# "Magnetized" black holes

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Physical aspects of the theory of black holes in an external electromagnetic field are reviewed. The "magnetized" black hole model is currently widely discussed in astrophysics because it provides a basis for the explanation of the high energy activity of galactic cores and quasars. The particular feature of this model is that it predicts unusual "gravimagnetic" phenomena that arise as a result of a natural combination of effects in electrodynamics and gravitation, namely, the appearance of an inductive potential difference during the rotation of a black hole in a magnetic field, the drift of a black hole in an external electromagnetic field, the change in the chemical potential of the event horizon, the creation of an effective ergosphere of a black hole in a magnetic field, and so on. Questions relating to the description of electromagnetic fields in Kerr space-time are examined, including their influence on the space-time metric, the interaction between a rotating charged black hole and an external electromagnetic field, the motion of charged particles near "magnetized" black holes, including their spontaneous and stimulated emission, and the influence of magnetic fields on quantum-mechanical processes in black holes.

**CONTENTS**



## 1. INTRODUCTION

Progress in the theory of gravitation and advances in observational astronomy during the last decade have ensured that the idea of the black hole has become an inseparable part of modern physics and astronomy. A presentation of the basic theoretical concepts of black-hole physics can be found in a number of books,[1-3] and a detailed development of the theory of black holes and of their interaction with the ambient medium is given in recent monographs.[4-6] Particularly intriguing are the predictions of the quantum-mechanical properties of microscopic black holes[7-9] that follow from Hawking's remarkable discovery of the quantum-mechanical instability of black holes.[10] Although there is, at present, no experimental evidence for the existence of microscopic black holes in the universe, their possible formation at the early stages of cosmological expansion and subsequent evaporation is a significant factor governing the choice of a cosmologically consistent model of the theory of elementary particles.[11] The quantum theory of black holes has revealed unexpected and profound connections between geometry, quantum-field theory, and thermodynamics, and has had a definite influence on the development of the quantum theory of gravitation. The idea of a black hole has thus become an organic part of the overall physical picture of the universe, so


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that the gathering of evidence for the existence of black holes of stellar mass in space, or of supermassive black holes at the center of galaxies, has ceased to be "a matter of life and death" for the theory of black holes. However, we seem to find, once again, a confirmation of the validity of Dirac's phrase (introduced in relation to the magnetic monopole): "It would be strange if nature did not use this possibility," and astrophysics of the last decade has provided new arguments in favor of the actual existence of black holes in space.

There are reasonably convincing data indicating that many of the x-ray sources that can be seen at present contain black holes. The basis for this conclusion is that the probable mass of a compact object is found to exceed the limiting mass of a neutron star, which is estimated in the most likely models as being not more than three solar masses. Apart from the well-known source Cygnus X-1, discovered as far back as 1971 (see the review given in Ref. 12), there is a number of "reliable" candidates for black holes. They include the x-ray source LMC X-3 in the Large Magellanic Cloud, in which the mass of the compact object is estimated to be ten solar masses,[13] and, probably, the source LMC X-1, whose mass exceeds three solar masses and whose x-ray spectrum has similar properties.[14,15] According to recent data, the binary system AO620-00 in the Monoceros constellation contains a black hole. Analysis of photometric observations of this system, performed in the course of the last few years, and of the shift of discrete absorption lines found in the spectrum of the optical star, has led to the conclusion that the probable mass of the invisible component is greater than seven solar masses.[16,17] A black hole may be present as an invisible component of the unique object called Geminga—a source of hard gamma rays whose observable properties can be reproduced with reasonable precision by assuming that the system has a total mass of about five solar masses and consists of a white dwarf and a black hole rotating relative to one another with a period of 59 s (Refs. 18 and 19). It is very probable that the unusual source of optical, x-ray, and radio emission, SS-433 in the Aquila constellation[20] is a black hole. Analysis of the light curve of the system, reported in Refs. 21 and 22, shows that, under different assumptions about the masses of the component objects, the mass of the invisible component is much greater than the critical mass of a neutron star.

Observational data show quite convincingly that there are black holes in the active cores of galaxies and quasars. The observed high energy activity of these objects is relatively naturally explained by the presence of supermassive black holes within them.[1,23-25] This model is supported by observations of the central part of the giant elliptic galaxy M87 in the Virgo constellation in which dark mass of the order of $3\times 10^9$ solar masses has been found.[26,27]

Studies of the mechanisms responsible for the extraction of energy from a black hole are important for astrophysical applications. One of these mechanisms is the Penrose process, in which a particle traveling in the ergosphere of a rotating black hole decays into a pair of particles, one of which has a negative total energy (relative to the observer at infinity). The particle leaving the ergosphere receives additional energy drawn from the rotational energy of the black hole.[28] However, calculations show that this process is not very probable under astrophysical conditions.[29] The wave analog of the Penrose process is the so-called "superradiation" predicted by Zel'dovich[30] and Misner.[31] The theory of this was constructed by Starobinskiĭ[32] in the case of the Kerr metric. In the superradiation regime, the scattering of a multipole with orbital angular momentum component $m$ along the symmetry axis of a rotating black hole and frequency $\omega < m\Omega_H$, where $\Omega_H$ is the angular velocity of the black hole, is accompanied by an increase in the field amplitude.

However, in a realistic model of gas accretion by a black hole, energy is released mostly at the expense of the binding energy of the particles and the strong gravitational field of the hole. It is well known that the maximum binding energy in the Schwarzschild field is about 5.7% of the rest energy but, in the Kerr black hole with maximum rotation, this can reach 42% (see Ref. 33 for further details). Further searches for mechanisms responsible for energy release which could, in particular, explain the high activity of the galactic cores and quasars, have led to the "magnetized" black hole model, in which a large-scale magnetic field is present around the rotating black hole. This model has recently been discussed within the framework of different approaches[34-37] and has attracted considerable interest in astrophysics. On the other hand, the intimate connection between electrodynamics and gravitation, which arises in this model and leads to a number of new gravimetric effects, is also of general interest in physics. This has stimulated the publication of this review which, in contrast to Refs. 36 and 37 (see also the presentation in the book in Ref. 5) is not explicitly astrophysical in character and is devoted to the physical aspects of the theory of "magnetized" black holes.

It is well known that an electrically neutral black hole can have no intrinsic "magnetic hair" (except for the hypothetical monopole hair). As far back as 1965, V. L. Ginzburg and L. M. Ozernoĭ examined the gravitational collapse of a static star with a frozen-in magnetic field and came to the important conclusion that, as the boundary of the body approaches the event horizon, the dipole magnetic field of the star must completely vanish.[38] This result was subsequently generalized to the case of a rotating star[39] and was confirmed elsewhere,[40-42] in accordance with Wheeler's well known dictum that "a black hole has no hair." However, a magnetic field can arise near a black hole due to external factors, e.g., the presence of a magnetic satellite (pulsar) that appears near a black hole as a result of accretion of the ambient plasma. Finally, a magnetic field of cosmological origin may be present.[43-45] The presence of a magnetic field around a black hole, which can frequently be regarded as quasiuniform near the event horizon, establishes the conditions for the realization of the electrodynamic mechanism of rotational-energy extraction from the black hole as a result of the interaction between charged particles and the induced electric field due to its rotation. The estimated energy release in this system suggests that it would be possible to use it as a model of active galactic cores and quasars.[34-37]

Time-independent axially-symmetric configurations of electromagnetic fields have been investigated in Refs. 46–54 in black-hole space-time. A solution of Maxwell's equations was found for a test electromagnetic field in the Kerr metric, which asymptotically corresponds to a uniform magnetic field.[46-48] As the black hole rotates in the uniform magnetic field, Faraday induction produces an electrostatic potential difference between the event horizon and an infinitely distant point, even when the electric charge of the black hole is zero. This makes it energetically advantageous for the hole to absorb the electric charge of the necessary sign during the accretion of the plasma. The process terminates when the Faraday potential difference is cancelled by the potential difference produced by the charge acquired by the black hole.

If the black hole rotates in a uniform electromagnetic field that is not axially symmetric, it eventually loses its angular momentum and, according to Hawking's theorem,[42] the configuration becomes either axially symmetric or static,[55–57] where, if the hole has an electric charge, this is accompanied by the precession of the angular momentum vector around the direction of the magnetic field.[57,58] A black hole rotating in an asymmetric electromagnetic field with a nonzero Poynting vector experiences a ponderomotive force due to the asymmetric absorption of the electromagnetic momentum flux by the black hole.[57,59] Electromagnetic fields near a rotating black hole, generated by time-independent axially symmetric forces (point electric charge, ring current) was investigated in Refs. 51, 52, and 54 and, in contrast to previous analyses,[48–50,53] solutions were obtained in a closed algebraic form and not in the form of multipole expansions, which meant that, in particular, the self-energy shift of a charge in the gravitational field of the black hole could be calculated.[60,61]

There is considerable interest in exact solutions of the Einstein-Maxwell equations for a black hole in an external magnetic field, taking into account the effect of the latter on the space-time metric. Such solutions can be constructed using the symmetry of the Einstein-Maxwell equations for axially-symmetric field configurations under some one-parameter group of transformations. Ehlers[62] was one of the first to suggest a way of finding these solutions, and the problem was subsequently discussed in Refs. 63–66. In particular, a proof was produced of a theorem[66] stating that the Einstein-Maxwell equations for the time-independent axially symmetric configurations of electrovacuum were invariant under the noncompact group SU(2, 1). Subsequent advances in this area led to the discovery of an infinite-parameter group of transformations[67] and, in the final analysis, to the formulation of different methods for the complete integration of the Einstein-Maxwell equations for axially-symmetric stationary fields. We shall not discuss in detail these interesting results, obtained in recent years, and refer the reader to the review literature,[68] since it will be sufficient for our purposes to use the one-parameter transformation group first suggested by Harrison.[63] Ernst[64,65] has formulated this theory in the language of complex potentials, and has used the above-mentioned theory to construct new exact solutions for a nonrotating, rotating, and specifically charged black hole, and for a slowly rotating and electrically neutral black hole in an external uniform magnetic field.[69–71] The characteristic features and physical properties of these solutions were investigated in Refs. 72–78 (Sections 2 and 3).

The motion of test particles near a black hole in the absence of external electromagnetic fields has now been investigated in adequate detail (see books,[1,79–81] reviews,[33,82] and the references cited therein). When an external asymptotically uniform magnetic field is present, analysis of the motion of neutral and charged particles in Schwarzschild and Schwarzschild-Ernst spaces and also in the Kerr metric has shown that the magnetic field produces a significant expansion of the region of existence and stability of circular orbits, and allows ultrarelativistic motion on stable orbits that are distant from a closed photon orbit.[83,84] There are stable circular orbits for which the particle mass defect approaches 100% (Section 4). The motion of charged particles in a magnetic field near a black hole has also been investigated numerically.[85–87]

Charged particles captured into ultrarelativistic circular orbits around a black hole in an external magnetic field emit radiation whose properties are very similar to those of the synchrotron radiation of electrons in accelerators.[88–90] This radiation has been investigated in the literature[83,84,91–93] (see Ref. 6 for further details). It is interesting to note that the radiation emitted by an ultrarelativistic charged particle moving along a geodesic[94] is the so-called "geodesic synchrotron radiation" (GSR), whose spectral composition and dependence of intensity on the Lorentz factor $\gamma = E/\mu$ are significantly different as compared with synchrotron radiation due to a charge moving largely under the influence of nongravitational forces. This is so because the geodesic along which the ultrarelativistic particle is moving is close to the world line of a light ray, so that the length within which a high-frequency pulse of radiation is formed under GSR conditions is greater by the factor $\gamma$ than the corresponding length for synchrotron radiation.[95] To describe the synchrotron radiation spectrum in curved space-time, we can use the method of local coordinates, which enables us to generalize the theory of synchrotron radiation (including quantum-mechanical effects) to the case of an arbitrary, slowly-varying gravitational field[96–98] (Section 5).

It is well known that the interaction between electrons moving in a magnetic field in flat space-time and electromagnetic waves can take the form of negative absorption, which is the principle of the cyclotron resonance maser.[99–101] The effect is also important under astrophysical conditions.[102] One way of producing negative absorption of waves is to introduce a nonlinearity into the system. An example is provided by a relativistic electron in a magnetic field. The amplification of waves is then related to the nonlinear dependence of momentum on velocity. Another negative absorption mechanism is found to operate in linear multiperiodic systems with several noncoincident frequencies.[103]

Both negative absorption mechanisms occur naturally in gravitating systems. Analyses based on the theory of stimulated emission near nonrotating or rotating black holes in a magnetic field (Refs. 104–106) have shown that the interaction between charged particles and electromagnetic waves takes the form of negative absorption at certain combination frequencies (maser effect); see Section 5.

An external electromagnetic field has a significant influence on quantum-mechanical processes in black holes.[107] In the case of a uniform magnetic field, including situations in which the effect of this field on the space-time metric is taken into account, or when an arbitrary (test) axially symmetric electromagnetic field that decreases at infinity is present, there is a change in the superradiation threshold due to the appearance of an additional potential difference between the event horizon and a distant point, and a change in the rate of dragging of the reference frames at the horizon of the charged black hole. There is also a change in the transmission coefficient of the potential barrier, where, during the development of boson instability in superradiant quasistationary levels of massive particles, the magnetic field generates an asymmetry in the sign of the electric charge and of the projection of the angular momentum of the created particles (Section 6).

## 2. BLACK HOLE IN AN EXTERNAL MAGNETIC FIELD

According to the theorems proved in Refs. 40–42, the most general family of asymptotically flat solutions of the Einstein-Maxwell equations that have a nonsingular event horizon is the same as the Kerr-Newman family.[108] In terms of the coordinates introduced by Boyer and Lindquist,[109] the corresponding space-time interval takes the form

$$\begin{aligned} ds^2 = & \left(1-\frac{2Mr-Q^2}{\Sigma}\right) dt^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma\, d\theta^2 \\ & - \sin^2\theta \left(r^2+a^2+\frac{2Mr-Q^2}{\Sigma} a^2 \sin^2\theta\right) d\varphi^2 \\ & + \frac{2(2Mr-Q^2)}{\Sigma} a \sin^2\theta\, d\varphi\, dt \end{aligned} \tag{2.1}$$

where

$$\Delta = r^2 + a^2 - 2Mr + Q^2, \quad \Sigma = r^2 + a^2\cos^2\theta,$$

and $M$, $Q$, and $a = J/M$ are, respectively, the mass, electric charge, and rotation parameter of the black hole. The electromagnetic 4-potential is then given by

$$A = A_\mu\, dx^\mu = \frac{Qr}{\Sigma}(dt - a\sin^2\theta\, d\varphi). \tag{2.2}$$

In the system of units that we are using $G = \hbar = c = 1$, the electric field has the dimensions of the reciprocal of length, and so has the quantity $1/M$. The latter determines the characteristic scale of the magnetic field strength:

$$B_M = \frac{1}{M} \approx 2.4\cdot 10^{19} \frac{M_\odot}{M}\,(\mathrm{G}), \tag{2.3}$$

where $M_\odot$ is the mass of the Sun, which has a significant influence on the metric near the event horizon of the black hole (in ordinary units, $B_M = c^4G^{-1}M^{-1}$). When the magnetic field is $B \ll B_M$, there is definitely a region near the black hole in which the space-time is not distorted by the external field, and it is sufficient to consider small field perturbations to describe it. We note that, for a charged black hole, electromagnetic and gravitational perturbations can be examined together because, in the set of equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2\left(F_{\mu\lambda}F^\lambda_\nu + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}\right), \tag{2.4}$$

$$F^{\mu\nu}_{;\nu} = 0, \tag{2.5}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, and the semicolon represents a covariant derivative, the expansions for the electromagnetic field tensor

$$F_{\mu\nu} = F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu} \tag{2.6}$$

and for the metric

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \tag{2.7}$$

lead to a coupled set of linear equations for $F^{(i)}_{\mu\nu}$ and $h_{\mu\nu}$.

If the black hole is uncharged ($Q = 0$), or its charge is small ($Q \ll M$), the external fields can be adequately described by constructing the corresponding solutions of the Maxwell equations against the background of the Kerr metric [(2.1) for $Q = 0$]. We emphasize that, strictly speaking, (2.4) and (2.5) do not allow the introduction of a current into the right-hand side of (2.5) because a corresponding contribution would have to be inserted into the right-hand side of (2.4) to satisfy the covariant conservation condition $G^{\nu}_{\mu;\nu} = 0$. However, if the change in the metric due to electromagnetic field sources leads to significantly smaller effects than those of the electromagnetic field itself, they can be neglected.

*2.1. Constant uniform axially symmetric magnetic field near a rotating black hole. Faraday induction.* There is an interesting internal connection between the Coulomb "hair" of a black hole and a uniform external magnetic field: both these fields can be obtained by taking the 4-potential $A_\mu$ as a definite linear combination of Killing vectors in Kerr space-time.[46] It can be shown that, in the case of the vacuum space-time, $R_{\mu\nu} = 0$, the Killing field vectors satisfy an equation that is identical with the equation for the electromagnetic 4-potential in this space-time.[110] Actually, in the general case ($R_{\mu\nu} \neq 0$), the homogeneous Maxwell equations have the following form in the covariant Lorentz gauge $A^{\mu}_{;\mu} = 0$:

$$A^{\mu;\nu}_{;\nu} - R^\mu_\nu A^\nu = 0. \tag{2.8}$$

On the other hand, from the Killing equations

$$\mathcal{K}^{\mu}_{;\nu} + \mathcal{K}_{\nu}^{;\mu} = 0 \tag{2.9}$$

we can show by covariant differentiation and commutation that

$$\mathcal{K}^{\mu;\nu}_{;\nu} + R^\mu_\nu \mathcal{K}^\nu = 0. \tag{2.10}$$

Comparisons of (2.8) with (2.10) proves the above proposition.

The Kerr space-time is stationary and axially symmetric, and this is reflected in the existence of two commuting vector Killing fields $\mathcal{K}^\mu_{(t)} = \{1,0,0,0\}$, $\mathcal{K}^\mu_{(\varphi)} = \{0,0,0,1\}$. It is found that the use of these fields for $A_\mu$ generates nontrivial electromagnetic fields that are superpositions of Coulomb and asymptotically uniform magnetic fields. We shall use the fact that any Killing vector can be written as a linear combination of other Killing vectors, and take the vector potential in the form

$$A^\mu = \alpha \mathcal{K}^\mu_{(t)} + \beta \mathcal{K}^\mu_{(\varphi)}. \tag{2.11}$$

Using the formulas [111]

$$\begin{aligned} 8\pi M &= -\oint_\infty \mathcal{K}^{\mu;\nu}_{(t)}\, d^2\Sigma_{\mu\nu}, \\ 16\pi J &= \oint_\infty \mathcal{K}^{\mu;\nu}_{(\varphi)}\, d^2\Sigma_{\mu\nu}, \\ 4\pi Q &= \oint F^{\mu\nu}\, d^2\Sigma_{\mu\nu}, \end{aligned} \tag{2.12}$$

which give the mass $M$ and angular momentum $J$ of a black hole in terms of surface integrals of covariant derivatives of Killing vectors, and the expression for the electric charge of the black hole, we can show that the relation for the parameters $\alpha$ and $\beta$ is

$$2\alpha M - 4\beta J = -Q. \tag{2.13}$$

For $Q = 0$, we put $\beta = B/2$ and, omitting the index on the 4-potential, we obtain

$$
\begin{aligned}
A_t &= aB\left[1-\frac{Mr}{\Sigma}(1+\cos^2\theta)\right],\\
A_\varphi &= \frac{B\sin^2\theta}{2\Sigma}[a^2(4Mr+\Delta\sin^2\theta)-(r^2+a^2)^2],
\end{aligned}
\tag{2.14}
$$

which, for $a = 0$, is identical with the 4-potential corresponding to a uniform magnetic field pointing along the symmetry axis. In general, when $Q \neq 0$, we obtain the expression for the 4-potential that determines the superposition of the Coulomb field and the external uniform magnetic field in Kerr space-time:

$$
A^\mu = \frac{B}{2}(\mathcal{K}^\mu_{(\varphi)}+2a\mathcal{K}^\mu_{(t)})-\frac{Q}{2M}\mathcal{K}^\mu_{(t)}. \tag{2.15}
$$

It is clear from this expression that the Coulomb part of the field is generated by a timelike Killing vector and that the "magnetic" part of $A^\mu$ contains an analogous contribution proportional to the rotation parameter $a$. The latter can be related to the electric field induced during the rotation of the black hole in the uniform magnetic field. A rotating black hole in an external magnetic field is thus seen to be a dynamo that creates the electrostatic potential difference between the event horizon and an infinitely distant point. The magnitude of this potential difference is the zero component in (2.14) and, as can be seen from this formula, it is given by

$$
\Delta\Phi_{\rm el} = \frac{Q-2aMB}{2M}. \tag{2.16}
$$

Faraday induction is a possible electrodynamic mechanism for the extraction of rotational energy from a black hole. It has recently been used in astrophysical models of galactic cores and quasars.[34–37]

Similarly, we can construct a "magnetostatic" 4-potential $B_\mu$ that generates the Maxwell dual tensor

$$
\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\tau}F^{\lambda\tau} = 2B_{[\nu;\,\mu]},
$$

that describes the uniform electric field $E$ pointing along the symmetry axis:

$$
B^\mu = -\frac{1}{2}E(\mathcal{K}^\mu_{(\varphi)}+2a\mathcal{K}^\mu_{(t)}), \tag{2.17}
$$

where the magnetic charge of the black hole is set equal to zero.

*2.2. Uniform constant electromagnetic field without axial symmetry.* When the directions of the electric and magnetic fields (or at least one of them) do not lie along the symmetry axis of the Kerr metric, the above method of solving the Maxwell equations is no longer valid. One can then use the effective method of Debye potentials.[6,112] We refer the reader to Ref. 6 for a detailed presentation, and confine our attention to a brief explanation. If we write the space-time metric in terms of the complex isotropic Newman-Penrose tetrad $\{l^\mu, n^\mu, m^\mu, m^{*\mu}\}$

$$
g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu m^*_\nu - m^*_\mu m_\nu, \tag{2.18}
$$

where

$$
\begin{aligned}
&(ln) = 1,\quad (mm^*) = -1,\quad (ll) = (nn) = (mm) = 0,\\
&l_\mu = \{1,\ -\Sigma/\Delta,\ 0,\ -a\sin^2\theta\},\\
&n_\mu = 1/2\ \Sigma^{-1}\{\Delta,\ \Sigma,\ 0,\ -\Delta a\sin^2\theta\},\\
&m_\mu = [\sqrt{2}\,(r+ia\cos\theta)]^{-1}\\
&\qquad\times\{ia\sin\theta,\ 0,\ -\Sigma,\ -i\sin\theta\,(r^2+a^2)\},
\end{aligned}
$$

and expand the self-dual electromagnetic-field bivector

$$
\begin{aligned}
\mathcal{F}_{\mu\nu} &= \frac{1}{2}(F_{\mu\nu}+i\tilde{F}_{\mu\nu})\\
&= 2[\Phi_0 m^*_{[\mu}n_{\nu]}+\Phi_1(n_{[\mu}l_{\nu]}+m_{[\mu}m^*_{\nu]})+\Phi_2 l_{[\mu}n_{\nu]}],
\end{aligned}
\tag{2.19}
$$

we obtain the following expressions for the complex projections $\Phi_0$, $\Phi_1$, $\Phi_2$ from the Maxwell equations

$$
\begin{aligned}
&\rho^2\mathcal{D}_0\rho^{-2}\Phi_1+\frac{1}{\sqrt{2}}\mathcal{L}_1\rho\Phi_0=0,\\
&\rho^2\mathcal{L}^+_0\rho^{-2}\Phi_1-\frac{\Delta}{\sqrt{2}}\mathcal{D}^+_1\rho\Phi_0=0,\\
&\mathcal{D}_0\rho^{-1}\Phi_2+\frac{1}{\sqrt{2}}\rho^2\mathcal{L}_0\rho^{-2}\Phi_1=0,\\
&\mathcal{L}^+_1\rho^{-1}\Phi_2-\frac{\Delta}{\sqrt{2}}\rho^2\mathcal{D}^+_0\rho^2\Phi_1=0,
\end{aligned}
\tag{2.20}
$$

where

$$
\begin{aligned}
&\rho = -(r-ia\cos\theta)^{-1},\\
&\mathcal{D}_n = \frac{\partial}{\partial r}+\frac{a}{\Delta}\frac{\partial}{\partial\varphi}+2n\frac{r-M}{\Delta},\\
&\mathcal{L}_s = \frac{\partial}{\partial\theta}+s\,\mathrm{ctg}\,\theta-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi},\\
&\mathcal{D}^+_n = \mathcal{D}_n(\varphi\to-\varphi),\quad \mathcal{L}^+_s = \mathcal{L}_s(\varphi\to-\varphi).
\end{aligned}
\tag{2.21}
$$

By the substitution

$$
\begin{aligned}
\Phi_0 &= \frac{1}{2}\mathcal{D}^2_0\psi,\\
\Phi_1 &= -\frac{1}{2\sqrt{2}}\left(\mathcal{D}_0\frac{\rho}{\rho_*}\mathcal{L}_1+ia\sin\theta\frac{\rho}{\rho^*}(\rho+\rho^*)\mathcal{D}_0\right)\rho^*\psi,\\
\Phi_2 &= \frac{1}{4}\rho^2\mathcal{L}_0\mathcal{L}_1\psi
\end{aligned}
\tag{2.22}
$$

this system reduces to the single equation for the Debye potential $\Psi$:

$$
(\Delta\mathcal{D}^+_0\mathcal{D}_0+\mathcal{L}^+_0\mathcal{L}_1)\psi=0. \tag{2.23}
$$

To construct a solution which describes a constant uniform electromagnetic field for $r\to\infty$, we note that, as $r\to\infty$,

$$
\begin{aligned}
\Phi_0(r\to\infty) &= -\frac{1}{\sqrt{2}}(F_\theta+iF_\varphi),\\
\Phi_1(r\to\infty) &= -\frac{1}{2}F_r,\\
\Phi_2(r\to\infty) &= -\frac{1}{2\sqrt{2}}(F_\theta-iF_\varphi),
\end{aligned}
\tag{2.24}
$$

from which we can readily obtain the boundary value for the Debye potential. The solution of (2.23) that satisfies the boundary conditions (2.24) is then constructed by the method of separation of variables, and has the simple form

$$
\psi = -\frac{\Delta}{2\sqrt{2}}\left(F_z\sin\theta-F_{(+)}\cos^2\frac{\theta}{2}e^{-i\tilde{\varphi}}+F_{(-)}\sin^2\frac{\theta}{2}e^{i\tilde{\varphi}}\right), \tag{2.25}
$$

where $\mathbf{F}=\mathbf{E}+i\mathbf{B}$, $F_{(\pm)}=F_x\pm iF_y$, $\tilde{\varphi}=\varphi+a(r_+-r_-)^{-1}\ln[(r-r_+)/(r-r_-)]$. The electromagnetic field tensor is then obtained from (2.22) and (2.19). When $F_x = F_y = 0$, we return to the solution constructed in the last Section. In general, the field configuration that we are considering is a simple example of an electromagnetic field without axial symmetry. Since, according to the well-known Hawking theorem,[41] a stationary black hole should be axially symmetric, one would expect the appearance of nontrivial ponderomotive effects in the interaction between a black hole and a field of this kind. We shall return to this question in Section 3.

*2.3. Nonuniform axially symmetric stationary fields.* Another case for which the Maxwell equations can be integrated in Kerr space-time. without resorting to the laborious formalism involving expansion in the spin spherical harmonics, is the case of axially symmetric and time-independent configurations with arbitrary dependence on the coordinates $r$ and $\theta$. For these configurations, the equation for the Debye potentials (2.23) then assumes the form

$$\Delta \frac{\partial^2}{\partial r^2}\,{}_{-1}\psi + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\,\psi\right) - (1+\operatorname{ctg}^2\theta)\,{}_{-1}\psi = -4\pi\Sigma\,{}_{-1}T, \tag{2.26}$$

where we have introduced the source ${}_{-1}T(r,\theta)$ which is related to the current $J^\mu$ by the Teukolsky relations.[113] Transforming to the Weyl coordinates

$$\xi = \Delta^{1/2}\sin\theta,\quad z = (r-M)\cos\theta,$$

we have

$$\tilde{\Delta}\,(\sin\theta\,{}_{-1}\Psi) = -4\pi\Sigma T\,[\Delta + (M^2 - a^2)\sin^2\theta]^{-1}\sin\theta, \tag{2.27}$$

where the generalized Laplace operator is given by[114]

$$\tilde{\Delta} = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial z^2} + \frac{1+2s}{\xi}\frac{\partial}{\partial\xi}\quad (s=-1). \tag{2.28}$$

The Green's function for this equation

$$\tilde{\Delta} G(z, z_0;\ \xi, \xi_0) = -\xi^{-3}\delta(z-z_0)\,\delta(\xi-\xi_0) \tag{2.29}$$

can be written in the form of the integral[6]

$$G = \frac{1}{2\pi}\int_{-1}^{+1} Z^{-3}(1-\varkappa^2)^{1/2}\,d\varkappa, \tag{2.30}$$

where

$$Z(z, z_0;\ \xi, \xi_0) = [(z-z_0)^2 + \xi_0^2 + \xi^2 - 2\xi_0\xi\varkappa]^{1/2}.$$

As a result, the solution of (2.26) assumes the form

$${}_{-1}\psi = 4\pi\int G\,\Sigma_0 \sin\theta_0\xi\xi_0 \left(\frac{\Delta}{\Delta_0}\right)^{1/2} {}_{-1}T_0\, d\theta_0\, dr_0. \tag{2.31}$$

As an example, let us find the electromagnetic field due to a point charge $e$ at rest on the symmetry axis of the Kerr metric at the point $\theta_0 = 0$:

$$J^\mu = \frac{e}{2\pi\Sigma}\delta^{\mu 0}\delta(r-r_0)\,\delta(\cos\theta - 1). \tag{2.32}$$

Using (2.31) and (2.22), we find that

$$\psi = -\frac{\sqrt{2}\,e}{r_0+ia}\frac{Z}{\sin\theta} + \psi_{\mathrm{Coul}}, \tag{2.33}$$

where $\psi_{\mathrm{Coul}}$ is the solution of the homogeneous equation (2.26), which must be added in order to ensure that the physical charge of the system is $e$. Using the asymptotic expansion

$$Z(r\to\infty) \approx r - M - (r_0 - M)\cos\theta,$$

we find that the Coulomb term is

$$\psi_{\mathrm{Coul}} = -\frac{\sqrt{2}\,e}{r_0+ia}\frac{1}{\sin\theta}[M(1-\cos\theta) - r - ia\cos\theta] \tag{2.34}$$

The explicit expression for the components of the electromagnetic vector potential of a point charge at rest on the symmetry axis of the black hole can now be found with the aid of (2.33):[54]

$$A_t = \frac{e}{(r_0^2+a^2)\Sigma}\Big[(r_0 r + a^2\cos\theta) \times \left(M + \frac{(r-M)(r_0-M) - (M^2-a^2)\cos\theta}{Z}\right) + a^2(r - r_0\cos\theta)Z^{-1}((r-M) - (r_0-M)\cos\theta)\Big], \tag{2.35}$$

$$A_\varphi = -\frac{ea}{r_0^2+a^2}\Big[(r - r_0\cos\theta)\frac{(r-M)-(r-M)\cos\theta}{Z} - Z - M(1-\cos\theta)\Big] - a\sin^2\theta A_t. \tag{2.36}$$

It is readily seen that the potential difference between the event horizon and an infinitely distant point is $\Delta\Phi_{\mathrm{el}} = e/2M$, independently of the position of the charge. A charged black hole with the same value of charge $e$ produces precisely the same potential difference. Other examples of constructing axially symmetric nonuniform fields can be found in the literature.[50,52,54]

*2.4. Influence of an external magnetic field on the space-time metric.* When the magnetic field near a black hole is strong enough, its effect on the geometry of space-time must be taken into account. The characteristic magnetic-field scale for a black hole of mass $M$ is, as already noted, the quantity $B_M = 1/M$. When $B \simeq B_M$, the space-time geometry near the horizon undergoes a significant change. However, even when $B \ll B_M$, it is interesting to consider the effect of a magnetic field on the metric because this enables us to analyze the influence of the magnetic field on the ergosphere parameters and the thermodynamic properties of the horizon.

The exact solution of the Einstein-Maxwell equations describing a black hole in an external magnetic field was constructed by Ernst[69] using the Harrison transformation[63,66] of the Schwarzschild solution.[69] This solution is not asymptotically flat, but its physical interpretation is quite simple because there is an intermediate asymptotic region $r_+ \ll r \ll B^{-1}$ in which the space-time is approximately of the Schwarzschild type ($r_+$ is the radius of the event horizon) and the magnetic field is uniform (this region exists for $B \ll B_M$). The solution for a charged rotating black hole can be constructed similarly.[115] However, the question of the physical interpretation is less trivial: the field is then found to contain an electric component, the conical singularity of the metric appears, and so on. Moreover, the parameters $M$ (mass), $a$ (angular momentum per unit mass), and $Q$ (electric charge) of the bare solution are found to be different from the corresponding parameters of the magnetized solutions.

Consider the following interval of stationary axially symmetric space-time in terms of cylindrical coordinates:

$$ds^2 = -f(d\varphi - \omega dt)^2 - f^{-1}[e^{2\nu}(d\xi^2 + dz^2) - \xi^2 dt^2], \tag{2.37}$$

where $f$, $\omega$, $\gamma$ are real functions. We now introduce the complex electromagnetic potential $\Phi = A_\varphi + iB_\varphi$ and the complex gravitational potential $\mathscr{E} = f - iH$, where $A_\varphi$ is the component of the 4-potential $A_\mu$ and $B_\varphi$, i.e., the component of the magnetostatic potential generating the dual tensor of the electromagnetic field whose existence, like that of the potential $H$, follows from the corresponding Einstein-Maxwell equations. It can then be shown, after some further manipulation, that the complete set of Einstein-Maxwell equations for axially symmetric stationary configurations reduces to a set of two nonlinear equations for the potentials[65] $\Phi$ and $\mathscr{E}$:

$$f\Delta\Phi = \partial_a\Phi\,(\partial^a\mathscr{E} - 2\Phi^*\partial^a\Phi),$$
$$f\Delta\mathscr{E} = \partial_a\mathscr{E}\,(\partial^a\mathscr{E} - 2\Phi^*\partial^a\Phi); \tag{2.38}$$

where

$$f = \operatorname{Re}\mathscr{E} - |\Phi|^2,$$
$$\Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{\xi}\,\frac{\partial}{\partial\xi},$$
$$\partial_a = \frac{\partial}{\partial x^a} \qquad (a \equiv \xi, z);$$

and the raising and lowering of indices are performed by the two-dimensional tensor

$$g^{ab} = g_{ab} = \operatorname{diag}\ (1,1).$$

The potentials $B_\varphi$ and $H$ satisfy the equations

$$\nabla A_t + \omega\nabla A_\varphi = -i\xi f^{-1}\nabla B_\varphi, \tag{2.39}$$

$$\nabla H - 2i\Phi^*\nabla\Phi = i\xi^{-2}f\nabla\omega, \tag{2.40}$$

where

$$\nabla = \frac{\partial}{\partial\xi} + i\frac{\partial}{\partial z}.$$

The set of equations given by (2.38) is invariant under the SU(2,1) group of transformations of complex potentials.[66] Applying one of the transformations from this group to a "bare" solution of the Einstein-Maxwell equations, we can construct a new solution that physically corresponds to the turning on of a uniform magnetic field $B$. The corresponding transformation is[63]

$$\mathscr{E}' = \Lambda^{-1}\mathscr{E}, \quad \Phi' = \Lambda^{-1}\left(\Phi - \frac{1}{2}B\mathscr{E}\right),$$
$$\Lambda = 1 - B\Phi + \frac{1}{4}B^2\mathscr{E}. \tag{2.41}$$

We then have $f \to f'$, $\omega \to \omega'$, where

$$f' = \operatorname{Re}\mathscr{E}' - |\Phi'|^2 = |\Lambda|^{-2}f, \tag{2.42}$$

$$\nabla\omega' = |\Lambda|^2\nabla\omega - \xi f^{-1}(\Lambda^*\nabla\Lambda - \Lambda\nabla\Lambda^*) \tag{2.43}$$

and the remaining quantities in (2.37) stay the same.

For the Kerr-Newman metric that describes space-time containing a black hole, the potentials $\Phi$ and $\mathscr{E}$, found from (2.39) and (2.40), are

$$\Phi = Q\left(i\cos\theta - \frac{a\sin^2\theta}{r + ia\cos\theta}\right), \tag{2.44}$$

$$\mathscr{E} = (r^2 + a^2)\sin^2\theta + Q^2\cos^2\theta - 2iaM\cos\theta\,(3 - \cos^2\theta)$$
$$+ \frac{2a\sin^2\theta}{r + ia\cos\theta}(Ma\sin^2\theta + iQ^2\cos\theta). \tag{2.45}$$

The corresponding magnetized metric is specified in the form

$$ds^2 = \left(\frac{\Delta}{A}dt^2 - \frac{dr^2}{\Delta} - d\theta^2\right)\Sigma|\Lambda|^2 - \frac{A\sin^2\theta}{\Sigma|\Lambda|^2}(d\varphi - \omega'\,dt)^2, \tag{2.46}$$

where

$$A = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta.$$

The quantity $\omega'$ satisfying (2.43) was found in Refs. 70, 71, and 115 for different values of the parameters $M$, $a$, $Q$, $B$.

The conical singularity on the symmetry axis[75] is an interesting property of the magnetized solutions. It can be avoided by expanding $\Lambda$ around the point $\theta = 0$, and introducing the new angular coordinate $\tilde\varphi = \varphi/|\Lambda_0|^2$, where

$$|\Lambda_0|^2 = \left(1 + \frac{1}{4}B^2Q^2\right)^2 + B^2(Q + aMB)^2. \tag{2.47}$$

The area of the surface of the event horizon of the magnetized black hole is then given by

$$S = \int_0^{2\pi|\Lambda_0|^2} d\varphi \int_0^\pi d\theta\,|g_{22}g_{33}|^{1/2} = 4\pi|\Lambda_0|^2(r_+^2 + a^2). \tag{2.48}$$

It is clear that a strong magnetic field would increase the area of the surface of the event horizon, where the change for $Q \neq 0$ is proportional to the square of the magnetic field, whereas for $Q = 0$ it is proportional to $(a/M)^2\,(B/B_M)^4$. For a nonrotating black hole, the area of the surface of the event horizon retains its Schwarzschild value.

A different interpretation of solutions with the conical singularity is also possible. In particular, the metric of an infinitely thin cosmic string is associated with the conical singularity.[116] The solution of (2.46) for $0 \leqslant \varphi \leqslant 2\pi$ can then be interpreted as the space-time of a magnetized black hole "pierced" by the infinitesimally thin cosmic string. We shall not take these ideas any further, but we emphasize that the Harrison transformation (2.41) will, in general, alter the conical structure of space-time. If the bare solution does not have a conical point, the transformation will, in general, have this singularity. On the contrary, to obtain the transformed solution without the conical singularity, we must take the bare space-time with such a singularity.

Let us now consider the physical significance of the parameters $Q_0$, $M$, $a$ of the magnetized solutions.

We shall calculate the electrical and magnetic charges $Q$ and $P$ of the black hole described by (2.46). It is convenient to express $F_{\mu\nu}$ in terms of its components in a locally nonrotating reference frame $B_{\hat r}$, $E_{\hat r}$. This gives

$$\int_0^{2\pi|\Lambda_0|^2} d\varphi \int_0^\pi d\theta\,\sin\theta A^{1/2}(B_{\hat r} + iE_{\hat r}) = 4\pi(iQ + P). \tag{2.49}$$

Since

$$B_{\hat r} + iE_{\hat r} = -(A^{1/2}\sin\theta)^{-1}\frac{\partial\Phi'}{\partial\theta}, \tag{2.50}$$

we find that

$$Q = Q_0 + 2aMB - \frac{1}{4}B^2Q_0^3, \quad P = 0. \tag{2.51}$$

The magnetic charge of a magnetized black hole is thus seen to be zero, and the electric charge is not equal to the original "bare" charge $Q_0$.

To elucidate the physical significance of $M$ and $a$, we can use the following expressions:[111]

$$\oint (2T^{\mu}_{\nu}-\delta^{\mu}_{\nu}T)\,\mathscr{K}^{\nu}_{(t)}\,d^3\Sigma_{\mu}$$

$$= -\frac{1}{8\pi}\oint_{\infty}\mathscr{K}^{\mu;\nu}_{(t)}\,d^2\Sigma_{\mu\nu}+\frac{1}{8\pi}\oint_{r_+}\mathscr{K}^{\mu;\nu}_{(t)}\,d^2\Sigma_{\mu\nu}, \tag{2.52}$$

$$-\oint T^{\mu}_{\nu}\mathscr{K}^{\nu}_{(\varphi)}\,d^3\Sigma_{\mu}$$

$$=\frac{1}{16\pi}\oint_{\infty}\mathscr{K}^{\mu;\nu}_{\varphi}\,d^2\Sigma_{\mu\nu}-\frac{1}{16\pi}\oint_{r_+}\mathscr{K}^{\mu;\nu}_{(\varphi)}\,d^2\Sigma_{\mu\nu}, \tag{2.53}$$

where $T^{\mu}_{\nu}$ is the energy-momentum tensor. In an asymptotically flat space-time, the surface integrals over the infinitely distant surface on the right-hand side of (2.52) and (2.53) determine the mass and angular momentum of the entire configuration, and the integrals over the surface of the event horizon determine the corresponding black hole parameters $(M_H, J_H)$. In our case, the metric is not asymptotically flat, and the surface integrals over the distant surface are found to diverge. However, when only the terms that are linear in $B$ are taken into account, the total mass of the configuration is found to be finite, and neither $M_\infty$ nor $M_H$ coincides with the mass of the "bare" Kerr-Newman solution. As far as the angular momentum is concerned, the quantity $J_\infty$ will, in general, diverge for $r\to\infty$, which is due to the insufficiently rapid reduction of $\omega'$ as $r\to\infty$, where the bare angular momentum $aM$ is not equal to $J_H$ either. It is therefore clear that, when the external magnetic field is present, the parameters $Q_0$, $M$, and $a$ can no longer be interpreted as the electric charge, mass, and specific angular momentum of the black hole.[135]

## 3. PONDEROMOTIVE EFFECT OF AN EXTERNAL ELECTROMAGNETIC FIELD ON A BLACK HOLE

We may suppose that a black hole with electric charge $Q$ experiences a force $Q\mathbf{E}$ in an external electric field $\mathbf{E}$, just as if it were an electric point charge. However, the charge of a black hole is located under the event horizon, so that the question requires further investigation. Moreover, the black hole is a nonlocal object characterized by a parameter with the dimensions of length, namely, the gravitational radius $r_+$. We may therefore suppose that the point force $Q\mathbf{E}$ will be obtained only when the electric field varies little over distances of the order of $r_+$, i.e., strictly speaking, only in the limit of the uniform field.

An additional force should act on the black hole as a result of its rotation. In particular, if the Poynting vector of the external electromagnetic field (density of the field momentum) is nonzero, the asymmetry in the absorption of field momentum by the rotating black hole should give rise to a force perpendicular to the angular momentum.[57] This force is now of purely gravitational origin, and is independent of the electric charge of the hole.

The motion of a charged rotating black hole in a magnetic field whose direction is at a certain angle to the axis of rotation should be accompanied by precession of the angular momentum because a charged rotating hole has an intrinsic magnetic moment $\mu = aQ$. Finally, in accordance with Hawking's theorem, a rotating hole (whatever its electric charge) located in an external field, and not axially symmetric, should lose its angular momentum (when the field is axially symmetric, but its symmetry axis does not lie along the axis of rotation, there is a loss of the angular momentum component perpendicular to the field symmetry axis).

All these features can be demonstrated without any particular difficulty by using the above formulas for the different external-field configurations. We shall use the device employed by Press[117] to calculate the relaxation of the rotation of a black hole in an external scalar field. We shall consider that the solutions of the homogeneous Maxwell equations found in Sections 2.1 and 2.2 against the space-time background of the rotating black hole are valid not in the entire space, but only in the interior of a sphere of radius $R\gg r_+$, and vanish outside this sphere. If we then use the Maxwell equations with the source $J^{\mu}$, we find that, in general, there are not only electric, but also (fictitious) magnetic currents on the surface of the sphere. If the self-dual bivector of the electromagnetic field is written in the form

$$\mathscr{F}_{\mu\nu}=2\,[\Phi_0 m^*_{[\mu}n_{\nu]}+\Phi_1\,(n_{[\mu}l_{\nu]}+m_{[\mu}m^*_{\nu]}) + \Phi_2 l_{[\mu}m_{\nu]}]\,\theta\,(R-r), \tag{3.1}$$

where $\theta$ is the Heaviside function, the Maxwell equations for the complex current density yield

$$J^{\mu}=J^{\mu}_{E}+iJ^{\mu}_{M}=\frac{1}{2\pi}\,\mathscr{F}^{\mu 1}\delta\,(r-R), \tag{3.2}$$

where $J^{\mu}_{E}$ is the electric current density and $J^{\mu}_{M}$ the magnetic current density (appearing in the equation for the dual tensor $\tilde{F}^{\mu\nu}_{;\nu} = -4\pi J^{\mu}_{M}$). The field $\mathscr{F}_{\mu\nu}$ acts on currents flowing in a shell with a force whose density can be calculated in a standard manner. Integration over the surface of the sphere leads to expressions that are finite as $R\to\infty$, i.e., they are independent of the position of the shell. Because of the global conservation of 4-momentum and angular momentum in asymptotically flat space, an equal and opposite force (moment of force) must be applied to the black hole.

*3.1. Charged black hole in a uniform electric field.* The resultant electromagnetic field of a black hole and the external field can be obtained by substituting the following expression in (3.1):

$$\Phi_1 = -\frac{Q\rho^2}{2}+\Phi_{1E},$$

where the first term represents the Coulomb field of the black hole (we assume that $Q\ll M$) and the second is an arbitrarily directed uniform electric field (found in Section 2.2). When the force is determined, we must confine our attention to terms proportional to the product of $Q$ and $E$, and the result for the force density is[57]

$$f^{\mu}=\frac{Q}{2\pi}\,\mathrm{Re}\left[\rho^2\Phi_{1E}\left(n^{\mu}-\frac{\Delta}{2\Sigma}\,l^{\mu}\right)-\Phi_{2E}m^{\mu}-\frac{\Delta}{2\Sigma}\,\Phi_{0E}m^{*\mu}\right]. \tag{3.3}$$

If we carry out the integration over the surface of the shell in the limit as $R\to\infty$, and take into account the fact that an oppositely directed force must act on the hole, we obtain

$$\mathbf{f} = Q\mathbf{E}, \tag{3.4}$$

as expected in the limit of a uniform field. The alternative derivation of this formula for a black hole was given by Bičák.[118] We note that, if we transform to the reference frame moving with velocity $\mathbf{v}$ relative to the frame in which the black hole is at rest, and apply the Lorentz transformation to the field $\mathbf{E}$ (a magnetic field $\mathbf{B}$ will then appear in this frame), we find that the force acting on the black hole is the resultant Lorentz force

$$\mathbf{f} = Q\,(\mathbf{E} + [\mathbf{vB}]). \tag{3.5}$$

*3.2. Drift of a rotating black hole in an electromagnetic field.* Let us now suppose that the uniform external field has both an electric and a magnetic component. The tetrad projections of the Maxwell tensor $\Phi_0$, $\Phi_1$, $\Phi_2$ must then be evaluated using (2.22) and the Debye potential found in Section 2.2. The force density acting on a shell is given by

$$f^{\mu} = \frac{1}{2\pi}\left\{|\Phi_1|^2\left(\frac{\Delta}{2\Sigma}\,l^{\mu} - n^{\mu}\right) + \frac{\Delta}{2\Sigma}\,|\Phi_0|^2 n^{\mu} - |\Phi_2|^2 l^{\mu} + \mathrm{Re}\left[\Phi_1^*\left(2\Phi_2 m^{\mu} - \frac{\Delta}{\Sigma}\,\Phi_0 m^{*\mu}\right)\right]\right\}, \tag{3.6}$$

into which we substitute the values of $\Phi_0$, $\Phi_1$, $\Phi_2$ just found. Integrating over the sphere and passing to the limit as $R \to \infty$, we obtain the expression for the force acting on the shell (the contributions to $f^{\mu}$ that are infinite as $R \to \infty$ are found to vanish after integration with respect to the angles). The counteracting force applied to the black hole is

$$\mathbf{f} = \frac{16\pi}{3}\,[\mathbf{SJ}], \tag{3.7}$$

where $\mathbf{S} = [\mathbf{E}\ \mathbf{B}]/4\pi$ is the Poynting vector of the external electromagnetic field and $\mathbf{J}$ is the angular momentum of the hole. As noted above, this force is associated with the asymmetry in the absorption of the field momentum of the rotating black hole. An analogous effect should arise when a rotating black hole is immersed in a uniform flux of particles traveling with velocity $\mathbf{v}$, whose direction is not the same as that of the angular momentum vector $\mathbf{J}$ of the hole. The asymmetry in momentum absorption that arises as a result of rotation must lead to the appearance of a transverse force $\mathbf{f} \sim \mu \mathbf{v} \times \mathbf{J}$, where $\mu$ is the mass density of the particle flux (we are assuming that $v \ll c$).

*3.3. Precession of the angular momentum of a charged black hole in a magnetic field.* The moment of the force acting on a spherical shell for $R \to \infty$ can be calculated from the formula for flat space–time

$$\mathbf{N} = \oint_{R\to\infty} [\mathbf{rf}]\, r^2\, \mathrm{d}\Omega, \tag{3.8}$$

where $\mathbf{f}$ is the force density. Repeating the discussion of Section 3.1, we find that the rate of change of the angular momentum of a charged black hole in an external magnetic field $\mathbf{B}$ is given by

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \frac{Q}{M}\,[\mathbf{JB}]. \tag{3.9}$$

This describes the precession of the magnetic moment $\boldsymbol{\mu} = Q\,\mathbf{J}/M$ in an external magnetic field with angular velocity $\Omega = QB/M$ that is equal to the cyclotron frequency of the charge $Q$.

Under the conditions prevailing in real astrophysical objects, the angular precession velocity is very small. However, when the magnetic field itself is produced by a lighter accretion disk, the latter will precess with frequency

$$\Omega_{\mathrm{disk}} = \frac{Q}{M}\,B\,\frac{J}{J_{\mathrm{disk}}}, \tag{3.10}$$

which can be somewhat greater for $J_{\mathrm{disk}} \ll J$. The possibility of observing this effect is discussed in Ref. 58. The critical factor is the charge of the hole: Astrophysically reasonable precession times can be obtained for high enough (and, in principle, admissible) values of the charge.

*3.4. Relaxation of the angular momentum in an asymmetric field.* The rate of change of the angular momentum of a black hole in a uniform magnetic field, whose direction is at an angle to the precession axis, has been discussed by several authors.[55,56,117,119] In the course of time, the black hole loses its angular momentum component perpendicular to the field, and the system becomes axially symmetric (because the angular momentum effectively rotates along the direction of the field), in accordance with the Hawking theorem.[42] When the external field does not have an axis of symmetry, the black hole can be expected to lose its angular momentum altogether. To evaluate the relaxation of the angular momentum, it will be sufficient to determine, within the framework of the above method, the moment of force acting on the electrically neutral black hole, which is due to the crossed external fields, described in Section 2.2, for arbitrarily oriented vectors **E** and **B**. The result is the following expression for the rate of change of the angular momentum of the black hole:

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = -\frac{2}{3}\,M\,([\mathbf{B}\,[\mathbf{JB}]] + [\mathbf{E}\,[\mathbf{JB}]]). \tag{3.11}$$

Diagonalization of the matrix, multiplied, on the right-hand side by the vector **J**, leads to a set of eigenvalues $1/\tau_1$, $1/\tau_2$, $1/\tau_3$ that are the reciprocals of the relaxation times of the components of the vector **J** along the eigenvectors $\xi^{(a)}$:

$$J_a = J_{a0}\, e^{-t/\tau_a}. \tag{3.12}$$

One of the eigenvectors $\xi^{(3)}$ is the Poynting vector of the external field and the corresponding eigenvalue is

$$\tau_3 = \frac{3}{2}\,M\,\frac{B_M^2}{E^2 + B^2}; \tag{3.13}$$

The two other eigenvectors lie in the plane containing **E** and **B**, and the eigenvalues are given by

$$\tau_{1,2} = 3MB_M^2\,\{(E^2 + B^2) \pm [(E^2 + B^2)^2 - 4\,[\mathbf{EB}]^2]^{1/2}\}^{-1}. \tag{3.14}$$

When **E** and **B** are collinear, the matrix is degenerate, $1/\tau_2 = 0$, and the other two eigenvalues are equal. This corresponds to the conservation of the angular momentum component along the common direction of **E** and **B**. In the case of mutually perpendicular vectors **E**, **B**, we have $\tau_1 = \tau_2 = 2\tau_3$. Under these conditions, the black hole initially loses its angular momentum component perpendicular to the plane containing **E** and **B**, and then the component lying in the **E**, **B** plane. In general, the relaxation process is such that[6]

$$\frac{1}{\tau_3} \geqslant \frac{1}{\tau_1} \geqslant \frac{1}{\tau_2}. \tag{3.15}$$

## 4. MOTION OF CHARGED PARTICLES

The motion of test particles in Kerr space-time is usually investigated with the aid of the Hamilton-Jacobi equation for which a complete separation of variables can be carried out.[120] However, it is found that, because of the symmetry of the problem, direct integration of the equations of motion is not only simpler, but also frequently has technical advantages. This approach is best investigated by the method of successive approximations, using nearly circular orbits, and taking the zero-order approximation to be the equatorial circular trajectory.[84,106]

*4.1. Circular orbits in the equatorial plane of a nonrotating black hole in a magnetic field.* The equation of motion for a particle of charge $e$ and mass $\mu$ traveling near a black hole in the presence of an electromagnetic field is

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=\frac{e}{\mu}F^\mu_\nu\frac{dx^\nu}{ds}. \quad (4.1)$$

It is clear from the symmetry of the problem that the circular motion

$$u^\mu=\frac{dz^\mu}{ds}=u^0(1,\ 0,\ 0,\ \omega_0)$$

is possible in the equatorial plane $\theta=\pi/2$. Substituting appropriate values of the Christoffel symbols and of the electromagnetic field tensor into (4.1), we find that the angular velocity $\omega_0$ is given by

$$\omega_0=\frac{1}{2}\omega_B\left[\pm\left(1+4\frac{\omega_S^2}{\omega_B^2}\right)-1\right], \quad (4.2)$$

where $\omega_s=M^{1/2}/r^{3/2}$ is the Kepler frequency and $\omega_B=eB/\mu u^0$ is the cyclotron frequency in the gravitational field. The two signs correspond to the "Larmor" (Lorentz force pointing toward a hole) and "anti-Larmor" (Lorentz force pointing in the opposite direction) rotations of the particles. Using the normalization conditions for the 4-velocity of the particle, $g_{\mu\nu}u^\mu u^\nu=1$, we find that

$$(u^0)^2\left(1-\frac{3M}{r}+r^2\omega_0\omega_B\right)=1. \quad (4.3)$$

Comparison of (4.2) and (4.3) readily shows that the Larmor motion ($\omega_0<0$) is possible only for $r>3M$, whereas the anti-Larmor motion occurs for $r>3M$ and $2M<r\leqslant 3M$. Let us now introduce the dimensionless parameter $\varepsilon=eBM/\mu$ that characterizes the relative effect of the magnetic field on the motion of the particle. We note that, even for small values of the magnetic field, $BM\ll 1$, the parameter $\varepsilon$ for a particle with high charge-to-mass ratio (for the electron $e/\mu\approx 10^{21}$) may not be small. Simultaneous solution of (4.2) and (4.3) shows that the energy of the particles is given by

$$E=\mu u_0=\frac{\Delta^{1/2}}{r}(1+\lambda^2_{(\pm)})^{1/2}, \quad (4.4)$$

where

$$\lambda_{(\pm)}=-\frac{\Delta}{2M(r-3M)}\left\{\varepsilon\pm\left[\varepsilon+4\frac{(r-3M)M^3}{\Delta^2}\right]^{1/2}\right\}. \quad (4.5)$$

Hence, it is clear that $\lambda_+$, which corresponds to the Larmor orbits, has a singularity at $r=3M$, but $\lambda_-$ does not have a singularity. It is clear from (4.5) that the region of existence of large trajectories extends right up to the event horizon for large enough $\varepsilon$. For $2M<r\leqslant 3M$, there are anti-Larmor orbits with radii $r\geqslant M(2+\varepsilon^{-1})$. Accordingly, the energy measured in the locally inertial frame is $\widetilde{E}=\mu/\sqrt{2}$, and the gravitational mass defect is

$$\Delta\mu=\frac{\mu-E}{\mu}=(1-\varepsilon^{-1/2}), \quad (4.6)$$

which approaches 100% for $\varepsilon\gg 1$. We note that ultrarelativistic trajectories ($\gamma\gg 1$) exist for $r>3M$, but this is unrelated to the fact that the orbit is close to a closed photon orbit. When $\varepsilon\Delta/M^2\gg 2[(r/M)-3]^{\frac{1}{2}}$, it follows from (4.4) that

$$\gamma^2_{(+)}=\left(1-\frac{2M}{r}\right)\left[1+\frac{\varepsilon^2\Delta^2}{M^2(r-3M)^2}\right]\gg 1. \quad (4.7)$$

*4.2. Small oscillations about circular orbits.* To describe nearly circular orbits, we modify the initial set of equations (4.1) and replace it with the set of equations for the deviations $\xi^\mu(s)=x^\mu(s)-z^\mu(s)$. Retaining terms that are linear in $\xi^\mu$ on the left-hand side, we obtain

$$\frac{d^2\xi^\mu}{dt^2}+\gamma^\mu_\alpha\frac{d\xi^\alpha}{dt}+\xi^a\frac{\partial}{\partial x^a}U^\mu=\mathcal{N}^\mu(\xi), \quad (4.8)$$

where

$$\gamma^\mu_\alpha=\left[2\Gamma^\mu_{\alpha\beta}u^\beta(u^0)^{-1}-\frac{e}{\mu u^0}F^\mu_\alpha\right]_{\theta=\pi/2},$$
$$\frac{\partial}{\partial x^a}U^\mu=\frac{1}{2}\frac{\partial}{\partial x^a}\left[\gamma^\mu_\alpha u^\alpha(u^0)^{-1}-\frac{e}{\mu u^0}F^\mu_\alpha u^\alpha\right]_{\theta=\pi/2} \quad (4.9)$$
$$(a\equiv r,\ \theta).$$

The term $N^\mu(\xi)$ on the right-hand side of (4.8) represents terms that are nonlinear in $\xi^\mu$. When the electromagnetic field $F_{\mu\nu}$ is axially symmetric, the only nonzero components in the equatorial plane ($\theta=\pi/2$) are the components $\gamma^1_0,\gamma^0_1,\gamma^1_3,\gamma^3_1,U^r,U^\theta$ of the quantities $\gamma^\mu_\alpha,U^\mu$, introduced above. Taking this into account, and integrating (4.8) with $\mu=0.3$, we obtain the following equation in the approximation that is linear in $\xi^\mu$:

$$\frac{d\xi^A}{dt}+\gamma^A_1\xi^r=0\quad (A\equiv t,\ \varphi=0.3). \quad (4.10)$$

Substituting this into the equation with $\mu=1$, we obtain the simple harmonic equation

$$\frac{d^2\xi^r}{dt^2}+\omega_r^2\xi^r=0, \quad (4.11)$$

where the angular frequency is given by

$$\omega_r=\left(\frac{\partial U^r}{\partial r}-\gamma^1_A\gamma^A_1\right)^{1/2}. \quad (4.12)$$

Equation (4.8) with $\mu=2$ gives

$$\frac{d^2\xi^\theta}{dt^2}+\omega_\theta^2\xi^\theta=0,\quad \omega_\theta=\left(\frac{\partial U^\theta}{\partial\theta}\right)^{1/2}. \quad (4.13)$$

Equations (4.11) and (4.13) describe the radial-phase and axial oscillations of the particle around a circular orbit, in which the axial oscillations are independent, whereas the radial oscillations

$$\xi^r=\text{const}\cdot\sin\omega_r t \quad (4.14)$$

are accompanied by azimuthal and "temporal" oscillations

$$\xi^A=\text{const}\cdot\gamma^A_1\frac{\cos\omega_r t}{\omega_r}. \quad (4.15)$$

These formulas enable us to investigate the stability of circular orbits. From (4.12) and (4.13), we then find that, near a nonrotating black hole,

$$\omega_r^2=\omega_S^2\left(1-\frac{6M}{r}\right)+\omega_B^2\left(1-\frac{2M}{r}\right),\quad \omega_\theta^2=\omega_S^2. \quad (4.16)$$

Hence, it is clear that motion in the vertical direction is stable ($\omega_\theta^2>0$), independently of the radius of the orbits. Radial motion is stable for $r>6M$, independently of the magnetic field. When $r<6M$, the regions of stability are different for the Larmor and anti-Larmor rotations. For large enough $\varepsilon$, the Larmor motion is stable up to $r\approx 4.3M$ and the anti-Larmor motion up to the event horizon.[83]

*4.3. Motion in a Schwarzschild-Ernst field.* For an arbitrary external magnetic field strength, the space-time of a nonrotating black hole is described by the Schwarzschild-Ernst metric[69]

$$ds^2 = \left[\left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} - r^2\,d\theta^2\right] \times \Lambda^2 - \frac{r^2\sin^2\theta}{\Lambda^2}\,d\varphi^2,$$

$$\Lambda = 1 + \frac{1}{4}B^2r^2\sin^2\theta. \tag{4.17}$$

The motion of neutral and charged particles in this metric was investigated in Ref. 83. In particular, for neutral particles in the $\theta = \pi/2$ plane, the normalization condition for the 4-velocity yields the following conditions for the radii of closed circular photon orbits:

$$(r-3M) = \frac{1}{4}B^2r^2(3r-5M). \tag{4.18}$$

Analysis of this equation shows that, when $B=0$, there is only one root $r=3M$, but, for large values of $B$, there are no circular orbits in the physical domain of the radius $r>2M$. The critical magnetic field

$$B_{cr} = 2\sqrt{3}\,B_M\,(169+38\sqrt{19})^{-1/2}, \tag{4.19}$$

corresponds to one closed photon orbit with $r_{ph} = \frac{1}{3}(8+\sqrt{19})M$. Thus, when $B<B_{cr}$, there are two closed photon orbits, whose radii are

$$r_1 \approx 3M + 9M\left(\frac{B}{B_M}\right)^2, \quad r_2 \approx \frac{2}{\sqrt{3}}\frac{1}{B}. \tag{4.20}$$

We note that the factor $\Lambda$ in (4.17) satisfies the condition $1 \leqslant \Lambda \leqslant \frac{4}{3}$ in the region of existence of circular orbits. Similar results were subsequently reported in Refs. 121 and 122.

*4.4. Motion in a magnetic field in the Kerr metric.* Consider the motion of a charged particle in a test uniform magnetic field defined by the vector potential (2.14) in Kerr space-time (Refs. 84, 106). From (4.1) with $\mu=1$, we then find that the angular frequency in the equatorial plane is given by

$$\omega_0 = \beta(1-a^2\omega_S^2)^{-1}\{\pm[1+\beta^{-2}\omega_S^2(1-a^2\omega_S^2)(1+a\omega_B)]^{1/2}-1\},$$

$$\beta = \frac{\omega_B}{2}(1+a^2\omega_S^2)+a\omega_S^2; \tag{4.21}$$

where the two signs correspond to the forward and reverse rotations of the particle in the Kerr field, and there are also Larmor and anti-Larmor motions.

On the other hand, it is readily shown that

$$(u^0)^2\left[1-\frac{3M}{r}(1-a\omega_0)^2-\omega_0^2a^2 + r^2\omega_0\omega_B(1+a^2\omega_S^2)-\frac{Ma\omega_B}{r}\right]=1. \tag{4.22}$$

In this case, the simultaneous exact solution of (4.21) and (4.22) cannot be obtained for $\gamma(r,\varepsilon)$ and $\omega_0(r,\varepsilon)$. However, assuming that $\omega_S \ll \omega_B$, i.e., that the electromagnetic force predominates, and comparing (4.21) with (4.22), it can be shown that the kinematic properties of the charged particles remain the same as in the uniform magnetic field in the Schwarzschild matrix. The regions of existence of circular orbits are shifted toward the event horizon, and ultrarelativistic motion is possible well away from the closed photon orbit. The explicit expressions for the frequencies of radial-phase and axial oscillations around the circular orbits are as follows:

$$\omega_r^2 = \omega_0^2\left(1-\frac{6M}{r}-\frac{3a^2}{r^2}-8a^2\omega_S^2\right)+\omega_0\omega_B\left(1-\frac{6M}{r}+\frac{3a^2}{r^2}\right) + 4a\omega_S^2(2\omega_0+\omega_B)+\omega_B^2\left(1-\frac{2M}{r}-r^2a^2\omega_S^4+2a^2\omega_S^2\right), \tag{4.23}$$

$$\omega_\theta^2 = \left(1+\frac{3a^2}{r^2}+4a^2\omega_S^2\right)\omega_0^2+\omega_0\omega_B\left(1+\frac{3a^2}{r^2}\right) - 2a\omega_S^2(2\omega_0+\omega_B). \tag{4.24}$$

Hence, it follows that the boundaries of the region of radial stability are shifted toward the event horizon, depending on the magnetic field strength. These conclusions are consistent with numerical calculations of the effective potential for radial motion in the Hamilton-Jacobi equation.[85]

## 5. SPONTANEOUS AND STIMULATED EMISSION DURING NONGEODESIC MOTION OF PARTICLES

The problem of radiation in curved space-time must be formulated so that the conditions under which observations are carried can be carefully defined. The scattering of radiation emitted by a particle in curved space-time makes this process essentially nonlocal. If the wavelength in a particular spectral interval is much shorter than the characteristic length over which there is a significant change in the gravitational field, it is in principle possible to isolate secondary scattering effects, but this cannot be done when the two lengths are of the same order. In asymptotically flat space-time, the radiation energy can be calculated at infinity and, in this global formulation of the problem, the effects of scattering and of the change in frequency in the gravitational field are automatically taken into account. For a nonrelativistic particle moving along a geodesic, the radiation wavelength is always of the same order as the scale of gravitational field inhomogeneity. Only the global formulation of the radiation problem is meaningful in this case. In nongeodesic motion and, especially, for ultrarelativistic particles, the characteristic wavelength may be much smaller than the inhomogeneity scale. The radiation problem can then be examined in the locally geodesic coordinate frame. In what follows, we shall consider both formulations of the problem with a view to determining the radiation emitted by an ultrarelativistic particle moving mostly under the influence of electromagnetic forces.

*5.1. Scalar wave model.* To avoid complications connected with the vector character of electromagnetic radiation, let us begin by considering the global formulation of the radiation problem, using the scalar wave model. We shall assume that the particle has mass $\mu$ and travels in Kerr space-time along a world line $x^\mu(s)$. It interacts with the massless real scalar field $\Phi(x)$ and has a scalar charge $q$. The field produced by the charge is the retarded solution of the d'Alembert equation in curved space-time

$$\frac{\partial}{\partial x^\mu}\left[(-g)^{1/2}g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right] = 4\pi q\int ds\,\delta^4(x-x(s)), \tag{5.1}$$

where $\delta$ is defined by $\int \delta^4(x)d^4x = 1$. The total energy loss $E$ by radiation consists of two parts, namely, radiation departing to infinity ($E_{out}$) and radiation absorbed by the black hole ($E_{in}$). The corresponding intensities are given by [6]

$$P_{\text{out}} = \frac{dE_{\text{out}}}{dt} = \lim_{r\to\infty} \oint (r^2+a^2)\, T^{\mu 1} \mathscr{K}_\mu \, d\Omega, \tag{5.2}$$

$$P_{\text{in}} = \frac{dE_{\text{in}}}{dt} = \lim_{r\to r_+} \oint (r_+^2+a^2)\, T^{\mu 1} \mathscr{K}_\mu \, d\Omega, \tag{5.3}$$

where $\mathscr{K}_\mu$ is a timelike Killing vector. The solution of (5.1) is constructed using an expansion in the spheroidal functions[123] $Z(\theta,\phi)$

$$\Phi(x) = \sum_{l,m} R_{lm}(r)\, Z_{lm}(\theta,\varphi)\, e^{-i\omega t}. \tag{5.4}$$

The radial functions $R_{lm}$ satisfy the equation

$$\frac{d}{dr}\left(\Delta \frac{dR_{lm}}{dr}\right) + \left(\frac{K^2}{\Delta} - \lambda_{lm}\right) R_{lm} = -\frac{4\pi q}{u^0} Z^*_{lm}\left(\frac{\pi}{2}, 0\right) \delta(r-r_0); \tag{5.5}$$

where $K = (r^2+a^2)\omega - am$, $\lambda = E - 2am\omega + a^2\omega^2$ and $E$ *is the separation constant. The subscripts carried by the radial functions will be omitted henceforth.* Substituting

$$R(r) = u(r)\,(r^2+a^2)^{-1/2},$$
$$r^* = r + \frac{M}{(M^2-a^2)^{1/2}}\left(r_+ \ln\frac{r-r_+}{r_+} - r_- \ln\frac{r-r_-}{r_-}\right) \tag{5.6}$$

we can transform from (5.5) to an equation without the first derivative. The corresponding homogeneous equation takes the form of the Schrödinger equation

$$\frac{d^2u}{dr_*^2} - V_{\text{eff}} u = 0 \tag{5.7}$$

with the effective potential

$$V_{\text{eff}} = -(r^2+a^2)^{-2}(K^2 - \Delta\lambda) + \frac{\Delta}{(r^2+a^2)^3}\left[2(Mr - a^2) + 3\Delta a^2 (r^2+a^2)^{-1}\right]. \tag{5.8}$$

As $r^* \to \infty$, the potential becomes $V_{\text{eff}} \to -\omega^2$, whereas near the event horizon, $r^* \to -\infty$, $V_{\text{eff}}$ assumes the value $k = (\omega - m\Omega_H)^2$. Accordingly, we can introduce two pairs of linearly independent solutions of (5.7), namely, $\chi^\pm(r^*)$ and $v^\pm(r^*)$ with the following asymptotic behavior:

$$\chi^\pm \sim e^{\pm i\omega r^*}, \quad v^\pm \sim e^{\pm ikr^*}. \tag{5.9}$$

We shall use these functions to construct the "right" ($u_R$) and "left" ($u_L$) solutions. The first describes waves departing to infinity as $r^* \to \infty$ and the second describes waves incident on the hole for $r^* \to -\infty$:

$$u_R \approx (2|\omega|)^{-1/2} e^{i\omega r^*}, \quad u_L \approx (2|k|)^{-1/2} \tau e^{-ikr^*}, \tag{5.10}$$

where the constant $\tau$ can be interpreted as the transmission coefficient of the potential barrier. Using these solutions, we obtain the radial function corresponding to the retarded solution of (5.1) in the form

$$R(r) = 4\pi q (r^2+a^2)^{-1/2} (r_0^2+a^2)^{-1/2} Z_l^{*m}\left(\frac{\pi}{2}, 0\right) \times \frac{i|\omega|}{\omega u^0}\left[u_R(r)\, u_L(r_0)\, \theta(r-r_0) + u_L(r)\, u_R(r_0)\, \theta(r_0 - r)\right]. \tag{5.11}$$

Substituting this in (5.2) and (5.3), we obtain the following expressions for the radiation intensity $P_{\text{out}}$ departing to infinity and the intensity $P_{\text{in}}$ incident on the black hole:

$$P_{\text{out}} = 4\pi \sum_{l,\, m>0} m\omega_0 q^2 (u^0)^{-2} (r_0^2+a^2)^{-1} \times \left|Z_l^m\left(\frac{\pi}{2}, 0\right)\right|^2 |u_L(r_0)|^2, \tag{5.12}$$

$$P_{\text{in}} = 4\pi \sum_{l,\, m>0} m\omega_0 q^2 \frac{|\omega k|}{\omega k} (u_0)^{-2} \times |\tau|^2 (r_0^2+a^2)^{-1} \left|Z_l^m\left(\frac{\pi}{2}, 0\right)\right|^2 |u_R(r_0)|^2. \tag{5.13}$$

It is clear from (5.13) that, when $\omega_0 < \Omega_H$, the power $P_{\text{in}}$ becomes negative, which corresponds to the superradiant state in the Kerr field.[30,31]

For the ultrarelativistic motion of a scalar particle, $P_{\text{in}}$ is exponentially small because the transmission coefficient $\tau$ is small. Since, in this case, the higher harmonics of the angular frequency are radiated, it is sufficient to determine the radiated power $P_{\text{out}}$ using the radial function in the high-frequency approximation. This can be done in the quasiclassical approximation, assuming that $l \gg 1, |m| \gg 1$ in (5.7). When the particle travels well away from the closed photon orbit, $\omega_0(r - r_{\text{ph}}) \gg 1$, the quasiclassical solutions of (5.7) must be joined at the point $r_t$ at which the descending segment of (5.8) crosses the abscissa axis. The coordinate $r_0$ that determines the position of the circular orbit of the ultrarelativistic particle differs from $r_t$ by a small amount because the potential $V_{\text{eff}}$ for the massless scalar field in the WKB approximation is close to the potential for a massive particle when $\gamma \gg 1$. This means that the value of the radial function at the point $r_0$ can be obtained by solving the parabolic equation, valid in the immediate vicinity of the point $r_t$, which arises when $V_{\text{eff}}$ in (5.7) is expanded in the vicinity of the point $r_t$:

$$\frac{d^2u}{dq^2} - qu = 0, \tag{5.14}$$
$$q = -\left\{\frac{m^2 \Delta r}{(r^2+a^2)^3}\left[\omega_0^2(3r^2+a^2) - 1\right]\right\}^{1/3}_{r=r_t} (r^* - r_t^*).$$

The solution of this equation that can be joined to the quasiclassical solution $u_L$ for $r \neq r_0$ is the Airy function $\Phi(q)$. Transforming from the Airy function to the Macdonald function $K_{\frac{1}{3}}$, and recalling that

$$r_t - r_0 \approx \frac{\Delta}{r} g^{-1}\left[\frac{r_0^2\Omega_0^2}{\gamma^2} + 2\frac{l-|m|}{|m|}(1 - a^2\Omega_0^2)^{1/2}\right], \tag{5.15}$$
$$\Omega_0 = \frac{2Ma \mp (\Delta_0)^{1/2} r_0}{r_0^3 + a^2 r_0 + 2Ma^2},$$

where

$$g = \Omega_0^2(3r_0^2 + a^2) - 1, \tag{5.16}$$

we finally obtain the following expression[84] for $\mu_L(r_0)$:

$$u_L(r_0) = \left[\frac{2}{3\pi}\frac{r_0\Omega_0^2}{\gamma^2} g^{-1}(r_0^2+a^2)(1+\psi^2)\right]^{1/2} K_{1/3}(z),$$
$$z = \frac{2|m|}{3\gamma^3} g^{-1}\Omega_0^3 r_0^3 \Delta_0^{1/2} (1+\psi^2)^{3/2}, \tag{5.17}$$
$$\psi^2 = \frac{2\gamma^2}{r_0^2\Omega_0^2}\frac{l-|m|}{|m|}(1 - a^2\Omega_0^2)^{1/2}. \tag{5.18}$$

Since the Macdonald function decreases exponentially for large values of its argument, it is clear from (5.18) that the main contribution to radiation is provided by the harmonics

$$m \leqslant m_{\max} = \frac{g\gamma^3}{2r_0^2\Delta_0^{1/2}\Omega_0^3}. \quad (5.19)$$

We see that the maximum number of radiated harmonics is proportional to the third power of the energy (just as in the case of flat space-time) and depends on the gravitational field parameters. However, as the closed photon orbit is approached, the $\gamma^3$ dependence is replaced with the $\gamma^2$ dependence. To demonstrate this, consider the product $\gamma g$. Formulas (4.21) and (4.22) then readily show that, as $r \to r_{\text{ph}}$ $(g \to 0)$ for given $\varepsilon$,

$$\gamma g \to \varepsilon \cdot \text{const}, \quad (5.20)$$

i.e., the product remains finite, whereas $\gamma \to \infty$. It is also interesting to compare the $l$ distributions in synchrotron radiation and geodesic synchrotron radiation spectra at given frequency $\omega = m\omega_0$, which is determined by the number $m$ of the harmonic. Since the effective range of variation of the parameter $\psi$ extends from zero to unity, we have $l - |m| \lesssim m/\gamma^2$, and a large number of multipoles, $l \sim \gamma$, contributes to synchrotron radiation for fixed $m$, whereas the principal contribution to geodesic synchrotron radiation is provided by terms with $l = |m|$ and $l = |m| + 1$. This is in agreement with conclusions reported elsewhere (Refs. 124, 125), based on numerical estimates. The approximate formula for the spheroidal harmonics $Z_l^m(\pi/2,0)$ for $l \gg 1, |m| \gg 1$ was found in Ref. 124. In our notation,

$$\left| Z_l^m\left(\frac{\pi}{2}, 0\right)\right|^2 = \frac{1+(-1)^{l-|m|}}{2} \frac{1}{\pi^2} \frac{\gamma}{\psi} \frac{(1-a^2\Omega_0^2)^{1/2}}{r_0\Omega_0}. \quad (5.21)$$

Substituting (5.17) in (5.21) and (5.12), and transforming from summation with respect to $l$ and $m$ to integration with respect to $\psi$ and the parameter

$$y = \frac{4\,|m|}{3\gamma^3} g^{-1} r_0^2 \Delta_0^{1/2} \Omega_0^3, \quad (5.22)$$

(because the spectrum is quasicontinuous), we find that the spectral distribution of the radiation is

$$\frac{dP_{\text{out}}}{dy} = \frac{3\sqrt{3}}{32\pi}\left(\frac{q^2\gamma^2}{r_0^2}\right)^2 \frac{g^2 y}{r_0^2\Delta_0^{1/2}\Omega_0^3} \int_y^\infty K_{1/3}(x)\,dx. \quad (5.23)$$

For small $y$, the radiation intensity increases as $y^{\frac{1}{3}}$ whereas for large $y$ it decreases exponentially. The spectral curve has a maximum corresponding to the critical harmonic (5.19). The total intensity of the scalar wave is

$$P_{\text{out}} = \frac{1}{12} \frac{q^2\gamma^4 g^2}{\Delta_0^{1/2}\Omega_0^3}. \quad (5.24)$$

We note that this intensity is greater by a factor of $\gamma^2$ than the intensity of the scalar geodesic synchrotron radiation.[124] It also depends on how close the radius $r_0$ of the particle orbit is to the radius $r_{\text{ph}}$ of the closed photon orbit. When $a = 0$, these formulas become identical with the result obtained in Ref. 83 for the Schwarzschild space-time.

*5.2. Synchrotron radiation from ultrarelativistic particles.* To calculate the electromagnetic radiation intensity due to ultrarelativistic charged particles in the Kerr metric, it is convenient to use the Newman-Penrose-Teukolsky formalism.[113,126,127] The radiation flux departing to infinity is given by

$$\frac{d^2E_{\text{out}}}{dt\,d\Omega} = \frac{1}{2\pi} \lim_{r\to\infty} r^2\,|\Phi_2|^2, \quad (5.25)$$

where $\Phi_2$ is the Newman-Penrose scalar defined by (2.22). Separation of variables in the Teukolsky equations for $\Phi_2$ leads to the following expansion in spheroidal spin functions of weight $s = -1$:

$$\Phi_2 = \rho^2 \int d\omega \sum_{l,\,m} {}_{-1}R(r)\,{}_{-1}Z_l^m(\theta,\varphi)\,e^{-i\omega t}. \quad (5.26)$$

The radial functions ${}_{-1}R(r)$ satisfy the equation

$$\Delta \frac{d^2}{dr^2}\,{}_{-1}R - {}_{-1}V\,{}_{-1}R = -\,{}_{-1}T, \quad (5.27)$$

where

$${}_{-1}V = -\frac{K^2}{\Delta} + \lambda - i\Delta\left(2\Delta\frac{dK}{dr} - K\frac{d}{dr}\Delta\right). \quad (5.28)$$

The source on the right-hand side of (5.27) has the following form for the above case of circular motion:

$$\begin{aligned} {}_{-1}T = {} & -\frac{e\Delta}{\sqrt{2}\,r}\int dt\,\exp\left[i(\omega - m\omega_0)\,t\right] \\ & \times\left\{\left(-i\omega\frac{r^2+a^2}{\Delta} + \frac{iam}{\Delta} - \frac{\partial}{\partial r}\right)\frac{iK}{m}\right. \\ & \left. - (1-a\omega_0)\left[m(1-a\omega_0) + \frac{ia}{r} - \frac{\partial}{\partial\theta}\right]\right\}_{-1} \\ & \times Z_l^{*m}\left(\frac{\pi}{2}, 0\right)\,\delta(r-r_0). \end{aligned} \quad (5.29)$$

The potential defined by (5.28) is complex, which complicates the procedure for joining the quasiclassical solutions. However, the formalism proposed in Ref. 128 can be used to avoid this difficulty by transforming (5.27) into an equation with a real potential. As a result, the problem of finding the radial function in the high-frequency approximation $l \gg 1$, $|m| \gg 1$ can be reduced to the above scalar case. Repeating the discussion given in the last Section, and transforming from spheroidal harmonics with $s = -1$ to functions with $s = 0$, we obtain the following expressions for the two independent polarization states of electromagnetic radiation departing to infinity ($\sigma$ and $\pi$ components):[84]

$$\begin{aligned} P_{\text{out}}^{(\sigma)} = {} & \sum_{l,\,m>0} \frac{8}{3}\frac{e^2}{\gamma^4}\,m g^{-1} r_0 \Delta_0 \Omega_0^5 (1+\psi^2)^2 \\ & \times\left|Z_l^m\left(\frac{\pi}{2}, 0\right)\right|^2 K_{2/3}^2(z), \end{aligned} \quad (5.30)$$

$$\begin{aligned} P_{\text{out}}^{(\pi)} = {} & \sum_{l,\,m>0} \frac{8}{3}\frac{e^2}{\gamma^2 m}\,g^{-1} r_0^{-1} \Delta_0 \Omega_0^3 (1+\psi^2) \\ & \times\left|\frac{dZ_l^m(\pi/2,\,0)}{d\theta}\right|^2 K_{1/3}^2(z). \end{aligned} \quad (5.31)$$

Replacing summation over $l$, $m$ with integration with respect to $\psi$, $y$, and using the asymptotic representation

$$\left|\frac{dZ(\pi/2,\,0)}{d\theta}\right|^2 = \frac{1-(-1)^{l-|m|}}{2}\cdot\frac{1}{\pi^2}\frac{\psi}{\gamma}\,r_0\Omega_0 m^2 (1-a^2\Omega_0^2)^{1/2}, \quad (5.32)$$

we obtain the following expression for the spectral distribution:

$$\frac{dP_{\text{out}}^{\sigma,\pi}}{dy} = \frac{3\sqrt{3}}{32\pi}\frac{e^2\gamma^4}{r_0^4\Delta_0^{1/2}\Omega_0^3}\,g^2 y\left(\int_y^\infty K_{5/3}(x)\,dx \pm K_{2/3}(y)\right). \quad (5.33)$$

It is readily seen that the spectral distribution of electromagnetic radiation emitted by an ultrarelativistic charge has similar properties to those noted in the last Section for scalar waves. In particular, the conclusions relating to the change in the limiting frequency in the spectrum and to the set of multipoles providing the main contribution to radiation as we pass from synchrotron radiation to geodesic synchrotron radiation remain valid. The total radiation intensity is

$$P=\frac{1}{6}\frac{e^2\gamma^4}{r_0^4\Delta_0^{1/2}\Omega_0^3}g^2. \tag{5.34}$$

The degree of polarization is

$$\Pi=\frac{P^{(\sigma)}-P^{(\pi)}}{P}=\frac{4}{3} \tag{5.35}$$

and is independent of the presence of the gravitational field. These formulas generalize existing results[88-90] on the spectrum and polarization of synchrotron radiation in flat space-time to the case of the strong gravitational field of a rotating black hole.

*5.3. Local description of synchrotron radiation in a gravitational field.* Since the high-frequency part of the radiation spectrum due to ultrarelativistic particles originates in a small segment of the trajectory, the method of local coordinates can be used to describe the spectrum. Consider the normal Riemann coordinates[79] $\xi^\alpha$, for which the space-time metric takes the form

$$g_{\mu\nu}=\eta_{\mu\nu}+\frac{1}{3}R_{\mu\alpha\nu\beta}\xi^\alpha\xi^\beta+\dots \tag{5.36}$$

and all the Christoffel symbols vanish at the origin $\xi^\alpha=0$. We can then write the Maxwell equation in terms of these coordinates, taking into account only the terms that are linear in $\xi^\alpha$:

$$\eta^{\alpha\beta}\frac{\partial^2A^\mu}{\partial\xi^\alpha\,\partial\xi^\beta}+\frac{2}{3}\left[R_\tau^\lambda\frac{\partial A^\mu}{\partial\xi^\lambda}+(R_{\lambda\tau}^{\mu\nu}+R_{\tau\lambda}^{\mu\nu})\frac{\partial A^\lambda}{\partial\xi^\nu}\right]+\frac{4}{3}R_\nu^\mu A^\nu=-4\pi J^\mu,\quad A^\mu_{;\mu}=0. \tag{5.37}$$

The main operator on the left-hand side is the usual d'Alembert operator in Minkowski space-time. The additional terms that are proportional to space-time curvature are of relative order $\xi\lambda/L^2$, where $L$ is the inhomogeneity scale of the gravitational field and $\lambda$ is the characteristic wavelength in the spectrum. For synchrotron radiation $\lambda_{\rm syn}\sim\Delta l/\gamma^2$, where $\Delta l\sim\rho/\gamma$ is the length within which the radiation from the ultrarelativistic particle is produced on a trajectory with instantaneous radius of curvature $\rho$, and $\gamma$ is the Lorentz factor in the same frame of reference (the wave zone begins at distances that are small in comparison with $\rho$). When $\xi\lesssim L$, the relative size of terms proportional to the space-time curvature is then small in a significant wavelength range even when $\rho\sim L$.

Confining our attention to the high-frequency part of the radiation and omitting, in view of the foregoing, all terms in (5.37) that are proportional to $\xi^\mu$ for a freely falling system, the spectral distribution of the radiation can be described by the formula for flat space-time

$$\frac{d\hat{P}}{dy}=-\frac{3\sqrt{3}}{4\pi}e^2\dot{u}^2y\int_y^\infty K_{5/3}(x)\,dx, \tag{5.38}$$

where

$$y=\frac{2}{3}(k\ddot{u})(-\dot{u}^2)^{-3/2},\quad \dot{u}^2=\dot{u}_\mu\dot{u}^\mu. \tag{5.39}$$

For ultrarelativistic particles, we have $k\ddot{u}=-\omega\dot{u}^2/u^0$, which is valid to within $\gamma^{-2}$. If we transform the 4-momentum of the photon $u\to Du/ds$ to an arbitrary frame, so that $k_\mu\Lambda^\mu_\nu=k_\nu$, where $\Lambda^\mu_n=\partial\xi^\mu/\partial x^\nu$ is the coordinate transformation matrix at the instantaneous position of the particle, we can rewrite (5.38) in terms of generally covariant quantities. The derivatives of the 4-velocity that are represented by dots then become covariant derivatives: $u\to Du/ds$ and so on. Correspondingly, the total intensity assumes the form

$$P=\frac{2}{3}\frac{e^2}{\mu^2}F_{\hat{\mu}\hat{\nu}}F^{\nu\lambda}u^\mu u_\lambda. \tag{5.40}$$

Consider synchrotron radiation emitted by ultrarelativistic particles traveling in a uniform magnetic field in the Kerr metric.

Transforming from local coordinates $\xi^\mu$ to an arbitrary frame with origin at the instantaneous position of the particle $X^\mu$ by the standard method,[129]

$$\xi^{\hat{\mu}}=\Lambda_\nu^{\hat{\mu}}(X)\left[(x-X)^\nu+\frac{1}{2}\Gamma^\nu_{\alpha\beta}(X)(x-X)^\alpha(x-X)^\beta+\dots\right], \tag{5.41}$$

we can readily show that the total radiation intensity expressed in terms of the time coordinate in the Boyer-Lindquist frame is the same as (5.34), the latter being obtained in the Newman-Penrose-Teukolsky formalism.

The foregoing discussion remains valid for the Dirac equation referred to the local inertial frame. The quantum spectrum of synchrotron radiation in a freely falling reference frame is therefore given by the corresponding expressions found in the case of flat space-time.[130] The total radiation intensity, including quantum corrections to the classical spectrum in the Kerr metric, is found to be[97]

$$P=P_{\rm cl}\left(1-\frac{55}{32}\frac{\gamma^2}{\mu}gr_0^{-2}\Delta_0^{-1/2}\Omega_0^{-2}+\dots\right), \tag{5.42}$$

where $P_{\rm cl}$ is given by (5.34) in which the orbital radius $r_0$ is a function of $\gamma$. The dependence of the quantum corrections on energy is therefore more complicated than in the case of flat space-time, in which the relation is linear. When $a=0$, (5.42) becomes identical with the expression obtained in Ref. 96 for the Schwarzschild space-time.

*5.4. Stimulated emission.* The interaction of radiation with a charged particle traveling in a uniform magnetic field in Kerr space-time can give rise to negative reabsorption of radiation. Consider the forced oscillations of a charged particle traveling on a circular orbit under the influence of weak electromagnetic waves described by the field tensor $f_{\mu\nu}$. With this in view, let us introduce the perturbing force $f_\mu=f^\mu_\nu u^\nu(u^0)^{-1}$ into the right-hand side of (4.10), (4.11), and (4.13), which describe the free radial-phase and axial oscillations of the particle about its circular orbit. Transforming to the Fourier expansions for the quantities $f^\mu$ and the perturbations $\xi^\mu$,

$$f^\mu(t,r,\theta,\varphi)=\int_{-\infty}^{\infty}d\omega\sum_{m=-\infty}^{\infty}f^\mu(r,\theta,\omega,m)\times\exp(-i\omega t+im\varphi), \tag{5.43}$$

$$\xi^\mu(t)=\int_{-\infty}^{\infty}d\omega\,\xi^\mu(\omega)\exp(-i\omega t), \tag{5.44}$$

we finally obtain

$$\xi^A(\omega,m)=\frac{1}{i\omega_m}\gamma_1^A\xi^r(\omega,m)-\frac{e}{\mu u^0}\frac{f^A(\omega,m)}{\omega_m^2}, \tag{5.45}$$

$$\xi^r(\omega,\ m)=\frac{1}{\omega_r^2-\omega_m^2}\,\frac{e}{\mu u^0}\left(f^r(\omega,\ m)-\frac{i}{\omega_m}\,\gamma_A^1 f^A(\omega,\ m)\right),\tag{5.46}$$

$$\xi^\theta(\omega,\ m)=\frac{1}{\omega_r^2-\omega_m^2}\,\frac{e}{\mu u^0}\,f^\theta(\omega,\ m),\tag{5.47}$$

where

$$\omega_m=\omega-m\omega_0,\quad f^\mu(r,\ \theta,\ \omega,\ m)\equiv f^\mu(\omega,\ m).$$

It follows from the form of these solutions that, when the spectrum of the perturbing force contains the frequencies

$$\omega=m\omega_0,\quad \omega=\Omega_r^\pm=m\omega_0\pm\omega_r,\quad \omega=\Omega_\theta^\pm=m\omega_0\pm\omega_0,\tag{5.48}$$

the forced oscillations exhibit resonance. To give (5.45)–(5.47) a physical meaning under resonance conditions, we must take dissipative processes into account by shifting the pole into the complex plane, $\omega\to\omega-i\nu$, and assuming that the collision frequency $\nu$ is small in comparison with $\omega$. If the Kerr space-time admits a timelike Killing vector $\mathscr{K}^\mu_{(t)}$, the work done by the field $f_{\mu\nu}(x)$ on the current $J^\nu(x)$ can be defined in a covariant manner as follows:

$$A=\int f_{\mu\nu}(x)\,J^\nu(x)\,\mathscr{K}^\mu_{(t)}\,(-g)^{1/2}\,\mathrm{d}^4x.\tag{5.49}$$

Assuming that there are no correlations between the particles, the wave power absorbed by the system can be determined by integrating the corresponding one-particle result over the particle distribution. Substituting the current

$$J^\nu(x)=e\int u^\nu(x)\,\delta^4(x^\alpha-x^\alpha(s))\,\frac{\mathrm{d}s}{(-g)^{1/2}},\tag{5.50}$$

into (5.49), we obtain

$$A=e\int f_{\mu\nu}(x(s))\,u^\nu(s)\,\mathscr{K}^\mu_{(t)}\,\mathrm{d}s,\tag{5.51}$$

where

$$x^\alpha(s)=z^\alpha(s)+\xi^\alpha(s).$$

Expanding the integrand in powers of $\xi^\alpha$, and using the lowest nonvanishing order of perturbation theory, we find that the absorbed wave power averaged over the phases is given by

$$P=e\left\langle\xi^\mu\frac{\partial f_\mu}{\partial t}\right\rangle.\tag{5.52}$$

We now introduce the correlation tensor for the electromagnetic field with random phases:

$$\langle f_\mu(\omega,\ m)\,f_\nu^*(\omega',\ m')\rangle=I_{\mu\nu}\delta_{mm'}\delta(\omega-\omega').\tag{5.53}$$

It is clear that the tensor $I_{\mu\nu}$ satisfies the following conditions:

$$I_{\mu\nu}(\omega,\ m)=I^*_{\nu\mu}(\omega,\ m)=I_{\nu\mu}(-\omega,\ -m).\tag{5.54}$$

Using the above formulas for the absorbed power averaged over the phases, we obtain[106]

$$P=\int_0^\infty \mathrm{d}\omega\sum_{m=-\infty}^{\infty}\Big[\sum_{\lambda=\pm1}P_r^\lambda(\omega,\ m)\,\frac{\nu}{(\omega-\Omega_r^\lambda)^2+\nu^2}$$
$$+\sum_{\lambda=\pm1}P_\theta^\lambda(\omega,\ m)\frac{\nu}{(\omega-\Omega_\theta^\lambda)^2+\nu^2}+\frac{\nu}{(\omega-m\omega_0)^2+\nu^2}\,P_0(\omega,\ m)\Big].\tag{5.55}$$

The spectral functions in this expression are given by

$$P_r^\pm(\omega,\ m)=\pm\frac{e^2}{\mu u^0}\,\frac{\Delta_0}{r_0^2}\,\frac{\omega}{\omega_r}\left(I_{11}+\frac{\varkappa^2}{\omega_r^2}\,I_{33}\pm\frac{2\varkappa}{\omega_r}\,\mathrm{Im}\,I_{13}\right),$$
$$P_\theta^\pm(\omega,\ m)=\pm\frac{e^2}{\mu u^0}\,\frac{1}{r_0^2}\,\frac{\omega}{\omega_\theta}\,I_{22},$$
$$P_0(\omega,\ m)=-\frac{2e^2}{\mu u^0}\left\{\frac{1}{\Delta_0}\left[1-\frac{2M}{r_0}(1-a\omega_0)-\omega_0^2(r_0^2+a^2)\right.\right.$$
$$\left.\left.-\frac{\Delta_0^2}{r_0^2}\,\frac{\varkappa^2}{\omega_r^2}\right]\frac{d}{d\omega}(\omega I_{33})+\frac{2\omega}{\omega_r^2}\,\varkappa\,\frac{\Delta_0}{r_0^2}\,\mathrm{Im}\,I_{13}\right\},\tag{5.56}$$

where

$$\varkappa=\gamma_1^3-\omega_0\gamma_1^0.$$

Similar expressions for the wave power absorbed during the motion of a charge in the Schwarzschild-Ernst field (4.17) were obtained in Ref. 105 and, in the Schwarzschild field, in Ref. 104. The typical feature of these expressions is the fact that $P^-_{r,\theta}$ is nonpositive and $P^+_{r,\theta}$ is nonnegative. We thus find that the power absorbed at the combination frequencies $\Omega_r^-$ and $\Omega_\theta^-$ is negative, and we have maser-type wave amplification, whereas at frequencies $\Omega_r^+$ and $\Omega_\theta^+$ the absorbed power is positive. The sign of $P_0$, which describes the resonance interaction for the harmonics $\omega=m\omega_0$, depends on the more detailed structure of the correlation functions and, in principle, can also be negative. It is interesting that the condition for wave amplification in our model is analogous to the condition for superradiation in the Kerr field. In point of fact, negative absorption occurs for $\omega<m\omega_0$ ($\Omega^-_{r,\theta}=m\omega_0-\omega_{r,\theta}$), and the condition for superradiation is $\omega<m\Omega_H$.

## 6. EFFECT OF A MAGNETIC FIELD ON QUANTUM PROCESSES IN BLACK HOLES

Quantum processes occurring in the strong gravitational field of a black hole are of interest because there may be microscopic remnant black holes that may have been produced at the early stages in the evolution of the Universe (see Ref. 11 for a review of the modern state of the problem). Historically, the first quantum effect discovered in the theory of black holes was the prediction by Zel'dovich[30] and by Misner[31] of "superradiation" in classical language. From the classical point of view, superradiation is the amplification of waves of different nature when they are scattered by a rotating black hole. Superradiation occurs for multipoles for which the frequency $\omega$ and azimuthal quantum number $m$ satisfy the condition

$$\omega<m\Omega_\mathrm{H},\tag{6.1}$$

where $\Omega_H=a/2Mr_+$ is the angular frequency of the event horizon of the black hole. From the point of view of quantum theory, superradiation is the stimulated creation of field quanta by a rotating black hole. The corresponding spontaneous process is the creation of particle pairs with opposite components of orbital angular momentum, due to the interaction between the orbital angular momentum and the angular momentum of the hole, which depends on their mutual orientation. For modes satisfying (6.1), this interaction is a repulsion which makes superradiation possible.

Somewhat later, Hawking[10] predicted the quantum evaporation of a black hole due to the creation of pairs in the gravitational field of the hole (including a nonrotating hole) as a result of tidal forces. This process ensures that the black hole loses energy (mass) and that this loss corresponds to the emission of radiation by the black hole at temperature $T=(\frac{1}{2}\pi)(r_+ - M)/(r_+^2 + a^2)$. For a Schwarzschild black hole (using usual units), we have

$$T=\frac{\hbar c^3}{8\pi GM} \approx 10^{-9}\frac{M_\odot}{M}\ \mathrm{K}. \tag{6.2}$$

The loss of mass (angular momentum) is described by the equation (in the case of bosons)

$$\frac{d}{dt}\begin{pmatrix} M \\ J \end{pmatrix} = -\frac{1}{2\pi}\sum_{lm}\int_0^\infty d\omega \frac{\Gamma_{lm}^{\omega}\begin{pmatrix} \omega \\ m \end{pmatrix}}{\exp\{(\omega - m\Omega_H)/T\}-1}, \tag{6.3}$$

where the "barrier factor" $\Gamma_{lm}^{\omega}$ is due to the gravitational interaction between the emitted quanta and the black hole. The temperature of the black hole is a purely quantum quantity. In the classical limit $T\to 0$, the Planck factor in (6.3) transforms into the Heaviside function $\theta(m\Omega_H - \omega)$ (at the same time, $\Gamma_{lm}^{\omega}\to -1$) and (6.2) describes spontaneous superradiation in the Kerr field.

The quantity $m\Omega_H$ in the argument of the exponential in (6.3) plays the part of the chemical potential $\mu_H$ of the horizon in the Bose distribution of the created particles. In the case of a charged black hole (charge $Q$), the creation of particles of charge $e$ is determined by the chemical potential of the horizon

$$\mu_H = m\Omega_H + eV_H, \tag{6.4}$$

where $V_H = Q/2M$ is the electrostatic potential of the horizon.

Massive particles can form quasistationary states around the black hole which decay by particle tunneling under the event horizon. When the gravitational radius $r_+$ of a hole is small in comparison with the Compton wavelength $\lambda_C$ of the particle, these massive states are found to be long-lived.[6,131,132,134] The real part of their energy is hydrogen-like in character (when relativistic corrections are neglected):

$$\mathscr{E}_n = \mu\left[1-\frac{(\mu M - eQ)^2}{2n^2}\right], \tag{6.5}$$

where $\mu$ is the particle mass and $n$ the principal quantum number $n = l + n_r + 1$. In this approximation, the rotation of the hole has no effect on the positions of the energy levels. The complex energy of the quasilevels is given by

$$\mathscr{E}_{lmn_r} = \mathscr{E}_n - \frac{1}{2}i\Gamma_{lmn_r}, \tag{6.6}$$

where the damping rate $\Gamma$, which depends on the quantum numbers $l$, $m$, and $n_r$ satisfies the condition

$$\Gamma \ll \mathscr{E}_n - \mathscr{E}_{n-1}, \tag{6.7}$$

which shows that the levels are long-lived. For a rotating black hole, there are superradiant quasistationary states for which $\mathscr{E}_n < \mu_H$. These do not decay but, on the contrary, are excited as a result of superradiation. When $\lambda_c \gg r_+$, the superradiant level $l = 1$, $m = 1$, $n_r = 0$ is most effectively excited (for $a \ll M$). For this level,

$$\Gamma_{110} = -\frac{2a\mu}{3M}\left(\frac{r_+}{2\sqrt{2}\lambda_c}\right)^8. \tag{6.8}$$

The significant difference between superradiation involving quasibound levels and "ordinary" superradiation is the exponential instability development. The number of particles occupying quasistationary levels (in the case of bosons) increases as $\exp(|\Gamma|t)$. This process involves the successive transfer of the angular momentum of the hole to increasingly higher energy levels, and the mass of the hole is lost much more slowly than its angular momentum.[6]

The remarkable fact is the possibility of "controlling" the above quantum processes by an external magnetic field. Let us examine this in greater detail.

*6.1. Variation in the chemical potential of the event horizon.* The chemical potential of the horizon is determined by the behavior of the effective potential in the radial equation (5.7) in the vicinity of the event horizon of the black hole. When the magnetic field is weak enough, so that it can be looked upon as a test field, we can use the results obtained in Section 5.1. As already noted, the rotation of the black hole in a uniform magnetic field produces a Faraday potential difference between the event horizon and an infinitely distant point. It is readily verified that a corresponding quantity appears in the effective potential in the radial equation, so that the chemical potential at the horizon is given by

$$\mu_H = m\Omega_H + \frac{e}{2M}(Q - 2aMB). \tag{6.9}$$

The creation of particles as a result of superradiation can give rise to a tendency for the chemical potential to fall to zero, while the reduction in the second term occurs as $Q\to 2aMB$. When the magnetic field becomes so strong that the induced electric field near the horizon exceeds the Schwinger value $\mu^2/e$, the electrodynamic mechanism of pair production comes into play.[133] This process is also found to cease when the hole acquires the electric charge $Q = 2aMB$. A discussion of thermodynamic relationships for rotating black holes in an external magnetic field can be found in Ref. 72.

The chemical potential of the event horizon in an external field of a more general form behaves similarly:

$$\mu_H = m\Omega_H + e(A_t + \Omega_H A_\varphi), \tag{6.10}$$

where $A_t$ and $A_\varphi$ are the components of the 4-potential of the external field at the horizon if the gauge is chosen so that $A_t \to 0$ at infinity.

*6.2. Magnetic ergosphere.* A new curious phenomenon occurs when the magnetic field is strong enough to ensure that its effect on the metric of space-time must be taken into account. As noted in Ref. 107, the ergosphere of a charged black hole in a strong magnetic field does not vanish even for $a = 0$. The value of the function $\omega'$ in (2.46) at the event horizon, which has the significance of the angular velocity with which inertial frames are dragged, is given by (to within terms linear in $B$)

$$\omega'_H = \Omega_H - \frac{2QBr_+}{r_+^2 + a^2}. \tag{6.11}$$

Thus, even for neutral particles, the chemical potential is modified[136]:

$$\mu_H = m(\Omega_H - 2BQr_+(r_+^2 + a^2)^{-1}). \tag{6.12}$$

In the case of charged particles, we must additionally take into account the change in the electrostatic potential of the event horizon.

*6.3. Variation in the spectrum of quasistationary states of massive charged particles.* If we assume that the external magnetic field is uniform up to the region of localization of finite orbits of massive particles, the effect of the field on the spectrum of quasistationary states of charged particles will be as follows. First, there is the Zeeman shift of levels, which can be described by introducing the effective mass

$$\mu^2 \to \mu^2 - eBm, \tag{6.13}$$

where $m$ is the magnetic quantum number. Second, because of the change in the chemical potential of the event horizon [in accordance with (6.9)], there is an asymmetry in the excitation threshold of quasistationary states with respect to the sign of the charge. The result is that an electric current is excited during the development of the boson instability.

## 7. CONCLUSION

The effects discussed above are interesting because they are associated with the simultaneous existence of gravitational and electromagnetic fields. Moreover, since the gravitational interaction and the Einstein law of gravitation are universal, the electromagnetic field is both affected by the gravitational field and itself acts on the latter. This gives rise to phenomena that are not a simple "sum" of electrodynamic and gravitational effects, but reflect their synthesis and internal interrelation.

All this applies to classical effects, such as the appearance of an induced potential difference during the rotation of a black hole in a magnetic field, the drift of a rotating black hole in the electromagnetic field with nonzero Poynting vector, and spontaneous and stimulated radiation processes accompanying the motion of charged particles in an electromagnetic field in the Kerr metric. There are also characteristic quantum effects associated with the change in the chemical potential of the event horizon of a rotating black hole in an external magnetic field and the appearance of an effective ergosphere of a charged black hole in a magnetic field.

The "gravimagnetic" phenomena probably play a definite role in astrophysics. For example, there has recently been considerable interest in the hypothesis that the extraction of energy from black holes (in particular, supermassive holes in galactic cores and quasars) is due precisely to the electrodynamic mechanism associated with the above inductive effects (see Refs. 5 and 34–37 for further details).

In our view, these phenomena constitute a methodologically interesting example of new properties ensuring from the synthesis and interrelation of two branches of physics, namely, electromagnetism and gravitation.


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