thomson scattering by electrons. Actually, the scattering was caused by electrons, but by those electrons which also belonged to the cloud, so the cross section for scattering by an ion turned out to be of the order of the Thomson cross section for electrons. Here again there was a "redistribution," but in this case a redistribution of the momentum transfer. One might say that distribution of this sort are consequences of the long-range nature of Coulomb forces and, in the relativistic case, of electromagnetic forces (the Breit interaction or the interaction of the current of particles). In terms of the momentum transfer in virtual processes, this situation is completely understandable, since the products of matrix elements of the types in Figs. 8(b) and 7 require, for the appearance of a δ-function of the type in (8.1), that a virtual line of the particles in Fig. 8(b) lie close to the mass shell, while the rest of the diagram be close to the diagram for simply an interaction of particles (Fig. 2), which is a long-range effect. We should reiterate that essentially all quantities are operators in a description of fluctuations and that these arguments are simply explanations.

Incidentally, the long-range nature of the interaction is supported directly by result (7.10). In the limit $q\to 0$ the first term in (7.8) tends towards ∞ as $1/q^3$; i.e., the integral over **q** is dominated by small values of **q**. In the terms with $\Phi_{\mathbf{p}'}^{\alpha}$, on the other hand, the quantity **q** is determined by the difference between the momenta of the two particles which are giving up and acquiring the momentum $\mathbf{q},\mathbf{q}=\mathbf{p}'-\mathbf{p}$, as in the case of the ordinary Coulomb or Breit interaction. It is the fact that the terms with $\Phi_{\mathbf{p}+\mathbf{q}}^{\alpha}=\Phi_{\mathbf{p}'}^{\alpha}$ in (7.9) and (7.10) contain $R_{\mathbf{p},\mathbf{p},\mathbf{q}+\mathbf{q}}=R_{\mathbf{p},\mathbf{p}}$, which points out this circumstance. One might say that the appearance of $\Phi_{\mathbf{p}'}^{\alpha}$ is the result of a summatiom over all possibilities for obtaining photons by "different" particles of the distribution $\Phi_{\mathbf{p}'}^{\alpha}$ with the absorption at one particle with a momentum **p** [the integral over $\mathbf{p}'=\mathbf{p}+\mathbf{q}$ or over **q** in (7.9)]. A summation of this sort is a consequence of the theorem for combining probabilities.

The S matrix method can also be used for a rigorous derivationof kinetic equations (7.9). Here it is sufficient to use the operator representation[39]

$$\hat{f}_{\mathbf{p},\mathbf{q}}(t)=S^{+}f_{\mathbf{p},\mathbf{q}}^{(0)}(t)\,S \tag{8.13}$$

and to find

$$\hat{f}^{\alpha}=\Phi^{\alpha}+\delta\hat{f}^{\alpha},\quad \frac{d}{dt}\left\langle\int \mathrm{Sp}\,\hat{f}_{\mathbf{p},\mathbf{q}}^{\alpha}\right\rangle d\mathbf{q}=\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}. \tag{8.14}$$

This approach is convenient in that it uses standard renormalization procedures. All the quantities in (8.13) and (8.14) enter through Lagrangian operators, and in this sense the assertion above regarding the role of fluctuations is more apparent. Actually, diagrams have to be constructed for the quantity $\Phi_{\mathbf{p}}^{\alpha}$, which is an average over the vacuum and over the statistical ensemble of the operators defined above.

Denoting by $S^{(i)}$ the ith order of the expansion of the S matrix in the field, we see that the following approximation is sufficient for deriving a quasilinear equation:

$$\frac{d\Phi_{\mathbf{p}}^{\alpha ql}}{dt}=\frac{d}{dt}\,\mathrm{Sp}\int_{t\to\infty} d\mathbf{q}'\,(\langle\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}S^{(2)}\rangle+\langle S^{(2)+}\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}\rangle+\langle S^{(1)+}\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}S^{(1)}\rangle). \tag{8.15}$$

From (8.15) we find Eq. (8.10), without making any further assumptions regarding the nature of the balance equation.

To derive fluctuation corrections we need to use the approximation[42]

$$\frac{d\Phi_{\mathbf{p}}^{\alpha fl}}{dt}=\frac{d}{dt}\,\mathrm{Sp}\int_{t\to\infty} d\mathbf{q}'\,(\langle\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}S^{(4)}\rangle+\langle S^{(4)+}\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}\rangle+\langle S^{(1)+}\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}S^{(3)}\rangle+\langle S^{(3)+}\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}S^{(1)}\rangle+\langle S^{(2)+}\delta f_{\mathbf{p},\mathbf{q}'}^{(0)}S^{(2)}\rangle). \tag{8.16}$$

In this case we should retain in (8.16) only terms of order no higher than $|E^{w}|_{k}^{2}$. This approach leads us[39] exactly to Eqs. (7.9). We have one final comment, which concerns the structure of (7.9). The diagram in Fig. 8(a) with two incoming lines can obviously be understood in an operator sense in the expression for $\Phi_{\mathbf{p}}^{\alpha}$ in (8.14), as in the case of the diagram in Fig. 8(b). In Fig. 8(a), only a partial pairing of the internal operators has been carried out. Consequently, all ends in $\Phi_{\mathbf{p}}^{\alpha}$ are ultimately paired through $\delta f_{\mathbf{p}}^{(0)}$. The operator in Fig. 8(a) corresponds to a definite term $S^{(3)}$, while the diagram in Fig. 8(b) correspond to $S^{(1)}$; i.e., it is contained in the third or fourth term in (8.16). It gives us

$$\langle\hat{a}_{\mathbf{p}-\mathbf{q}}^{+}\gamma_{\mu}^{t}\hat{a}_{\mathbf{p}}\delta f_{\mathbf{p}}^{(0)}\hat{a}_{\mathbf{p}}^{+}\gamma_{\mu}^{t}\hat{a}_{\mathbf{p}+\mathbf{q}}\beta\hat{a}_{\mathbf{p}+\mathbf{q}-\mathbf{k}}\hat{a}_{\mathbf{p}+\mathbf{q}-\mathbf{k}}^{+}\beta\hat{a}_{\mathbf{p}-\mathbf{q}}\rangle. \tag{8.17}$$

The operators $\hat{a}_{\mathbf{p}}$ are closed through $\delta\delta f_{\mathbf{p}}^{(0)}$. The result in (8.17) is of course proportional to $\Phi_{\mathbf{p}-\mathbf{q}}^{\alpha}$; i.e., it describes an effect with a transfer of momentum to other particles. An important point is that the square matrix element for Fig. 8(b) would contain the product $\Phi_{\mathbf{p}}^{\alpha}$ and $\Phi_{\mathbf{p}-\mathbf{q}}^{\alpha}$. The product of the diagrams in Figs. 8(a) and 8(b), in contrast, should also contain a term which is linear in $\Phi_{\mathbf{p}}^{\alpha}$, since the diagram in Fig. 7 contains only a single incoming line, i.e., either $\Phi_{\mathbf{p}}^{\alpha}$ [if we add to Fig. 8(b) the same diagram, with an initial momentu of **p**] or $\Phi_{\mathbf{p}-\mathbf{q}}^{\alpha}$.

9. ELECTROMAGNETIC FLUCTUATIONS IN A SYSTEM OF PARTICLES WHICH ARE INTERACTING IN A QUASILINEAR FASHION

We will show that the momentum transfer in fluctuations occurs through fluctuation electromagnetic fields. Although this would seem to be a trivial point, a calculation of the energy of the fluctuations of electromagnetic fields of very high frequencies in the presence of a quasilinear interaction sheds light from yet another angle on the collective effect described above, in which momentum and energy are transferred from a large number of low-energy particles to a small number of high-energy (possibly very-high-energy) particles. If the energy of the electromagnetic fluctuations in a system of particles is not zero, if there are even fluctuations with very high frequencies, and if the energy of these fluctuations is furthermore a function of the time, this energy would have to be transferred to particles of one sort or another. In speaking of fluctuations in a system of particles which are interacting in a quasilinear fashion, we mean that we are considering (for simplicity) only very high-frequency fluctuations, i.e., fluctuations for which the refractive index is essentially unity. We are of course discussing only those fluctuations which are proportional to the number of particles which are interacting in a quasilinear fashion (the zero-point vacuum fluctuations do not vary in time, and no energy can be taken from them). We find the energy of the fluctuations by a perturbation theory in the approximation linear in the number of particles, $\Phi_{\mathbf{p}}^{\alpha}$, but in the general relativistic and quantum-mechanical case.

We write the following expression for the energy of the high-frequency fields:

$$W^t = \int \exp\left[i(\mathbf{k}+\mathbf{k}')\mathbf{r}\right] \frac{d\mathbf{k}\,d\mathbf{k}'}{8\pi} \times \left(\langle \hat{A}_{\mathbf{k}'}(t)\hat{A}_{\mathbf{k}}(t)\rangle \mathbf{k}^2 + \left\langle \frac{\partial \hat{A}_{\mathbf{k}'}(t)}{\partial t}\frac{\partial \hat{A}_{\mathbf{k}}(t)}{\partial t}\right\rangle \right) - 4\pi \int_{-\infty}^{t} dt' \left(\frac{\partial \hat{A}_{\mathbf{k}'}(t')}{\partial t'}\hat{j}_{\mathbf{k}}(t') + \hat{j}_{\mathbf{k}}(t')\frac{\partial \hat{A}_{\mathbf{k}'}(t')}{\partial t'} \right), \tag{9.1}$$

where the operator $\hat{A}$ is the vector potential (the Coulomb gauge). Any operator can be written in the form $\langle \hat{L}\rangle = \langle S^{+}\hat{L}^{(0)}S\rangle$, where $\hat{L}^{(0)}$ is a noninteracting operator. The contribution from $\Phi_{\mathbf{p}}^{\alpha}$ is dealt with by perturbation theory. Consequently, $\Phi_{\mathbf{p}}^{\alpha}$ appears as an expectation value of $\langle a^{+}a\rangle$ and makes the first corrections in terms of the number of particles. The operators $\hat{L}^{(0)}$ of course do not contain ε, and in the zeroth approximation the field A is the field of the zero-point oscillations. The result of an evaluation of (9.1) is[41]

$$W^t = \int \frac{d\mathbf{k}}{(2\pi)^3}\,\omega\frac{\partial}{\partial\omega}\left(\omega\varepsilon^{t(+)}(\omega,\mathbf{k}) + \omega\varepsilon^{t(-)}(-\omega,\mathbf{k})\right)\Big|_{\omega=|\mathbf{k}|}. \tag{9.2}$$

Expression (9.2) thus gives us the energy of the fluctuation fields in the presence of particles [the part which is independent of $\Phi_{\mathbf{p}}^{\alpha}$ must be discarded from (9.2); it represents zero-point vacuum fluctuations]. Naturally, in using perturbation theory we can express the fields which are linear in $\Phi_{\mathbf{p}}^{\alpha}$ in terms of the vacuum fields. The substitution $\omega = |\mathbf{k}|$ is thus used in (9.2).

Expression (9.2) contains "pieces" of the general expression for the permittivity (which incorporates relativistic and quantum-mechanical effects), which was found many years ago, by different methods, in Ref. 43:

$$\varepsilon^t(\omega,\mathbf{k}) = 1 + \varepsilon^{t(+)}(\omega,\mathbf{k}) + \varepsilon^{t(+)}(-\omega,\mathbf{k}) + \varepsilon^{t(-)}(\omega,\mathbf{k}) + \varepsilon^{t(-)}(-\omega,\mathbf{k}), \tag{9.3}$$

where

$$\varepsilon^{t(\pm)}(\omega,\mathbf{k}) = 1 \mp \sum_{\alpha}\frac{\pi e_{\alpha}^2}{\omega^2}\int\frac{\Phi_{\mathbf{p}}^{\alpha}\,d\mathbf{p}}{(2\pi)^3}\frac{\mathrm{Sp}\left[\Lambda_{\mathbf{p}}^{+}\left(\boldsymbol{\gamma}-\frac{\mathbf{k}(\mathbf{k}\boldsymbol{\gamma})}{k^2}\right)\Lambda_{\mathbf{p}-\mathbf{k}}^{\pm}\left(\boldsymbol{\gamma}-\frac{\mathbf{k}(\mathbf{k}\boldsymbol{\gamma})}{k^2}\right)\right]}{\omega-\varepsilon_{\mathbf{p}}\pm\varepsilon_{\mathbf{p}-\mathbf{k}}}. \tag{9.4}$$

The expression for ε^t which was derived in Ref. 43 was found by a Green's-function method, and in a slightly different form. It is easy to verify, however, that that other expression is the same as (9.3).

For yet another test of the results of all the calculations, we can calculate the energy W^t in the classical limit, under conditions such that there are transverse electromagnetic waves, which are described by the number of photons, $N_{\mathbf{k}} \gg 1$.

We find

$$\frac{\partial W^t}{\partial t} = \int \frac{N_{\mathbf{k}}^t\,d\mathbf{k}}{(2\pi)^3}\,\omega\frac{\partial}{\partial\omega}\,\omega\,\frac{\partial\varepsilon^t(\omega,\mathbf{k})}{\partial t}\Big|_{\omega=|\mathbf{k}|}. \tag{9.5}$$

The complete expression for ε^t appears here, as in (9.3)—not simply the pieces $\varepsilon^{t(+)}(\omega,\mathbf{k})$ and $\varepsilon^{t(-)}(\omega,\mathbf{k})$ of this expression. It is clear why only pieces of ε^t are involved in the absence of photons: A photon must first be emitted and then absorbed, but when photons are present there can also be an absorption followed by an emission.

We now consider how the energy in (9.2) varies in time because of the quasilinear variation in $\Phi_{\mathbf{p}}^{\alpha}$ in time in the classical limit. The quantity W^t depends on $\Phi_{\mathbf{p}}^{\alpha}$. However, differentiating W^t with respect to the time and substituting $d\Phi_{\mathbf{p}}^{\alpha}/dt$ from the quasilinear equation is not sufficient to describe the entire effect.

This fact was established in the nonrelativistic, nonquantum limit for the case of a large number of waves, $N_{\mathbf{k}}^t \gg 1$ (by a classical approach), in Ref. 44 (see also Ref. 45). The essence of the matter is that the derivative dW^t/dt found from (9.5) is proportional to $N_{\mathbf{k}}^t$ and $|E^{\mathrm{w}}|_k^2$. In other words, it corresponds to a nonlinear process which is proportional to both the intensity of the electromagnetic waves, $\sim N_{\mathbf{k}}^t$, and the intensity of the resonant longitudinal fields, $|E^{\mathrm{w}}|_k^2$. In the calculation of W^t, however, the only terms which were taken into account in the currents **j** were those linear in the amplitudes of the electromagnetic fields. In general, **j** would also contain nonlinear terms, proportional to both the amplitude of the electromagnetic field and the amplitude of the resonant field. This additional nonlinear component must also be taken into account. It, together with the result found by differentiating W^t with respect to the time (and using the quasilinear equation), gives us the conserved numbers of electromagnetic-wave photons in the high-frequency limit. To show how this result can be generalized to the relativistic, quantum-mechanical case, we rewrite (9.5) as

$$\frac{dW^{t,ql}}{dt}=\int\frac{N_{\mathbf{k}}^{t}\,d\mathbf{k}}{(2\pi)^3}\left(-\omega\frac{\partial\varepsilon^t(\omega,\mathbf{k})}{\partial t}+\frac{\partial}{\partial\omega}\omega^2\frac{\partial\varepsilon^t(\omega,\mathbf{k})}{\partial t}\right)_{\omega=|\mathbf{k}|}. \tag{9.6}$$

Substituting (9.4) into the second term, using quasilinear equation (3.24) for $d\Phi_{\mathbf{p}}^{\alpha}/dt$, incorporating the nonlinear terms in the S matrix in (9.1), and adding the results, we find that the second term in (9.6) is canceled out exactly by the nonlinear terms, and the final result is

$$\frac{dW^{t,\mathrm{tot}}}{dt}=\frac{dW^{t,\mathrm{ql}}}{dt}+\frac{dW^{t,\mathrm{Nl}}}{dt}=\int\frac{N_{\mathbf{k}}^{t}\,d\mathbf{k}}{(2\pi)^3}\left(-\omega\frac{\partial\varepsilon^t(\omega,\mathbf{k})}{\partial t}\right)_{\omega=|\mathbf{k}|}. \tag{9.7}$$

We note that the terms with $\Phi_{\mathbf{p}}^{\alpha}$ are being dealt with in first-order perturbation theory. We use the dispersion relation

$$\omega_{\mathbf{k}}^2(t)\,\varepsilon^t(\omega_{\mathbf{k}}(t),\mathbf{k},t)=\mathbf{k}^2. \tag{9.8}$$

In the adiabatic limit we have

$$\frac{\partial}{\partial\omega}\omega^2\varepsilon^t(\omega,\mathbf{k},t)\Big|_{\omega=\omega_{\mathbf{k}}}\frac{d\omega_{\mathbf{k}}(t)}{dt}+\omega_{\mathbf{k}}^2(t)\frac{\partial\varepsilon^t(\omega_{\mathbf{k}}(t),\mathbf{k},t)}{\partial t}=0. \tag{9.9}$$

Noting that $d\omega_{\mathbf{k}}(t)/dt$ depends on $\Phi_{\mathbf{p}}^{\alpha}$, we must set $\varepsilon^t=1$ in the coefficients of the first term in (9.9) in this approximation, but in the second term we must set $\omega_{\mathbf{k}}=|\mathbf{k}|$. We then find $\omega_{\mathbf{k}}=|\mathbf{k}|+\delta\omega_{\mathbf{k}}(t)$ and

$$\frac{d}{dt}\delta\omega_{\mathbf{k}}(t)=-\omega\frac{\partial\varepsilon^t(\omega,\mathbf{k},t)}{\partial t}\Big|_{\omega=|\mathbf{k}|}, \tag{9.10}$$

so the field energy in (9.7) varies only to the extent that the frequency varies. In other words, the number of photons, $N_{\mathbf{k}}$, is conserved.

We now assume $N_{\mathbf{k}}=0$. This is the case in which we are interested; we discussed the case with $N_{\mathbf{k}}\neq 0$ for illustration. Our starting point in this case should be relation (9.2). Taking the nonlinear effects and the renormalization into account, we find[41]

$$\frac{dW^{t,\mathrm{tot}}}{dt}=-\sum_{\alpha}\pi^2e_{\alpha}^4\int\frac{d\mathbf{p}\,d\mathbf{q}\,dk}{(2\pi)^6}\frac{|E^{\mathrm{w}}|_k^2}{k^2}(\varepsilon_{\mathbf{p+q}}-\varepsilon_{\mathbf{p}})R_{\mathbf{p,p+q}}\times\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega-\mathbf{kv})\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\Phi_{\mathbf{p}}^{\alpha}. \tag{9.11}$$

Here we see the same expression for $R_{\mathbf{p,p+q}}$ as was found above in Secs. 7, 8. It is easy to verify that this change in the energy is the same as the energy which is acquired by the particles in accordance with the first two terms of Eq. (7.9). The last two terms of (7.9) make a contribution which is associated with an additional change in the energy of the longitudinal fields. A calculation of this energy leads to[41]

$$\frac{dW^l}{dt}=-\sum_{\alpha}e_{\alpha}^4\pi^2\int\frac{d\mathbf{p}\,d\mathbf{q}\,dk}{(2\pi)^6}\frac{|E^{\mathrm{w}}|_k^2}{k^2}\omega\delta(\omega-\mathbf{kv})\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\times(R_{\mathbf{p,p+q}}\Phi_{\mathbf{p}}^{\alpha}-R_{\mathbf{p+q,p}}\Phi_{\mathbf{p+q}}^{\alpha}). \tag{9.12}$$

If the number of low-energy quasilinear particles, $\Phi_{\mathbf{p+q}}^{\alpha}$, is large, while the number of fast particles is very small, $\Phi_{\mathbf{p+q}}^{\alpha}\gg\Phi_{\mathbf{p}}^{\alpha}$, the generation of high-energy particles in (7.12) and (7.9) can be described by the first term of the equation:

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}\simeq\pi e_{\alpha}^2\int R_{\mathbf{p',p}}\hat{I}_{\mathbf{p'}}^{\mathrm{ql}}\Phi_{\mathbf{p'}}^{\alpha}\frac{d\mathbf{p'}}{(2\pi)^3}. \tag{9.13}$$

The asymptotic expression for $R_{\mathbf{p',p}}$ for $p\gg p'$ and $p\gg m$ gives us, in the isotropic case,

$$R_{\mathbf{p',p}}=\frac{R_{\mathbf{p'}}}{p^5},$$

i.e.,

$$\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}=\frac{\pi e_{\alpha}^2}{p^5}\int R_{\mathbf{p'}}\hat{I}_{\mathbf{p'}}^{\mathrm{ql}}\Phi_{\mathbf{p'}}^{\alpha}\frac{d\mathbf{p'}}{(2\pi)^3}. \tag{9.14}$$

We have found a specific expression for $R_{\mathbf{p'}}$. That result, however, is not of particular importance; what is of importance is that the integral in (9.14) depends on neither the details of the distribution of resonant fields nor the details of th particle distribution $\Phi_{\mathbf{p}}^{\alpha}$. In other words, result (9.14) is of universal applicability.

The energy spectrum found in the ultrarelativistic limit,

$$\frac{d\Phi_{\varepsilon}^{\alpha}}{dt}\sim p^2\frac{d\Phi_{\mathbf{p}}^{\alpha}}{dt}\sim\frac{1}{\varepsilon^3},\qquad\varepsilon\approx p, \tag{9.15}$$

is very nearly the same as the observed spectrum of cosmic-ray electrons and ions. These questions are discussed in more detail in Refs. 12, 14, and 15. Here we would like to emphasize that the power-law nature of spectrum (9.14) is a direct consequence of the power-law dependence of the energy spectrum of the electromagnetic fluctuations on the frequency $\omega=|\mathbf{k}|$ [see (9.11)].

The total energy which is transferred to the fast particles is not large. It cannot be calculated from the first term in (7.9) (the result diverges at small values of $\mathbf{p}$); the second term in (7.9) must also be taken into account. Assuming that the distribution of the resonant particles $\mathbf{p}'$ is nonrelativistic, we find ($\hbar\neq 1$, $c\neq 1$)

$$\frac{dE^{\mathrm{fl}}}{dt}=\int\frac{d\Phi_{\mathbf{p}}^{\mathrm{fl}}}{dt}\varepsilon_p\frac{dp}{(2\pi)^3}\approx\frac{8e_{\alpha}^2}{3\pi\hbar c}\left(\ln 2-\frac{11}{24}\right)\frac{dE^R}{dt}, \tag{9.16}$$

where

$$E^R=\int\frac{p'^2}{2m}\frac{\Phi_{\mathbf{p}'}^{\alpha}d\mathbf{p}'}{(2\pi)^3}$$

is the energy of the resonant particles. Since the average energy of the fast particles is of the order mc^2, their density is smaller by a factor of $10^{-3}v_T^2/c^2$ than the density of the resonant particles (v_T is their thermal velocity). The entire effect is therefore of the order of $e_{\alpha}^2/\hbar c$, and the generation of the tail is even weaker, of the order of $e_{\alpha}^2/v_T^2/\hbar c^3$.

A collective effect of this sort, involving a transfer of momentum and energy to the particles of a tail, is nevertheless important because of its universal nature and because of the power-law nature of the energy distribution of the fast particles. This distribution falls off fairly slowly (in comparison with an exponential function) with the energy, and we can expect to see observable effects stemming from the degradation of the plasma confinement, dragging currents, etc.[13]

Finally, we should say a few words about the theory of measurements as it applies to the description of particles by means of the equations which we have been using. The quantity $\Phi_{\mathbf{p}}^{\alpha}$ has a completely unambiguous and measurable meaning: the expectation value of an occupation number. At any rate, this point is particularly transparent for noninteracting particles. In the presence of an interaction, the corresponding generalization is $\Phi_{\mathbf{p}}^{\alpha} = \int \mathrm{Sp}\langle \hat{f}_{\mathbf{p},\mathbf{k}}^{\alpha} \rangle d\mathbf{k}$. An important point, however, is that after the interaction $\Phi_{\mathbf{p}}^{\alpha}$ recovers its previous meaning for free particles. In this regard the appearance of accelerated particles after a quasilinear interaction which operates for a finite time interval is a completely clear and unambiguous prediction. With regard to $\Phi_{\mathbf{p}}^{\alpha}$ during the actual interaction with the resonant field, we note that a theory of quantum measurements has not been developed for this case; this is a matter for future work.

10. CONCLUSION

This entire description tells us that an ensemble of particles such as a rarefied plasma behaves in a manner which is completely different from that of individual particles. The changes are seen as a radical change in the cross sections for various processes, a combining of particles into particle-plus-cloud complexes, and a transfer of energy to other particles. In our opinion, a correct description of all the processes would hardly be possible by any approach other than one incorporating fluctuations, as has been demonstrated.

An important point is that the diagrams which actually arise correspond to definite expansions in charges and field (but not to an expansion in e^2n in transition scattering and transition bremsstrahlung). The reason why such an expansion is possible is that a plasma is a system of weakly interacting particles, and a theory of this sort, involving an expansion in fluctuation and collective fields, is appropriate for the problem. We wish to stress that all the results are valid for a highly nonequilibrium plasma, with arbitrary distributions of particles, $\Phi_{\mathbf{p}}^{\alpha}$, and of collective fields, $|E^{w}|_{k}^{2}$.

All these results could of course have been derived by a Green's-function method.[46] However, the generalization of the corresponding Keldysh equations[46,47] to the relativistic, quantum-mechanical case is complicated, and the use of the diagram technique here could hardly be of much assistance in seeing the physics of the situation, although that approach would have been more elegant from the mathematical standpoint. Here we have chosen a simpler method for presenting the material.

Finally, under conditions of thermal equilibrium one could of course use the standard Matsubara technique.[48,49] In that case it is curious to note how the contributions from the nonlinear vertices associated with transition scattering and transition bremsstrahlung are modified (they become thermal fluctuations). To the best of our knowledge, a reformulation of the Matsubara Green's-function technique of that sort has so far attracted little interest.

We wish to stress that taking an average over fluctuations leads to a new and informative picture of a system of charged particles. For emission, scattering and collision processes, the system consists in a sense of "neutral atoms" which are surrounded by dynamically polarized clouds. Both the electrons and the ions have such clouds, and the number of dressed particles (electrons and ions) is equal to the number of these particles in the system. In the processes of scattering and bremsstrahlung involving heavy particles, the polarization of these clouds may play an important role (transition scattering and transition bremsstrahlung), but the recoil momentum is acquired by the "central particle" which is scattered by them or, more precisely, by the dressed particle as a whole. It is for this reason that the ions in a plasma may scatter waves with a cross section of the order of the Thomson cross section for scattering of electrons in vacuum. In precisely the same way, the transfer of momentum and energy in radiation effects may ocur to "a different" particle, and the radiation effects in a system of charged particles may thus differ substantially from those in vacuum for individual charged particles in external fields. Here we are seeing a manifestation of collective processes in a system of many charged particles.

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