Superfluidity of helium II near the λ point[1)]

V. L. Ginzburg and A. A. Sobyanin

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow
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The present status of theoretical and experimental research on the superfluidity of helium II near the λ point is outlined. Attention is focused on the state of the phenomenological Ψ theory of superfluidity. Questions concerning the applicability limits of this theory and its underlying assumptions are discussed. Results found through the solution of several problems are discussed. Experimental data are also discussed. The need for a further comparison of theory and experiment and the actual feasibility of such a comparison are emphasized. The relationship with other approaches is pointed out.

CONTENTS

1. INTRODUCTION

Research on the superfluidity of helium II near the λ point continues to attract much interest for several reasons. One is that conditions are particularly favorable for studying critical fluctuation phenomena near the λ transition in liquid ^{4}He. In fact, it was this circumstance which has served as the main driving force for the experimental and theoretical research on the properties of helium II near the λ point over the past two decades (see, for example, Refs. 1–3 and the bibliography cited in these reviews).

The second reason for the unflagging interest in the λ transition in liquid ^{4}He is that near the λ point Landau's superfluidity theory (including both its component parts: the concept of a gas of elementary excitations and a phenomenological two-fluid hydrodynamics) breaks down; in order to solve various steady-state and time-dependent problems it becomes necessary to incorporate the spatial variations and the relaxation of the order parameter—the macroscopic wave function $\Psi = \eta e^{i\varphi}$, which describes the superfluid state[4,5]—from the very outset.

A final reason is that research on superfluidity of helium II near the λ point has just recently acquired further urgency because of the mechanisms for high-temperature superconductivity which are being discussed widely today. In a metal, some strongly coupled two-electron formations—so-called local pairs, which are different from ordinary Cooper pairs—can exist under certain specific conditions. It may be that, at least in certain high-temperature superconductors, condensed local pairs of this sort carry the superconducting current (Refs. 6 and 7, for example). If this is the case, then the properties of superconductors with a superconductivity mechanism of this type should be very similar to the properties of liquid ^{4}He. In particular, to describe the behavior of superconductors with local pairs at temperatures near the critical temperature, the ordinary Ψ theory of superconductivity[8] may be inapplicable, and it may become necessary to use a generalized version[2)] of this theory which is analogous to the generalized Ψ theory of the superfluidity of liquid ^{4}He (Refs. 9 and 10). Our purpose in the present paper is to draw a picture of the present state of the latter theory.

The generalized Ψ theory of superfluidity which we will be discussing combines the simplicity and graphic value of Landau's classical (self-consistent) theory of phase transitions with the results of the extremely new fluctuation theory of phase transitions, which is based on the concept of gauge invariance of critical phenomena in a field-theory approach using renormalization groups. While the corresponding field-theory approach[1-3] has so far been developed for problems in which the order parameter is spatially homogeneous or only slightly inhomogeneous, the Ψ theory of superfluidity[9,10] is intended primarily for solving problems in which it cannot be assumed that the spatial variations in the order parameter are slight (the distribution of the order parameter near a solid wall or a free surface of helium II; the particular features of the λ transition of helium in films, slits, and capillaries; a vortex filament; the smearing of the λ transition in a gravitational force field; etc.). Consequently—

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and we wish to stress this point from the very beginning—the fluctuation theory[1-3] and the Ψ theory of superfluidity[9,10] are by no means contradictory. It is more accurate to say that they complement and enrich each other.

Admittedly, experimental research on effects in which irregularities in the spatial distribution of an order parameter play an important role is still in its infancy. The reasons are both the difficulty in fabricating capillaries and slits of uniform thickness, on the one hand, and the relatively small magnitude of the effects and the need to work very close to T_λ, on the other. However, some important steps have already been taken in this direction. For example, we might mention some very accurate measurements of the surface tension of helium II near the λ point,[11,12] quantitative studies of the λ transition of helium in films[13] and plane-parallel slits,[14,15] measurements of the temperature dependence of the mutual friction force near the λ point,[16] and the observation[17] of the theoretically predicted[9,10] anomalous contribution to the Kapitsa boundary thermal resistance.

In §2 below, we outline the foundation of the generalized Ψ theory of the superfluidity of helium II near the λ point[4,9,10,18] for the steady-state case, with $\mathbf{v}_n = 0$, and we discuss its range of applicability. In §3 we then present the results of the solution of several spatially nonuniform equilibrium problems (such as the calculation of the dependence of the temperature T_λ on the film thickness d), and we compare these results with the experimental data available. In §4 we briefly discuss the foundations of the Ψ theory of superfluidity for the general case ($\mathbf{v}_n \neq 0$, with a time dependence) and a description of certain nonequilibrium and dissipative effects on the basis of this theory. Finally, in §5 we summarize the discussion and offer a list of problems for future research.

Most of the questions touched on in the present paper are discussed systematically, and considerably more comprehensively, in the review in Ref. 19. In addition, some recent results which could not be incorporated in Ref. 19 will be reflected below.

2. FOUNDATIONS OF THE Ψ THEORY OF SUPERFLUIDITY (HELIUM AT REST; PURELY SUPERFLUID FLOW)

2.1. Macroscopic wave function

The one-particle macroscopic wave function

$$\Psi(\mathbf{r}, t) = \eta(\mathbf{r}, t) \exp[i\varphi(\mathbf{r}, t)] \quad (1)$$

plays a fundamental role in this theory. This function is constructed by taking an average of the exact microscopic wave function of the ground state of the system over all the coordinates of the particles which lie within a physically small volume with typical dimensions

$$L \simeq \frac{\xi_M}{Q},$$

where ξ_M is the correlation length of Ψ, and $Q \gtrsim 1$ is a universal numerical factor which serves as a parameter of the theory (Refs. 10 and 19; see also Subsection 2.3 below).

The phase of wave function (1), φ, is related to the velocity of the superfluid motion by

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\varphi, \quad (2)$$

where $m \equiv m_{He}$ is the mass of the ^4He atom. The square of the amplitude of function (1) characterizes the number density of ^4He atoms in the multiparticle ground state:

$$|\Psi|^2 = \eta^2 = n_{gr} = n - n_{ex}; \quad (3)$$

here n is the total number density of the particles of liquid ^4He, and n_{ex} is the total density of particles in excited states. The amplitude of the function Ψ can also be related to the density of the superfluid component:

$$\rho_s = m^*|\Psi|^2, \quad (4)$$

where the mass m^* is generally different from that (m_{He}) of the ^4He atom and may be a function of the temperature and/or the pressure. In the self-consistent version of the Ψ theory,[4] i.e., far from the λ point, the corresponding temperature dependence is inconsequential, and the mass m^* can be set equal to m. Near T_λ, on the other hand, we can write[20,21]

$$m^* = m\tau^{-\sigma}, \quad (5)$$

where $\tau = (T_\lambda - T)/T_\lambda$, and σ is a critical index which is related by the relation $\sigma = \hat{\eta}\nu$ to the critical index ν of the correlation length ($\xi_M \propto \tau^{-\nu}$) and the critical index $\hat{\eta}$ of the correlation function $\langle\Psi(r)\Psi^*(0)\rangle \propto r^{-1-\hat{\eta}}$ at $\tau = 0$.

Recent field-theory calculations of the values of the exponents $\hat{\eta}$ and ν show[1-3] that for the λ transition in ^4He the value of the exponent $\hat{\eta}$ and also the product $\hat{\eta}\nu$ do not exceed $2\cdot 10^{-2}$. Such a weak temperature dependence of the mass m^* is unimportant for all practical purposes. Accordingly, we will usually ignore it below, setting $m^* \equiv m_{He}$, as in the self-consistent Ψ theory.

2.2. Partial thermodynamic potential

The most important expression in the Ψ theory of superfluidity is that for the density of the incomplete thermodynamic potential ("incomplete" here means that it has not been integrated over the fluctuations of Ψ with a scale L larger than or of the order of ξ_M/Q):

$$\tilde{\Omega} = \int \Omega(\Psi, \nabla\Psi; \mu, T)\, dV. \quad (6)$$

This potential depends on Ψ, ∇Ψ, and two ordinary thermodynamic variables (e.g., the chemical potential μ and the temperature T). We write an expression for potential density (6) in the following simple form:

$$\Omega \equiv \Omega_{II} = \Omega_I(\mu, T) + \frac{3\Delta C_\mu(\mu) T_\lambda(\mu)}{3+M}\left(-\tau|\tau|^{1/3}\left|\frac{\Psi}{\Psi_{00}}\right|^2 + \frac{1-M}{2}|\tau|^{2/3}\left|\frac{\Psi}{\Psi_{00}}\right|^4 + \frac{M}{3}\left|\frac{\Psi}{\Psi_{00}}\right|^6\right) + \frac{\hbar^2}{2m}|\nabla\Psi|^2; \quad (7)$$

where $\tau = (T_\lambda(\mu) - T)/T_\lambda(\mu)$, $\Omega_I(\mu, T)$ is the density of the thermodynamic potential of equilibrium helium I (the regular part and the term proportional to $\tau^2 \ln|\tau|$, which is responsible for the logarithmic anomaly in the heat capacity, are incorporated here), ΔC_μ is the jump in the heat capacity $C_\mu \approx C_p$ at the λ transition (i.e., the difference between the values of the heat capacity C_μ at identical relative distances from the λ point), $\Psi_{00}(\mu)$ is the coefficient in the temperature dependence of the equilibrium value of Ψ below the λ point ($\Psi_e = \Psi_{00}\tau^{1/3}$), and M is a universal numerical parameter ("universal" in the sense that it does not depend on T, μ, or, say, the ^3He density), which characterizes the relative contribution of the term with $|\Psi|$ in (7).[6]

Expression (7) holds at temperatures $|\tau| \lesssim 10^{-2}$, where similarity theory can be applied. The critical exponents α and $\hat{\eta}$ for the heat capacity $C_\mu \propto |\tau|^{-\alpha}$ and the correlation function of the order parameter, $\langle \Psi(r)\Psi^*(0)\rangle \propto r^{-1-\hat{\eta}}$, have been set equal to zero in the case $\tau = 0$, since these exponents are very small for helium ($|\alpha| \sim \hat{\eta} \lesssim 10^{-2}$).

It is clear from this discussion that expression (7) is not exact. It does, however, correctly convey the temperature dependences $C_\mu(\tau)$ and $\Psi_e(\tau)$. It also satisfies all the requirements of similarity theory at both small and large values of the ratio $\Psi/\Psi_e(\tau)$. It furthermore gives a correct description (within a small critical exponent $\hat{\eta}$) of the decay law of the correlation function $\langle \Psi(\mathbf{r})\Psi^*(0)\rangle$ at both large and small values of r. We therefore assume that expression (7) does not contradict experimental data, and from the theoretical standpoint it can be thought of as an acceptable interpolation expression suitable for describing the spatial variations in Ψ over distances comparable to or even slightly smaller than the correlation radius ξ_M below the λ point[9,10]:

$$\xi_M \equiv \xi_M^- = \xi_0 \left(\frac{3+M}{6+6M}\right)^{1/2}, \quad \xi_0 = \xi_{00}|\tau|^{-2/3},$$
$$\xi_{00} = \left(\frac{\hbar^2\Psi_{00}^2}{2m\Delta C_\mu T_\lambda}\right)^{1/2} = 1{,}63\cdot 10^{-8}\ \text{cm}. \tag{8}$$

In addition to $\xi_M^-(\tau)$ we can introduce a correlation length $\xi_M^+(\tau)$ above the λ point; here

$$\xi_M^+(\tau) = \xi_0(\tau)\left(1 + \frac{M}{3}\right)^{1/2}. \tag{8'}$$

The choice of the coefficient of $|\nabla\Psi|$ in the form $\hbar/2m$ in (7), and the neglect of the higher-order derivatives and the powers of the derivatives of Ψ correspond to the normalization of Ψ which we mentioned earlier:

$$m\,|\Psi|^2 = m\eta^2 = \rho_s(\mathbf{r},\ t), \tag{9}$$

where ρ_s is the density of the superfluid part of the liquid. By virtue of (1), (2), and (9), the last term in (7) has the form $\rho_s v_s^2/2$ in the case $|\Psi| = \text{const}$. We would like to point out, however, that in more-general situations [e.g., if the exponent $\hat{\eta}$ is nonzero or if (7) contains terms of the type $|\nabla\Psi|^4$ or $|\Delta\Psi|^4$] normalization (9) will not hold.[30]

2.3. Basic equation

An equilibrium equation for Ψ is found by varying (6) with respect to Ψ^*. This equation is

$$\xi_{00}^2\nabla^2\Psi = \left[-\tau|\tau|^{1/3} + (1-M)\,|\tau|^{2/3}\left|\frac{\Psi}{\Psi_{00}}\right|^2 + M\left|\frac{\Psi}{\Psi_{00}}\right|^4\right]\Psi. \tag{10}$$

An extremely important question is that of the boundary conditions on Eq. (10). In a general phenomenological approach to the question of the boundary conditions,[22,23] we would add a surface energy

$$\tilde{\Omega}_S = \int \Omega_S(|\Psi_S|^2)\,dS \tag{11}$$

to functional (6), where dS is an area element of the surface bounding the given volume, and Ψ_S is the value of the order parameter on this surface. The resultant functional $\tilde{\Omega} + \tilde{\Omega}_S$ must be minimized with respect to $\Psi(\mathbf{r})$ and Ψ_S simultaneously. Near the λ point, where $|\Psi|$ and $|\Psi_S|$ are small, the function Ω_S can be expanded in a power series in $|\Psi_S|^2$, and only the first nonvanishing term need be retained:

$$\Omega_S = \Omega_{S0} + \frac{\hbar^2}{2ml_S}|\Psi_S|^2 + \ldots, \tag{12}$$

where the parameter l_S, which has the dimensionality of a length, characterizes the properties of the boundary and is frequently called an "extrapolation length."[23] A variation of (6) and (11) with respect to Ψ^* and Ψ_S^* leads in this case to the fairly general boundary condition

$$\Psi = l_S\frac{\partial\Psi}{\partial z}\quad \text{at the boundary}, \tag{13}$$

where the z axis runs normal to the surface, into the helium II.

If the helium is bounded by a solid, the length l_S is small ($l_S \lesssim a$), where $a \approx 3\cdot 10^{-8}$ cm is an interatomic distance), and condition (13) takes the form[4]

$$\Psi = 0\quad \text{at the boundary}. \tag{13'}$$

(See the discussion at the end of Subsection 3.3 for more details regarding this condition at the boundary with a solid.)

For a free boundary of helium (a boundary with the vapor), however, the length l_S is about 20 Å, and taking the difference between l_S and zero into account has a significant effect on the temperature dependence of the surface-tension coefficient at $\tau \gtrsim 10^{-3}$ (see Ref. 24 and Subsection 3.2 of the present paper).

2.4. Accuracy of the Ψ theory and its relationship with renormalization-group theory

The accuracy of the conclusions reached on the basis of expression (7) for the density of the incomplete thermodynamic potential is determined, on the one hand, by the small values of the critical exponents α and $\hat{\eta}$ (we have already mentioned that these exponents are small) and, on the other, by the small value of the contribution [ignored in (7)] of the long-wavelength Fourier components $\Psi_\mathbf{k}$ of the fluctuations $\delta\Psi = \sum_\mathbf{k}\Psi_\mathbf{k}e^{i\mathbf{kr}}$ with wave vectors $|\mathbf{k}| \leqslant k_m = Q/\xi_M$. As a criterion for judging the contribution of the long-wavelength fluctuations to be small at $T < T_\lambda$ we could use the condition[9,19,25]

$$\frac{\langle(\delta\eta)^2\rangle}{\eta_e^2} = \frac{\sum_{|\mathbf{k}|<k_m}\langle|\eta_\mathbf{k}|^2\rangle}{\eta_e^2} = \frac{k_B T_\lambda m^2}{2\pi^2\hbar^2\rho_{se}(\tau)\,\xi_M(\tau)}\,(Q - \operatorname{arctg} Q) \ll 1. \tag{14}$$

It is easy to see that, by virtue of the equality of the critical exponents of the density of the superfluid part and the correlation length, the left side of inequality (14) does not depend on τ, and its numerical value at the saturation vapor pressure is

$$0.15\left(\frac{3+3M}{3+M}\right)^{1/2}(Q - \operatorname{arctg} Q) = 3\cdot 10^{-2}$$

with $Q = 1$ and $M = 0$. Since the exponents α and $\hat{\eta}$ differ from zero by no more than $3\cdot 10^{-2}$, we can expect that the Ψ theory of superfluidity will be successful in describing the experimental data within an error of no worse than a few percent.

In renormalization-group theory, the equilibrium static and dynamic properties of helium near the λ point are found[1-3] by calculating the contribution of, again, only the most important part of the fluctuations in Ψ, with scales ranging from a microscopic scale $\xi_0(0)$ (of the order of interatomic distances) all the way to scales of the order of the correlation radius $\xi_0(\tau)$. The accuracy of these calculations is determined by the extent to which the so-called dimensionless renormalized coupling constant $u \sim \frac{\langle|\delta\Psi|^2\rangle}{|\Psi_e|^2}$ is small. The analytic expression for this constant is essentially the same as the left side of inequality (14) (according to Dohm,[3] we have $u \to u^* = 0.0362$ in the limit $\tau \to 0$).

Consequently, as we stressed back in the Introduction, calculations based on the Ψ theory of superfluidity and on the basis of the renormalization-group theory are by no means in contradiction of each other. The results derived in the renormalization-group theory[1-3] could in fact have been used to find the temperature dependence of the coefficients in the density of the incomplete thermodynamic potential, (6), not only in the region close to the λ point but also at some distance from this point, where these coefficients tend toward their original (unrenormalized) values, which correspond to Landau's theory of phase transitions. As a result, the range of applicability of that theory could have been extended considerably, especially at elevated pressures, where, according to Refs. 3 and 26, the temperature width of the range of applicability of the similarity theory on which expression (7) is based contracts significantly, amounting to only $\sim 10^{-4} T_\lambda$ (instead of $10^{-2} T_\lambda$ at the saturation vapor pressure).

We restrict the discussion below to an analysis of all the effects which occur in the similarity region, i.e., on the basis of specific expression (7) and on the basis of Eq. (10), which follows from it.

3. EQUILIBRIUM PROBLEMS (INCLUDING PURELY SUPERFLUID FLOW)

Armed with an equation for Ψ and corresponding boundary conditions, we can solve a variety of problems. Among them are size-effect problems: the shift of the λ point, $\Delta T_\lambda(d) = T_\lambda - T_\lambda(d)$ as a function of the film thickness d (we are speaking in terms of a film only for definiteness; we could also speak in terms of a slit, a capillary, a droplet, and so forth) and the changes in the density ρ_S and other thermodynamic properties as functions of the dimensions, i.e., as functions of d in the case of films. In the group of size-effect problems we might also include a calculation of the temperature dependence of the surface-tension coefficient at an interface between HeII and the vapor and between HeII and solid helium.[24,27]

A second category of equilibrium nonuniform problems is made up of problems in which the density ρ_S and other quantities are varied by external fields (gravitational, electric, and magnetic fields and the fields of van der Waals forces).[9,28,29] We could also include here the question of the nature of the change in the density ρ_S near the nuclei of positive and negative ions.[10,19]

A third and final category of problems which can be dealt with on the basis of expressions (7) and (10) concerns the dependence of ρ_S on the velocity of a purely superfluid flow, $\mathbf{v}_S$, and calculations of the maximum (critical) velocity of the superfluid motion of helium II in films and slits,[30] the distribution of the density ρ_S near the axis of a vortex filament,[1,31] the relationship between the λ transition of helium in a film or slit and the Berezinskiĭ-Kosterlitz-Thouless transition,[19] etc.

We will briefly discuss some of these effects below, referring the reader interested in the details and a corresponding bibliography to Refs. 9, 10, and 19.

3.1. Size effects

The nonuniformity of the distribution of the order parameter near boundaries [conditions (13), (13′)] leads to a shift $\Delta T_\lambda(d) = T_\lambda - T_\lambda(d)$ of the temperature of the λ transition of helium in films, slits, and capillaries. This effect was recognized experimentally a very long time ago (Ref. 32, for example), but a quantitative study of it has been complicated by the circumstance that for helium in films or slits there are generally two, closely spaced phase transitions, rather than a single phase transition. The first (thermodynamic) transition, which is the transition in which we are actually interested here, involves the appearance of a nonzero equilibrium value of Ψ, i.e., the appearance of a macroscopic number of particles in the energy ground level.[3)] If the parameter M which figures in (7) satisfies[19] $M < M_c \approx 2$, however, this thermodynamic λ transition does not yet lead to superfluidity, because of the spontaneous appearance of vortices in the film, which lead to a dissipation of the superfluid flow.[33-35] The corresponding vortices connect in pairs, and the dissipation disappears only as a result of the second ("topological") phase transition [the Berezinskiĭ-Kosterlitz-Thouless transition], which occurs at a lower temperature $T_{BKT}(d) < T_\lambda(d)$, which is found from the condition[34,35]

$$\bar{\rho}_s(T)d = \frac{2}{\pi}\left(\frac{m}{\hbar}\right)^2 k_B T, \qquad (15)$$

where $\bar{\rho}_S = d^{-1}\int \rho_S(z)dz$ is the mean density of the superfluid part over the cross section of the film.

In the case of sufficiently thick films (with $d \gtrsim \xi_M$), we can write[9]

$$\bar{\rho}_s d = \rho_{sb}(T)\, d - \Delta = \rho_{s0}\, d\tau^{3/2} - \Delta, \qquad (16)$$

where Δ is a τ-independent quantity which characterizes the surface deficiency of superfluid mass. From these two equations we find

$$T_\lambda - T_{BKT} = k_1 d^{-3/2}. \qquad (17)$$

Using homogeneous boundary condition (13′) on Ψ, we find that the numerical value of the coefficient k_1 in (17) is $k_1 = 3.82 \cdot 10^{-11}$ K·cm$^{3/2}$ in the case $M = 0$ or $k_1 = 3.57 \cdot 10^{-11}$ K·cm$^{3/2}$ at $M = 1$.

At small values $M < 1$, the shift of the point of the "thermodynamic" λ transition, in which we are interested, depends on the film thickness d in precisely the same way (this "thermodynamic" transition corresponds to the appearance of a spontaneous nonzero value of Ψ in the film):

$$T_\lambda - T_\lambda(d) = k_2 d^{-3/2}, \qquad (18)$$

where the constant k_2 is of course different, given by

$$k_2 = 2.53 \cdot 10^{-11}\left(\frac{3+M}{3}\right)^{3/4} \text{K/cm}^{3/2} \text{ for } M < 1. \quad (19)$$

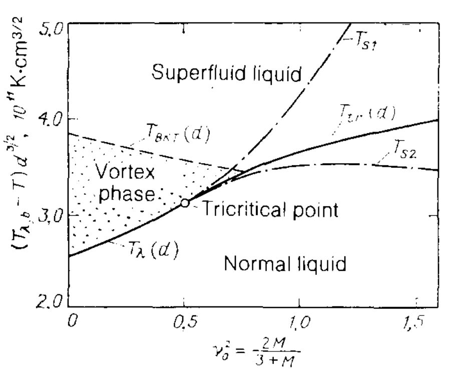

FIG. 1. Temperatures of phase transitions in helium films as functions of the parameter M. Solid line—Line of equilibrium phase transitions from helium II to helium I; dashed line—line of Berezinskiĭ-Kosterlitz-Thouless transitions; dot-dashed lines—lines of absolute superheating and supercooling, respectively, of the helium I phase and the helium II phase at $M > 1$. The value $M = 1$ on the line of equilibrium phase transitions corresponds to the tricritical point.

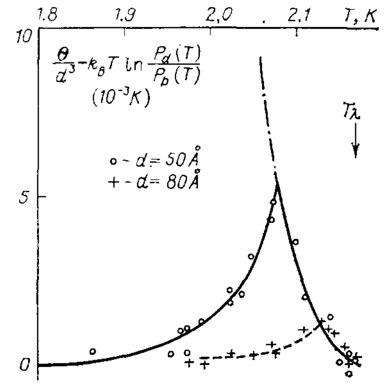

FIG. 2. Temperature dependence of the difference between the vapor pressures above a helium film with a fixed thickness d and above bulk helium. Circles and plus signs—Experimental data[13]; solid and dashed lines—results of theoretical calculations carried out in Ref. 13 on the basis of the Ψ theory of superfluidity.

At $M > 1$, the λ transition of helium turns out to be a first-order phase transition in films.[9,10] Along with temperature (18), which now determines the point of absolute supercooling of the normal phase, we are interested in two other characteristic temperatures: the temperature $T_{tr}(d)$, of the equilibrium λ transition (the point at which the thermodynamic potentials of the normal and superfluid phases are equal), and the temperature $T_{S_2}(d)$, which is the temperature of the absolute superheating of the superfluid phase (Fig. 1). Below, however, we will not discuss certain curious effects which might occur in the case $M > 1$, since the overwhelming majority of the experimental results available indicate that the λ transition in films remains a second-order transition, so we have $M < 1$ (this comment does not apply to ^{3}He-^{4}He solutions, in which, at sufficiently high ^{3}He concentrations, we have a parameter value $M > 1$; Ref. 44).

Experimentally, the temperature $T_{BKT}(d)$ is found from the beginning of a dissipationless superfluid flow, while the thermodynamic λ transition of a film (from the helium I phase to the helium II phase) has its most obvious effect on the behavior of the difference $\Delta C_\mu = C_{\mu,d}(T) - C_{\mu,b}(T)$ between the heat capacities of a helium film of thickness d and a bulk sample and on the behavior of the corresponding difference in the equilibrium vapor pressures $\Delta p = p_d(T) - p_b(T)$ and the difference between the mean equilibrium values of the total density, $\Delta\bar{\rho} = \rho_d(T) - \rho_b(T)$ [in contrast, at the point $T_{BKT}(d)$ the thermodynamic properties of helium films exhibit essentially no anomalies[35]].

Figure 2 shows examples of the temperature dependence of the difference between the vapor pressure above a film and above bulk helium, according to the measurements of Ref. 13. The dependences are seen to have a sharp maximum at $T = T_\lambda(d)$. This maximum can be explained easily, since it is at $T = T_\lambda(d)$ that the difference between the properties of helium in a film and the properties of bulk helium should be at a maximum.[4)] The solid lines in Fig. 2 correspond to the results of theoretical calculations[13] based on Eq. (10) with $M = 0$, and also with van der Waals forces (see also Refs. 28 and 36 and Subsection 3.3 below regarding the incorporation of these forces). Unfortunately, and despite the good agreement between the experimental and calculated curves, the experimental data of Ref. 13 cannot be used for a detailed quantitative test of the theory since they refer primarily to temperatures $T_\lambda - T > 10^{-2}$ K, where the assumption of a simple power-law temperature dependence for the coefficients in expression (7) for the thermodynamic potential density is no longer valid.

A detailed quantitative test of the predictions of the λ theory of superfluidity was recently undertaken in Ref. 14, where the difference $\Delta\bar{\rho}_d$ between the mean values of the total helium density in a narrow slit (with a thickness $d = 0.28$–$0.54\,\mu$m) and in a wide slit, with a well-controlled geometry, was measured. We are reproducing two figures from Ref. 14 here (Figs. 3 and 4). It can be seen from these figures that calculations based on Eq. (10) reproduce all the features of the experimental curve, including the presence of a poorly defined maximum on the $\Delta\bar{\rho}_d(T)$ dependence at $T = T_\lambda(d)$ and an increase in $\Delta\bar{\rho}_d$ in the superfluid phase. The set of experimental data of Ref. 14 yields the following value for the parameter M:

$$M = 0.6 \pm 0.3, \qquad (20)$$

According to the second paper in Ref. 14, the most probable value of M is close to 0.5.

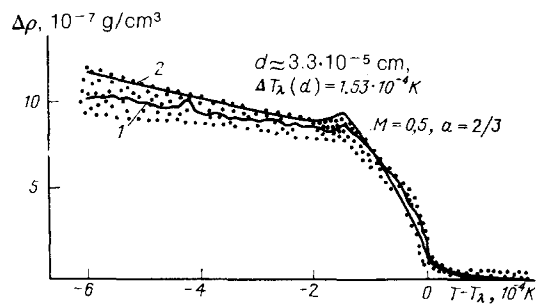

FIG. 3. Temperature dependence of the difference between the density of helium in a narrow slit ($d = 3.3 \cdot 10^{-5}$ cm) and in a wide slit ($d_0 = 5.2 \cdot 10^{-3}$ cm) near the λ point.[14] 1—One recording of an experimental signal; 2—results of theoretical calculations based on the Ψ theory of superfluidity, carried out under the assumption that for helium in a slit the logarithmically diverging part of the thermal expansion coefficient is cut off in the temperature interval between $T_\lambda - \alpha\Delta T_\lambda(d)$ and $T_\lambda + \alpha\Delta T_\lambda(d)$, where $\alpha = 2/3$. The dots show the typical error band, within which 80% of the experimental curves fall.

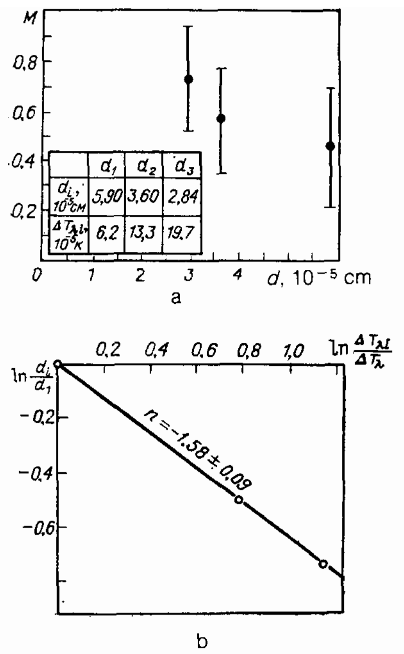

FIG. 4. The parameter M (a) and the shift of the λ point (b), $\Delta T_\lambda(d)$ as functions of the slit width d according to the data of Ref. 14.

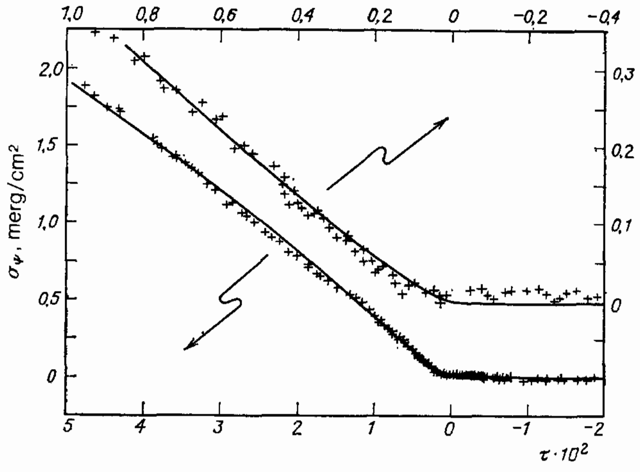

FIG. 5. Temperature dependence of the surface tension of liquid ^{4}He near the λ point, minus the regular (quadratic) function, as a function of $\tau = (T_\lambda - T)/T_\lambda$. Solid lines—theoretical[24]; plus signs—experimental data.[12]

To the best of our knowledge, the experiments of Refs. 13 and 14 have been the only experiments in which only a single slit or a film with a well-known, given thickness was used. In all other cases, the experiments have been carried out in systems containing a very large number of films, channels, or pores with unknown size distributions. The corresponding experimental results can therefore be of only extremely limited value. The recent experiments in Ref. 15 are apparently also not an exception to this rule. Those experiments dealt with the dynamic properties of liquid ^{4}He in gaps between the turns of a roll of a long Mylar tape. The distance between the turns of tape was controlled only on the average in those experiments, and it was apparently not uniform, as is implied by the presence of a tail on the temperature dependence of the density of the superfluid part (the intrinsic oscillation period of the system) at $T > T_{\mathrm{BKT}}(d)$ (see Fig. 1 in Ref. 15). This circumstance, combined with the use of a method for normalizing the $\bar{\rho}_s(d,T)$ dependence which we regard as improper (involving a matching with the $\rho_{sb}(T)$ dependence in a large volume), would make it premature to draw any conclusions about the violations supposedly discovered in Ref. 15 of the predictions of the Ψ theory and the theory of gauge invariance (see also the comments in Refs. 3 and 36 in this connection).

3.2. Surface tension

In good agreement with the conclusions of the Ψ theory of superfluidity[24,28] and the data from earlier experiments[11] are the results of some very accurate recent studies[12] of the temperature dependence of the surface tension of liquid ^{4}He below and near the λ point. Just how good this agreement is can be seen in Fig. 5, which is taken from Ref. 24. The curves in Fig. 5 show the results of a solution of Eq. (10) for a half-space with $M = 1/2$ and boundary condition (13), in which the extrapolation length l_S was taken to be

$$l_S = 22 \pm 2\,\text{Å}. \tag{21}$$

This value of l_S, which was found in Ref. 24 from the experimental data of Refs. 11 and 12, seems completely reasonable, since it is approximately equal to the mean value of the distance between ^{4}He atoms in the saturated vapor and larger by a factor of only three than the value of the thermal de Broglie wavelength $l_T = (2\pi\hbar^2/mk_B T) \approx 6$ Å at $T = T_\lambda$.

3.3. Effect of external fields

The following class of problems, which can be solved on the basis of the Ψ theory of superfluidity, consists of problems involving studies of the effect of various external fields on the λ transition in helium: gravitational, electric, and magnetic fields and the fields of van der Waals forces.

In expression (7) and Eq. (10), the presence of an external field with a potential $V(\mathbf{r})$, acting on a unit mass of helium, can be allowed for by taking account of the dependence of the field $V(\mathbf{r})$ on the temperature of the λ transition:

$$T_\lambda(\mu_0 - V(\mathbf{r})) \approx T_\lambda(\mu_0) - \frac{dT_\lambda}{d\mu} V(\mathbf{r}), \tag{22}$$

where μ_0 is the chemical potential of helium in the absence of a field, and $dT_\lambda/d_\mu \approx \rho_\lambda^{-1} dT_\lambda/d\mathbf{p}$ is the slope of the λ curve.

Because of correlation effects, which are frequently called "proximity effects" [the terms with spatial derivatives in (7) and (10)], the interface between HeI and HeII in a field is diffuse. In the case of a gravitational field ($V = -gz$), for example, the scale width of this diffuse interface is[28]

$$l_g = \xi_{00}^{3/5} \left(\left| \frac{dT_\lambda}{d\mu} \right| g \right)^{-2/5} \approx 6.7 \cdot 10^{-3}\ \text{cm}, \tag{23}$$

and the shape of the distribution of the function $\Psi(z)$ in the transition layer (more precisely, its second derivative $d^2\Psi/dz^2$) is determined directly by the right side of Eq. (10) (Fig. 6). In other words, by varying the shape of the distribution $\Psi(z)$ in a gravitational field, we can find direct information on the form of the thermodynamic potential density $\Omega(|\Psi|^2)$ [see (7)]. So far, there have been no successful attempts to measure the width of the interface between HeI and HeII in a gravitational force field, not to mention at-

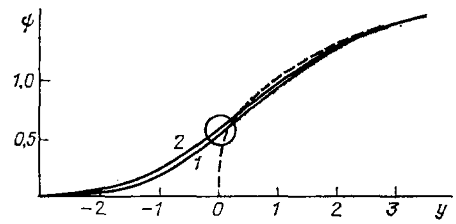

FIG. 6. Distribution of the order parameter $\psi = \Psi/\Psi_g$, where $\Psi_g = \Psi_{00}(\xi_{00}/l_g)^{1/2}$, in a gravitational force field.[28] Solid lines—found through a numerical solution of Eq. (10) with $M = 0$ (line 1) and $M = 1$ (line 2); dashed line—distribution $\psi(y) = y^{1/3}$, which would prevail in the absence of correlation effects [terms with spatial derivatives in Eq. (10)]; circle—the region in which the term with $|\Psi|^6$ plays an important role in Eq. (10).

tempts to find the function $\Psi(z)$. Studies of this sort are in principle completely feasible, by the method of the refraction of second-sound waves, for example.[29,37]

The effect of electric and magnetic fields on the λ transition in helium is studied in a corresponding way. For this purpose, it is sufficient to substitute the corresponding electric or magnetic striction potential into Eq. (22) as $V(\mathbf{r})$:

$$V_E(\mathbf{r}) = -\frac{1}{2}\alpha_E E^2(\mathbf{r}), \quad V_H(\mathbf{r}) = -\frac{1}{2}\alpha_H H^2(\mathbf{r}), \qquad (23')$$

where $\alpha_E = 3.1 \cdot 10^{-2}$ cm^3/g is the polarizability per 1 g of helium, and $\alpha_H = 0.47 \cdot 10^{-6}$ cm^3/g is the diamagnetic susceptibility.

The use of electric and magnetic fields expands the experimental capabilities, since the width of the transition region between helium I and helium II can be varied (reduced or increased), and one can create local regions with a reduced or elevated density of the superfluid part, which would serve as converging or diverging lenses for second sound (there are also other possibilities).[28,29] We would also like to point out that the gradient of an electric field or, especially, a magnetic field could be used to compensate for the nonuniformity of a column of a liquid in a gravitational field and thereby to avoid the use of satellites for such experiments. At the same time, we know quite well that here on the earth it is the presence of gravitational forces which is blocking research on critical phenomena near the λ point at $|\tau| \lesssim 10^{-7}$.

Van der Waals forces with a potential

$$V_{\text{v-d-W}} = -\theta/z^3, \quad \theta = 10^{-13} - 10^{-14}\ \text{erg}\cdot\text{cm}^3/\text{g} \qquad (24)$$

distort the function Ψ near a boundary between HeII and a solid. The effective manifestation of the distortion is that the surface on which Ψ vanishes is displaced a certain distance b from the surface of the solid, into helium.[28,36] The magnitude of the displacement b is essentially independent of τ, as was shown in Refs. 28 and 36, and ranges from 2 to 10 Å, depending on the value of the parameter θ. Nevertheless, as we move away from T_λ (and as ξ_M therefore decreases) the effects which stem from van der Waals forces generally become important.[13,28,36]

3.4. Contribution of ions and impurities to the thermodynamic functions of helium II

Electrostrictive potential (33) and boundary condition (13) lead to a decrease in the order parameter near the boundary of the nucleus of a positive ion and at the boundary of a bubble in the case of a negative ion or electron. Associated with this decrease are the contribution of the ion to the mass of the normal component,[28,10,19]

$$\Delta M_n = 8\pi\rho_{sb}\xi_M^2 R\,\frac{1+(R/\xi_M)}{1+(l_S/R)+(l_S/\xi_M)} \propto \tau^{-2/3} \qquad (25)$$

and the additional energy

$$\Delta\tilde{\Omega} = 2\pi\left(\frac{\hbar}{m}\right)^2 \rho_{sb} R\,\frac{1+(R/\xi_M)}{1+(l_S/R)+(l_S/\xi_M)} \propto \tau^{2/3}, \qquad (26)$$

where R is the radius of the nucleus or bubble, ξ_M is the correlation length given by (8), and l_S is the extrapolation length which figures in boundary condition (13). In this case there is also an additional contribution to the entropy and heat capacity of helium II. Furthermore, the existence of energy (26) causes a slight change in the equilibrium radius of the nucleus of the ion or bubble, and it increases their effective mass:

$$\Delta m_{eff} \approx \frac{4\pi}{3}\left(\frac{\hbar}{m}\right)^2 \rho_{sb} R\,\frac{1}{T_\lambda - T}\left|\frac{dT_\lambda}{d\mu}\right|\frac{1}{1+(l_S/R)} \propto \tau^{-1/3}. \qquad (27)$$

The latter effect could apparently be seen most easily, since at $\tau = 10^{-6}$ the increase in the effective mass of a negative ion would be about $60 m_{He}$.

If we set $R \approx a$, where a is the interatomic distance, we can use expressions (25)–(27) to estimate the contribution of microscopic impurity particles such as ^{4}He atoms to the thermodynamic functions of helium. The density of these atoms, n_3, however, must satisfy the condition

$$3n_3 a^2 \xi_M^2 l_S^1 \ll 1. \qquad (28)$$

In other words, expressions (25)–(27) can be used only for very dilute solutions. At high impurity concentrations (these impurities may be ions), it becomes necessary to use the results of Ref. 54.

3.5. Superfluid flows ($v_n = 0$)

A third category of problems to which the Ψ theory can be applied effectively involves analysis of the flow and rotation of helium. The corresponding list of questions is very long (see Refs. 9, 19, and 38, for example). Here we can only touch on some of them.

The appearance of a flow (as in the presence of boundaries) alters the equilibrium values of ρ_S and of other thermodynamic quantities. For a purely superfluid flow, which is constant over time (a steady-state flow), the relative magnitude of these changes is proportional to the square of the ratio $v_S/v_0(T)$, where

$$v_0(T) = \frac{\hbar}{m\xi_0(T)} = 9.74 \cdot 10^3 \tau^{2/3}\ \text{cm/s}. \qquad (29)$$

In particular, with $v_s > v_{s,c2} \approx v_0(T)\sqrt{3}$ a laminar superfluid state becomes unstable in general.[9,10,19] More-accurate expressions for the maximum possible (critical) velocities of a laminar superfluid flow in films and massive helium, which depend on d and M, are given in Refs. 9, 19, and 30. The behavior of $\bar{\rho}_S$ as a function of v_s^2 and d was also studied in Ref. 30.

For rotating helium II, an isolated vortex filament is the simplest and at the same time the most important entity. The problem of the structure of a vortex filament in helium II near the λ point was in fact studied a fairly long time ago on

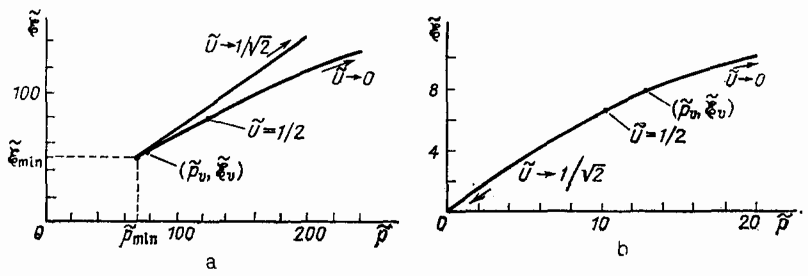

FIG. 7. a—The dimensionless energy of a vortex ring, $\tilde{\mathscr{E}} = \mathscr{E}/(\Omega_{\rm I} - \Omega_{\rm II})$, as a function of its dimensionless momentum $\tilde{p} = p/mv_0(T)$ and the velocity of the translational motion, $\tilde{U} = U/v_0(T)$ (according to the data of Ref. 39b) at the point $(\tilde{p}_v, \tilde{\mathscr{E}}_v)$, where $\tilde{U} \approx 0.62$, the ring loses its circulation and converts into a vortex-free excitation with $|\Psi|^2 = 0$ at no spatial point; vortex-free excitations correspond to a part of the lower branch {to the left of the point $(\tilde{p}_v, \tilde{\mathscr{E}}_v)$} and the entire upper branch {to the left of the point $(\tilde{p}_v, \tilde{\mathscr{E}}_v)$ and the entire upper branch of the spectrum}; b—the same for a vortex pair the part of the curve from the origin to the point $(\tilde{p}_v, \tilde{\mathscr{E}}_v)$ with $\tilde{U} = 0.43$ corresponds to vortex-free excitations.

the basis of the Ψ theory of superfluidity by Ginzburg and Pitaevskiĭ.[4] This problem was solved in Ref. 31 with the subsequent modification of that theory. In addition, the contribution of a unit length of a filament to the entropy, the heat capacity, the density of helium, and the mass deficit of the superfluid component was calculated in Ref. 31; the cross sections for the scattering of light and second sound by individual vortex filaments were also estimated. All this work is interesting from the standpoint of possible experiments.

Yet another important problem, which was mentioned in Ref. 9 and which is of fundamental importance to an understanding of the nature of the critical rates of vortex formation, is the problem of the motion in helium II of a quantized vortex ring with a radius R comparable to $\xi_{\rm M}(T)$.

An important step was recently taken in Refs. 39 and 40 (see also Ref. 41) toward the solution of this problem. That step was the derivation of an exact localized (soliton-like) solution of the equation of motion for Ψ in the absence of dissipation [see Eq. (30) below with $\Lambda = 0$]. This forward step generated some interesting and rather unexpected results. It turned out that at low velocities U of a corresponding axisymmetric soliton the energy $\mathscr{E}$, the radius R, and the momentum p of the soliton decrease with increasing U, as they should in the case of a vortex ring. When the velocity U reaches a certain critical value $U_{\rm c} = 0.62\, v_0(T)$, however, where $v_0(T)$ is given by (29), the energy and momentum of the soliton begin to increase again (Fig. 7a). The reason, as was shown in Ref. 40, is that in the limit $U \to U_{\rm c}$ a vortex ring (i.e., a circle on which the equation $|\Psi| = 0$ holds) contracts to a point, while at $U > U_{\rm c}$ it "jettisons" circulation and converts into a vortex-free excitation, for which the amplitude of the function Ψ no longer vanishes, at any point in space.

Iordanskiĭ and Smirnov[40] showed that a pair of parallel quantized vortex filaments with opposite circulation signs (a "vortex pair") in translational motion exhibits a similar behavior. In that case, however, in contrast with a vortex ring, the energy and momentum of the vortex pair continue to decrease below the point $U = U_{\rm c} \approx 0.43\, v_0(T)$, at which the pair jettisons its circulation (Fig. 7b).

That picture is extremely instructive, but we would like to emphasize that incorporating dissipation and the relaxation of Ψ may cause some fundamental changes in this picture.[5)]

Unfortunately, there have been almost no experimental studies of the thermodynamic properties of moving helium II near the λ point, not to mention studies of the structure and properties of individual vortex filaments and vortex rings in this region. We are thus not able to illustrate with experimental data the results discussed above.

4. GENERAL Ψ THEORY OF SUPERFLUIDITY (WITH NORMAL FLOW, DISSIPATION, AND A TIME VARIATION)

4.1. Equation of motion for a macroscopic wave function

In the steady state, with a given nonzero velocity of the normal flow, $\mathbf{v}_{\rm n}$, and with a constant total density ($\nabla\rho = 0$), it is a straightforward matter to generalize expression (7) and equilibrium equation (10) (Ref. 9). This generalization is achieved through the substitution

$$\nabla \to \nabla - \frac{im}{\hbar}\mathbf{v}_{\rm n}.$$

In the general time-dependent case, the following equation has been proposed[5] (see also Refs. 9, 10, and 19) for Ψ:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + (\mu + \mu_{\rm s})\, m\Psi - i\Lambda\left[\frac{1}{2}\left(\frac{i\hbar}{m}\nabla + \mathbf{v}_{\rm n}\right)^2 + \mu_{\rm s}\right] m\Psi. \qquad (30)$$

This equation was mentioned back in Subsection 3.5. Here $\mu_{\rm S} = m^{-1}(\partial\Omega/\partial|\Psi|^2)$ is the chemical potential of the superfluid component, and the dimensionless kinetic coefficient Λ determines the relaxation time of Ψ, i.e., the density $\rho_{\rm S}$:

$$t_\Psi = \frac{\hbar}{2\Lambda m}\left(\frac{\partial\mu_{\rm s}}{\partial\rho_{\rm s}}\right)^{-1}_{\mu,T} = \frac{2m\xi_{\rm M}^2}{\hbar\Lambda}. \qquad (31)$$

A complete system of hydrodynamic equations for helium II near the λ point was derived in Ref. 5 and can also be found in Refs. 9, 10, and 19. The system includes, along with Eq. (30), conservation equations for the mass and the momentum and a heat balance equation. Furthermore, some similar but slightly different systems of equations were derived in Refs. 42 and 43. The corresponding equations are quite lengthy, and we will not reproduce them here. We would simply like to note that they contain, in addition to Λ, several other kinetic coefficients, among which those which play the most important role near the λ point are the thermal conductivity κ and the viscosity of the normal component, $\eta_{\rm n}$.

The temperature dependence of the coefficients Λ, $\varkappa$, and $\eta_{\rm n}$ must be determined, as in the static case (§2), from experimental data, or it must be found from calculations by the renormalization-group theory.[1-3] Such calculations can be carried out for the most part by using the Ψ theory to analyze the propagation of various types of sound in HeII near the λ point. Unfortunately, this analysis has not yet been completed. From a comparison[9,10] with experimental data on the absorption of first and second sound, we can only conclude that in a certain narrow interval ($10^{-5} \lesssim \tau \lesssim 10^{-3}$) the temperature dependence of Λ at the saturation vapor pressure can be described approximately by

$$\Lambda = \Lambda_0 \tau^{-1/3} \text{ with } \Lambda_0 = 0.3 \pm 0.1. \quad (32)$$

The experimental value of the viscosity coefficient η_n at the λ point at the saturation vapor pressure is[44]

$$\eta_n = (27.4 \pm 1.4) \cdot 10^{-6} \text{ P}, \quad (33)$$

but the temperature dependence of η_n near T_λ has not been measured accurately.

With regard to the thermal conductivity $\varkappa$, we note that below T_λ we can cite only an extremely crude order-of-magnitude estimate, based on the theory of dynamical gauge invariance[45]:

$$\varkappa \sim \frac{\hbar \Lambda C_\mu}{2m} \sim 10^4 \tau^{-1/3} \text{ erg/(cm} \cdot \text{g} \cdot \text{K)}. \quad (34)$$

We turn now to the results of a solution based on the complete system of hydrodynamic equations for helium II near the λ point[5,9,10,19] for certain specific problems.

4.2. Mutual friction force

The force of mutual friction between the superfluid and normal components in rotating helium II, caused by the presence of quantized vortex filaments in it, was calculated on the basis of the Ψ theory of superfluidity in Refs. 46–48. The most comprehensive quantitative analysis was carried out by Sonin,[47] who found the following results:

$$g = g_0 \tau^{1/3} = 1.0 \tau^{1/3}, \quad g' = g'_0 \tau^{1/3} = 1.48 \tau^{1/3}. \quad (35)$$

Here the coefficients g and g' are related to the mutual friction coefficients B and B', which are ordinarily used, by

$$B - iB' = \frac{2}{g - ig'}. \quad (36)$$

Relations (35) correctly convey the observed temperature dependence of the longitudinal and transverse components of the mutual friction force,[16] but the numerical values of the coefficients (the "amplitudes") in these relations are slightly different from the experimental values (experimentally,[16] these amplitudes are found to be $g_0 = 2.8$ and $g'_0 = 2.02$). These discrepancies are not surprising in view of the crudeness of the numerical estimates of the coefficients in Ref. 47, which were furthermore carried out for a version of the theory with $M = 0$. Furthermore, the viscosity and the thermal conductivity of the normal component were ignored in Ref. 47, but these effects may turn out to be extremely important at temperatures at which the measurements were carried out in Ref. 16.

4.3. Boundary thermal resistance

As heat propagates across an interface between two media, a temperature drop forms at the interface. The magnitude of this drop is proportional to the heat flux density ($\Delta T = R_K q$), and the corresponding proportionality coefficient R_K is called the "boundary thermal resistance" or "Kapitsa resistance": In a superfluid liquid the ordinary surface temperature drop is accompanied by an additional temperature drop which results from the conversion of the diffusion heat flux into a countercurrent of the normal and superfluid components. This conversion is localized in a layer near the boundary of liquid helium II with a typical thickness[0]

$$l_T = \left(\frac{\varkappa \Sigma}{S^2 T} \right)^{1/2}, \quad (37)$$

where S is the entropy density, and $\Sigma = \frac{4}{3}\eta_n + \zeta_2 - 2\rho\zeta_1 + \rho^2\zeta_3$ is a combination of the coefficients of the first and second viscosities which appears in the expression for the attenuation coefficient for second sound.

Near the λ point the anomalous part of the combination of second-viscosity coefficients can be expressed[9] in terms of the same kinetic coefficient Λ:

$$\zeta_2 - 2\rho\zeta_1 + \rho^2\zeta_3 = \frac{\hbar\rho(1+\Lambda^2)}{2\Lambda m \rho_{sb}(T)}. \quad (38)$$

In the limit $T \to T_\lambda$ it increases in accordance with $\sum \propto \tau^{-1}$. Correspondingly, in the limit $T \to T_\lambda$ the length l_T ($l_T \propto \xi_M \propto \tau^{-2/3}$) increases without bound, and the additional component of the resistance, R_K, which is related to this length also increases[9,10]:

$$\begin{aligned} \delta R_K &= l_T \varkappa^{-1} && \text{for } l_T \gg \xi_M \\ &= 2.2 (l_T \xi_M)^{1/2} \varkappa^{-1} && \text{for } l_T \ll \xi_M. \end{aligned} \quad (39)$$

An experimental detection of the component δR_K might be of assistance both in testing the predictions of the Ψ theory of superfluidity and in determining the temperature dependence of $\varkappa$, which is not known at $T < T_\lambda$, as we have already mentioned. The presence of a singular component of the resistance R_K was recently found experimentally.[17] However, a detailed comparison of the data of Ref. 17 has not yet been made with expressions (37)–(39), in particular, because the coefficient $\varkappa(T)$ is not known, as we have just mentioned.

4.4. Mobility of ions in helium II near the λ point

Incorporating the diffusion heat flux, which exists over length scales of the order of l_T, and the nonuniformity of the distribution of the order parameter near the core of an ion (Subsection 3.5) causes the expression for the friction force $F_\pm$ which acts on the positive (+) and negative (−) ions in helium II to be different from the hydrodynamic Stokes formula.[49,50] Near the λ point, within terms of the order of ρ_s/ρ, we find[49,10]

$$F_+ = 6\pi\eta_n R u \left\{ 1 - \frac{\rho_{sb}(T)}{\rho(\Lambda^2+1)} \left[\frac{c}{2} - \frac{2}{15}c^2 + \frac{\Lambda}{\omega_0}\left(1 - c + \frac{c^2}{3}\right) - \frac{11}{105}\Lambda\omega_0 c^2 \right] \right\}, \quad (40)$$

$$F_- = 4\pi\eta_n R u \left\{ 1 - \frac{2\rho_{sb}(T)}{3\rho(\Lambda^2+1)} \left[c - \frac{c^2}{3} + \frac{\Lambda}{\omega_0}\left(1 - c + \frac{c^2}{3}\right) - \frac{1}{3}\Lambda\omega_0 c^2 \right] \right\},$$

where u is the velocity, R is the radius of the nucleus of the ions, $\omega_0 = \hbar\rho_\lambda / m\eta_n = 0.846$, $c = R/(R + l_S)$, and l_S is the extrapolation length which figures in boundary condition (13).

Expressions (4) with $\Lambda = 0.3\,\tau^{-1/3}$ and $c = 1$ and 0.5, for positive and negative ions, respectively, agree with experimental data.[51,52] For a more reliable test, however, the corresponding measurements should have been pursued into a temperature interval closer to the λ point. Furthermore, account should have been taken of the nonuniformity of the total-density distribution near the ions and of the slight singular dependence of the radii of the ion cores near T_λ.

4.5. Transverse acoustic impedance[6)]

Yet another interesting method for testing the Ψ theory would be to study the spatial distribution of the order parameter (the density of the superfluid component) near a solid wall by probing this distribution with an aperiodic viscous wave excited by transverse (shear) oscillations of the wall material.[55,58]

We know that the depth to which an aperiodic viscous wave penetrates into a normal liquid depends on the frequency; if the liquid is homogeneous, this depth is $\delta = (2\eta\omega/\rho)^{1/2}$. For homogeneous helium II, we should set $\rho = \rho_n$ and $\eta = \eta_n$ in this formula. If, however, the density ρ_n and the viscosity coefficient η_n vary significantly over length scales less than or comparable to δ, the contribution of the helium to the real and imaginary parts of the transverse acoustic impedance of an oscillating object differs from that calculated from the standard hydrodynamic formula (Ref. 55, for example):

$$Z = R - iX = (1-i)\left(\frac{\eta_n\rho_n\omega}{2}\right)^{1/2}.$$

The corresponding corrections, which contain information on the dependence $\rho_n(z) = \rho - \rho_S(z)$ and $\eta_n(z)$ near the wall, were calculated and measured in Refs. 55–58. Those investigators reached the conclusion that the temperature dependence of the correlation lengths $\xi_M^-(\tau)$ and $\xi_M^+(\tau)$ can be described well by Eqs. (8) and (8a), but the coefficients in these dependences are only about 0.5 Å, i.e., only half the values which have been estimated from other experiments (and, above T_λ, only a third of these other values).

We believe that the reason for this discrepancy might be the use of the greatly simplified approach of Ref. 57 in the interpretation of the experimental data in Refs. 55–58. The influence of van der Waals forces on the distribution $\rho_n(z)$ and, especially, $\eta_n(z)$ was ignored in Ref. 57. Near the λ point, the viscosity of helium is a very strong function of the total density of the liquid, ρ (Ref. 44), which increases near a wall. Furthermore, at the frequencies $\omega \gtrsim 10^7$ Hz which were used in Refs. 55–58 the frequency dispersion of the shear viscosity coefficient η_n might also play a certain role.

These examples illustrate only some of the possible applications of the general Ψ theory of superfluidity. On the whole, we can say that with regard to time-dependent and dissipative problems we are merely at the beginning of a long road.

5. CONCLUDING REMARKS

We listed several problems above which have been studied theoretically or could be analyzed on the basis of the Ψ theory of superfluidity. Progress toward a comparison of this theory with experimental data must nevertheless be regarded as slow. Considerably less is being done in this field than one might wish and than appears possible in view of the experimental facilities available at present. There is the possibility that part of the difficulty is that experimentalists are being overloaded with a stream of new and trendy questions (the properties of ^{3}He, etc.). However, it appears to us that a more important factor is a distrust of the Ψ theory of superfluidity, which has been the subject of this review. Such a distrust would have been justified if we had been talking about the original self-consistent version of the theory,[4,5] but in its present form[19] the theory is—as we have repeatedly stressed above—free of the shortcomings of the mean-field approximation, and it organically incorporates the results of the fluctuation theory of phase transition which have been derived for a homogeneous substance.

The use of the Ψ theory of superfluidity near the λ point thus appears to us to be completely natural and legitimate. In this regard, this theory is analogous to the Ψ theory of superconductivity.[8] Several convincing experimental confirmations of the predictions of the Ψ theory of superfluidity have already been obtained. It would, on the other hand, be premature to say that a specific Ψ theory of superfluidity (with the set of values of critical indices and coefficients used above) is completely successful. Everything depends on how well it corresponds to experiment. Further effort in this direction is extremely desirable and might be helpful both in determining the place of the Ψ theory of superfluidity and in clarifying the situation with regard to the behavior of helium near the λ point in general.

1)Invited paper presented at the Eighteenth International Conference on Low-Temperature Physics (LT-18), Kyoto, Japan, August 1987, Proceedings, pp. 1785–1797.

2)This comment had already been made by one of us (V.L.G.) in a discussion (on 8 July 1987) at a conference on high-temperature superconductivity in Trieste, Italy (see also Ref. 6).

3)We wish to emphasize that we are talking here about a condensate of particles specifically in the ground level, not in a state with a momentum $\mathbf{p} = 0$ [this statement means that the Fourier expansion of the microscopic Ψ function of the ground state contains a term $n_0\delta(\mathbf{p})$]. As we know quite well, however, there is no term $n_0\delta(\mathbf{p})$ in the case of quasi-two-dimensional (and, especially, quasi-one-dimensional) systems. In other words, a state (Fourier component) with $\mathbf{p} = 0$ for quasi-two-dimensional systems is not macroscopically filled anywhere down to $T = 0$. With regard to the ground level, we note that even in the strictly two-dimensional case in an interacting system of Bose particles a macroscopic number of particles will necessarily appear in this level even if the temperature $T_\lambda(0)$ deviates only slightly from zero. The reason is that in an interacting system of Bose particles (in contrast with a noninteracting system) the initial part of the spectrum of excited states is linear ($\varepsilon = up$, where u is the sound velocity), and here the integral

$$N_{ex} = \frac{V}{(2\pi\hbar)^2}\int\frac{d\mathbf{p}}{\exp(\varepsilon/k_BT) - 1},$$

which determines the total number of particles in excited states, does not diverge in the long-wavelength limit. Furthermore, it vanishes as $T\to 0$. Consequently, in a two-dimensional Bose gas at low temperatures, only a relatively small fraction of the total number of particles can be in excited states. This statement also means that in it there is a condensate of particles in the ground level.

4)In the interval $T_\lambda > T > T_\lambda(d)$, the difference $p_d(T) - p_b(T)$ increases in proportion to the difference between the thermodynamic potentials of the normal and superfluid phases: $\Delta p_d = \Omega_I - \Omega_{II} = T_\lambda \Delta C_\mu \tau^2$. At $T < T_\lambda(d)$, on the other hand, it again begins to decrease because of the decrease in the ratio $2\xi_M(\tau)/d$, which characterizes the effective fraction of the volume of helium near the boundaries which remains in the normal phase.

[5)]The Ψ theory of a nonideal Bose gas,[41] which is widely used in the literature, in particular, in Refs. 39 and 40, is actually a particular case of the Ψ theory of the superfluidity of helium II which we have developed. Specifically, in the theory of Ref. 41, which pertains to $T=0$, the temperature dependence of the coefficients in expressions of the type (7) and (10) is ignored, and the relaxation and dissipation of Ψ are also ignored.

[6)]This subsection (4.5) of the paper is not included in the English text of this report, since we learned about Refs. 55–58 only at the LT-18 conference itself.

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