Polarization waves and super-radiance in active media

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The macroscopic electrodynamics of time-dependent coherent processes in active media is reviewed. Dicke super-radiance in the two-level model and cyclotron super-radiance in a model consisting of an electron beam in a magnetic field are investigated. Similar phenomena in other media are discussed. Particular attention is devoted to polarization waves, i.e., a type of normal wave that coexists with electromagnetic waves and in many ways determines the character of coherent effects. Super-radiance is considered as a dissipative instability of negative-energy polarization waves, which occurs in an active sample as a result of losses by emission of positive-energy electromagnetic waves into the ambient space. The method of phenomenological quantization is developed for unstable modes in active samples, and directly describes the quantum-mechanical properties of collective excitations. The procedure is used to analyze macroscopic quantum-mechanical fluctuations of super-radiance.

1. INTRODUCTION. CONTINUUM ELECTRODYNAMICS AS A METHOD OF DEALING WITH PROBLEMS IN QUANTUM RADIOPHYSICS AND OPTICS

1.1. Collective coherent processes in active media

Classic textbooks on macroscopic electrodynamics[1-6] are mostly devoted to transparent or absorbing equilibrium media. On the other hand, the enormous interest in wave instabilities in modern physics, i.e., in the physics of lasers, nonlinear optics, electronics, solid state physics, plasma physics, and astrophysics, has gradually identified the fundamentals of the general electrodynamics of nonequilibrium media. Our review is concerned with that part of the electrodynamics of active continuous media that deals with coherent processes in media consisting of excited particles (molecules). Until quite recently, this part of electrodynamics was essentially confined to processes that were slow in comparison with the relaxation times in the medium, and it is only now that the concepts and methods of macroscopic electrodynamics, including the phenomenological approach, have began to permeate research into fast transient processes occurring faster than energy (T_1) and phase (T_2) relaxation in active particles of the medium, i.e.,

$$\Delta t \ll T_1,\ T_2. \qquad (1.1)$$

In contrast to slow processes, described by the inequality opposite to (1.1), such coherent processes cannot be described by rate equations with time-dependent radiation intensity and time dependent stimulated transition probability. In this respect, collective processes involving a large number of particles of the medium are particularly interesting. Dicke super-radiance was among the first of these processes to be predicted and discovered experimentally, and a considerable proportion of the present review is devoted to the methods of macroscopic electrodynamics as applied to this phenomenon.

The microscopic method, which starts with the quantum electrodynamics of cold field modes and individual molecules in vacuum (Sect. 1.2), can in principle take into account all the effects of the interaction between molecules and the field. In particular, this applies to effects due to the spatial inhomogeneity of radiation and the internal energy of particles in the active medium.[13b] However, considerable difficulties are encountered in the transition from micro to macro characterization of the relevant processes, and a number of simplifications and approximations have had to be resorted to. The direct numerical method of investigation is difficult to use as a basis for developing physical ideas capable of describing and explaining these phenomena. Neither approach can reveal the place of different collective coherent processes in quantum radiophysics and optics in the overall picture of unstable wave processes, or point to analogous processes in other branches of physics.

The macroscopic approach of continuum electrodynamics simplifies the solution of these problems and often provides a unified physical interpretation of the phenomena under consideration.[1)] It effectively utilizes the concepts of permittivity, dispersion, anisotropy, energy-momentum flux, normal waves (modes), phase and group velocities, and convective and absolute instabilities in problems of growth rate, the sign of energy, and the linear and nonlinear interaction between waves. It does so by applying, whenever possible, the Hamiltonian method, the method of Green's functions, the phenomenological quantization of collective excitations in the active medium, and so on. Problems treated by the continuum electrodynamics of active media also include the reflection, refraction, and propagation of normal waves in inhomogeneous and time-dependent media, the emission of radiation by particles moving in such media, the scattering of waves, the van der Waals interaction, and the different processes that occur in cavity resonators and waveguides filled with active media. The lack of the usual general phenomenological relationships for the active media has meant that models of such media have had to be introduced.

In this paper, we shall review the results achieved in the macroscopic electrodynamics of coherent processes (above all super-radiance) by using the very popular two-level model of a medium (Sects. 1–6) and the model consisting of an electron beam in a magnetic field (Sect. 7). Similar processes in other active media will be briefly reviewed in Sect. 8. Our presentation will focus on the concept of polarization waves, i.e., a particular type of normal wave that coexists with normal electromagnetic modes in an active medium and in many ways determines the picture of collective coherent processes and super-radiance.

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1.2. Quantum electrodynamics of cold modes and individual molecules in vacuum

Consider a sample of volume V that consists of two-level molecules with concentration N. Transitions between energy levels 1 and 2 of a molecule are characterized by a frequency ω_0 and dipole moment $\mathbf{d}$. In the *microscopic approach*,[5,9-21] we directly take into account the interaction of each of the molecules ($l = 1,2,...,NV$) with each free-space mode $\mathbf{g_k}(\mathbf{r}) = (2\pi\hbar c^2/\omega_\mathbf{k}\mathscr{V})^{1/2}\mathbf{e_k}\exp(i\mathbf{kr})$ of frequency $\omega_\mathbf{k} = \omega(\mathbf{k})$, wave vector $\mathbf{k}$, and polarization unit vector $\mathbf{e_k}\perp\mathbf{k}$ (the quantization volume is $\mathscr{V}\to\infty$). The dynamics of the molecules + field system is determined by the Hamiltonian

$$\hat{H} = \sum_\mathbf{k} \hbar\omega_\mathbf{k}\left(\hat{a}_\mathbf{k}^+\hat{a}_\mathbf{k} + \frac{1}{2}\right) + \sum_{l=1}^{NV}(\hbar\omega_0\hat{R}_{l3} - i\omega_0 c^{-1}\mathbf{d}_l\hat{\mathbf{A}}_l(\hat{R}_{l+} - \hat{R}_{l-})) \qquad (1.2)$$

via Heisenberg equations of the form $\hbar d\hat{G}/dt = i[H,\hat{G}]$. Hence, the photon creation and annihilation operators ($a_\mathbf{k}^+$ and $\hat{a}_\mathbf{k}$) and the operators for the excited states of the molecules ($\hat{R}_{l+}$ and $\hat{R}_{l-}$) obey the equations

$$\frac{d\hat{a}_\mathbf{k}^+}{dt} = i\omega_\mathbf{k}\hat{a}_\mathbf{k}^+ + \omega_0\hbar^{-1}c^{-1}\sum_{l=1}^{NV} g_\mathbf{k}(\mathbf{r}_l)\,\mathbf{d}_l(\hat{R}_{l+} - \hat{R}_{l-}), \qquad (1.3)$$

$$\frac{d\hat{R}_{l+}}{dt} = i\omega_0\hat{R}_{l+} + 2\omega_0\hbar^{-1}c^{-1}\hat{R}_{l3}\,\mathbf{d}_l\hat{\mathbf{A}}_l, \qquad (1.4)$$

$$\frac{d\hat{R}_{l3}}{dt} = -\omega_0\hbar^{-1}c^{-1}(\hat{R}_{l+} + \hat{R}_{l-})\,\mathbf{d}_l\hat{\mathbf{A}}_l; \qquad (1.5)$$

where $\mathbf{d}_l(\hat{R}_{l+} + \hat{R}_{l-}) = \hat{\mathbf{d}}_l$ is the dipole moment operator, $\hat{\mathbf{A}}_l = \Sigma_\mathbf{k}(\hat{a}_\mathbf{k}\mathbf{g_k} + \hat{a}_\mathbf{k}^+\mathbf{g}_\mathbf{k}^*)|_{r=r_l}$ is the vector potential operator at the point occupied by the lth molecule $\mathbf{r}_l$, and

$$\vec{\mathscr{E}} = i\sum_\mathbf{k}(\hat{a}_\mathbf{k}\mathbf{g_k} - \hat{a}_\mathbf{k}^+\mathbf{g}_\mathbf{k}^*)\,\omega_\mathbf{k}c^{-1} \qquad (1.6)$$

is the electric field. According to (1.3) and (1.4), there is a linear relationship between the field oscillators and the molecular polarization oscillators, i.e., they are partial and not normal. The variation in the operator for the half-difference between the populations of the lth molecule, $\hat{R}_{l3}$, which is described by (1.5), characterizes the nonlinear relationship between these oscillators. The creation and annihilation operators satisfy the canonical commutation relations

$$[\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k}'}^+] = \delta_{\mathbf{kk}'},\quad [\hat{a}_\mathbf{k}^+, \hat{a}_{\mathbf{k}'}^+] = [\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k}'}] = 0, \qquad (1.7)$$

$$[\hat{R}_{l3}, R_{l'\pm}] = \pm\,\delta_{ll'}\hat{R}_{l\pm},\quad [\hat{R}_{l+}, \hat{R}_{l'-}] = 2\delta_{ll'}\hat{R}_{l3}; \qquad (1.8)$$

and the field operators commute with the molecular operators $\delta_{ll'}$ is the Kronecker symbol).

In the quantum-electrodynamic approach, the individual molecules and cold modes of free space are not directly related either to the geometry of the macro system or the collective excitations within it. In this approach, it is difficult to take account of diffraction effects and variations in field, polarization, and inversion over the specimen,[13b] so that one has to resort to some form of the "mean field approximation"[10-18]. Nevertheless, the approach is very widely used in quantum radiophysics and optics, and has led to a number of new important results. For example, the evolution of induced intermolecular correlations during the spontaneous formation of a super-radiant state by photon exchange was examined explicitly in Ref. 21.

1.3. The semiclassical approximation. Normal waves in a medium

A different approach is employed in macroscopic electrodynamics. Here we start with the semiclassical equations for the interaction between the field and the continuous medium, i.e., the classical Maxwell equations, and the equations for the mean polarization $\mathscr{P}$ and population difference $\Delta N = N_2 - N_1$ per unit volume that follow from the quantum-mechanical description of the two-level medium:[11,22]

$$\mathrm{rot}\,\vec{\mathscr{E}} = -c^{-1}\frac{\partial\vec{\mathscr{B}}}{\partial t},\quad \mathrm{rot}\,\vec{\mathscr{B}} = c^{-1}\frac{\partial(\vec{\mathscr{E}} + 4\pi\vec{\mathscr{P}})}{\partial t} + 4\pi\sigma c^{-1}\vec{\mathscr{E}}, \qquad (1.9)$$

$$\frac{\partial^2\vec{\mathscr{P}}}{\partial t^2} + 2T_2^{-1}\frac{\partial\vec{\mathscr{P}}}{\partial t} + (\omega_0^2 + T_2^{-2})\vec{\mathscr{P}} = \frac{\omega_c^2\vec{\mathscr{E}}}{4\pi}, \qquad (1.10)$$

$$\frac{\partial\Delta N}{\partial t} = -(\Delta N - \Delta N_p)T_1^{-1} + 2\hbar^{-1}\omega_0^{-1}\vec{\mathscr{E}}\frac{\partial\vec{\mathscr{P}}}{\partial t}. \qquad (1.11)$$

Possible resistive dissipation due to the conductivity σ of the "background" medium is taken into account in (1.9) (for bounded samples, there are similar diffractive losses; see Sects. 3.1 and 4.2). The coupling coefficient between the polarization and the field in (1.10) is given by the square of the "cooperative" frequency of the medium[2)]

$$\omega_c^2 = -8\pi d^2\Delta N\omega_0\hbar^{-1}. \qquad (1.12)$$

In an inverted medium we have $\Delta N > 0$ and $\omega_c^2 < 0$. As a rule, in real samples, the cooperative frequency is low as compared with the transition frequency, i.e., $|\omega_c| \ll \omega_0$.

The usual procedure is to employ the rotating wave approximation (RWA),[12,23] i.e., the equations are truncated at high optical frequency ω_0. The resulting Maxwell-Bloch equations for the slowly-varying population difference ΔN and the complex amplitudes of the field $\mathbf{E}$ and polarization $\mathbf{P}$ have the following form in the case of plane waves $\propto\exp(-i\omega_0 t + i\omega_0 z/c)$:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z} + 2\pi\sigma\right)\mathbf{E} = 2\pi i\omega_0\mathbf{P}, \qquad (1.9')$$

$$\left(\frac{\partial}{\partial t} + T_2^{-1}\right)\mathbf{P} = i\omega_c^2(8\pi\omega_0)^{-1}\mathbf{E}, \qquad (1.10')$$

$$\frac{\partial\Delta N}{\partial t} = -(\Delta N - \Delta N_p)T_1^{-1} + \mathrm{Im}(\mathbf{E}^*\mathbf{P})\hbar^{-1}. \qquad (1.11')$$

The derivation of these equations is discussed, for example, in Refs. 13–15 and 20. The semiclassical equations describe only quantities averaged over the ensemble, or over a macroscopically large volume, and do not directly include the quantum-mechanical fluctuations. However, phenomenological second quantization of these macro equations offers an effective method of analyzing quantum-mechanical fluctuations that does not require direct solution of the original microscopic equations (see Sect. 5).

We emphasize the classical origin of not only the Maxwell equations (1.9), but also of the equation for the population difference (1.11). The latter describes the variation in the energy density $\hbar\omega_0\Delta N/2$ in the medium that is due to the work done $\mathscr{E}\partial\mathscr{P}/\partial t$ by the electric field $\mathscr{E}$ on the current density $\partial\mathscr{P}/\partial t$ and the effect of the pump $\hbar\omega_0\Delta N_P/2$. The quantum-mechanical properties of molecules determine only the specific constitutive equation of the medium (1.10); in the two-level model, it takes the form of the equation for the damped oscillator excited by an electric force. The initial conditions for the field and the polarization, which are determined by the quantum-mechanical fluctuations in the absence of the microfield, are also of quantum-mechanical origin. In all other respects, the evolution of macro observables (fields, polarizations, energies, etc.) of systems consisting of a large number of particles is described by the classical electrodynamics of continuous media, as expected for a macroscopic specimen in the presence of a large number of photons.

For slow processes obeying the inequality opposite to (1.1), e.g., for quasistationary generation in lasers, we can transform from the semiclassical equations (1.9)–(1.11) to the simpler and easier to interpret rate equations[12,22–27] by neglecting the intrinsic space-time dynamics of oscillations in the polarization of the medium. However, for fast coherent processes (1.1), which include super-radiance, this simplification cannot be justified.

According to (1.10), the linear response (for ω_c = const) to the harmonic field

$$\mathscr{E}=\frac{1}{2}E\exp(-i\omega t+i\mathbf{k}\mathbf{r})+(\text{c.c.}) \tag{1.13}$$

in an infinite two-level system is characterized by the susceptibility

$$\chi(\omega)\equiv PE^{-1}=-\omega_c^2\{4\pi[(\omega+iT_2^{-1})^2-\omega_0^2]\}^{-1}. \tag{1.14}$$

For the permittivity, which is related to $\chi(\omega)$, we find near resonance that

$$\varepsilon(\omega)=1+i4\pi\sigma\omega^{-1}+4\pi\chi$$
$$\approx 1+i4\pi\sigma\omega_0^{-1}-\omega_c^2[2\omega_0(\omega-\omega_0+iT_2^{-1})]^{-1},\quad \omega\approx\omega_0. \tag{1.15}$$

This expression determines the properties of normal waves, i.e., the harmonic eigensolutions of the linearized equations of electrodynamics in the medium.[1–5] The dispersion relation satisfied by $\omega(\mathbf{k})$ for transverse and longitudinal normal waves in an isotropic medium is defined by the following two equations:

$$\omega^2\varepsilon=c^2k^2\quad(\mathbf{E}\perp\mathbf{k}), \tag{1.16}$$

$$\varepsilon=0\quad(\mathbf{E}\parallel\mathbf{k}). \tag{1.17}$$

It is precisely these normal waves (photons in the medium) that constitute the collective excitations. In solid state physics they are called polaritons (lumoexcitons) and plasmons. A significant point that must be noted[29] is the qualitative difference between normal waves in highly inverted ($-\omega_c^2 \gg 4T_2^{-2}$) and uninverted or weakly inverted ($-\omega_c^2 \ll 4T_2^{-2}$) media. The last case usually occurs in lasers. Strong inversion in gases or solids has been achieved only very recently, e.g., in super-radiance experiments. The qualitative change in the spectral, energy, and other characteristics of normal waves in highly inverted media, including the appearance of negative-energy polarization waves, is responsible for the observed properties of collective coherent processes. This will be discussed below.

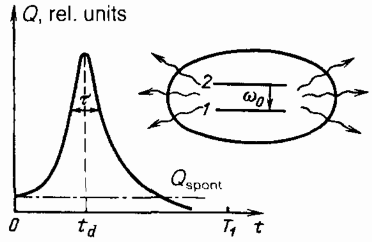

FIG. 1. Super-radiance from the $2\to1$ transition of frequency ω_0 and radiated power $Q \ll Q_{spont}$.

1.4. Dicke super-radiance: spontaneous collective emission

Super-radiance was predicted by Dicke[10] in 1954 and was observed experimentally in the infrared and optical ranges in gases[13–15,23,30,31] and in activated crystals[32–35]; it has also been seen in the radiofrequency range in nonequilibrium spin systems.[169] The effect occurs in macroscopic samples with high enough concentration N of molecules that have been inverted in a preliminary step. The internal energy of the molecules in the sample is spontaneously emitted in the form of a short electromagnetic pulse whose power QV exceeds by several orders of magnitude the power transported by incoherent spontaneous radiation emitted by the same number of isolated molecules (Fig. 1). The formation of the phased super-radiant state of a system of molecules occurs with a delay time t_d that exceeds the pulse length τ by roughly an order of magnitude. Initially ($t \ll t_d$), the phasing process is quantum-mechanical. However, when a sufficiently large number of photons has been produced, the electromagnetic field and the polarization of the medium become classical in character, and super-radiance can be treated in the semiclassical approximation.[1,14,23,27,28] When we speak of a spontaneous process under these conditions, we have in mind only the corresponding formulation of the problem in the absence of the external or initial macrofield, whereas in relation to each individual molecule we are, of course, concerned with stimulated emission under the influence of the collective, self-consistent field of all the other molecules.

We shall now illustrate the approach based on macroscopic electrodynamics to the description of coherent collective processes by considering the simple example of super-radiance by a spherical particle. First, we shall show that, for a particle of an inverted medium of small radius $a \ll \lambda_0 = 2\pi c/\omega_0$, in which relaxation can be neglected ($T_2^{-1}=\sigma=0$), the initial high-frequency polarization $\mathscr{P}(t=0)$ grows exponentially at the rate $\omega'' \propto NV$. To solve this initial-value problem, we use the well-known solutions of the electrodynamic problem of emission by a high-frequency point dipole $\mathscr{P}V$ in a vacuum and the electrostatic problem of polarization of a sphere in an external field. The former solution[11] shows that the particle is in the quasistatic radiation reaction field $\mathscr{E}_{\text{rad}}=(2V/3c^3)d^3\mathscr{P}/dt^3$, whereas the second solution[1] shows that the resultant field inside the sphere is $\mathscr{E}=\mathscr{E}_{\text{rad}}-(4\pi/3)\mathscr{P}$. For dielectrics with $\varepsilon\mathscr{E}\equiv\mathscr{D}=\mathscr{E}+4\pi\mathscr{P}$ in electrostatics ($\omega\to0$), this leads to $\mathscr{P}=\mathscr{E}_{\text{rad}}3(\varepsilon-1)/4\pi(\varepsilon+2)$ and shows that the permittivity has the resonant value $\bar{\varepsilon} = -2$ for which the sphere can become self-excited, i.e., it can assume finite uniform polarization in any field, however weak. If we now substitute for the field $\mathscr{E}$ in the constitutive equation (1.10), neglect the Lorentz correction to the field acting on the molecules (see Sect. 6.1), and put $\mathscr{P} = 1/2P\exp(-i\omega t) + \text{c.c.}$, we obtain the following characteristic equation with fixed inversion ΔN for the complex frequency $\omega = \omega' + i\omega''$ of an unstable mode in the particle: $\varepsilon(\omega) = -2 - 3i\omega^3V/2\pi c^3$ [see (1.15)]. Its solution gives the required growth rate:

$$\omega'' \approx -2\pi^2\omega_c^2V(3\omega_0\lambda_0^3)^{-1}, \quad \omega' \approx \omega_0. \tag{1.18}$$

To determine the type of instability, we must obtain the growth rate (1.18) by an energy method. The energy density in the particle is determined by the well-known electrodynamic formula for dispersive media[1-3]:

$$w = \frac{|E|^2}{16\pi}\frac{d\omega\varepsilon(\omega)}{d\omega} \approx \frac{|E|^2}{8\pi}\cdot\frac{9\omega_0^2}{\omega_c^2} < 0. \tag{1.19}$$

The resonant value $\bar{\varepsilon} = -2$ is assumed for the sphere in this equation. As can be seen, the inverted medium has a negative energy (because of the large negative contribution of the energy associated with polarization oscillations), i.e., the energy of the medium + field system in the presence of the h.f. oscillations is less than in the absence of these oscillations. Moreover, because of the radiative loss $QV = \mathscr{E}_{\text{rad}}V\,d\mathscr{P}/dt$ by the h.f. dipole into the ambient space, the energy of the particle becomes more and more negative, and its absolute magnitude increases: $dw/dt = -Q$. This in turn signifies an increase in the amplitude of the polarization oscillations in the particle. Substituting the loss power density $Q = \omega_0^4|P|^2V/3c^3 > 0$ into this expression, and recalling that the field $E \approx -(4\pi/3)P$ is large in comparison with E_{rad}, we find that the growth rate $\omega'' = -Q/2\omega$ is again given by (1.18). The instability of systems with negative oscillatory energy, which arises when this energy loss is present, is referred to as the *dissipative instability*.[37,38-41] The special instabilities that occur in the case of super-radiance[29,36] are due to the fact that radiative loss introduces dissipation into the system.

In the adiabatic approximation, the subsequent dynamics of the system, i.e., its super-radiance, is described by the following equations:

$$\frac{dQ}{dt} = 2\omega''Q, \quad \frac{d\,\Delta N}{dt} = -\frac{2Q}{\hbar\omega_0}, \tag{1.20}$$

where the coefficient ω'' on the nonlinear stage is given by expression (1.18) for the linear growth rate, in which the inversion ΔN is regarded as time dependent. The function $\Delta N(t)$ decreases during this stage and then changes sign as the molecules undergo a transition from the upper to the lower energy level. The solution of (1.20) leads to the following well-known shape of the super-radiant pulse (see Fig. 1):

$$\Delta N = -N\,\text{th}\,[(t-t_d)(2\tau)^{-1}],$$
$$Q = \hbar\omega_0N\{4\tau\,\text{ch}^2[(t-t_d)(2\tau)^{-1}]\}^{-1}. \tag{1.21}$$

The pulse length $\tau = 1/2\omega''(t=0) = T_1/NV$ is smaller by the factor NV than the spontaneous emission time of an isolated molecule $T_1 = 3\hbar c^3/4d^2\omega_0^3$. Hence the maximum power $Q_{\max} = \hbar\omega_0N/4\tau$ is proportional to the square of the concentration of the active molecules, N^2, and exceeds the initial level of incoherent spontaneous emission $Q(t=0) \sim \hbar\omega_0N/T_1$ by a factor of about NV. The power $Q_{\max} = \omega_0^4|dN|^2V/3c^3$ is reached after the delay time $t_d = \tau\ln[Q_{\max}/Q(t=0)]$ and corresponds to coherent emission by a dipole with total dipole moment of all the molecules in the particle $P_{\max}V = dNV$. This clearly demonstrates the coherent collective character of the process.

Thus, macroscopic electrodynamics shows that Dicke super-radiance is based on the dissipative instability of polarization oscillations with negative energy.[29] It will become clear later that this is opposite to the traditional mechanism of laser instability of electromagnetic waves in masers and lasers, which have positive energy ($w > 0$) and grow as a result of negative loss ($Q < 0$) introduced by stimulated emission by active molecules.

More complicated wave instabilities in a distributed active medium of size $L \gg \lambda_0$ will be discussed below, but the essential role of negative-energy polarization normal waves (modes) in coherent processes, including super-radiance, will remain unaffected.

2. POLARIZATION WAVES IN A HOMOGENEOUS TWO-LEVEL MEDIUM

Before we consider the properties of coherent processes such as super-radiance in a bounded sample, let us examine the solution of the corresponding initial-value problem in an infinite medium. As a first step, let us consider normal waves in a homogeneous medium consisting of two-level molecules. These waves differ from one another by the ratio of polarization to electric field, and also by their spectra, i.e., the dependence of the complex frequency $\omega = \omega' + i\omega''$ on the wave number $k = \text{Re}\,k$.

2.1. Polariton spectrum of an active medium. Effect of resistive losses

When $kc \sim \omega_0$, the dispersion relation (1.16) together with (1.15) determine two normal transverse waves in the two-level medium, namely, an electromagnetic wave and a polarization wave[29,36]:

$$\omega_{c,p} = \omega_0 - iT_2^{-1} + \frac{1}{2}[ck - \omega_0 + i(T_2^{-1} - 2\pi\sigma)]$$
$$\times [1 \pm \{1 + \omega_c^2[ck - \omega_0 + i(T_2^{-1} - 2\pi\sigma)]^{-2}\}^{1/2}]. \tag{2.1}$$

In the polariton resonance region, i.e., for $\omega \sim \omega_0$ and $kc \sim \omega_0$, this is usually referred to in optics as the polariton spectrum (see Refs. 2 and 36–42). Its generalization to the case of Doppler broadening is discussed in Sect. 6.1. The shape of the spectrum (2.1) undergoes a qualitative change between the uninverted and the inverted medium or, more precisely, when the inequality sign in $\omega_c^2 \gtrless (T_2^{-1} - 2\pi\sigma)^2$ is reversed, where $\omega_c^2 \propto -\Delta N$ (Fig. 2). The designation *polarization wave* is introduced because, for a given amplitude of the field E, the amplitude of the polarization of the medium in the polarization wave is greater (usually much greater) than in the electromagnetic wave. It is clear from (2.1) that, for a given wavelength $\lambda = 2\pi/k$, only one of these two waves can be unstable. The maximum wave

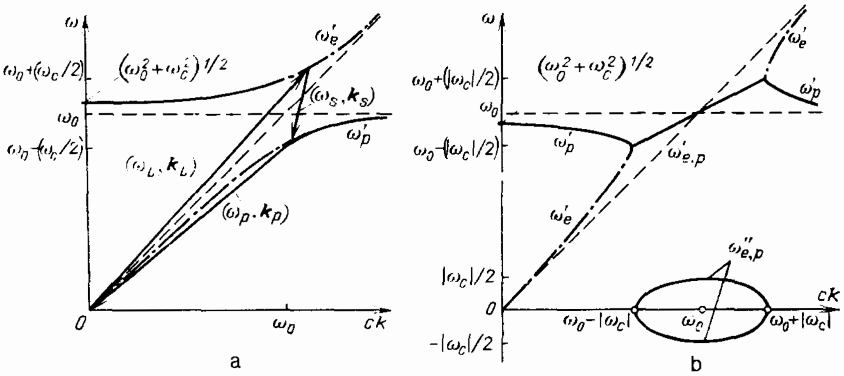

FIG. 2. Dispersion curves for electromagnetic waves and polarization waves (polaritons) $\omega_{e,p}(k)$ in uninverted ($\omega''_{e,p} \equiv 0$) (a) and inverted (b) media for $T_2^{-1} = \sigma = 0$. Arrows show Raman scattering by polaritons.

growth rates are achieved at the line center $ck = (\omega_0^2 + T_2^{-2})^{1/2} = |\omega_{e,p}|$ and are given by

$$\omega''_{e,p} = -T_2^{-1} + \frac{1}{2}(T_2^{-1} - 2\pi\sigma) \times \{1 \pm [1 - \omega_c^2 (T_2^{-1} - 2\pi\sigma)^{-2}]^{1/2}\}. \qquad (2.2)$$

The usual concepts of maser (induced) instability[11,12,26] refer to the electromagnetic wave $\omega_e(k)$ under the conditions of strong relaxation of polarization and weak field dissipation $T_2^{-1} > |\omega_c|/2 > 2\pi\sigma$. According to (2.2), when $T_2^{-1} \gg |\omega_c|/2$, the growth rate is given by

$$\omega''_e = -\frac{\omega_c^2 T_2}{4} - 2\pi\sigma \equiv 2\pi\omega_0 T_2 d^2 \Delta N \hbar^{-1} - 2\pi\sigma. \qquad (2.3)$$

The polarization wave $\omega_p(k)$ is then rapidly damped out at the rate T_2^{-1}, and ceases to be of any interest. On the other hand, the instability of the electromagnetic wave corresponds to stimulated amplification of the field by practically unphased molecules, i.e. the relatively slow and incoherent superluminescence process that can be described by the rate equations.

A totally different situation occurs in the case of strong inversion and weak relaxation that is typical for super-radiance: $T_2^{-1} < |\omega_c|/2$ (Fig. 3). When the dissipation is small, $2\pi\sigma < T_2^{-1} \ll |\omega_c|/2$, we have anomalous saturation of the growth rate (2.2):.

$$\omega'' = \frac{1}{2}|\omega_c| - \frac{1}{2}T_2^{-1} - \pi\sigma \approx \left(2\pi\omega_0 d^2 \frac{\Delta N}{\hbar}\right)^{1/2}. \qquad (2.4)$$

In contrast to the maser growth rate (2.3), which is proportional to the inversion ΔN, the anomalous growth rate[28] (2.4) becomes $\omega'' \propto (\Delta N)^{1/2}$. If field dissipation in the medium predominates over the relaxation of polarization ($T_2^{-1} < 2\pi\sigma$), the polarization wave becomes unstable (instead of the electromagnetic wave) in the range $2\pi\sigma < -\omega_c^2 T_2/4$. So long as $2\pi\sigma \ll |\omega_c|/2$, this instability within the line $|ck - \omega_0| \lesssim |\omega_c|$ smooths the spectrum of the polarization wave $\omega_p(k)$ (see Fig. 3b) and reduces the instability: at the line center, we have

$$\omega''_p = -\frac{\omega_c^2}{8\pi\sigma} - T_2^{-1} \qquad (2.5)$$

(for $2\pi\sigma \gg |\omega_c|/2$). When $2\pi\sigma \gg T_2^{-1}$ it follows from (2.1), that the wave-number interval for which the polarization-wave instability develops with a growth rate of the order of the maximum value (2.2) is

$$\Delta k \sim \frac{1}{c}[(2\pi\sigma)^2 - \omega_c^2]^{1/2} \left[\frac{-(\omega_c^2 T_2/4) - 2\pi\sigma}{-(\omega_c^2 T_2/4) + 2\pi\sigma}\right]^{1/2}. \qquad (2.6)$$

In the above situation, the field cannot be represented by a set of incoherent components because the process is so fast that the width of the spectral line of the medium, T_2^{-1} is smaller than the minimum possible width of the emission spectrum $\Delta\omega \sim \max \omega''_{e,p}$, which is equal to the growth rate of the most rapidly growing spatial harmonic of the field (see Sect. 3.5 for further details).

This is a convective (drift) and not an absolute instability for $-\omega_c^2 T_2/4 < \omega_0$ It is concentrated within the wave-number interval

$$k_{1,2} = \frac{1}{c}\left\{\omega_0^2 + T_2^{-2} + \left[\frac{i}{2} + (4\pi\sigma T_2)^{-1}\right] \times [-\omega_c^2 - 8\pi\sigma T_2^{-1} \mp (-\omega_c^2 - 8\pi\sigma T_2^{-1})^{1/2} \times (-\omega_c^2 + 8\pi\sigma T_2 \omega_0^2)^{1/2}]\right\}^{1/2}. \qquad (2.7)$$

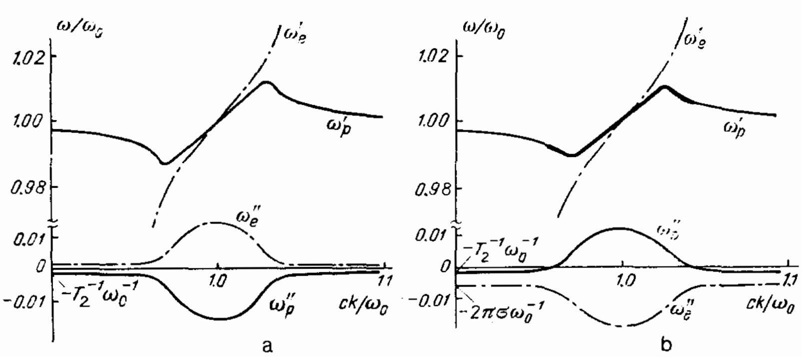

FIG. 3. Polariton spectrum of an inverted two-level medium in the absence (a) and in the presence of dissipation (b) $2\pi\sigma = 10^{-2}\ \omega_0/2 \sim |\omega_c|/2$. The values $1/T_2\omega_0 = 10^{-3}$, $-\omega_c^2/\omega_0^2 = 10^{-3}$ were chosen to ensure that $|\omega_c|/2 \gg T_2^{-1}$. Thick lines show regions of wave instability on the dispersion curves $\omega'_{e,p}$.

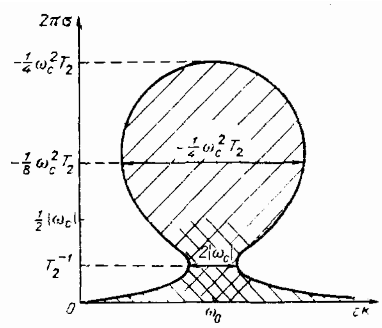

FIG. 4. Region of unstable wave numbers (2.7) for different values of conductivity σ and $4\sqrt{2}\,T_2^{-1} < |\omega_c|$. The electromagnetic wave is unstable below the level $2\pi\sigma = T_2^{-1}$; the polarization wave is unstable above this level. Doubly cross-hatched region shows the anomalous instability (2.4).

It is clear from Fig. 4 that, when $4\sqrt{2}\,T_2^{-1} < |\omega_c| c$ an increase in the conductivity σ in the range $2\pi\sigma > T_2^{-1}$ is accompanied by expansion of the interval (2.7) in which the polarization waves are unstable, which differs from the maser instability of electromagnetic waves for $2\pi\sigma < T_2^{-1}$. In other words, as dissipation grows, we have instability in the line wings $|ck - \omega_0| \gtrsim |\omega_c|$ (see Sect. 2.3). The polarization waves do not play an appreciable role in most lasers. The point is that lasers employ cavity resonators with a high quality factor q, so that $2\pi\sigma \equiv \omega_0/2q < T_2^{-1}$ and, according to (2.3), generation occurs for a relative low level of inversion $|\omega_c| < 2T_2^{-1}$, for which weakly damped polarization waves do not exist. Certain quantum oscillators employing a high degree of inversion[24,43,44] and low-Q resonators $q \lesssim T_2\omega_0/2$. are an exception. Examples include molecular-beam masers (NH_3) as well as gas (He–Xe) and chemical (HF) continuously operating lasers. Other examples include pulsed lasers that employ metal vapor (Cu) and pulsed molecular lasers using a high-pressure gas (>0.1 atm) and exploiting vibrational-rotational (CO_2–TEA), electronic vibrational (N_2), and excimer (Xe_2) transitions. For such systems, $|\omega_c| \gtrsim 2T_2^{-1}$, so that the corresponding generation dynamics, including stochastic generation, is largely determined by coherent processes involving polarization waves. However, the part played by these waves in this situation has not as yet been analyzed.

2.2. Observation of the polariton spectrum

Intensive experimental studies of polaritons began after the advent of the quantum theory of exciton-photon coupling in solid state physics, although the first attempts were based on the classical theory of the dispersion of light.[2,8,45–48] For a long time, these experiments were concerned exclusively with absorbing crystals. The spectrum was examined by both optical (interference, reflection, luminescence) and nonoptical techniques (neutron scattering, electron scattering, etc.) Raman scattering of laser radiation (ω_L) into the Stokes radiation (ω_s) with the generation of polaritons (ω_p), which was developed in the 1970s, has been particularly effective (see Fig. 2a).

Direct observations of the polariton spectrum of active media in which time dependent coherent processes are taking place have not as yet been carried out. This has been due to the following factors.

First, a high concentration of active molecules is necessary if the polariton effect is to be produced, since the cooperative frequency must be greater than the linewidth, i.e., $|\omega_c| > 2T_2^{-1}$. If this were not so, the spectrum would not be very different from the partial-wave spectrum. In crystals, in which the interaction between the particles is strong, this condition could not be satisfied until the 1980s when experiments on super-radiance in diphenyl containing pyrene,[32] KCl:O_2^- (Refs. 33 and 34), Nd:YAG and ruby[35] at low temperatures were carried out.

Second, the above condition is more readily satisfied,[3)] for example, in the case of super-radiance in cesium vapor.[14] However, studies of polaritons in gases were greatly delayed. The first paper, published in 1967, on excitons in gases[48] did not elicit a notable response. Evidently, in the case of gas electrodynamics, the use of the analogy with ordinary phonons or excitonic polaritons in crystals was impeded by the absence of translational invariance and the fact that polaritons in a gas, due to electron transitions in free molecules, were unusual. It is only in the last few years that the idea of polaritons in gases, and also in glasses and amorphous materials, has begun to emerge[45,46] and is being used in practice. The parametric polariton laser employing sodium vapor is an example.[49]

Third, fast detection is essential because the shape of the polariton spectrum changes as the nonlinear stage of nonlinearity is reached in a time $\sim t_d, 1/\omega''_{c,p}$. Measurement of the parameters of ultrashort small-area pulses, transmitted by an active medium,[35] is a promising method (see Sect. 3.6). Ultrashort pulses have long been used in the spectroscopy of absorbing media,[47] e.g., for the direct measurement of picosecond lifetimes of polaritons, using coherent Raman scattering,[50] and also for the determination of the group velocity $d\omega_p/dk$ of polaritons in crystals (CuCl, CdS).[51]

2.3. Negative energy of polarization waves

We shall show that, just as in the case of a particle (see Sect. 1.4), polarization waves in a distributed inverted medium have negative energy and their instability is dissipative.[29,36] According to (1.9) and (1.10), the rate of change of the total field and polarization energy density w for $\Delta N = \text{const}$ is determined by the relaxation of molecular polarization, resistive dissipation, and the inhomogeneity of the energy flux: $\partial w/\partial t = -Q - c\,\mathrm{div}[\mathscr{E},\mathscr{B}]/4\pi$, i.e.,

$$\frac{\partial}{\partial t}\left\{\frac{\vec{\mathscr{E}}^2+\vec{\mathscr{B}}^2}{8\pi}+\frac{2\pi}{\omega_c^2}\left[\left(\frac{\partial\vec{\mathscr{P}}}{\partial t}\right)^2+\left(\omega_0^2+\frac{1}{T_2^2}\right)\vec{\mathscr{P}}^2\right]\right\}$$
$$=-\frac{8\pi}{\omega_c^2T_2}\left(\frac{\partial\vec{\mathscr{P}}}{\partial t}\right)^2-\sigma\vec{\mathscr{E}}^2-\frac{c}{4\pi}\,\mathrm{div}\,[\vec{\mathscr{E}},\ \vec{\mathscr{B}}]. \qquad (2.8)$$

Hence, if we know the susceptibility χ (1.14), we can find the average energy per h.f. period and the inhomogeneous-wave power loss in the active medium:[4)]

$$w=|E|^2[(1+|\varepsilon(\omega)|)(16\pi)^{-1}$$
$$+\pi\omega_c^{-2}(|\omega|^2+\omega_0^2+T_2^{-2})|\chi(\omega)|^2]\exp(2\omega''t), \qquad (2.9)$$

$$Q=\frac{1}{2}|E|^2[\sigma+8\pi\omega_c^{-2}T_2^{-1}|\omega\chi(\omega)|^2]\exp(2\omega''t). \qquad (2.10)$$

Using (2.9) and (2.1), we can verify that a polarization wave in an inverted medium with $\omega_c^2 < 0$, has negative energy

$w_p = w(\omega_p) < 0$. Hence, when energy is removed by field dissipation, and $Q_p = Q(\omega_p) > 0$, the field amplitude must increase (the growth rate is $\omega_p'' = -Q_p/2w_p > 0$), i.e., the polarization wave exhibits dissipative instability.[5)]

The energy of the electromagnetic wave, on the other hand, is positive: $w_e = w(\omega_e) > 0$. Its maser instability ($\omega_e'' = -Q_e/2w_e > 0$; Sect. 2.1) arises as a result of negative loss: $Q_e = Q(\omega_e) < 0$. In an uninverted medium, the energy and loss are positive for both waves,[2,3] and there is no instability.

The reason for negative energy becomes clearer if we turn to the Hamiltonian h of the system, i.e., the total energy density of the medium and of the field in the nonlinear case $\Delta N \neq \text{const}$. We shall use the following law of variation for the Bloch vector that follows from (1.10) and (1.11)[12]:

$$S = (\vec{\dot{\mathscr{P}}}[1 + (\omega_0 T_2)^{-2}]^{1/2}, -\vec{\mathscr{P}}\omega_0^{-1}, \Delta N d):$$

$$\frac{\partial}{\partial t}[\Delta N^2 d^2 + (\dot{\vec{\mathscr{P}}}\omega_0^{-1})^2 + (1 + \omega_0^{-2}T_2^{-2})\vec{\mathscr{P}}^2]$$
$$= -4\dot{\vec{\mathscr{P}}}^2 (T_2\omega_0^2)^{-1} + \omega_c^2 h (\Delta N - \Delta N_{\text{H}})(4\pi T_1\omega_0)^{-1}, \quad (2.11)$$

where the dot represents partial differentiation with respect to time $\partial/\partial t$. To be specific, we shall consider a lossless inverted medium, and assume that $t \ll T_{1,2}, \sigma^{-1}$. The length of the Bloch vector is then conserved: $|\mathbf{S}| = \Delta N_{\text{B}} d$, and inversion is expressed in terms of polarization as follows:

$$\Delta N = \Delta N_{\text{B}}\{1 - [\vec{\mathscr{P}}^2 + (\dot{\vec{\mathscr{P}}}\omega_0^{-1})^2](\Delta N_{\text{B}} d)^{-2}\}^{1/2}. \quad (2.12)$$

Eliminating ΔN from (1.10) with the aid of (2.12), we write the equations for the transverse field and the polarization in the Lagrange form

$$\frac{\partial}{\partial t}\frac{\partial l}{\partial \dot{A}} + \frac{\partial}{\partial z}\frac{\partial l}{\partial A'} - \frac{\partial l}{\partial A} = 0,$$

$$\frac{\partial}{\partial t}\frac{\partial l}{\partial \dot{\mathscr{P}}} + \frac{\partial}{\partial z}\frac{\partial l}{\partial \mathscr{P}'} - \frac{\partial l}{\partial \mathscr{P}} = 0,$$

$$l = \frac{\mathscr{E}^2 - \mathscr{B}^2}{8\pi} + \mathscr{P}\mathscr{E} - \frac{\hbar\omega_0\Delta N}{2} - \frac{\hbar\dot{\mathscr{P}}}{2d}\arcsin\frac{\dot{\mathscr{P}}/\omega_0}{[(\Delta N_{\text{B}} d)^2 - \mathscr{P}^2]^{1/2}}$$
$$\equiv p_1\dot{q}_1 + p_2\dot{q}_2 - h(p_1, q_1; p_2, q_2) \quad (2.13)$$

where ΔN is given by (2.12), $\mathscr{E} = -\dot{\mathbf{A}}/c$, $\mathscr{B} = \text{rot}\,\mathbf{A}$ and the one-dimensional problem is being considered for the sake of simplicity. The prime represents the differentiation $\partial/\partial z$. The canonical variables other than the vector potential $A(z,t) = q_1$ and induction $\mathscr{D}(z,t) \equiv \mathscr{E} + 4\pi\mathscr{P} = -4\pi c p_1$ (linearly polarized for the sake of simplicity) are taken to be $q_2 = \mathscr{P}(z,t)$ and $p^2 = -(\hbar/2d)\times\arcsin\{\dot{\mathscr{P}}[\omega_0(\Delta N_{\text{B}}^2 d^2 - \mathscr{P}^2)^{1/2}]^{-1}\}$. We thus find that the Hamiltonian for the molecules + field system does not contain their binding energy $\mathscr{P}\mathscr{E}$, and is given by

$$h = \frac{1}{8\pi}\mathscr{E}^2 + \frac{\mathscr{B}^2}{8\pi} + \frac{1}{2}\hbar\omega_0\Delta N. \quad (2.14)$$

In addition, the basic equations (1.9)–(1.11) lead to the expression for the rate of change of the energy density in the general (lossy) case:.

$$\frac{\partial}{\partial t}\left(\frac{\vec{\mathscr{E}}^2 + \vec{\mathscr{B}}^2}{8\pi} + \frac{\hbar\omega_0\Delta N}{2}\right)$$
$$= -\frac{\hbar\omega_0(\Delta N - \Delta N_{\text{H}})}{2T_1} - \sigma\vec{\mathscr{E}}^2 - \frac{c}{4\pi}\text{div}\,[\vec{\mathscr{E}}, \vec{\mathscr{B}}]. \quad (2.15)$$

In the linear approximation, in which $\partial(\Delta N^2)/\partial t \approx 2\Delta N_{\text{B}}\partial\Delta N/\partial t$, this result leads to (2.8) by virtue of (2.11) and if we take into account the possible arbitrariness in the choice of the origin for the wave energy. In an inverted medium, it is precisely the "linearized" and "shifted" energy of the waves w (2.9) that can become negative. Actually, according to (2.12), oscillations in the polarization of inverted molecules ($\mathscr{P} \neq 0$; $\dot{\mathscr{P}} \neq 0$), can only be produced by partially taking the molecules from the upper to the lower state ($\Delta N < \Delta N_{\text{B}}$). When $\Delta N_{\text{B}} = N$, all this is obvious because the molecules can acquire h.f. polarization $\int (C_1\psi_1)^* e\mathbf{r}(C_2\psi_2) d^3\mathbf{r}$ only in the mixed state $\psi = C_1\psi_1 + C_2\psi_2$, for $C_{1,2} \neq 0$, in which both levels are populated. If this removal of energy from the upper level results in fields of moderate energy, which occurs in the case of polarization waves, then negative work is done on the medium as a whole, and the medium acquires negative wave energy. In an uninverted medium, on the other hand, the polarization and the field are produced by taking the molecules from the lower to the upper level, thus communicating positive energy to the medium. The result is that the wave energy becomes positive.

2.4. Absolute instability in a conducting inverted medium. Plasma-dipole resonance

The dissipative instability of polarization waves can be used to generate radiation not only under convective conditions (Sect. 2.1), but also under absolute conditions when, despite the absence of reflections, the dispersive spreading of a wave packet and its nonuniform amplification are found to compensate convective drift.[52] A homogeneous inverted medium then becomes formally "opaque" ($\text{Re}\,\varepsilon(\omega') < 0$), and strong field dissipation ensures that the group velocity is small: $|d\omega_p'/dk| \ll c$.

The criterion for absolute instability can be formulated as follows[37,38]: at the point ω_a at which the two roots $k(\omega)$ of the dispersion relation $c^2k^2 = \omega^2\varepsilon(\omega)$ that correspond to counterpropagating waves are found to coincide, we must have $\omega_a'' \equiv \text{Im}\,\omega_a > 0$. This corresponds to the crossing of the ordinate axis in Fig. 4 by the instability (2.12) ($k_1^2 < 1$) and to the growth rate

$$\omega_{a0}'' = -2\pi\sigma\omega_c^2(\varepsilon_0^2\omega_0^2 + 16\pi^2\sigma^2)^{-1} - T_2^{-1} > 0. \quad (2.16)$$

The permittivity of the "background" medium was taken into account in this expression for ω_{a0}'' by replacing unity in (1.15) with ε_0. The attainment of absolute instability is facilitated by $\varepsilon_0 < 1$ and by choosing the optimum conductivity $\sigma_{\text{opt}} \approx |\varepsilon_0|\omega_0/4\pi$. When $\varepsilon_0 < 0$, so that the opacity of the phonon medium precludes the polariton resonance $c^2k^2 \approx \varepsilon_0\omega_0^2$, absolutely unstable longitudinal and long-wave ($k \to 0$) transverse polarization waves are found to grow with a growth rate ω_{a0}'', which is greater than the growth rate of the convectively unstable waves with $k \neq 0$.

Absolute dissipative instability can probably be attained when there is a strong inversion of rotational molecular transitions in a partially ionized gas,[6)] in which case we have the plasma-dipole resonance $\omega_0 \approx \omega_{\text{L}} = (4\pi N_e e^2/m_e)^{1/2}$ in the far infrared. Here, the conducting background medium is replaced by the low-temperature ($T \sim 10^3$ K) plasma in which $\varepsilon_0(\omega) = 1 - (\omega_{\text{L}}^2/\omega^2)$ and $4\pi\sigma \approx \nu_{ei} \sim 10 N_e T^{-3/2} \ll \omega_{\text{L}}$ (for electron and ion concentrations $N_e \sim N_i \lesssim 10^{17}$ cm^{-3} and degree of ionization equal to 10^{-2}–10^{-7}). Allowance for the additional plasma dispersion $\varepsilon_0(\omega)$ is equivalent to the replacement $c^2k \to c^2k^2 + \omega_L^2$ in (1.16). Thus, when $|\varepsilon_0| \ll 1$, the absolute instability of waves with $k \approx 0$ replaces the convective instability in the region of the polariton resonance $k \approx \omega_0/c$, and becomes the dominant process. Its growth rate ω_p'' is a maximum for $\omega_L^2 = \omega_0^2 + T_2^{-2}$ and is given by (2.2). Estimates show that the condition for absolute instability $-\omega_c^2 > 8\pi\sigma/T_2$, is satisfied when the relative inversion is $\Delta N/N \gtrsim (10^{-18}N_e)^{1/2}$.

2.5. Birefringence and polarization of waves in an anisotropic medium

The concept of normal waves in a medium arises naturally in the analysis of the polarization properties of radiation in anisotropic (including active) media.[1-3] Their anisotropy $\varepsilon_{\gamma\delta}(\omega)$ may be related, for example, to the background medium (crystal[32-35]), a polarizing pump,[14,53,54] or the anisotropic distribution of the population difference $\Delta N(\mathbf{e})$ between the active molecules over the orientations of the dipole moments $\mathbf{d} = d\mathbf{e}$; $|\mathbf{e}| = 1$. In the last of these, we need the tensor generalization of the cooperative frequency:

$$(\omega_c^2)_{\gamma\delta} = -8\pi\omega_0\hbar^{-1} \iint_{|\mathbf{e}|=1} d_\gamma d_\delta^* \Delta N(\mathbf{e})\, d^2\mathbf{e} \qquad (\gamma, \delta = x, y, z). \tag{2.17}$$

In general, the polarization and dispersion of normal waves is determined by the set of equations[1,2]

$$\sum_{\delta=1}^{3} A_{\gamma\delta}E_\delta = 0, \quad A_{\gamma\delta} = \omega^2\varepsilon_{\gamma\delta}c^{-2} - k^2\delta_{\gamma\delta} + k_\gamma k_\delta$$

and the Fresnel dispersion equation $\det(A_{\gamma\delta}) = 0$. In the absence of spatial dispersion, these lead to two values of the refractive index $n_{1,2}(\omega)$ for transverse waves with different polarization coefficients $K_{1,2} \equiv iE_y/E_x = -i\,\mathrm{ctg}(\phi + i\theta)$, which determine the ratio of semiaxes of the polarization ellipse, $\tanh\theta$, and the angle φ between its principal axis and the y axis:

$$\begin{aligned} \mathrm{th}\,\theta &= -4\,\mathrm{Re}\,K \cdot (|K-1| + |K+1|)^{-2}, \\ \phi &= \frac{1}{2}\arg[(K-1)(K+1)^{-1}]. \end{aligned} \tag{2.18}$$

The result is that, according to the dispersion relation $\omega^2 n_{1,2}^2(\omega) = c^2k^2$, which replaces (1.16), there are two differently polarized electromagnetic waves $\omega_e^{(1),(2)}(k)$ and two polarization waves $\omega_p^{(1),(2)}(k)$. Birefringence and polarization of radiation have been examined in detail for quasistationary boundary-value problems,[1-3] including the linear interaction between waves in inhomogeneous media[55] and the transformation of the polarization ellipse in isotropic media with nonlinear anisotropy, due to, for example, the reorientation of molecules, the degeneracy of levels, and the nonisotropic saturation of $\Delta N(\mathbf{e})$ (Refs. 54, 56, and 57). A number of experimental and theoretical publications have appeared on time-dependent polarization effects accompanying the propagation of picosecond pulses in passive media.[54,56]

The analogous range of problems for coherent and time-dependent processes in active media that involve the participation of polarization waves is only just beginning to be investigated. As an example, let us consider super-radiance based on the two-level transition $j \to j'$, that is degenerate in the components j_z of the total angular momentum j. Suppose that the anisotropy is determined by a polarized pump pulse that determines the density matrix $\hat\rho(t=0)$ of the coherent mixture of Zeeman sublevels $|jj_z\rangle$ of the upper state by populating them from the lower state[53] with angular momentum j'':

$$\begin{aligned} \varepsilon_{xx,yy} &= 1 + i\frac{4\pi\sigma}{\omega} - \frac{\omega_c^2\{\mp \mathrm{Re}\,U + [(C_+ + C_-)/2]\}}{(\omega + iT_2^{-1})^2 - \omega_0^2}, \\ \varepsilon_{xy,yx} &= -\frac{\omega_c^2\{\mathrm{Im}\,U \pm i[(C_+ - C_-)/2]\}}{(\omega + iT_2^{-1})^2 - \omega_0^2}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} C_\pm &= d^{-2}\sum_{j_z, j_z'} \langle j'j_z'|\hat d_{\pm 1}|jj_z\rangle^2 \langle jj_z|\hat\rho(t=0)|jj_z\rangle, \\ U &= d^{-2}\sum_{j_z, j_z'} \langle j'j_z'|\hat d_{+1}|jj_z\rangle \langle j'j_z'|\hat d_{-1}|jj_z+2\rangle \\ &\quad \langle jj_z+2|\hat\rho(t=0)|jj_z\rangle. \end{aligned} \tag{2.20}$$

In the above presentation, we used the standard expansion of the dipole moment operator $\hat{\mathbf{d}} = \hat d_{+1}\mathbf{e}_{+1} + \hat d_{-1}\mathbf{e}_{-1}$ in terms of circular polarizations $\mathbf{e}_{\pm 1} = \mp(\mathbf{x}^0 \pm i\mathbf{y}^0)/\sqrt{2}$. According to (2.19) and the equation for the field of normal waves $\Sigma_\delta A_{\gamma\delta}E_\delta = 0$, super-radiance develops on the linear stage with birefringence

$$\begin{aligned} n_{1,2}^2(\omega) &= 1 + i\frac{4\pi\sigma}{\omega} - \frac{\tilde\omega_{c1,2}^2}{(\omega + iT_2^{-1})^2 - \omega_0^2}, \\ \tilde\omega_{c1,2}^2 &= \omega_c^2\left\{\frac{C_+ + C_-}{2} \pm \frac{C_- - C_+}{2}\left[1 + \frac{4|U|^2}{(C_+ - C_-)^2}\right]^{1/2}\right\}, \end{aligned} \tag{2.21}$$

which corresponds to the presence of waves with two polarization ellipses $K_{1,2} = i(n_{1,2}^2 - \varepsilon_{xx})/\varepsilon_{xy}$. When $(C_+ - C_-^2) + 4|U|^2 \neq 0$, one of the waves predominates and has a high growth rate. This has been observed in an experiment[53] with Rb gas and a linearly polarized pump $\mathbf{e}_L \perp \mathbf{z}^0$, using the $j'' = 1/2$ level and the working transition $j = 3/2 \to j' = 1/2$) ($C_\pm = 5/48$, $U = -\exp(2i\alpha)/16$, and α is the angle between $\mathbf{e}_L$ and the axis $\mathbf{x}^0$). In these expressions $\tilde\omega_{c1}^2 = 4\tilde\omega_{c2}^2$ and super-radiance is always linearly polarized in the direction of $\mathbf{e}_L$. Moreover, there is no anisotropy for the $j = 1/2 \to j' = 1/2$ transition ($C_\pm = 1/6$, $U = 0$): The directions of the orthogonal polarizations of normal waves can be chosen arbitrarily, and their growth rates are equal. This degeneracy enables us to explain the fluctuations in the polarization of super-radiance observed in this case[53] (see Sect. 5.3).

3. DISSIPATIVE INSTABILITY IN THE PROBLEM OF UNIDIRECTIONAL SUPER-RADIANCE

The electrodynamic picture of Dicke super-radiance (Sect. 1.4) and the associated phenomena in inverted two-level media are presented in Sects. 3–5.

3.1. Evolution of a packet of unstable normal waves and their Green's functions

We begin with an analysis of super-radiance in the one-dimensional model of a plane layer whose normal is the z axis and whose thickness in the z direction is $L \gg \lambda = 2\pi/k$. More precisely, we shall confine our attention to the propagation of plane waves (1.13) in the direction of the positive z axis[14,23,27,58,59] and postpone the discussion of counterpropagating waves to Sect. 3.7 and Sect. 4. In reality, the one-dimensional model works best in the case of a cylindrical sample with a small Fresnel number $F \equiv S/\lambda L \lesssim 1$, where S is the cross sectional area ($\perp z$). According to this model, super-radiance can be described by the unidirectional Maxwell-Bloch equations (1.9′)–(1.11′) in which the distributed dissipation σ includes resistive absorption and diffractive "leakage" of radiation through the side surface of the sample ($\sigma_{\text{diff}} \sim c\lambda/6\pi S$) (Refs. 14, 23, 60), thus specifying a unified dissipation scale $L_\sigma = c/2\pi\sigma$. Their general solution on the linear super-radiance stage (for which $\Delta N = N$) has the following form for arbitrary initial conditions: $P(z,t=0) = P_0(z)\theta(z)$, $E(z,t=0) = E_0(z)\theta(z)$ where the field $E(z=0,t) = E_{\text{in}}(t)\theta(t)$ is assumed to be incident from the left on the surface of the medium ($z=0$).

$$E(z,\ t) = \theta(\tilde{t})\, e^{-z/L_\sigma} \Big\{ E_{\text{in}}(\tilde{t}) + \int_0^{\tilde{t}} E_{\text{in}}(t')\, e^{-(\tilde{t}-t')/T_2} \frac{\Omega_c}{2} \left(\frac{z/c}{\tilde{t}-t'}\right)^{1/2} \times I_1\left(\frac{\Omega_c}{c}\, |c(\tilde{t}-t')z|^{1/2}\right) dt' \Big\}$$
$$+ \frac{2\pi i\omega_0}{c} e^{-t/T_2} \int_0^z P_0(z') \exp\left[(z-z')\left(\frac{1}{cT_2} - \frac{1}{L_\sigma}\right)\right] \times \theta\left(\tilde{t} - \frac{z'}{c}\right) I_0(\xi')\, dz'$$
$$+ E_0(z-ct)\, e^{-ct/L_\sigma} + \frac{1}{2} e^{-t/T_2} \int_0^z \Big\{ E_0(z') \exp\left[(z-z')\left(\frac{1}{cT_2} - \frac{1}{L_\sigma}\right)\right] \times \theta\left(\tilde{t} - \frac{z'}{c}\right) \xi' I_1(\xi')\, (c\tilde{t} - z')^{-1} \Big\}\, dz'; \tag{3.1}$$

where $\Omega_c = (8\pi d^2 N\omega_0/\hbar)^{1/2} > 0$ is the cooperative frequency ω_c in a medium that is completely inverted at $t=0$, $\xi' = (\Omega_c/c)\{(z-z')\,[ct-(z-z')]\}^{1/2}$, $\tilde{t} = t - (z/c)$, θ is the Heaviside step function, and I_k represents the modified Bessel functions. The length of the sample does not appear in the solution when there are no reflections; it simply defines the right-hand boundary of the medium: $z = L$. Different special solutions of (3.1) are discussed in Refs. 14, 15, 42, and 61.

To explain the instability in the inverted medium, consider the amplification of a packet of normal unstable waves of a given type, in which the field amplitudes and the polarizations are related by the following expression for each Fourier harmonic $q = i[k - (\omega_0/c)]$:

$$P_0(q) = E_0(q)\,(p_{e,p}(q) + cq + 2\pi\sigma)\,(2\pi i\omega_0)^{-1}, \tag{3.2}$$
$$p_{e,p}(q) = -i\,(\omega_{e,p}(k) - \omega_0).$$

According to the Fourier-Laplace method, the packet evolves for $E_{\text{in}} = 0$ in accordance with the following law:

$$P_{e,p}(z,\ t) = \int_{-\infty}^{\infty} P_0(z')\, D_{e,p}(z-z',\ t)\, dz',$$
$$D_{e,p}(z,\ t) = \int_{-i\infty}^{i\infty} \exp(qz + p_{e,p}(q)\,t)^{-1}\, dq\, (2\pi i)^{-1}, \tag{3.3}$$

where, for unstable e and p normal waves, the Green's function is

$$D_{e,p}(z,\ t) = \exp\left(-\frac{t}{T_2} - \frac{z}{L_\sigma} + \frac{z}{cT_2}\right) \times \left(\delta(z_{e,p}) + \frac{t\Omega_c^2}{2c}\,\theta(z_{e,p})\,\frac{I_1(\xi)}{\xi}\right), \tag{3.4}$$
$$\xi = \frac{\Omega_c}{c}\,[z(ct-z)]^{1/2},$$

in which $z_e = ct - z$, $z_p = z$. Obviously, in the absence of polarization relaxation ($T_2^{-1} = 0$), and for $\omega''_{e,p} > 0$, the instability is absolute independently of the field dissipation σ:

$$D_{e,p}(z,\ t) \underset{t\to\infty}{\longrightarrow} \infty,$$

since $I_k(\xi) \sim (\exp\xi)/(2\pi\xi)^{1/2} \to_{\xi\to 0} \infty$. If, on the other hand, $T_2^{-1} \neq 0$, the factor $\exp(-t/T_2)$ is found to suppress the growth of the Green's function for times $t \gtrsim t_{\max} = \Omega_c^2 T_2^2 z/4c$:

$$D_{e,p}(z,\ t) \underset{t\to\infty}{\longrightarrow} 0$$

and the instability becomes *convective*.[37,38]

3.2. Absolute and convective instability of polarization waves

We must now examine the propagation of a packet of dissipatively unstable polarization waves for $2\pi\sigma > T_2^{-1}$. The initial δ-pulse $P_0\delta(z)$ corresponding to the polarization Green's functions (3.3) and (3.4) and the field

$$E_p(z,\ t) = \int_{-\infty}^{\infty} P_0(z')\, D'_p(z-z',\ t)\, dz',$$
$$D'_p(z,\ t) = \frac{2\pi i\omega_0}{c}\,\theta(z)\left(I_0(\xi) + \frac{z}{ct-z} I_2(\xi)\right) \times \exp\left(-\frac{t}{T_2} + \frac{z}{cT_2} - \frac{z}{L_\sigma}\right) \tag{3.5}$$

is converted into an amplified pulse propagating with the group velocity $v_b = c[1 - \nu(1+\nu^2)^{-1/2}]/2$, where $\nu = (2\pi\sigma - T_2^{-1})/\Omega_c$. The pulse is illustrated in Fig. 5. The field maximum at z_E occurs in advance of the polarization maximum (z_p) by an amount proportional to the cooperative length[7)] of Arecchi and Courtens[62] $L_c = c/\Omega_c$. More precisely, $z_p \approx v_{rp} t - [3L_c \nu/2(1+\nu^2)]$ and $z_E - z_p \approx L_c/(1+\nu^2)^{1/2}$ for $t\Omega_c \gg 2(1+\nu^2)^{1/2}$.

Polarization waves with negative energy density $w < 0$ have a positive energy flux density: $S_z = c|E|^2/8\pi > 0$. It would appear that the energy of a packet of such waves should propagate in the opposite direction to that of the phase velocity, i.e., in the direction of the negative z axis, since the average velocity of the energy flux is negative:

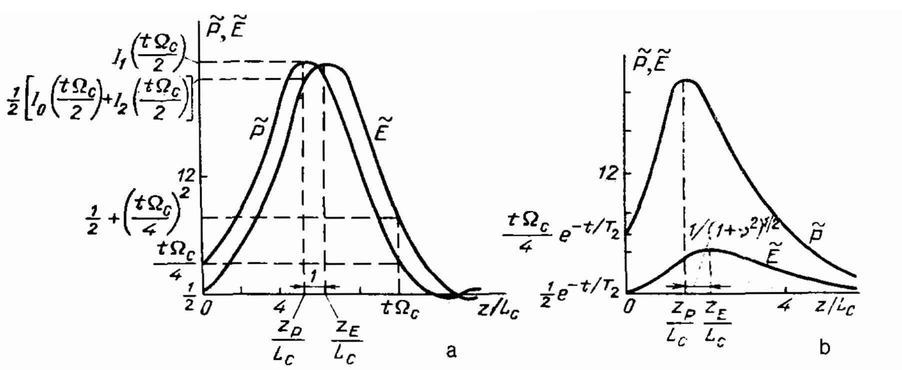

FIG. 5. Evolution of a δ-pulse of unstable polarization waves in accordance with (3.3)–(3.5) in the case of absolute instability ($T_2^{-1}=0$, $\sigma=0$, $t\Omega=10$) (a) and convective instability ($\nu=(2\pi\sigma-T_2^{-1})/\Omega_c=2$; $T_2\Omega_c=80$; $t\Omega_c=40$). (b) The following notation is employed: $\tilde{P}=cP_p/\Omega_c P_0$, $\tilde{E}=cE_c/4\pi i\omega_0 P_0$.

$$v_w = \int_{-\infty}^{\infty} S_z \, dz \left(\int_{-\infty}^{\infty} w \, dz \right)^{-1} < 0.$$

However this conclusion is not correct because $\partial w/\partial t + \partial S_z/\partial z + Q = 0$, and the energy center z_w of the packet in the active medium obeys the more complicated equation

$$\frac{dz_w}{dt} = v_w + \gamma (z_w - z_Q), \quad z_w = \frac{\int_{-\infty}^{\infty} zw\, dz}{\int_{-\infty}^{\infty} w\, dz}, \quad z_Q = \frac{\int_{-\infty}^{\infty} zQ\, dz}{\int_{-\infty}^{\infty} Q\, dz},$$

$$\gamma = \frac{\int_{-\infty}^{\infty} Q\, dz}{\int_{-\infty}^{\infty} w\, dz}. \tag{3.6}$$

In the polarization wave, the center of losses z_Q, determined mostly by field dissipation, propagates in the direction of the positive z axis with the velocity of light c, and travels in advance of the center of energy z_w, determined mostly by oscillations in the polarization of the immobile medium. The result is that $\gamma(z_w - z_Q) > 0$, and this leads to a positive rate of transport of the energy of the packet. For large times, the latter is equal to the group velocity $d\omega_p'/dk$ at the line center.

For an arbitrary initial distribution $P_0(z)$ in a finite layer $0<z<L$, the evolution of the packet is described by a convolution of $P_0(z)$ and the Green's functions (3.3)–(3.5). When the dissipation is large, $L_\sigma \ll L_c, cT_2$, and ($z \gg L_\sigma$) the main contribution to the convolution for $ct \gg z, L_\sigma^2/z, c/\omega_p''$, is provided by the neighborhood of the extremal point $z_0' = z - (ctL_\sigma^2/4L_c^2)$. So long as $z_0' > 0$, the amplitudes of polarization and field in the packet are determined by the amplified signal arriving at a given point z from the "source" point z_0', and slowly traveling from left to right with the group velocity[8)] $v_{gr} \approx cL_\sigma^2/4L_c^2 \ll c/2$:

$$P_p(z, t) = P_0(z) \exp\left(-\frac{t}{T_2}\right) + P_0(z_0') \exp(\omega_p'' t), \tag{3.7}$$

$$E_p(z, t) = 2\pi i \omega_0 c^{-1} L_\sigma P_0(z_0') \exp(\omega_p'' t).$$

When $t \gg t_\sigma = 4zL_c^2/cL_\sigma^2$, and the "source" point z_0' on the left-hand edge of the layer ($z_0'=0$) has been reached, the space-time evolution of the packet has an asymptotic behavior determined by the "boundary source" $P_0(0)$:

$$P_p(z, t) = \frac{E_p c\xi}{4\pi i \omega_0 z} = \frac{P_0(0)}{\sqrt{2\pi\xi}} \exp\left(\xi - \frac{z}{L_\sigma} - \frac{t}{T_2}\right),$$
$$\xi = [(ct - z)z]^{1/2} L_c^{-1}. \tag{3.8}$$

Thus, after a certain instant of time, the field and the polarization follow the same law of evolution at all points in the active medium, and this law does not depend on the initial (smooth) distribution $P_0(z)$.

3.3. Self-similar description of the oscillator regime

We now turn to the nonlinear stage of super-radiance. We consider a short sample ($L \lesssim L_c$), initially without dissipation or relaxation: $\sigma = T_{1,2}^{1} = 0$. In accordance with (2.12), the transformation to the polar Bloch angle $\varphi = \mathrm{Re}\,\varphi$

$$P = -idN \sin\varphi, \quad E = \frac{\hbar}{d}\frac{\partial\varphi}{\partial t}, \quad \Delta N = N\cos\varphi \tag{3.9}$$

enables us to reduce (1.9′)–(1.11′) to the sine-Gordon equation

$$\frac{\partial^2\varphi}{\partial z\, \partial \tilde{t}} = \frac{c}{4L_c^2} \sin\varphi, \quad \tilde{t} = t - \frac{z}{c}, \quad L_c = \frac{c}{\Omega_c}. \tag{3.10}$$

This equation has self-similar solutions[63–65] that obey the equation

$$\frac{d^2\varphi}{d\xi^2} + \xi^{-1}\frac{d\varphi}{d\xi} = \sin\varphi, \quad \xi = [(ct - z)z]^{1/2} L_c^{-1}. \tag{3.11}$$

Super-radiance is described by the one-parameter family of solutions of (3.11) that are nonsingular at the origin ($\varphi'(0) = 0$) and depend on the asymptotically small initial angle $\varphi(0)$, given by [14,20] $\varphi_0 \sim (NV)^{-1/2} \ll 1$. This solution transforms to the sequence of lightly damped pulses shown in Fig. 6. This is the concluding stage of the development of absolute instability of waves with the anomalous growth rate (2.4).

3.4. Transition from the oscillator to the single-pulse regime

An oscillator regime similar to (3.11) is observed, for example, in infrared experiments[15] with cesium vapor in samples with $F \lesssim 1$ and $L_\sigma \sim L \sim 3$ cm ($\tau \sim 1$ ns and the super-radiant power is $Q_{max} V \sim 1$ mW). However, the recorded damping of super-radiant oscillations occurs much more rapidly[14,15,23] than indicated by (3.11) and, when the length L_c is increased, or the initial inversion reduced, only the

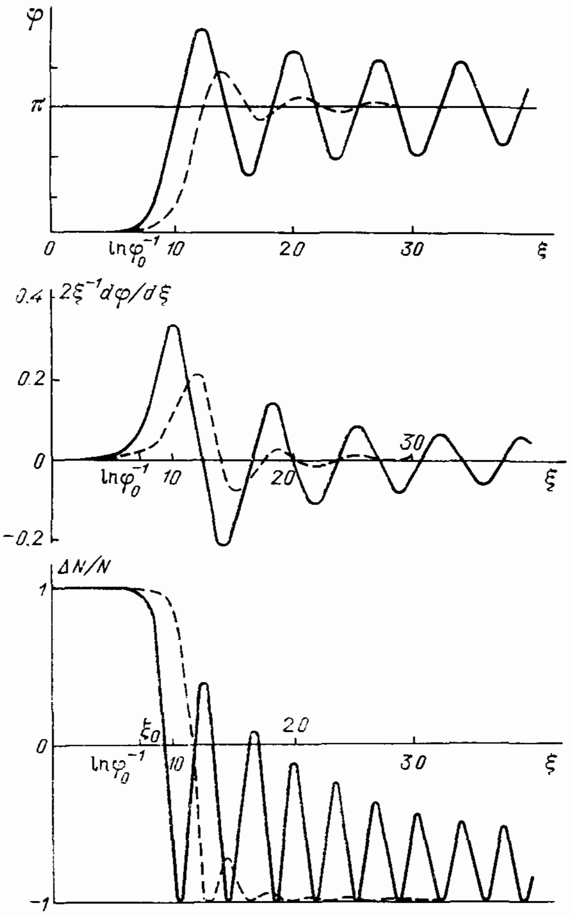

FIG. 6. Solution of the sine-Gordon equation: self-similar without dissipation–(3.11) (solid curves) and quasi-self-similar with dissipation–(3.15) for $Z/T = 1/6$ (dashed). The graphs for the Bloch angle φ, the field amplitude $2\varphi'/\xi$, and the inversion $\Delta N/N = \cos\varphi$ are all plotted for the initial condition $\varphi_0 = \varphi(0) = 10^{-3}$, $\varphi'(0) = 0$.

single-pulse regime is observed. This is due to factors that have been ignored in the model (3.10), namely, field dissipation, Lorentz and Doppler line broadening, transverse inhomogeneity of the pump and of radiation field, and so on. The relative importance of these factors has not been properly investigated experimentally, especially in the case of dissipation.

The effect of relatively strong dissipation ($2\pi\sigma > T_2^{-1}$) on super-radiance can be described theoretically as follows. The super-radiance profile is formed on the nonlinear stage of the dissipative instability of polarization waves, beginning with the time t_d at which inversion is removed at $z = L$. Dissipation traps super-radiance, reduces the growth rate ω_p'', and increases the delay t_d. Actually, for the asymptotic surface source (3.8) and a relatively short sample, we find from (1.11′) that, when $t_{max} \gg T_2$, t_d, we have

$$t_d \sim \frac{L_c(\ln\eta)^2}{4\Omega_c L},\quad \eta = \left|\frac{Nd}{P_0(0)\exp[-L(L_\sigma^{-1}-c^{-1}T_2^{-1})]}\right|^2 \gg 1, \tag{3.12}$$

where, for simplicity, $T_1^{-1} = 0$. In a long sample with $L \gg L_\sigma > 8L_c^2/cT_2$, the removal of inversion begins with the "propagating source" regime (3.7): $t_d \sim (\omega_p'')^{-1}\ln|Nd/\bar{P}_0| < t_\sigma = 4LL_c^2/cL_\sigma^2$). This is followed by the emission of a random sequence of pulses: Each segment of length $\sim Lt_d/t_\sigma \sim L_\sigma \ln|Nd/\bar{P}_0|$ produces its own super-radiant pulse.

The effect of dissipation is greater for $L_\sigma \ll L_c$. The replacement (3.9) and the assumption that $T_2^{-1} = 0$ then lead to the sine-Gordon equation with dissipation:[60]

$$\frac{\partial^2\varphi}{\partial Z\,\partial T} + \frac{\partial\varphi}{\partial T} = \sin\varphi,\quad Z = \frac{z}{L_\sigma},\quad T = \frac{L_\sigma(ct-z)}{4L_c^2}. \tag{3.13}$$

This enables us to extend the solution of the linear problem of Sect. 3.2 to the nonlinear stage (Fig. 7). For $z \gg L_\sigma \ln\varphi_0^{-1}$, the asymptotic form of the "propagating source" (3.7) gives rise to the single-pulse regime (the region BB_1B_2), which smooths $\varphi_0(z) \approx iP_0(z)/Nd$ to $\bar{\varphi}_0(z)$:

$$\frac{\partial\varphi}{\partial T} \approx \sin\varphi \Rightarrow \varphi(Z, T) = 2\,\mathrm{arctg}\left[\frac{1}{2}\bar{\varphi}_0(Z-T)\exp T\right]. \tag{3.14}$$

For $z \ll L_\sigma \ln\varphi_0^{-1}$, the asymptotic super-radiance in the case of $\ln\varphi_0^{-1} \gg 1$ and $[(1/\xi) + (Z/T)^{1/2}]^2 \ll 1$ yields the quasi-self-similar approximation $\partial/\partial(Z/T)^{1/2} \approx 0$ for $\varphi[\xi,(Z/T)^{1/2}]$:

$$\frac{\partial^2\varphi}{\partial\xi^2} + \left[\xi^{-1} + \left(\frac{Z}{T}\right)^{1/2}\right]\frac{\partial\varphi}{\partial\xi} = \sin\varphi,\quad \xi = 2(TZ)^{1/2}. \tag{3.15}$$

The one-parameter family of solutions

$$\varphi(\xi) = \varphi_0(0)\exp\left[-\frac{\xi}{2}\xi\left(\left(\frac{Z}{T}\right)^{1/2} + \left(4+\frac{Z}{T}\right)^{1/2}\right)\right] \times \Phi\left(\frac{1}{2} + \left(\frac{T}{Z}+\frac{1}{4}\right)^{1/2};\ 1;\ \xi\left(4+\frac{Z}{T}\right)^{1/2}\right) \tag{3.16}$$

of the linearized equation (3.15) joins smoothly to the asymptotic "surface source" (3.8), where Φ is the confluent hypergeometric function. The quasi-self-similar solutions of (3.15) generated by them (see Fig. 6) describe a highly

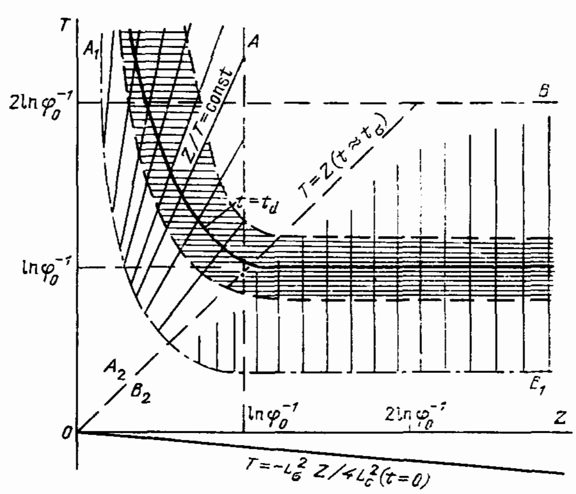

FIG. 7. Subdivision of the coordinate-time plane into zones of different unidirectional super-radiant regimes in accordance with the sine-Gordon equation with dissipation (3.13). The double-hatched area represents the nonlinear zone without inversion ($t = t_d$ corresponds to maximum super-radiance). Zones in which the linear theory is valid in inverted and uninverted media, respectively, are shown below and above. Curve A_1A_2 represents the beginning of the range of validity of the intermediate asymptotic form of the "surface source" (3.8), and curve B_2B_1 represents the same situation for the "traveling source" (3.7).

damped oscillator regime (the region AA_1A_2). The approximations given by (3.14) and (3.15) differ from the well-known "mean-field approximation"[12-15,25,66,67] for which $\partial/\partial z = 0$ in (3.13) and propagation effects are ignored:

$$\frac{\partial^2\varphi}{\partial t^2} + 2\pi\sigma\frac{\partial\varphi}{\partial t} = \frac{\Omega_c^2}{4}\sin\varphi. \tag{3.17}$$

Oscillations and the transition to single-pulse super-radiance (for $L_\sigma \sim L_c$) are similar within the framework of equation (3.17) to the oscillations of a damped pendulum and are different in character. They are due to beats between the e and p modes, and not to effects associated with the propagation of radiation (see Sects. 4.3 and 4.5).

3.5. Super-radiance and the instability of electromagnetic waves. Superluminescence

Dicke's original treatment[10] of super-radiance as an aperiodic collective spontaneous relaxation of excited molecules via the emission of the entire energy stored in the sample in the form of radiation is closest to the single-pulse regime generated by the dissipative instability of polarization waves. We shall therefore begin by focusing on this idea (see Sects. 1.4 and 3.4, and also Sects. 4.2 and 4.4 in which reflections are taken into account). As dissipation is reduced to $2\pi\sigma < T_2^{-1}$, the dissipative instability of polarization waves is replaced by the anomalous instability of electromagnetic waves (see Sect. 2.1). Super-radiance does not then vanish, but smoothly changes its properties, while remaining a coherent transient process of stimulated emission. The reduction in dissipation σ and, in particular, the reduction in diffractive emission of energy out of the sample ($\sigma_{\rm diff}$), leads to a delay and reabsorption of the emission by the active medium. The result is that, instead of a single pulse, we have a long train of peaks, i.e., the oscillator regime.[23,62,68]

In the case of strong relaxation of polarization, $T_2^{-1} \gtrsim \Omega_c/2$, the instability of electromagnetic waves described by (2.3) is possible only for a very low level of dissipation ($2\pi\sigma < -\omega_c^2 T_2/4$) and has the maser-like character, generating not super-radiance but superluminescence, i.e., quasi-stationary induced amplification of spontaneous emission.[12-15,23,26,67] The last of these is described by the rate equations for the transfer of radiation intensity because, owing to strong relaxation, the polarization of the medium no longer exhibits its own dynamics, but follows the field adiabatically: $P = (E\omega_c^2/8\pi\omega_0)/(\omega_0 - \omega - iT_2^{-1})$ [see (1.10')]. If the amplification coefficient is large ($\Omega_c^2 T_2 L/2c \gg 2\ln\varphi_0^{-1}$), then a short pulse of superluminescence[9)] appears after a short delay $t_d \sim (4\ln\varphi_0^{-1})/\Omega_c^2 T_2 \gg T_2$. The duration of this pulse is of the order of one mean free transit time in the sample, L/c. For a lower degree of amplification, the exponential intensity profile $I_{\rm spout}\exp(2\omega_c''(k)z/c)$ is established throughout the sample, and the duration of this weakly amplified spontaneous emission by the inverted sample is determined by the ratio of stored internal energy of the molecules to the emitted power. Similar regimes of superluminesce of duration $\Delta t \gg T_2$, in which the radiated intensity is proportional to the concentration N of the active molecules (and not to N^2 as in super-radiance) have long been known and used in electronics, both quantum and classical (see, for example, Refs. 69–72). In contrast to super-radiance, their characteristic feature is that they cannot radiate more than one-half of the energy stored in the inverted molecules: induced amplification gives way to the equalization of level populations, $N_1 = N_2 = N/2$.

To avoid misunderstanding, we note that, when we speak of superluminescence, we have in mind the quasistationary emission of photons by isolated molecules, due to incoherent vacuum fluctuations of field and polarization (spontaneous emission) or the incoherent wave field produced independently and amplified by the remaining molecules (induced emission).[11,12,72] The last process is characterized by the Einstein coefficient $B_\omega = n_\omega A_\omega$, which can be calculated from perturbation theory as the probability of a stimulated transition of a molecule per unit time in a given monochromatic field containing N_ω normal free-space modes of frequency ω (A_ω is the corresponding Einstein coefficient for spontaneous emission). At the same time, throughout this review, whenever we are concerned with the semiclassical theory of super-radiance, the concept of an induced process will be understood to be wider and not exhausted by the above rate equations based on Einstein's coefficients.

Let us explain the foregoing points in terms of the energy balance relation

$$\frac{2\omega''|E^2|}{8\pi} = \hbar\omega_0\Delta N\rho - \frac{1}{2}\sigma|E^2|.$$

The rate of increase in the field energy is thus seen to be determined by the competition between resistive losses and stimulated emission by the molecules. The probability ρ of a stimulated transition in a given molecule between the upper and lower states per unit time can be found from (1.10) and (1.11):

$$\rho = \frac{\mathrm{Im}(-\omega E^*P)}{2\hbar\omega_0\Delta N} = -\frac{\omega_c^2|E^2|}{16\pi\omega''\hbar\omega_0\Delta N},$$

where, for the sake of simplicity, we have neglected incoherent relaxation of polarization ($T_2^{-1} \ll \omega''$) and have confined our attention to the resonant case: $\omega' = \omega_0$. As can be seen, the probability of stimulated emission is determined by the spectral density of the radiation, which is inversely proportional to the growth rate ω''. If we substitute for ρ in the energy balance equation, we obtain $\omega'' = -\omega_c^2/4\omega'' - 2\pi\sigma$. Hence, when $|\omega_c| \ll 2\pi\sigma$, we obtain the expression for the growth rate of dissipative instability (2.5) (without T_2^{-1}), whereas for $|\omega_c| \gg 2\pi\sigma$, we obtain the anomalous growth rate (2.4). The departure from the corresponding superluminescence growth rate ω_c'' (2.3), which is well-known in laser theory, is due to the wide spectrum of super-radiance ($\Delta\omega \sim \omega_p''$), which exceeds the relaxation width of the transition, T_2^{-1} (since otherwise the expression for ρ would contain ω'' instead of T_2^{-1}). It is precisely the coherence, i.e., short duration, of the super-radiance (collective spontaneous emission) that distinguishes it from superluminescence.

Super-radiance is therefore a process of induced emission of the internal energy of the molecules, which is due to their interaction with the self-consistent coherent radiation field (electric field of the polarization wave). Hence the word "spontaneous" in the commonly used phrase "collective spontaneous emission" refers, strictly speaking, only to the absence of external radiation or the formation of super-radiance from quantum noise. On the other hand, when external radiation is incident on an active sample, this gives rise to the so-called initiated super-radiance in which there is no quantum dynamics at the initial stage of the Dicke super-radiance.

3.6. Initiated super-radiance. Polariton π-pulse in a long amplifier

The super-radiant regime in the presence of strong relaxation of polarization is more readily established by using an initiating electromagnetic pulse of short duration $\tau_{in} \ll t_d$ and area

$$\varphi_0 = \frac{d}{\hbar}\int_0^{\tau_{in}} E_{in}(t)\,dt \gg \left|\frac{\bar{P}_0}{Nd}\right|$$

[see (3.1) and (3.12); Refs. 15, 33b, 35, and 73]. When this device is used to produce the initial macropolarization, the result is that the delay time t_d is reduced by the factor $(\ln \varphi_0^{-1})^2 \gg 1$, down to a value $t_d < T_2$. This has been used[35] to produce super-radiance in ruby crystals and in Nd:YAG at 100 K.

A closely related problem is that of the propagation of the initiating pulse, preceded by a pump, in a long coherent amplifier ($L_c \ll L \ll cT_2$). When there is no dissipation and $z \to \infty$, this produces a time-dependent pulse of duration $\tau \propto 1/z$, amplitude $E \propto z$, and area $d\int E(t)dt/\hbar \approx \pi$. This π-pulse takes up the entire energy stored in the medium.[60,64,74] Its asymptotic behavior is quasi-self-similar with $E = (\hbar cz/2dL_c^2\xi)\,\partial\varphi/\partial\xi$, and is determined by the solution (3.11) of the sine-Gordon equation (3.10) in which, for each value of the self-similar variable φ, there are two coordinates $z_{1,2} = \{ct \mp [(ct)^2 - 4\xi^2 L_c^2]^{1/2}\}/2$. The inversion ΔN first vanishes for $\xi = \xi_0 \sim \ln \varphi_0^{-1} \gg 1$ (see Fig. 6), i.e., in a time $t_0 = 2\xi_0 L_c/c$ at the point $z_0 = \xi_0 L_c$, where $z_1 = z_2$. Eventually, two relaxation waves are emitted in opposite directions by this point, and the inversion on their wave fronts $z_{1,2}^{(0)} = \{ct \mp [(ct)^2 - 4\xi_0^2 L_c^2]^{1/2}\}/2$ is $\Delta N(z_{1,2}^{(0)}) = 0$ (Fig. 8). As $t \to \infty$, the wave front propagating to the left approaches the beginning of the amplifier $z_1^{(0)} \approx \xi_0^2 L_c^2/ct \to 0$), whereas the wave front propagating to the right approaches the light cone ($z_2^{(0)} \approx ct$). The former generates the self-similar super-radiant pulse in the short sample ($L \lesssim L_c$; Sect. 3.3) and the latter produces a time-dependent π-pulse in the long amplifier.[65]

The growth of the π-pulse is limited by dissipation. The result is a time-independent π-pulse[24,60] that, in accordance with (3.13), assumes the form $\partial/\partial Z = 0$, for $E = \hbar cL_\sigma/4dL_c^2\,\mathrm{ch}(T - T_0)$, which is analogous to the single-pulse super-radiance (1.21) and (4.11). It can be referred to as a polaritonic soliton, since it is associated with waves in the polariton spectrum (2.1) and arises as a consequence of the dissipative instability of polarization waves.

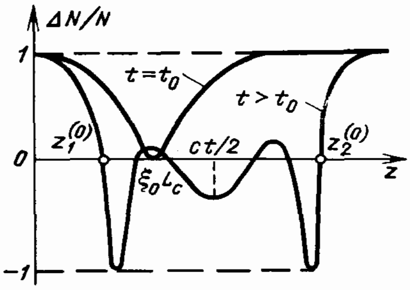

FIG. 8. Spatial structure of inversion in a long amplifier,[65] corresponding to the self-similar solution (3.11) for times $t_0 = 2\xi_0 L_c/c$ and $t > t_0$.

3.7. Super-radiance from a three-dimensional sample in the absence of reflections from boundaries

The approach used in macroscopic electrodynamics is convenient as a means of generalizing the problem of unidirectional super-radiance to a real three-dimensional situation.[10] The linear stage of super-radiance ($\Delta N = N$), that determines the character of the instability is described by a partial differential equation for the Laplace time transform $\mathbf{E}(\omega,\mathbf{r})$

$$\left(\frac{\omega}{c}\right)^2 \varepsilon \mathbf{E}(\omega, \mathbf{r}) - [\nabla, [\nabla, \mathbf{E}(\omega, \mathbf{r})]] = \mathbf{\Pi}(t = 0, \mathbf{r}). \quad (3.18)$$

This follows from the Maxwell-Bloch equations. The right-hand side $\mathbf{\Pi}$ is specified by the initial distribution of the field, the polarization, and their time derivatives in the active sample. Unstable solutions are characterized by the "integral" growth rate $\omega'' = W/2W > 0$, which, according to (2.8), is

$$\omega'' = -\left(\Sigma_{\text{изл}} + \int_V Q dV\right)\frac{1}{2W},$$

$$W = \int_V w\, dV, \quad \Sigma_{\text{изл}} = \frac{c}{4\pi}\int_{S_0} [\vec{\mathscr{E}}, \vec{\mathscr{B}}]\, d\mathbf{S}. \quad (3.19)$$

Hence, it follows again that super-radiance is associated with the dissipative instability, since for $\omega'' > 0$ and small volume sample losses Q, the presence of the energy flux $\Sigma_{rad} > 0$ across the sample surface S_0 is compatible only with a negative energy ($W < 0$).

In the interior of a uniformly inverted sample, the solution $\mathbf{E}(\omega,\mathbf{r})$ of (3.18) can be expanded in terms of the complete set of known multipole fields with a continuous spectrum of complex frequencies.[4] However, the more informative is the Green's function for the Maxwell-Bloch wave equations (1.9) and (1.10), which is determined by the initial condition $\delta(\mathbf{r} - \mathbf{r}')$ and the radiation condition, and has a definite time dependence. It takes the form of an outgoing spherical wave $\propto \theta(ct - |\mathbf{r} - \mathbf{r}'|)$. As in the one-dimensional problem, analysis of the three-dimensional Green's function shows that its dissipative instability with the "integral" growth rate (3.19) leads to the Dicke super-radiance. We note that, in a very long sample ($L \gg L_c$), the super-radiance is frequently reabsorbed by the medium on the nonlinear stage, so that the loss factor representing emission through the sample surface becomes unimportant, and super-radiance transforms from the periodic to the oscillator or irregular regime that consists of a random sequence of pulses (see the beginning of Sect. 3.5).

The scheme presented above for the solution of the electrodynamic vector problem has been implemented systematically in the case of a sphere.[76] The problem has also been solved for the scalar model[77] in the case of a sphere and a cylinder with $F \gtrsim 1$, for which the replacement $-[\nabla,[\nabla,\mathbf{E}]] \to \Delta E$ in (3.18) yields

$$E(\omega, \mathbf{r}) = \int_V \Pi(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}') \, d^3\mathbf{r}',$$

$$G_0(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{\exp(i\omega\varepsilon^{1/2} |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}. \tag{3.20}$$

This model transition to the Green's function of the scalar wave equation for infinite space corresponds to the retention of only the first term of the Debye expansion in the theory of diffraction, i.e., the geometric-optics approximation with reflection.[11)] The inverse Laplace transformation in the approximation defined by $|\varepsilon - 1| \ll 1$, and the treatment of $E(t,\mathbf{r})$, $P(t,\mathbf{r})$ as Heisenberg operators with initial conditions that are δ-correlated in space, has enabled the above authors to provide an analytic description of the linear stage of super-radiance in three-dimensional samples. As in the case of a progressive pump in a cylinder with $F \gg 1$, the mean super-radiant intensity for $\sigma = T_{1,2}^{-1} = 0$ and $\xi \gg 1$ [see (3.11)] is as given in Refs. 77 with $\langle \mathscr{E}^2 \rangle / 4\pi = \pi^2 c N d^2 F \exp(2\xi)/\lambda^3 \xi^2$. The incoherence of spontaneous noise, amplified during propagation along geometric-optics rays from one end of the cylinder to the other, ensures that the super-radiance at the exit end S of the rod has the characteristic coherence area $\sim \lambda L/F$, i.e., of the order of the area of the first Fresnel zone divided by the Fresnel number $F = S/\lambda L$. This conclusion is in agreement with the well-known optical theorem of van Cittert and Zernike,[77,170] and means that the super-radiance is due to a large number $F^2 \gg 1$ of diffraction modes (rays) whose relative intensities fluctuate from one shot to the next, but the total intensity is almost constant. The investigations presented above show that the methods of macroscopic electrodynamics are effective in super-radiance problems.

The analytic continuation of linear solutions to the nonlinear stage of the three-dimensional problem is still an open question.[13,14] Numerical studies have shown that, for random initial conditions and nonuniform inversion, the spatial structure and angular distribution of super-radiance exhibit considerable fluctuations. This means that, when they are recorded with high angular resolution, oscillations that occur at different points on the exit cross section are not synchronized (see Sects. 3.3 and 3.4). Moreover, the resultant intensity is averaged out and corresponds to a super-radiant single pulse. All this has been confirmed directly by experiment.[79]

4. REFLECTIONS IN BOUNDED SAMPLES

Because of high gain, even weak reflections can have a significant effect on super-radiant power and dynamics, since they transform the continuous super-radiance spectrum to a discrete spectrum. Reflections have attracted increasing attention[23,36,66,80-82] during the 1980s in connection with super-radiance experiments involving resonators and waveguides,[14,23,31] and also impurity crystals.[32-35]

4.1. Hot modes with a discrete spectrum

Reflections from the boundaries of a dielectric sample of an active medium, including the abrupt increase in the background permittivity ε_0 (which is taken into account below together with the susceptibility of the two-level medium), give rise to hot modes[12)] with a discrete spectrum, i.e., the "natural modes" of Refs. 83 and 84. In general, hot modes are introduced for a time-independent structure of inversion $\Delta N(\mathbf{r})$ as the eigensolutions of the *homogeneous* equations of macroscopic electrodynamics (3.18) (for $\Pi = 0$) in a sample of volume V, which satisfy the well-known conditions on the sample boundary S_0 and the radiation conditions.[1-4,84] These eigensolutions

$$\mathbf{E}_m(\mathbf{r}), \quad \mathbf{B}_m(\mathbf{r}), \quad \mathbf{P}_m(\mathbf{r}) = \chi(\omega_m, \mathbf{r}) \mathbf{E}_m(\mathbf{r}) \tag{4.1}$$

have a discrete spectrum of complex frequencies $\omega_m = \omega'_m + i\omega''_m$, which is determined by the characteristic equation that is identical with the traditional equation used in the time-independent eigenmode theory of diffraction.

The solution $\mathbf{E}(\omega,\mathbf{r})$ of the *inhomogeneous* equations (3.18) in the interior of the sample contains a set of hot modes $\Sigma_m a_m(\omega) \mathbf{E}_m(\mathbf{r})$ with amplitudes $a_m(\omega)$ determined by the initial condition Π. (The completeness of this set has so far been proved only for special cases.[83,84]) The field outside the sample is formed by the emission of hot modes from its surface.

The solution of the initial-value problem of super-radiance on the linear stage is given by the Laplace transform of $\mathbf{E}(\omega,\mathbf{r})$. It includes contributions due to the pole ω_m of the hot-mode amplitude $a_m(\omega) \propto (\omega - \omega_m)^{-1}$, and other singularities of the integrand, e.g., the essentially singular points. The former lead to the super-radiance of modes with a discrete spectrum $\propto \mathbf{E}_m(\mathbf{r}) \exp(-i\omega_m t)$, and the latter appear even in the limit of super-radiance with a continuous spectrum in the absence of reflections (cf. unidirectional super-radiance). The rapid removal of inversion $\Delta N(t,\mathbf{r})$ on the nonlinear stage leads to time-dependent hot modes (time-dependent frequencies and structure; see Sect. 4.2), their nonadiabatic interaction (Sect. 4.3) and mixing with continuous-spectrum waves (Sect. 4.5).

4.2. Self-excitation of polariton modes in a one-dimensional layer

In a one-dimensional layer of an inverted medium of length L, hot modes take the form of the sum of two counter-propagating waves radiated outwardly into the vacuum (Fig. 9):

$$\mathbf{E}_m = \mathbf{x}^0 (E_m^{(1)} + E_m^{(2)}) e^{-i\omega_m t},$$
$$E_m^{(1),(2)} = \frac{\varepsilon^{1/2} \mp 1}{2\varepsilon^{1/2}} E e^{\pm i\omega_m \varepsilon^{1/2} z/c}, \tag{4.2}$$
$$\omega_m \varepsilon^{1/2}(\omega_m) = \frac{m\pi c}{L} - i\nu_m, \qquad \nu_m = \frac{c}{L} \ln \frac{1 + \varepsilon^{1/2}(\omega_m)}{1 - \varepsilon^{1/2}(\omega_m)}$$

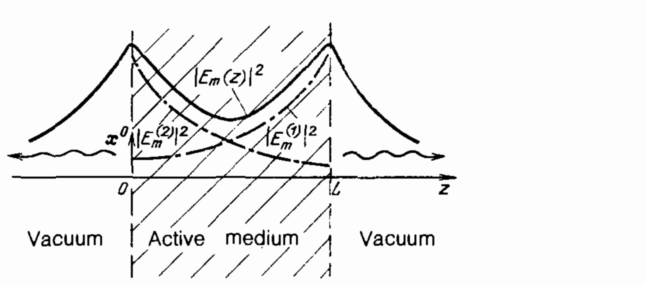

FIG. 9. Spatial structure of hot modes in the one-dimensional model of super-radiance.

In the resonance approximation, their frequencies are equal to $\omega_{e,p}(k_m)$, (2.1) if we assume that the wave number is discrete and given by $k_m = m\pi/L$, and we if we introduce the replacement $2\pi\sigma \to 2\pi\sigma + \nu_m$. The quantity ν_m represents losses by radiation into ambient space.

We thus have hot (m, e) modes of positive energy and (m, p) modes of negative energy.[22,29] Their properties are analogous to those of electromagnetic waves and polarization waves in an infinite medium. However, in bounded samples, σ appears together with field dissipation due to radiation via the sample boundaries, i.e., $\sigma_{\rm rad} = (c/4\pi L)\ln R^{-1}$, where $R = |(\varepsilon - 1)/(\varepsilon + 1)|^{1/2}$ is the reflection coefficient. Thus, for short samples ($L \ll L_c$) with large inversion ($\Omega_c \gg 2/T_2$), which is typical for super-radiance, we find that for $R \ll 1$ and $2\pi\sigma \ll 2\pi\sigma_{\rm rad} \ll \Omega_c^2 T_2/4$ the dissipative instability develops for the polariton (m, p) mode closest to the line center, and its growth rate is $\omega_p'' \approx -\omega_c^2/8\pi\sigma_{\rm rad}$. In the mean-field adiabatic approximation ($\omega_p'' \propto \overline{\Delta N}(t)$), this leads to a super-radiant single pulse described by a formula such as (1.21) with $\tau = 1/2\omega_p''(t=0)$.

The conclusions of Sect. 3.5 can be extended to hot-mode super-radiance. In particular, for small inversion, there are no super-radiant modes, and quasistationary superluminescence with a continuous spectrum is found to develop. Its power and time of emission are given by

$$Q_{\rm SL} V \sim \frac{\hbar\omega_0 NV}{\tau_{\rm SL}},$$
$$\tau_{\rm SL} \sim 2\pi^2 T_1 L^2 S^{-1} (\mu_0 L)^{3/2} \exp(-\mu_0 L) \gtrsim \frac{2}{\Omega_c^2 T_2} = \frac{1}{\mu_0 c},$$

if the amplification coefficient at the line center is $\mu_0 L = 2\omega_c'' L/c \gtrsim 2$. For a high enough initial inversion, we have a transition to super-radiant modes with a discrete spectrum. The super-radiant power at the pulse maximum is $Q_{\rm SL} V \sim \hbar\omega_0 NV\omega_p''/2 \gg Q_{\rm SL} V$. However, in contrast to the situation without reflections, the transition that occurs when the threshold for the generation of polariton modes with $\omega_p'' = \Omega_c^2/8\pi\sigma_{\rm rad} - T_2^{-1}$ is reached (Fig. 10) is accompanied by a much more rapid increase in the emission rate and power. This sharp reduction in the duration τ of the pulse is also found to occur for small reflection coefficients $R \ll 1$. It is characterized by a large "jump" $\tau_{\rm SL}^*/T_2 \gg 1$, where the time $\tau_{\rm SL}^*$ is defined in terms of the mode generation threshold, i.e., the condition $\omega_p'' \sim 1/2\tau_{\rm SL}$, and is given by $\tau_{SL}^* \sim T_1/\{1 + [S/2\pi^2 L^2 R(\ln R^{-1})^{3/2}]\}$.

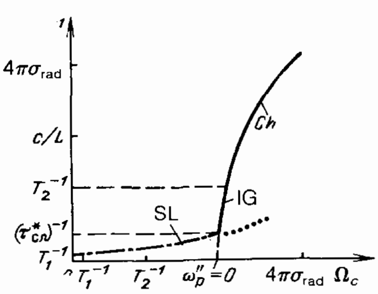

FIG. 10. Rapid variation of the rate of emission after transition from superluminescence (SL) to incoherent ($\tau^{-1} < T_2^{-1}$) generation of polariton modes (IG) and then to their super-radiance in a short sample ($L \ll cT_2 \ln R^{-1}$) with reflection coefficient $R \ll 1$ (τ is the duration of the pulse of radiation).

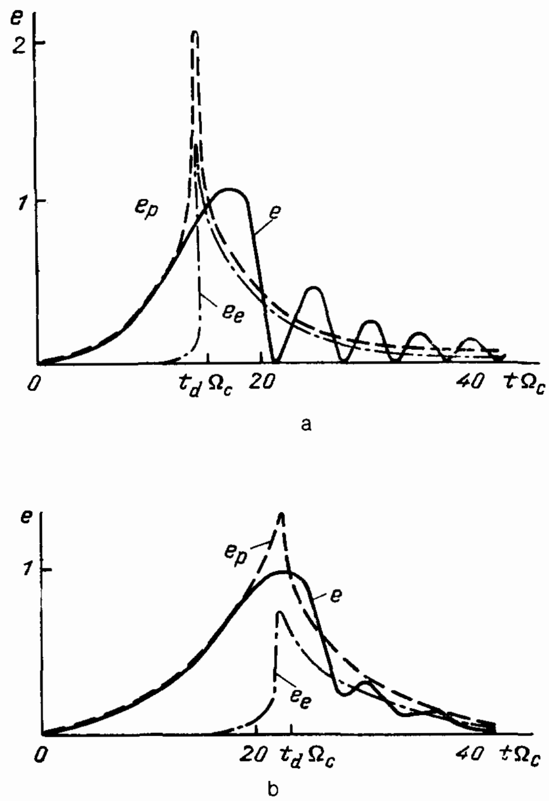

FIG. 11. Replacement of self-consistent optical nutation [oscillator regime of superabsorption (a): $2\pi\sigma = \Omega_c/4$, $ck - \omega_0 = \Omega_c/32$ with irreversible collective relaxation [single-pulse superabsorption (b): $2\pi\sigma = \Omega_c/2$, $ck - \omega_0 = \Omega_c/8$ as the dissipation σ is increased in the mean field model. The resultant field $e = |E|(8\pi\hbar\omega_0 N)^{1/2}$ is a superposition of the e and p mode fields; $T_2^{-1} = 10^{-2}\Omega_c$.

4.3. Superabsorption and optical nutation in a cavity resonator

An increase in the reflection coefficient R and in the sample length L obviously impede the escape of radiation from the sample. When the inequality $2\pi\sigma_{\rm rad} \lesssim |\omega_c|$ (Sect. 2.1) is first satisfied, the instability of the polariton modes becomes anomalous and has the high growth rate $\omega_{m,p}'' \propto (\Delta N)^{1/2}$ (for $|\omega_c| \gg T_2^{-1}$). As a result, the self-excited (m, p) modes rapidly remove inversion and nonadiabatically generate damped (m,p) modes of comparable amplitude. Their nonlinear beats generate oscillator super-radiance. In a closed resonator ($R \sim 1$), this essentially coincides with an effect that is well-known in quantum radiophysics and involves the periodic transfer of energy between the electromagnetic field and the two-level medium, i.e., self-consistent optical nutation.[67,86-88] Under the conditions of strong resistive dissipation that replace radiative losses and lead to the dissipative instability of the polariton modes with growth rate $\omega_{m,p}'' \lesssim 2\pi\sigma$, this effect reduces to the aperiodic collective transformation of the energy stored in the inverted medium into heat in a time $\tau \sim 1/2\omega_p'' \ll T_1$ (Fig. 11). This superabsorption effect[25,29] is completely analogous to single-pulse super-radiance.

Polariton modes are excited not only in supper-radiance or superabsorption by an inverted medium, but also in an absorbing (noninverted) two-level medium in the cavity of a laser that contains an active element with negative losses

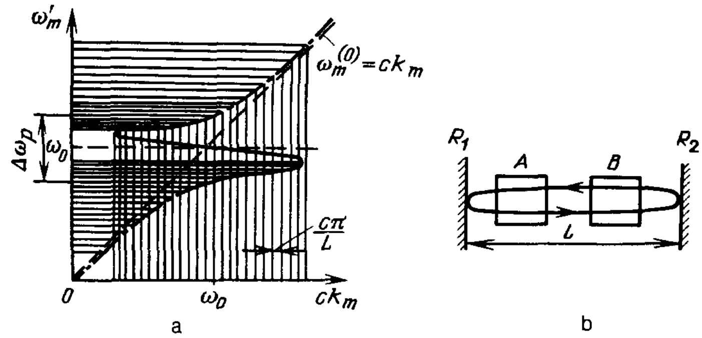

FIG. 12. a—Polariton spectrum of the quasistationary generation by a laser ($\omega''_m \approx 0$, $\Delta\omega_p \sim \omega_c$). b—Laser with mirrors $R_{1,2}$, active element A ($\sigma_a < 0$), and two-level absorber B.

$\sigma_a < 0$ (Fig. 12). Laser generation may then be largely determined by strong frequency dispersion of polaritons (see, for example, Refs. 89).

Optical nutation accompanied by the excitation of polariton modes in a system of excitons and photons in semiconductors was investigated in Ref. 137.

4.4. Super-radiance by a three-dimensional sample with reflecting boundaries

Hot-mode super-radiance by three-dimensional samples differs from one-dimensional super-radiance by the large number of simultaneous unstable polariton modes of different structure. This multimode situation leads to a number of static and dynamic super-radiant features. and is responsible for the angular distribution of the super-radiance. For example, possible effects include nonsimultaneous emission in different directions (see Ref. 59), partial depolarization of radiation, and smoothing of oscillations.[14]

Hot modes and their super-radiance in the three-dimensional case have been examined in detail but only for a sphere.[29] In the limit of a small particle ($a \ll \lambda$), the growth rate has been determined only for the single electrodipole mode of the sphere. Its dissipative instability gives the Dicke super-radiance discussed in Sect. 1.4. For a sphere of radius $a \gg \lambda$, the number of unstable modes with a discrete spectrum is such that $M \sim (\omega_0 a/c)^2 \gg 1$, and their super-radiance on the linear stage is described by

$$VQ_{\rm mod}(t) \sim M\hbar\omega_0 \cdot 2\omega''_{\rm p} \exp(2\omega''_{\rm p} t),$$

$$t_{\rm d}^{\rm mod} \sim \frac{L_{\rm c}^2 \ln R^{-1}}{2ac} \ln\frac{NV}{2M}, \quad \omega''_{\rm p} = \frac{\Omega_{\rm c}^2}{8\pi\sigma_{\rm rad}}. \tag{4.3}$$

This is the situation that actually occurs, and is responsible for the emission of the energy of the medium, if the mode delay time $t_{\rm d}^{\rm mod}$ is less than the delay time $t_{\rm d}^{v}$ for waves in the continuous spectrum. This occurs if the reflection coefficient is less than the critical value: $R > R_{\rm st} \sim (M/NV)^{1/4}$. It can be estimated by comparing (4.3) with the solution[76] of the vector problem for the linear stage of continuous-spectrum super-radiance if we neglect the reflection and the delay of waves in the sphere

$$VQ_v(t) \sim \frac{\hbar\omega_0 (\omega_0 a/c)^2}{8\pi t} \exp\frac{2(2act)^{1/2}}{L_{\rm c}},$$

$$t_{\rm d}^{v} \sim \frac{L_{\rm c}^2}{8ac}[\ln(\eta\ln\eta)]^2, \quad \eta = \frac{2\pi NV}{(\omega_0 a/c)^2}. \tag{4.4}$$

We must now compare the hot-mode super-radiance in an open sphere with the "single-mode" cylinder with Fresnel number $F \equiv S/\lambda L \sim 1$ (which is relatively close to the one-dimensional model; Sect. 4.2).[13,14] For a given size $L \sim 2a \gg \lambda$, reflection coefficient R, and inversion density ΔN, the duration of the super-radiant pulse in these samples is the same: $\tau = 1/2\omega''_{\rm p}$. The peak power $VQ_{\max}$ emerging from the sphere is greater by the factor $V_{\rm sph}/V_{\rm cyl} \sim 2a/\lambda$ as compared with the cylinder. However, under typical conditions ($\sigma = 0$, $8c \ln R^{-1}/\Omega_{\rm c}^2 T_2 \ll 2a \lesssim L_{\rm c} \ll cT_2$, the emission by the sphere is a superposition of a large number N of modes with the multipole angular distributions. Hence, super-radiance by the sphere is almost isotropic when averaged over all the modes, and its intensity per unit solid angle is the same as for the "single-mode" cylinder radiating into a narrow solid angle $\sim\lambda/2a$ containing its axis.

We note that it was assumed in Refs. 16, 76, and 90 that each individual super-radiant shot from a sphere was confined to a narrow solid angle $\lesssim 4\pi/(\omega_0 a/c)^2 \ll 1$. The opposite point of view is examined in detail in Refs. 29, 77, and 91 (see Sect. 3.7) and is in agreement with experiments[33b,92,93] and numerical results [94,95] on super-radiance by three-dimensional samples $F \gg 1$.

4.5. Super-radiance of modes with a discrete spectrum and waves with a continuous spectrum

In general, the instability of waves with a continuous spectrum (Sects. 2 and 3) and modes with a discrete spectrum (Sect. 4) develop simultaneously. To investigate how the super-radiance of waves is replaced by super-radiance of modes when the reflection coefficient R is allowed to increase, we can consider the simple example of the unidirectional model in the form of a thin-ring sample with feedback coefficient R (Fig. 13).[82] This model is simpler than the one-dimensional plane-layer model (see Fig. 9) because it enables us to avoid additional complications associated with the interaction between counterpropagating waves on the nonlinear stage of super-radiance. For the sake of simplicity, we shall neglect distributed losses ($\sigma = T_{1,2}^{-1} = 0$) and the delay (1.9'), and will assume that the sample is short: $L/L_{\rm c} \ll \ln R^{-1}$, $\ln\varphi_0^{-1}$. The Maxwell-Bloch equations (1.9')–(1.11') then reduce to the sine-Gordon equation with the boundary condition [see (3.13)]

$$\frac{\partial^2\varphi}{\partial T\,\partial Z} = \frac{\ln R^{-1}}{4}\sin\varphi, \quad \left.\frac{\partial\varphi}{\partial T}\right|_{Z=0} = R^{1/2}\left.\frac{\partial\varphi}{\partial T}\right|_{Z=1},$$

$$Z = \frac{z}{L}, \quad T = 2\omega''_{\rm p} t = \frac{\Omega_{\rm c}^2 t}{4\pi\sigma_{\rm rad}}. \tag{4.5}$$

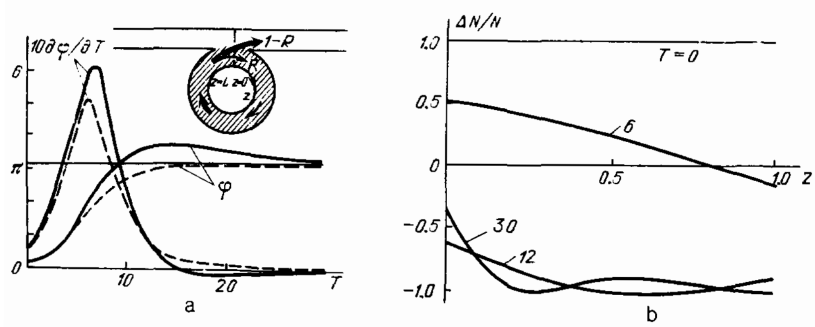

FIG. 13. Numerical solution of (4.5) (solid curve) and the mean-field approximation (4.11) (dashed curve) for $R = 1/e = 0.37$: dependence of the Bloch angle φ and field amplitude $\partial\varphi/\partial T$ at exit from the sample ($Z = z/L = 1$) (a) as functions of the time $T = 2\omega_p'' t$, and the evolution of the spatial structure of inversion $\Delta N/N = \cos\varphi$. (b) Unidirectional super-radiance in the ring model is shown at the top.

On the linear stage, when $\sin\varphi \approx \varphi$, the Laplace time transformation gives

$$\varphi(p, Z) = \left\{ \frac{\varphi(T=0, Z=0) - R^{1/2}\varphi(T=0, Z=1) + R^{1/2}\int_0^1 e^{\frac{(1-Z')\ln R^{-1}}{4p}} \frac{\partial\varphi(T=0, Z')}{\partial Z'} dZ'}{1 - R^{1/2}\exp[(\ln R^{-1})/4p]} + \int_0 e^{-\frac{Z'\ln R^{-1}}{4p}} \frac{\partial\varphi(T=0, Z')}{\partial Z'} dZ' \right\} \frac{e^{\frac{Z\ln R^{-1}}{4p}}}{p}. \tag{4.6}$$

The inverse Laplace transformation then leads to the two super-radiant components

$$\varphi = \varphi_c + \varphi_d, \quad \varphi_c(T, Z) = \operatorname*{res}_{p=0}[\varphi(p, Z)\exp(pT)],$$

$$\varphi_d(T, Z) = \sum_{m=-\infty}^{\infty} \frac{4p_m}{\ln R^{-1}} \exp\left[p_m T + Z\left(2\pi i m + \frac{1}{2}\ln R^{-1}\right)\right]$$

$$\times \left\{\varphi(T=0, Z=0) - R^{1/2}\varphi(T=0, Z=1) + R^{1/2}\int_0^1 e^{\frac{(1-Z')\ln R^{-1}}{2}} \frac{\partial\varphi(T=0, Z')}{\partial Z'} dZ'\right\}; \tag{4.7}$$

where the dimensionless Laplace variable is $p = -i(\omega - \omega_0)/2\omega_p''$, φ_c is the residue at the essentially singular point $p = 0$, and φ_d is the sum of the residues at all the other discrete singular points (poles)[58] $p_m = [2 + (8\pi i m/\ln R^{-1})]^{-1}$. The latter give the frequencies of the polariton modes (4.2).

The solution is easier to analyze if we write it in a different form by evaluating the Green's function and its convolution with the initial distribution $\varphi(T = 0, Z)$. We thus obtain the general solution, equivalent to (4.7), in the form of a sum of unidirectional solutions corresponding to n-fold propagation in the ring and involving the Z-shifted self-similar variable $\xi_n = [T(Z+n)\ln R^{-1}]^{1/2}$:

$$\varphi(T, Z) = \sum_{n=0}^{\infty} R^{n/2}\left\{(\varphi(0,0) - R^{1/2}\varphi(0,1)) I_0(\xi_n) + \int_0^1 \frac{\partial\varphi(0, Z')}{\partial Z'}\theta(Z - Z' + n) \times I_0([T(Z - Z' + n)\ln R^{-1}]^{1/2})\, dZ'\right\}. \tag{4.8}$$

In the absence of reflections ($R \to 0$), only the $n = 0$ term survives in (4.8) and describes the limiting super-radiance of waves with a continuous spectrum in the absence of reflections. Its asymptotic form is $\propto \exp(TZ\ln R^{-1})^{1/2}$. Modes $R \neq 0$ with the highest growth rate begin to emerge from the overall background for $T > T_R$ when, as it turns out, we can make the following replacement in (4.8):

$$\sum_n \ldots \to \int \ldots dn.$$

This integral can be evaluated:

$$\varphi(T, Z) \approx \varphi_d \approx \exp\left(\frac{T}{2} + \frac{Z\ln R^{-1}}{2}\right) \times \int_0^1 \varphi(T=0, Z') e^{-Z'\ln R^{-1/2}} dZ',$$

$$T > T_R \sim 4\ln R^{-1} + \frac{1}{4}(\ln R^{-1})^2. \tag{4.9}$$

Comparison of (4.9) with (4.7), shows that, in the asymptotic case, the $m = 0$ polariton mode has the maximum growth rate ω_p'' and begins effectively to remove inversion after the delay time $T_d \sim \ln(NV)$. The final result is super-radiance of modes with a discrete spectrum, which occurs for $T_d > T_R$, i.e., when $R > R_{cr}$

$$\ln R_{cr}^{-1} \sim \ln(NV)\left[2\left\{1 + \left[1 + \frac{1}{16}\ln(NV)\right]^{1/2}\right\}\right]^{-1}. \tag{4.10}$$

This criterion takes account of the increase in the rate of growth of super-radiance in the presence of feedback, and therefore gives a slightly greater value for R_{cr} than the coarse estimate $R_{cr} \sim (NV)^{-1/4}$ of Sect. 4.4 for $M \sim 1$. The earlier condition for the absence of the effect of feedback on super-radiance,[23] i.e., $R \ll 1/\pi^{2/3}\ln[\pi(NV)^{1/2}](NV)^{1/3}$, is not a criterion for the selection of modes with a discrete spectrum, and is not, therefore, in conflict with (4.10).

The dynamics of the nonlinear stage of super-radiant

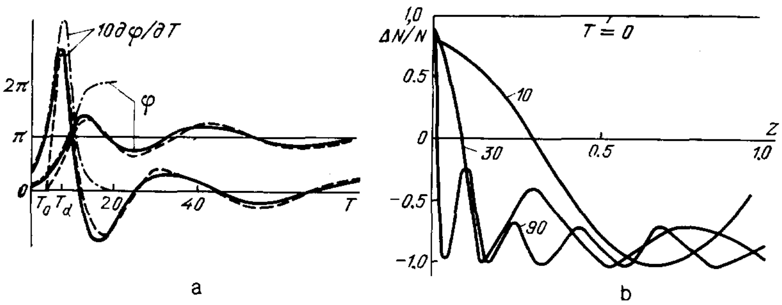

FIG. 14. *a*, *b*—Same as Fig. 13 in the case of the numerical solution of (4.5) with the 2π pulse (4.12) (dot-dash curve) and the self-similar solution of (3.11), (4.13) (dashed lines) for $R = 1/e^4 \approx 1.8 \cdot 10^{-2}$.

modes ($R > R_{cr}$) is significantly different for strong ($R \sim 1$) and weak ($R \ll 1$) reflections. When $R \sim 1$, equation (4.5) reduces to the equation of the mean-field model [cf. (3.17) and Ref. 66]

$$\frac{d\varphi(T, Z=1)}{dT} = \frac{1}{2}\sin\varphi(T, Z=1)$$
$$\Rightarrow \varphi(T, Z=1) = 2\,\mathrm{arctg}\exp\left[\frac{1}{2}(T - T_d)\right]. \tag{4.11}$$

The result is single-pulse polariton-mode super-radiance (Fig. 13).

When $R \ll 1$, the mode (4.9) is pushed against the exit end at $Z = 1$. Initially, inversion is removed only at this end, and the presence of the left-hand boundary at $Z = 0$ has no effect. The first super-radiant pulse is therefore very similar to the 2π-pulse in infinite space, i.e., a soliton of the sine-Gordon equation

$$\varphi \approx 4\,\mathrm{arctg}\exp\left(\frac{T + Z\ln R^{-1}}{2} - \frac{T_d}{2}\right),$$
$$E \propto \frac{\partial\varphi}{\partial T} = \left[\mathrm{ch}\left(\frac{T + Z\ln R^{-1}}{2} - \frac{T_d}{2}\right)\right]^{-1}. \tag{4.12}$$

Thereafter, the value $\varphi(T,Z = 1) \sim \pi$ is established at the exit end, so that, in accordance with the boundary condition (4.5), a small polarization appears at entry: $\varphi(T,Z=0) \sim \pi\sqrt{R} \ll 1$. The final effect[82] is that the first powerful super-radiant polariton-mode pulse is followed by the self-similar super-radiant regime, i.e., a continuous-spectrum "afterglow" (Fig. 14) described by a solution of (3.11) that is a function of the variable

$$\xi = [(T - T_0) Z \ln R^{-1}]^{1/2}, \quad T_0 \approx T_d - \frac{\xi_a^2}{\ln R^{-1}},$$
$$\xi_a \sim \frac{1}{2}\ln(2\pi\xi_a R^{-1}) \approx \frac{1}{2}\ln[\pi R^{-1}\ln(2\pi R^{-1})]. \tag{4.13}$$

4.6. Effect of weak reflections

For a macroscopic sample, (4.10) gives $R_{cr} \ll 1$. For example, when $NV \sim 10^{10}$, we have $R_{cr} \sim 10^{-2}$. This means that even weak (and occasionally uncontrolled) reflections can alter the super-radiant regime by increasing the power, reducing the duration τ of the first pulse, and reducing its delay time t_d. This is important if we wish to produce super-radiance in the short-wave range in which good mirrors are not available. In addition to initiation (see Sect. 3.6), reflections can also be used to ensure that the super-radiance condition $t_d \lesssim T_2$ is satisfied in media with fast phase relaxation, e.g., in crystals and glass fibers (laser generation in activated fiber lightguides is discussed, for example, in Ref. 113).

We also note that, while in the absence of reflections by the boundaries, the super-radiant frequency shift and its variation during the emission of the pulse are small, they may become considerable[78] when reflections occur in a particle, resonator, or waveguide. Moreover, this collective Lamb frequency shift is difficult to detect experimentally.

The appearance of symmetric polariton modes due to reflections during super-radiance (Fig. 9) may be signaled by the synchrotron emission of identical pulses from both ends of the cylindrical sample. This type of super-radiance, and also the super-radiance of modes with a discrete spectrum in a disk-shaped active sample, have been observed experimentally[33] in the case of $KCl:O_2^-$ crystals. From the experimental point of view, the effect of reflections on super-radiance is still a largely unresolved problem. There are also some unsolved theoretical problems, e.g., it is not clear whether weak reflections by boundaries ($R \ll 1$) can result, during super-radiance, in lattice inversion $\Delta N(\mathbf{r})$ with strong Bragg reflection that produces the self-trapping of some of the radiation in the sample. Super-radiance under the conditions of "given" Bragg diffraction is discussed in Refs. 13a, 59, 75, and 96.

5. PHENOMENOLOGICAL QUANTUM ELECTRODYNAMICS OF ACTIVE MEDIA AND THE QUANTUM-STATISTICAL PROPERTIES OF SUPER-RADIANCE

In phenomenological quantum electrodynamics (PQED), the normal waves are taken to be photons in the medium (quanta of transverse waves) and plasmons (quanta of longitudinal waves) with energy $\hbar\omega_j(\mathbf{k})$ and momentum $\hbar\mathbf{k}$. In contrast to microscopic quantum electrodynamics, PQED starts not with the equations for the quantum interaction between individual particles and photons in vacuum, but with the classical (non-operator) equations of continuum electrodynamics for the local macroscopic field and the polarization in the medium.[3,11,97-103] The transformation of these equations to the Hamiltonian form and their subsequent canonical quantization immediately enables us to use the quantum statistics of collective excitations in the medium.

We shall show in this Section how PQED can be constructed for active media. This is not a trivial question because PQED is generally accepted only for transparent media in which $\mathrm{Im}\,\omega_j(\mathbf{k}) = 0$ (Sect. 5.1). Attempts to generalize PQED to absorbing media $\mathrm{Im}\,\omega_j < 0$), which are reviewed in Ref. 99, have not resulted in a physically satisfactory theory. On the other hand, there is no doubt that PQED reduces in the case of absorbing media to the quantization of damped oscillators with positive-definite energy.[104,105] The situation is found to be qualitatively different in the case of active media ($Im\,\omega_j > 0$). It has become clear, following the publication of Refs.29 and 91, that the evolution of quantum fluctuations in an active medium, e.g., during super-radiance, must be described as an instability occuring during the interaction between quantum oscillators (modes or waves) with different signs of energy. Broadly speaking, this may be referred to as a dissipative instability because, relative to a selected dynamic subsystem of unstable oscillators, the remaining oscillators play the role of a dissipative subsystem in one way or another (see Sect. 5.2). The dissipative character of the instability as a macroscopic phenomenon means that we have to introduce into quantum theory the Hermitian Hamilton operator that is not positive-definite (in the linear approximation). This approach enables us to establish a general quantization scheme and to describe the dynamic evolution of fluctuations in collective excitation between the micro- and macro-levels. This provides a much simpler way of taking into account frequency and spatial dispersion, nonlinearity and inhomogeneity, anisotropy, and the presence of sources in the medium.

As a simple example of the application of PQED to active media, we can consider the statistics of the delay time and the polarization elipse for discrete-mode super-radiance (see Sect. 5.3). The efficacy of PQED in the analysis of quantum-statistical phenomena in amplifiers and oscillators relies, in the first instance, on the fact that the well-tried methods of solution of the truncated equations of classical wave theory can be extended to the quantum theory of Heisenberg operators for slowly-varying macrofield amplitudes. A closely related approach has been developed for problems involving the interaction and propagation of photons and excitons (via the kinetic equation approximation),[107] stimulated parametric scattering (based on the parabolic operator equation),[98] and super-radiance (using the Maxwell-Bloch operator equations).[42,53,76,77,78,90,91]

5.1. Quantum electrodynamics of transparent dispersive media

PQED has its origin in a 1940 paper by Ginzburg, and has been the subject of extensive development ever since (see Ref. 3 for the relevant citations). The theory starts with Maxwell's equations for a medium

$$\mathrm{rot}\,\vec{\mathscr{E}} = -c^{-1}\frac{\partial\vec{\mathscr{B}}}{\partial t},\qquad \mathrm{rot}\,\vec{\mathscr{B}} = c^{-1}\frac{\partial\vec{\mathscr{E}}}{\partial t} + 4\pi c^{-1}\frac{\partial\vec{\mathscr{P}}}{\partial t},\tag{5.1}$$

where $\vec{\mathscr{P}}$ is the polarization of the medium. PQED is constructed by the Hamilton method whereby the field is expanded in terms of normal waves (modes) in the medium, $g_{\mathbf{k}j} \propto \exp(i\mathbf{kr})$, labeled by the subscript j (see Sect. 1.2). The frequencies of these modes in a transparent medium are positive ($\omega_{\mathbf{k}j} > 0$), and the permittivity is an even function of frequency, $\varepsilon_{\gamma\delta}(\omega,\mathbf{k}) = \varepsilon_{\gamma\delta}(-\omega,\mathbf{k})$, since there is no relaxation and the equations for the polarization of the medium are reversible in time.[1,2]

The quantization of the free field in the transparent linear medium is performed by analogy with quantization in vacuum. The creation and annihilation operators $\hat{a}^+_{\mathbf{k}j}$ and $\hat{a}_{\mathbf{k}j}$ for photons in the medium and plasmons must satisfy the canonical commutation relations (1.7). The free-field Hamiltonian is equal to the total energy of the normal field oscillators with generalized coordinates $\hat{q}_{\mathbf{k}j}$ and momenta $\hat{p}_{\mathbf{k}j}$:

$$\hat{H}_0 = \sum_{\mathbf{k},j}\hbar\omega_{\mathbf{k}j}\left(\hat{a}^+_{\mathbf{k}j}\hat{a}_{\mathbf{k}j} + \frac{1}{2}\right) \equiv \frac{1}{2}\sum_{\mathbf{k},j}(\hat{p}^2_{\mathbf{k}j} + \omega^2_{\mathbf{k}j}\hat{q}^2_{\mathbf{k}j}),$$
$$\hat{q}_{\mathbf{k}j} = \left(\frac{\hbar}{2\omega_{\mathbf{k}j}}\right)^{1/2}(\hat{a}^+_{\mathbf{k}j} + \hat{a}_{\mathbf{k}j}),\qquad \hat{p}_{\mathbf{k}j} = i\left(\frac{\hbar\omega_{\mathbf{k}j}}{2}\right)^{1/2}(\hat{a}^+_{\mathbf{k}j} - \hat{a}_{\mathbf{k}j}).\tag{5.2}$$

In accordance with the energy of the quasimonochromatic field in a dispersive medium, well known in classical electrodynamics,[1,2] the eigenfunctions $g_{\mathbf{k}j}$ are normalized to the energy of the jth mode quantum:

$$\frac{1}{4\pi}\int_{\mathfrak{B}}\left\{\frac{\omega^2}{c^2}\sum_{\gamma,\delta=1}^{3}\frac{d\omega\varepsilon_{\gamma\delta}}{d\omega}(\mathbf{g}^*_{\mathbf{k}j})_\gamma(\mathbf{g}_{\mathbf{k}j})_\delta + |[\mathbf{k},\,\mathbf{g}^*_{\mathbf{k}j}]|^2\right\}_{\omega=\omega_j(\mathbf{k})}$$
$$\times\ d^3\mathbf{r} = \hbar\omega_{\mathbf{k}j}\tag{5.3}$$

When the medium is nonlinear, the Maxwell equations can also be reduced to an equivalent set of Hamilton equations, and then quantized. A systematic presentation of PQED for transparent media and its applications to the generation and propagation of waves in media can be found in Refs. 1, 3, and 97–103 and in the citations given therein. The fact that PQED had to be used in the analysis of, for example, the bremsstrahlung emitted by electrons interacting strongly with neighboring atoms in the medium, was well understood by Ter-Mikaelyan, Landau, and Pomeranchuk.[97]

5.2. Quantum theory of dissipative instability

The PQED of active media, which we shall develop here for super-radiance, is based on the quantum theory of dissipative instability. Its simplest variant, i.e., the dynamic dissipative instability of two coupled oscillators with *different signs of the quanta of energy* (see the second footnote in Sect. 2.3) is described by the Hamiltonian[13)]

$$\hat{H} = -\hbar\omega_1^{(0)}\hat{a}^+_1\hat{a}_1 + \hbar\omega_2^{(0)}\hat{a}^+_2\hat{a}_2 + \frac{1}{2}\hbar(\eta\hat{a}_1\hat{a}_2 + \eta^*\hat{a}^+_2\hat{a}^+_1),$$
$$\frac{d\hat{a}_j}{dt} = \frac{[\hat{a}_j,\,\hat{H}]}{i\hbar};\tag{5.4}$$

where $\omega_j^{(0)} > 0$ and the partial-oscillator creation and annihilation operators $\hat{a}_j^+$ and $\hat{a}_j$ ($j = 1, 2$) satisfy the canonical commutation relations (1.7). For example, this variant describes the anomalous instability (2.4) in an inverted two-level medium during the interaction ($\eta = \omega_c$) between the partial oscillations of polarization ($\omega_1^{(0)} = \omega_0$) and the electromagnetic field ($\omega_2^{(0)} = ck$) in the single-mode model of super-radiance or optical nutation. The complex transformation of the creation and annihilation operators

$$\tilde{\hat{a}}^+_1 = \hat{a}^+_1 - \hat{a}_2\eta[2(\omega_1 - \omega_2^{(0)})]^{-1},$$
$$\tilde{\hat{a}}_2 = -i\eta\{\hat{a}^+_1\eta^*[2(\omega_1 - \omega_2^{(0)})]^{-1} - \hat{a}_2\}\{2[|\eta^2|$$
$$-(\omega_1^{(0)} - \omega_2^{(0)})^2]^{1/2}\}^{-1}\tag{5.5}$$

leads to noncommuting normal oscillators with the following Hamiltonian:

$$\hat{H} = \hbar\omega_1\hat{a}_2^+\hat{a}_1^+ + \hbar\omega_2\hat{a}_1\hat{a}_2 + \hbar\omega_1^{(0)},$$
$$\omega_{1,2} = \frac{1}{2}\{\omega_1^{(0)} + \omega_2^{(0)} \pm i[|\eta^2| - (\omega_1^{(0)} - \omega_2^{(0)})^2]^{1/2}\}, \qquad (5.6)$$

and to the following cross-commutation relations[99,104] instead of the canonical relations (1.7):

$$[\hat{a}_1^+, \hat{a}_2^+] = 1, \quad [\hat{a}_j^+, \hat{a}_{j'}] = 0 \quad (j = 1, 2, \ j' = 1, 2). \quad (5.7)$$

The Heisenberg equations of motion for the new oscillators are $\dot{\hat{a}}_1^+ = -i\omega_1\hat{a}_1^+, \dot{\hat{a}}_2 = -i\omega_2\hat{a}_2$ where, and henceforth, we assume that $\omega_1'' > 0$.

The solution of the problem with the Hamiltonian (5.6) is elementary and enables us to perform a complete investigation of the statistics of the process. According to the conservation law for the difference between the numbers of partial oscillators $d/dt(\hat{n}_1 - \hat{n}_2) = 0$, which follows from (5.4), where $\hat{n}_{1,2} = \hat{a}_{1,2}^+\hat{a}_{1,2}$, the dynamic dissipative instability develops as a result of the exchange of excitation quanta between these oscillators. The fundamental quantum-mechanical result is that the coupled oscillators, which need not be initially excited [$\rho(n,t=0) = \delta(n)$], can become excited as a result of spontaneous fluctuations, i.e., we can have the spontaneous creation of pairs of quanta from the vacuum state. The mean number of quanta is then found to rise from the initial value of zero: $\bar{n}(t) = |\eta/2\omega_1''|^2 \operatorname{sh}^2(\omega_1'' t)$. The asymptotic form ($t \to \infty$) of the statistical distribution of the number n of quanta of the partial oscillators is found to be exponential and Gibbs-like:

$$\rho(n, t) = \frac{1}{\bar{n}}\exp\left(-\frac{n}{\bar{n}}\right) \qquad (n \geqslant 0),$$
$$\int_0^\infty \rho(n, t)\,dn = 1, \qquad \bar{n}(t) = n_{\text{eff}}\exp(2\omega_1'' t), \qquad (5.8)$$

which corresponds to the Gaussian statistics of fluctuations in the field amplitude. If we start with thermal fluctuations at temperature T, the asymptotic form of $\rho(n,t)$ is again given by (5.8), but with a larger mean number of quanta:

$$n_{\text{eff}} = \left|\frac{\eta}{4\omega_1''}\right|^2 \left(\frac{1}{2}\operatorname{cth}\frac{\hbar\omega_1^{(0)}}{2\varkappa T} + \frac{1}{2}\operatorname{cth}\frac{\hbar\omega_2^{(0)}}{2\varkappa T}\right).$$

Another form of dissipative instability occurs during the interaction between a dynamic subsystem of negative energy (the oscillator $\hat{a}_1$) and a dissipative subsystem of positive energy (*thermostat consisting of a continuum of oscillators* $\hat{b}_k$ that are not directly coupled):

$$\hat{H} = -\hbar\omega_1^{(0)}\hat{a}_1^+\hat{a}_1 + \sum_k \hbar\omega_k\hat{b}_k^+\hat{b}_k + \sum_k \frac{1}{2}\hbar(\beta_k\hat{a}_1\hat{b}_k + \beta_k^*\hat{b}_k^+\hat{a}_1^+). \qquad (5.9)$$

The fact that the thermostat is macroscopic in the limit of the continuous spectrum of frequencies $\omega_k\,[\Sigma_k\ldots \approx \int d\omega g(\omega)\ldots]$ ensures that the process is irreversible in time. This model of the thermostat, i.e., the particular phenomenological description of the quantum dynamics of the initial classical system,[105] can be justified by the fact that the

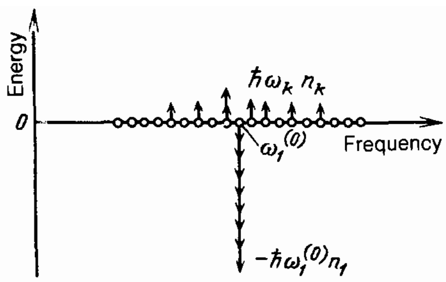

FIG. 15. Exchange of negative and positive energy quanta between dynamic oscillators ($-\hbar\omega_1^{(0)}$) and thermostat oscillators ($+\hbar\omega_k$) during the spontaneous development of dissipative instability.

macroscopically observed results do not depend on the choice of the microscopic parameters $g(\omega)$ and $\beta_k = \beta(\omega)$.

The above problem can be solved analytically in the Weisskopf-Wigner approximation. It follows from this solution that the dissipative instability again spontaneously develops even from the unexcited vacuum state: $\bar{n}_1(t) = \exp(2\omega_1'' t) - 1 (t \geqslant 0)$ (Fig. 15). The observed (modified by the thermostat) complex frequency of the dynamic oscillator is

$$\omega_1 = \omega_1^{(0)} + \Delta\omega_1' + i\omega_1'',$$
$$\Delta\omega_1' = \text{V. p.}\int_{-\infty}^{\infty}\left|\frac{\beta(\omega)}{2}\right|^2 \frac{g(\omega)}{\omega - \omega_1^{(0)}}\,d\omega, \qquad (5.10)$$
$$\omega_1'' = \pi g(\omega_1^{(0)})\left|\frac{\beta(\omega_1^{(0)})}{2}\right|^2 \ll \omega_1^{(0)}.$$

In general, if we start with spontaneous and/or thermal fluctuations, we again have the asymptotic form (5.8) with effective number of quanta $n_{\text{eff}} = \operatorname{cth}(\hbar\omega_1^{(0)}/2\varkappa T) \geqslant 1$.

The quantum theory based on the models defined by (5.4) and (5.9) can be extended to the more general case of dissipative instabilities that include both the interaction between dynamic oscillators with energies of different sign and the irreversible removal of oscillator energy by the thermostats. The coupling between a dynamic oscillator and a thermostat consisting of oscillators whose energy has the same or different sign [which modifies its frequency by analogy with (5.10)] will then describe relaxation and incoherent amplification, respectively. Thus, for fluctuations in the polarizations of a two-level medium, this gives the relaxation T_2^{-1}, whereas for the electromagnetic mode, we have the positive resistive dissipation $2\pi\sigma$ or the negative dissipation $2\pi\sigma_a$ in lasers (see Sect. 4.3).

The PQED quantization procedure for active linear media may be summarized as follows. First, we must find the normal waves and their frequencies $\omega_j(\mathbf{k})$. Next, we must set to zero all the relaxation and dissipation constants, and proceed to the Hamiltonian equations for the dynamic modes, which are found to be split into pairs of modes with complex conjugate frequencies $\Omega_\alpha^{(0)} = \Omega_\alpha^{(0)'} \pm \Omega_\alpha^{(0)''}$ and individual stationary modes with real frequencies $\Omega_\beta^{(0)}$. Each pair of modes $\Omega_\alpha^{(0)}, \Omega_\alpha^{(0)*}$ is then represented by a set of two interacting partial oscillators with energies $-\hbar\omega_{\alpha1}^{(0)}$ and $+\hbar\omega_{\alpha2}^{(0)}$ of different sign, by analogy with (5.4). The amplitudes of the resulting mode oscillators are normalized to the energy of one quantum [cf. (5.3)]. Next, in the Hamiltonian, we add to each dynamic partial oscillator $\Omega_\beta^{(0)}$ and $-\omega_{\alpha 1}^{(0)}, +\omega_{\alpha 2}^{(0)}$ a pair of thermostats of partial oscillators with different sign of energy, which interact with it, so that the frequencies $\Omega_\alpha^{(0)}$ and $\Omega_\beta^{(0)}$ modified by the thermostats are equal to the original frequencies ω_j [cf. (5.10)].

Finally, quantization is performed by replacing the canonical coordinates and momenta of all the above partial oscillators with the corresponding operators obeying the canonical commutation relations [cf. (1.7) and (5.2)]. Of course, the canonical modes with $\Omega_\alpha^{(0)''} \neq 0$ are then found to obey the cross-commutation relations (5.7). This means that the Hamiltonian for the set of oscillators with energies of different sign in the active medium is not diagonalized by the analog of the canonical Bogolyubov transformation[6] that preserves the commutation relations. This is in contrast to the situation for the set of oscillators of positive energy in a transparent medium (see Sect. 5.1). Unless we understand this fundamental point, we cannot extend our discussion beyond the framework of PQED for transparent media, and systematically quantize the field in active and absorbing media. It is therefore clear that the analysis of the quantum-statistical properties of unstable macrofield oscillators reduces in PQED to the quantum theory of dissipative instability.[14)]

5.3. Macroscopic manifestations of quantum fluctuations of super-radiance

Spontaneous quantum fluctuations are amplified in the course of super-radiance to the macroscopically observable level, and this in turn leads to strong fluctuations in the parameters of the super-radiant pulse that cannot be predicted between successive shots.

This remarkable phenomenon has been investigated in a few experimental[15,33a,34,53,92,93] and theoretical[14–16,19,28,42,76–108] papers. We shall illustrate how it can be described by the PQED of active media by considering the statistics of the delay time t_d in discrete-mode super-radiance (Sect. 4). For simplicity, we shall take into account only M modes with growth rates ω_m'' ($m = 1,...,M$) of the order of the maximum value $1/2\tau$. According to Sect. 5.2, the numbers n_m of quanta in different modes are independent random quantities with the asymptotic probability distribution (5.8). The probability that the super-radiant pulse will be emitted within the time interval $0 < t < t_d$ is equal to the probability that the total number of quanta

$$n = \sum_{m=1}^{M} n_m$$

with the probability distribution

$$\rho(n) = \left(\frac{n}{\bar{n}_1}\right)^{M-1} [(M-1)!\,\bar{n}_1]^{-1} \exp\left(-\frac{n}{\bar{n}_1}\right)$$

at time t_d will exceed one-half of the number of inverted molecules[15)], $NV/2$. If we write the above condition in the form

$$\int_0^{t_d/\tau} f\left(\frac{t_d'}{\tau}\right) d\frac{t_d'}{\tau} = \int_{NV/2}^{\infty} \rho\left(n, \frac{t_d}{\tau}\right) dn$$

and differentiate it with respect to t_d, we obtain the required distribution for the normalized delay time[91]

$$f\left(\frac{t_d}{\tau}\right) = \frac{u^M}{(M-1)!} \exp\left(-M\frac{t_d}{\tau} - ue^{-t_d/\tau}\right),$$

$$u \equiv \frac{NV}{2n_{\text{eff}}} \gg 1. \tag{5.11}$$

This result means that the super-radiance statistics depends on the shape of the sample, which reduces to the dependence of the number M of unstable polariton modes on the sample shape (Fig. 16a). This number is determined by the solution of the corresponding electrodynamic problem (see Sect. 4). For example, for a sphere of radius $a \gg \lambda$, the number of modes is $M \sim (\omega_0 a/c)^2 \gg 1$, whereas for a cylinder with Fresnel number F, estimates[14] show that $M \sim [(F^2 + 1 + (1/F)]/3$. As M increases, the mean delay time is reduced in accordance with the formula $t_d = \tau \ln(u/M)$, whereas experiment shows that fluctuations decline[92,93] in accordance with the expression (Figs. 16 and 16b)

$$\sigma^2(M) \equiv [\overline{t_d^2} - (\bar{t}_d)^2](\bar{t}_d)^{-2}$$

$$= \left(\frac{\pi^2}{6} - \sum_{m=1}^{M-1} \frac{1}{m^2}\right)\left(\ln\frac{u}{M}\right)^{-2} \approx M^{-1}\left(\ln\frac{u}{M}\right)^{-2}. \tag{5.12}$$

The investigations[77,94,95] cited above refer to an experiment on the super-radiance of waves with a continuous spectrum in low-pressure cesium vapor[92] in which $L \lesssim L_c$ and reflections by the boundaries were very weak (in contrast to the super-radiance of modes with a discrete spectrum in resonators and activated solids). Nevertheless, the behavior of super-radiance statistics when the scale of the cylindrical sample is varied is similar in the two super-radiance regimes (Fig. 16b). In the special case of the single-mode model ($M = 1$), the result given by (5.11) agrees at $T = 0$ with the risetimes obtained for single-mode lasers[109] by other methods in super-radiance theory[14] and in the theory of fluctuations.

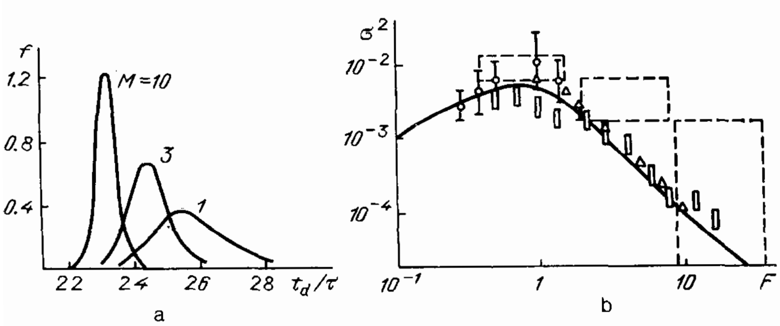

FIG. 16. Dependence of the statistics of super-radiance delay time on sample shape. *a*—Probability distribution (5.11) for $u = 10^{11}$. *b*—Relative variance σ^2 of the delay time t_d as a function of the Fresnel number $F = S/\lambda L$ for cylindrical samples ($N\lambda L^2 = 10^8$, L = const): solid line—envelope (5.12), dashed rectangles—experiment,[92] triangles, full rectangles, and vertical segments—numerical calculations reported in Refs. 77, 94, and 95, respectively.

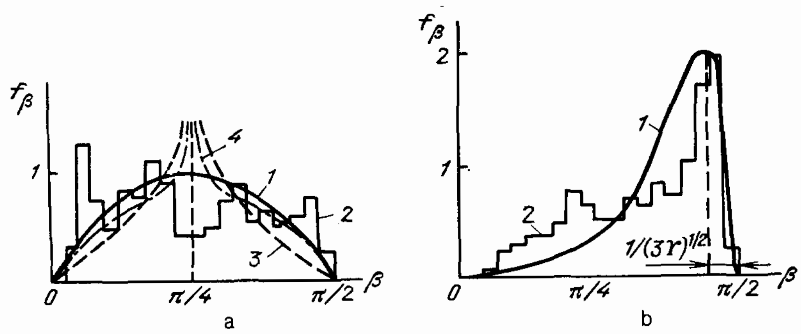

FIG. 17. Probability distribution for the angle β representing the orientation and eccentricity of the super-radiance polarization ellipse in the case of polarization degeneracy ($\omega_1'' = \omega_2''$, $\Upsilon = 1$) (a) and weak anisotropy, determined by the linear polarization of the pump at angle $\beta = \pi/2$ ($\omega_1'' \neq \omega_2''$, $\Upsilon = 10$) (b): 1—linear PQED—formula (5.13), 2—typical experimental histogram,[53] 3 and 4—linear and nonlinear theory[53].

When the properties of orthogonally polarized super-radiant modes in an isotropic medium (Sec. 2.5) are taken into account, the PQED of active media is also capable of explaining the statistics of fluctuations in the polarization ellipse, observed experimentally in Refs. 14 and 53. The usual procedure is to measure the probability distribution f_β for the angle $\beta \in [0,\pi/2]$ deduced from the ratio of intensities (number of photons) detected by two receivers with orthogonal polarizations: $\mathrm{tg}^2\beta = I_2/I_1 \equiv n_2/n_1$. For example, consider two unstable super-radiant modes with orthogonal linear polarizations and approximately equal growth rates, $\omega_1'' \approx \omega_2''$ $j = 1/2 \to j' = 1/2$ emitted as a result of the transition (see Sec. 2.5). The numbers of photons, $n_{1,2}$, are statistically independent, and their distributions are given by (5.8). The distributions of the ratio n_2/n_1 and of the angle β are therefore of the form

$$f\left(\frac{n_2}{n_1}\right) = \frac{\Upsilon}{[\Upsilon + (n_2/n_1)]^2}, \qquad f_\beta = \frac{\Upsilon \sin 2\beta}{(\Upsilon \cos^2\beta + \sin^2\beta)^2},$$
$$\Upsilon = \frac{\bar{n}_2}{\bar{n}_1} \propto \exp[2(\omega_2'' - \omega_1'')t]. \tag{5.13}$$

The theory of fluctuations in the polarization of super-radiance was given in Ref. 53, but it was based on a distribution of mode photons that was different from (5.8), and the analysis was confined to the case of exact degeneracy ($\omega_2'' = \omega_1''$). It predicted an infinite peak in f_β at $\beta = \pi/4$ (Fig. 17a), which has not been observed. The result given by (5.13) and its generalization to arbitrary elliptic polarization of modes disagree with the theoretical predictions of Ref. 53, but are in qualitative agreement with observations of Ref. 53 such as the considerable change (including total irreproducibility) in histograms for a small change in experimental conditions, e.g., the direction and polarization of the pump, geometric parameters, external fields, and so on. The point is that large changes in f_β are due to the exponentially large enhancement of the manifestations of even a weak anisotropy (and gyrotropy) during super-radiance. Even a small difference between the growth rates of orthogonally polarized modes can have a considerable effect on the statistics of polarization, and can emphasize the contribution due to the mode with the higher growth rate (Fig. 17b).

6. EFFECTS OF SPATIAL DISPERSION IN A GAS OF TWO-LEVEL MOLECULES

So far, we have confined our attention to active media consisting of immobile particles. Allowance for the translational velocity distribution of the particles leads to qualitatively new effects in the electrodynamics of coherent wave processes. We then have to consider not only frequency dispersion, but also the nonlinearity of spatial dispersion, i.e., the dependence of permittivity ε on the wave number k. Spatial dispersion is due to the different reaction of particles with different velocity to the field with a given spatial structure.[1,2,37]

6.1. Polariton spectrum of a gas with allowance for the thermal motion of the molecules

Consider an isotropic gas of two-level molecules with the Maxwellian velocity distribution $F(v/v_T) = \pi^{-1/2}\exp(-v^2/v_T^2)$, where M is the mass of a molecule, $v_T = \sqrt{2\varkappa T/M}$, and T is the gas-kinetic temperature. The constitutive equation (1.10) must now be replaced[5,9,110] with the transport equation[16)]

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)^2 \mathbf{p} + 2T_2^{-1}\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)\mathbf{p} + (\omega_0^2 + T_2^{-2})\mathbf{p}$$
$$= -\frac{2\omega_0 d^2 n}{3\hbar}\vec{\mathscr{E}}_m(\mathbf{r}, t), \tag{6.1}$$

where $\mathbf{p}(\mathbf{v},\mathbf{r},t)d^3\mathbf{r}d^3\mathbf{v}$ and $n(\mathbf{v},\mathbf{r},t)d^3\mathbf{r}d^3\mathbf{v}$ are, respectively, the mean h.f. polarization and the difference between the populations of molecules within the volume element $d^3\mathbf{r}$ and velocity interval $d^3\mathbf{v}$. Assuming that $n = \Delta N v_T^{-1}F$, and introducing the Kramp function[5,111]

$$w(Z) = e^{-Z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^Z \exp s^2\, ds\right), \tag{6.2}$$

for fields $\alpha \exp(-i\omega t + i\mathbf{k}\mathbf{r})$, we find that the permittivity is given by[45]

$$\varepsilon(\omega, k) = \frac{1+2\theta}{1-\theta} + \frac{i4\pi\sigma}{\omega}, \tag{6.3}$$
$$\theta = \frac{i\sqrt{\pi}\,\omega_c^2}{18\omega_0 k v_T}\left[w\left(\frac{\omega + iT_2^{-1} - \omega_0}{kv_T}\right) - w\left(\frac{\omega + iT_2^{-1} + \omega_0}{kv_T}\right)\right]$$

where we have taken account of the difference between the acting field and the mean field, i.e., the Lorentz correction $\mathbf{E}_a - \mathbf{E} = (4\pi/3)\mathbf{P}$. Hence, for frequencies that are very distant from the Doppler line, for which $|\omega + iT_2^{-1} \mp \omega_0| \gg kv_T$, $\omega'' + T_2^{-1} > -|\omega' \mp \omega_0|$ and $\omega(Z) \approx i\pi^{-1/2}Z^{-1}$, the value of ε for $\omega \sim \omega_0$ differs from (1.15) by only the small shift of the resonance frequency: $\omega_0 \to \omega_0 - \omega_c^2/9\omega_0$ (for $|\theta| \ll 1$).[8,112]

The resonance spatial dispersion effects[45] that are of interest to us here produce a qualitative change in the shape of the polariton spectrum (2.1) and arise in the region of the Doppler line

$$\operatorname{Im} Z + |\operatorname{Re} Z| \lesssim |Z|^{-1},$$
$$Z = (\omega + iT_2^{-1} - \omega_0)(kv_T)^{-1}. \tag{6.4}$$

The character of the spectrum of homogeneous transverse waves (1.16) for $ck \sim \omega_0$ then depends mostly on the ratio of the two parameters $2kv_T$ and $|\omega_c|$. If Doppler broadening is small in comparison with the cooperation frequency of the gas, i.e., $2kv_T \ll |\omega_c|$, the effect of the thermal motion on wave dispersion $\omega_{e,p}(k)$ (2.1) is small because, almost everywhere,

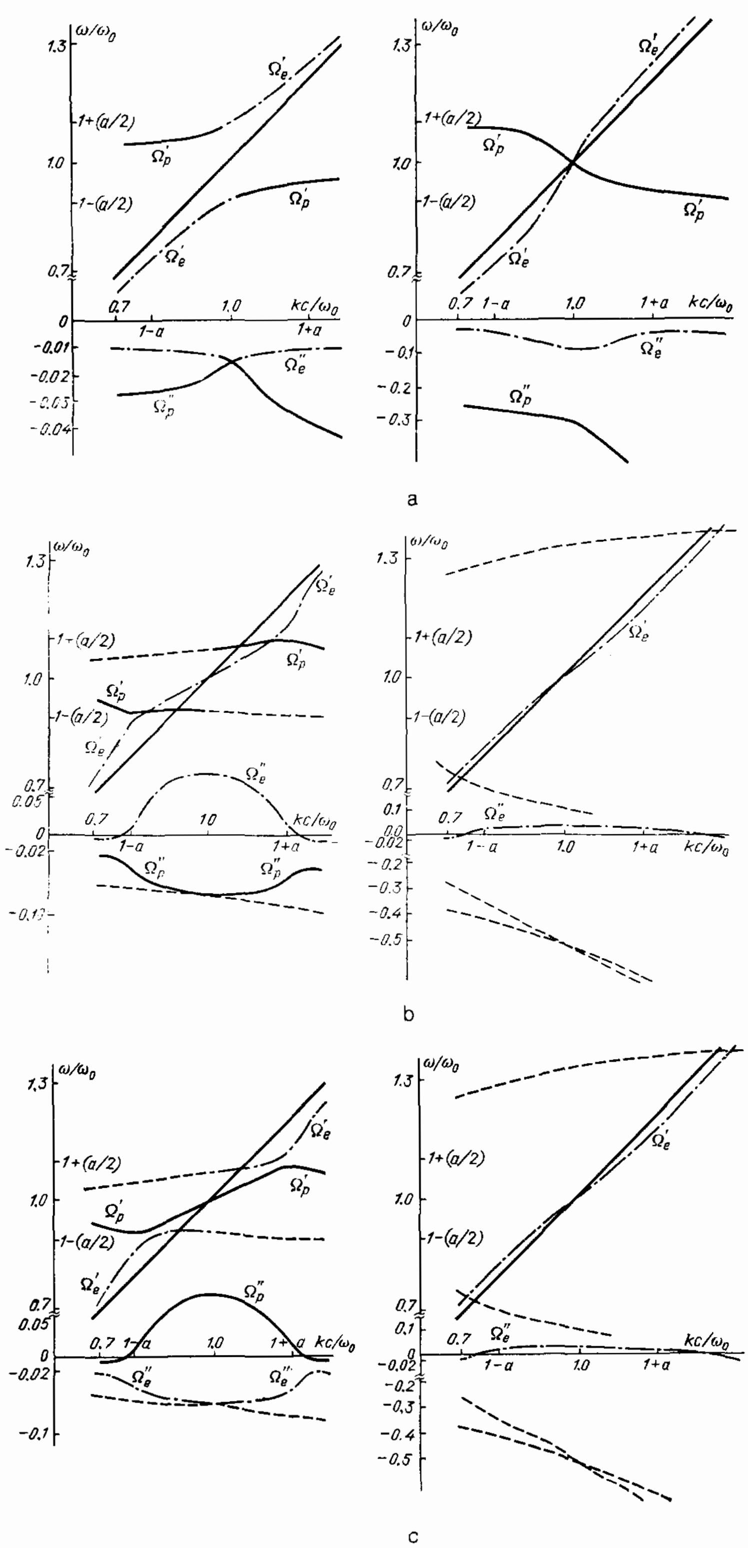

FIG. 18. a—Polariton spectrum $\Omega = (\omega' + i\omega'')/\omega_0$ as a function of kc/ω_0 for a gas, with allowance for the Doppler broadening for $|2\pi\sigma - T_2^{-1}| = |\omega_c^2|/12\omega_0 = 10^{-2}\omega_0$. Left—graphs for $2v_T/c = 0.4a$, right—$2v_T/c = 3a$, where $a \equiv |\omega_c|/\sqrt{3}\,\omega_0 = 0.2$. a—Univerted gas with $T_2^{-1} = 4\pi\sigma = \omega_c^2/6\omega_0$, b—inverted gas with $T_2^{-1} = 4\pi\sigma = -\omega_c^2/6\omega_0$, c—conducting inverted gas with $2\pi\sigma = T_2^{-1} = -\omega_c^2/6\omega_0$. Solid curves—polarization waves, dot-dash curves—electromagnetic waves. Dashed lines show segments of dispersion curves with strong collisionless damping.

$$|Z| \gg 1,$$

$$w(Z) = i\pi^{-1/2}Z^{-1}\left(1 + \frac{1}{2}Z^{-2} + \dots\right).$$

This asymptotic behavior gives rise to small corrections and is invalid for (6.4), i.e., for $|Z| \lesssim 1$ and/or within the cone$|(\pi/2) + \arg Z| < \pi/4$. For the polarization wave $\omega_p(k)$, this corresponds to the part of the Doppler line for which one of the following inequalities is satisfied:[17)]

$$\begin{gathered} |ck - \omega_0 - i(2\pi\sigma - T_2^{-1})| \ngtr \frac{|\omega_c^2|c}{3\omega_0 v_T}, \\ (ck - \omega_0)^2 + \frac{\omega_0^2 v_T^2}{c^2} \lesssim -\frac{\omega_c^2}{6} + \left(2\pi\sigma - \frac{1}{T_2}\right)^2. \end{gathered} \tag{6.4'}$$

The spatial dispersion effect manifests itself in both variants in (6.4′) as *strong collisionless damping* that occurs with the rate $-\omega'' \gtrsim kv_T + T_2^{-1}$ and is due to the dephasing of the ordered fluctuations in the polarization of the molecules in the course of their thermal motion (see Fig. 18 and Sec. 6.2).

When the Doppler broadening is large, $2kv_T \gg |\omega_c|$, the above suppression of the polarization wave by thermal motion occurs throughout the polariton spectrum. The only weakly damped (or growing) wave is the electromagnetic wave with the dispersion relation

$$\omega_V = ck - i2\pi\sigma - \frac{i\sqrt{\pi}\, c\omega_c^2}{12\omega_0 v_T}\, w\left(\frac{ck - \omega_0 - i(2\pi\sigma - T_2^{-1})}{kv_T}\right),$$

$$2\pi\sigma - T_2^{-1} \lesssim |ck - \omega_0| + kv_T. \tag{6.5}$$

This is also valid for large Lorentz broadening, $|2\pi\sigma - T_2^{-1}| \gg |\omega_c$, and is obtained by applying the perturbation method to the dipersion relation (1.16), (6.3). When $kv_T \ll \omega_0$ and $2\pi|\chi| = 1.5|\theta| \ll 1$, we find that

$$\omega - \omega_0 = ck - \omega_0 - i\,\frac{2\pi\sigma ck}{\omega} - i\,\frac{\sqrt{\pi}\, c\omega_c^2}{12\omega_0 v_T}\, w\left(\frac{\omega - \omega_0 + iT_2^{-1}}{kv_T}\right). \tag{6.6}$$

Only the limiting case defined by (6.5) is well known in the spectroscopy of gases. This is the "Voigt" dispersion law, which takes the form of a correction to the "vacuum" law[5,9,110]

$$\frac{ck}{\omega} - 1 = 2\pi\chi_0,$$

$$\chi_0\left(\frac{\omega}{\omega_0}\right) - = \frac{d^2\Delta N}{3\sqrt{\pi}\,\hbar\omega_0 v_T} \int_{-\infty}^{\infty} \frac{\exp(-v^2/v_T^2)\, dv}{(\omega/\omega_0) + (i/T_2\omega_0) - 1 - (v/c)}, \tag{6.7}$$

and implies that the replacement $k \to \omega_0/c$ is possible in (6.3) before the dispersion equation (1.16) is solved. However, this replacement, which distorts the spatial dispersion effects associateed with the Doppler spread of moleculear frequencies $\omega_0 + \mathbf{kv}$, is not always admissible.[37,45] This applies, above all, to the case $|\omega_c \gtrsim 2kv_T \gtrsim |2\pi\sigma T_2^{-1}|$. for which spatial dispersion becomes significant, but does not as yet preclude the polariton resonance. To find the correct polariton spectrum $\omega(k)$, we must therefore turn to equation (6.6) which allows for the dependence of ε on k, and is found to contain a whole series of solutions, most of which are highly damped.[18)] A numerical analysis of the evolution of the polariton spectrum with increasing Doppler broadening is shown in Fig. 18 from which highly damped waves have been removed.

6.2. Collisionless damping in the Doppler line wings

Strong collisionless damping of polarization and electromagnetic waves is due to the relaxation of the initial polarization of the gas as the molecules traverse the distance $\sim\lambda = 2\pi/k$ in the course of their thermal motion. The damping is formally related to the new normal waves $\omega_j^{(\pm)}(k)$ that appear near the polariton spectrum. These waves correspond to solutions of the dispersion equation (1.16), that are localized near the zeros of the Kramp function($w(Z) = 0$), which lie along the bisectors of the third and fourth quadrants of the complex plane of Z. We shall not reproduce here the corresponding solutions of (6.6) (Ref. 45); they are illustrated in Fig. 19.

We note that, in the most interesting case for which $|\omega_c| \gtrsim 2kv_T \gg T_2^{-1}$, the damping rate $\omega_j^{(\pm)''}(k) \sim 2kv_T$ for waves with collisionless damping and low values of j is found to be less than $|\omega_c|/2$ or $2\pi\sigma$, which determine the damping rate ($\omega''_{e,p}(k)$ (2.2) of waves in the polariton spectrum if we neglect spatial dispersion [in the wave number range defined by the second inequality in (6.4′)]. In particular, in a highly conducting gas, in which $2\pi\sigma \gg kv_T$, we can therefore have prolonged (on the time scale σ^{-1}) existence of molecular polarization transported by waves with collisionless damping. According to Fig. 19b, the Voigt dispersion curve for electromagnetic waves is significantly deformed and shows a break in the Doppler line wings, so that the standard solution (6.5), (6.7) ceases to be valid. Spatial dispersion is thus seen to alter the spectrum and the damping of normal waves and, consequently, the evolution of field and polarization perturbations in the gas at frequencies close to the strong electrodipole lines.

6.3. Longitudinal polarization waves. Analogy with plasma waves

Polaritons $\omega_{e,p}(k)$ with $k \sim \omega_0/c$ are not the only phenomena found beyond the Doppler line limits of the gas. There are also long polarization waves with the dispersion relation

$$\begin{gathered} \omega_p = 1 + \frac{\omega_c^2}{9\omega_0} + \frac{9k^2 v_T^2 \omega_0}{2\omega_c^2} + \frac{k^2 c^2 \omega_c^2}{6\omega_0^3} - iT_2^{-1} - i\,\frac{2\pi\sigma\omega_c^2}{3\omega_0^2} \\ - i\sqrt{\pi}\, kv_T \left(\frac{\omega_c^2}{9\omega_0 kv_T}\right)^2 \exp\left[-1 - \left(\frac{\omega_c^2}{9\omega_0 kv_T}\right)^2\right], \\ k \ll |\omega_c^2| \cdot \frac{1}{9\omega_0 v_T} \lesssim \frac{\omega_0}{c} \end{gathered} \tag{6.8}$$

[see (6.6) and Fig. 19].[45] These transverse waves have $\varepsilon = k^2c^2/\omega_p^2 \ll 1$, so that they are not very different from longitudinal waves with $\varepsilon = 0$, for which $\omega_\parallel = \omega_p - k^2c^2\omega_c^2/6\omega_0^3$ because of the absence of the magnetic field. [A dispersion relation for the longitudinal waves in an uninverted gas has been obtained by Kazantsev[48] for $\sigma = 0$. However, his transverse excitons in a gas do not correspond to the actual polarization waves described by (6.8)].

The fact that the damping of polarization waves (6.8) becomes exponentially small for $T_2^{-1} \ll kv_T$ is due to the frequency shift of collective molecular polarization oscilla-

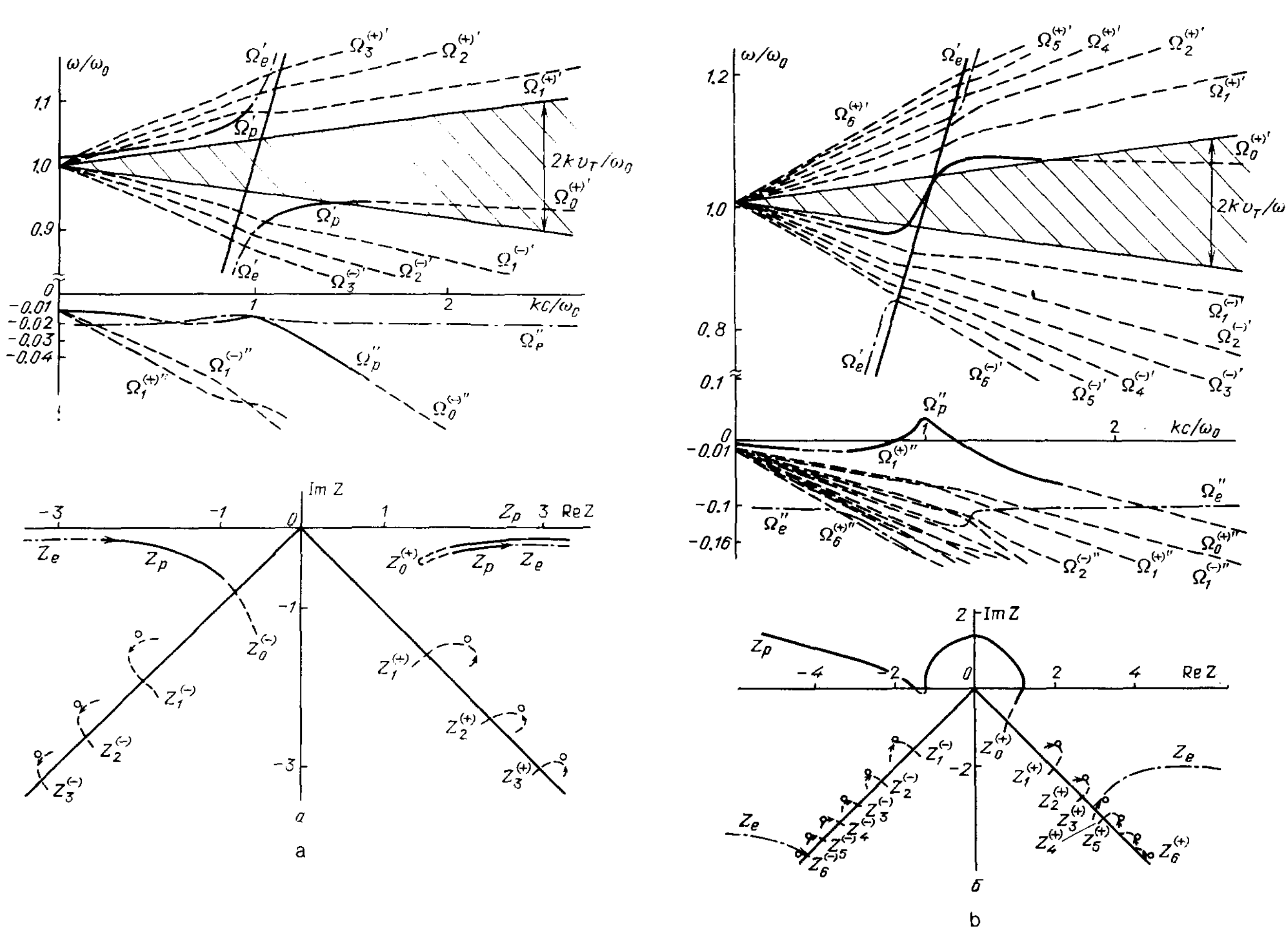

FIG. 19. *a*, *b*—Spectrum of normal transverse waves $\Omega = (\omega' + i\omega'')/\omega_0$ and the corresponding functions $Z(k) = (\omega + iT_2 - \omega_0)/kv_T$ for the dispersion curves on the plane of the complex argument Z of the Kramp function in the region of the Doppler line (6.4) for $0 < k < \omega_0/10v_T$. The graphs are constructed for $v_T\omega_0/c = |\omega_c^2|/3\omega_c = 4T_2^{-1} = 0.04\omega_0$; weakly conducting uninverted gas ($v_T\omega_0/c > 2\pi\sigma = \omega_c^2/6\omega_0$) (a) and a highly conducting inverted gas ($v_T\omega_0/c < 2\pi\sigma = |\omega_c|/2\sqrt{3} = -5\omega_c^2/6\omega_0$. (b) The inner part of the Doppler line $|\omega' - \omega_0| < kv_T$ is shown shaded; the direction of increasing k is indicated by arrows at points $Z(k = \omega_0/c)$; circles mark the zeros of the Kramp function $[w(Z) = 0]$.

tions due to the Lorentz correction which takes most of the molecules out of the Doppler resonance with the wave. The thermal velocity spread of the molecules then gives a spatial dispersion effect that is quadratic in k, i.e., there is an additional shift of the wave frequency [cf. the second and third terms in (6.8)]. This leads to a nonzero group velocity $d\omega_p'/dk \approx 9v_T^2 k\omega_0/\omega_c^2$. In the inverted gas, the ohmic conductivity reduces the long-wave damping and, for $2\pi\sigma > -3T_2^{-1}\omega_0^2/\omega_c^2$, leads to dissipative instability, since the wave energy density is negative:

$$\mathcal{W} = \frac{|E^2|}{16\pi}\omega\frac{\partial\varepsilon(\omega, k)}{\partial\omega} \approx \frac{3|E^2|\omega_0^2}{8\pi\omega_c^2}\left(1 + \frac{\omega_c^2}{9\omega_0^2}\right) \times \left[1 + \frac{1}{2}\left(\frac{9\omega_c kv_T}{\omega_c^2}\right)^2\right]. \qquad (6.9)$$

If we ignore the last point, we can readily see an analogy between these waves in a gas of two-level molecules ($\omega_0 \neq 0$; Secs. 3.1–6.3) and cyclotron waves in magnetoactive plasma ($\omega_0 \approx \omega_B$) or Langmuir waves in isotropic plasma[37,39,45,111] ($\omega_0 = 0$). The difference between them is essentially (and only) due to the different reasons for the frequency shift of collective oscillations. In a gas, the Lorentz correction ensures that it is determined by the cooperative frequency of the medium ($\omega_c^2/9\omega_0 = -8\pi d^2\Delta N/9\hbar$), whereas in magnetoactive and isotropic plasma it is due to the gyrofrequency ω_B and the electron Langmuir frequency ω_L. As for the Debye length $r_D = v_T/\omega_L$, which determines the short-wave limit of the exponentially weak collisionless Landau damping of Langmuir waves in Cherenkov resonance, it is clear from (6.8) that the scale $\lambda_d = 18\pi v_T\omega_0/|\omega_c^2|$ plays the analogous part for longitudinal waves and long polarization waves.

6.4. "Beam" instabilities in a gas of active molecules

Let us now develop further the analogy with plasma phenomena and consider a gas containing a beam of neutral molecules of the same kind as the molecules of the main gas, but with opposite inversion: $\Delta N_b = -\rho\Delta N, \rho \ll 1$. Analysis of this situation in the polariton part of the spectrum $k \sim \omega_0/c$ shows[45] that, at the Doppler resonance $kv_b = \omega - \omega_0$, a beam moving with velocity $v_b \gtrsim v_T$ can excite transverse waves of frequency $\omega = \omega_{e,p}(k)$. When

$|\omega''_{e,p} + T_2^{-1}| \ll kv_{Tb}$, the presence of the beam is equivalent to the conductivity $\sigma_b = -\rho\omega_c^2/24\sqrt{\pi}kv_{Tb}$, where v_{Tb} is the thermal velocity spread of the molecules in the beam. The inverted beam has $\sigma_b < 0$ and gives rise to the maser-type instability: $\omega''_{e,p} \approx -(\pi\sigma_b + \pi\sigma + T_2^{-1}/2)$ (we are assuming that $|\omega_c| \gg 2kv_T, 2\pi\sigma, T_2^{-1}$). The uninverted beam has $\sigma_b > 0$ and, according to Sec. 2.1, produces either an anomalous instability of the electromagnetic wave or the dissipative instability of the polarization wave.

The beam of inverted molecules or, more precisely, the "beam" distribution of inversion with velocity, $\Delta N(v)$, can be produced in the gas by, say, a quasimonochromatic pump, using the well-known three-level scheme and the Doppler effect. In gas lasers with Doppler broadening, it is also possible to produce two-hump or more complicated velocity distributions of inversion by exploiting the nonlinear saturation effect. According to Ref. 44, the instabilities of the new normal waves that accompany this can have a significant effect on the emission spectrum and on the operation of the laser. It is quite possible that these "beam" instabilities develop in the course of spontaneous coherent fluctuations in gas lasers operating well above the generation threshold.[43] More complicated spatial dispersion effects may be expected in partially ionized gases[114,115] in the case of the plasma-dipole resonance for which $\omega_0 \approx \omega_L$ (see Sec. 2.4), or during the excitation of molecular polarization oscillations by an electron beam.[116-118] However, studies of these questions, and searches for new (other than electromagnetic) high-frequency waves are only just beginning.

6.5. Effect of Doppler (inhomogeneous) broadening on super-radiance

We have seen that an increase in linewidth reduces the growth rate of normal waves in an inverted medium. This applies both to the inhomogeneous broadening[19)] ($2/T_2^*$) (including the Doppler broadening ($2kv_T$) and homogeneous, Lorentz broadening ($2/T_2$). The result is a reduction in the super-radiant power, an increase in its delay t_d and duration τ, a loss of super-radiant oscillations, an asymmetric pulse, and a partial release of the energy of the medium by emission of radiation. These conclusions are based mostly on numerical calculations[13-15,17,18,26,27] and are in agreement with experiments on gases and crystals.[15,23,30-35] On the whole, the effect of inhomogeneous broadening on the properties of super-radiance is significantly smaller than the effect of homogeneous broadening.[42] When inhomogeneous broadening is present, the analytic solution of the problem, which includes the nonlinear stage of super-radiance, can be obtained, at least in principle, by the inverse-problem method of the theory of scattering.[65,119] However, simple general formulas for the super-radiance parameters have not as yet been obtained.

The coherence criterion for a time-dependent optical process in general, and super-radiance in particular, has attracted considerable attention in the literature. In our view, in accordance with (1.1), the general criterion is $|E^{-1}dE/dt| > 1/T_2$, i.e., it involves the rate of change of the field amplitude acting on the molecules. This means that the polarization of the molecule does not follow the field adiabatically, i.e., it depends on the form of the process $E(t, \mathbf{r})$. The growth of the plane wave $E \propto \exp(\omega'' t)$ with $k = \omega_0/c$ and growth rate $\omega'' \sim |\omega_c^2|T_2^*/4 \ll 1/T_2^*$ in the case of the inhomogeneous line of Lorentzian shape (see footnote 18) is a coherent process if $|E^{-1}dE/dt| = \omega'' > 1/T_2$, i.e., $|\omega_c| > 2/(T_2T_2^*)^{1/2}$. This is a generalization of the criterion $|\omega_c| > 2/T_2$ of Sec. 2.1. For polariton-mode super-radiance with a discrete spectrum (Sec. 4), the criterion for coherent super-radiance assumes the form $\omega''_p > 1/T_2$. For unidirectional super-radiance of waves with a continuous spectrum and inhomogeneous broadening with a Lorentz frequency distribution of the molecules, $f^*(\omega_0) = T_2^*/\pi\{1 + [(\omega_0 - \bar{\omega}_0)T_2^*]^2\}$, the asymptotic form of the field on the linear stage $E \propto \exp[|\omega_c|(tL/c)^{1/2} - (t/T_2^*)]$, which corresponds to (3.8) after the replacement $1/T_2 \to 1/T_2^* \gg 1/T_2$. In this case, we obtain the condition $(|\omega_c^2|L/4ct_d^*)^{1/2} > 1/T_2^*$, i.e., $(t_d^*\tau_R)^{1/2} < T_2^*$, where $\tau_R = 2T_2^*/\mu_0 L$ and μ_0 is the amplification factor at the line center. The observed delay t_d^* is greater than t_d (3.12) and, in particular,[15] $t_d^* \approx t_d[1 + (t_d\tau_R)^{1/2}(T_2^*)^{-1}]$ for $(t_d\tau_R)^{1/2} \ll T_2^*$. It is indeed the criterion $(t_d^*\tau_R)^{1/2} < T_2^*$ and not a stronger criterion $t_d^* < T_2^*$ that is often thought[15,17,34] to distinquish super-radiance from superluminescence (see Sec. 3.5).

6.6. Soft mode and the antiferroelectric gas crystal

Apart from the h.f. waves ($\omega \sim \omega_0$) that occur in a dense gas of two-level molecules, there are also soft modes that correspond to low-frequency, self-consistent oscillations of polarization $|\omega| \ll \omega_0$. There is particular interest in the possible instability of the transverse soft mode $\omega_t(k)$ in an uninverted equilibrium gas in which $\omega_c^2 \propto N\,\mathrm{th}(\hbar\omega_0/2\varkappa T)$. This mode is due to the strong frequency shift of collective polar-

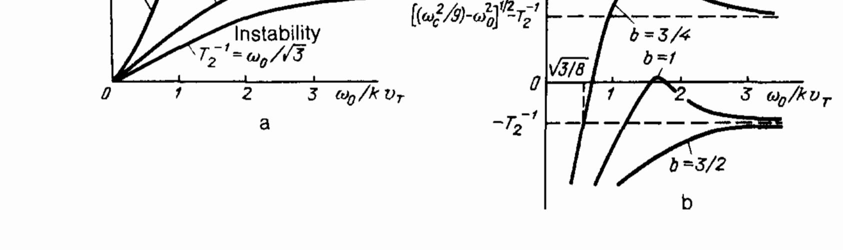

FIG. 20. Instability of the transverse soft mode for the two-level gas model in equilibrium. *a*—Stability threshold (6.10), $b_{cr} = (9\hbar\omega_0/8\pi d^2N_{cr})\,\mathrm{cth}\,(\hbar\omega_0/2\varkappa T_{cr})$ as a function of the concentration N_{cr} and temperature T_{cr}. *b*—Growth rate ω''_t as a function of ω_0/kv_T. The quantity b is defined by $b = 9\omega_0^2/\omega_c^2$.

ization oscillations of molecules, due to the Lorentz correction (see Sec. 6.1). The instability threshold and the growth rate near it can be determined from (1.16) and (6.3):

$$\left(\frac{3\omega_0}{\omega_c}\right)^2 = \frac{\sqrt{\pi}\omega_0}{kv_T}\,\mathrm{Im}\,w(Z),$$
$$\omega_t'' = \frac{kv_T\,[\mathrm{Im}\,w(Z) - (9kv_T\omega_0/\sqrt{\pi}\omega_c^2)]}{2\,\mathrm{Re}\,[Zw(Z)]}, \qquad Z = \frac{\omega_0 + iT_2^{-1}}{kv_T} \tag{6.10}$$

(Fig. 20). It is clear from Fig. 20a that, as the temperature T is reduced and/or the concentration N is raised to the critical value $(\omega_0^2)_{cr} = 4\sqrt{3}\,\omega_c^2$, the instability starts at the finite wavelength $\lambda \sim 3\pi v_T/\omega_0$. This spatial dispersion effect is thought to be the reason for a possible antiferroelectric phase transition in a homogeneous gas to the coherent gas crystal state with the static transverse polarization density wave. The antiferroelectric phase transition and the corresponding soft mode were previously considered only for anisotropic media (crystals). Earlier analyses of the isotropic gas were confined to the ferroelectric phase transition ($\lambda = \infty$) in the appproximation of stationary molecules[5,13]) (cf. papers on the super-radiant phase transition, mentioned in Refs. 5, 13, and 19). We emphasize, however, that the result given by (6.10) is only a model, and its applicability to real gases is still unclear.

7. CYCLOTRON SUPER-RADIANCE IN PLASMA PHYSICS AND ELECTRONICS

So far, we have been concerned with polarization waves and Dicke super-radiance in a set of quantum oscillators (two-level molecules) with space-time dispersion of a special form and saturation-type nonlinearity. Since, from the electrodynamic point of view, the phenomenon of super-radiance is, under certain conditions, a dissipative instability of negative-energy waves, and the dissipation is due to the emission of energy by the sample into ambient space, it is clear that similar super-radiant effects can occur in other systems, including classical systems with different types of dispersion and nonlinearity. When we turn to the classical analog of super-radiance, we extend to the case of coherent processes (1.1) the analogs of masers and lasers that are well-known in electronics and culminate in the work of Gaponov[120] and Lamb.[121] They include, for example,the cyclotron-resonance maser[122-124] and the free-electron laser.[69-71,125]

7.1. Dissipative instability in a beam of electrons in a magnetic field

Broadly speaking, spontaneous collective emission (super-radiance) can include any time-dependent coherent emission process that develops spontaneously in an open nonequilibrium system of initially nonvibrating particles (in the absence of a resonant external field) in a time shorter than the incoherent relaxation time T_2 of the oscillations of the individual particles.[20]) In a narrower sense, and by analogy with the Dicke super-radiance, super-radiance can also be understood as a coherent process that is associated with the dissipative instability of negative-energy waves that develop from spontaneous noise.

It will be clear from the ensuing analysis that the latter situation can occur in a set of classical harmonic oscillators. From the standpoint of classical macroscopic electrodynamics, it is precisely this specific situation that distinguishes the phenomenon of super-radiance[52] from other instabilities of systems of weakly nonlinear oscillators that interact as a result of induced emission.[72,124] The super-radiant regime has a direct relationship to microwave electronics in which different ways of generating powerful coherent pulses of electromagnetic radiation that do not require the use of resonators or are based on low-Q resonators are being extensively studied at present.[85,123] Attempts to abandon high-Q resonators, which facilitate the attainment of the generation threshold, have been dictated by a number of factors. They include the difficulty of developing such generators in certain wavelength ranges (e.g., x-ray or submillimeter ranges) and the necessity for higher pump power and shorter radiation pulse lengths that would ensure higher output power.[21]) It is therefore interesting to consider the possibility that phased oscillations of radiating particles could be produced as a result of interaction between them via the time-dependent intrinsic super-radiant field[52,125] rather than an external pump or long-lived quasistationary field, built up in the resonator over a long interval of time[120-122] $\Delta t \gg T_2$.

To be specific, let us consider a set of classical oscillators, e.g., electrons, in a uniform magnetic field $\mathbf{B}_0 \| \mathbf{z}^0$. The electrons can circulate around the lines of force with relativistic cyclotron frequency $\omega_B = eB_0/m_e c$ that is a function of electron energy $\mathscr{E}_e = m_e c^2$. The variance and nonlinearity of the system are determined by the well-known structure of "transverse" Landau energy levels[111] and the relativistic variation of the velocity of the electrons $v_\|$ in the direction of the magnetic field under the influence of the radiation reaction.

Let us consider the evolution of an unbounded rectilinear beam of monoenergetic electrons traveling in the direction of the magnetic field. We know that transverse h.f. oscillations can occur in this beam under the conditions of the *anomalous Doppler effect*, $v_\| > c_0 = c/\varepsilon_0^{1/2}$ (Fig. 21), for which the initial population of the lower Landau level corresponds to the excited state of the electron[128,129] in a medium with effective permittivity $\varepsilon_0 > 1$. To explain the nature of the h.f. instability in the presence of resistive losses, consider the waves $\vec{\mathscr{E}} = (E_\perp/2\sqrt{2})(\mathbf{x}^0 \mp i\mathbf{y}^0)\exp(-i\omega t + ikz)$ + c.c. that propagate along the magnetic field and are left and right polarized, respectively. In this one-dimensional formulation of the problem, the Maxwell equations and the relativistic equation of motion of an electron in a uniform rectilinear beam of concentration N_e in the laboratory frame leads to the following dispersion relation:[52]

$$[\omega(1 + i4\pi\sigma\omega^{-1})^{1/2} + (c_0^2k^2 + \omega_L^2\varepsilon_0^{-1})^{1/2}][\omega(1 + i4\pi\sigma/\omega)^{1/2} - (c_0^2k^2 + \omega_L^2\varepsilon_0^{-1})^{1/2}][\omega + iT_2^{-1} - (kv_\| \mp \omega_B)] = (\mp\omega_B - iT_2^{-1})\,\omega_L^2\varepsilon_0^{-1}; \tag{7.1}$$

where $\omega_L = (4\pi e^2 N_e/m_e)^{1/2}$ is the plasma frequency and $\sigma = \sigma_0/\varepsilon_0$. The quantity σ_0 represents the dissipation of the field by the effective conductivity of the "background medium" with permittivity $\varepsilon_0 > 1$. Apart from a dielectric, this medium can be a slowing-down electrodynamic system (waveguide). In the low-density limit, $\omega_L^2 \to 0$, the beam

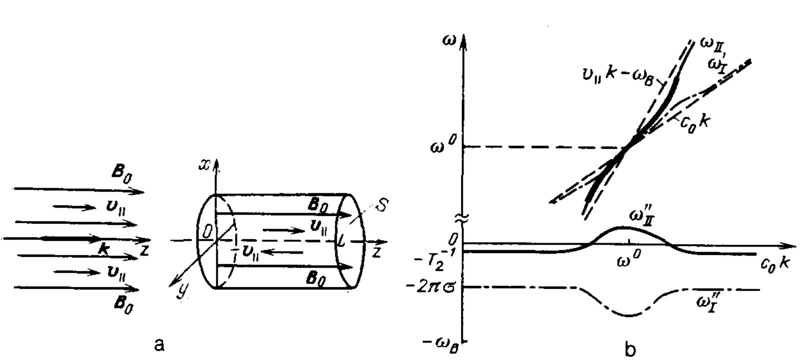

FIG. 21. *a*—"Superluminous" beams of electrons in a uniform magnetic field and in a homogeneous trap of length $L \gtrsim S/\lambda$. *b*—Dispersion curves for accompanying waves, slow cyclotron ω_{II} and electromagnetic ω_I waves.

maintains only damped partial waves, namely, electromagnetic waves $\omega = c_0|k| - i2\pi\sigma$ and cyclotron waves $\omega = kv_\parallel \mp \omega_B - iT_2^{-1}$ (slow and fast). However, the wave spectrum changes when the finite electron density is taken into account, and an instability becomes possible (see Fig. 21b). Actually, substituting $\omega_{\mp 0} = kv_\parallel \mp \omega_B$ and assuming that $2\pi\sigma \ll \omega_{\mp 0}$ and T_2^{-1}, $\omega_L \ll \omega_B$, we find that in the region of resonance with the accompanying waves $\omega_I' \approx \omega_{II}'$ and for $c_0 k \approx \omega_{\mp 0} \gg \omega_L (2\omega_B/\varepsilon_0\omega_{\mp 0})^{1/2}$, we obtain the normal electromagnetic waves $\omega_I(k)$ and the cyclotron waves $\omega_{II}(k)$ (slow and fast) that are analogous to the electromagnetic wave and the polarization wave in a medium consisting of two-level molecules, described by (2.1) with the following replacements:

$$ck \to (c_0^2k^2 + \omega_L^2\varepsilon_0^{-1})^{1/2},\quad \omega_0 \to \omega_{\mp 0},$$
$$\omega_c^2 \to \omega_{\mp c}^2 = \mp\frac{2\omega_L^2\omega_B}{\varepsilon_0\omega_{\mp 0}} \approx \frac{2\omega_L^2(c_0 - v_\parallel)}{c_0\varepsilon_0}. \tag{7.2}$$

The instability develops only for one of the waves, $\omega_I(k)$ or $\omega_{II}(k)$. It is convective and arises under the conditions of the anomalous Doppler effect if the square of the "electronic cooperative frequency" ω_{-c}^2 is negative and its modulus exceeds $8\pi\sigma/T_2$ (electron density threshold). The maximum growth rate [see (2.2)] is achieved for the wave number $k^0 = \omega_B/(v_\parallel - c_0)$ at the frequency $\omega^0 = c_0 k^0$. When $T_2^{-1} > 2\pi\sigma$, this instability is related to the negative dissipation of the positive-energy electromagnetic wave.

The dissipative instability of the slow cyclotron wave, which is due to its negative energy,[22]) develops when the field dissipation is large enough, i.e., $2\pi\sigma > T_2^{-1}$. This means that the growth of the slow cyclotron wave, i.e., the excitation of the transverse oscillations of electrons, is accompanied by a reduction in their total velocity as compared with the velocity in the undisturbed beam. The development of this type of instability is impossible under the conditions of the normal Doppler effect ($v_\parallel < c_0$) when there is no inversion of the Landau levels ($\omega_{+c}^2 > 0$) in the monoenergetic beam and resonance occurs only for positive-energy waves, i.e., electromagnetic and fast cyclotron waves.

7.2. Cyclotron super-radiance by electrons in a magnetic trap

This dissipative instability can evolve spontaneously in a time much shorter than T_2 and can lead to cyclotron super-radiance in a uniform magnetic trap with electrostatic or magnetic plugs at the ends at $z = 0$ and $z = L$. If the slowing-down electrodynamic system or medium is also bounded $\varepsilon_0 > 1$ and is confined to the layer $0 < z < L$ (see Fig. 21a), the slowed-down cyclotron waves will be partially reflected from its ends together with the electrons. In the one-dimensional model[23]) this leads to circularly polarized modes with a discrete spectrum $\vec{\mathscr{E}}_m = (E_m(z)/2\sqrt{2})(\mathbf{x}^0 - i\mathbf{y}^0) \times \exp(-i\omega_m t) + \text{c.c.}$ (see Sec. 4.2). The characteristic equation for the frequencies of the electromagnetic and cyclotron waves is given by (7.1) with discrete wave numbers $k = k_m - i(\ln R^{-1})/2L$, where $k_m = \pi m/L$. The imaginary part of the wave number, determined by the field reflection coefficient R of the ends, determines the inhomogeneous structure of the modes $E_m(z)$ along the trap. This can be taken into account in (2.1) and (7.2) by introducing the following additional replacement:

$$2\pi\sigma \to 2\pi\tilde{\sigma} = 2\pi\sigma + \frac{c_0}{2L}\ln R^{-1},$$
$$T_2^{-1} \to \tilde{T}_2^{-1} = T_2^{-1} + \frac{v_\parallel}{2L}\ln R^{-1}. \tag{7.3}$$

It is readily shown[52] that, when effective reflections $R \gtrsim 1/2$ occur in a sufficiently short trap of length $L \lesssim c_0/|\omega_{-c}|$, the super-radiant regime is due to the instability of modes with a discrete spectrum, since the instability of waves with a continuous spectrum is then found to be weak. It is precisely this situation that we shall examine below for the following special case:

$$2\pi\sigma \gg \frac{|\omega_{-c}|}{2} \gg \tilde{T}_2^{-1},\quad \omega_{II}''(k^0) = -\frac{\omega_{-c}^2}{8\pi\sigma} \gg \tilde{T}_2^{-1}, \tag{7.4}$$

for which the inhomgeneity of the modes structure along the trap is small. In this "mean-field approximation" (cf. Sec. 3.4), the mode growth rates and the attendant polarizability of the electron beam are given by

$$\omega_{II}''(k_m) = -\tilde{T}_2^{-1} - \frac{\pi\sigma\omega_{-c}^2/2}{(2\pi\sigma)^2 + [\omega_B - k_m(v_\parallel - c_0)]^2}$$
$$\chi_{II}(k_m) \equiv \frac{P_{\perp m}}{E_{\perp m}} = \frac{\varepsilon_0[\omega_B - k_m(v_\parallel - c_0) - i2\pi\sigma]}{2\pi\omega^0}. \tag{7.5}$$

We emphasize that cyclotron super-radiance is a transient process that is fundamentally different both from the slow quaistationary processes of instability development in resonators for times $\Delta t \gg T_2$, commonly encountered in electronics (for example, in the case of cyclotron masers using the anomalous Doppler effect[123]) and the well-known plasma physics transient processes involving the development of kinetic instabilities that end in quasiperiodic oscillations due to the nonlinear Landau damping.[39] In contrast to this damping of oscillations in plasma, which appears in the absence of true energy dissipation and is associated with the dephasing of particles with different velocities and energies, the damping of cyclotron super-radiant oscillations, and their transition to the a periodic regime in the system of monoenergetic particles that we are considering, occurs as a result of the strong dissipation of field energy due to the escape of radiation out of the trap.

7.3. Nonlinear stage

The growth of the cyclotron modes, which is initially exponential, terminates when they are no longer in resonance with the electron beam that is being slowed down by the resulting h.f. mode field: $v_{\parallel}(t) = v_{\parallel 0} - \Delta v_{\parallel}(t)$. Suppose that, initially, a beam of electrons of energy $\mathscr{E}_{e0}$ is in resonance with the mode $m = r$, i.e., $k_r = k^0(v_{\parallel 0})$. Since the instability band $\Delta k = (2\pi\sigma\tilde{T}_2)^{1/2}|\omega_{-c}|/(v_{\parallel} - c_0)$ is narrow for an individual mode (7.5), we shall confine ourselves to the case $\Delta v_{\parallel} \ll v_{\parallel 0} - c_0$ when we examine the shape of the super-radiant pulse. We shall also neglect plasma effects and the longitudinal bunching of electrons, assuming that $\omega_L \ll \omega_B$. We begin with the single-mode regime, assuming that the mode spacing is large: $k_m - k_{m-1} = \pi/L > \Delta k$. In the adiabatic approximation, we then obtain the following equations for the square of the mode polarization $\overline{P}_r^2$ averaged over the trap (or over the h.f. period)[52]

$$\dot{\overline{P_r^2}} = 2\omega''_{11}(t)\overline{P_r^2},\ 2N_e\dot{\mathscr{E}}_e = -\sigma_0\overline{P_r^2}|\chi_{11}(t)|^{-2},$$
$$\Delta\dot{v} = -(c\varepsilon_0^{1/2} - v_{\parallel 0})\,\dot{\mathscr{E}}_e\mathscr{E}_{e0}^{-1}. \qquad (7.6)$$

In view of (7.5), this set of equations is self-consistent and includes relativistic effects. Its solution is shown in Fig. 22 in which $\bar{t} = t/\tau$ is normalized to the reciprocal of twice the growth rate $\tau = -4\pi\sigma/\omega_{-c}^2$. The length of the pulse of cyclotron super-radiance $Q_r = \sigma_0|\overline{E_r^2(t)}|$ is $\tau_p \approx 10\tau \ll \tilde{T}_2$, and the maximum power (per unit trap volume) is $Q_{r,\max} = N_e\mathscr{E}_{e0}\omega_L^2(v_{\parallel} - c_0)^2/\omega_B(\varepsilon_0 - 1)\varepsilon_0c_0^2 \propto N_e^2$. According to (7.6), the pulse shape can be expressed in terms of the electron slowing-down function $U = \Delta v_{\parallel}\omega^0(\varepsilon_0 - 1)/2\pi\sigma(c\varepsilon_0^{1/2} - v_{\parallel 0})$ and has the form

$$Q_r(\bar{t}) = Q_{r,\max}\cdot 2U(U_{\max}^2 - U^2)(1 + U^2)^{-1}(U_{\max}^2 - 1)^{-1},$$
$$\bar{t} - \bar{t}_d = \frac{1}{2}\ln U^2 - \frac{1}{2}(U_{\max}^2 + 1)$$
$$\times\ \ln[(U_{\max}^2 - U^2)(U_{\max}^2 - 1)^{-1}],$$
$$U_{\max} = \left(\frac{3\tilde{T}_2}{2\tau}\right)^{1/2}.$$

Next, let us consider the *multimode regime* in which $\pi/L \ll \Delta k$ and the number of modes M within the strong instability band in which $\omega''_{11}(k_m) \sim \max\omega''_{11}$ is large is given by $M = 2L\sigma/(v_{\parallel 0} - c_0) \gg 1$. In the adiabatic approximation analogous to (7.6), the slowing down of the electron beam is now determined by the total mode power and does not end when the beam is no longer in resonance with one of the unstable modes $m = r$. New and higher frequency modes, which grow and continue to slow it down, enter resonance with the beam. The result of all this is that mode sequences corresponding to equally slowed-down electrons are established and are defined by $\Delta\dot{v}_{\parallel} = -a\omega_{-c}^2(c\varepsilon_0^{1/2} - v_{\parallel 0})/2\omega^0(\varepsilon_0 - 1)$ $a = \text{const}$. For $t \ll \tilde{T}_2$, the radiated mode power $Q_m = \sigma_0|\overline{E_m^2(t)}|$ is then described by a function[52] of the running variable $\zeta = a\bar{t} - (\Delta m/M)$:

$$\overline{Q}(\zeta) = \left(\frac{2a}{e}\right)^2 \frac{N_e\mathscr{E}_{e0}|\omega_{-c}^2|}{M(\varepsilon_0 - 1)\omega^0}\,\frac{1 + (2a)^{-2}}{1 + \zeta^2}$$
$$\times\ \exp\left[\frac{1}{a}\left(\operatorname{arctg}\zeta - \operatorname{arctg}\frac{1}{2a}\right)\right], \qquad (7.7)$$

where $\Delta m = (m - r)$ is an integer and $e = 2.71\ldots$. In the most interesting case, in which $(2\tau/\tilde{T}_2)^{1/2} \ll a \ll 1$, we have $a \approx \pi/\ln[(\tau e^2/2\tilde{T}_2a^2)\overline{Q}(1/2a)/Q_r(0)]$ and the total quasi-stationary radiated power is $\Sigma_m Q_m = 2aQ_{R,\max}$. The pulse length for an individual mode is $\tau_p \approx \tau/a^2$, its total time within the instability band $\omega''_{11}(k_m) > 0$ is $\Delta t \sim (\tau\tilde{T}_2)^{1/2}$, and the time to establish this mode-sequence regime $t_d \sim \tau\ln[Q_{r,\max}/MQ_r(0)]$ is short in comparison with the time $\tilde{T}_2$ for a random dephasing of the rotation of an electron in the trap.

Let us now estimate the maximum cyclotron super-radiant power, e.g., for electrons with energy $\mathscr{E}_{e0} \sim 1$ MeV and concentration $N_e \sim 3\cdot 10^{10}$ cm^{-3} ($\omega_L \sim 10^{10}$ s^{-1}) in a trap with slowing-down factor $\varepsilon_0 = 1.5$ and magnetic field $B_0 \sim 20$ kG ($\omega_B \sim 2\cdot 10^{11}$ s^{-1}). To be specific, we assume that $v_{\parallel} - c_0 = 0.1\,c_0$ and $a = 1/6$. For a trap with perfectly reflecting ends ($R \sim 1$), we find that, in the sequential mode state, $\Sigma_m Q_m \sim 10$ kW/cm^3 at $\omega^0 = 10\omega_B$ (wavelength ~ 1 mm). When $L \sim 30$ cm and $S \sim 0.3$ cm^2, the total power transported by diffraction super-radiance across the side surface of the trap is $LS\Sigma_m Q_m \sim 100$ kW, so that, during the time of emission of an individual mode, i.e., $\tau_p \sim 30$ ns, about 3% of the kinetic energy of the electrons is radiated. Cyclotron super-radiance should cease after a time $t \gtrsim \tilde{T}_2 \gg \tau_p$, or before, if the conditions for the anomalous Doppler effect

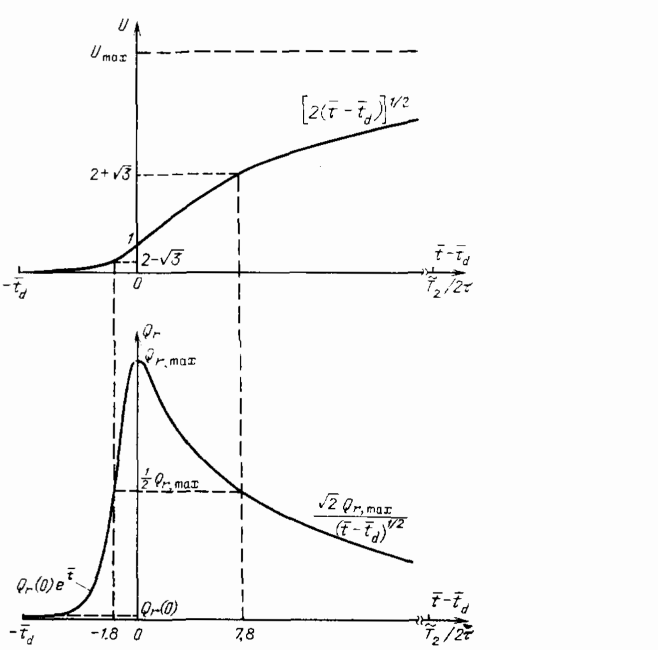

FIG. 22. Single-mode cyclotron super-radiance regime: time dependence of the slowing-down of the electron beam, $\Delta v_{\parallel} \propto U(t)$ and the shape of the radiation pulse $Q_r(\bar{t})$ for $\bar{t}_d \equiv t_d/\tau = \ln[Q_{r,\max}/Q_r(0)] = 10$, $\tilde{T}_2/\tau = 54$.

are violated as a result of the rapid deceleration of the electrons. However, the long-term course of the process, including the transition from the anomalous Doppler effect to the normal effect, and the possibility of quasiperiodic energy transfer between the field and the electrons in the presence of partial emission, has not been adequately investigated. Some aspects of the dynamics of this type of process are examined in Refs. 129–132.

7.4. Dicke super-radiance by a system of molecules under the conditions of the anomalous Doppler effect

To complete the classical analogy that we have developed, let us compare cyclotron super-radiance in a beam of harmonic oscillators (electrons) under the conditions of the anomalous Doppler effect with the corresponding problem for Dicke super-radiance by a "superluminous" beam of two-level oscillators (molecules).[133,134] Suppose that the velocity of the molecules is $v > c_0 = c/\sqrt{\varepsilon_0}$ in a homogeneous medium of conductivity $\sigma_0 > 0$. The molecules are aligned at right angles to the direction of motion (the z axis). In the one-dimensional case, and if we use the equations of Sec. 1.3 together with the Lorentz transformation, we then find that the complex frequencies of the two counterpropagating electromagnetic waves $\omega_{\pm e}(k)$ and the two accompanying fast and slow polarization waves $\omega_{\pm p}(k)$ satisfy the following dispersion relation:

$$|\omega(1+i4\pi\sigma\omega^{-1})^{1/2}+c_0k][\omega(1+i4\pi\sigma^{-1})^{1/2}-c_0k]\times\left(\omega-vk+\frac{i}{T_2\gamma}-\frac{\omega_0}{\gamma}\right)\left(\omega-vk+\frac{i}{T_2\gamma}+\frac{\omega_0}{\gamma}\right)=\frac{(\omega-vk)^2\omega_c^2}{\varepsilon_0\gamma}, \quad (7.8)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ [cf. (7.1)]. The solution in the resonance case is similar to (2.1), and is shown in Fig. 23. For the branches ω_{+e} and ω_{-p}, the part played by the square of the cooperative frequency (1.12) is now taken up by the quantity $\omega_{c1}^2 = \omega_c^2(1 - v/c_0)/\varepsilon_0\gamma$, whose sign is opposite to that of ω_c^2. Consequently, as well as both the slow cyclotron wave, the slow polarization wave has a negative energy in the beam of uninverted molecules, and it is precisely there that its dissipative instability and super-radiance become possible for $-\omega_{c1}^2 > 8\pi\sigma/T_2\gamma$.

Conversely, inversion in the "superluminous" beam precludes the instability of the slow polarization wave. However, the dissipative instability of the fast polarization wave ω_{+p} is then possible (in the region of the normal Doppler effect, i.e., for $k < 0$ in Fig. 23a). For this wave and for the counterpropagating electromagnetic wave ω_{-e}, the role of the square of the cooperative frequency is taken up by $\omega_{c2}^2 = \omega_c^2(1 + v/c_0)/\varepsilon_0\gamma$, so that the character of the super-radiance is essentially no different from the usual case with $v = 0$, for which the solution of the dispersion relation (7.8) reduces to (2.1). Hence, it is clear that, for times $t \lesssim T_1$, the "superluminous" sample of two-level molecules (but not classical electrons) can spontaneously generate a sequence of super-radiant pulses that are emitted as a result of the successive instability of waves in the two resonances indicated in Fig. 23. The process of unidirectional super-radiance after each radiated pulse repeats itself on the next resonance because of the change in the sign of the population difference ΔN that occurs as a result of the development of the instability on the previous resonance. Thus, the kinetic energy of the molecules is the source of the two asymmetric trains of super-radiant pulses having different frequencies and propagating in opposite directions.

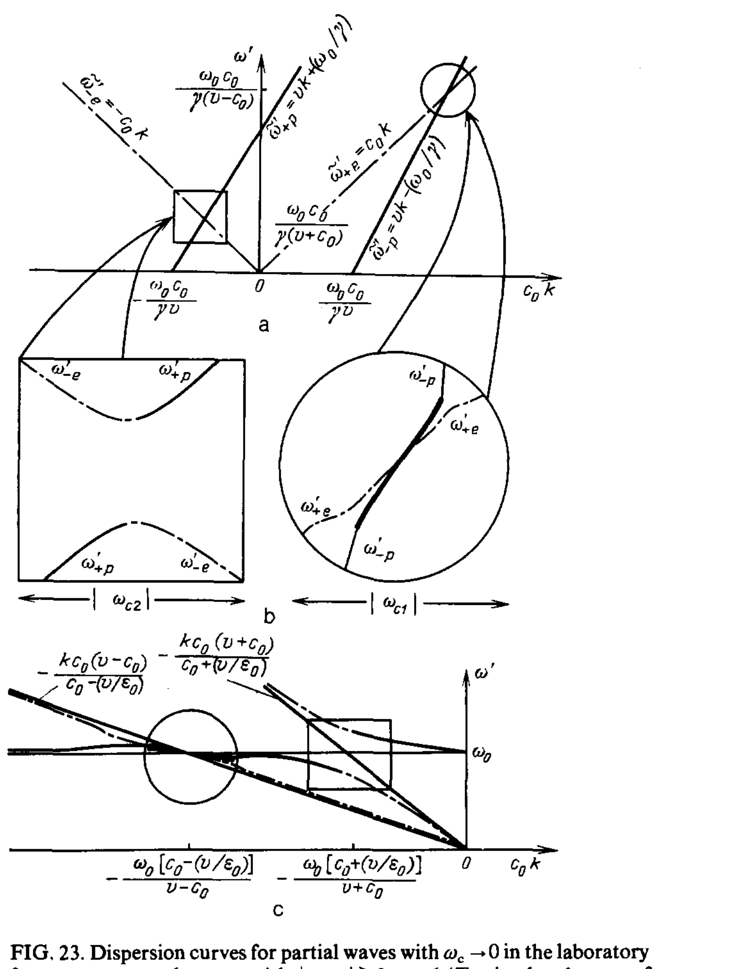

FIG. 23. Dispersion curves for partial waves with $\omega_c \to 0$ in the laboratory frame (a), normal waves with $|\omega_{c1,2}| \gtrsim 2\pi\sigma \ll 1/T_2\gamma$ in the absence of inversion $\Delta N < 0$ (thick line shows the region of instability of the slow polarization waves; for $\Delta N > 0$, the other, fast polarization wave is unstable) (b), and the same normal waves in the accompanying frame attached to the beam of uninverted molecules (c).

We emphasize that the established specific analogy between Dicke super-radiance in a set of quantum (two-level) molecular oscillators and cyclotron super-radiance in a set of classical electron oscillators does not exhaust the problem of collective spontaneous emission in the classical physics of plasma and in electronics, but actually merely poses it.[135] In particular, it has been suggested[125,136] that it may be possible to produce super-radiance from moving electron bunches also in the free-electron laser. It is then interesting to consider the emission by an electron bunch of both discrete modes and waves with a continuous spectrum.

8. ANALOGS OF DICKE SUPERRADIANCE IN MORE COMPLICATED SYSTEMS

The example of cyclotron super-radiance examined in the last Section showed that our electrodynamic approach is an effective means of finding analogies between collective coherent processes in different quantum and classical systems. In this Section, we review some of the more complicated systems in which Dicke super-radiance type processes have been produced. To do this, we shall need to extend the theory of polarization waves to new special situations. In many cases, this generalization has not been carried out. The brief description of collective coherent phenomena, given below, is intended to draw attention to problems that can be usefully and successfully investigated by the methods available in the electrodynamics of continuous active media.

8.1. Super-radiance in a three-level medium. Subradiance

The transition from the two-level to the three-level medium already provides a number of interesting effects (Fig. 24). The successive super-radiance of two pulses of different color in the $3\to2\to1$ cascade Σ-scheme was observed in Refs. 14, 32, and 138. For the V and Λ schemes (see Fig. 24b), we have typically partial super-radiance, i.e., the incomplete emission of stored energy as a result of competition between pulses of different color.[13,14,33,108] Super-radiance produced as a result of the weaker transition is quenched without succeeeding in fully developing because of the removal of inversion by rapid super-radiance via the adjacent transition. For closely spaced sublevels, there are quantized intensity beats, time-dependent polarization ellipse of the super-radiant pulse, and very strong macroscopic quantum fluctuations.[13-15,53]

The self-consistent excitation of low-frequency coherence (off-diagonal element of the density matrix ρ_{32}) in the V scheme during the development of super-radiance from an incoherent initial state is a nontrivial fact. The result is the destructive interference between h.f. polarizations of the optical 3–1 and 2–1 transitions that removes the resultant macroscopic polarization, and super-radiance is limited and terminated: the three-level medium goes over to the metastable state of subradiance[24)] (see Fig. 24b).[140] The point is that the nonlinearity of the three-level medium that is responsible for the excitation of low-frequency coherence may come into play for a lower field amplitude than the ordinary nonlinearity of the two-level medium that is responsible for the complete removal of population inversion by Dicke super-radiance. This means that we can have coherent trapping of populations on the upper sublevels and the termination of super-radiance for an incomplete removal of inversion (this type of effect in initially uninverted media is discussed in Ref. 141). The molecules are subsequently slowly de-excited only as a result of incoherent relaxation. The phenomenon of subradiance was first established experimentally in 1985.[138]

Similar phenomena occur during super-radiance by two uncoupled degenerate or almost degenerate transitions in a medium of one species of atoms or two different isotopes

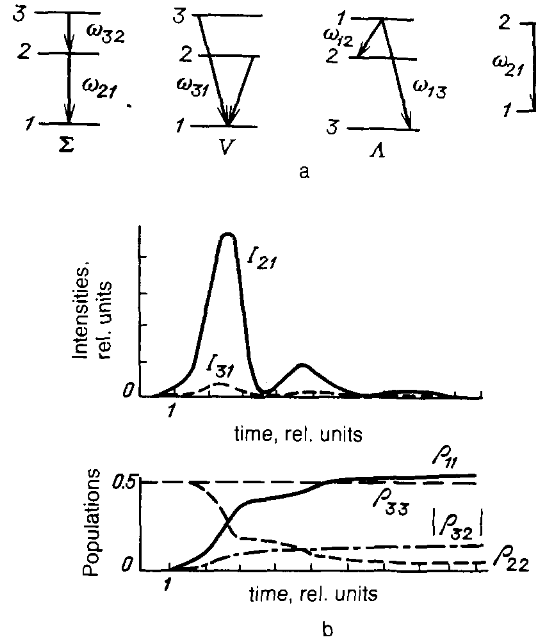

FIG. 24. Super-radiance in a three-level medium. *a*—Possible schemes (Σ, *V*, Λ), and also super-radiance from two uncoupled transitions. *b*—Numerical example[140] of partial superradiance and subradiance in the V scheme; I_{21} and I_{31}—superradiance intensities from the $2\to1$ and $3\to1$ transitions. ρ_{ii}—populations of levels $i = 1, 2, 3, \ldots$. $|\rho_{32}|$—low-frequency coherence.

(see Ref. 14 and the references therein). In this case, as for the three-level medium, the presence of two partial oscillations of polarization, coupled to the electromagnetic field, leads to the existence of three normal waves (Fig. 25). Two of them are different types of polarization wave and the third is an electromagnetic wave. This situation is known experimentally in gases and in crystals.[137]

8.2. Nonresonant coherent Raman scattering

Super-radiance and the excitation of polarization waves are typical not only for single-photon (resonance) processes, but also for two-photon and multiphoton (nonresonant) processes. Nonresonant cooperative Raman scattering of light, also called coherent scattering, has been particularly well investigated. This is a form of stimulated Raman scattering of a powerful pump of frequency ω_L, which differs from the resonance frequencies ω_{mn} ($m, n = 1, 2, 3, \ldots$) of molecular transitions, to the Stokes ($\omega_S = \omega_L - \omega_0$) and anti-Stokes ($\omega_A = \omega_L + \omega_0$) components (Fig. 26a).

It is common to distinguish two states of coherent [in

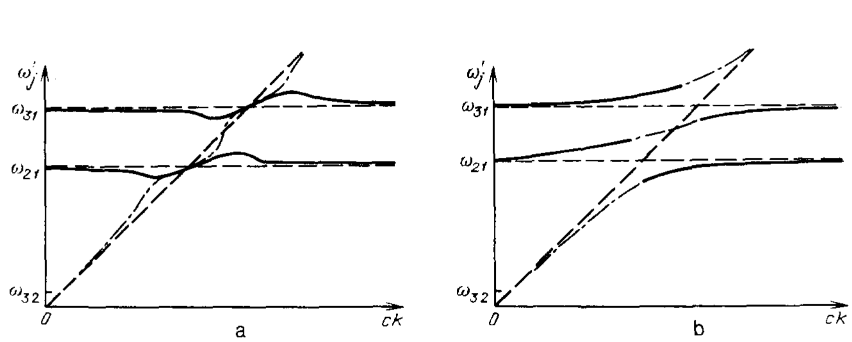

FIG. 25. Typical form of the polariton spectrum of inverted (a) and uninverted (b) three-level medium in the V scheme.

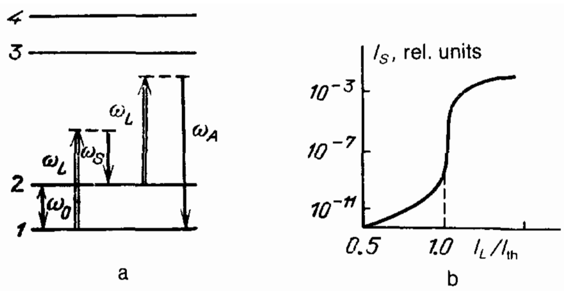

FIG. 26. *a*—Stokes and anti-Stokes SRS. *b*—Experimental dependence[152] of the intensity I_S of the Stokes component of SRS for liquid nitrogen on the pump intensity I_L/I_{th}.

the sense of (1.1)] stimulated Raman scattering (SRS). First, *time-dependent* SRS is discussed, for example, in Refs. 54, 142, and 143. This occurs when the pump length is $\tau_L \ll T_2$, but its intensity is insufficient to ensure that, during the generation of the Stokes components, there is a significant increase in the population of the upper level 2 during the time τ_L. Second, intrinsic cooperative Raman scattering (CRS) is also referred to as *super-radiant Raman scattering* and occurs when the pump intensity is high enough and independent of the pulse length $\tau_L \lesssim T_2$, so that the growth rate of the Stokes radiation is so high, $\omega_p'' \gg T_2^{-1}, \tau_L^{-1}$, that the lower level 1 is almost completely depleted during this process in a time $\Delta t \ll T_2$ and populates level 2. This process was first observed in the case of SRS in hydrogen[114] and, subsequently, in other experiments.[93,145–147] Ordinary quasitationary SRS, that was well-known before the 1970s,[151] differs from cooperative Raman scattering in roughly the same way that superluminescence differs from super-radiance (cf. Sec. 3.5). Several authors have given theoretical descriptions of different aspects of cooperative Raman scattering,[13,77,78,145,148–150] but many of the possibilities of the methods employed in the electrodynamics of continuous active media have not as yet been fully exploited.

For example, it may be shown by analogy with resonant super-radiance (Sec. 4, Fig. 10) that slight reflections can lead to cooperative Raman scattering into the Stokes polariton modes with a discrete spectrum, and to a rapid discontinuous rise in the Stokes radiation intensity when the threshold for its generation has been reached. A similar anomalous rise in cooperative Raman scattering by a factor of 10^6 or more, was observed, for example, in Ref. 152 (Fig. 2b). Other interpretations of this phenomenon have also been put forward.[13,153,154]

8.3. Resonant CRS and super-radiance using prolonged coherent pumps

Nonresonant CRS becomes the corresponding resonant effect when the pump is in resonance with some transition in the molecule ($\omega_L \approx \omega_{31}$; cf. Fig. 26a). Once a prolonged coherent pump ($t_L = L/c$ in Fig. 27a) has been turned on and has traversed the sample, Stokes radiation is produced by the combined effect of $3 \to 2$ Dicke super-radiance and the CRS of the pump ($\omega_L \to \omega_S$). Its intensity I_S is doubly modulated: It consists of slow damped oscillations due to propagation effects (as in the case of Dicke super-radiance; see Sections 3.3–3.5) and fast optical nutations between the initially populated level 1 and level 3 under the influence of the pump field E_L with Rabi frequency $\omega_R = d_{13}E_L/\hbar$. A more detailed discussion of resonant CRS is given in Refs. 27, 115,

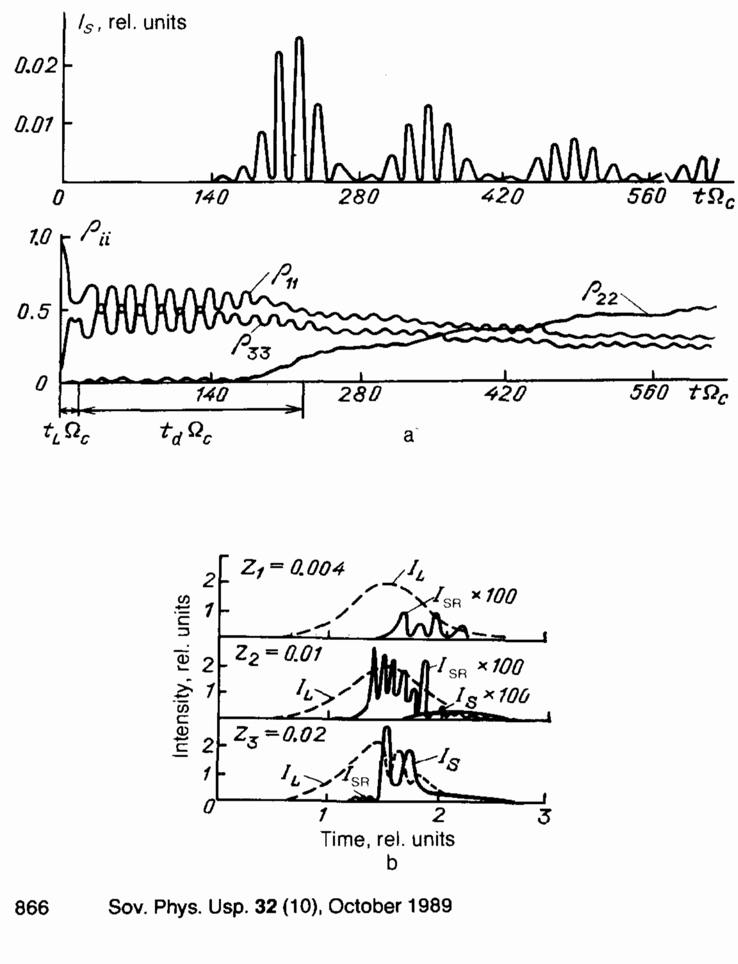

FIG. 27. a—Oscillogram of the resonant Stokes CRS intensity I_S and integral level populations ρ_{ii} ($i = 1, 2, 3$) for a sample of a three-level medium of length $L = 2\sqrt{2}c/\Omega_c$; pump field amplitude $E_L = \hbar\Omega_c/2\sqrt{2}d_{13}$ (numerical calculation in the unidirectional model in the absence of relaxation[27]). b—Numerical solution[157] for the evolution of the pump pulse I_L, super-radiance I_{SR}, and Stokes CRS I_S during propagation in the medium; Z—optical thickness of the medium at pump frequency.

and 156. In particular, one possibility is the self-consistent Stokes transformation of the 2π-pump pulse into a Stokes pulse.[156]

In the general case super-radiance and Stokes CRS are different radiation components with different frequencies ($\omega_{32} \neq \omega_S = \omega_L - \omega_{21}$) and different type of evolution. For example, a sequence of super-radiant 2π pulses (Fig. 27b, Z_1) may appear and rapidly develop on the trailing edge of the pump pulse, and this may be followed by more slowly growing Stokes radiation (Fig. 27b, Z_2) that increases the population of level 2, quenches the $3 \to 2$ super-radiance, and begins to deplete the pump, leading to two-photon self-modulation of pulses (Fig. 27b, Z_3).[157]

In addition to CRS by spontaneous or thermal fluctuations, there is also CRS by macroscopic coherent polarization of the medium, produced by external field sources (see initiated super-radiance; Sec. 3.6). The latter includes CRS by nonlinear polarization waves[158] accompanying the propagation of simultons, SRS solitons, and other pulses in three-level media. This soliton-like CRS regime was observed in Ref. 147.

The analogy between CRS and super-radiance can be extended to a number of coherent three-wave interactions in the pump-wave field in a medium with a quadratic nonlinearity. This includes, for example, stimulated parametric scattering in the presence of a pulsed pump[98] and second-harmonic generation.[159]

8.4. Other analogs

First, we note that spontaneous collective emission can be produced by spontaneous phasing of a set of radiators[19,160-162] that is unrelated to their interaction via the intrinsic radiation fields. For example, there is the coherent radiation emitted when the dipole moments of the unit cells of a crystal undergo a transition to the ferroelectric or ferromagnetic[161] state when the crystal is rapidly cooled below the Curie point T_c, or when a depolarizing pulse of an external field (or pressure) is applied at $T < T_c$ via the direct (and not via the photons) intermolecular interaction.[160] Another example[162] is the radiative decay of the exciton mode that had been filled macroscopically by nonequilibrium Bose condensation of excitons generated in a crystal by an incoherent pump. A similar explanation has been offered for the narrow emission lines observed experimentally in Ref. 163. Essentially, this is not Dicke super-radiance, since the radiators are correlated by the direct dipole-dipole interaction, or some other mechanism, and not by the self-consistent super-radiant field. Such emission processes are closer to the radiative damping of free polarization[12] produced by extraneous sources.

Negative-energy waves, analogous to polarization waves, and their dissipative instability, exist not only among electromagnetic phenomena in electrodynamics, optics, radiophysics, electronics, and plasma physics, but also in other areas of physics, including hydrodynamics and acoustics (see Refs. 41, 126, and 164, and the literature cited therein). The generation of sound by supersonic hydrodynamic shear flow, i.e., a tangential discontinuity, is an example of this. It is related to the phenomenon of super-reflection of a wave incident on a tangential discontinuity, which amplifies the reflected waves. A negative-energy wave then travels into the moving medium, whereas a positive-energy wave enters the stationary medium. The presence of the boundary leads to acoustic feedback and a dissipative instability in which the negative-energy wave grows at the expense of the positive-energy wave radiated by the system.

Similar processes are possible for electromagnetic waves near a rotating conducting cylinder,[166] for sound propagating near a rotating viscous vortex,[164] and for gravitational and electromagnetic waves near a rotating black hole.[6] A direct analog of electromagnetic super-radiance in acoustics is the super-radiance of ultrasound (phonons) by a set of atoms prepared in excited electronic states.[32,165,167]

Coherent collective emission of fermions (neutrinos, neutrons, etc.) by a set of identically excited nuclei is discussed in nuclear physics and is analogous to the super-radiance of photons by a set of excited molecules.[168] Unfortunately, the power that can be produced in existing accelerators is as yet insufficient for the super-radiance of fermions.

9. CONCLUSION

We have tried to demonstrate the efficacy of the phenomenological approach of macroscopic electrodynamics in the investigation of coherent collective processes in active media. In particular, the above description of super-radiance as a dissipative instability of negative-energy polarization waves enabled us to establish not only the physical mechanism responsible for this phenomenon, but also to move forward its analysis, both in the examples considered above and, probably, in other cases, too. Moreover, direct phenomenological quantization of modes and normal waves in active samples (rather than field oscillators in a vacuum) enabled us to provide a simpler description of the quantum-mechanical properties of collective excitations of the medium, and to facilitate the solution of problems involving macroscopic quantum fluctuations. One example, is the derivation of the quantum statistics of the delay time and the parameters of the polarization ellipse of super-radiant pulses. On the whole, this approach leads to the correct allowance for the way super-radiance depends on geometric factors and the three-dimensional character of the problem, the inhomogeneity of the active medium, reflections from the ends of the sample, self-excitation of modes with a discrete spectrum and of waves with a continuous spectrum, nonlinear interaction between modes and/or waves, and so on.

The concepts and descriptions employed in the macroscopic electrodynamics of continuous media are exceptionally useful in the comparative analysis of different coherent collective processes, and in estimating their place in the overall picture of wave processes in active media. The inherent possibilities of this approach are far from having been fully exploited.

[1)] Its advantages for transparent and equilibrium absorbing media were noted a long time ago[7-9].

[2)] For the sake of simplicity, we assume that $\vec{\mathscr{E}} \parallel \mathbf{d}_l$ and omit from (1.12) the factor $\sim 1/3$ which is associated with averaging over the orientations of the molecules. It is taken into account in Sec. 6 [see Sec. (6.1)].

[3)] In a gas with $N \gtrsim \omega_0^3/c^3$, a minimum occurs for $T_2^{-1} \sim |\omega_c^2|/10\omega_0 \ll |\omega_c|$. It is determined by the dipole–dipole h.f. interaction between the molecules in Weisskopf collisions.[45,59] In particular, this limits the resonance susceptibility $|\chi| = |\varepsilon - 1|/4\pi \ll 1$ for $\omega'' = 0$ even when $\omega' \approx \omega_0$.

[4]The expression for the energy (2.9) differs from the modulus or the real part of the usual electrodynamic expression[1-3] $(|E^2|/16\pi\omega)\mathrm{d}(\omega^2\varepsilon)/\mathrm{d}\omega$ because the susceptibility of the medium is now complex. It follows that $\mathrm{d}(\omega^2\varepsilon)/\mathrm{d}\omega$ is not sufficient to enable us to determine the energy.

[5]In the limit of no relaxation or dissipation ($Q = 0$), for which the instability is anomalous in character (see Fig. 2b), the normal waves with $\omega''_{e,p} \neq 0$ have zero energy because $w = -Q/2\omega''$. The nonzero growth rate in this case is due to the transfer of energy from the partial oscillations of polarization to partial oscillation of the electromagnetic field due to the interaction between these dynamic subsystems. This type of dynamic excitation of the oscillations of the two subsystems with opposite sign of energy is the limiting case of dissipative instability.

[6]Or active centers in semiconductors and metals in the optical and ultraviolet ranges (at the limit of transparency, $\omega_0 \approx \omega_L \sim 4\pi\sigma|\varepsilon_0^{-1}| \gg T_2^{-1}$), where the field dissipation is due to free electrons.

[7]The Arecchi-Courtens length $L_c = c/\Omega_c$ determines (apart from a logarithmic factor) the maximum size of regions in the medium in which the instability with maximum growth rate $\Omega_c/2$ occurs in a causally connected manner: light traverses the length L_c during its development.

[8]Strictly speaking, $P_0(z_0')$ in (3.7) must be understood to represent the quantity $\overline{P_0(z_0')} = \int_\infty^0 P_0(z')\exp\{-[(z_0'-z')(\Delta z')^{-1}]^2\}\,\mathrm{d}z'$ averaged over the scale $\sim \Delta z' = (L_\sigma/L_c)(ctL_\sigma)^{-1/2}$.

[9]In the original formulation of the problem, there are no pump sources or incident external fields.

[10]Super-radiance in the two-dimensional problem is discussed in Refs. 28, 59, and 75.

[11]The second term in the Debye expansion, which represents waveguide effects in super-radiance from a cylinder due to reflections from the side surface, is discussed in Ref. 78.

[12]The concept of "hot modes" is widely used in microwave electronics when the introduction of a dense enough electron beam into an empty "cold" resonator (or waveguide) produces a significant change in the structure and spectrum of its electromagnetic modes, transforming them into new "hot" modes (see, for example, Ref. 85).

[13]By adding to the quadratic Hamiltonian (5.4), (5.9) further terms of a higher order, which make it positive-definite, we can describe the subsequent nonlinear intability state, as well. However, in many cases, including super-radiance, the macrofluctuations already occur on the linear stage, so that we can confine our attention to the quadratic Hamiltonian in the case in which we are interested here.

[14]The PQED of macrofields of normal oscillators can be compared with the QED of cold modes and individual molecules in vacuum apparently by using the exact solutions for the quantum models of super-radiance, recently obtained by means of the Bethe substitution.[106].

[15]This criterion determining for the delay time is based on the linear approximation for the description of super-radiance, and is analogous to that used in Refs. 14, 42, and 77.

[16]This is valid for wavelengths exceeding the dimensions of the molecules, but not greater than the mean free path: $r_m \ll k^{-1} \ll l$. The resonance properties of the gas in relation to the long wave fields with $k^{-1} \gtrsim l \sim v_T T_2$ are due to the diffusion (Brownian) motion of molecules that is described by the collision integral omitted from (6.1). These collisions lead to a narrowing and replacement of the Doppler line (6.7) with the Lorentzian line of half-width $\sim k^2 l v_T$.

[17]Under the conditions of effective dissipation $2\pi\sigma > T_2^{-1}$ in an inverted medium with $2kv_T \ll |\omega_c|$, the last variant in (6.4) refers only to strongly damped electromagnetic wave $\omega_e(k)$, whereas a weakly damped or growing polarization wave (2.1) will leave the Doppler line because the frequency has an imaginary part $\omega_p'' + T_2^{-1} \gg kv_T$ although $|\omega_p' - \omega_0| \lesssim kv_T$.

[18]As in the case of plasma, the new solutions do not arise if instead of the Maxwell distribution we use the Lorentz distribution $F_L(v/v_T) = [\pi(1 + v^2 v_T^{-2})]^{-1}$. According to (6.1), this leads to the Lorentz type permittivity (1.14), (1.15) with the replacement $T_2^{-1} \to T_2^{-1} + kv_T$, so that the dispersion relation (1.16) for transverse waves is completely exhausted by the polariton solution (2.1), whereas equation (1.17) for longitudinal waves is exhausted by the solution $\omega_\parallel^{(L)} = \omega_0 + (\omega_c^2/9\omega_0) - i(T_2^{-1} + kv_T)$ (for $\sigma = 0$).

[19]One could consider the spread of the resonance frequencies of active particles not only in a gas but also in a condensed medium, for example, a crystal host.

[20]In electronics, the time T_2 is determined not only by collisions between particles and by spontaneous transitions, but mostly by the electron transit time in the electrodynamic system (resonator).

[21]The attainment of this aim in different generators is in one way or another related to an increase in the concentration of active electrons, and, consequently, leads to an enhancement of coherent processes such as super-radiance, which ensure maximum power and minimum radiation pulse length.

[22]In plasma physics and in electronics, the possible development of the so-called radiative instability of a beam of classical oscillators under the conditions of the anomalous Doppler effect and the corresponding "dissipative amplification" along a beam of negative-energy waves was investigated independently of the super-radiance effect in Refs. 30, 40, 118, 126, 127, etc.

[23]After some modification, the one-dimensional model can be used to describe a magnetic trap with transversely bounded electron beams in the form of a cylinder with Fresnel number $F = S/\lambda L \lesssim 1$. The "diffractive dissipation" of the field along the side surfaces is described by replacing σ_0 with the effective quantity $\sigma_0 + c_0\varepsilon_0/6\pi FL$ (Refs. 14, and 29).

[24]The word "subradiance" was first used in Ref. 139 to denote a nonradiating coherent (phased) state of a set of two-level molecules (the reverse of the super-radiant state).

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