# The quantum chaos problem

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This review describes the present state of the quantum chaos problem: The problem of determining the specific properties of autonomous and nonautonomous quantum systems with few degrees of freedom whose classical analogs have an unstable (stochastic) motion and also that of determining the relationships between these properties and the characteristics of the classical stochastic situation. The criteria for quantum chaos which have been established to date are examined and compared for autonomous systems. These criteria make use of properties of the energy spectrum, the wave functions in various representations, the matrices of operators other than the Hamiltonian, and particular features of the evolution of time-varying states (wave packets) in such systems. For nonautonomous systems, the conditions for the applicability of classical dynamics for describing observables and certain specific quantum-mechanical effects (tunneling through a separatrix and a global quantum resonance) are analyzed.

**CONTENTS**



The concept of stochastic (chaotic) motion in classical dynamic systems with few degrees of freedom was established in theoretical physics about a quarter of a century ago, but interest in research on related phenomena continues to grow. The primary feature of a stochastic situation is a sensitivity of the phase paths of the motion to the initial conditions. This sensitivity makes the behavior of the system essentially unpredictable, and it requires a new approach to a description of the motion—an approach which focuses on the instability itself, i.e., its magnitude and the conditions under which it occurs. Today, this approach has been developed well, and it has a rigorous foundation.

The quantum chaos problem is the problem of studying manifestations of a stochastic nature of dynamic systems in their quantum-mechanical properties. Such research is of fundamental interest: The difference between the methods used for describing quantum and classical systems raises the problem of the practical realization of the correspondence principle. The task is quite general, since a classical stochastic nature is inherent in nearly any Hamiltonian system. The task is thus one of studying the quantum-mechanical properties of semiclassical systems of a general type with a few degrees of freedom. Among the possible applications, we would first mention generating a description of highly excited states of atoms (in external fields) and of polyatomic molecules.

Research on the quantum chaos problem is rapidly picking up speed. Vector computers have emerged as an effective tool for an empirical study of the quantum properties of model systems. Several general features have been established and interpreted theoretically. The first experimental studies appeared recently. Our purposes in the present review are to describe the present state of affairs in the quantum chaos problem, to call attention to some existing contradictions, and to list the problems to be taken up next.

The terms "chaos" and "stochastic nature," and terms derived from them are presently being used nearly synonymously to describe an unstable, random motion in a dynamic system. The term "chaotic" is ordinarily used to describe a random motion in a dissipative system, while "stochastic" is

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applied more commonly to Hamiltonian systems.[130] We will be discussing only Hamiltonian systems in the present review, but for classical systems we will speak in terms of "stochastic" properties, while for quantum systems we will speak in terms of "chaotic" properties, intending to make use of this difference in terminology to emphasize the distinctions. A second argument in favor of this choice of words is a convention which is materializing: in the papers which we have cited, the combination "quantum chaos" is encountered twice as often as "quantum stochastic nature."

## 1. INTRODUCTION. STOCHASTIC BEHAVIOR IN CLASSICAL MECHANICS

The purpose of this section of the review is to summarize briefly the definitions and the concepts of the classical theory of stochastic motion of Hamiltonian systems, which we will need in discussing the quantum chaos problem. Most of the material in this section of the paper is contained in textbooks[9,42] and monographs,[129,130] to which we will be referring continually. The reader who is already acquainted with the theory of a classical stochastic situation can go directly to Sec. 2.

### 1.1 Regular and stochastic motions

The dynamics of a classical Hamiltonian system with $N$ degrees of freedom is governed by the canonical equations of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad (1 \leqslant i \leqslant N), \tag{1.1}$$

where $H(\mathbf{p},\mathbf{q})$ is a Hamiltonian.[1)] A function of the dynamic variables $F(\mathbf{p},\mathbf{q})$ such that we have $\{F,H\} = 0$, where $\{\,,\}$ denotes the Poisson brackets (§42 in Ref. 9), is called an "integral of the motion": $dF/dt = 0$. If there exist $N$ independent integrals $F_i\,(1 \leqslant i \leqslant N)$ such that we have $\{F_i,F_j\} = 0$, then the system (1.1) is *integrable*. For an integrable system one can introduce canonical action-angle variables $(I_i,\theta_i)$ such that the Hamiltonian will be a function of the action variables alone: $H(\mathbf{p},\mathbf{q}) = H(I)$ (§49 in Ref. 9 and §50 in Ref. 42). The motion of an integrable system is quasiperiodic: The angular coordinates $\theta = (\theta_1,\ldots,\theta_N)$ vary in time at a constant rate:

$$\frac{d\theta_i}{dt} = \omega_i(\mathbf{I}), \quad \omega_i = \frac{\partial H}{\partial I_i}. \tag{1.2}$$

The quantities $\omega_i$ are the "frequencies" of the quasiperiodic motion. The paths in the phase space of an integrable system are bounded by an $N$-dimensional surface in a $2N$-dimensional phase space. A particular case of an integrable system is a system with separable variables (§48 in Ref. 9). We will also refer to integrable systems as *regular*.

In general, a system with two or more degrees of freedom is not integrable, and it may execute both a quasiperiodic (regular) motion and a stochastic motion. A distinctive characteristic of stochastic motion is an instability, manifested by an exponential divergence of close-lying paths. If $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are two paths in phase space which are close at $t = 0$, then in the case of a stochastic motion we will have

$$\Delta(t) = |\,\mathbf{x}(t) - \mathbf{x}'(t)\,| \approx \Delta(0)\, e^{\sigma_1 t} \tag{1.3}$$

if $\Delta(0)$ is sufficiently small. The quantity $\sigma_1$ is the maximal *Lyapunov index* and is defined formally by

$$\sigma_1(\mathbf{x}) = \lim_{\substack{t\to\infty \\ \Delta(0)\to 0}} \left(\frac{1}{t} \ln \frac{\Delta(t)}{\Delta(0)}\right) \quad (\mathbf{x} = \mathbf{x}(0)) \tag{1.4}$$

[for nearly all directions of the vector $\mathbf{w} = \mathbf{x}(0) - \mathbf{x}'(0)$]. A system with a $D$-dimensional phase space is characterized by a set of $D$ Lyapunov indices $\sigma_i$, which are numbered in decreasing order. They are defined in Ref. 130 (§5.2). Points which belong to a common path have identical values of $\sigma(\mathbf{x})$. For Hamiltonian systems, the Lyapunov indices are symmetric: $\sigma_i = -\sigma_j$ for $i + j = 2N + 1$. Two of the $\sigma_i$ are exactly zero in this case.

The motion of a system is called *stochastic* if the relation $\sigma_1 > 0$ holds on a path, while it is regular if $\sigma_1 = 0$.

Related to the Lyapunov indices is the Kolmogorov entropy $h$ (§1.6 in Ref. 129 and §5.2 in Ref. 130). For a given stochastic path, $h$ is the sum of the positive Lyapunov indices:

$$h = \sum_{\sigma_i > 0} \sigma_i. \tag{1.5}$$

For stochastic motion, the correlation function of the dynamic quantities $f(t) = f(\mathbf{p},\mathbf{q})$ and $g(t) = g(\mathbf{p},\mathbf{q})$, defined by

$$B(f,\, g;\, \tau) = \langle f(t+\tau)\, g(t)\rangle_T - \langle f(t)\rangle_T \langle g(t)\rangle_T, \tag{1.6}$$

(the angle brackets $\langle\ldots\rangle_T$ mean a time average), tends to zero with increasing $\tau$:

$$\lim_{\tau\to\infty} B(f,\, g;\, \tau) = 0. \tag{1.7}$$

This property is called *mixing* (§1.5 in Ref. 129).

The power spectrum $S(\omega)$ of the dynamic quantity $f(t)$ is given by

$$S(f;\, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(f,\, f;\, \tau)\, e^{-i\omega\tau}\, d\tau. \tag{1.8}$$

For motion with mixing, the power spectrum $S(\omega)$ is continuous, while for regular motion it is discrete:

$$S(f;\, \omega) = \sum_k A_k \delta(\omega - \Omega_k); \tag{1.9}$$

here the $A_k$ form a countable sequence, and the frequencies $\Omega_k$ are combinations of the frequencies of the quasiperiodic motion, $\omega_i$.

### 1.2. Measure of the stochastic component

For a general—nonintegrable—autonomous Hamiltonian system, there is a single integral of the motion: the energy $F_1 = H$. Accordingly, any path belongs entirely to an energy surface: a set of dimensionality $2N - 1$ on which the relation $H(\mathbf{p},\mathbf{q}) = E$ holds. This surface contains regions of regular motion, with $\sigma_1 = 0$, and also stochastic components, i.e., regions with $\sigma_1 > 0$. In a system with two degrees of freedom, there may be several stochastic components (or even an infinite number of such components). We distinguish them by means of an index $k$. Each stochastic component generally has its own set of Lyapunov indices $\sigma_k = (\sigma_{1k},\ldots,\sigma_{2N-1,k})$.

An important characteristic of stochastic motion is the *measure* $\mu_k(E)$: that fraction of the volume of the (compact) energy surface which is occupied by the $k$ th stochastic component. We will be numbering the stochastic components in order of decreasing $\mu$. Systems with $\mu_1 = 1$, for which the sole stochastic component fills the entire energy

surface, are *ergodic* (§1.5 in Ref. 129 and §5.2 in Ref. 130).

The value of $\mu(E)$ can be found by partitioning the energy surface into cells and by counting the number of cells which are penetrated by the stochastic-motion path found through a numerical integration of Eqs. (1.1). Another approach uses a calculation of Lyapunov indices for a large number of paths, with randomly selected initial conditions. The measure $\mu_k(E)$ is estimated as the fraction of paths with identical sets $\sigma_k(\mathbf{x})$ $(\sigma_1 > 0)$.

If $\mu_1 \ll 1$, a stochastic motion is essentially indistinguishable from a regular motion. Modern computers [e.g., the IBM 370/168 (Ref. 202), the CDC Cyber 172 (Ref. 123), and the HITAC S810/20 (Ref. 191)] are capable of determining $\mu$ within an error $\Delta\mu \approx (1\text{–}3)\cdot 10^{-2}$ in a reasonable time for a system with two degrees of freedom. We will accordingly speak in terms of a significant stochastic behavior if the relation $\mu_1 \gtrsim 0.1$ is satisfied under the conditions of the problem, and we will call the region of parameter values in which this inequality does not hold the "region of regular motion." For estimates of $\mu(E)$, the quantity $\bar{\mu}(E)$ is frequently used; this is the fraction of the area of one of the two-dimensional cross sections of the energy surface which is occupied by the stochastic component.[8,123]

The stochastic aspects of the motion of a Hamiltonian system are characterized if we know the measure $\mu_k(E)$ and the values of the Lyapunov indices $\sigma_k$ for each of the stochastic components. We will call these quantities the "parameters of the stochastic nature."

### 1.3. Basic models

Of greatest importance to the quantum chaos problem today are the properties of autnonmous systems with two degrees of freedom and those of nonautonomous systems with a single degree of freedom. In the present review we will be dealing exclusively with such systems. For them, the instability is determined by the single positive Lyapunov index $\sigma_1$, which we will write below as simply $\sigma$. The following classes of systems have been studied in greatest detail.

*1.3.1. Billiards.* These are systems in which a point executes a free motion in a two-dimensional region bounded by one or several closed curves, being reflected elastically from the boundaries. Examples of billiards in which the motion is stochastic are Sinaĭ billiards,[7,14] whose boundaries consist of a square of side $L$ and a circle of radius $R < L/2$, with coincident centers, and stadium billiards,[22,41] whose boundaries are parallel line segments of length $2a$ which are linked by semicircles of radius $r$ (Fig. 1).

These definitions specify single-parameter families of systems (with the parameters $\gamma = R/L$ and $\gamma = a/r$, respectively). For arbitrary $\gamma$ and $E$, the motion in these billiards is ergodic ($\mu_1 = 1$), and the Lyapunov index $\sigma$ depends on $\gamma$ (Fig. 2).

*1.3.2. Nonlinear oscillators.* We call a "nonlinear oscillator" a system with a Hamiltonian $H(\mathbf{p},\mathbf{q})$ which can be approximated in some region of phase space by a positive definite quadratic form

$$H_0 = E_0 + \sum_{i=1}^{2} \left( \frac{p_i^2}{2m_i} + \frac{m_i\omega_i^2}{2} q_i^2 \right), \tag{1.10}$$

but which does not coincide with it. Since the Hamiltonian of a nonlinear oscillator contains at least three parameters of

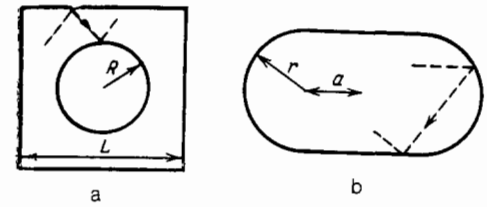


FIG. 1. a—Sinaĭ and b—stadium billiards. The dashed lines show particle paths.

independent dimensionalities (the particle mass $m$, the small-oscillation frequency $\omega$, and the nonlinearity length $l$, which is determined by the nonlinear term $V = H - H_0$), these scale values are convenient to adopt as a system of units. In the discussion below, we will write the Hamiltonians of nonlinear oscillators in dimensionless form, and we will put the origin of our energy scale at $E_0$.

Here are some examples of the nonlinear oscillators which are used in stochastic theory.

1) The Thiele-Wilson model,[6]

$$H = \sum_{i=1}^{2} \frac{1}{2}\left[p_i^2 + (1 - e^{-q_i})^2\right] + \alpha p_1 p_2. \tag{1.11}$$

2) The Henon-Heiles model[8] (Fig. 3),

$$H = \sum_{i=1}^{2} \frac{1}{2}(p_i^2 + q_i^2) + q_1^2 q_2 - \frac{1}{3} q_2^3. \tag{1.12}$$

A generalization of this model is the Eastes-Marcus model[23]

$$H = \sum_{i=1}^{2} \frac{1}{2}(p_i^2 + \omega_i^2 q_i^2) + \lambda\,(q_1^2 q_2 + \eta q_2^3), \tag{1.13}$$

which represents a three-parameter family, to which system (1.12) also belongs.

3) The Pullen-Edmonds model,[76]

$$H = \sum_{i=1}^{2} \frac{1}{2}(p_i^2 + q_i^2) + k q_1^2 q_2^2. \tag{1.14}$$

A distinctive feature of the last model (and an advantage over examples 1 and 2) is that the motion is finite at any energy. The Pullen-Edmonds model can be used to describe (classical) massive Yang-Mills fields which depend on a sigle variable.[67] The stochastic properties of Hamiltonian (1.14) and its generalizations (with a potential which does not contain terms of the type $q_i^2$ and/or which does contain terms of the type $q_i^4$) have been the subject of many studies.[68,141,203,216,220,228]

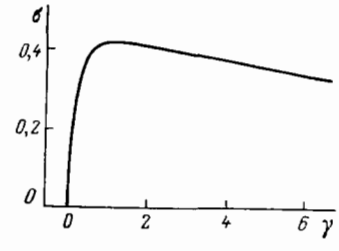


FIG. 2. The Lyapunov index $\sigma$ as a function of the shape parameter $\gamma$ of a stadium billiard.[41]

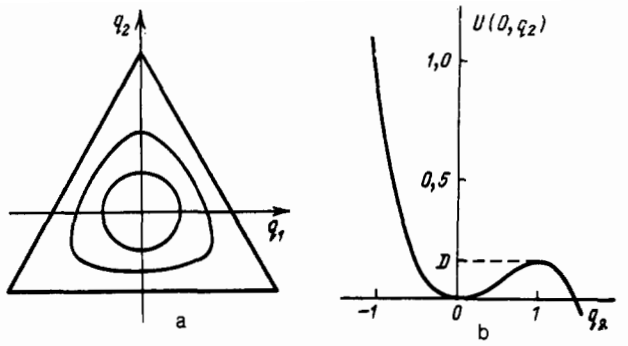


FIG. 3. a—Contour map of the potential energy; b—cross section of the potential, for Henon-Heiles model (1.12). Here $D$ is the threshold for an infinite motion.

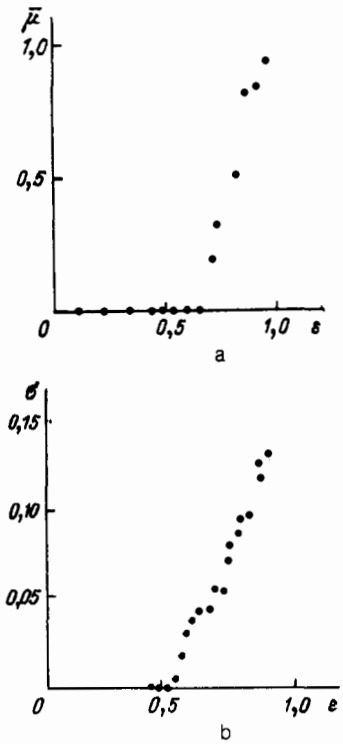


FIG. 4. Energy-dependence (a) of the cross section of the stochastic component $\bar{\mu}$ (Ref. 8); and (b) of the Lyapunov index $\sigma$ (Ref. 29) for the Henon-Heiles model. The parameter $\varepsilon$ is the ratio of the energy to the threshold for infinite motion: $\varepsilon = E/D$.

It is clear from the definition of nonlinear oscillators that under the condition $E \ll 1$ the motion in such systems is approximately regular, since the system is integrable in the limit $E \to 0$. Accordingly, the stochastic nature is noticeable only at high energies, $E \sim 1$. For such systems, in a region with a significant stochastic behavior, one observes essentially only one stochastic component. We will denote its measure by simply $\mu$. Both $\mu$ and $\sigma$ increase with increasing energy for nonlinear oscillators (Fig. 4).

*1.3.3. Nonautonomous systems with jolts.* Among nonautonomous systems, those which are important for the discussion below are systems whose Hamiltonians have the following form in terms of action-angle variables:

$$H = \frac{I^2}{2} + Kv(\theta)\,\tilde{\delta}(t), \tag{1.15}$$

where

$$\tilde{\delta}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n), \tag{1.16}$$

and $\delta(t)$—is the Dirac $\delta$-function. Such a system can be represented as a plane rotator which is acted upon at constant time intervals $T$ by an instantaneous pulsed moment of force [the moment of inertia of the rotator, $J$, and the period of the jolts, $T$, have been set equal to unity in (1.15)]. For model (1.15), the relationship between the values of the dynamic variables $(I,\theta)$ and $(I',\theta')$, taken at the times $t = n + 0$ and $t' = n + 1 + 0$, can be specified analytically. It is

$$I' = I + K\frac{\partial v}{\partial \theta}, \quad \theta' = \theta + I'. \tag{1.17}$$

This mapping of a cylinder ($-\infty < I < \infty$, $0 \leqslant \theta < 2\pi$) into itself contains comprehensive information about the motion of the system after a long time. With $v(\theta) = -\cos\theta$, mapping (1.17) is called *standard* and has been studied in extreme detail.[45] We call model (1.15) with a periodic but otherwise arbitrary $v(\theta)$ a *rotator with jolts*, while with $v(\theta) = -\cos\theta$ it is a *standard rotator with jolts.*

In the model of a standard rotator with jolts, at small values[47] $K < K_C = 0.9716$, sequential points $(I_n,\theta_n)$ lie on lines which run around the cylinder or in narrow stochastic layers; the range of $I$ is limited in magnitude: $|I_n - I_0| < 4K^{1/2}$ for all $n$. In the case $K > K_C$, the system goes into a regime of a global stochastic behavior, where there exists a stochastic component which is not bounded in terms of the action variable $I$.

In this regime, the behavior of the distribution function $f(I,n)$ for an ensemble of systems can be described by a Fokker-Planck equation (§6.2 in Ref. 129 and §5.2 in Ref. 130)

$$\frac{\partial f}{\partial n} = \frac{\partial}{\partial I}\left(D(I)\frac{\partial f}{\partial I}\right). \tag{1.18}$$

For the model of the standard rotator with jolts, under the condition $K \gg 1$, the diffusion coefficient is[45] $D \approx K^2/2$. Diffusion in action space leads to an increase in the average energy of the rotator, $E = I^2/2$, over time:

$$\langle E(t)\rangle \approx \frac{K^2 t}{4}. \tag{1.19}$$

We call this form of the dependence $\langle E(t)\rangle$ a "diffusive growth."

## 2. STATEMENT OF THE PROBLEM

For what question is 9W the answer?
*G. J. Morgan*
[after translation into Russian and back]

The quantum chaos problem is the problem of determining the specific properties of quantum-mechanical systems whose classical analogs exhibit an unstable (stochastic) motion.

If a classical system specified by a Hamiltonian $H(\mathbf{p},\mathbf{q},t)$ is integrable, we call the quantum-mechanical system with the Hamiltonian $\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{q}},t)$ "regular." In many cases, an integrable classical system remains integrable in quantum mechanics—i.e., it has a set of $N$ operators which commute with each other. If the classical integral of motion $F$ contains terms of the type $p_i^{\alpha}q_i^{\beta}$, however, the circumstance that $\hat{p}_i$ and $\hat{q}_i$ do not commute in quantum mechanics means

that the operator $\hat{F}$ constructed by the familiar rules from the classical integral $F$ may not commute with the Hamiltonian. One such example was studied in Refs. 94 and 114.

If the classical system $H(\mathbf{p},\mathbf{q},t)$ can execute a stochastic motion (in phase space there are regions in which the maximum Lyapunov index is positivie: $\sigma_1 > 0$), we call the quantum system with the Hamiltonian $\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{q}},t)$ "chaotic."

The quantum chaos problem has two basic aspects: comparing the properties of quantum chaotic and regular systems and comparing the dynamics of an observable classical system and its quantum-mechanical analog.

### 2.1. Comparison of regular and chaotic quantum systems

The features which distinguish quantum chaotic systems from quantum regular systems can be brought out by asking three questions:

1. What properties distinguish a quantum chaotic system from a quantum regular system?

2. Which parameters $\zeta$ of the quantum chaos describe these properties?

3. How are the parameters $\zeta$ of the quantum chaos related to the parameters $\sigma$ and $\mu$ of the stochastic behavior of the corresponding classical system?

In this regard, one analyzes primarily the properties of autonomous systems which execute a finite motion. For such systems, these questions were first posed in 1973 by Zaslavskiĭ and Filonenko[19] and also by Percival.[21] Even earlier, in 1971, Pukhov and Chernavskiĭ[17] asked whether an unstable infinite motion would be possible in a quantum-mechanical system.

The first of these questions is not trivial because one cannot literally import the classical stochastic parameters $\sigma$ and $\mu$ into a quantum theory: These parameters are defined in terms of the properties of phase-space paths of the system, which have no precise quantum-mechanical analog. By convention, each statement which establishes a difference between the properties of quantum regular systems and quantum chaotic systems is called a *quantum chaos criterion*.

Many such criteria have now been proposed, but all are applicable only in the semiclassical region, where the following inequality holds for stationary states of the system, $\psi_n$:

$$\xi_n = \left(\frac{\hbar^2}{2mE_n a_n^2}\right)^{1/2} \ll 1. \tag{2.1}$$

Here $E_n$ is the energy of the state, and $a_n$ is a typical size of the classically accessible region at the energy $E_n$. Deep in the quantum region, where $\xi_n \sim 1$, the stochastic nature of the corresponding classical system is not manifested significantly in the properties of the quantum system.

We will be using the dimensional quantities which appear in the classical Hamiltonian as units of measure in the discussion below. If there are three independent scales (as there are for nonlinear oscillators), the system of units is determined unambiguously, and Planck's constant $\hbar$ becomes a dimensionless parameter of the problem. For nonlinear oscillators with $\hbar \ll 1$, we have $\xi_n \sim 1$ for low-lying states ($E_n \ll 1$), while for highly excited states ($E_n \sim 1$) we have $\xi_n \sim \hbar^{1/N}$, where $N$ is the number of degrees of freedom. If there are only two independent scales in the classical Hamiltonian (as in the cases of billiards and uniform power-law potentials[9]; see §10), the energy $E_n$ can be used as a third scale. In this case, $\hbar$ is the same as $\xi_n$ and can become arbitrarily small.

The second question—about the parameters of the quantum chaos, $\zeta$—is not identical to the first, since quantitative characteristics have not been established for anything like all the quantum chaos criteria. Some criteria are formulated in a semiquantitative language ("large" or "small"), while others make use of the reader's ability to distinguish between regular and irregular systems of points and lines. For certain criteria, the quantitative characteristics $\zeta$ are determined in several ways, and the relationships among the various definitions are not clear.

Finally, the third question has been pursued only extremely slightly so far. The uncertainty clouds not only the nature of the relationship $\zeta = f(\sigma,\mu)$ but even the question of which stochastic characteristics of a classical system will be the arguments of this relationship. The situation is that the properties which have been studied in greatest detail are those of nonlinear oscillators with two degrees of freedom. For such systems, both $\mu$ and $\sigma$ increase with increasing energy (Fig. 4). Accordingly, the increase with increasing $E$ in the strength of a characteristic which underlies a criterion can be linked with an increase in the "extent of the stochastic behavior," but it cannot be related unambiguously to a dependence on $\mu$ or $\sigma$. Most authors tend to regard $\mu$ but not $\sigma$ as important for quantum chaos (Sections 3 and 8).

The quantum chaos criteria for autonomous systems can be classified in four groups, which are based on the properties of (1) the energy spectrum, (2) the stationary states, (3) operators other than $\hat{H}$, and (4) states which are not stationary. This is the order in which the criteria will be discussed in Sections 3–6. We first note the following point: Any criterion must resolve the question of which of the following assertions is to be chosen:

1. Quantum chaos is a property of one stationary state.

2. Quantum chaos is a property of a group of staionary states with approximately equal energies.

Percival[21] proposed the first resolution of this choice, postulating the existence of stationary states of two types in a quantum chaotic system:

•Regular states (which belong to a regular spectrum), which are related to a quasiperiodic motion of the classical system. They have a set of quantum numbers in the number of degrees of freedom. For regular states, the matrix elements of the operators are limited by selection rules.

•Irregular states (which belong to an irregular spectrum), which are related to a stochastic motion of the classical system. Such states do not have a definite set of $N$ quantum numbers; for irregular states there are no selection rules for operators. This scheme has defined a paradigm, within which the theory of quantum chaos has developed. Today, the universal validity of this distinction is under doubt: A large number of criteria use the second resolution of the choice between alternatives.

### 2.2. Quantum effects in the dynamics of observables

Since quantum chaos is associated with the semiclassical region, the quantum chaos problem has another aspect: comparison of the dynamics of an observable classical system and that of its quantum analog at a small but nonzero value of $\hbar$.

For autonomous systems, the qualitative difference in the dynamics follows immediately from the circumstance that the energy spectrum is discrete: Mixing property (1.7) and the related continuous nature of the power spectrum $S(\omega)$ do not hold for such systems, by definition. On the other hand, a dense discrete spectrum $S(\omega)$ may differ from a continuous spectrum only in an observation over a very long time. There is accordingly a time scale $\tau(\hbar)$ ($\tau \to \infty$ as $\hbar \to 0$) such that at $t \ll \tau(\hbar)$ the classical and quantum observables are approximately the same, while at $t \gtrsim \tau(\hbar)$ there is a substantial difference between them. In the quantum chaos problem, there are accordingly the tasks of seeking $\tau(\hbar)$ and determining the dynamics of the quantum system at $t \gtrsim \tau(\hbar)$.

Even more important is the question of how quantum effects influence the dynamics for nonautonomous systems. For them, there can be qualitatively different types of motion in the classical theory: motion with a bounded change in the action variables during a regular and locally stochastic motion, and motion with an unbounded diffusive growth of these variables during a globally stochastic motion (Subsection 1.3.3). The qualitative changes made in this picture by quantum effects were found by Casati, Chirikov, Izrailev, and Ford[46] in 1977 and have been the subject of active research since then. The work in this direction is discussed in Section 7.

### 2.3. Some applications

Who ordered chaos? More specifically, what use is it?
*M. C. Gutzwiller*
[after translation into Russian and back]

Stochastic behavior is a general property of dynamic systems, while integrability is a rare exception, associated with a very high symmetry, which can easily be destroyed by some perturbation. It is true that the perturbation would have to be fairly strong in order to make the stochastic nature noticeable, but this is a condition limiting the parameter values, not the type of system. Essentially any semiclassical ($\hbar \ll 1$) quantum system with few degrees of freedom and a known Hamiltonian $\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{q}},t)$ can, for certain parameter values, exhibit the characteristics of quantum chaos. The most important actual examples of these systems are the following:

1. Atoms in highly excited (Rydberg) states ($n = \hbar^{-1} \sim 100$) in the presence of external fields—a magnetic field[65,199] or an alternating electric field.[142,176,192,213]

2. Triatomic molecules ($\hbar^{-1} \sim 100$). Nonlinear oscillators (Subsection 1.3.2), which have been the subject of most of the papers in the theory on quantum chaos, are studied for the purpose of describing the vibrational spectra of molecules in the region of pronounced excitation, where the anharmonicity of the vibrations and the interactions between modes are important.

Other examples would include electrons which are grazing along the surface of a metal in a magnetic field,[19] an atom or group of atoms interacting with a resonant mode of a quantized electromagnetic field,[43,149,193] electrons in semiconductors with an anisotropic mass tensor in the Coulomb field of an impurity,[91] electrons above the surface of liquid helium in an alternating electric field,[159] a magnetic moment in an alternating magnetic field,[189] and electrons in layered structures.[195] Theory has not yet been compared with experiment in these problems.

## 3. PROPERTIES OF THE ENERGY SPECTRUM

There are two advantages in using the energy spectrum to establish the criteria of quantum chaos. First, the spectrum does not depend on the choise of representation, so the criteria are not afflicted by an arbitrariness. Second, a spectrum is amenable to experimental determination.

### 3.1. Sensitivity of levels to perturbations

Percival[21] offered the following suggestion:

**A1.** The levels of the spectrum of a quantum chaotic system are more sensitive than the levels of the spectrum of a quantum regular system to the magnitude of a perturbation applied to the system. #[2)]

This hypothesis has been tested in numerical experiments.[25] The Hamiltonian of the Henon-Heiles model, (1.12), was written in the form

$$H = \frac{1}{2}\sum_{i=1}^{2}(p_i^2+q_i^2)+\alpha\left(q_1^2q_2-\frac{1}{3}q_2^3\right)=H_0+\alpha V \qquad (3.1)$$

(in a system in which the units are the scales of the classical Hamiltonian, $\hbar = \alpha^2$). The susceptibilities of the eigenvalues $E_n(\alpha)$ to changes in the parameter $\alpha$ were calculated:

$$\chi_\alpha(n) = \frac{d^2E_n}{d\alpha^2}. \qquad (3.2)$$

With $\hbar^{-1} = 129$, it was found that at energies $\varepsilon \geqslant 0.73$, where the classical system has a noticeable stochastic behavior ($\bar{\mu} > 0.2$), there are, along with small values of $\chi_\alpha$ (which monotonically continue the energy dependence of the susceptibility out of the region of regular motion), values of $\chi_\alpha$ which exceed these small values by about an order of magnitude (in absolute value). Their appearance was discussed as a confirmation of Criterion **A1** (Fig. 5).

The value of $\chi_\alpha$ can be calculated by perturbation theory. It is expressed in terms of the matrix elements $V_{mn}$ of the operator $\overline{V}$ between eigenfunctions of $H$ and the energy values $E_n$ in the following way:

$$\chi_\alpha(n) = 2\sum_m{}' \frac{V_{nm}^2}{E_n-E_m}. \qquad (3.3)$$

Percival himself related the increase in $\chi_\alpha$ to the circumstance that a large number of terms contribute to the sum in (3.3). A detailed analysis of system (3.1) with $\hbar^{-1} = 80$ showed that in fact the opposite was true.[62,75] Large values of

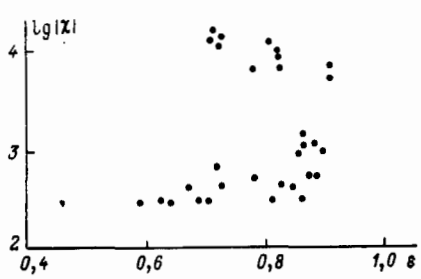


FIG. 5. The susceptibility of levels to a change in the parameter $\chi_\alpha$ in the Hamiltonian as a function of the normalized level energy $\varepsilon = E/D$ for the Henon-Heiles model.[25]

$\chi_\alpha$ arise because of, and in the vicinity of, level quasicrossings ("avoided crossings") as $\alpha$ is varied and are determined by the one anomalously large term on the right side of (3.3) (Fig. 6). The results of these studies were subjected to criticism based on some new numerical calculations[80,102] for system (3.1) with $\hbar^{-1} = 138$. That criticism, however, is not convincing. In the first place, the critics restricted their study to the first derivative, $dE_n/d\alpha$, which, as we see in Fig. 6, is not anomalously large near an avoided crossing. On the contrary, it is anomalously small. Second, the accuracy of the numerical calculations by those authors seems totally inadequate for drawing conclusions about fine characteristics of a spectrum, as was pointed out in Ref. 87.

For values $\hbar^{-1} \sim 100$, the values of the regular ("small") susceptibilities and the positions of the avoided crossings (but not the magnitudes of the splittings $\Delta_{mn}$) can be determined fairly accurately in low orders of perturbation theory.[77,87]

Criterion **A1** has been tested by numerical calculations also for systems with two[76,193] and three[196,221] degrees of freedom other than the Henon-Heiles model. In all cases, large values of $\chi_\alpha$ were found in energy regions corresponding to stochastic motion. These large values were manifested as sharp spikes above the background of the typical values.

Criterion **A1** resolves the question in Subsection 2.1 by choosing the second alternative: The quantum chaos is manifested as a property of a group of levels, since the susceptibility $\chi_\alpha$ becomes large only in comparison with the neighboring small values.

Nakamura *et al.*[196] have suggested using the statistical characteristics of the values of $\chi$ for levels from a given energy iterval $E_l < E < E_h$—the mathematical expectation of the magnitude of the susceptibility, $|\chi|$, and the variance $D\chi = \overline{\chi}^2 - (\overline{\chi})^2$. According to Nakamura *et al.*,[196,221] large values of $|\chi|$ and $D\chi$ are not by themselves indicators of quantum chaos. They suggested modifying criterion **A1** in the following way:

**$A1_1$.** The sensitivity of the spectra of quantum regular systems and quantum chaotic systems to perturbations differs in the $\hbar$ dependence of the relative size of the susceptibility fluctuations in a given energy range:

$$g(E_1, E_2; \hbar) = (D\chi)^{1/2} / \overline{\chi}^{-1}. \qquad (3.4)$$

In the limit $\hbar \to 0$, this function increases for quantum chaotic systems but decreases for quantum regular systems. #

So far, this assertion has been tested for only a single model in the region $\hbar^{-1} \lesssim 20$; a dependence $g(\hbar) \sim \hbar^{\nu}$ has been found. The difference between small and large susceptibilities in this example is of record size (more than three orders of magnitude), while the $\hbar$ remains comparatively large. We are forced to suspect that this example is an exceptional case, and we must postpone evaluating the generality of criterion **$A1_1$**.

That there are large values of $\chi$ in a quantum chaotic system at $\hbar^{-1} \lesssim 100$ is beyond doubt. However, the connection between these values and a classical stochastic nature cannot be regarded as unambiguously established. Noid *et al.*[61] have pointed out that an isolated avoided crossing of levels may be analogous not to a stochastic component with a large value of $\mu$ but to a resonance: a regular motion with commensurable frequencies. The relationship between avoided crossings and classical resonances was studied in Refs. 119 and 122. According to Noid,[61] a distinctive feature of a quantum chaotic system is not the presence of avoided crossings but their high density:

**$A1_2$.** The picture of the dependence of the levels of a family of Hamiltonians $H(\alpha)$ on the parameter $\alpha$ has isolated level crossings (or avoided crossings) for a quantum regular system and overlapping avoided crossings for a quantum chaotic system (Fig. 7). #

This criterion is supported qualitatively by numerical calculations; quantitative characteristics have not been established for it. In particular, there is no definition of the width in $\alpha$ of an avoided crossing. The relationship between an overlap of avoided crossings and an overlap of resonances (which leads to the appearance of a global stochastic behavior, according to the Chirikov criterion[45]) which was proposed by Uzer *et al.*[127] is apparently not a profound one. An overlap of resonances may occur at fixed parameter values of a Hamiltonian (with a change in, for example, the energy), while an overlap of avoided crossings depends in a fundamental way on the variability of $\alpha$. Furthermore, **$A1_2$** has the limitation that it refers not to a given Hamiltonian but to a family, and it cannot answer the question of the degree of quantum chaos for a given $\hat{H}(\alpha_0)$. Finally, we note that there is a direct contradiction between **$A1_1$ and $A1_2$**: For a pattern of overlapping avoided crossings, in the limit $\hbar \to 0$ we would be left with $g(\hbar) \sim 1$.

It is possible that the distance between neighboring levels will increase to the extent that avoided crossings overlap (see Section 4), with the result that "large" susceptibilities decrease. With increasing degree of stochastic behavior of a classical system, the value of $|\overline{\chi}|$ for an energy interval of a given width would thus initially increase and then decrease. This behavior of $|\overline{\chi}|$ as a function of the extent of the stochastic behavior was reported in Ref. 179; it is compatible with the data of Ref. 25, but it requires further confirmation.

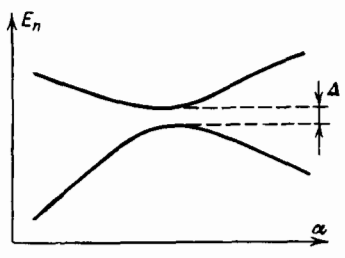


FIG. 6. Avoided crossing of energy levels upon a change in the parameter in the Hamiltonian. Here $\Delta$ is the magnitude of the splitting.

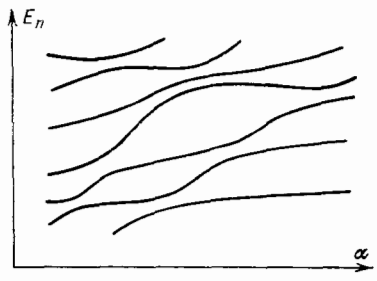


FIG. 7. Pattern of levels $E_n$ of a single-parameter family of Hamiltonians, $H(\alpha)$, in a region of overlapping avoided crossings.

It is thus most likely that the increase in the sensitivity of the eigenvalues of a quantum system to the magnitude of a perturbation in a region of classical stochastic behavior is a transitional property, which disappears in the semiclassical limit $\hbar \to 0$. This circumstance strips **A1** of its value as a criterion, but not of its physical interest. The region $\hbar^{-1} \sim 100$ is typical of systems which are accessible experimentally. Accordingly, the susceptibility to physically realizable perturbations (e.g., a polarizability) may thus also have anomalously large values.

### 3.2. Statistics of the energy spectrum

The basis for the study of the statistics of the energy spectrum of quantum chaotic systems has been the statistical theory of spectra which was derived in order to describe the structure of the spectra of complex nuclei. A bibliography of the basic work on the statistical theory of spectra is given in the review by Zaslavskiĭ,[43] which is devoted to the statistics of an energy spectrum in connection with the quantum chaos problem (see also the review by Brody *et al.*[72]). This theory examines the properties of the distribution of eigenvalues for matrices with random elements which belong to certain statistical ensembles. It has recently been established that the conclusions of the statistical theory are also valid for the spectra of complex atoms[108] and triatomic molecules.[110]

The basic prerequisite for using the methods and models of the statistical theory of spectra to describe quantum chaotic systems is the concept of a complex, pseudorandom structure of the energy spectrum. This concept also agrees qualitatively with the concepts developed in Subsection 3.1: If the positions of levels vary in a complex way upon a change in the parameters of the Hamiltonian, there is no physical interest in a detailed description of the system of levels. More important are the stable average characteristics of the system of levels. The study of the statistics of an energy spectrum thus utilizes the second of the alternatives in Subsection 2.1, treating quantum chaos as a property of a group of states.

*3.2.1. Distribution of level spacings.* The simplest characteristic of the structure of an energy spectrum is the distribution function of the relative size of the spacings between levels, $S_n$:

$$S_n = (E_n - E_{n-1})\,\rho(E_n), \tag{3.5}$$

where $\rho(E)$ is the density of levels. It is assumed here that $\rho(E)$ varies only slightly over distances of the order of the distance between levels: $d\ln\rho(E)/dE \ll 1$. This condition holds in the semiclassical case. The distribution function $P(S)$ of the random quantity $S$ is normalized by the conditions

$$\int_0^\infty P(S)\,dS = 1, \quad \int_0^\infty SP(S)\,dS = 1. \tag{3.6}$$

If the level positions are uncorrelated, $P(S)$ is given by the Poisson distribution

$$P_P(S) = \exp(-S). \tag{3.7}$$

In this case, in the limit $S \to 0$ we have $P(S) \neq 0$; this behavior of $P(S)$ is called "level bunching."

For Hamiltonians with random matrix elements $H_{ij}$, the probability for finding two closely spaced levels ($S \ll 1$) is small, along with $S$. The origin of this *level repulsion* can be seen in the very simple example of a Hamiltonian $H$ which is a second-rank matrix. The distance ($\Delta$) between the levels of this Hamiltonian is given by $\Delta^2 = (H_{11} - H_{22})^2 + 4|H_{12}|^2$. If the system is invariant under time reversal, one can choose a basis in which the matrix $\hat{H}$ is real: $H_{12} = H_{21}$. The distribution $\Delta$ will then be determined by the statistical properties of the variables $z_1 = H_{12}$ and $z_2 = (H_{11} - H_{22})/2$. If we assume that the $z_i$ are independent random quantities for which the probability distribution $W_i(z)$ is nonzero in the limit $z \to 0$ [$W_i(0) \neq 0$], then we find $P(\Delta) \sim \Delta$ in the limit $\Delta \to 0$. If we assume that the $z_i$ are normally distributed quantities with identical variances, the normalized level spacing $S = \Delta/\overline{\Delta}$ ($\overline{\Delta}$ is the mean distance between levels) has a Wigner distribution (Fig. 8)

$$P_W(S) = \frac{\pi}{2} S \exp\left(-\frac{\pi}{4} S^2\right). \tag{3.8}$$

Expression (3.8) gives a good description of the properties of the series of levels of complex systems with fixed exact integrals of motion (the total angular momentum $J$ and the parity $\pi$ in the cases of atoms and nuclei).

A key role is played in the statistical theory of spectra by the concept of an ensemble of matrices—Hamiltonians which have statistically independent components and whose statistical properties remain invariant under any transformations compatible with the discrete symmetries of the Hamiltonian. For the models which we mentioned in Section 1, used to describe the motion of spinless particles in the absence of a magnetic field, the Hamiltonian is invariant under time reversal. A class of systems of this sort is associated with an ensemble of real symmetric matrices which are invariant under arbitrary orthogonal transformations: Gaussian orthogonal ensemble (GOE).

Although the $P(S)$ distribution for a Gaussian orthogonal ensemble formally differs from a Wigner distribution,[12] quantitatively the difference is small, and it can be ignored in view of the amount of data available today. It was shown in Ref. 120 that a reliable discrimination of these distributions would require a set of $N \sim 10^5$ levels, while series of several hundred levels are available for quantum chaotic systems today. However, the model of Gaussian orthogonal ensembles makes it possible to establish characteristics of the spectrum other than $P(S)$ (Subsections 3.2.2. and 3.2.3).

The question of the properties of the distribution of level spacings for quantum chaotic systems was first taken up

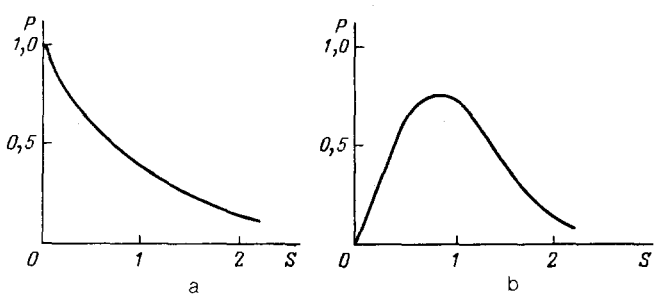


FIG. 8. Limiting forms of the distribution of level spacings, $P(S)$: a—Poisson distribution; b—Wigner distribution.

by Zaslavskiĭ and Filonenko,[19] who proposed a dependence of $P(S)$ on the Kolmogorov entropy $h$ of a model system. This dependence was subsequently generalized by Zaslavskiĭ[31] to the case of arbitrary semiclassical stochastic systems. We will not dwell on those papers here for two reasons: First, Zaslavskiĭ's theory has been presented in detail in several places in the literature (Refs. 31, 43, 81, and 129). Second, the proposed dependence [in the limit $S \to 0$, $P(S) \sim S^{\nu}$, where $\nu \sim h^{-1}$] has not found support in numerical calculations. The question of a dependence of the properties of the spectra of quantum chaotic systems on the extent of the instability of a classical system (i.e., on $\sigma_1$ or $h$) is not completely clear at this point (Subsection 8.3).

The results of a numerical study of the distribution $P(S)$ for spectra of various quantum systems which are invariant under time reversal are compatible (at least semiquantitatively) with the criterion

**A2.** The distribution of level spacings $P(S)$ is (a) a Poisson distribution, (3.7), for integrable systems of a general type in the semiclassical limit, $\hbar \to 0$, and (b) a Wigner distribution, (3.8), for highly stochastic systems ($\mu \approx 1$). #

Position **A2-a** was advanced by Berry and Tabor. It has been supported by numerical calculations (for two models), and a theory has been derived for it on the basis of a semiclassical quantization.[35]

The occurrence of level bunching for integrable systems has been confirmed qualitatively by most studies. Quantitatively, the convergence of $P(S)$ to a Poisson distribution may be rather slow. For example, numerical calculations for a rectangular billiard have shown that again at a number of levels $\mathcal{N} \sim 10^5$ the behavior of $P(S)$ at $S < 0.1$ contains statistically significant deviations from (3.7) (Ref. 180); these deviations fade away with a further increase in $\mathcal{N}$ and become negligible at $\mathcal{N} \approx 2.5 \cdot 10^6$ (Ref. 186). This slow approach to asymptotic behavior appears to be a specific feature of a rectangular billiard.[226] The significant deviations of $P(S)$ from a Poisson distribution which are observed for certain integrable systems[204] and certain approximately integrable systems[193] can probably be explained on the basis that the corresponding models are only slightly semiclassical.

Not included in the category of systems of a general type are multidimensional harmonic oscillators for which, because of the equidistant nature of the partial-oscillation spectrum, an effective level repulsion arises, with characteristics which depend on the arithmetic properties of the frequency ratio. The nature of $P(S)$ in such systems was studied in Refs. 35 and 198.

Position **A2-b** was advanced by Casati *et al.*[55] This position has been supported by numerical calculations for ergodic systems: a stadium billiard[50-55] and a Sinaĭ billiard.[71,139] The results reveal a similarity between $P(S)$ and a Wigner distribution; particularly good agreement was found in Ref. 139. The distribution $P(S)$ takes this form for pseudointegrable billiards[78] and related billiards[136] (polygons). A strong level repulsion has been observed in many cases for nonlinear oscillators also. The Pullen-Edmonds model—two harmonic oscillators coupled by a fourth-degree potential, (1.14) (Refs. 153, 161, 174, and 203)—and generalizations of this model which include terms of the form $q_1^4$ and $q_2^4$ in the potential (Refs. 184, 200, and 232) have become the most popular models for research on the statistical properties of energy spectra. In some papers, the authors have simply asserted that level repulsion occurs.[160,185,203] We will examine some approaches to a quantitative description of repulsion below.

In several systesm, changes in the parameter values are accompanied by a change in the motion from nearly regular ($\mu \approx 0$) to nearly ergodic ($\mu \approx 1$). During such a change, the form of $P(S)$ should change continuously from a Poisson distribution to a Wigner distribution. Several methods have been proposed for an interpolation of the transition. Chronologically the first was the criterion

**$A2_1$.** The distribution $P(S)$ for a quantum chaotic system can be described by the Brody distribution[20]

$$P_B(\beta;\ S) = AS^{\beta} \exp\left(-BS^{1+\beta}\right), \tag{3.9}$$

where we have $\beta \to 0$ in the limit $\mu \to 0$ and $\beta \approx 1$ at $\mu \approx 1$. #

The constants $A$ and $B$ in (3.9) are determined by normalization conditions(3.6):

$$A = (1+\beta)B, \quad B = \Gamma^{1+\beta}\left(\frac{2+\beta}{1+\beta}\right), \tag{3.10}$$

where $\Gamma(z)$ is the (Euler) gamma function. In the case $\beta = 0$, the Brody distribution is a Poisson distribution, while at $\beta = 1$ it is a Wigner distribution. A Brody distribution was introduced in Refs. 86 and 110 for describing the statistics of level spacings in quantum chaotic systems (Fig. 9). The natural assumption that the functional dependence $\beta(\mu)$ is monotonic agrees with the results of the calculations of Refs. 153 and 205, but it is doubtful that the relationship $\beta(\mu)$ is of universal applicability (Section 8; Fig. 10).

Another interpolation method was proposed in Refs. 135 and 174:

**$A2_2$.** The distribution $P(S)$ for a quantum chaotic system is given by an expression $P_M(S)$ which depends on the values of the measures $\mu_i$ for all $M$ components of the regular and stochastic motions at the given energy . #

Three postulates are used in the derivation of an expression for $P_M(S)$: 1) The level density $\rho(E)$ is a superposition of partial densities $\rho_i(E)$ for all regions of the regular and stochastic motion; i.e.,

$$\rho(E) = \sum_{i=1}^{M} \rho_i(E), \tag{3.11}$$

where $M$ is the number of such regions. Here $\rho_i \sim \mu_i$, where $\mu_i$ is that measure of the phase-space region which corresponds to the given component. 2) The positions of levels belonging to different components $\rho_i(E)$ are not correlated. 3) In the stochastic components, $P(S)$ is given by a Wigner

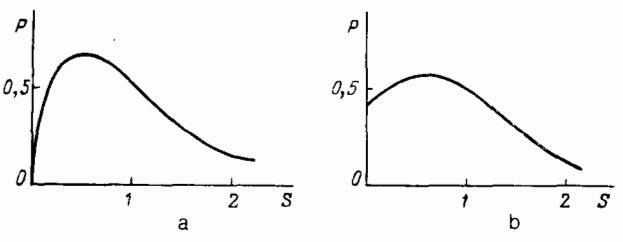


FIG. 9. Interpolation forms of the distribution of level spacings $P(S)$. a—Brody distribution with $\beta = 0.5$; b—Berry-Robnik distribution $P_2$ with $\mu_q = 0.25$.

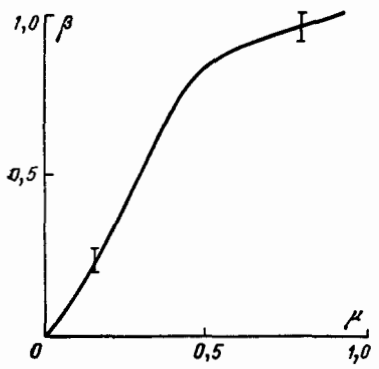


FIG. 10. The Brody parameter $\beta$ as a function of the measure of the stochastic component $\mu$, for the (smoothed) Pullen-Edmonds model.[153,161] The vertical bar shows the size of the error.

distribution, while in the regular components it is given by a Poisson distribution.

These postulates lead to the following expression for $P_M(S)$ under the assumption that there is only a single region of regular motion, with a measure $\mu_r$:

$$P_M(S) = \frac{d^2}{dS^2}\left(e^{-\mu_r S}\prod_{i=2}^{M}\operatorname{erfc}\left(\frac{\sqrt{\pi}}{2}\mu_i S\right)\right), \qquad (3.12)$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \qquad (3.13)$$

Position $\mathbf{A2}_2$ is amenable to a quantitative test. Such tests were carried for various models in Refs. 161, 191, and 200. The results showed that the value $\mu_q$ of the (unique) measure of the stochastic component which is found by fitting $P(S)$ by a distribution $P_2(S)$ differs from the value $\mu_c$ which is found through numerical calculations for a classical system. The difference is particularly large at small values of $\mu_c$ (Fig. 11). Consequently, criterion $\mathbf{A2}_2$ is only semiquantitatively valid for the models which have been studied. The reason for the limited accuracy may be that the systems which have been considered have not been sufficiently semiclassical.[232] Alternatively, the pertinent point may be that in Refs. 161 and 191 $\mu$ was taken to be the resultant measure of the stochastic components, without an analysis of the possibility of a partitioning of the stochastic region into several components.

Attempts have been made to generate a theoretical description of the level-repulsion effect on the basis of an isomorphism between, on the one hand, the behavior of the levels of a system with a Hamiltonian $H = H_0 + \varepsilon V$ upon a change in $\varepsilon$ and, on the other, the time evolution (with a time $t \equiv \varepsilon$) of a one-dimensional gas of classical particles which interact through a time-dependent repulsive potential.[120,163,208,231] Although these investigators have managed, by means of various assumptions, to bring the original problem to a point close to the problem of the thermodynamic equilibrium of a one-dimensional "Coulomb" gas, which is equivalent to the Gaussian orthogonal ensemble,[12] and although they declared on this basis that $P(S)$ takes a Wigner form at sufficiently large values of $\varepsilon$, the procedure which they have used is not very convincing. In Refs. 120 and 163 the properties of the matrix elements of the perturbation $\hat{V}$ were postulated, while in Refs. 208 and 231 they were not specified at all. As a result, the conclusions reached there also apply to integrable systems. So far, there have been no theoretical calculations of $P(S)$ for a specific Hamiltonian which are amenable to a comparison with numerical calculations.

*3.2.2. Spectral rigidity.* A characteristic of the extent of ordering of the levels in a spectrum over a scale large in comparison with the level spacing is the spectral rigidity $\Delta_3(L)$, which is defined below.

For a sequence of levels $\varepsilon_n$ normalized to a unit density ($\varepsilon_n = \varepsilon_{n-1} + S_n$), one introduces a step function $n(\varepsilon)$, which is equal to the number of levels with $\varepsilon_n \leqslant \varepsilon$. By construction, $n(\varepsilon)$ is a ladder with a unit average slope. The function $\Delta_3(x,L)$ is defined as the minimum square deviation of $n(\varepsilon)$ from a straight line on the interval $(x, x+L)$:

$$\Delta_3(x, L) = \frac{1}{L}\min_{A,B}\int_x^{x+L}(n(\varepsilon) - A\varepsilon - B)^2\,d\varepsilon. \qquad (3.14)$$

The value $\langle\Delta_3(x,L)\rangle_x$, averaged over the values of $x$ from a region in which the nature of the fluctuations in the spectrum can be regarded as fixed, depends on $L$ alone and is denoted by $\Delta_3(L)$.

The function $\Delta_3(L)$ describes the ordering of a spectrum over large regions: The slower the increase in $\Delta_3(L)$ with increasing $L$, the less likely it is that the spectrum will contain closely spaced clusters of levels and lacunas with a

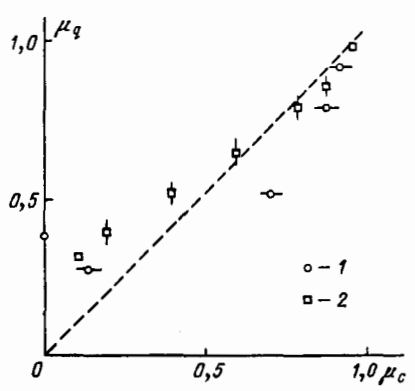


FIG. 11. The parameer $\mu_q$ found from a fit of $P(S)$ by a Berry-Robnik distribution $P_2$ as a function of the measure of the stochastic component, $\mu_c$. 1—For an oval billiard[191]; 2—for a nonlinear oscillator.[153]

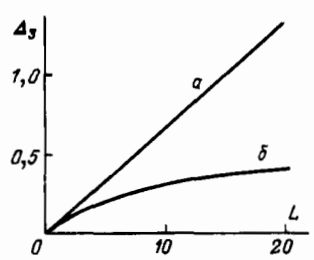


FIG. 12. The spectral rigidity $\Delta_3(L)$. *a*—For a regular system; *b*—for a Hamiltonian belonging to a Gaussian orthogonal ensemble.

reduced level density. For an equidistant sequence of levels we would have $\Delta_3(L) = 1/12$. For randomly distributed uncorrelated levels we would have (Fig. 12)

$$\Delta_3(L) = \frac{L}{15}. \tag{3.15}$$

For a Gaussian orthogonal ensemble, $\Delta_3(L)$ takes the following asymptotic form at $L \gg 1$:

$$\Delta_3(L) \approx \frac{1}{\pi^2} \ln L - 0.00687. \tag{3.16}$$

The use of $\Delta_3(L)$ as a characteristic of a quantum chaotic system was first proposed by Haller *et al.*[110] Here is the corresponding criterion:

**A3.** The spectral rigidity $\Delta_3(L)$: (a) for integrable systems has the form in (3.15), characteristic of a sequence of uncorrelated levels, and (b) for highly stochastic quantum chaotic systems ($\mu \approx 1$) is given by expression (3.16), which corresponds to the matrices of a Gaussian orthogonal ensemble. #

The nature of the onset of dependence (3.15) with increasing $L$ was studied for integrable systems in Refs. 180 and 186. For chaotic systems with $\mu \approx 1$—the Sinaĭ billiard[139] and a nonlinear oscillator[110]—good agreement with expression (3.16) was found.

The transitions between the limiting forms of the spectral rigidity were studied in Refs. 168 and 202. For this purpose, a new method for parametrizing the statistics of the spectrum in intermediate cases was introduced. That new method uses the model of a Gaussian orthogonal ensemble as the starting point. The matrices $\hat{X}$, which belong to a Gaussian orthogonal ensemble and which have the matrix elements $X_{ij}$, are associated with the matrices $\hat{Y}(a,k)$, with the elements

$$Y_{ij} = X_{ij} \exp\left[ -\left( \frac{|i-j|}{a} \right)^k \right], \tag{3.17}$$

where $a$ and $k$ are parameters. In the limit $a \to 0$, the ensemble $\{\hat{Y}\}$ transforms into a set of diagonal matrices with random elements and a Poisson-statistics spectrum. In the limit $a \to \infty$, the ensemble $\{\hat{Y}\}$ is the same as a Gaussian orthogonal ensemble. The calculations of Refs. 168 and 202 show that the values of $a$ found by fitting $\Delta_3(L)$ to the results of the numerical calculations lead to good agreement between the histograms of $P(S)$ and the distribution $P_Y(S)$ calculated for the matrices $\hat{Y}$ with the same value of $a$ (in the cases studied, the values were $3.5 \leqslant a \leqslant 7.5$ and $k = 2$; a variation of $k$ had no significant effect on the results). The results lead to the criterion

**$A2_3$-$A3_1$.** The spectral properties of a quantum chaotic system—the distribution of level spacings, $P(S)$, and the spectral rigidity $\Delta_3(L)$—are the same as the properties of the spectra of an ensemble of matrices $\{\hat{Y}\}$; we have $a \to 0$ in the limit $\mu \to 0$ and $a \to \infty$ at $\mu \approx 1$. #

The specific functional dependence $a(\mu)$ has not been determined for this criterion. Seligman and Verbaarschot[200] have introduced another method for parametrizing the transition of the spectral rigidity between limiting forms, on the basis of the prerequisites of criterion **$A2_2$**. The function $\Delta_3(L)$ found by that method agrees at $L \lesssim 25$ with the results of numerical calculations, with a relative error $\delta\Delta_3 \lesssim 0.05$.

*3.2.3. Correlations in level spacings.* Yet another set of parameters characterizing the structure of an energy spectrum is given by the correlation coefficients $C(n)$ describing the correlation among the sizes of the energy intervals separated by a fixed number of levels:

$$C(n) = \frac{\sum_i (S_{i+n}-1)(S_i-1)}{\left[\sum_i (S_{i+n}-1)^2 \sum_i (S_i-1)^2\right]^{1/2}}. \tag{3.18}$$

The simplest of this set is the quantity $C(1)$: The inequality $C(1) < 0$ means a tendency for neighboring intervals to be systematically larger and smaller than the mean, while $C(1) > 0$ means that the sizes of the neighboring intervals deviate predominantly in a single direction from the mean. For the matrices of the Gaussian orthogonal ensemble we would have $C(1) = -0.271$. Calculations for a quantum chaotic system have yielded $-0.14 \geqslant C(1) \geqslant -0.32$ (Ref. 110) and $C(1) \approx -0.30$ (Ref. 139). Although the absence of correlations has not been studied for integrable systems, by analogy with **A2** and **A3** it is natural to postulate the criterion

**A4.** The correlation coefficient of neighboring level spacings, $C(1)$, (a) is zero for integrable systems and (b) has the same value as for the spectra of a Gaussian orthogonal ensemble for highly stochastic ($\mu \approx 1$) quantum chaotic systems: $C(1) = -0.271$. #

There has been no study of the nature of the transition of $C(1)$ betwen limiting values.

Hirooka *et al.*[156] have proposed using a plot $y(x)$ which images points with coordinates $x_n = \Delta_n, y_n = \Delta_{n+1}$, where $\Delta_n = E_{n+1} - E_n$, to analyze the structure of a spectrum.[3)] The following has been proposed:

**$A4_1$.** A plot of the functional dependence $\Delta_{n+1}(\Delta_n)$ is a regular array of points for the spectra of integrable systems and a random distribution for the spectra of quantum chaotic systems. #

This approach was discredited in Refs. 134 and 147, where various examples were used to demonstrate that the plot is disordered for completely integrable systems. We would note, however, that a plot of this sort (with an appropriate normalization) contains all the information which is used in criterion **A4**.

*3.2.4. Systems which are not invariant under time reversal.* All the examples discussed in Subsections 3.2.1–3 have referred to the case in which the Hamiltonian is invariant under time reversal. Systems which do not have this invariance are associated in statistical spectrum theory with a Gaussian unitary ensemble of Hamiltonians: an ensemble of Hermitian matrices with independent elements whose statistical properties remain invariant under arbitrary unitary transformations. The distribution of level spacings for a Gaussian unitary ensemble, $P_U(S)$, is given (approximately) by

$$P_U(S) \approx \frac{32}{\pi^2} S^2 \exp\left( -\frac{4}{\pi} S^2 \right), \tag{3.19}$$

and the spectral rigidity at $L \gg 1$ has the asymptotic form

$$\Delta_3(L) \approx \frac{1}{2\pi^2} \ln L + 0.058. \tag{3.20}$$

It was shown in Ref. 201 for a system with the Hamiltonian

$$H = \frac{1}{2}(p_1 - aq_2^3)^2 + \frac{1}{2}(p_2 + aq_1^3)^2 + \alpha_1 q_1^6 + \alpha_2 q_2^6 - \alpha_{12}(q_1 - q_2)^6 \tag{3.21}$$

that the statistics of a spectrum can have the properties of both a Gaussian orthogonal ensemble (in the case $\alpha_{12}=0$, in which the Hamiltonian can be written in the form of real symmetric matrices) and a Gaussian unitary ensemble (in the case $\alpha_{12}\neq 0$) in the region of parameter values corresponding to a highly stochastic motion ($\mu = 0.98 \pm 0.02$).

The conditions for the transformation of the statistics of a spectrum from a Gaussian orthogonal ensemble to a Gaussian unitary ensemble upon a violation of the invariance of a Hamiltonian under time reversal were studied in Ref. 210 for a highly stochastic billiard penetrated by a filament carrying a magnetic flux $\Phi$. That model has the advantage that the classical dynamics is independent of the magnitude of the flux, while the spectrum and wave functions of the quantum system depend on $\Phi$ (the Aharonov-Bohm effect). It has been shown that the statistics of the low-lying levels with indices $n \ll n_0 = 0.44(e\Phi/\hbar c)^{-4}$ ($e$ is the charge of an electron, and $c$ is the velocity of light) preserve the nature of the Gaussian orthogonal ensemble but transform into the statistics of a Gaussian unitary ensemble with a further increase in $n$.

The types of statistics of a spectrum when a system has composite symmetries which include the operation of time reversal as one of the elements were studied in Ref. 224.

## 4. PROPERTIES OF THE WAVE FUNCTIONS OF STATIONARY STATES

It was shown in the preceding section that the properties of the spectra of the Hamiltonians of quantum chaotic systems are similar to the spectral properties of certain classes of random matrices. Accordingly, again in the case of the wave functions[4)] of the stationary states of these systems we would naturally expect to see manifestations of randomness properties—in contrast with the regular properties of the wave functions of integrable systems. The choice of the wave functions of a stationary state as a subject of study predetermines the choice of the first of the alternatives in Subsection 2.1: The concept of a quantum chaotic nature applies to a single state.

In contrast with a spectrum, the form of a wave function depends on the basis in which it is determined. The following two representations are the ones of most importance for the quantum chaos problem:

1. In many problems the Hamiltonian is of the form $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is the Hamiltonian of an integrable system. For such systems it is natural to use the (countable) basis $\{\varphi\}$—the complete (orthonormal) system of eigenfunctions of the Hamiltonian $\hat{H}_0$—and to study the properties of the coefficients $a_{nm}$ in the expansion of $\psi$, which is a normalized eignfunction of the Hamiltonian $\hat{H}$:

$$\psi_n = \sum_m a_{nm}\varphi_m. \tag{4.1}$$

We call this form of a wave function the "$H_0$ representation."

2. The second is the coordinate representation, since in it the behavior of $\psi(\mathbf{q})$ can be clearly compared with the picture of the orbits of the classical motion of a particle in coordinate space.[157]

In addition to these representations, the quantum chaos problem uses a description of states by means of a Wigner function[1,104] $W(\mathbf{p},\mathbf{q})$. A Wigner function has several properties in common with the classical distribution function $W_C(\mathbf{p},\mathbf{q})$ in phase space. This similarity between $W$ and $W_C$ was pointed out in Ref. 34 and verified by numerical calculations in Refs. 59 and 229. In the limit $\hbar \to 0$, this similarity follows in a natural way from the correspondence principle. In particular, it has been proved[190] that in the limit $\hbar \to 0$ the eigenfunctions of a Hamiltonian are localized in phase-space regions which contain invariant sets: surfaces of regular motion or stochastic omponents.

### 4.1. Properties of wave functions in the $H_0$ representation

The properties of the wave functions of quantum chaotic systems in the $H_0$ representation were first studied by Nordholm and Rice.[24] For several nonlinear oscillators (including the Henon-Heiles model), the coefficients in the expansion of the wave functions of the stationary states $\psi_n(q_1,q_2)$ in the basis $|kl\rangle = \varphi_k(q_1)\varphi_l(q_2)$ were found numerically; here $\varphi_m(q)$ is the wave function of level $m$ of a harmonic oscillator:

$$\psi_n(q_1, q_2) = \sum a_{n;\,kl}\varphi_k(q_1)\varphi_l(q_2). \tag{4.2}$$

Used as a characteristic of the properties of the set of coefficients $\{a\}$ was the degree of *distributedness* of $\psi$ among the various basis functions, ranked qualitatively from locality (only one of the coefficients $a$ is large) to globality (all the coefficients $a$ from the region of quantum numbers $k + l = \text{const}$ are identical in order of magnitude) (Fig. 13). The calculations made it possible to formulate criterion

**B1**. As the behavior of the classical motion becomes increasingly stochastic there is an increase, on the average, in the degree of distributedness of the wave functions of the corresponding quantum chaotic system. #

Parameters which quantitatively determine the degree of distributedness of wave functions were not introduced in Ref. 24. Such parameters can be taken from the theory of disordered systems in solid state physics, e.g., the participation ratio[15]

$$P_n = \sum_m |a_{nm}|^4 \tag{4.3}$$

or the entropy[64]

$$S_n = -\sum_m |a_{nm}|^2 \ln |a_{nm}|^2. \tag{4.4}$$

A similar approach—describing the degree of distributedness of $\psi$ by means of an entropy calculated in a basis of coherent states—was taken in Ref. 133. The complexity $C_n$ of the state $\psi_n$—the effective number of basis states making up $\psi_n$—can be determined from $C_n = P_n^{-1}$ or $C_n = \exp S_n$. Although a large value of $C$ does not give us a sufficient

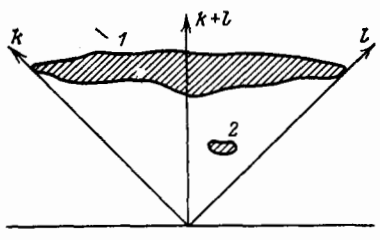


FIG. 13. Diagram used in determining the distributedness of the wave function $\psi$. The hatching shows the region of the quantum numbers $k$ and $l$ which contribute substantially to $\psi$. 1—Global state; 2—local state.

condition for a quantum chaotic nature, it is an important characteristic of $\psi$ in the given basis.

One approach to the study of the degree of distributedness of the eigenfunctions of a system with a Hamiltonian

$$\hat{H} = \hat{H}_1(p_1, q_1) + \hat{H}_2(p_2, q_2) + \hat{V}(q_1, q_2) \tag{4.5}$$

involves a conversion of stationary states of $\hat{H}_1$ into unstable quasistationary states and a determination of how the decay rate ($\Gamma$) of state $\psi_n$ depends on the energy $E_n$. If $\Gamma(E)$ calculated for quasistationary states of $H$ is a monotonic function, this circumstance is taken as an indication of the globality of the states in the region under consideration (Fig. 14). On the contrary, a significant nonmonotonic behavior of $\Gamma(E)$ would indicate a pronounced difference among the states in terms of the degree to which the energy is concentrated in a given partial oscillator (mode selectivity).

Calculations of the $\Gamma(E)$ dependence for various models[63,77] have shown that a change in the coupling constant describing the coupling of oscillators is accompanied by a transition from a local to a global behavior of the wave functions, simultaneously throughout the entire energy region. The existence of a region of a transition of the system to a significant classical stochastic behavior and the position of this region play no role in the behavior of $\Gamma(E)$. A similar approach was taken in Ref. 101, where a study was made of a Henon-Heiles system with a term added to the Hamiltonian to describe irreversible transitions into the continuum under the influence of a constant perturbation. The main conclusion drawn from the numerical calculations was that the decay time $\Gamma^{-1}$ increases with increasing $E$ in an approximately exponential manner, exhibiting no anomalies in the transition to a significant stochastic behavior near $\varepsilon = 0.6$. Further studies showed that the decay rate of local states may be either less than or greater than the decay rate of global states, depending on the choice of $\hat{H}_0$ (Ref. 105). The dependence $\Gamma(E)$ is thus not very convenient for determining the degree of distributedness of $\psi$.

The conclusion drawn from this research was that there is no causal relationship between stochastic behavior and a distributedness of the wave functions in the $\hat{H}_0$ representation: globality serves as a necessary but not sufficient condition for quantum chaotic behavior. This result is understandable: The globality of the wave functions is evidence of nothing more than the circumstance that the eigenfunctions of $\hat{H}$ and $\hat{H}_0$ are not similar, and this situation can be arranged even for integrable systems. As an example we might take $\hat{H}$ and $\hat{H}_0$ to be the Hamiltonians of linear oscillators

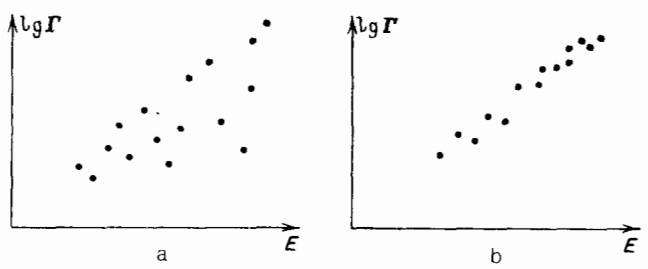


FIG. 14. Sketch of the decay rate of a quasistationary state, $\Gamma$, as a function of the energy $E$. a—With mode specificity (the eigenfunctions of $H$ differ markedly in composition); b—without mode specificity (all the states are global).

which differ in frequency, the direction of the normal oscillations, and equilibrium position (shifted oscillators).

A criterion which is close to **B1** was proposed by Stratt *et al.*[51] That criterion uses the distributedness of $\psi$ in the basis of natural orbitals—with the functions $\psi_m(q_i)$ in expansion (4.2) chosen in such a way that the expansion converges extremely rapidly. This requirement leads to the diagonal representation

$$\psi(q_1, q_2) = \sum_m w_m^{1/2} \theta_m(q_1) \varphi_m(q_2), \tag{4.6}$$

for which the following position is established:

**B1$_1$.** In an expansion of the eigenfunction $\psi$ in natural orbitals (4.6), the sequence $w_m$ becomes convergent more slowly in the transition from a regular motion to a stochastic motion of the classical system; $C(\psi)$ increases. #

Criterion **B1$_1$** is supported by numerical calculations. This approach overcomes the counterexample of shifted oscillators. The sequence $w_m$ is not universal, however; it depends on the variables $q_i$ in terms of which the system is described. A transformation from, say, Cartesian coordinates to polar coordinates alters $w_m$ and thus the extent of the quantum chaotic nature of the given state.

It has been proposed[86] that what is important for quantum chaos is not the fact that a large number of coefficients $a_{nm}$ differ from zero, but the pseudorandum behavior of these coefficients; more specifically,

**B2.** For the wave function $\psi_n$ of a chaotic state, the coefficients $a_{nm}$ are random quantities, with a normal distribution. #

This assertion was made by Buch *et al.*,[86] who carried out a numerical study of the statistical behavior of $a_{nm}$ for a nonlinear oscillator. It was found that there is satisfactory agreement between their distribution $W(a)$ and a normal distribution under conditions such that the classical motion is approximately ergodic.

Again, an approximate agreement of the distribution $W(a)$ with a Gaussian distribution does not qualify as a sufficient condition for quantum chaos, since this approximate agreement is also possible for integrable systems (for example, let $H_0$ and $H$ be the Hamiltonians of one-dimensional linear oscillators with shifted equilibrium positions). However, there evidently is a pseudorandomness of $a_{nm}$ in the region of pronounced stochastic behavior.

It was shown in Ref. 182 that the values calculated for the maximum coefficients $w_1 = \max(|a_{nm}|^2)$ for a group of close-lying states at the known value of $C$, under the assumption that the random coefficients $a_{nm}$ have a normal distribution, agree with the results of quantitative calculations for a single series of levels of a complex system (the $J^\pi = 1^+$ series of the neutral Ce atom). In this approach, one studies the properties of a group of states which form an ensemble in which the distribution $w_1$ is calculated, so one chooses the second of the alternatives in Subsection 2.1. Unfortunately, this quantum system has no simple classical analog.

The question of transforming from regular to stochastic motion of a quantum system in order to determine the conditions for "quantum integrability" has been studied[96,112,158] for Hamiltonians of the form $H = H_0 + V$. Those authors believe that a criterion for chaotic nature can be based on the square of the maximum scalar product of the unperturbed and perturbed wave functions: $\eta_n = \max|\langle\psi_n|\varphi_m\rangle|^2$.

**B3**. For regular states of a quantum chaotic system we have $\eta_n > 1/2$, while for chaotic states we have $\eta_n < 1/2$. #

Criterion **B3** essentially establishes a quantitative boundary between local and global states. The satisfaction of the condition

$$\eta_n > \frac{1}{2} \tag{4.7}$$

makes it possible unambiguously to assign $N$ quantum numbers to the state of the perturbed system; it also ensures the convergence of iterative solutions of a Brillouin-Wigner perturbation theory for $\psi$ and $E$ (Ref. 113).

The threshold involved in the appearance of quantum chaos for a given state which was postulated in **B3** seems doubtful: It was mentioned in Ref. 118 that as $\eta$ varies from 0.51 to 0.49 there are no qualitative changes in the physical properties in system $H$. Criterion **B3** looks inadequate for two more reasons: 1) Condition (4.7) may be violated for many states, while the system will remain integrable (an example is provided by the same oscillators, $H_0$ and $H$, with shifted equilibrium positions); 2) condition (4.7) may also be satisfied by chance in a case in which the classical system is highly stochastic.[112] Even for random $a_{nm}$, with $\psi_n$ which are not very complex ($C_n \leqslant 10$), there is a significant probability for the satisfaction of condition (4.7).

The nature of the relationship between the coefficients $a_{nm}$ describing the wave functions in the $H_0$ representation and the parameters of the classical stochastic situation thus has not been established. An increase in the complexity of $C$ is a necessary but not sufficient condition for classifying a state as chaotic; it has not been found possible to introduce an objective definition of the degree of randomness of a set of numbers $\{a_{nm}\}$.

### 4.2 Properties of wave functions in the coordinate representation

*4.2.1. Integrated properties of wave functions.* The difference between the properties of regular and stochastic states for wave functions in the coordinate representation, $\psi(\mathbf{q})$, was first mentioned by Berry[33] on the basis of a semiclassical approach. An analysis of a regular quantum state as an analog of classical motion on an $N$-dimensional torus led to the conclusion that there is a singular (in the limit $\hbar \to 0$) behavior of $\psi(\mathbf{q})$ near caustics—boundaries of the region of classical motion in coordinate space. According to Ref. 107, the integrals

$$I_m = \int |\psi(\mathbf{q})|^{2m} d\mathbf{q} \tag{4.8}$$

($m$ is an integer) increse without bound in the limit $\hbar \to 0$ for regular states for $m \geqslant 2$. In contrast, for chaotic states under conditions corresponding to an ergodic classical motion, in which the classical density $W_C(\mathbf{p},\mathbf{q})$ uniformly fills an energy surface in phase space ($W_C(\mathbf{p},\mathbf{q}) \sim \delta[E - H(\mathbf{p},\mathbf{q})]$), the probability density $\rho(\mathbf{q}) = |\psi(\mathbf{q})|^2$ vanishes near the boundary of the classically accessible region in the limit $\hbar \to 0$ (in the case $N > 2$; it is a constant in the case $N = 2$). Integrals (4.8) remain finite (Fig. 15).

**B4**. Near the boundaries of the classically accessible region, the probability density exhibits the behavior $\rho(\mathbf{q}) \to \infty$ for regular states and remains bounded for highly chaotic states. #

This criterion refers to the case $\hbar \to 0$. It is thus difficult to verify numerically, and no such attempts have been made.

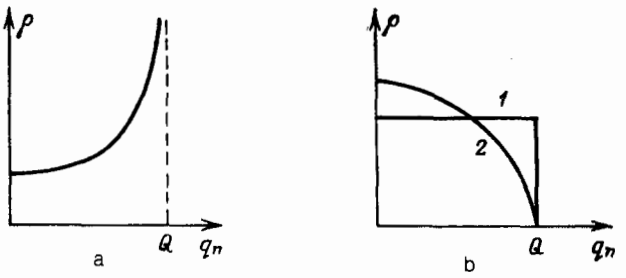


FIG. 15. The behavior $\rho(\mathbf{q})$ near the boundary $Q$ of the classically accessible region. Here $q_n$ is the distance along the normal to the boundary. a: For a regular state. b: For an ergodic state. 1—$N = 2$; 2—$N = 3$.

On the basis of a qualitative intepretation of the wave functions of a chaotic state as a superposition of a large number of plane waves with uncorrelated phases, Berry[33] hypothesized that $\psi(\mathbf{q})$ in such states was a random function of $\mathbf{q}$ with a Gaussian distribution of $\psi$. There is a similarity between this suggestion and criterion **B2**.

An attempt to test this assertion was made by Shapiro and Goelman,[169] who calculated the distribution function of values of the wave function, $W(\psi)$, for a stationary state $\psi_n$ of a stadium billiard ($n \sim 10^3$). It turns out to be bell-shaped with a maximum near $\psi = 0$, providing a qualitative verification of Berry's hypothesis. Information about the shape of $W(\psi)$, however, cannot yet be used as a criterion for quantum chaos, since the form of this distribution for the wave functions of integrable systems is not known.

*4.2.2. Local properties of wave functions.* A criterion based on the topography of a $\psi(\mathbf{q})$ contour map, primarily of the nodal lines, on which the condition $\psi(\mathbf{q}) = 0$ holds, has become quite popular because of the clarity of representation.[5)] The nodal lines of the wave functions of a two-dimensional ergodic system (a stadium billiard) were first calculated by McDonald and Kaufman,[50] who pointed out an irregularity in the directions of the nodal lines (Fig. 16). The corresponding criterion is

**B5**. For regular states $\psi_n$, the system of nodal lines $\psi_n(\mathbf{q})$ is a lattice of quasiorthogonal curves (or approximately so—having narrow avoided crossings of nodal lines), while for a chaotic state such a representation is not allowed. #

This criterion was proposed by Stratt *et al.*[51] and illustrated by examples for nonlinear oscillators. The properties of the map of nodal lines of a chaotic state are described as a random behavior of lines of nodes.

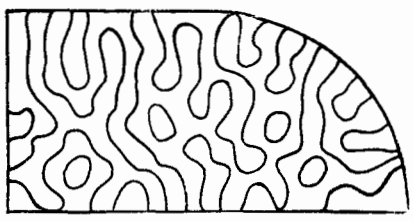

FIG. 16. Nodal lines for a single quadrant of the eigenfunction $\psi_n(\mathbf{q})$ ($n \sim 600$) in a stadium billiard.[50]

Criterion **B5** does not hold for all systems. Regularity of nodal lines is not an exclusive property of regular systems. Even for a stadium billiard, in which the classical motion is ergodic, calculations[126,170] have revealed a regular pattern of $\psi_n(\mathbf{q})$ contour lines. The reasons for this regularity were studied in Ref. 175. Another type of regular characteristics $\psi_n(\mathbf{q})$ of this system is the category of "scars," which are regions of an anomalously high probability density along the contours of closed classical paths. This type was observed in Ref. 154. For a classical ergodic system also $\psi(\mathbf{q})$ may thus have regular features in the quantum case.

On the other hand, the complex nodal structure of the wave functions may result from something other than a stochastic behavior. In particular, some of the states which were classified as chaotic in Ref. 51 were interpreted in Ref. 145 as corresponding to a quasiperiodic classical motion.

The pattern of nodal lines is a form of representation of the properties of $\psi(\mathbf{q})$; its relationship with other characteristics of a system has not received much study. We might note, however, that Korsch[116] has established a relationship between the change in structure of nodal lines upon a change in a parameter in a Hamiltonian with an avoided crossing of levels. The sole advantage of that picture is its clarity, and then only in the case of two-dimensional configuration space. The situation with regard to lines of force in electrostatics is similar.[6)] It therefore appears less promising to develop criteria based on the properties of nodal lines.

The common deficiencies of all the criteria based on a study of wave functions are obvious. In the first place, $\psi$ is not an observable, so such criteria cannot be applied to experimental data (although they could be used in numerical calculations). Second, there is some arbitrariness in the choice of representation. Third, a quantitative measure of the extent to which a given eigenfunction is chaotic has yet to be established for any of the criteria which we have presented. The advantage of clarity does not make up for these disadvantages.

## 5. PROPERTIES OF OPERATORS OTHER THAN HAMILTONIANS

A criterion based on the behavior of the magnetic moments of operators other than $\hat{H}$ follows from the absence of selection rules for operators in a quantum chaotic system, which was pointed out by Percival.[21]

**C1**. If the motion of a classical system in a given energy region is regular (quasiperiodic), the matrix elements $A_{mn}$ of the "good" operator $\hat{A}$, calculated between the eigenfunctions $\psi_m$ and $\psi_n$ of states with energies belonging to this region, and zero—except for a few combinations of $m$ and $n$. If the classical motion is highly stochastic, then we have $A_{mn}=0$ for nearly all states, and the values of $A_{mn}$ vary randomly with $m$ and $n$. #

Criterion **C1** is a direct corollary of **B2** (if $A$ is the operator which projects onto one of the states of the basis $\{\varphi\}$, then **C1** is the same as **B2**). However, **C1** is more closely related to experimental data. The concept of a good operator includes, for example, polynomials of $p_i$ and $q_i$, in particular, the dipole moment $\mathbf{d}=e\mathbf{q}$. The intensities of lines in the emission spectrum can be used to learn about the magnitudes of the dipole-moment matrix elements. At the transition from the regular motion to stochastic motion, many lines, with an irregularly varying intensity, appear in the emission spectrum. This conclusion was supported by numerical calculations[51] for a nonlinear oscillator (Fig. 17).

One of the corollaries of **C1** which were found in Ref. 165 is the conclusion that the temporal fluctuations of the expectation values $\langle A(t)\rangle$ of operators which do not commute with $\hat{H}$ are small in the stochastic region. Peres[164] traced the transition from a regular behavior to a random behavior of the diagonal matrix elements of a simple operator as the degree of stochastic nature of the classical motion was varied. Similar properties have been found for the Green's function $G_{mn}$, where $m$ and $n$ are the indices of the basis functions.[188] The distribution of the matrix elements, $W(A)$, was studied in Ref. 148. In agreement with **C1**, this distribution has a sharp peak at $A=0$ in the region of regular motion. This peak spreads out as the system goes into the region of stochastic motion.

It was suggested in Ref. 115 that the inequalities $|A_{mn}|\ll|A_{mm}|, |A_{nn}|$ hold for chaotic states $\psi_m$ and $\psi_n$ which are nearly equal in energy and for nearly any operators $A$. This suggestion has yet to be verified.

Criterion **C1** is so far only qualitative, since parameter values which would make it possible to describe the extent to which a given set of matrix elements are chaotic have not yet been established.

## 6. PROPERTIES OF NONSTATIONARY STATES

Important information on the properties of quantum chaotic states emerges from a study of the evolution of nonstationary states: wave packets. A wave packet localized in a small region of phase space ($\Delta p_i \Delta q_i \sim \hbar$) is a close analog of a point in phase space which describes the state of a system in the classical theory.

In a classical system, a continuous power spectrum of the dynamic variables arises in the transition from a regular motion to a stochastic one. This continuous spectrum causes a damping of correlations. This is not the case for a quantum system with a discrete energy spectrum. Any state $\Psi(0)$ in such a system exhibits a quasiperiodic behavior[2]: For any small $\delta$, one can find a return time (recurrence time) $t_\Psi(\delta)$ such that at the time $t_\Psi$ the system returns to its original state within an error better than $\delta$:

$$1-|\langle\Psi(t_\Psi)|\Psi(0)\rangle|<\delta. \tag{6.1}$$

It is true that this quasiperiodicity is largely formal in nature. The return times become exceedingly long for complex

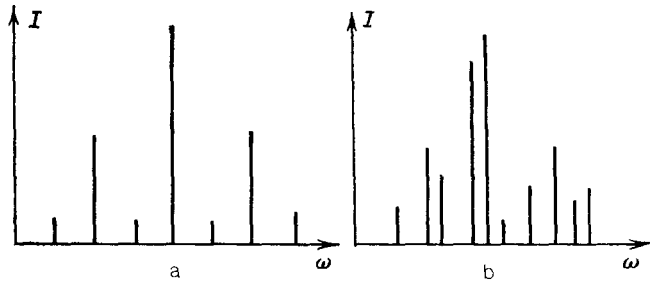


FIG. 17. The emission spectrum $I(\omega)$ of quantum chaotic systems in (a) regular and (b) chaotic states.

packets. One estimate was given in Ref. 99 for a packet of the type

$$\Psi(t)=\frac{1}{C^{1/2}}\sum_{n=1}^{C}\psi_n(q)\,e^{-i\omega_n t}, \tag{6.2}$$

which consists of $C$ stationary states $\Psi_n$ with equal weights. In this case the return time is, in order of magnitude,

$$t_\Psi(\delta)\sim(\bar{\omega})^{-1}\delta^{1/2}\left(\frac{\gamma}{\delta}\right)^{C/2}, \tag{6.3}$$

where

$$\bar{\omega}=(C^{-1}\sum_n\omega_n^2)^{1/2}$$

is the mean square frequency, and $\gamma$ is a constant of the order of unity [$\gamma=2\pi\exp(-1)=2.31$]. For the example with $\delta=0.1$, $C=50$ and $\bar{\omega}=10^{15}\,\mathrm{s}^{-1}$, the return time turns out to be $t_\Psi(\delta)\sim10^{18}$—longer than the age of the universe. Similar estimates were found in Refs. 111 and 212.

The only return times which are of interest in practice are those which lie inside the region of applicability of the model of a Hamiltonian system with few degrees of freedom, with $t_\Psi\lesssim\tau$, where $\tau$ is the relaxation time. This condition imposes a restriction on the values of $C$ for which recurrences can be observed. Specifically, these values must be a few units. Under the conditions of the numerical example given above, we would have $C\leqslant1.46\,\mathrm{l\,g}(\bar{\omega}\tau)$. A packet consisting of a sufficiently large number of stationary states may thus evolve in an essentially irreversible way over time intervals of physical interest.

A consequence of the quasiperiodic nature of the evolution of states for systems with a discrete spectrum is quasiperiodicity of the entropy[5] and of coarse observables.[219] However, no estimates of the recurrence times $t_A(\delta)$ of the values of the observables $\hat{A}(\mathbf{p},\mathbf{q})$ are available. On the basis of general considerations, we would expect that the times $t_A(\delta)$ might be much shorter than the return times for the state, $t_\Psi(\delta)$.

The quantum chaos criteria can be based on the properties of the motion of wave packets over time intervals small in comparison with $t_R$. Packets which occupy a small volume in phase space are made up in the case $\hbar\ll1$ of many eigenfunctions of the Hamiltonian- $C\gg1$. Such criteria accordingly use the second of the alternatives in Subsection 2.1—that which assigns quantum chaos to a group of states.

### 6.1. Autocorrelation function of a wave packet

Brumer and Shapiro[54] have pointed that a packet autocorrelation function

$$P(t)=|\langle\Psi(t)\,|\,\Psi(0)\rangle|^2 \tag{6.4}$$

can be used to establish a criterion for quantum chaos. The behavior of $P(t)$ has been studied in most detail for a Gaussian wave packet: a state with the wave functions

$$\Psi(q_1,q_2,0)=\frac{1}{\sqrt{\pi}}\prod_{i=1}^{2}\exp\left[-\frac{1}{2\hbar}(q_i-\bar{q}_i)^2+\frac{i}{\hbar}\bar{p}_iq_i\right]. \tag{6.5}$$

We will call the phase-space point with the coordinates ($\bar{p}_1$, $\bar{p}_2$, $\bar{q}_1$, $\bar{q}_2$) the "center of the packet." The form of the wave functions in (6.5) was not chosen at random. In the first place, such a packet can be prepared experimentally (e.g., in molecules, in the course of electronic transitions from a vibrational ground state[92]). Second, it is specifically the form (6.5), which is responsible for the minimum in the product $\Delta p_i\Delta q_i$. The autocorrelation function of a Gaussian packet was studied in Ref. 56 for a Henon-Heiles model with $\hbar^{-1}=80$. It turned out that even if the initial packet had comparatively large phase-space dimensions the behavior of the autocorrelation function would differ sharply depending on whether the center of the packet belonged to a regular or a stochastic region (Fig. 18).

**D1**. The autocorrelation function $P(t)$ of a Gaussian wave packet (a) exhibits pronounced fluctuations of a damped-recurrence type if the center of the packet belongs to a region dominated by regular paths or (b) has small, irregular fluctuations if the center belongs to a stochastic component. #

These conclusions were confirmed in Refs. 74 and 102. A qualitative analysis revealed how the functional dependence $P(t)$ in the regular region depends on the ratio of frequencies of the quasiperiodic motion for paths on which the center of the packet lies. The difference in the time evolutions of Gaussian packets in a quantum system and their classical analogs—Gaussian density distributions in phase space—was studied in Ref. 144.

Completely different results are found in the initial shape of the packet is specified by an expansion

$$\Psi(\mathbf{q},0)=\sum a_n\psi_n(\mathbf{q}), \tag{6.6}$$

where $\psi_n$ are the eigenfunctions of the Hamiltonian $\hat{H}$, and the $a_n$ are a set of coefficients which either depend on $E$ in a regular way[54,100] or are random.[98] This case, however, is of little interest from the physical standpoint. First, as was shown in Ref. 93, for a quantum chaotic system a packet of

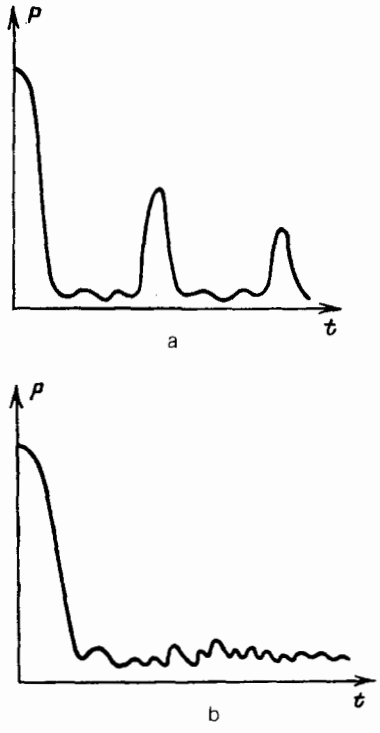


FIG. 18. Behavior of the autocorrelation function of the wave packet, $P(t)$. a—The center of the packet belongs to a regular region; b—the center of the packet belongs to a stochastic region.

the type in (6.6) with a regular envelope $a_n = a(E_n)$ has, at values of $E$ in the region a pronounced stochastic behavior, an extremely complex, pseudorandom density distribution in configuration space $\rho(\mathbf{q}) = |\psi(\mathbf{q})|^2$. In a sense, it is already in a chaotic state. Second, no method is known for preparing states of the type in (6.6) experimentally.

### 6.2. Transition of a packet into another state

Additional information about the properties of quantum chaotic states can be found by examining the evolution of the overlap of a Gaussian packet with states $\Phi$ which differ from $\Psi(0)$. One aspect of this problem was first taken up in Ref. 48, where it was shown that the time average of the overlap of states $\Phi$ and $\Psi$,

$$P(\Psi \mid \Phi) = \lim_{t\to\infty} \left( \frac{1}{t} \int_0^t | \langle \Psi(t') \mid \Phi(0) \rangle |^2 \, dt' \right), \tag{6.7}$$

will differ depending on whether $\Phi$ is taken to be the initial state of the packet, $\Psi(0)$, or the function $\Psi' = \hat{R}\Psi(0)$, which is found from $\Psi(0)$ by using one of the symmetry operations $\hat{R}$ which leave the Hamiltonian invariant:

$$P(\Psi \mid \Psi) > P(\Psi \mid \hat{R}\Psi). \tag{6.8}$$

Inequality (6.8) means that there is no ergodic behavior (equality of time averages and averages over phase space) for the motion of a packet in a quantum system with symmetry also in the case in which the classical system is ergodic. The time evolution of the transitions of packet (6.5) between given regions of phase space was also studied in Ref. 60.

Stationary states $\psi_n(\mathbf{q})$ were studied in the role of $\Phi$ in Refs. 57, 58, and 92. In this case the overlap probabilities

$$p_n = | \langle \Psi(\mathbf{q}, t) \mid \psi_n(\mathbf{q}) \rangle |^2 \tag{6.9}$$

will be constant over time (Fig. 19). The following criterion has been proposed:

**D2**. For a Gaussian wave packet, the overlap coefficients $p_n$ exhibit the following behavior as $n$ varies: a) They vary markedly and nonmonotonically if the packet belongs to a region of regular motion; b) they vary only slightly, and the variation is monotonic, if the packet belongs to a region of stochastic motion. #

Criterion **D2** is similar to **D1**, since it presents information on the behavior of the autocorrelation function $P(t)$ expressed in spectral language:

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(t) e^{i\omega t} \, dt = \sum p_n \delta(\omega - \omega_n). \tag{6.10}$$

The spectrum $S(\omega)$, which contains several strong lines, leads to pronounced recurrences of $P(t)$, while for a dense spectrum of a random motion the fluctuations in $P(t)$ are small.

As a measure of the complexity of the sequence $p_n$, the following entropy has been proposed[92,171]:

$$S = -\sum_n p_n \ln p_n.$$

The quantity $S$, however, tells us about the complexity of the packet, not about the behavior of $p_n$ as a function of $n$. Furthermore, the assumption that $p_n$ varies slowly does not agree well with the results of Refs. 183 and 214, where the specificity of the composition of a Gaussian packet in an ergodic quantum chaotic system (in a stadium billiard) was established. It was shown in those studies that a packet which executes a motion along the normal to rectilinear sections of a boundary at small values of $t$ consists primarily of the wave functions $\psi_n(\mathbf{q})$ of a pseudoregular stucture (Subsection 4.2). On the whole, criterion **D2** does not seem to have a really convincing foundation, and it has no apparent advantages over **D1**.

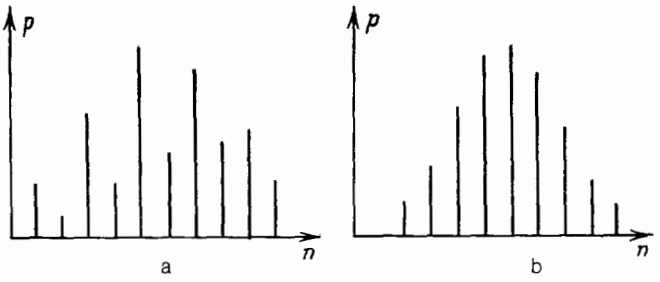


FIG. 19. Presumed form of the overlap coefficients $p_n$ for a Gaussian packet in (a) a regular region and (b) a stochastic region.

A different approach to a description of the transition of a Gaussian packet into other states was proposed in Ref. 117. For a state whose initial form is (6.5), the parameter

$$\Omega(t) = \prod_{i=1}^{2} \Delta q_i(t) \, \Delta p_i(t), \tag{6.11}$$

is calculated, where $\Delta z(t)$ is the mean square deviation of $\hat{\mathbf{z}}$ from its expectation value at the time $t$. The quantity $\Omega$ may be thought of as a coarse phase volume occupied by a Gaussian packet at the time $t$. Here is the corresponding criterion:

**D3**. The coarse phase volume $\Omega(t)$ increases much more rapidly in a stochastic region than in a regular region. #

The numerical calculations carried out in Ref. 117 support **D3** for the Henon-Heiles model on a small time interval ($t \leqslant 30$, i.e., about five periods).

### 6.3. Stability of the evolution of packets

The evolution of wave packets consisting of eigenfunctions of a Hamiltonian $H$ was studied above. How the nature of this evolution changes when a small perturbation is imposed on the system was studied in Ref. 166, where a system with a Hamiltonian $H' = H + V$ was studied along with $H$. Used as a parameter was the quantity

$$R_V(t) = | \langle \Psi_H(t) \mid \Psi_{H'}(t) \rangle |^2, \tag{6.12}$$

where $\Psi_H(t)$ and $\Psi_{H'}(t)$ are solutions of Schrödinger equations with the respective Hamiltonians $H$ and $H'$ and with an identical initial condition $\Psi(0)$. The assumption that the behavior of the quantity $R_V(t)$ is different in regular and stochastic cases is based on the circumstance that a perturbation $V$ mixes the wave functions of many states in the initial packet, since there are no selection rules in a quantum chaotic system. Being richer in spectral composition, $R_V(t)$ will fluctuate less.

**D4**. For a given $V$, the expectation value $\langle R(t) \rangle$ will be greater, and the fluctuations in $R(t)$ more pronounced, for a state which belongs to a regular region than for a state which belongs to a stochastic region. #

This criterion has been verified at a qualitative level by calculations for a model of rotators with nonlinear coupling.

Analysis of the evolution of wave packets is a method which is sensitive enough to establish not only the difference in the properties of quantum regular systems and quantum chaotic systems but also the structure of different regions in the phase space of a quantum chaotic system in a narrow energy interval. The information obtained by this approach complements the concept (Section 4) of the structure of the wave functions of stationary states of a quantum chaotic system. On the other hand, a study of nonstationary states also involves a second aspect of the quantum chaos problem: comparing the dynamics of classical and quantum systems with identical Hamiltonians.

## 7. QUANTUM CHAOS IN NONAUTONOMOUS SYSTEMS

Nonautonomous systems with a single degree of freedom and a periodic perturbation

$$H = H_0(I) + V(I, \theta) f(t), \tag{7.1}$$

where $f(t) = f(t+T)$ and $T = 2\pi/\omega$, are extremely convenient models for studying the appearance and development of stochastic motion. In the classical theory, one can distinguish three problems:

1) The motion in the stochastic layer near the separatrix of an isolated nonlinear resonance (Chapter 5 in Ref. 129 §3.5 in Ref. 130).

2) The stochastic motion which arises upon the overlap of two nonlinear resonances (§4.5 in Ref. 130).

3) The transition to a global stochastic motion which arises upon the overlap of an infinite number of resonances which span an unbounded region of the action variable (§4.2 in Ref. 129 and Chapter 4 in Ref. 130).

These problems also have analogs in quantum theory.

For systems with a Hamiltonian which is periodic in time, as in (7.1), there exists a complete set of quasienergy states: solutions of the Schrödinger equation which are of the form

$$\Psi_\varepsilon(q, t) = \exp\left(-i\frac{\varepsilon t}{\hbar}\right)\varphi_\varepsilon(q, t), \tag{7.2}$$

where $\varphi_\epsilon(q,t) = \varphi_\epsilon(q,t+T)$ and $\epsilon$ is the "quasienergy."[16] The quasienergies $\epsilon$ and the quasienergy wave functions $\varphi_\epsilon$ are largely similar to the energy spectrum and eigenfunctions of autonomous Hamiltonian systems. In particular, the discrete nature of the quasienergy spectrum means that the evolution of an arbitrary state $\Psi(q,0)$ will be quasiperiodic. It also rules out an unbounded growth of the action and energy variable.[95,111]

The establishment of the fact that the quasienergy spectrum is discrete would make it possible to extend to the case of nonautonomous quantum systems the criteria which were formulated in Sections 3–6 for autonomous systems. The first two of the problems listed above can be examined in models with a Hamiltonian $\hat{H}$ which has a finite number of levels of a discrete spectrum; the effect is to guarantee that the quasienergy spectrum will be discrete. For models which allow formulation of the third problem, the nature of the quasienergy spectrum is known only in particular cases.

The chaotic properties of nonautonomous quantum systems with a periodic perturbation have been the subject of comparatively few studies. The existence of a motion with mixing in such systems was originally observed in Refs. 109 and 125; it was studied for various models in Refs. 152, 162, and 225. A continuous power spectrum and a damping of correlations are apparently exhibited even by a very simple and by no means semiclassical system: a two-level atom in a bichromatic field.[222] In the balance of this section of the review we will consider only systems with a periodic perturbation.

### 7.1 Nonlinear quantum resonance

For a slightly anharmonic system (a nonlinear oscillator), whose spectrum can be represented locally as

$$E_n = \hbar\omega_0(n + \beta n^2) \qquad (n \gg 1,\ |\beta| \ll 1), \tag{7.3}$$

a nonlinear quantum resonance consists of an approximate agreement of the frequency $\omega$ of the field which is acting on the system and one of the transition frequencies in the system:

$$\omega_n = \frac{dE_n}{\hbar\, dn} = \omega_0(1 + 2\beta n). \tag{7.4}$$

The condition $\omega_r = \omega$ determines the resonance level $r$. The properties of quasienergy states of an anharmonic oscillator (7.3) in the presence of a resonance of this sort, for a model with a perturbation of the form

$$\hat{V}(q, t) = \hat{v}(q)\cos\omega t, \tag{7.5}$$

were first studied by Berman and Zaslavsky.[32] In a system of this sort, there can be a packet, localized in energy space, consisting of states which have been captured into a nonlinear quantum resonance.[7] If the perturbation $V$ has nonzero matrix elements $v_{nn} = v_0 = \text{const}$ only for transitions between neighboring levels, $n$ and $n+1$, one can determine the form of the quasienergy solutions by reducing the problem to a one-dimensional Schrödinger equation in the energy representation.[32] There exist about $(v_0/\beta)^{1/2}$ different solutions $\varphi_\epsilon(n,t)$ which are localized near the resonance, i.e., which are made up of functions with indices close to $r$. The number $(C)$ of levels which contribute substantially to these states (which are captured into the nonlinear resonance) ranges from a minumum $C_{\min} \sim (v_0/\beta)^{1/4}$ (Ref. 128) to a maximum $C_{\max} \sim (v_0/\beta)^{1/2}$ (Ref. 32).

The solutions which have been studied to date correspond to a regular motion of the classical system within a resonance, far from the separatrix. The solutions which correspond to a motion near the separatrix—and only here is there stochastic motion in a classical system—have essentially escaped study. The one attempt which has been made in this direction has been that by Petrosky and Schieve.[197]

### 7.2 Interaction of nonlinear quantum resonances

The problem of the interaction of nonlinear quantum resonances is conveniently studied using a two-resonance model, whose Hamiltonian can be written as follows in terms of action-angle variables:

$$H = \frac{I^2}{2J} + V[\cos(\theta + \nu\tau) + \cos(\theta - \nu\tau)] \tag{7.6}$$

(below we assume $J = 1$ and $\nu = 1$). Hamiltonian (7.6) also comes under study in the problem of the behavior of a slightly anharmonic system as in (7.3) when subjected to a bichromatic perturbation.[66] The degree of interaction of the resonances is determined by the dimensionless parameter

$s = 2(V)^{1/2}$ [in the original units, $s = 2(V/J\nu^2)^{1/2}$]. In the classical theory, the stochastic layers near the separatrix of the main resonances [$\omega(I) = \pm 1$] overlap at $s = s_C = 0.70$. In the region of an overlap of resonances, numerical calculations have been carried out to study the following topics for the system (7.6):

1) The spreading of packets prepared from the stationary states localized between resonances.[66,84]

2) The behavior of the correlation functions of the coefficients in the expansion

$$\Psi(\theta, t) = \sum_n A_n(t)\, e^{in\theta} \tag{7.7}$$

and the nature of the damping of the correlations for various values of $s$ (Refs. 66, 84, and 194).

3) The power spectrum of a two-resonance system.[84,106]

Qualitatively, the results of these studies reduce to the assertion that the dynamics of a quantum system becomes increasingly complex as nonlinear quantum resonances overlap. This conclusion also follows, in a natural way, from the correspondence principle. Quantitative characteristics of the degree of complexity have not been established. Corresponding results were found in Ref. 223 for an interaction of resonances in another model (a particle between rigid walls, with a sinusoidal field as perturbation).

A qualitative feature of a quantum two-resonance system is the possibility of a transition from a state which is localized near one resonance to the vicinity of the other as the result of a *tunneling through a separatrix*: a process which involves a large number (on the order of $2/\hbar$) of quanta of the external field. This process is possible at any $s < s_C$, although it is improbable. The transition probability is exponentially small; at $s \ll 1$, it is

$$W \sim \exp\left[-\frac{4}{h}\left(1 - \frac{s}{2}\right)\ln\frac{4}{s^2}\right]. \tag{7.8}$$

The ratio of the probability densities near the first and second resonances is of the same order of magnitude. A tunneling between resonances under the condition $s \lesssim s_C$ has been observed[206] in numerical calculations.

### 7.3 Quantum rotator with jolts

The particular features of the evolution of a nonautonomous quantum system in the transition of its classical analog to a global stochastic motion have been studied in most detail for the model of a rotator with jolts (Subsection 1.3.3). There are basically three such features: a limitation of the diffusive energy growth by the finite time interval; the possibility of an unbounded growth of the energy in the absence of a classical global stochastic behavior, if the parameters are chosen in a special way; and an instability of a regime of accelerating modes in the quantum case.

*7.3.1. Limitation on the energy growth.* Numerical calculations on the time evolution of a standard quantum rotator with jolts,

$$\hat{H} = \frac{1}{2}\hat{I}^2 - K\cos\hat{\theta}\cdot\tilde{\delta}(t), \tag{7.9}$$

which were first carried out in Refs. 36 and 46, have shown that the energy growth $\langle E\rangle \sim t$ which is characteristic of the classical system occurs in the quantum system only on the limited time interval $t \ll \tau_1$: At $t \sim \tau_1$, the rate of diffusion among levels decreases, while at $t \gg \tau_1$ the growth in the energy comes to an essentially complete halt (Fig. 20). An estimate of the time taken for the transition from the classical regime to the quantum regime,[69]

$$\tau_1 \approx \left(\frac{K}{\hbar}\right)^2 \tag{7.10}$$

has been confirmed satisfactorily by numerical calculations[125] for the interval $5 \leqslant K/\hbar \leqslant 100$. Expression (7.10) refers to the case $K \gg 1$. Near the threshold for a global stochastic behavior, at $K \lesssim K_C = 0.9716$, the transition time is $\tau_1 \sim \hbar^{-\gamma}$ in order of magnitude, where $\gamma = 3.04$ (Ref. 150). A quantum kinetic equation for systems with jolts was studied in Refs. 40 and 44. In general, the correlation functions of the dynamic quantities for a standard quantum rotator with jolts fall off no more rapidly than a power law[70]: $B(\tau) \gtrsim (K\tau/\hbar)^{-1/2}$. A corresponding result was found in Ref. 83 for a system with a slight nonlinearity subjected to a pulsed perturbation. The question of saturation of the growth of $E$ is still open: In the numerical calculations of Ref. 146, the tendency toward a slow growth of $E$ (against the background of intense fluctuations) persists up to the longest times which were studied, $t \sim 10^3$.

*7.3.2. Global quantum resonance.* The very first studies[36,46] found a superdiffusive growth of the energy of a standard quantum rotator with jolts. This phenomenon has subsequently been studied in detail[52,146]:

$$E(t) \sim \eta t^2. \tag{7.11}$$

This growth arises under the global resonance condtion $\hbar = 4\pi(b/a)$, where $a$ and $b$ are mutually simple integers. Such a value of $\hbar$ makes the energy of the excitation quantum, $\hbar\omega = \hbar\cdot 2\pi$, commensurable with any distances between the energy levels of the rotator, $E_n = \hbar^2 n^2/2J = \hbar^2 n^2/2$. It also gives rise to a continuous quasienergy spectrum. The value of the coefficient $\eta$ in (7.11) depends on both the value of $K$ and the order of the resonance, $a$. The estimates

$$\eta \approx \left(\frac{K}{a\hbar}\right)^{2a} \quad \left(a \gg \frac{K}{\hbar}\right) \qquad \eta \approx 0.2a\left(\frac{K}{a\hbar}\right)^2 \quad \left(a \ll \frac{K}{\hbar}\right), \tag{7.12}$$

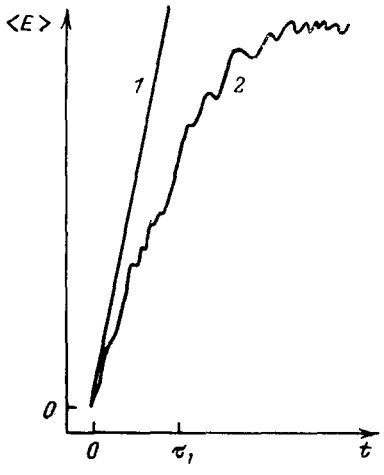


FIG. 20. Growth over time of the average energy in the model of a standard quantum rotator with jolts. 1—According to the classical theory; 2—according to numerical calculations.[125]

found in Ref. 52, have been supported by numerical calculations.[52,138]

A global quantum resonance in a standard quantum rotator with jolts sets the stage for an unbounded growth of the energy of the system even in the case $K < K_c$, i.e., in a case such that the energy of the classical system can vary over only a restricted interval. However, a global quantum resonance is a specific property of system (7.9), associated with the exact multiplicity of all the level spacings. A very simple example of global quantum resonance is the unbounded growth in the energy of a linear oscillator in a resonant field.[16] The same property is exhibited by a model with a spectrum which is equidistant but not bounded from below; this model was studied in Ref. 133. These examples show that a global quantum resonance is also possible for regular systems, so it is not related to the quantum chaos problem.

*7.3.3 Tunneling from accelerating modes.* Under the condition $K \gtrsim 2\pi$, a growth of the action which is linear in the time ($|I| \approx Kt$) and a growth of the energy which is quadratic in the time ($\langle E \rangle \approx (Kt)^2/2$) arise in the classical model of a standard rotator with jolts under special initial conditions ($I_0 \approx 0$, $\theta_0 \approx \pm \pi/2$). This motion regime, which is stable with respect to small perturbations, is called "motion in an accelerating mode." Farrelly[147] has shown that in the quantum model the tunneling through a separatrix from an accelerating mode into other states leads to a decrease in the probability for being in the accelerating mode, $P_a(t)$, in accordance with

$$P_a(t) = P_a(0)\, e^{-\gamma t}, \quad \gamma = \exp\left(-\frac{C(K)}{\hbar}\right). \tag{7.13}$$

### 7.4. Relationship with the Anderson localization theory

The saturation of the energy growth in the model of a standard quantum rotator with jolts forces the assumption that in the absence of a global quantum resonance the quasienergy spectrum of this model is discrete, and all the quasienergy states are localized in the $H_0$ representation; i.e., their amplitudes fall off exponentially outside a certain range of values $n$. This circumstance can be illustrated by an analogy, pointed out in Refs. 86 and 87, with Anderson localization in one-dimensional chains.[8)]

Localization is the appearance of a discrete spectrum for a quantum system which represents a particle in a random potential $U(\mathbf{q})$, which fills an infinite region of space. The possibility of such a localization was pointed out by Anderson[3] for a strong-coupling model [a particle on a lattice with random node energies $E_i$, which are distributed in a known way, with a characteristic distribution width $W$ and constant matrix elements for the transition ($V$) between neighboring nodes]. In a one-dimensional chain of this sort, all the states are localized if there are any (arbitrarily slight) disorder (a small value of $W/V$).[4] The theory of Anderson localization has been the subject of the reviews of Refs. 26, 39.

We denote by $\varphi(\theta)$ half the sum of the values of the periodic part of the quasienergy solution for a quantum rotator with jolts respectively before and after a jolt:

$$\bar{\varphi}(\theta) = \frac{1}{2}[\varphi(\theta, -0) + \varphi(\theta, +0)]. \tag{7.14}$$

The Fourier amplitudes $u_n$— the amplitudes of $\bar{\varphi}(\theta)$ in the $H_0$ representation—

$$u_n = \frac{1}{2\pi}\int \bar{\varphi}(\theta)\, e^{-in\theta}\, d\theta \tag{7.15}$$

satisfy the equation

$$T_n u_n + \sum_{k\neq 0} U_k u_{n+k} = \bar{E} u_n, \tag{7.16}$$

where

$$U_k = U_{-k} = \int U(\theta)\, e^{ik\theta}\, d\theta, \quad U(\theta) = -\operatorname{tg}\frac{Kv(\theta)}{2\hbar},$$
$$T_m = \operatorname{tg}\left(\frac{\varepsilon}{2\hbar} - \frac{\hbar m^2}{4}\right), \quad \bar{E} = U_0. \tag{7.17}$$

Equation (7.16) may be interpreted as an equation for the amplitudes of the wave functions in the strong-coupling approximation in a one-dimensional chain with node energies $T_m$ and a matrix element $U_k$ for a transition to the $k$ th neighbor. The theory of Anderson localization deals with models with random $T_m$. In Eq. (7.16), the $T_m$ are pseudorandom, although in many regards they are similar to the values which would be obtained from a random-number generator.[151]

In the absence of a global resonance, with an irrational $\hbar/2\pi$, the values of the "energies"! $T_m$ at the nodes have Cauchy distribution

$$W(T) = \frac{1}{\pi(1+T^2)}. \tag{7.18}$$

For model (7.16), with a distribution of this type for the diagonal matrix elements, and with a transition matrix element which is nonzero only for nearest neighbors,

$$U_k = \varkappa\delta_{1,k} \tag{7.19}$$

(the so-called Lloyd model[13]), an explicit expression is available for the localization index $\gamma(\bar{E},\varkappa)$, which determines the rate of the exponential decay of the wave function at large distances[18]:

$$\gamma = \operatorname{Arch}\left[\frac{1}{2\varkappa}\{[(\bar{E}+\varkappa)^2+1]^{1/2} + [(\bar{E}-\varkappa)^2+1]^{1/2}\}\right]. \tag{7.20}$$

If $v(\theta)$ is chosen in a special way, one can satisfy (7.19) in a model of a rotator with jolts. Expression (7.20) then gives a satisfactory description of the decay law which has been found in a numerical simulation for the wave functions in the $H_0$ representation (Refs. 89 and 90; see Fig. 21 of the present paper). The case of a discontinuous $v(\theta)$ leads to a long-range interaction among nodes ($|U_k| \geqslant \text{const}\cdot|k|^{-1}$), a de-

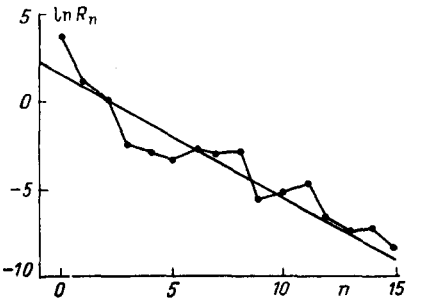


FIG. 21. The normalized amplitude $R_n = |\mathrm{Re}\, u_n u_n^*|$ of a quasienergy state in the model of a quantum rotator with jolts as a function of $n$. Straight line—Theoretical; broken line—connects the points found by numerical calculation.[89]

localization of states, and an unbounded growth in the energy of the rotator.[124] In the case $\gamma \ll 1$, the localization index is related to the classical diffusion coefficient $D_C$ by[142,227] $\gamma \approx 2\hbar^2/D_C$. This relation has been supported by numerical calculations. The structure of a wave function in the case of a global resonance was studied in Ref. 181.

Although the analogy between a quantum rotator with jolts and one-dimensional chains is not mathematically rigorous (certain aspects of it, associated with the specific differences between quasienergies and energies, are discussed in Ref. 146), and although we do not have a rigorous proof that the spectrum is discrete (certain limitations on the nature of the spectrum were established in Refs. 143 and 178), essentially all the available data provide evidence in favor of a discrete nature of the quasienergy spectrum in the model of a standard quantum rotator with jolts in the absence of a global resonance.

Feingold *et al.*[187] have studied the statistics of the quasienergy levels of this system. They showed that the distribution function of the level spacings $P(S)$, does not generally exhibit level repulsion; it is similar to a Poisson distribution function, as in the case of the Anderson model. In the case of global resonance, under the auxiliary condition of periodicity in $I$, the level statistics change from Poisson to Wigner.[217] A corresponding transition for another model (an infinitely deep potential well whose width varies periodically) was established in Ref. 218.

### 7.5 Stochastic ionization and quantum effects

Of major interest from the standpoint of the possibility of comparing the conclusions of the quantum chaos theory with experiment is the problem of the behavior of the hydrogen atom in a monochromatic external field. Such a system is described by the Hamiltonian

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r} + \mathscr{E} z \cos \omega t \qquad (7.21)$$

(in this section, we are using a system of units with $e$, $\hbar$, $m = 1$). The possibility of treating a highly excited hydrogen atom (with a principal quantum number $n \gg 1$) as a classical system, and of treating the process of ionization in a field of frequency $\omega \ll n^{-2}$ as a consequence of a diffusion among energy levels (in the action variable $I = n\hbar$), was first pointed out by Leopold and Percival.[49] This approach has been taken in many studies (see the review by Delone *et al.*[103]).

Many important features of (7.21) are retained in the simpler system in which $H_0$ has a single degree of freedom:

$$H = \frac{1}{2}p_z^2 - \frac{1}{z} + \mathscr{E} z \cos \omega t \quad (z \geqslant 0). \qquad (7.22)$$

Such a model can describe the evolution of system (7.21) if the initial state has parabolic quantum numbers, $n_1 \gg n_2 \sim 1$, $m = 0$.

The classical theory predicts[103,159] that for the system (7.22), at a value of the field $\mathscr{E}$ exceeding the critical value $\mathscr{E}_c$,

$$\mathscr{E}_c \approx \frac{1}{50} n_0^{-5} \omega^{-1/3} \qquad (7.23)$$

($n_0$ is the principal quantum number of the initial state), diffusion among levels arises, with a coefficient which depends on $n$:

$$D \approx 2\mathscr{E}^2 n^3 \omega^{-7/3}. \qquad (7.24)$$

This diffusion leads to ionization of the atom over a time of the order of $\tau_I \sim (n_0^2 D)^{-1}$ [expressions (7.23) and (7.24) were derived under the assumption $\tilde{\omega} \equiv \omega n_0^3 \geqslant 1$]. The time restriction which arises in quantum mechanics on the applicability of the classical evolution laws of expectation values (Subsection 2.2) naturally suggests the following question: Under what conditions can the ionization process in model (7.22) be described by the classical theory? The question of the quantum corrections to stochastic ionization was posed in Ref. 97.

At a qualitative level the situation is clear. On the one hand, diffusion among levels is possible in the system described here (under the condition $\tilde{\omega} \geqslant 1$) if the amplitude of the alternating field, $\mathscr{E}$, exceeds the quantum localization threshold[142,213]

$$\mathscr{E}_q \approx 0{,}4 n_0^{-4} \omega^{7/6}. \qquad (7.25)$$

On the other hand, in a system of this type ionization is possible in the quantum case, first, because of direct multiphoton transitions to the continuum and, second, because of multiphoton transitions from $n_0$ to states with $n$ large enough to ensure global stochastic nature and subsequent classical ionization (see "tunneling through a separatrix" in Subsection 7.2). The role played by multiphoton transitions in the time evolution of the ionization of system (7.22) was studied in Refs. 142, 176, and 192. Depending on the particular combination of parameter values, the dominant ionization mechanism may be either diffusion among levels (which is amenable to a classical description) or purely quantum excitation mechanisms.

The first experimental data on the excitation by a microwave field of hydrogen atoms polarized by an auxiliary static electric field $\mathscr{E}_S$ (with the result that the geometry of the motion of the electron is effectively one-dimensional) were obtained by Bayfield and Pinnaduwage[177] in 1984. With the parameter values $n_0 = 60$, $\tilde{\omega} = 0.19 - 0.26$ and $\mathscr{E} \gtrsim 10\mathscr{E}_c$, the distribution among levels of the atoms excited by the field over a time $T = 3.7 \cdot 10^{-7}$ s is of a smooth, nonresonant nature. This circumstance has been discussed as an indication of diffusion among levels. The ionization threshold differs by a factor of two from that found in the classical approach: The discrepancy can be attributed to a contribution from four- and five-photon transitions.[176] Subsequent experiments by Van Leeuwen *et al.*[207] ($n_0 = 32$–$74$, $\tilde{\omega} = 0.05$–$0.6$; $\mathscr{E} \gtrsim 10\mathscr{E}_c$) revealed that the ionization threshold (corresponding to the ionization of 10% of the atoms over a fixed time) agrees well with results calculated on the basis of the classical model.[159] This agreement can be explained in terms of proximity of the thresholds for global classical stochastic behavior and for substantial delocalization of the quasienergy wave functions under the conditions of such experiments.[209] For other values of the parameters, quantum effects may be important.

Another entity which can be described by the Hamiltonian (7.22) is an electron above the surface of liquid helium[53] in an alternating field. The possibility of using such a system to study quantum chaos was pointed out in Ref. 97. The behvaior of an electron above the surface of liquid helium during pulsed excitation [the $\cos \omega t$ in (7.22) is replaced by $\tilde{\delta}(t)$] was studied theoretically in Refs. 137 and 138. An

additional feature here—not found in the case of the hydrogen atom—is the possibility of imposing a static electric field [adding a term $V_s = \mathscr{E}_s z$) to the Hamiltonian (7.22)] to squeeze the electron toward the surface and to limit the possibility of ionization. Classical and quantum calculations on the properties of such a system were carried out in Refs. 132 and 173. It was shown that in the region of parameter values accessible experimentally some nonlinear resonances may appear, spanning a large number of levels ($C \sim 10^3$). This circumstance would make it possible to study the properties of quantum chaos in a highly semiclassical regime. Corresponding experiments, however, have yet to be carried out.

## 8. CONCLUSION. THE NEXT PROBLEMS

The quantum chaos problem is far from being exhausted. The number ($N$) of papers on this problem which are published per year has been increasing in accordance with $N \sim \exp(\lambda t)$, where $\lambda = 0.23\ \text{yr}^{-1}$, since this problem was formulated. Such a value of $\lambda$ is much larger than the growth index of the entire number of scientific publications ($\lambda = 0.046\ \text{yr}^{-1}$; Ref. 10), but it is close to the growth index of the cheapness of computer calculations ($\lambda = 0.25\ \text{yr}^{-1}$; Ref. 172). This circumstance is not surprising: Many of the studies (about 70% of those cited in this review) have been devoted to numerical "experiments" on quantum chaotic systems. The rapid accumulation of numerical data is of a basically extensive nature (new models of quantum chaotic systems are always being studied), and to some extent it is depreciated by the way in which the results are most commonly presented—in a way which is appealing to the eye and which shows only qualitative characteristics. Suffice it to say that up to 1985 the papers on the quantum chaos problem did not use statistical criteria to test hypotheses: The $\chi^2$ test was first used in the context of this problem in Refs. 180 and 201. The very next problem should thus be to make the transition from a qualitative to a quantitative description of the properties of quantum chaotic systems. Making this transition will require answering the following question:

### 8.1. What parameters describe quantum chaos?

Further progress in numerical research on the properties of quantum chaotic systems requires introducing good parameters. As an example, we can cite the distribution of level spacings $P(S)$, which is the characteristic of the spectrum of quantum chaotic systems which has been studied in most detail. Under the condition $\mu \lesssim 1$ the parameters, $\beta$, $\mu_q$, and $a$ (Subsection 3.2.1) correlate well with that fraction ($\mu$) of the phase space which is occupied by stochastic paths. In no case, however, it is possible to establish a functional relationship $\zeta(\mu)$, since these quantities are found by comparing $P(S)$ with the functions of a certain family, while the question of whether $P(S)$ belongs to this family has not been solved.

For a hypothesis-free parametrization of $P(S)$, we need a quantity which should be (1) an inherent characteristic of the array of numerical values of the energy levels, (2) of an integrated nature and amenable to highly accurate calculations at the existing sizes of the series, $\mathscr{N}$, and (3) sensitive to level repulsion.

Using the moments

$$M_n = \int_0^\infty S^n P(S)\, dS, \tag{8.1}$$

as parameters of $P(S)$, as proposed in Refs. 131, 167, and 191, is ineffective. The moments meet the first of these requirements completely; with some labor they meet the second (since the $M_n$, especially the higher ones, up to $M_8$ in Ref. 191, are determined by the maximum values of $S$, which are of a fluctuating nature); but they fail completely to satisfy the third requirement.

These requirements are met best by the quantity $\Lambda$, which is defined for a given $P(S)$ by

$$\Lambda = \int_0^\infty \ln S \cdot P(S)\, dS + C. \tag{8.2}$$

There is a similarity between $\Lambda$ and the parameter $Q(n)$ in statistical spectrum theory.[72] The Euler constant $C = 0.5772$ has been added to the right side of (8.2) to generate $\Lambda = 0$ in the case of a Poisson distribution. For a Wigner distribution we would have $\Lambda = 0.409$. At $\mathscr{N} = 500$, the quantity $\Lambda$ is determined within an absolute error $\Delta\Lambda \lesssim 10^{-2}$. By way of comparison, the Brody parameter $\beta$ at the same vlaue of $\mathscr{N}$ is calculated within an error larger by an order of magnitude.[153,205]

The problem of parametrizing the properties of wave functions has not been solved. As was mentioned back in Section 4, the complexity $C_n$ is insufficient to describe the properties of the eigenfunctions $\psi_n$: We still need a characteristic of the degree of randomness of the coefficients $a_{mn}$. The situation is the same with regard to the parametrization of the properties of the matrix elements of arbitrary operators and the evolution of wave packets.

Clearly, until some good parameters are established in these approaches we will also be unable to speak in terms of the identification of a quantitative relationship between quantum chaos and classical stochastic nature. Along the approach to the quantum chaos problem, which has been pursued to the greatest extent so far, and which starts from a study of the structure of the energy spectrum, where the parameters have been determined in one way or another, the next question to be taken up is the following:

### 8.2. Is the model of a Gaussian orthogonal ensemble universal?

As was shown back in Section 3, many numerical calculations confirm that the spectral properties of highly stochastic quantum chaotic systems are approximately the same as those of the matrices of a Gaussian orthogonal ensemble. On the other hand, there are counterexamples, which prevent us from concluding that this approximate agreement is a universal property of quantum chaotic systems.

Another question which is not completely clear is that of the structure of the spectrum of a hydrogen atom in a strong magnetic field $\mathscr{H}$. The dependence $E_n(\mathscr{H})$ which was found in Ref. 65 for the region of stochastic classical motion is a regular pattern of levels with exponentially small splittings at avoided crossings [$\Delta \sim \exp(-\alpha n)$, where $n$ is the principal quantum number], similar to the pattern of the spectrum of integrable systems.[88,230] Again in the stochastic case, the distribution $P(S)$ here is similar to a Poisson distribution[9] (Ref. 199). An attempt has been made[73] to relate the sizes of the splittings to the properties of the stochastic nature of the classical system.

A distribution $P(S)$ with a large peak in the limit $S \to 0$

was found in Ref. 184 for a highly stochastic nonlinear oscillator. That result is based on a misunderstanding: The assertion that $P(S)$ is of a Wigner nature refers to a series of levels with identical exact quantum numbers. In the construction of $P(S)$, doubly degenerate states referring to different series were erroneously taken into account in Ref. 184. However, any arbitrarily weak perturbation which disrupts the exact symmetry of the Hamiltonian will lead to a small splitting of the degenerate levels and will make $P(S)$ double-humped. The parameters of the stochastic motion of the classical motion, on the other hand, will not change substantially (Fig. 22).

The possibility of an excess of nearby levels with an exponentially small splitting, associated with tunneling between classically isolated regions, was discussed in Ref. 121.

On the other hand, one can cite integrable systems which are characterized by level repulsion. For example, for a Hamiltonian with two degrees of freedom,

$$H=\frac{\omega}{2}\{I_1+I_2+[(I_1+I_2)^2+4\hbar I_1]^{1/2}\} \tag{8.3}$$

(the $I_i$ are action variables), the semiclassical spectrum which is obtained through the substitution $I_i=n_i\hbar$, $n_i\geqslant 0$, is

$$E(n_1, n_2)=\frac{\hbar\omega}{2}\{n_1+n_2+[(n_1+n_2)^2+4n_1]^{1/2}\}. \tag{8.4}$$

This spectrum exhibits absolute level repulsion[101]:

$$\lim_{\mathcal{N}\to\infty} P(S)=\delta(S-1). \tag{8.5}$$

These counterexamples show that there is no general relationship between the shape of $P(S)$ and the stochastic motion of a classical system. There is a way to combat counterexamples: They have to be made "local" (i.e., made to contradict individual lemmas of the proof) and eliminated by adding reservations under the conditions of the theorem.[11] With regard to the problem of the structure of the energy spectrum of a quantum chaotic system, this approach poses the problem of determining the conditions under which criteria **A2**, **A3**, and **A4** hold. This problem has another aspect, associated with the following question:

### 8.3. What role does $\sigma$ play in quantum chaos?

It is presently believed that a transition of the spectral structure of a quantum chaotic system between limiting shapes is associated with a change in $\mu$ but not in $\sigma$. The main argument against a dependence on $\sigma$ is the approximate agreement between the spectrum of many systems with $\mu\approx 1$ and the spectra of Gaussian orthogonal ensembles.[202] There are two objections to this argument. First, as we have already mentioned, this approximate agreement is not a general property of quantum chaotic systems. Second, a change in the structure of the energy spectrum of a quantum chaotic system [an increase in the Brody parameter $\beta$ of the $P(S)$ distribution] has also been seen in a region in which the classical motion has remained ergodic ($\mu=1$) but the Lyapunov index $\sigma$ has continued to grow.

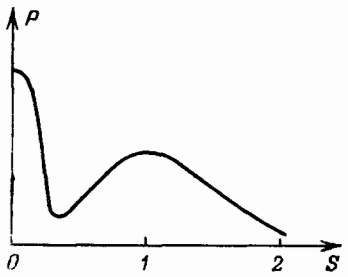


FIG. 22. Sketch of the distribution function of level spacing, $P(S)$, in a stochastic system when there is a slight violation of a discrete symmetry.

Identifying a dependence of the spectral properties of a quantum chaotic system on $\sigma$ (or proving an independence) will require studying the spectra of single-parameter families of ergodic ($\mu=1$) billiards and two-parameter families of nonlinar oscillators, in which there is the possibility of independent variations in $\mu$ and $\sigma$. The model of a standard quantum rotator with jolts is also extremely convenient, under the further condition of periodicity in $I$ (Refs. 181 and 217), since for it the index $\sigma$ behaves in accordance with $\sigma\approx\ln(K/2)$ at $K\gg 1$ and can be made arbitrarily large. So far, no calculations of this sort have been carried out.

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Research on the quantum chaos problem is presently concentrated around the large store of data from numerical calculations on various physical systems. On the whole, these calculations are mutually consistent. The nature of the differences between the quantum properties of chaotic and regular systems (particularly the properties of their energy spectra) has been established extremely thoroughly by these calculations. In general, today we have a better understanding of the properties of ergodic systems than of the nature of the changes in the parameters at the transition from a regular motion to an ergodic motion. In the future, we can expect numerical calculations to provide, in adition to a "quantification" of the description of the results and progress toward more semiclassical systems ($\hbar\lesssim 10^{-3}$), an increase in the activity of research on systems with $N\geqslant 3$ degrees of freedom.

In the theoretical approach, it can be asserted that we have a qualitative understanding of most of the basic properties of quantum chaotic systems. Much has been achieved at the level of semiquantitative estimates, which are in satisfactory agreement with the results of the numerical calculations. An important role is being played here by the heuristic aspect of a randomness of the wave functions and matrix elements of simple operators for highly stochastic systems. This heuristic aspect makes it possible to invoke the ideas and methods of the theory of random matrices. A quantitative description, however, especially one based on the first principles, has so far been developed only for a small number of isolated parameters of particularly simple models. Making further progress in the theoretical description of quantum chaotic systems appears to require a parallel development of methods for calculating the characteristics of the stochastic motion ($\sigma$ and $\mu$) in classical systems.

Finally, experimental research on the properties of quantum chaos lags well behind the numerical and theoretical work and is essentially still in an initial stage. The difficulties which are being encountered here appear to be technical ones, but they derive from a fundamental circumstance: Quantum chaos is an asymptotic property, so a study of it reduces to determining the quantum characteristics of systems which are definitely close to being classical systems. Hence the experimental accuracy must be quite high. The theoretical work which has already been done sug-

gests that the efforts of experimentalists in the immediate future will be concentrated on research on stochastic ionization, the dynamics of a system of levitated electrons, and the properties of highly excited states of atoms in a strong magnetic field.

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[1)]Up to Subsection 1.3.3 we will consider only autonomous systems for which the equation $\partial H/\partial t = 0$ holds. Formally, a nonautonomous system with $N$ degrees of freedom can be represented as an autonomous system with $N+1$ degrees of freedom (§1.2 in Ref. 130).

[2)]Here and below, the # means the end of the formulation of a criterion.

[3)]As a point of curiosity, we note that diagrams of this sort are used in quantitative art to analyze works of music. In that application, $\Delta_n$ is chosen to be the frequency interval between successive notes.[233] For works by Bach, the plot looks chaotic.

[4)]We will be using the abbreviations WF for wave function and EF for eigenfunction.

[5)]It is to be understood here that $\psi(\mathbf{q})$ can be chosen to be real. For complex wave functions of systems which are not invariant under time reversal, an analog of nodal lines would be dislocation points of wave functions, at which the condition $|\psi(\mathbf{q})| = 0$ holds. A growth of the number of such points with the level index $n$ for quantum chaotic systems was established in Ref. 211.

[6)]According to Feynman *et al.*,[234] the field-line concept does not contain the most profound of the principles of electrodynamics: the superposition principle. With appropriate modifications, this comment also applies to the $\psi(\mathbf{q})$ nodal lines.

[7)]The problem of the excitation of a nonlinear oscillator from the ground state to the vicinity of a nonlinear quantum resonance has been studied in extreme detail.[27,28,30,37,38]

[8)]The relationship between stochastic nature of a classical system and Anderson localization was studied in another context in Ref. 82.

[9)]In a later study[235] it was shown that these features disappear as the system becomes increasingly semiclassical. Calculations with $I \gtrsim 10^3$ levels have shown that $P(S)$ makes a transition from a Poisson distribution to a Wigner distribution. The calculations also reveal agreement of $\Delta_3$ and $C(1)$ in the ergodic limit with the values of the model of a Gaussian orthogonal ensemble.

[10)]This result *does not* contradict position **A2-a,** since in the limit $\hbar \to 0$ the Hamiltonian (8.3) drops out of the class of systems of the general type. We might add that a spectrum with property (8.5) was regarded in Ref. 98 as an attribute of a quantum system of an ultimately chaotic nature.

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This review of the literature covers up to June 1986 inclusively. The items are listed in order of year of publication.


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