Quantization of the electromagnetic field

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The question of a canonical quantization of the electromagnetic field as a system with two primary constraints is analyzed. Attention is focused on the following aspects of the problem: 1) the conditions on physical states; 2) the "old" and "new" Gupta formalisms; 3) the interpretation of the operators $\hat{a}_0, \hat{\alpha}_0^+$; 4) the vacuum in quantum electrodynamics. The starting point for the analysis is Dirac's quantization recipe for systems with constraints. It is shown that the Fermi formalism with a condition on the physical states is completely correct.

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Since the electromagnetic field is observable classically, it, above all other fields, should be quantized according to the canonical procedure. It is ironic that of the fields we shall consider, it is the most difficult to quantize.

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1. INTRODUCTION

Although the electromagnetic field was the first dynamic system with an infinite number of degrees of freedom to be subjected to quantization,[1] it seems that we still lack a description of this procedure which is completely satisfactory from the modern standpoint.

The difficulties in a quantum description of the electromagnetic field are rooted in a combination of the pseudo-Euclidean nature of space-time and the property of gauge invariance of the theory, i.e., in the presence of nonphysical degrees of freedom for the vector potential $A_\mu(x)$. These nonphysical degrees of freedom are allowed in the theory so that it can be made explicitly relativistically invariant. In classical mechanics, gauge invariance is eliminated by adding to the equations of motion a suitable gauge condition, e.g.,

$$\partial_\mu A_\mu(x) = 0. \tag{1.1}$$

This is the relativistically invariant Lorentz gauge. In quantum theory, $\check{A}_\mu(x)$ are operators which obey certain commutation relations[2]

$$[\hat{A}_\mu(x), \hat{A}_\nu(y)] = ig_{\mu\nu}D(x-y) \tag{1.2}$$

($g_{\mu\nu}$ is a metric tensor, and D is the commutation function of a massless scalar field; i.e., it is assumed that the fields are free), so conditions (1.1) cannot be transferred to these operators. Specifically, if we apply the operator ∂_μ to (1.2) and use (1.1), we find $\partial_\mu D(x-y) = 0$, which is wrong. The latter point was understood quite clearly by the creators of quantum electrodynamics (QED). The only way to retain condition (1.1) in the classical limit without running into conflict with the postulates of quantum mechanics is to require that this condition hold only on the physical vectors of the Hilbert space:

$$\partial_\mu \hat{A}_\mu(x)\psi_{\text{phys}} = 0; \tag{1.3}$$

This equality is written in the Heisenberg picture. Condition

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(1.3) is linked with the name of Fermi[3-5] (Refs. 3–5 have been translated into Russian[6]; see also Ref. 7 and Subsection 6.5 of the present paper). It was this condition which was subsequently used by Dirac.[8,9] The question of the acceptability of this condition was raised during the development of modern QED.

Several authors[10-12] have stated that physical states cannot be normalized in a formalism using condition (1.3) (see the criticism in Subsections 4.2 and 6.4). Accordingly, Gupta[13] replaced this condition by the weaker condition

$$\partial_\mu \hat{A}_\mu^{(+)}(x)\psi_{\text{phys}} = 0 \tag{1.4}$$

($\hat{A}_\mu^{(+)}$ contains annihilation operators). In the same paper, Gupta[13] formulated a rather lengthy quantization procedure which incorporated condition (1.4). Gupta essentially postulated the existence of an auxiliary Hilbert space, to which he then assigned an indefinite metric. The operation of taking the Hermitian adjoint was redefined (see Ref. 14 and also Subsection 6.1 of the present paper). Immediately after the appearance of Gupta's paper,[13] Bleuler[15] generalized it to the case of interacting fields. It appears that Gupta subsequently recognized that an indefinite metric is inherent in QED, and he rejected the artificial introduction of an auxiliary Hilbert space with a definite metric.[16] Since the original formalism[13] also lacked an explicit relativistic invariance (it has been asserted[17] that is formalism is Lorentz-noninvariant in general), Gupta devoted a separate paper[18] to proving the relativistic invariance of his final construction. A (harmless) deficiency of that construction is the presence of vectors with a zero norm in the physical subspace. States of this sort obviously cannot have a physical meaning. The overall procedure is now referred to as "Gupta-Bleuler quantization." The first version of this procedure[13,15] is the one usually reproduced in textbooks.[2,19,20] In their book, Jauch and Rohrlich[21] present both quantization schemes, i.e., the schemes using conditions (1.3) and (1.4).

On the whole, a reading of the literature on this problem does not leave the impression of complete clarity. This question, however, is of more than methodological interest. Some typical systems of this sort are Yang-Mills fields and the gravitational field. A corresponding problem arises in the quantization of a string. A Hamiltonian mechanics of systems with constraints has been developed, and general rules for the canonical quantization of these systems have been formulated, in an effort to construct a quantum-mechanical description of gravity.[22-24] According to the theory, electrodynamics is a system with two primary constraints. Below we analyze the problem of canonical quantization of the electromagnetic field in detail.

Notation. The metric $g_{\mu\nu}$ $(+ - - -)$ is adopted. The Greek indices run over the values 0, 1, 2, 3; the Latin indices run over the values 1, 2, 3, unless otherwise stipulated. A streamlined notation is used for differentiation operators, $\partial_\mu = \partial/\partial x^\mu$, and for the d'Alembertians, $\Box \equiv -\partial_\mu^2 = -g^{\mu\nu}\partial_\mu\partial_\nu$. Repeated Greek indices which take on identical values mean a summation with the appropriate metric tensor, e.g., $q_\mu x_\mu \equiv g^{\mu\nu} q_\mu x_\nu = q_\mu x^\mu \equiv qx$. Physical state vectors are denoted by Φ, the physical vacuum by Φ_0, and the mathematical vacuum by ψ_0. States generated by the operators $\hat{a}_0^+$, $\hat{a}_3^+ = \hat{a}_1^+(q)q_i/|\mathbf{q}|$ are called states with "time-like" and "longitudinal" photons. The quantities δ_{ik}, δ_i^k, $\delta_{\mu\nu}$, δ_μ^ν are Kronecker deltas; {,} are classical Poisson brackets; and $\{x^i, p_k\} = \delta_k^i$. We are using the Heaviside (rationalized) system of units. Everywhere we are setting $\hbar = c = 1$; e is the electric charge.

2. CLASSICAL THEORY

2.1. Lagrangian formalism

The dynamics of a free electromagnetic field is specified by the Lagrangian functional density

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{2}K^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma$$
$$= -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}(\partial_\mu A_\mu)^2 + \frac{1}{2}\partial_\mu(A_\nu\partial_\nu A_\mu - A_\mu\partial_\nu A_\nu), \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $K^{\mu\nu\rho\sigma} = g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}$. The tensor $F_{\mu\nu}$ is invariant under gauge transformations $A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)$, where Λ is an arbitrary function of the coordinates. In other words, the theory is invariant under an infinite-dimensional transformation group, and according to Noether's second theorem[25] an identity relation holds among the equations of motion. This assertion means that there are fewer equations of motion than there are unknowns; i.e., the time evolution of some of the variables is not contained in (2.1). The Langrange equations of motion which follow from (2.1) are

$$\partial_\mu F_{\mu\nu} = 0, \tag{2.2}$$

and the identity relation which we just mentioned among the four equations in (2.2) for the four functions A_μ is $\partial_\mu\partial_\nu F_{\mu\nu} \equiv 0$ (the Noether identity). Gauge invariance is eliminated by supplementing (2.2) with requirement (1.1). The rest of Maxwell's equations $(\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0)$ hold identically for arbitrary A_μ.

2.2. Hamiltonian formalism

Gauge invariance complicates making the switch to a Hamiltonian formalism. If L is the Lagrangian, we need to solve the N equations

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \qquad (i = 1, 2, \ldots, N), \tag{2.3}$$

for the velocities $\dot{q}$ and then eliminate them from the expression $H = p_i\dot{q}^i - L(q,\dot{q})$ in order to make this switch. System (2.3) can be solved only if the matrix

$$T_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \tag{2.4}$$

is nondegenerate. This is not the case in electrodynamics, where, according to (2.1), the matrix (2.4),

$$T^{\nu\sigma} = \frac{\partial^2 \mathscr{L}}{\partial \dot{A}_\nu \partial \dot{A}_\sigma} = K^{0\nu 0\sigma} = g^{00}g^{\nu\sigma} - g^{0\nu}g^{0\sigma} \tag{2.5}$$

is degenerate since, being diagonal, it has a zero on its diagonal: $T^{00} = 0$. It is this circumstance which leads to the appearance of constraints.[24] When Eqs. (2.3) are solved, the velocities drop out of some of the equations, and they convert into relations between coordinates and momenta of the type

$$\varphi_k(q, p) = 0 \qquad (k = 1, 2, \ldots, s, \quad s < N), \tag{2.6}$$

which are called "constraints" (the functions φ_k themselves are also called "constraints"). Dirac[24] carried out a general analysis of systems of this sort. For electrodynamics, Eqs. (2.3) are

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}, \tag{2.7}$$

from which we conclude that one of the canonical momenta π^μ is zero (a primary constraint).

$$\pi^0 = 0. \tag{2.8}$$

This is how the presence of nonphysical degrees of freedom is manifested in the Hamiltonian formalism. Calculating the Hamiltonian, we find

$$H_0 = \int d^3x \left[\frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2) - A_0 \operatorname{div} \mathbf{E} + \pi^0 \dot{A}_0\right], \tag{2.9}$$

$$E^k = F^{k0},\ \mathbf{H}^2 = \frac{1}{2} F_{ik}^2,\quad \operatorname{div} \mathbf{E} \equiv \partial_k E^k \qquad (i,\ k = 1,\ 2,\ 3).$$

By virtue of (2.8), and also since the Hamiltonian formalism must not contain velocities, the last term in (2.9) is usually discarded. We now need to ensure that the dynamic system is noncontradictory, i.e., that constraint (2.8) holds at all times and does not contradict the equations of motion. For this purpose we need the equality $\dot{\pi}^0 = 0$. Calculating $\dot{\pi}^0 = \{\pi^0, H_0\} = \operatorname{div}\mathbf{E}$, where

$$\{f,\ g\} = \int d^3x \left(\frac{\delta f}{\delta A_\mu(x)} \frac{\delta g}{\delta \pi^\mu(x)} - \frac{\delta f}{\delta \pi^\mu(x)} \frac{\delta g}{\delta A_\mu(x)}\right)$$

are the classical Poisson brackets, we find the new condition

$$\operatorname{div} \mathbf{E} = 0. \tag{2.10}$$

Since the electric field $E^k \equiv \pi^k$ is a canonical momentum according to (2.7), Eq. (2.10) is also a constraint (a secondary constraint). There are no other constraints, since we have $\{\operatorname{div}\mathbf{E}, H_0\} = 0$. For the analysis below it is important to determine the values of the Poisson brackets of both constraints. We have $\{\pi^0, \operatorname{div}\mathbf{E}\} = 0$; i.e., the constraints are in an involution (primary constraints). This fact plays an important role in the derivation of quantum theory,[24] since constraints of this sort may be thought of as certain generalized momenta. Conditions (2.8) and (2.10) cannot be extended to operators without running into conflict with the canonical commutation relations.[7,24]

It was mentioned above that the presence of constraints in the theory reflects the presence of nonphysical variables. The number of nonphysical variables (if there are no secondary constraints) is equal to the number of constraints. Consequently, an electrodynamics formulated in terms of a vector field A_μ is a system with two nonphysical degrees of freedom. The two other physical components of A_μ describe two possible polarization states of the photon. The fact that the solutions of equations of motion (2.2) contain only a single arbitrary function means that the time evolution of one of the nonphysical degrees of freedom is determined entirely by the time evolution of the other.

The introduction of an interaction with charged fields does not cause any substantial changes in this analysis.[8,9,24] The theory retains its gauge invariance, and in the Hamiltonian formulation the dynamics is again characterized by the presence of two primary constraints:

$$\pi^0 = 0, \quad \operatorname{div} \mathbf{E} = j_0 \tag{2.11}$$

($j_\mu = -\partial \mathcal{L}/\partial A^\mu$ is the 4-current). In other words, only the secondary constraint changes.

3. QUANTUM THEORY

Going over to a quantum description of the dynamic system reduces to replacing the canonical variables q,p by the operators $\hat{q},\hat{p}$ in all the expressions. These operators obey the commutation relations

$$\hat{q}\hat{p} - \hat{p}\hat{q} = i\{q,\ p\}. \tag{3.1}$$

This recipe is valid only in Cartesian coordinates. It is unambiguous if there is no problem with the ordering of the operators in the Hamiltonian.[8] The presence of constraints aggravates the difficulties.[22–24] Electrodynamics is a typical system with constraints. In addition to the physical variables, it also contains nonphysical variables. There are two ways to deal with them in quantum theory.

1) Banish all nonphysical variables. In this approach, the theory loses its explicit covariance.

2) Allow nonphysical as well as physical degrees of freedom. This approach is usually preferred because of its explicit relativistic invariance.

3.1. Elimination of nonphysical variables

To illustrate the first of these methods, we consider the example of the scalar electrodynamics specified by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + [(\partial_\mu + ieA_\mu)\Phi][(\partial_\mu + ieA_\mu)\Phi]^* - V(2\Phi\Phi^*), \tag{3.2}$$

where V describes the self-effect of the complex field $\mathbf{\Phi}$. It is convenient to transform from $\mathbf{\Phi}$ to a pair of real scalar fields: $\mathbf{\Phi} = (\varphi_1 + i\varphi_2)/\sqrt{2}$. We will treat these fields as constituting a two-dimensional vector $\boldsymbol{\varphi} = (\varphi_1,\varphi_2)$. We can then rewrite (3.2) as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(\nabla_\mu \boldsymbol{\varphi})^2 - V(\boldsymbol{\varphi}^2). \qquad \nabla_\mu \equiv \partial_\mu + eTA_\mu; \tag{3.3}$$

where $T = -i\tau_2$, τ_2 is a Pauli matrix, and $(T\boldsymbol{\varphi})_i = T_{ij}\varphi_j$. Switching to a Hamiltonian formalism,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}, \qquad \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\varphi}}} = \dot{\boldsymbol{\varphi}} + eA_0 T\boldsymbol{\varphi},$$

we find

$$H = \int d^3x \left\{\frac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2) + \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}[(\partial_k + eA_k T)\boldsymbol{\varphi}]^2 + V(\boldsymbol{\varphi}^2) - A_0 G\right\}. \tag{3.4}$$

$$G = \partial_k E^k + e\mathbf{p}T\boldsymbol{\varphi} = \operatorname{div} \mathbf{E} - j_0, \quad E^k = F^{k0}. \tag{3.5}$$

As we have already mentioned, a primary constraint remains the same ($\pi^0 = 0$) when an interaction is incorporated, while a secondary constraint changes. In place of (2.10) we have $\dot{\pi}^0 = G = 0$. Here π^0 and G are generators of gauge transformations. Calculating the Poisson brackets of G with $\mathbf{p}$ and φ, we see that G (more precisely, its density) is a generator of rotations in a plane. To get a better understanding of the physical content of this model, we single out the physical variables[26] from among the canonical variables φ, $\mathbf{p}$, and A_k, E^k. We transform from the Cartesian variables φ_1, φ_2 to the polar variables ρ, θ ($\rho^2 = \varphi^2$) with the canonical-conjugate momenta $p_\rho = (\boldsymbol{\varphi},\mathbf{p})/\rho, p_\theta = (\mathbf{n}_\theta,\mathbf{p})\rho = \mathbf{p}T_\varphi$ (n_θ is a unit vector). We resolve the electromagnetic fields into components:

$$E^k = \varepsilon^k + \partial^k \pi, \qquad \partial_k \varepsilon^k = 0, \qquad \pi = -\Delta^{-1}\partial_k E^k,$$
$$A_k = \alpha_k + \Delta^{-1}\partial_k \alpha, \qquad \partial^k \alpha_k = 0, \qquad \alpha = -\partial^k A_k, \tag{3.6}$$

where $\Delta \equiv -\partial_k \partial^k = \partial_k \partial_k$. It is clear that α_k, ε^k and α,π form pairs of canonically conjugate variables. It follows from (3.5) that we have $G = -\Delta\pi_\theta + ep_\theta$, so according to (3.6) and the definition of ρ, θ the gauge-invariant variables are the pairs $\alpha_k, \varepsilon^k \rho, p_\rho$, while θ and α change under displacements generated by G: $\alpha \to \alpha + \Delta\omega$, $\theta \to \theta - e\omega$. Here $\omega(x)$ is an arbitrary infinitesimal function. It follows from these relations that the combination $\Delta^{-1}\alpha + \theta/e$ is gauge-invariant. Consequently, the two "nonphysical" variables α and θ, which do vary under gauge transformations, are actually linear combinations of a physical component and a nonphysical component. We accordingly transform to the new canonical variables η, p_η and ϑ, p_θ:

$$\eta = \Delta^{-1}\alpha + \frac{\theta}{e}, \qquad p_\eta = \frac{1}{2}(\Delta\pi + ep_\theta),$$
$$\vartheta = -\Delta^{-1}\alpha + \frac{\theta}{e}, \qquad p_\vartheta = \frac{1}{2}(-\Delta\pi + ep_\theta). \tag{3.7}$$

We obviously have $p_\theta = G/2$, and ϑ, p_θ form a pair of nonphysical variables, while η, p_η form a pair of physical variables (they do not change under gauge transformations).[1)] We write Hamiltonian (3.4) in terms of the new variables:

$$H = \int d^3x \left\{ \frac{1}{2}\left[\varepsilon^2 - (p_\eta - p_\vartheta)\Delta^{-1}(p_\eta - p_\vartheta) + \mathbf{H}^2\right] + \frac{1}{2}\left[p_\rho^2 + \frac{(p_\eta + p_\vartheta)^2}{e^2\rho^2}\right] + \frac{1}{2}\left[(\partial_k\rho)^2 + e^2(\alpha_k + \partial_k\eta)^2\rho^2\right] + V(\rho^2) - 2A_0 p_\vartheta \right\}. \tag{3.8}$$

As we have assumed, the nonphysical degree of freedom ϑ turns out to be cyclic. It has been shown[26] that the phase space of the variables ρ, p_ρ is a cone which has been rolled out into a half-plane.

Although we have singled out the physical variables, we cannot make the switch to operators in (3.8), since we are using curvilinear coordinates. Strictly speaking, furthermore, we cannot eliminate the nonphysical degrees of freedom before performing the quantization, e.g., by setting them equal to zero in (3.8), since the operations of eliminating variables and quantization generally do not commute.[26,27] In order to derive the correct expression for the energy operator without auxiliary nonphysical degrees of freedom, we need to go back to Eq. (3.4), written in terms of Cartesian coordinates, replace the canonical variables by operators and recall that according to the general theory,[22–24] the constraints vanish only on vectors Φ of the physical subspace:

$$\hat{\pi}^0\Phi = 0, \quad \hat{G}\Phi = 0 \qquad (G = \dot{\pi}^0). \tag{3.9}$$

We accordingly write the term $\mathbf{E}^2$ in (3.4) in the form see (3.6)]

$$\int d^3x \mathbf{E}^2 = \int d^3x\left[\varepsilon^2 + (\partial\pi)^2\right] = \int d^3x(\varepsilon^2 - \pi\Delta\pi). \tag{3.10}$$

In integrating by parts in (3.10), we ignored the terms outside the integrals. Noting that the zeroth component of the current is $j_0 = -\partial\mathscr{L}/\partial A_0 = -e\mathbf{p}T\boldsymbol{\varphi} = -ep_\theta$, and noting that we have $\pi = -\Delta^{-1}(-ep_\theta + G)$, according to (3.7), we draw the following conclusion: When we use (3.10) and conditions (3.9), we can describe the physical Hamiltonian $\hat{H}_{\rm phys}$ corresponding to (3.4) by

$$\hat{H}_{\rm phys} = \int d^3x \left\{ \frac{1}{2}\left[\hat{\varepsilon}^2 + \hat{\mathbf{H}}^2 - \hat{j}_0\Delta^{-1}\hat{j}_0\right] + \frac{1}{2}\hat{\mathbf{p}}^2 + \frac{1}{2}\left[(\partial_k + e\hat{\alpha}_k T)\hat{\boldsymbol{\Phi}}\right]^2 + V(\hat{\boldsymbol{\Phi}}^2) \right\}, \tag{3.11}$$

where

$$\hat{\boldsymbol{\Phi}}(x) = \exp\left(-e\int \frac{d^3y}{4\pi}\frac{\partial_k \hat{A}_k(y)}{|\mathbf{x}-\mathbf{y}|}T\right)\hat{\boldsymbol{\varphi}}(x).$$

The third term here describes the Coulomb interaction of the charges. We have written α_k in place of A_k in (3.11), since the "nonphysical" component of the vector A_k (i.e., α) is assigned to the field phase $\hat{\boldsymbol{\varphi}}$, where it combines with θ and converts into η. In making the transformation to operators in (3.4) and in the constraints, we have no problem with ordering, so the operator (3.11) is the Hamiltonian which we have been seeking.

3.2. Fixing of nonphysical variables (choice of gauge)

If we wish to retain the explicit relativistic invariance of the theory after we quantize, we need to construct a formalism which involves all four components of the vector $A_\mu(x)$, including the nonphysical components. However, nonphysical canonical variables in classical physics are, generally speaking, totally arbitrary,[24] so we must ask just which operators should be associated with them in a quantum theory.

In quantum theory, in contrast with the classical case, condition (3.1) for nonphysical variables q, p itself imposes a definite restriction on these variables, since not every pair of operators will satisfy this condition. Requirement (3.1) means that nonphysical variables, like physical variables, are related at different times by a unitary transformation, the generator of which for the former is completely arbitary.[2)] It follows that in deriving a specific theory we must fix the nonphysical variables, i.e., specify their time evolution, in some way. Obviously, the fixing of a law of this sort violates the explicit gauge invariance of the theory: This approach is equivalent to choosing a gauge. On the whole, the theory of course remains gauge-invariant in the sense that a change in gauge condition does not affect the physics (the situation here is analogous to the choice of a coordinate system in mechanics). It would be desirable to construct the theory in such a way that the physical and nonphysical components of A_μ are formally equivalent. Corresponding to this goal would be the addition to the Lagrangian $\mathscr{L}$ in (2.1) (or (3.2)] of some term $\mathscr{L}'$ which lifts the degeneracy of matrix (2.5) (see Subsection 6.3 regarding the requirements imposed on $\mathscr{L}'$). We consider the specific example with[28] $\mathscr{L}^1 = -(\partial_\mu A_\mu)^2/2\xi$.

We thus assume

$$\mathscr{L} \to \mathscr{L} + \mathscr{L}' = \mathscr{L} - \frac{1}{2\xi}(\partial_\mu A_\mu)^2. \tag{3.12}$$

The added term lifts the degeneracy for any parameter value $\xi^{-1} \neq 0$, and it violates gauge invariance. The case $\xi = 1$ corresponds to the Feynman gauge. For an arbitrary ξ, one would speak in terms of the class of Fermi gauges. This class of gauges was first introduced by Heisenberg and Pauli.[28] In the case $\xi = 1$, the addition cancels out with the second term in (2.1), and the equations of motion take the particularly simple form

$$\Box A_\mu = -j_\mu, \tag{3.13}$$

so we will set $\xi = 1$ at this point.

3.2.1. Free electromagnetic field. We first consider a free field. We construct a Hamiltonian formalism. We have

$$\pi^\mu = F^{\mu 0} - g^{\mu 0}\partial_\nu A_\nu, \tag{3.14}$$

$$H_0 = \int d^3x\, (\pi^\mu \dot{A}_\mu - \mathcal{L} - \mathcal{L}')$$

$$= \int d^3x \left\{\frac{1}{2}\,\mathbf{E}^2 + \mathbf{H}^2 - (\pi^0)^2] - A_0 \operatorname{div}\mathbf{E} - \pi^0 \operatorname{div}\mathbf{A}\right\}. \tag{3.15}$$

In order to write H_0 in a form convenient for making the transformation to creation and annihilation operators, we express the functional (3.15) in terms of potentials:

$$H_0 = -\frac{1}{2}\int d^3x\, [\dot{A}_\mu^2 + (\partial_i A_\mu)^2 + \partial_i\,(A_k\partial_k A_i - A_i\partial_k A_k)]. \tag{3.16}$$

With $j_\mu = 0$, the vector A_μ satisfies d'Alembert's equation, so we have

$$A_\nu(x) = \int d\mu(q)\, (a_\nu(q)\, e^{-iqx} + a_\nu^*(q)\, e^{iqx}). \tag{3.17}$$

where

$$d\mu(q) \equiv \frac{d^3q}{(2\pi)^3\, 2\omega_q}, \quad \omega_q = |\,\mathbf{q}\,|, \quad q^2 = 0. \tag{3.18}$$

Substituting expression (3.17) into (3.16), and carrying out the necessary integrations, we find

$$H_0 = -\frac{1}{2}\int d\mu(q)\, \omega_q\, [a_\nu^*(q)\, a_\nu(q) + a_\nu(q)\, a_\nu^*(q)]. \tag{3.19}$$

We now go over to a quantum description. Since the classical Poisson brackets of the canonical variables A_μ, π^ν are

$$\{A_\mu(x),\, \pi^\nu(y)\} = \delta_\mu^\nu \delta(\mathbf{x}-\mathbf{y}), \quad x^0 = y^0, \tag{3.20}$$

we have the following result for the corresponding operators, according to (3.1):

$$[\hat{A}_\mu(x),\, \hat{\pi}^\nu(y)] = i\delta_\mu^\nu \delta(\mathbf{x}-\mathbf{y}), \quad x^0 = y^0. \tag{3.21}$$

Using (3.14) in (3.21), and noting that the fields at points separated by a space-like interval commute, we conclude that the following commutation relations hold for the fields:

$$[\hat{A}_\mu(x),\, \dot{\hat{A}}_\nu(y)] = -ig_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}), \quad x^0 = y^0. \tag{3.22}$$

Commutation relations for the operators $\hat{a}_\mu, \hat{a}_\mu^+$ follow from (3.22) if the latter are expressed in terms of $\hat{A}_\mu$ by means of

$$\hat{a}_\mu(q) = (\chi_q^{(+)},\, \hat{A}_\mu), \quad \hat{a}_\mu^+ = -(\chi_q^{(-)},\, \hat{A}_\mu). \tag{3.23}$$

where $\chi_q^{(\pm)} = \exp(\mp iqx)$ are solutions of d'Alembert's equation with positive ($\chi_q^{(+)}$) and negative ($\chi_q^{(-)}$) energies which have the properties $(\chi_q^{(\pm)}, \chi_{q'}^{(\pm)}) = \pm\tilde{\delta}(\mathbf{q},\mathbf{q}')$, $(\chi_q^{(\pm)}, \chi_{q'}^{(\mp)})$. In (3.23) and in the equations which follow, we are using the standard scalar product

$$(\chi_1,\, \chi_2) = i\int d^3x\, \chi_1^*(x)\overleftrightarrow{\partial_0}\chi_2(x). \tag{3.24}$$

and the "invariant" δ-function $\tilde{\delta}(\mathbf{q},\mathbf{q}') \equiv (2\pi)^3 2\omega_q \delta \times (\mathbf{q}-\mathbf{q}')$. Using (3.22)–(3.24), we find

$$[\hat{a}_\mu(q),\, \hat{a}_\nu^+(q')] = -g_{\mu\nu}\tilde{\delta}(\mathbf{q},\, \mathbf{q}'). \tag{3.25}$$

The energy operator of a free field is found from expression (3.19) by making the substitutions $a_\mu^* \to \hat{a}_\mu^+, a_\mu \to \hat{a}_\mu$:

$$\hat{H}_0 = -\frac{1}{2}\int d\mu(q)\, \omega_q\, [\hat{a}_\mu^+\hat{a}_\mu + \hat{a}_\mu\hat{a}_\mu^+]. \tag{3.26}$$

The basic difficulties of QED stem from (3.25) and (3.26).

1) According to (3.25), the theory has states with a negative norm (since we have $[\hat{a}_0(q),\ \hat{a}_0^+(q')] = -\tilde{\delta}(\mathbf{q},\mathbf{q}')$ and for $\psi_1 = \int d\mu(q) f(q)\hat{a}_0^+(q)\psi_0$ we have the inequality $(\psi_1,\psi_1) = \|\psi\|^2 = -\int d\mu(q)\, |f(q)|^2 < 0$ if $(\psi_0,\psi_0) = 1, \hat{a}_0(q)\psi_0 = 0$).

2) According to (3.26), states with a negative energy may appear (since the form $\hat{a}_\mu^+\hat{a}_\mu = \hat{a}_0^+\hat{a}_0 - \hat{a}_k^+\hat{a}_k$ is of variable sign).

Both these difficulties stem from time-like photons. We can show that if we restrict the discussion to the physical subspace $\mathcal{H}_{\rm phys}$ of the complete Hilbert space ($\mathcal{H}_{\rm phys} \subset \mathcal{H}$), which is fixed by conditions (3.9), and if we correctly define the operation of taking the "Hermitian adjoint" for the operator $\hat{a}_0$ (Subsection 4.2), we can eliminate the difficulties. According to (3.14) we have

$$\hat{\pi}^0 = -\partial_\mu\hat{A}_\mu = i\int d\mu(q)\, (\hat{L}(q)\, e^{-iqx} - \hat{L}^+(q)\, e^{iqx}), \tag{3.27}$$

$$\hat{G} = \dot{\hat{\pi}}^0 = \int d\mu(q)\, \omega_q\, (\hat{L}(q)\, e^{-iqx} + \hat{L}^+(q)\, e^{iqx}), \tag{3.28}$$

where

$$\hat{L}(q) \equiv q_\mu\hat{a}_\mu(q), \quad q^2 = 0. \tag{3.29}$$

Using (3.27) and (3.28), we can rewrite conditions (3.9) as

$$\hat{L}(q)\,\Phi = 0, \quad \hat{L}^+(q)\,\Phi = 0. \tag{3.30}$$

Equations analogous to (3.23) can be used to derive (3.30). By virtue of the equality

$$[\hat{L}(q),\, \hat{L}^+(q')] = 0 \tag{3.31}$$

conditions (3.30) do not contradict each other.

It is now a straightforward matter to show that the norm of vectors $\Phi \in \mathcal{H}_{\rm phys}$ is always positive and that the energy of corresponding states is nonnegative. Specifically, the only states which could have a negative norm are those which have an odd number of time-like photons. States of the type $(\hat{a}_0^+)^n\Phi$, however, do not belong to $\mathcal{H}_{\rm phys}$, since according to (3.25) and (3.29) we have

$$[\hat{L}(q),\, \hat{a}_\mu^+(q')] = -q_\mu\tilde{\delta}(\mathbf{q},\, \mathbf{q}'), \tag{3.32}$$

and $\hat{L}(q)(\hat{a}_0^+)^n\Phi \neq 0$, $n \geqslant 1$. In other words, the first of conditions (3.30) does not hold. If we ignore the second condition in (3.30), we see that nonphysical states with a zero norm of the type $(\hat{L}^+)^n\,\Phi$ are retained in the theory. After they are eliminated, we are left with exclusively states with a positive norm.

To demonstrate that the physical subspace has no negative-energy states, we write the operator $\hat{h} = -(1/2)(\hat{a}_\mu^+\hat{a}_\mu + \hat{a}_\mu\hat{a}_\mu^+)$ in the form

$$\hat{h}(q) = \frac{1}{2}[\hat{\mathbf{a}}_\perp^+(q)\, \hat{\mathbf{a}}_\perp(q) + \hat{\mathbf{a}}_\perp(q)\, \hat{\mathbf{a}}_\perp^+(q)]$$

$$-\frac{1}{\omega_q}\left\{\hat{a}_0(q)\, \hat{L}^+(q) + \hat{a}_3^+(q)\, \hat{L}(q)\right.$$

$$+\frac{1}{2}[\hat{L}^{+}(q),\ \hat{a}_0(q)]+\frac{1}{2}[\hat{L}(q),\ \hat{a}_3^{+}(q)]\Big\},\qquad(3.33)$$

where $\hat{a}_3(q) = (\mathbf{q},\hat{\mathbf{a}}(q))/\omega_q, (\mathbf{q},\hat{\mathbf{a}}_\perp(q)) = 0$. Using commutation relations (3.32) and their complex conjugates $[\hat{L}+,\hat{a}_\mu] = q_\mu\tilde{\delta}$, we find that the commutators cancel out in (3.33), and the terms containing the operators $\hat{L},\hat{L}^{+}$ vanish on physical vectors. Only physical photons thus contribute to the energy; nonphysical photons make no contribution even to the energy of the ground state (an infinite constant which appears after $\hat{h}$ is reduced to normal form).

This essentially completes the proof that the theory is self-consistent. Nevertheless, we wish to verify that the Hamiltonian $\hat{H}_0$ does not send physical vectors outside the physical subspace in the course of the motion. This circumstance follows from the equalities

$$[\hat{L}(q),\ \hat{H}_0]=\omega_q\hat{L}(q),\ [\hat{L}^{+}(q),\ \hat{H}_0]=-\omega_q\hat{L}^{+}(q),\quad(3.34)$$

according to which we have $\hat{L}\hat{H}_0\Phi = \hat{L}+\hat{H}_0\Phi = 0$. The theory thus satisfies all the physical requirements.

3.2.2. Interacting electromagnetic field. We have been discussing the quantization of a free electromagnetic field. It is easy to see that incorporating an interaction does not change the quantization procedure. Specifically, the Lagrangian of (for example) spinor electrodynamics is[2]

$$\mathscr{L}=-\frac{1}{4}F_{\mu\nu}^2+\bar{\psi}(i\partial_\mu-eA_\mu)\gamma_\mu\psi-m\bar{\psi}\psi.\qquad(3.35)$$

The modified Lagrangian $\mathscr{L}+\mathscr{L}'$ in (3.12) with $\xi = 1$ leads to equations of motion (3.13), from which it follows,[3] by virtue of the conservation of electric current, $\partial_\mu\hat{j}_\mu = 0$, that the operator $\partial_\mu\hat{A}_\mu$ satisfies d'Alembert's equation $\Box\partial_\mu A_\mu = 0$. Equations (3.14), (3.20), and (3.21) remain completely the same. Consequently, the equations for the constraints remain superficially the same:

$$\hat{\pi}^0=-\partial_\mu\hat{A}_\mu,\ \hat{G}=\dot{\hat{\pi}}^0=-\partial_\mu\dot{\hat{A}}_\mu.$$

We have already seen, however, that secondary constraint (2.10) changes when an interaction is incorporated [see (2.11) and (3.5)]. We can show that there is no contradiction here. Using the equation of motion $\ddot{A}_0+\partial_k\partial^k A^0 = j^0$, we find

$$\partial_k E^k-j^0=\partial_k(\partial^k A^0-\partial^0 A^k)-j^0$$
$$=j^0-\ddot{A}^0-\partial_k\dot{A}^k-j^0=-\partial_\mu\dot{A}_\mu.\qquad(3.36)$$

We simply need to verify that the evolution operator $\hat{U}_{tt'} = \exp[-i\hat{H}(t-t')]$ does not send physical states outside $\mathscr{H}_{\rm phys}$; i.e., we need to verify that $\mathscr{H}_{\rm phys}$ is an invariant subspace under the application of the operator $\hat{U}_{tt'}$. However, this point follows in a trivial way from the fact that the theory is noncontradictory:

$$[\hat{\pi}^0,\ \hat{H}]=i(\mathrm{div}\,\hat{\mathbf{E}}-j_0)\approx 0,$$
$$[\mathrm{div}\,\hat{\mathbf{E}}-j_0,\ \hat{H}]=i\partial_k\partial^k\hat{\pi}^0\approx 0,\qquad(3.37)$$

since, according to (3.37), we have $\hat{H}\Phi\in\mathscr{H}_{\rm phys}$ if $\Phi\in\mathscr{H}_{\rm phys}$. The symbol $\approx$ in (3.37) means equality on vectors from $\mathscr{H}_{\rm phys}$.

Let us take a brief look at the form of the conditions on physical states in various pictures. In the Heisenberg picture, physical vectors are fixed by condition (1.3). Since the operator $\partial_\mu\hat{A}_\mu$ satisfies d'Alembert's equation, (1.3) holds if only two conditions are satisfied at the time $t = 0$: $\partial_\mu\hat{A}_\mu\Phi = 0, \partial_\mu\dot{\hat{A}}_\mu\Phi = 0$. By virtue of (3.37), these conditions are equivalent to the two conditions in (3.9) in the Schrödinger picture [even when an interaction is incorporated; see (3.36)]. The relationship between Fermi quantization with the sole condition (1.3) and quantization by Dirac's recipe,[24] with the two conditions in (3.9) on the vectors Φ, thus becomes clear.

In order to write (3.9) in the interaction picture, we need to transform to free fields in the equations for π^0 and $G = \partial_k E^k - j^0$. In other words, we need to use the equation $\ddot{A}^0+\partial_k\partial^k A^0 = 0$ in (3.36). As a result we find[29]

$$\hat{\pi}^0=-\partial_\mu\hat{A}_\mu,\qquad \hat{G}=\dot{\hat{\pi}}=-\partial_\mu\dot{\hat{A}}_\mu-\hat{j}^0.\qquad(3.38)$$

These are the constraints in the interaction picture. It follows that condition (1.3), which hold for arbitrary x^0, is rewritten in the following form in the interaction picture[29,30]

$$\Big[\partial_\mu\hat{A}_\mu(x)+\int d^3yD(x-y)\,\hat{j}_0(y)\Big]\Phi=0.\qquad(3.39)$$

The reason is that according to (1.2) and (3.22) with $x^0 = 0$ we would have $D(x) = 0$ and $\dot{D}(x) = \delta(x)$.

4. STRUCTURE OF THE HILBERT SPACE

4.1. Hilbert space in the Heisenberg-Pauli-Fermi gauge

If all the components of the vector A_μ are allowed in the formalism, the structure of the complete Hilbert space is determined by commutation relations (3.25) for the operators $\hat{a}_\mu\hat{a}_\mu^{+}$, and it bears the imprint of the structure of Minkowski space. We will now prove these assertions.

We first consider the question of the vacuum. If the theory is given by Eqs. (3.25) and (3.26), without any restrictions on the state vectors, the mathematical vacuum ψ_0 is formally fixed by the conditions

$$\hat{a}_\mu(q)\psi_0=0.\qquad(4.1)$$

For the operators $\hat{a}_k(q)$, $k = 1,2,3$, requirement (4.1) is obvious. The only point which might raise some doubt is the assumption that (4.1) also holds for $\hat{a}_0(q)$ (see Subsection 6.4). However, this circumstance seems unavoidable if we postulate that the vector ψ_0 is relativistically invariant, since the Lorentz-transformed operator $\hat{a}_k$ may also contain a zeroth component. A difficulty (an imaginary one) is caused by the anomalous sign of the commutator $[\hat{a}_0,\hat{a}_0^{+}] = -\tilde{\delta}$ in comparison with the commutators of spatial components, since it would appear that $\hat{a}_0$ should play the role of a creation operator. Actually, the question reduces to the definition of the operator $\hat{a}_0^{+}$ which is the conjugate of $\hat{a}_0$. It is easy to see that if $\hat{a}_\mu^{+}(q)$ is the operator which performs multiplication by the function a_μ (q) then according to (3.25) we would have $\hat{a}_\mu(q) = -g_{\mu\nu}\beta_q\delta/\delta a_\nu(q)$ [see (4.7)] and condition (4.1) would hold. This form of the operator a_μ is dictated by relativistic invariance.

We thus postulate the existence of a Lorentz-invariant cyclic vector ψ_0 which satisfies condition (4.1). The vectors which are obtained by applying all possible operators of the type $(\hat{a}_\mu^{+})^n$, $n = 0,1,2,...$ to it form the basis of the Hilbert space. Among the basis vectors there are some which have a negative norm, since the Hilbert space has an indefinite metric. With physical applications in mind, we resolve the vector $a_\mu(q)$ along the basis vectors[31]:

$$e_\mu^{(1)}(q),\quad e_\mu^{(2)}(q),\quad q_\mu,\quad \tilde{q}_\mu, \tag{4.2}$$

where

$$e_\mu^{(i)}(q)\,q_\mu = e_\mu^{(i)}(q)\,\tilde{q}_\mu = 0,\qquad e_\mu^{(i)}(q)\,e_\mu^{(k)}(q) = -\delta^{ik}, \tag{4.2'}$$

$$q_\mu^2 = \tilde{q}_\mu^2 = 0,\quad q_\mu\tilde{q}_\mu = 2\omega_q^2 \qquad (i,\ k = 1,\ 2).$$

In other words, if $q_\mu = (\omega_q; 0,0,q)$ is a standard vector, then we have $\tilde{q}_\mu = (\omega_q; 0,0,-q)$. Since there are isotropic vectors among vectors (4.2), they form a nonorthogonal basis. We construct the operators

$$\hat{a}_i(q) = e_\mu^{(i)}(q)\,\hat{a}_\mu(q),\qquad \hat{L}(q) = q_\mu\hat{a}_\mu(q),\qquad \hat{K}(q) = \tilde{q}_\mu\hat{a}_\mu(q) \tag{4.3}$$

and their Hermitian adjoints $\hat{a}_i^+$, $\hat{L}^+$, $\hat{K}^+$. The operators $\hat{K}, \hat{K}^+$ obviously have the same properties as the operators $\hat{L}, \hat{L}^+$:

$$[\hat{K}(q),\ \hat{K}^+(q')] = 0,\qquad [\hat{a}_\mu(q),\ \hat{K}^+(q')] = [\hat{K}(q),\ \hat{a}_\mu^+(q')] = -\tilde{q}_\mu\tilde{\delta}(\mathbf{q},\ \mathbf{q}'), \tag{4.4}$$

and

$$[\hat{L}(q),\ \hat{K}^+(q')] = [\hat{K}(q),\ \hat{L}^+(q')] = -2\omega_q^2\tilde{\delta}(\mathbf{q},\ \mathbf{q}'). \tag{4.5}$$

Applying the operators $\hat{a}_i^+$, $\hat{L}^+$, $\hat{K}^+$ to ψ_0, we find that the Hilbert space $\mathscr{H}$ breaks up into subspaces $\mathscr{H}_\varphi, \mathscr{H}_\zeta, \mathscr{H}_\eta, \mathscr{H}_{\varphi\zeta}, \mathscr{H}_{\varphi\eta}, \mathscr{H}_{\zeta\eta}, \mathscr{H}_{\varphi\zeta\eta}$. The basis of subspace $\mathscr{H}_\varphi$ is found by applying to ψ_0 the operators $\hat{a}_i^+$, $\psi_0 \in \mathscr{H}_\varphi$, the bases $\mathscr{H}_\zeta$ and $\mathscr{H}_\eta$ are found by applying the operators $\hat{L}^+$, $\hat{K}^+$, respectively. The basis $\mathscr{H}_{\varphi\zeta}$ is found by applying the operators $(\hat{a}_i^+)^n(\hat{L}^+)^m$, $n \geqslant 1$, $m \geqslant 1$, etc. The vectors $\varphi \in \mathscr{H}_\varphi$ have a positive norm, while the norms of (for example) vectors from $\mathscr{H}_\zeta, \mathscr{H}_\eta, \mathscr{H}_{\varphi\xi}, \mathscr{H}_{\varphi\eta}$ are zero—by virtue of (3.31), (4.1), (4.4), and the commutation relations which follow from (4.2'),

$$[\hat{a}_i(q),\ \hat{L}^+(q')] = [\hat{a}_i(q),\ \hat{K}^+(q')] = [\hat{a}_i^+(q),\ \hat{L}(q')] = [\hat{a}_i^+(q),\ \hat{K}(q')] = 0. \tag{4.6}$$

The subspaces $\mathscr{H}_\zeta$ and $\mathscr{H}_\eta$ are not orthogonal by virtue of (4.5).

It is now clear just which states are eliminated by conditions (3.30). The first of them, by virtue of (4.1), (3.31), (4.5), and (4.6), eliminates states which are generated by the operators $\hat{K}^+$. In other words, the subspaces $\mathscr{H}_\eta, \mathscr{H}_{\varphi\eta}, \mathscr{H}_{\zeta\eta}, \mathscr{H}_{\varphi\zeta\eta}$ are cut out. We thereby obtain a Hilbert space in Gupta's formalism: $\mathscr{H}_G = \mathscr{H}_\varphi \oplus \mathscr{H}_\zeta \oplus \mathscr{H}_{\varphi\zeta}$. Vectors from the subspace $\mathscr{H}_\zeta \oplus \mathscr{H}_{\varphi\zeta}$ are orthogonal to vectors φ and have a zero norm. The second of conditions (3.30) eliminates nonphysical states with a zero norm. We are left with the subspace of states of physical photons, $\mathscr{H}_\Phi$, with a positive-definite matrix. To find its structure, we find the physical vacuum.

4.2. Vacuum in the Fermi formalism

In the representation in which $\hat{a}_\mu^+(q)$ is the operator which performs multiplication by the function $a_\mu(q)$ the operator $\hat{a}_\mu(q)$ is, according to (3.25), the operator which performs a variational differentiation:

$$\hat{a}_\mu^+(q) \to a_\mu(q),\quad \hat{a}_\mu(q) \to -g_{\mu\nu}\beta_q\frac{\delta}{\delta a_\nu(q)},$$
$$\beta_q \equiv (2\pi)^3\,2\omega_q. \tag{4.7}$$

Using

$$\hat{L}(q) = q_\mu g^{\mu\nu}\hat{a}_\nu(q) = -\beta_q q_\mu\frac{\delta}{\delta a_\mu(q)}, \tag{4.8}$$

we can rewrite conditions (3.30) for the vacuum vector Φ_0:

$$q_\mu\frac{\delta}{\delta a_\mu(q)}\Phi_0 = 0,\qquad q_\mu a_\mu(q)\,\Phi_0 = 0. \tag{4.9}$$

Furthermore, the vacuum must satisfy the condition that there are no physical photons:

$$\hat{a}_i(q)\,\Phi_0 = 0\qquad (i = 1,\ 2). \tag{4.9'}$$

The validity of representation (4.8) can be verified directly. Since we have $q_\mu\hat{a}_\mu(q) \equiv \omega_q(\hat{a}_0 - \hat{a}_3)$ for a standard vector q, and since we have $\hat{a}_0 = -\beta_q\delta/\delta a_0$, $\hat{a}_3 = \beta_q\delta/\delta a_3$, according to (4.7), we find[3)] $q_\mu\hat{a}_\mu(q) = -\beta_q\omega_q(\delta/\delta a_0 + \delta/\delta a_3) = -\beta_q q_\mu\delta/\delta a_\mu(q)$. Transforming to the variables $a^-(q) = q_\mu a_\mu(q)$, $a^+(q) = \tilde{q}_\mu a_\mu(q)$, $\delta a^-(q)/\delta a^+(q') = 0$ in (4.9), we find the pair of equations

$$\frac{\delta}{\delta a^+(q)}\Phi_0 = 0,\qquad a^-(q)\,\Phi_0 = 0. \tag{4.10}$$

Using (4.9'), we see that the solution of the first of these equations is an arbitrary functional of $a^-(q)$; i.e., $\Phi_0 = F[a^-(q)]$. The second condition in (4.10) fixes the functional F:

$$\Phi_0 = \prod_{\mathbf{q}}\delta(a^-(q))\qquad (\text{or}\quad \prod_{\mathbf{q}}\delta(\hat{L}^+(q))\,\psi_0). \tag{4.11}$$

Solution (4.11) can be interpreted easily: The physical vacuum is a state which has no transverse (physical) photons with an indefinite number of superimposed time-like and longitudinal photons for each value of the vector q, $q^2 = 0$. The state Φ_0 is unrenormalizable. However, this circumstance is not a flaw of the theory, since it is clear that the unrenormalizability of Φ_0 is of the same nature as the unrenormalizability of states with a definite momentum in scattering theory (monochromatic plane waves—generalized eigenvectors[32]). Just as unrenormalizability of plane waves in scattering theory can be eliminated by making a transition to a space of finite volume, we can get rid of the unrenormalizability of Φ_0 here by (for example) transforming to a "smeared" δ-function $\delta(x) \to 2\pi\varepsilon)^{-1/2}\exp(-x^2/2\varepsilon)$, $\varepsilon > 0$. The theory thus allows nonphysical states with a zero norm, but these states are harmless. At the end of the calculations, we can let ε go to zero. It was shown by direct calculations in Ref. 11 that unrenormalizability of Φ_0 does not affect the physical results.

A different realization of algebra (3.25) is ordinarily used in the literature[10–12,20–21]:

$$\begin{aligned} \hat{a}_i^+(q) &\to a_i(q), &\qquad \hat{a}_i(q) &\to \beta_q\frac{\delta}{\delta a_i(q)}, \\ \hat{a}_0^+(q) &\to \beta_q\frac{\delta}{\delta a_0(q)}, &\qquad \hat{a}_0(q) &\to a_0(q). \end{aligned} \tag{4.12}$$

This representation of algebra (3.25) violates the explicit relativistic invariance of the theory (there is no equality $\hat{a}_\mu\psi_0 = 0$). Accordingly, the solution $\tilde{\Phi}_0$ of the equations

$$\hat{L}(q)\,\tilde{\Phi}_0 \equiv \omega_q\left(a_0(q) - \beta_q\frac{\delta}{\delta a_3(q)}\right)\tilde{\Phi}_0 = 0. \tag{4.13}$$

$$\hat{L}^+(q)\,\tilde{\Phi}_0 \equiv \omega_q \left(\beta_q \frac{\delta}{\delta a_0(q)} - a_3(q)\right)\tilde{\Phi}_0 = 0,$$

which is given by[21]

$$\tilde{\Phi}_0 = c\exp\left(\int d\mu(q)\, a_3(q)\, a_0(q)\right), \tag{4.14}$$

is also not invariant under Lorentz transformations. The vector $\tilde{\Phi}_0$, like Φ_0, is unrenormalizable. In contrast with realization (4.7), realization (4.12) yields an unrenormalizable vacuum (4.14) even if we discard the second condition in (3.30) or in (4.13)]. It was the unrenormalizability of specifically $\tilde{\Phi}_0$ (Refs. 10–12 and 20) which motivated the search for new quantization paths.[13,30] It is clear from this discussion that $\tilde{\Phi}_0$ can be identified with the vacuum state only as a result of a misunderstanding.

Armed with the vector Φ_0, we can find a basis in the physical subspace. This basis is formed by vectors of the type $(\hat{a}_1^+(q))^{n_1}(\hat{a}_2^+(q'))^{n_2}\Phi_0, n_1, n_2 \geqslant 0$.

4.3. Scalar product in Hilbert space

Up to this point we have adhered to the traditional approach to Hilbert space in QED, without planning the definition of a scalar product in it. This question deserves a separate discussion.

Let us examine the quartet of Hilbert-space vectors $\psi_\mu = \hat{a}_\mu^+(q)\psi_0$ as an example. These vectors form a vector of Minkowski space. The element which is the conjugate of ψ_μ, i.e., $\psi_0^*\hat{a}_\mu(q)$, also transforms as a vector under Lorentz transformations. Consequently, a Lorentz-invariant scalar product of vectors of this sort can be formed only by means of the metric tensor $g_{\mu\nu}$ e.g.,

$$\begin{aligned}(\psi_1, \psi_2) &= -g^{\mu\nu}(\psi_{1\mu}, \psi_{2\mu}) \\ &= -g^{\mu\nu}\int d\mu(q)\, d\mu(q')\, \psi_1^*(q)\, \psi_2(q') \\ &\quad \times (\psi_0, \hat{a}_\mu(q)\,\hat{a}_\nu^+(q')\,\psi_0) = 4\int d\mu(q)\,\psi_1^*(q)\,\psi_2(q).\end{aligned} \tag{4.15}$$

Expression (4.15) incorporates relation (3.25). Generalizing this definition to multiphoton states, we conclude that the Lorentz-invariant norms of all elements of $\mathscr{H}$ are positive. This mathematical apparatus must be used with caution. For example, it may turn out that the norm of the state

$$\psi_L = \int d\mu(q)\,\psi(q)\,\hat{L}^+(q)\,\psi_0$$

is Lorentz-invariant. On the other hand, we have $\|\psi_L\| = 0$, contradicting the assertion that all invariant norms are positive. Actually, $\hat{L}^+(q)$ is a creation operator in basis (4.2). Indeed the metric tensor $\tilde{g}^{AB}$ in this basis is

$$\tilde{g}^{AB} = g^{\mu\nu}e_\mu^A e_\nu^B, \quad e_\mu^A = \left(\frac{q_\mu}{\sqrt{2}\omega_q};\; e_\mu^{(1)}(q),\; e_\mu^{(2)}(q),\; \frac{\tilde{q}_\mu}{\sqrt{2}\,\omega_q}\right),$$

$$\tilde{g}^{AB} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A, B = 0, 1, 2, 3. \tag{4.16}$$

In other words, we have $\hat{L}^+(q) = \sqrt{2}\omega_q e_\mu^0 \hat{a}_\mu^+(q) \equiv \sqrt{2}\omega_q \hat{a}_0^+(q)$, so that (ψ_L^0, ψ_L^0), where $\psi_L^0 = \hat{a}_0^+\psi_0$, is not invariant, but it does transform as the component $\tilde{g}^{00}$ of the tensor $\tilde{g}^{AB}$. In this connection we wish to emphasize that although Lorentz-invariant norms are positive-definite the Hilbert space is pseudo-Euclidean, and nonphysical states are to be eliminated. The circumstance that the concepts of covariant and contravariant vectors (covariant and contravariant under transformations in Minkowski space) should also be introduced for elements of the Hilbert space in electrodynamics was first demonstrated by Konisi and Ogimoto[33] (see also Ref. 34).

4.4. Is the Fermi formalism contradictory?

One of the proofs of the contradictory nature of the second condition in (3.30) is the following discussion,[14] which is taken from Bogolyubov and Shirkov's book,[2] where this proof is given in coordinate space. The vacuum is a state without photons, so a corresponding vector is annihilated by annihilation operators. Since the vacuum is a physical state, we have the following results according to (3.30):

$$\hat{L}^+(q)\,\psi_0 = 0. \tag{4.17}$$

Applying the operator $\hat{K}(q')$ from (4.3) to (4.17), and using (4.1) and (4.5), we find the contradiction

$$\hat{K}(q')\,\hat{L}^+(q)\,\psi_0 = -2\omega_q^2\tilde{\delta}(\mathbf{q}, \mathbf{q}')\,\psi_0 = 0 \tag{4.18}$$

(we find the equality $\psi_0 = 0$). The error in this conclusion stems from the identification of the mathematical vacuum ψ_0, defined by condition (4.1), with the physical vacuum Φ_0, given by (4.11). The assertion that the physical vacuum must obey condition (4.1) is totally arbitrary and does not follow from the formalism. The vector Φ_0 must, first of all, satisfy the condition that there are no physical photons, (4.9'). Another requirement which it must meet is that conditions (3.30), which are common to all the vectors of the physical subspace, are satisfied. Clearly, when Φ_0 is defined in this way we run into no contradiction: The operators $\hat{a}_i$ commute with $\hat{L}, \hat{L}^+$ [see (4.6)], and Eq. (4.18), in which the replacement $\psi_0 \to \Phi_0$ has been made, become inapplicable, since we have $\hat{K}(q)\Phi_0 \neq 0$, according to (4.5) and (4.11).

Another "proof" that condition (1.3) is contradictory is based on the assertion that commutation relations (1.2), taken between physical states,

$$(\Phi, [\partial_\mu \hat{A}_\mu(x), \hat{A}_\nu(y)]\,\Phi) = i\partial_\nu D(x-y)\,(\Phi, \Phi), \tag{4.19}$$

do not hold, since the left side of (4.19) is zero. The latter assertion is wrong because in its proof the operator $\partial_\mu \hat{A}_\mu$ in the expression $(\Phi, \partial_\mu \hat{A}_\mu \hat{A}_\nu \Phi)$ was brought over to the left bracket, and this manipulation is generally illegal. Pursuing this logic, we easily reach the conclusion that (for example) states with a definite momentum are not allowed in quantum mechanics. Let us assume $\hat{p}\psi_p = p\psi_p$ and $[\hat{x}, \hat{p}] = i$. We then have

$$(\psi_p, [\hat{x}, \hat{p}]\,\psi_p) = p\,[(\psi_p, \hat{x}\psi_p) - (\psi_p, \hat{x}\psi_p)] = i\,(\psi_p, \psi_p), \tag{4.20}$$

from which we conclude that the right side of (4.20) is zero. These arguments are incorrect because (first) the states ψ_p are unrenormalizable and (second) we have the obvious inequality $(\psi_p, \hat{p}\hat{x}\psi_p) \neq p(\psi_p, \hat{x}\psi_p)$: When $\hat{p}$ is brought over to the left bracket in the coordinate representation, nonintegral terms arise.

Finally, one may encounter the assertion that conditions (3.25) and (4.1) are contradictory for the zeroth components of operators.[2,14] Actually, if we multiply the vector $\psi_1 = \int d\mu(q) f(q) \hat{a}_0^+(q)\psi_0$ by its complex conjugate, we find

$$(\psi_1, \psi_1) = -\int d\mu(q) \,|f(q)|^2 < 0, \tag{4.21}$$

while there should be a positive quantity on the left. However, (4.21) actually demonstrates only the negativity of the norm of a state with a nonphysical photon, as follows from the specific realization (4.7) of algebra (3.25) and from the content of Subsection 4.3. We have been led astray by the application of the words "complex conjugate" to ψ_1. Actually, the vector which is the conjugate of ψ_1 is defined by the equality $\psi_1^* \int d\mu(q) \psi_0^* \hat{a}_0(q) f^*(q)$, and the sign of the square norm $\|\psi_1\|^2$ is dictated by commutation relations (3.25). The difficulty here is basically psychological, since we are convinced that the square norms of all states are positive. However, in agreeing to allow nonphysical entities in the theory we cannot insist that they have the same properties as physical dynamic variables. All that we can require in this case is that the formalism be noncontradictory.

5. CONCLUSION

Let us summarize. There are two equivalent procedures for quantizing the electromagnetic field. One is the so-called new Gupta formalism, which is set forth in Subsection 6.2. That formalism is based exclusively on condition (1.4) [or on the first condition in (3.30)]. If we adopt the relativistically invariant realization (4.7) of algebra (3.25), we find that the physical vacuum Φ_0 in this procedure is the same, within some term $\zeta, \|\zeta\| = 0$, which is orthogonal to it, as the mathematical vacuum ψ_0. This formalism has the disadvantage that it contains nonphysical states of zero norm. Realization (4.12) of algebra (3.25) does not have explicit relativistic invariance, and it gives us an unrenormalizable, relativistically noninvariant vacuum (4.14), even in a formalism with condition (1.4) alone.

The other procedure is Fermi's original formalism,[3-6] to which Dirac always adhered.[8,9] That formalism uses both conditions (3.30) on physical vectors, which follow from a general analysis of the dynamics of an electromagnetic field as a mechanical system. By incorporating the second condition in (3.30), we banish from the theory nonphysical states with a zero norm. The physical vacuum Φ_0 has a clear mathematical meaning: It is a generalized eigenvector of the operators $\hat{\pi}^0, \hat{\pi}^0$; in other words, it is unrenormalizable. However, this point causes no difficulties in the calculations (Ref. 11 and Sec. 4 of the present paper). In neither of these procedures is it necessary to introduce deliberately an indefinite metric[13] by means of an operator $\hat{\eta}$ (Subsection 6.1).

6. APPENDIX

6.1. "Old" Gupta formalism[13]

For the convenience of the reader, we will briefly review the essential features of Gupta's suggestions. The difficulties which arose in the quantization of the electromagnetic field occurred not only because the energy and the metric of the Hilbert space were not positive definite (Sec. 3) but also because of the interpretation of the operators $\hat{a}_0, \hat{a}_0^+$. Since their commutator $[\hat{a}_0, \hat{a}_0^+] = -\tilde{\delta}$ has the sign opposite that of the commutator $\hat{a}_k, \hat{a}_k^+$, according to (3.25), the standard analysis[8] shows that an operator which increases the eigenvalues of $\hat{a}_0\hat{a}_0^+$ is $\hat{a}_0$ [if $\hat{a}_0\hat{a}_0^+|n\rangle = n|n\rangle$, then $\hat{a}_0\hat{a}_0^+ \hat{a}_0|n\rangle = (n+1)\hat{a}_0|n\rangle$; for simplicity, we have taken the quantum mechanical harmonic oscillator here, $[\hat{a}_0, \hat{a}_0^+] = -1$; see Subsection 6.4 for more details]. That interpretation, however, violates the explicit relativistic invariance, since it follows from the invariance of the vacuum ψ_0 and from the condition $\hat{a}_0^+\psi_0 = 0$ that we have $\hat{a}_\mu^+\psi_0 = 0$ for all μ, but this result is unacceptable. Gupta's first paper was one attempt to resolve these difficulties. In that paper it was essentially postulated that commutation relations (3.25) are a consequence of the introduction of an indefinite metric in some Hilbert space with a positive-definite metric. Specifically, it is assumed that the theory is based on the operators $\hat{a}_\mu(q), \hat{a}_\mu^*(q)$, which obey the commutation relations

$$[\hat{a}_\mu(q), \hat{a}_\nu^*(q')] = \delta_{\mu\nu}\tilde{\delta}(\mathbf{q}, \mathbf{q}'), \tag{6.1}$$

where the operator $\hat{a}_\mu^*$ is the Hermitian adjoint of $\hat{a}_\mu$ and $\delta_{\mu\nu}$ is the Kronecker delta. All the operators $\hat{a}_\mu$ in this auxiliary space can be regarded as annihilation operators, while the $\hat{a}_\mu^*$ are creation operators. The vacuum is defined by the customary equation

$$\hat{a}_\mu\psi_0 = 0. \tag{6.2}$$

The scalar product changes in this space. Specifically, an indefinite metric is introduced, and this circumstance can obviously change the concept of the Hermitian adjoint of certain operators. The new metric is chosen in such a way that those operators which are Hermitian adjoints in the sense of the new metric satisfy commutation relations (3.25). The technical procedure for doing this can be outlined as follows. One chooses some metric operator $\hat{\eta}$, which is used to define a new scalar product of vectors:

$$(\psi_1, \psi_2) \to (\psi_1, \psi_2)_\eta \equiv (\psi_1, \hat{\eta}\psi_2). \tag{6.3}$$

The operator which is the adjoint of $\hat{A}$ is defined in accordance with the rule

$$(\hat{A}\psi_1, \psi_2)_\eta = (\psi_1, \hat{A}^+\psi_2)_\eta = (\psi_1, \hat{A}^*\hat{\eta}\psi_2) = (\psi_1, \hat{\eta}\hat{\eta}^{-1}\hat{A}^*\hat{\eta}\psi_2), \tag{6.4}$$

i.e.,

$$\hat{A}^+ = \hat{\eta}^{-1}\hat{A}^*\hat{\eta}. \tag{6.5}$$

It remains to choose an operator $\hat{\eta}$ in such a way that conditions (3.25) are satisfied, Clearly, the following equations must hold,

$$[\hat{\eta}, \hat{a}_k] = [\hat{\eta}, \hat{a}_k^*] = 0, \quad [\hat{\eta}, \hat{a}_0]_+ = [\hat{\eta}, \hat{a}_0^*]_+ = 0, \tag{6.6}$$

where $[\hat{A}, \hat{B}]_+ \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$. In other words $\hat{\eta}$ anticommutes with $\hat{a}_0, \hat{a}_0^+$. We thus find

$$\hat{a}_k^+ = \hat{a}_k^*, \qquad \hat{a}_0^+ = -\hat{a}_0^*, \tag{6.7}$$

and if relations (6.1) hold then the operators $\hat{a}_\mu, \hat{a}_\mu^+$ satisfy commutation relations (3.25). It is not difficult to find an explicit expression for the operator $\hat{\eta}$:

$$\eta = (-1)^{\hat{N}_0}, \tag{6.8}$$

where the operator $\hat{N}_0$ represents the number of timelike photons. In this procedure, use is made of only the auxiliary condition (1.4).

The shortcomings of this approach stand out like sore thumbs. We know that nonphysical photons are permitted in the formalism for the sake of making it explicitly covariant. The operator (6.8), on the other hand, does not have this property; i.e., the theory actually loses its explicit Lorentz invariance. (Although implicitly, of course, it is Lorentz-invariant). Another troublesome point is the fact that the derivation of the theory begins with explicitly noncovariant relations (6.1). Finally, the entire construction seems contrived: Even commutation relations (3.25) are evidence that the theory contains from the outset states with a negative norm. Gupta subsequently pruned the unnecessary elements from the construction.

6.2 "New" Gupta formalism[16,18]

The formalism which finally emerged is essentially equivalent to that set forth in Sec. 3 and Sec. 4, but without the second of conditions (3.30). The operators $\hat{a}_\mu, \hat{a}_\mu^+$ obey relations (3.25), and the mathematical vacuum ψ_0 is defined by equality (4.1). The physical subspace is delimited by condition (1.4); i.e., the mathematical vacuum also belongs to the physical subspace. The basis of the complete Hilbert space is formed by vectors which are found by applying to ψ_0 an arbitrary number of operators $\hat{a}_\mu^+$. Among them there are states with a negative norm (which contain an odd number of timelike photons). The Hilbert space also contains states with a zero norm. The latter are obtained by applying operators $\hat{L}^+$ to the physical states Φ:

$$\|\hat{L}^+\Phi\|^2 = (\hat{L}^+\Phi,\ \hat{L}^+\Phi) = (\Phi,\ \hat{L}\hat{L}^+\Phi) = 0, \tag{6.9}$$

since, according to (3.31), $\hat{L}(q)$ commutes with $\hat{L}^+(q)$, and we have $\hat{L}\Phi = 0$. States with a zero norm belong to the physical subspace because of (again) relation (3.31), $\hat{L}(\hat{L}^+)^n\Phi = (\hat{L}^+)^n \hat{L}\Phi = 0$, and they are orthogonal to vectors from $\mathscr{H}_{\text{phys}}$: $(\Phi_1, \hat{L}^+\Phi_2) = (\hat{L}\Phi_1, \Phi_2) = 0$. The physical Hilbert space of Gupta's formalism, $\mathscr{H}^G_{\text{phys}}$, is thus the direct sum of the subspace $\mathscr{H}^{(0)}$ of vectors with a zero norm and the subspace (orthogonal to it) of vectors with a positive norm, $\mathscr{H}^{(+)}$: $\mathscr{H}^G_{\text{phys}} = \mathscr{H}^{(+)} \oplus \mathscr{H}^{(0)}$. An arbitrary physical vector in this formalism is of the form $\Phi_G = \Phi^{(+)} + \zeta$, where $\zeta \in \mathscr{H}^{(0)}$. Nonphysical vectors ζ can be eliminated by transforming to the factor space[35] $\mathscr{H}_{\text{phys}} = \mathscr{H}^G_{\text{phys}}/\mathscr{H}^{(0)}$.

6.3. Fixing the gauge

What requirements should we impose on the gauge-fixing term $\mathscr{L}'$ [the "gauge fixer"; see (3.12)]. With regard to perturbation theory, these requirements fall in a natural way into two groups. In the first group are the requirements which must be met unconditionally. Let us list these requirements. The term $\mathscr{L}'$ which is to be added to $\mathscr{L}$ must

1) lift the degeneracy of the Lagrangian,
2) be relativistically invariant,
3) not alter the equations for physical variables (Maxwell's equations), and
4) not lead to a contradiction.

One requirement which we would like to satisfy is the following:

5) The term $\mathscr{L}'$ must satisfy all the conditions which are customarily imposed on Lagrangians (locality, renormalizability, and the absence of derivatives of fields higher than the first).

The first of these conditions is self-evident. The point of adding $\mathscr{L}'$ to the Lagrangian is to specify equations of motion for nonphysical degrees of freedom (to fix the gauge). This condition is not equivalent to the condition

1') that the term $\mathscr{L}'$ must violate gauge invariance. The lifting of degeneracy of course automatically means a violation of gauge invariance, but the inverse is not true.

For example,

$$\mathscr{L}' = \frac{A_\mu^2}{2} \tag{6.10}$$

violates gauge invariance, but it does not lift degeneracy, since when we go over to a Hamiltonian formalism we again find constraints, but now they are secondary.[36]

The second condition might be regarded as not absolutely necessary. However, that view would be inconsistent, since the nonphysical variables were admitted into the theory in order to make it explicitly covariant.

The third and fourth conditions are self-evident. The first of these eliminates the term (6.10). The need for the latter follows from the example $\mathscr{L}' = x_\mu A_\mu/4$. The equations of motion in this case take the form $\partial_\mu F_{\mu\nu} = j_\nu + x_\nu/4$. Applying the operator ∂_ν to them, we find the absurdity $1 = 0$.

The motivation for satisfying the other conditions is obvious, at any rate when perturbation theory is used. If we do not introduce any auxiliary fields, then these requirements lead in an essentially unambiguous way to an $\mathscr{L}'$ from Fermi class (3.12) (more precisely, we should speak in terms of the class of Heisenberg-Pauli-Fermi gauges; cf. Subsection 6.5).

6.4. "Relativistic" oscillator

Let us examine the problem of interpreting the operators $\hat{a}_0(q), \hat{a}_0^+(q)$ in the following simple example. We consider the system specified by the Lagrangian

$$L = -\frac{1}{2} g^{\mu\nu} (\dot{x}_\mu \dot{x}_\nu - x_\mu x_\nu), \tag{6.11}$$

where $g^{\mu\nu}$ is the metric tensor of Minkowski space, x_μ is a 4-vector in it, and, $\dot{x}_\mu = dx_\mu/d\tau$, τ is an invariant parameter (a "time"). This model serves as a good illustration of the essence of the difficulties which arise in a quantization of the electromagnetic field in the Heisenberg-Pauli-Fermi gauge, since here again the dynamic variable is a 4-vector. Switching to a Hamiltonian formalism, with $p^\mu = \partial L/\partial \dot{x}_\mu = -g^{\mu\nu}\dot{x}_\nu$, we find

$$H = p^\mu \dot{x}_\mu - L = -\frac{1}{2}(g_{\mu\nu} p^\mu p^\nu + g^{\mu\nu} x_\mu x_\nu). \tag{6.12}$$

Clearly, the energy is not positive definite.

We turn now to a quantum description. The classical Poisson brackets $\{x_\mu, p^\nu\} = \delta_\mu^\nu$ define the commutator of operators

$$[\hat{x}_\mu, \hat{p}^\nu] = i\delta_\mu^\nu. \tag{6.13}$$

We express the Hamiltonian $\hat{H} = -(\hat{p}^\mu \hat{p}^\mu_+ \hat{x}_\mu \hat{x}_\mu)/2$ in terms of the operators

$$\hat{a}_\mu = \frac{1}{\sqrt{2}}(g_{\mu\nu}\hat{p}^\nu + i\hat{x}_\mu), \quad \hat{a}_\mu^+ = \frac{1}{\sqrt{2}}(g_{\mu\nu}\hat{p}^\nu - i\hat{x}_\mu),$$

$$[\hat{a}_\mu, \hat{a}_\nu^+] = -g_{\mu\nu}; \qquad (6.14)$$

We find

$$\hat{H} = -\frac{1}{2}(\hat{a}_\mu^+\hat{a}_\mu + \hat{a}_\mu\hat{a}_\mu^+). \qquad (6.15)$$

Since one of the four independent oscillators is anomalous, we examine the question of its ground state ψ_0. If we set

$$\hat{a}_k\psi_0 = 0, \quad \hat{a}_0^+\psi_0 = 0 \qquad (k = 1, 2, 3). \qquad (6.16)$$

then by solving these equations in the x representation, $(\partial/\partial x_k + x_k)\psi_0 = (\partial/\partial x_0 + x_0)\psi_0 = 0$, we find

$$\psi_0 = c\exp\left[-\frac{1}{2}(x_0^2 + \mathbf{x}^2)\right]. \qquad (6.17)$$

The function (6.17) is normalizable (it is for this reason that $\hat{a}_0^+, \hat{a}_k$ were chosen as annihilation operators[2)]), but it does not offer relativistic invariance (and it is for this reason that this choice is unacceptable). If we instead require

$$\hat{a}_\mu\psi_0 = 0 \qquad (\mu = 0, 1, 2, 3), \qquad (6.18)$$

we find that the solution of these equations

$$\psi_0 = c\exp\frac{x_\mu^2}{2} \qquad (6.19)$$

is Lorentz-invariant but unrenormalizable. It is this circumstance which motivated the adoption of realization (4.12) in QED an analog of (6.16)] and which was the original cause of the difficulties. The dilemma is resolved by switching to a physical subspace, which is distinguished by condition (3.9) or (3.30). In this model, we correspondingly have the conditions $(\hat{a}_0 - \hat{a}_3)\Phi = 0, (\hat{a}_0^+ - \hat{a}_3^+)\Phi = 0$. These conditions determine a "physical" subspace which is formed by vectors of the type $\Phi = \delta(x_0 - x_3)\varphi(x_1, x_2)$, where φ is a square-integrable function. The function Φ is a well-defined mathematical entity (a generalized eigenvector[32]).

6.5. Brief history of the question

A quantum description of the electromagnetic field was first offered by Dirac.[1] He considered the radiation field, which he treated as a set of independent operators. Dirac's pioneering study stimulated further research in this direction. Jordan and Pauli[37] (Refs. 7, 28, and 37 have been translated into Russian[38]) established relativistically invariant commutation relations for the operators of the electromagnetic field, and they introduced functionals and functional derivatives of fields. The fundamental paper by Heisenberg and Pauli[28] contained nearly all of the basic elements of modern spinor electrodynamics. That theory was formulated in a relativistically invariant way. The appearance of constraints (the conditions $\pi^0 = 0$ and div$\mathbf{E} = 0$) was noted. A class of gauges known today as "Fermi gauges" was introduced. The same year saw the appearance of Fermi's paper[3] in which the conditions $\partial_\mu A_\mu = 0, \partial_\mu A_\mu = 0$ first appeared, although the meaning of these conditions in quantum theory was not discussed there. In a subsequent paper,[7] Heisenberg and Pauli pointed out quite unambiguously that these conditions should be understood as conditions on wave functions (or functionals). Fermi[4,5] treated them in exactly the same way. He found a wave function which satisfied these conditions. We thus see that the foundations of quantum electrodynamics had already been laid by the beginning of the 1930s.

At the end of the 1950s, when modern QED was constructed, condition (1.3) was at the center of attention. It was observed that this condition leads to the unrenormalizability of the vacuum (4.14) (Refs. 10–12). We should point out that solution (4.14), which is based on realization (4.12) of algebra (3.25), can be found in papers by many authors.[10-12,39-41] However, nowhere has it been pointed out that this solution is relativistically invariant. Dirac eliminated it from the fourth edition of his book.[8] It did not reappear in later studies by Dirac.[9]

The difficulties in interpreting the operators $\hat{a}_0, \hat{a}_0^+$ and the unrenormalizability of vacuum (4.14) spurred Gupta on to a search for a new formulation of the theory (Subsection 6.1) and led him to reject condition (1.3).[13] As motivation for rejecting (1.3), Gupta[13] offered only the comment that it was too restrictive to be satisfied by any states of radiation fields. Although it is evident that he subsequently[16,18] saw that an artificial construction with a metric tensor $\hat{\eta}$ was not necessary, and he abandoned that path, it is that path which found its way into the textbooks. As it happened, the basic treatises on quantum field theory and QED[2,19-21] were written in the period between the first[13] and second[16] of Gupta's publications. Although the monographs and textbooks have been reissued since then, Gupta's original construction has remained untouched. (To be fair, we should acknowledge that the fourth edition of the monograph in Ref. 20, which appeared in 1981, omitted the construction with the operator $\hat{\eta}$, but the reason for the change was not specified.) For the past two decades, there has been almost no discussion of the subtleties of the quantization of the electromagnetic field. It has been assumed that the problem was solved theoretically by Gupta and that everything required for practical calculations is contained in the rules postulated by Feynman. The equivalence of these rules (in the Feynman gauge) to a quantum field theory with Fermi condition (1.3) has been demonstrated by Ning Hu.[31]

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[1)]Gauge invariance is a necessary characteristic of a physical quantity, but it is by no means a sufficient characteristic. For example, $p_\vartheta = G/2$ is gauge-invariant, since $\mathbf{E}$ and j_0 are invariant.

[2)]Another limitation is the requirement that they must commute with physical quantities.

[3)]We wish to call attention to the circumstance that $\hat{a}_0$, as a variational-differentiation operator, does not have the same sign as $\hat{a}_3$. This sign is dictated by commutation relations (3.25).

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