The Casimir effect and its applications

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The Casimir effect is analyzed. This effect consists of a polarization of the vacuum of quantized fields which arises as a result of a change in the spectrum of vacuum oscillations when the quantization volume is bounded or the topology of the space is non-Euclidean. Calculations of the effect for manifolds of various configurations and for fields with various spins are reported. Various definitions of the Casimir vacuum energy in the presence of walls are discussed. The quantum field theory of Casimir forces is generalized to incorporate the dispersion properties of the medium. Applications of the Casimir effect in various branches of physics are reviewed: from the theory of molecular forces to cosmology and elementary particle physics, including the bag model and supersymmetry.

CONTENTS

1. INTRODUCTION

Recent years have seen continuously increasing interest in physical phenomena known collectively as the "Casimir effect." This effect is essentially a polarization of the vacuum of quantized fields which arises because of a change in the spectrum of vacuum oscillations when the quantization volume is bounded or the topology of the space is non-Euclidean. The list of the branches of physics in which the Casimir effect is manifested is a very long one, ranging from statistical physics to elementary particle physics and cosmology.

Historically the first prediction of the effect was made by Casimir in 1948 (Refs. 1 and 2): An attractive force $F = -(\pi^2/240)(\hbar c/a^4)$, which can be expressed in terms of Planck's constant $\hbar$, the velocity of light c, and the distance between the plates, a, should act on a unit area of two conducting plane-parallel plates in vacuum. An attraction of this sort was subsequently observed experimentally.[3-6] For plates 1 cm^2 in area with $a = 0.5\ \mu m$, the force was ≈ 0.2 dyn, in accordance with the theoretical prediction.

In quantum field theory the appearance of a vacuum energy density and thus a force between ideally conducting plates at absolute zero is explained in terms of a change in the spectum of zero-point vibrations due to the vanishing of the tangential component of the electric field at the plates.

The Casimir effect can be discussed as a manifestation

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of the van der Waals molecular-attraction forces at large distances, where the retardation of the electromagnetic interaction becomes important.[32,37] The Casimir force is calculated in this case in terms of the characteristics of the fluctuation electromagnetic field over the entire volume, including not only the region between the bounding objects but also these objects themselves. In several cases, however, it is simpler to describe the effect by taking an approach which makes it possible to replace the analysis of the field within objects by some effective boundary condition.

Since 1948, several hundred papers on the Casimir effect have appeared in the literature, most of them in the past decade. In addition to the interest in connection with research on the vacuum forces between solid objects, the Casimir effect has attracted interest because of the possibility of a non-Euclidean topology of space-time. As in the presence of material boundaries, the spectrum of zero-point vibrations in topologically nontrivial spaces differs from that in the case of Minkowski space, with the result that a nonzero vacuum energy-density arises. This fact is important to the problem of the cosmological constant[45] and in connection with inflationary cosmological scenarios.[46]

The Casimir effect has turned out to be extremely important in hadron physics, in the construction of a bag model[65,66] in which quarks are confined as a result of the postulated absence of a current through the surface of a bag which is bounding a hadron. The Casimir energy of the quark and gluon fields must be incorporated in the total energy of a bag in a calculation of the properties of hadrons.

The Casimir effect has extremely timely applications in the supersymmetry field theories of the Kaluza-Klein type. Here the Casimir effect must be taken into account in analysis of the mechanisms for the spontaneous compactification of the additional spatial dimensions (dimensional reduction).[74]

Finally, Casimir forces have turned out to be extremely sensitive to the values of the supersymmetry breaking parameter and the masses of hypothetical light particles—properties in elementary particle physics whose values need to be refined.[78] Atomic force microscopy holds great promise here.[102]

The present review covers all the research directions listed above. In contrast with a recent review by Plunien *et al.*,[10] the present review focuses on the physical side of the questions, including the application of calculation methods using effective boundary conditions to real models of boundaries, an analysis of the numerous applications of the effect, and a comparative analysis of the various approaches to the interpretation of the effect.

In §2 we use the very simple example of a scalar field for a detailed analysis of the fundamental physical and mathematical aspects of the Casimir effect. That section of this review is a self-contained introduction to the subject. In §3 we present some specific results for real three-dimensional problems for various types of fields. In §4 we analyze the changes which are introduced by the specific properties of a medium and the experiments which have been carried out. In §5 we discuss some thornier questions: the Casimir energy in a space-time with a non-Euclidean topology, including applications in cosmology. Finally, in §6 we describe applications of the Casimir effect to various topics in elementary particle physics. The appendices contain certain details of the regularization of the vacuum energy-momentum tensor. We set Planck's constant $\hbar$ and the velocity of light c equal to unity. Any exceptional cases will be specified.

We dedicate this review to the memory of our friend and colleague Sergeĭ Georgievich Mamaev, who did a great deal to develop the theory of the Casimir effect.

2. ALPHABET OF THE EFFECT: MODEL OF A ONE-DIMENSIONAL SCALAR FIELD

2.1. Boundary conditions: idealized version of an external field. As was mentioned in the Introduction, many interesting quantum effects arise in the interaction of a vacuum of quantized fields with external fields. The Casimir effect may be counted as one of these effects. Let us recall a very simple problem from classical mechanics: the reflection of a particle from an elastic wall. Such a wall is evidently a limiting form of a potential which increases rapidly along the coordinate as the interval of the increase approaches zero. The potential is then "remembered" only in the reflection condition, and the motion between walls is free and can be described in an elementary way. An analog of this problem in quantum mechanics is the problem of a square potential well of infinite depth, in which case the analysis of the limiting transition (in terms of the well depth) makes it possible to eliminate the step of finding wave functions for a given potential and to restrict the analysis to solving a free Schrödinger equation with homogeneous boundary conditions at the walls.

Replacing an external field by boundary conditions is a suitable approach in many problems, including relativistic problems. At the same time it allows one to concentrate on the fundamental side of the effects, since the technical difficulties are eased. One must of course bear in mind the limits on the applicability of the method, as in any idealization; e.g., in using the model of a well of infinite depth one cannot in principle deal with a continuous spectrum.

An important point for the discussion below is that replacing an external field by a boundary condition is also possible for systems which have an infinite number of degrees of freedom. A well-known example is a string whose ends are fixed. Instead of explicitly incorporating in Newton's equation the forces which prevent a displacement of the boundary points of the string, $x = 0, a$, one usually imposes boundary conditions

$$y(0, t) = y(a, t) = 0 \tag{2.1}$$

on the displacement y of the points of the string which are parametrized by the coordinate x. The equation of motion, on the other hand, is free:

$$\rho \frac{\partial^2 y}{\partial t^2} = g \frac{\partial^2 y}{\partial x^2}, \tag{2.2}$$

where ρ is the linear density of the string, and g is the elastic constant.

The imposition of constraints which fix the distance between the particles of a system can also be reduced to the limiting form of a potential which increases rapidly upon a violation of the conditions of the constraint.

2.2. Quantum mechanics of a string. Let us take a detailed look at the quantization of the displacement field $y(x, t)$ of a string as a transitional stage toward a description of relativistic quantized fields in bounded spatial regions. This subsection is basically pedagogical and can be skipped by those well-versed in quantum theory.

We rewrite (2.2) as a standard wave equation:

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = 0; \tag{2.3}$$

where $c = (g/\rho)^{1/2}$ is the propagation velocity of the vibrations in the string.

We seek a general solution of (2.3) under conditions (2.1) as an expansion in the orthonormal system $y_n(x)$:

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t)\, y_n(x), \qquad y_n(x) = \left(\frac{2}{a}\right)^{1/2} \sin k_n x, \quad k_n = \frac{\pi n}{a}. \tag{2.4}$$

Substituting (2.4) into (2.3), we find that each amplitude $a_n(t)$ satisfies a harmonic-oscillator equation

$$\ddot{a}_n + \omega_n^2 a_n = 0, \qquad \omega_n = c k_n = \frac{c\pi n}{a}. \tag{2.5}$$

The linear energy density ε, which is the sum of kinetic and potential components, determines the total energy of the string, E:

$$E = \int_0^a \varepsilon(x,t)\, dx = \int_0^a \left[\frac{\rho \dot{y}^2}{2} + \frac{g}{2}\left(\frac{\partial y}{\partial x}\right)^2\right] dx. \tag{2.6}$$

For brevity we set $\rho = g = c = 1$ at this point. Substituting expansion (2.4) into (2.6), and making use of the orthonormality of the functions $y_n(x)$, we find

$$E = \frac{1}{2}\sum_{n=1}^{\infty} (\dot{a}_n^2 + \omega_n^2 a_n^2). \tag{2.7}$$

The representation of the string displacement field $y(x, t)$ in terms of a set of oscillators a_n as in (2.4) and (2.5) and the representation of the energy E as the sum of oscillator energies as in (2.7) make it possible to construct a quantum-mechanical description of the motion of the string extremely quickly. We find the Hamiltonian H of the system, as in the case of an oscillator, from expression (2.7) for the energy, by replacing $\dot{a}_n$ by a momentum operator[83]:

$$\hat{H} = \frac{1}{2}\sum_n (\hat{p}_n^2 + \omega_n^2 a_n^2), \qquad \hat{p}_n = -\frac{i\partial}{\partial a_n}. \tag{2.8}$$

The Schrödinger equation

$$\hat{H}\psi(a_1, a_2, \ldots) = E\psi(a_1, a_2, \ldots)$$

determines the wave functions ψ and the possible energy levels. Since the oscillators in (2.8) are independent, we evidently have

$$E = E(m_1, m_2, \ldots) = \sum_{n=1}^{\infty} \omega_n \left(m_n + \frac{1}{2}\right). \tag{2.9}$$

The numbers $m_n \geqslant 0$ are called the "occupation numbers" of mode n; the same set of numbers, $\{m_n\}$, specifies the eigenfunctions: $\psi\{m_n\}(a_1, a_2, \ldots)$.

A unique feature of an oscillator—the linear dependence of the energy of each mode on the occupation number—makes it possible to represent an oscillator in an excited state ($m_n > 0$) as a set of m_n excitation quanta, each having an energy ω_n. For the case of a string, one speaks of a state with m_n quanta (phonons) in each of the modes. An important point is that even the ground state, with the lowest energy ($m_n = 0$ for all n) has a nonzero energy[83]:

$$E = E_0 = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n \neq 0. \tag{2.10}$$

A state with $\{m_n\} = 0$ is called the "vacuum state" or simply the "vacuum" of the phonon field, while E_0 is the zero-point energy of the vacuum oscillations.

It is useful to note that the set ω_n depends on the boundary conditions. For example, under the boundary conditions

$$\frac{\partial y(0,t)}{\partial x} = \frac{\partial y(a,t)}{\partial x} = 0 \tag{2.11}$$

we find, in place of frequencies (2.5),

$$\omega_n = \frac{2n+1}{2a}\pi, \tag{2.12}$$

while the expressions for the energy remain the same as before. A string can serve as a one-dimensional model of oscillations in a solid. The vacuum state with an energy E_0 is then reached at absolute zero[1]: $T = 0$.

We wish to call attention to the circumstance that in a macroscopic medium there is a length scale—the wavelength of the vibrations, $\lambda_n = 2\pi/k_n$—which cannot be shorter than the distance between atoms, d. The summation in (2.10) is thus actually restricted to frequencies $\omega < \Omega \approx c/d$, so the value of E_0 turns out to be finite. In the sections which follow we will deal with infinite values of E_0, as in (2.10), with a summation over all $n < \infty$. It is thus necessary to develop procedures for calculating a finite difference between infinite quantities. In particular, one can give meaning to the difference between the values of E_0 with frequency sets (2.5) and (2.12) by introducing, more or less explicitly, a cutoff parameter of the c/d type in the intermediate stages of the calculations.

The arguments of wave function (2.8) are the displacements of the oscillators, a_n; i.e., a coordinate representation has been used. When the fields are quantized, it is more convenient to use another representation: the occupation-number representation (which is also called the "second-quantization representation"). We introduce the operators

$$a_n^{\pm} = \left(\frac{\omega_n}{2}\right)^{1/2}\left(\hat{a}_n \mp \frac{i\hat{p}_n}{\omega_n}\right) \tag{2.13}$$

(in the coordinate representation, the action of the operator $\hat{a}_n$ reduces to multiplication by the number a_n, and $\hat{p}_n = -i\partial/\partial a_n$; in the momentum representation we have $\hat{p}_n = p_n$, $\hat{a}_n = i\partial/\partial p_n$). Working from the known commutator $[\hat{a}_n, \hat{p}_n] = i$, we easily find

$$[a_n^-, a_n^+] = 1, \quad [a_n^{\pm}, a_n^{\pm}] = 0, \tag{2.14}$$

and all operators with different indices commute.

For the time being we will assume that there is only a single oscillator, a_n (the nth mode). The wave function of such an oscillator, $\psi m_n(a_n) \equiv |m_n\rangle$, which corresponds to a number of quanta m_n, is well known.[83] By directly applying operators (2.13) to it one can verify the following relations[83]:

$$a_n^+ |m_n\rangle = (m_n+1)^{1/2} |m_n+1\rangle, \quad a_n^- |m_n\rangle = m_n^{1/2} |m_n-1\rangle, \quad a_n^+ a_n^- |m_n\rangle = m_n |m_n\rangle. \tag{2.15}$$

The fact that the number of quanta is changed by the operators $a_n^\pm$ justifies calling the latter "creation and annihilation operators," respectively (in our problem, the entity which is created or annihilated is a phonon), while the operator $N_n = a_n^+ a_n^-$ would naturally be called a "number-of-quanta operator" or a "number-of-(quasi-)particles operator." Let us assume that the operators $a_n^\pm$ also act on the wave function $|\{m_n\}\rangle = \psi_{m_1, m_2,...}(a_1, a_2,...)$ of the set of oscillators. In (2.15), the occupation numbers of the oscillators with indices different from n do not change.

It remains to express the Hamiltonian operator $\hat{H}$ in (2.8) in terms of $a_n^\pm$. Solving (2.13) for $\hat{p}_n$, $\hat{a}_n$, we find [using (2.14)]

$$\hat{H} = \sum_n \frac{\omega_n}{2}(a_n^+ a_n^- + a_n^- a_n^+) = \sum_n \omega_n \left(\hat{N}_n + \frac{1}{2}\right). \quad (2.16)$$

The application of $\hat{H}$ in (2.16) to the function $|\{m_n\}\rangle$ immediately gives us the earlier result, (2.9), for the energy.

After all the operators are expressed in terms of $a_n^\pm$, relations (2.14) and (2.15) completely determine the properties of the system, and we can forget about specific realization (2.13) of the operators $a_n^\pm$.

In the occupation-number representation, only the number-of-particles operator $\hat{N}_n$ and functions of it reduce to ordinary numbers (c-numbers). The other quantities, including a_n and the displacement field $y(x, t)$ which is related to it in a linear way [see (2.4)], become operators (the system of functions y_n of course remains a c-number system).

2.3. Complete sets of solutions of the Klein-Fok equation. In calculations on the Casimir effect, extensive use is made of eigenfunctions and eigenvalues of the corresponding field equation. We begin our analysis of relativistic problems with the very simple case of an uncharged scalar field which is defined on an axis (the dimensionality of the space-time is $D = 2$). This field obeys the Klein-Fok equation (below we set $\hbar = c = 1$)

$$\frac{\partial^2\varphi(x, t)}{\partial t^2} - \frac{\partial^2\varphi(x, t)}{\partial x^2} + (m^2 + V(x))\,\varphi(x, t) = 0; \quad (2.17)$$

here m is the mass of the field, i.e., of the particles which make up the field (for the phonons in Subsection 2.2 we obviously have $m = 0$), and $V(x)$ is an external scalar field which interacts with the field φ. Taking the same approach as in the preceding subsections, we set $V \equiv 0$, taking account of the interaction by means of a boundary condition of "impenetrability" at the edges of the segment:

$$\varphi(0, t) = \varphi(a, t) = 0. \quad (2.18)$$

Solutions of (2.17) with $V = 0$ and conditions (2.18) can be found easily:

$$\varphi_n^\mp(x, t) = \frac{1}{(a\omega_n)^{1/2}} e^{\pm i\omega_n t} \sin k_n x,$$
$$\omega_n = (m^2 + k_n^2)^{1/2}, \qquad k_n = \frac{\pi n}{a} \qquad (n = 1, 2, \ldots). \quad (2.19)$$

The functions (2.19) have been orthonormalized in accordance with the well-known indefinite scalar product for Eq. (2.17) (Ref. 83):

$$(\varphi_n^\pm, \varphi_{n'}^\pm) = \mp\,\delta_{nn'}, \qquad (\varphi_n^\pm, \varphi_{n'}^\mp) = 0,$$
$$(f, g) \equiv \int [f^*\partial_t g - g^*\partial_t f]\,dx. \quad (2.20)$$

Any arbitrary solution of (2.17), (2.18) can be expanded in $\varphi_n^\pm$, and the coefficients can be determined with the help of (2.20):

$$\varphi(x, t) = \sum_n (a_n^-\varphi_n^-(x, t) + a_n^+\varphi_n^+(x, t)). \quad (2.21)$$

On an infinite axis, i.e., in the absence of boundary conditions (2.18), the solutions of (2.17) obviously are not quantized; i.e., all values $0 \leqslant \omega < \infty$ are allowed. These solutions are conveniently written in the form of traveling waves,

$$\varphi_k^\mp = \frac{1}{(4\pi\omega_k)^{1/2}} \exp[\pm i(\omega t - kx)] \qquad (-\infty < k < \infty), \quad (2.22)$$

which are normalized by means of the replacement $\delta_{nn'} \to \delta(k - k')$ in (2.20). Expansion (2.21) is then replaced by

$$\varphi(x, t) = \int_{-\infty}^{\infty} (a_k^-\varphi_k^- + a_k^+\varphi_k^+)\,dk. \quad (2.23)$$

It is also useful to find solutions for a periodic boundary condition

$$\varphi(x, t) = \varphi(x + a, t),$$
$$\partial_x\varphi(x, t) = \partial_x\varphi(x + a, t), \quad (2.24)$$

which reflects the identical nature of the boundary points or, more clearly, the coiling of segment $(0, a)$ into a ring. The length of the segment must now be equal to an integer number of wavelengths [a half-integer number would also be permissible for (2.18)].

One family of solutions is the same as (2.19) in the case $k_n = 2\pi n/a$. Now, however, there can be another family, with $\varphi \neq 0$ at the points $x = 0, a$. This family has the same frequencies $k_n = 2\pi n/a$ and reduces to the replacement $\sin k_n x \to \cos k_n x$ in (2.19). The number of modes has thus doubled from that corresponding to conditions (2.18) (in addition to the change in the spectrum).

These solutions can be represented by traveling waves instead of standing waves:

$$\varphi_n^\mp = \frac{1}{(2\omega_n a)^{1/2}} \exp[\pm i(\omega_n t - k_n x)],$$
$$\omega_n = (m^2 + k_n^2)^{1/2}, \quad k_n = \frac{2\pi n}{a} \qquad (-\infty < n < \infty). \quad (2.25)$$

It is a simple matter to take the limit $a \to \infty$, of an unbounded space, in system (2.25).

2.4. The infinite energy of zero-point oscillations depends on the boundary conditions. The clearest characteristics of the Casimir effect are the average values of the energy-momentum tensor of the quantized field under consideration in the vacuum state. The energy-momentum tensor of a real scalar field in a two-dimensional space-time is[7,83]

$$\begin{matrix} T_{00} \\ T_{11} \end{matrix} = \frac{1}{2}[(\partial_t\varphi)^2 + (\partial_x\varphi)^2 \pm m^2\varphi^2],$$
$$T_{01} = T_{10} = \frac{1}{2}(\partial_t\varphi\,\partial_x\varphi + \partial_x\varphi\,\partial_t\varphi). \quad (2.26)$$

In the occupation-number representation, the amplitudes $a_n^\pm$ in field expansion (2.21) must be regarded as creation and annihilation operators with commutation relations (2.14). When we then substitute field expansion (2.21) into (2.26), we find the energy-momentum-tensor operator:

$$T_{00}=\frac{1}{2}\sum_n[(\omega_n^2+m^2)\varphi_n^+\varphi_n^-+(\partial_x\varphi_n^+)(\partial_x\varphi_n^-)](a_n^+a_n^-+a_n^-a_n^+),$$ (2.27)

with corresponding expressions for the other components. We have immediately discarded from (2.27) products of operators with unlike indices as well as terms of the type $a_n^+a_n^+$, $a_n^-a_n^-$, since they vanish when the energy-momentum tensor is averaged over any state, by virtue of relations (2.15) and the orthogonality of states with noncoincident sets $\{m_n\}$.

The unrenormalized energy density is found as the expectation value $\langle 0|T_{00}|0\rangle$; using (2.16), we find that the last set of parentheses in (2.27) gives us $\langle 0|2N_n+1|0\rangle=1$ after the expectation value is taken. The distinctive features of a particular problem are contained in the choice of the functions $\varphi_n^{\pm}$. The result is simplest for periodic conditions (2.24), since $\varphi_n^+\varphi_n^-=\text{const}$ follows from (2.25):

$$\langle 0|T_{00}|0\rangle=\frac{1}{2a}\sum_n\omega_n. \quad (2.28)$$

Since (2.28) is independent of the coordinate (as we would expect in view of the translational symmetry) the total energy E can be found by simply multiplying (2.28) by the length of the segment, a.

When there are walls, set (2.19) gives rise to terms with both $\sin 2k_nx$ and $\cos 2k_nx$ in (2.27). Since the sum of these quantities is equal to unity, we can put the results in the form

$$\langle 0|T_{00}|0\rangle=\frac{1}{2a}\sum_n\omega_n+\Delta(m^2,x), \quad (2.29)$$

$$\Delta(m^2,x)\equiv-\frac{m^2}{2a}\sum_n\frac{\cos 2k_nx}{\omega_n}.$$

The tension is given by the same expression in the two cases:

$$\langle 0|T_{11}|0\rangle=\frac{1}{2a}\sum_n\frac{k_n^2}{\omega_n}. \quad (2.30)$$

Finally, for the empty space (the axis) we find from (2.22)

$$\langle 0_M|T_{ij}|0_M\rangle=\frac{1}{2\pi}\int_0^\infty k_ik_j\delta_{ij}\frac{dk_1}{\omega}, \quad (2.31)$$

where we have introduced the 2-vector $k_0=\omega$, $k_1=k$.

In all cases, the off-diagonal components T_{01} and T_{10} have vanishing expectation values.

It can be seen from (2.29) that the presence of boundaries gives rise to an additional term in the energy density, which depends on the coordinates. if the field has a nonvanishing mass. The contribution of this term to ε will be discussed in Subsection (2.9); here we would simply like to note that Δ has a nonzero value at all interior points of the segment $(0,a)$, while the expectation value over the segment is $\Delta=0$. Accordingly, regardless of the mass of the field, its energy in the vacuum state is

$$E=\int_0^a dx\,\langle 0|T_{00}|0\rangle=\frac{1}{2}\sum_n\omega_n.$$

Since the sets of frequencies depend on the boundary conditions, even if the values of the length a are the same, it is tempting to state that the energies of the vacuum oscillations also depend on the boundary conditions.

One must proceed with caution here, however. The reason is that the frequencies ω_n increase without bound in the limit $n\to\infty$, so the quantities in (2.28)–(2.31) diverge, and there is no finite value which corresponds to them. One can hope to give meaning to the assertion that the values of the vacuum energies are unequal by proceeding in the following manner: We adopt some function $f_\alpha(\omega)\leqslant 1$ of such a nature that we have $f_\alpha(\omega)\to 0$ as $\omega\to\infty$ for all $\alpha\neq 0$, while for $\alpha=0$ the value is $f_\alpha(\omega)=1$, regardless of ω. As an example we could use $f_\alpha(\omega)=\exp(-\alpha\omega)$. Calculating the vacuum energy for each of the sets of frequencies, $\omega_n^{\mathrm{I,II}}$, we make the replacement $\omega_n\to\omega_nf_\alpha(\omega_n)$ in the sums. The sums for $\alpha\neq 0$ then converge and have a finite value which depends on α. The "difference between the vacuum energies" is naturally taken to be the limit

$$\Delta E=\lim_{\alpha\to 0}\frac{1}{2}\left(\sum_n\omega_n^{\mathrm{I}}f_\alpha(\omega_n^{\mathrm{I}})-\sum_n\omega_n^{\mathrm{II}}f_\alpha(\omega_n^{\mathrm{II}})\right). \quad (2.32)$$

Generally speaking, we should except $\Delta E\neq 0$. The question of the existence of a limit in (2.32) and the question of its independence from the nature of the function f_α of course require a separate analysis.

2.5. An observable quantity: the regularized energy difference of the zero-point oscillations. If we are interested in the value of the vacuum energy or its density, we would naturally subtract the contribution of Minkowski space from the vacuum expectation value of the 00 component of the energy-momentum tensor, in order to give it meaning. In other words, we define a renormalized energy density ε by

$$\varepsilon=\lim_{\alpha\to 0}(\langle 0|T_{00}|0\rangle_\alpha-\langle 0_M|T_{00}|0_M\rangle_\alpha), \quad (2.33)$$

where the index α is used to specify the same procedure as in (2.32). The renormalized energy $\mathscr{E}$ is defined as an integral of ε. The forms of the spectra in the two components in (2.33) may be different: e.g., denumerable if there are boundaries and continuous in empty space.

We can show that ε from (2.33) does indeed have a finite value; at this point we will restrict the discussion to the case of a massless scalar field. Here it is convenient to use the Abel-Plana summation formula[88,93]

$$\sum_{n=1}^\infty F(n)=\int_0^\infty F(y)\,dy-\frac{F(0)}{2}+i\int_0^\infty\frac{F(it)-F(-it)}{e^{2\pi t}-1}\,dt. \quad (2.34)$$

As the function $F(n)$ we adopt $f_\alpha(\omega_n)\omega_n$. It is not difficult to see that the integral in (2.34) corresponds to the contribution of an unbounded space, (2.31), and it drops out of difference (2.33). Consequently, the following regularized value should be assigned to the diverging series after the cutoff is eliminated, $\alpha\to 0$:

$$\operatorname{reg}\sum_{n=1}^\infty F(n)=-\frac{F(0)}{2}+i\int_0^\infty\frac{F(it)-F(-it)}{e^{2\pi t}-1}\,dt. \quad (2.35)$$

Because of the rapid convergence of the integral in (2.35), we can immediately set $f_\alpha=1$, i.e., $\alpha=0$, and the result does not depend on the particular choice of f_α.

In the important case of a massless field with boundaries, we have $\omega_n = \pi n/a$, and it is an elementary matter to carry out the summation:

$$\varepsilon = \frac{\pi}{2a^2}\,\mathrm{reg}\sum_{1}^{\infty} n = -\frac{\pi}{a^2}\int_0^\infty \frac{t\,dt}{e^{2\pi t}-1} = -\frac{\pi}{24a^2}. \quad (2.36)$$

Calculating the renormalized tension (or pressure) $P = \langle 0|T_{11}|0\rangle_{\mathrm{reg}}$ from (2.26) in a corresponding way, we find $P = \varepsilon$. For the renormalized energy $\mathscr{E} = a\varepsilon = -\pi/24a$ we thus find

$$P = -\frac{\partial \mathscr{E}}{\partial a},$$

i.e., the usual thermodynamic relation for elastic strain. One might say that the vacuum behaves as a Maxwellian elastic ether (with a negative energy and a negative tension).

As the distance between the walls changes, the energy of the internal region, $\mathscr{E}$, changes. The energy of the external regions, on the other had, does not change as the walls move, even if this energy is nonzero because of the presence of a boundary, as can be seen simply from the absence of a finite parameter. The total energy of the system thus falls off as $\mathscr{E} \sim -a^{-1}$ as the boundaries (the "plates") close on each other. In this manner a force (pressure) equal to P arises and tends to move the plates (boundaries) together.

It is an effect of this sort which has been observed experimentally as metal plates which are (nearly) impermeable to an electromagnetic field are moved toward each other.[3]

2.6. The boundary conditions model the topology. For periodic conditions (2.24) modeling a circle S^1, i.e., the simplest space with a nontrivial topology, it is a simple matter to derive the following expression in place of (2.37):

$$\varepsilon = -\frac{\pi}{6a^2}.$$

In this case the fourfold difference from the version with homogeneous conditions, (2.36), can be explained in a naive way: as a consequence of the doubling of the number of modes and the doubling of the frequency in each nth mode which we mentioned back in Subsection 2.3. The next example, however, shows that qualitative arguments concerning infinite sums can lead to errors. Let us impose the antiperiodicity condition

$$\varphi(x + a, t) = -\varphi(x, t). \quad (2.37)$$

The allowed frequencies are then

$$\omega_n = \frac{2\pi[n + (1/2)]}{a} = k_n. \quad (2.38)$$

In order to sum over the half-integers in (2.38) we need to use not Abel-Plana formula (2.36) but its analog[12]

$$\mathrm{reg}\sum_{n=0}^{\infty} F\left(n + \frac{1}{2}\right) = -i\int_0^\infty \frac{F(it) - F(-it)}{e^{2\pi t}+1}\,dt, \quad (2.39)$$

which differs, in particular, in the sign in front of the integral. A calculation from (2.39) yields

$$\varepsilon = \frac{\pi}{12a^2}, \quad \mathscr{E} = \varepsilon a = \frac{\pi}{12a}; \quad (2.40)$$

i.e., even the sign of ε changes for frequency set (2.38).

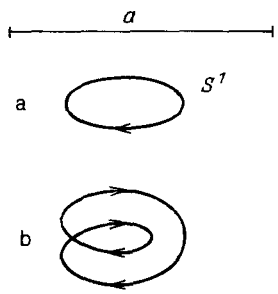

FIG. 1. a, b—Versions of the topology in the one-dimensional case.

Configurations of fields with condition (2.37) are called "twisted." They correspond to a so-called globally nontrivial stratification.[14,89] A case of a field with a self-effect is discussed in Ref. 23, among other places.

Let us attempt to clarify condition (2.37), which we have introduced in a formal way. As we have already mentioned, the ordinary periodicity condition $\varphi(x + a, t) = \varphi(x, t)$ corresponds to the diagram of circle S^1 in Fig. 1a. Under condition (2.37), we return to the previous value of the field, $\varphi(x, t) = \varphi(x + 2a, t)$, only after we have traveled a path $2a$, i.e., only after we have made "two revolutions" (see the diagram in Fig. 1b). In reality, a continuous path of this sort is traced out by a pencil on a two-sided surface: a Möbius sheet. We might also recall that a spinor wave function also returns to its previous value after two revolutions, i.e., after a rotation through 4π. We will see below that antiperiodicity conditions and summations over half-integer values do indeed arise in a natural way in problems involving a field on a Möbius sheet and problems involving spinor fields (these problems are of course not one-dimensional).

2.7. The Casimir effect as a polarization of vacuum. The examples above show that the vacuum of a quantized field has a certain nonzero energy when there are boundaries. In standard field theory for an unbounded space we know that the energy of the vacuum is assumed to be zero; this assumption reduces to a change of $\omega/2$ in the origin for the energy scale for each mode.[2)] The motivation for making such a transformation is that the energy is determined entirely, within an additive constant, and it is formalized by the operation of a normal ordering of the operators of physical quantities. This operation reduces to the replacements $H \to N(H)$, $T_{ik} \to N(T_{ik})$, etc., where N means the transposition of operators $a_n^- a_n^+ \to a_n^+ a_n^-$ as commuting operators. It can easily be seen from (2.16) that in this case the half-frequencies disappear. A further argument in favor of assigning zero values to the energy and other observables in vacuum is the circumstance that only for these values is the vacuum state invariant under the Poincaré group (i.e., displacements and Lorentz transformations).

In the circle of problems which we are considering here there is—even under fixed boundary conditions—an infinite set of different vacuum states for different volumes (different values of a) of the system. These states transform into each other in the course of an adiabatic change (i.e., a change which does not involve the excitation of quanta) in the parameters of the system (the value of a). This is precisely how the force acting between boundary plates is determined. Clearly, it is physically incorrect to assign prespecified values (even identical zero values) to the energies of several states between which transitions can occur. Furthermore, the argument based on Poincaré invariance does not work: Clearly, there is no such invariance if there are boundaries.

We thus see that it is logically unavoidable to use as a characteristic of the vacuum state the changes caused by the external field (or the boundary conditions) in the energy-momentum-tensor expectation values themselves rather than the expectation values of the normally ordered energy-momentum tensor. The possibility that the vacuum will have quantized fields of a nonzero energy-momentum tensor was derived in detail in Ref. 45 (see also Subsection 5.4 in connection with the problem of the cosmological constant).

It should be kept in mind that in problems involving calculations of forces, i.e., derivatives of $\mathscr{E}$, there is still some arbitrariness in the values of $\mathscr{E}$: A result will not change if all of the $\mathscr{E}$ are changed by the same constant. When gravitation is taken into account, however, and the renormalized energy-momentum tensor itself is substituted into the right side of Einstein's equations, even an arbitrariness of this type is impermissible. In such problems, characterized by a nonzero curvature of the space-time, rather than by boundaries, logically the only possibility is to subtract the vacuum of the tangent Minkowski space during the renormalization of the energy-momentum tensor (§5).

The appearance of nonvanishing local observables (the energy-momentum tensor) in a vacuum state is naturally called a "polarization of vacuum" by analogy with well-known effects in quantum electrodynamics, e.g., the polarization of the electron-positron vacuum in the Coulomb field of a nucleus.[86]

The global nature of the vacuum is clearly manifested in the problems which we are considering here. The oscillation spectrum and, along with it, the local energy density $\varepsilon(x)$ are functionals of the values of the external field $V(x)$ throughout the space, including at the boundaries. For this reason, even inside the region, i.e., at points with $V(x) = 0$, the energy density is sensitive to the behavior of $V(x)$ at all remote points. This circumstance turns out to be important in §5 in an analysis of cosmological applications of the Casimir effect.

2.8. A more realistic model: semipermeable walls. Boundary conditions (2.8) have absolutely impermeable walls, so the external space is completely isolated from the volume between walls. We now incorporate the external field $V(x)$ explicitly by means of Eq. (2.17) (Ref. 40).

We assume that the potential $V(x) = V_\lambda(x)$ is zero at all internal points of a segment and depends on the parameter λ in such a way that in the limit $\lambda \to \infty$ we have complete impenetrability: i.e., the regions outside and inside the segment $(0, a)$ are independent. These conditions obviously do not fix $V_\lambda(x)$ unambiguously; the arbitrariness can be utilized to simplify the calculations.

We choose $V_\lambda(x)$ to be the sum of two δ-functions. For symmetry considerations, it is convenient to shift the origin of coordinates to the center of segment $(0, a)$:

$$V_\lambda(x) = \lambda\left(\delta\left(x - \frac{a}{2}\right) + \delta\left(x + \frac{a}{2}\right)\right). \qquad (2.41)$$

As has been established, impermeable walls at the points $x = \pm a/2$ correspond to $\lambda \to \infty$. At all finite values of λ the walls $x = \pm a/2$ are semipermeable; i.e., solutions of wave equation (2.17) with potential (2.41) span the entire space and do not necessarily vanish at the walls.

In this problem we have only a continuous spectrum $0 < \omega < \infty$. On the basis of symmetry considerations we should seek solutions in the form of two families: even and odd in x, with wave numbers $k \geqslant 0$. For a massless field we would have $\omega = k$, so we will write ω in place of k everywhere:

$$\varphi^{\pm}_{j\omega}(x, t) = (2\pi\omega)^{-1/2} e^{\pm i\omega t}\chi_{j\omega}(x) \qquad (j = 1, 2), \qquad (2.42)$$

$$\chi_{1\omega} = A_1 \sin \omega x, \quad |x| \leqslant \frac{a}{2},$$
$$= \sin(\omega x + \delta_1 \varepsilon(x)), \quad |x| \geqslant \frac{a}{2}. \qquad (2.43)$$

The second family, $\chi_{2\omega}$, differs from the first by the substitutions $\sin \to \cos$, $A_1 \to A_2$, and $\delta_1 \to \delta_2$ [we have $\varepsilon(x) = \pm 1$ at $x \gtrless 0$]. The unit coefficient of the sine function in the region $|x| > a/2$ provides the correct normalization (in terms of the flux).

In the complete absence of a wall ($\lambda = 0$) there is no phase shift at the boundary ($\delta_j = 0$), so the continuity of $\chi_{j\omega}$ at the points $x = \pm a/2$ (which also holds in the case $\lambda \neq 0$) tells us that we have $A_j = 1$ in the empty space.

Using the general equations of Subsections 2.4 and 2.5, we can find an expression for T_{00} and for the renormalized energy density ε between the walls, which are determined by families of functions with index j:

$$\varepsilon_j = \frac{1}{\pi a^2}\int_0^\infty \Omega(A_j^2 - 1)\, d\Omega. \qquad (2.44)$$

Here and below we are using the dimensionless parameters $\Omega = a\omega/2$ and $\Lambda = \lambda a/2$. Expression (2.44) corresponds to (2.33): Subtracting the contribution of the vacuum of the empty space gives rise to the term -1 in the parentheses. According to the discussion above, in the case $\lambda = 0$ we have $A_j = 1$ and thus $\varepsilon_j = 0$, as we should.

The quantities A_j, like δ_j, can be found by elementary but slightly tedious calculations, by substituting (2.42) and (2.41) into the wave equation. After some manipulations, we can write A_1, for example, in the form[40]

$$A_1^2 - 1 = -\Lambda\, \mathrm{Im}\, \frac{e^{i\Omega}}{\Omega e^{i\Omega} + \Lambda \sin\Omega}.$$

An expression of this form makes it a simple matter to calculate ε_1, by rotating the integration contour through $\pi/2$ (the infinite fourth quadrant of the circle makes a vanishing contribution):

$$\varepsilon_1 = -\frac{\Lambda}{\pi a^2}\int_0^\infty \frac{y e^{-y}\, dy}{y e^y + \Lambda\, \mathrm{sh}\, y}. \qquad (2.45)$$

The total energy density is $\varepsilon = \varepsilon_1 + \varepsilon_2 < 0$ for all Λ, since ε_2 differs from (2.45) by a sign and by the replacement $\sinh \to \cosh$, and for all arguments we have $\sinh y < \cosh y$.

The switch to impenetrable walls is made by taking the limit $\Lambda \to \infty$. It is easy to see that in this case we have $\varepsilon = \varepsilon_1 + \varepsilon_2 = -\pi/24\, a^2$, in accordance with (2.36).

An advantage of this method is that spectra of identical form (continuum) are being compared. Furthermore, since the spectrum stretches out to the same region, $0 < \Omega < \infty$, for all values of Λ, including empty space ($\Lambda = 0$), we can carry out a renormalization in a mode-by-mode fashion, first subtracting the vacuum contribution of Minkowski space in each mode [this step reduces to the replacement $\Omega A_j^2 \to \Omega(A_j^2 - 1)$] and only then carrying out the integration over frequency. When the operations are carried out in this order, no divergent quantities appear at all. Clearly, a mode-by-mode subtraction of this sort would not be possible in a comparison of denumerable and continuous spectra in (2.34)—or in a comparison of different denumerable spectra (2.32). Here we are obliged to deal with infinite sums and integrals as entire entities and to give some meaning to their difference by introducing some regularization method or other.

It is instructive to derive from (2.45) the energy density for a weak boundary potential. If Λ is so small that the condition $|\ln \Lambda| \gg 1$ holds, then we have[40]

$$\varepsilon \approx -\frac{\Lambda^2}{2\pi a^2} \ln \Lambda. \tag{2.46}$$

We have $\varepsilon \to 0$ in the absence of walls, i.e., $\Lambda \to 0$, as we should have. The fact that (2.46) is not analytic at $\Lambda = 0$ is a signal that the expression for ε is not valid at $\Lambda < 0$ (where we have $\operatorname{Im} \varepsilon \neq 0$). For a massless field, any negative potential will lead at small values $k \approx 0$ to a negative square frequency $\omega^2 < 0$. This point is particularly clear in the case of a wide potential well of small depth $V(x) = -V_0$. If we forget about the edges of the well in a first approximation, with $k \approx 0$ and $m = 0$ we find the value $\omega^2 \approx -V_0 < 0$ from (2.17). One of the functions $\varphi^{\pm}$ now increases as $e^{|\omega| t}$ with the time. This instability leads to a boson condensation: the creation of a macroscopically large number of particles and a change in the structure of the vacuum.[86] This creation of particles is favored from the energy standpoint. This phenomenon, which is related to phase transitions accompanied by the formation of an order parameter $\langle 0|\varphi|0\rangle \neq 0$, was studied in Ref. 40 for the problems in which we are interested here.

The renormalization procedure which we have described, is not sufficient for finding a finite value of ε at points with $V(x) \neq 0$, i.e., in the wall region. Here we also need to subtract terms of the type $J_1 V(x) + J_2 \Box V(x)$, where J_n are certain infinite integrals over the spectrum.[42]

For a massless field, $\varepsilon = P = 0$, we have $x < 0$, and $x > a$ throughout the external region, regardless of the value of Λ, even in the case of impenetrable walls, $\Lambda \to \infty$.

2.9. Massive field. We will first find the renormalized total energy $\mathscr{E}$. As we mentioned back in Subsection 2.4, the additional term Δ in (2.29), which arises with $m \neq 0$ and when there are boundaries, disappears when we integrate over the volume, i.e., over the segment $(0, a)$, so it makes no contribution to the energy.

Calculations with frequencies $\omega_n = (m^2 + b^2 n^2)^{1/2}$, where $b = \pi/a$, for a massive field lead to the following results, which are different from those in the massless case. First, the term $F(0) = m$ in (2.35) is now nonzero. Second, the functions $F(\pm it) = [m^2 + b^2(\pm it)^2]^{1/2}$ must now be assumed to be the same except at $t > m/b$. As a result we have

$$\mathscr{E} = -\frac{m}{4} - \frac{1}{4\pi a} \int_{2\mu}^{\infty} \frac{[y^2 - (2\mu)^2]^{1/2}\, dy}{e^y - 1}, \tag{2.47}$$

where $\mu = ma$. With $m = \mu = 0$, we reproduce (2.36).

The first term in (2.47) does not depend on the geometry. It can be omitted in any problem in which we are involved with forces, i.e., derivatives $\partial \mathscr{E}/\partial a$. After a finite renormalization of this sort, we find the following result from integral (2.47) with $\mu = ma \gg 1$

$$\mathscr{E} \approx -\frac{\mu^{1/2}}{4\sqrt{\pi} a} e^{-2\mu}. \tag{2.48}$$

The fact that the vacuum energy is exponentially small at $\mu \gg 1$, i.e., when the dimensions (a) of the system are much larger than the Compton length m^{-1}, is a general property. In the opposite case of small masses we have, at logarithmic accuracy,

$$\mathscr{E} \approx -\frac{\pi}{24a} + \frac{\mu^2 \ln \mu^2}{32\pi a}. \tag{2.49}$$

The deviation of (2.49) from analyticity in the case $\mu^2 = 0$ is of the same nature as in (2.46).

We would now like to evaluate the component of the energy density ε which comes from Δ in (2.29), which is nonzero only for a massive field in regions with boundaries.

We find a simple asymptotic expression for Δ under the condition $\mu = ma \ll 1$:

$$\Delta(m, x) \approx \frac{m^2}{2\pi} \ln\left(2 \sin \frac{\pi x}{a}\right). \tag{2.50}$$

It follows from (2.50) that the sign of Δ changes at $x = a/6$ and $5a/6$; near the walls, the function Δ has an integrable singularity. For example, near the $x = 0$ wall we have

$$\Delta \approx \frac{m^2}{2\pi} \ln \frac{2\pi x}{a} < 0. \tag{2.51}$$

The nature of singularity (2.51) stays the same regardless of the value of μ. In a case with $\mu = ma \gg 1$, by replacing the summation over the slowly varying variable n/μ by an integration in (2.29) we find, near the $x = 0$ wall ($mx \ll 1$),

$$\Delta \approx -\frac{m^2}{2\pi} K_0(2mx), \tag{2.52}$$

where K_0 is the modified Bessel function.

Singularity (2.51) arises because of the replacement of a potential by an impenetrable wall. For semipermeable walls of the type in (2.41), for example, a new dimensionless parameter, $\zeta = m/\lambda$, appears. Using the method of Subsection 2.8, we easily find that $\ln(x, a)$ is replaced by $\ln \zeta$, i.e., by a constant in (2.51) under the condition $x/a < \zeta$. For all nonzero values of ζ, i.e., for $\lambda \neq \infty$, the growth of Δ and, along with it, the energy density near the wall comes to a halt.

2.10. Physical meaning of the Casimir vacuum energy. As was mentioned in Subsection 2.7, for a given configuration of boundaries the total Casimir energy is determined within an additive constant which does not depend on the distance and thus does not influence the value of the force. In Subsections 2.8 and 2.9 it was shown in some specific examples that this component of the energy stems from walls which are totally or partially impenetrable. An integration of the vacuum energy density (2.32) over the entire space gives us, generally speaking, the total Casimir energy of the vacuum, including the energy of the walls.

In calculations of the vacuum energy it is frequently convenient to carry out subtraction procedures which eliminate not only the leading divergence $\sim p^D$ (p is the cutoff momentum, and D is the dimensionality of the space-time), but also—partially or completely—the energy of the walls. For the analysis below, we need to consider how the values found for the Casimir energy in this manner are related to each other and to discuss their physical meaning.

We introduce the shorthand

$$t(m, \lambda, a; x) = \langle 0 \mid T_{00}(x) \mid 0 \rangle$$

for the expectation value of T_{00} in the vacuum state, with a specification of all the parameters of the problem. As before, we denote by $\varepsilon(m,\lambda,a;x)$ the renormalized value of the energy density, found through a subtraction of the contribution of Minkowski space, (2.33). We recall that this normalization is actually carried out through a mode-by-mode subtraction, (2.44), or, for walls which are initially not transparent ($\lambda = \infty$), by discarding the divergent integral in (2.34).

The total energy of the vacuum over the entire space when there are semitransparent walls at the points $x = \pm a/2$ is

$$\mathscr{E}^{\text{tot}}(m, \lambda, a; x) = \int_{-\infty}^{\infty} \varepsilon(m, \lambda, a; x)\,\mathrm{d}x$$
$$= \int_{-\infty}^{\infty} (t(m, \lambda, a; x) - t(m, \lambda = 0, a; x))\,\mathrm{d}x. \qquad (2.53)$$

Here we have allowed for the circumstance that with $\lambda = 0$ we return to an empty Minkowski space, so the difference between the values of t in the parentheses in (2.53) is the same as in (2.33). The energy $\mathscr{E}^{\text{tot}}(\lambda)$ may be interpreted as the work performed on the deformation of the vacuum during an adiabatic decrease in the transparency of the walls at a constant distance a. In the limit $\lambda \to \infty$ the entire space is broken up into three independent regions, and we can replace (2.53) by

$$\mathscr{E}^{\text{tot}} = \lim_{\lambda \to \infty} \mathscr{E}^{\text{tot}}(\lambda) = 2\mathscr{E}_{1/2} + \mathscr{E}, \qquad (2.54)$$

$$\mathscr{E}_{1/2} = \int_{-\infty}^{-a/2} \varepsilon(m, \infty, a; x)\,\mathrm{d}x, \qquad \mathscr{E} = \int_{-a/2}^{a/2} \varepsilon(m, \infty, a; x)\,\mathrm{d}x,$$

where $\mathscr{E}_{1/2}$ is the renormalized energy of the half-space bounded by the wall, and we have made use of the circumstance that ε is an even function of x.

The quantity $\mathscr{E}$ is the Casimir energy minus the "outer" part of the wall energy, $2\mathscr{E}_{1/2}$. This energy was actually encountered above [see, for example, (2.47) for the massive case]. The quantity $\mathscr{E}^{\text{tot}}(\lambda)$ is the most natural characteristic of the vacuum energy of a massive field when there are semitransparent walls, since in this case the vacuum energy density is nonzero throughout the space for all values $\lambda \neq 0$, including $\lambda \to \infty$ (Ref. 40). The introduction of $\mathscr{E}^{\text{tot}}$ is also necessary for a massless field in problems with curved boundaries (Subsection 3.3).

Obviously associated with an isolated impenetrable wall is the energy of the two half-spaces:

$$\mathscr{E}_{\text{w}} = 2\mathscr{E}_{1/2} \equiv Jm, \qquad (2.55)$$

where J is a dimensionless quantity. The proportionality between $\mathscr{E}_{\text{w}}$ and the mass of the field follows from the absence of any other dimensional parameters for an isolated wall [correspondingly, the energy density outside the opaque walls in (2.54) is also actually independent of a].

One can introduce yet another physically grounded procedure for deriving the vacuum energy. This procedure contains neither the "outer" nor "inner" parts of the wall energy:

$$\mathscr{E}^{\text{f}}(m, a) \equiv \lim_{a' \to \infty} (\mathscr{E}(m, a) - \mathscr{E}(m, a')). \qquad (2.56)$$

The quantity $\mathscr{E}^{\text{f}}$ has the meaning of the work which is performed when walls originally an infinite distance apart are brought together adiabatically to a distance a [here we are assuming that the walls are opaque, but (2.56) is actually also meaningful for any $\lambda = \text{const}$].

In the one-dimensional case the difference between the quantities $\mathscr{E}$ and $\mathscr{E}^{\text{f}}$ reduces to terms of the type $-m/4$ in (2.47). In discussing massive fields in multidimensional regions below, however, the quantity $\mathscr{E}$ generally turns out to be infinite since the energy density renormalized in accordance with (2.33) has a nonintegrable singularity, $\varepsilon(x) \to \infty$, as the walls are approached. In the multidimensional case, subtraction procedure (2.56) thus becomes nontrivial and makes it possible to avoid an infinite energy of the walls, which does not influence the values of the Casimir forces.

To find the relationship between $\mathscr{E}$ and $\mathscr{E}^{\text{f}}$, we move the walls apart in such a way that the condition $ma' \gg 1$ holds in (2.56); i.e., the walls are moved to an essentially infinite separation (Fig. 2). The energy $E(m,a')$ is thus, at an asymptotic accuracy, the independent sum of the energies of the two regions bordering the walls on the inside; i.e., we have $\mathscr{E}(m, a') \to 2\mathscr{E}_{1/2} = \mathscr{E}_{\text{w}}$ (see Fig. 2b; we recall that by definition $\mathscr{E}$ ignores the energy of the external region).

As a result we find

$$\mathscr{E}^{\text{f}} = \mathscr{E} - \mathscr{E}_{\text{w}}, \qquad \mathscr{E}^{\text{tot}} = \mathscr{E} + \mathscr{E}_{\text{w}} = \mathscr{E}^{\text{f}} + 2\mathscr{E}_{\text{w}}.$$

The vacuum energy $\mathscr{E}^{\text{tot}}$ is thus the additive sum of the "topological" energy of formation of the walls at an infinite-

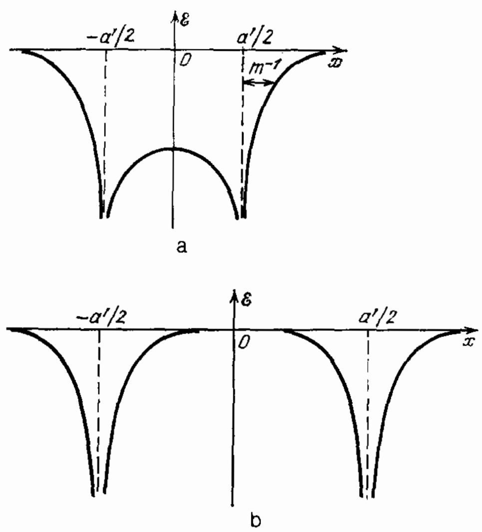

FIG. 2. Vacuum energy density (in versions a and b).

ly great distance, $2\mathscr{E}_w$, and the energy acquired as the walls are brought together, $\mathscr{E}^f$.

Definition (2.56) of $\mathscr{E}^f$ can be put in a form more convenient for practical applications. Specifically, according to dimensionality considerations we can write

$$\mathscr{E}(m, a) = m\varphi(ma) = \frac{\chi(ma)}{a},$$

where φ and χ are dimensionless functions whose explicit form is not important here. It follows that we have $\mathscr{E}(m, ba) = \mathscr{E}(bm, a)/b$ for any positive b and for $m \neq 0$. We can thus set $a' = ba$ in (2.56) and rewrite $\mathscr{E}^f$ in the form

$$\mathscr{E}^f(m, a) = \lim_{b\to\infty}\left(\mathscr{E}(m, a) - \frac{1}{b}\mathscr{E}(bm, a)\right), \qquad (2.57)$$

so that now we are comparing identical intervals but different masses of the quantized field.

Writing $\mathscr{E}^f$ in the form in (2.57) makes it possible to use a spectral representation for $\mathscr{E}$ to carry out a mode-by-mode subtraction procedure.

For calculating the forces in one-dimensional problems and also in multidimensional problems with plane boundaries, all the energy definitions given in this subsection of the paper are equivalent, since they differ by only a constant of the type in (2.55). In practice it is more convenient to use the quantity $\mathscr{E}^f$, since it does not contain a contribution of the wall energy. For problems with curved boundaries the quantity $\mathscr{E}^{tot}$ is an adequate characteristic of the vacuum energy. This is true, in particular, because moving curved boundaries off to infinity, like taking the limit in (2.56), is unavoidably accompanied by a deformation of these boundaries, so $\mathscr{E}^f$ is stripped of any direct meaning.

In cosmological problems, on the other hand, where it is necessary to substitute an unambiguously defined vacuum energy-momentum tensor into the Einstein equations, there are no boundaries, so there is no problem with the energy of boundaries.

3. THE CASIMIR EFFECT FOR VARIOUS FIELDS AND SPATIAL REGIONS

3.1. Scalar field between plates. In this section of the review we examine the Casimir effect for fields with spins of 0, 1/2, and 1 and for various spatial regions: the space between plane-parallel plates, a sphere, a parallelepiped, etc. We begin with the case of scalar field between plates.

Let us examine the energy-momentum tensor of a real scalar field for a configuration of two parallel plates at $x = \pm a/2$ with an arbitrary transparency λ given by potential (2.41). The corresponding expression is found from (2.26) by adding derivatives with respect to all three coordinates. It is called a "canonical energy-momentum tensor" T_{ik}^{can}. It turns out, however, that a more appropriate quantity for multidimensional problems is the so-called metric energy-momentum tensor[90]

$$T_{ik} = T_{ik}^{can} - \xi(\partial_i\partial_k - g_{ik}\partial_l\partial^l)\varphi^2, \qquad (3.1)$$

where $\xi = \xi_c = (D-2)/4(D-1)$, and D is the dimensionality of the space-time (in one-dimensional problems we have $\xi_c = 0$; at $D = 4$ we have $\xi_c = 1/6$).

Formally, all values of ξ are permissible in (3.1), since energy-momentum tensors (3.1) for different values of ξ differ by a 4-divergence. The basis for the choice $\xi = \xi_c$ is discussed in §5 and also in Ref. 78.

For all values $\lambda \neq \infty$ the spectrum of the problem is continuous, and by proceeding as in the one-dimensional case (Subsection 2.8) we find the following expression for the energy density in the region $|x| < a/2$ (Ref. 40):

$$\varepsilon = \frac{1}{8\pi^2}\int_0^\infty dq\,q\int_0^\infty \frac{dk}{\omega}\left[\omega^2\sum_{j=1}^2 (A_j^2 - 1) + (\omega^2 - 2\xi_c k^2)\times(A_2^2 - A_1^2)\cos 2kx\right]; \qquad (3.2)$$

here $\omega^2 = \mathbf{q}^2 + m^2 + k^2$, $\mathbf{q}$ is a two-dimensional momentum vector in the plane of the plates, $\xi_c = 1/6$, and the values of A_j are found by a procedure like that in Subsection 2.8.

We now rotate the integration contour through $\pi/2$ and first calculate the integrals over q. Here is the result in the limit $\lambda \to \infty$ (opaque plates):

$$\varepsilon = \varepsilon^f + \Delta(m, a, x),$$

where

$$\varepsilon^f = -\frac{1}{6\pi^2 a^4}\int_\mu^\infty \frac{(\zeta^2 - \mu^2)^{3/2}\,d\zeta}{e^{2\zeta} - 1}, \qquad (3.3)$$

$$\Delta(m, a, x) = -\frac{2m^4}{3\pi^2}\int_1^\infty (y^2 - 1)^{1/2}\,\frac{\mathrm{ch}\,2\mu(x/a)\,y}{\mathrm{sh}\,\mu y}\,dy \qquad (3.4)$$

($\mu = ma$). The integral of ε over the range $-a/2 \leqslant x \leqslant a/2$ evidently gives us the energy $\mathscr{E}$ per unit area of the plates (since the volume of the system is infinite, we cannot speak in terms of a total energy). The arguments of Subsection 2.10 remain in force for the quantities $\mathscr{E}$, $\mathscr{E}^f$, and $\mathscr{E}^{tot}$ per unit area. Because of the altered dimensionality, we should simply assume $\mathscr{E}(m, ab) = \mathscr{E}(bm, a)/b^3$ in (2.57) and $\mathscr{E}_w = Jm^3$ in (2.55).

For a massless field we would have $\Delta \equiv 0$, since we have $\mathscr{E} = a\varepsilon^f$. From (3.3) with $\mu = 0$ we find the following expressions for $\mathscr{E}$ and for the force F acting on a unit area of the plates (the pressure):

$$\mathscr{E} = -\frac{\pi^2}{1440a^3}, \qquad F = -\frac{\partial\mathscr{E}}{\partial a} = -\frac{\pi^2}{480a^4} \qquad (3.5)$$

(these results were derived by another method in Ref. 15).

In the opposite case, $\mu = ma \gg 1$, we find the exponentially small value $\mathscr{E} \sim -\exp(-2\mu)$. Essentially all the results written here can be extended to plates of finite area S under the condition $a \ll S^{1/2}$.

We can also write all components of the vacuum energy-momentum tensor of the scalar field in the case $m = 0$ (Ref. 15):

$$\langle T_{ik}\rangle = \frac{\pi^2}{1440a^4}\begin{pmatrix} -1 & & & \\ & -3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \qquad (3.6)$$

As was discussed in Subsection 2.10, in the general case $m \neq 0$ the quantity $\mathscr{E}^f$ derived by recipe (2.56) is an adequate characteristic of the vacuum energy. Since the quantity $\mathscr{E}^f$ from (3.3) falls off exponentially in the limit $a' \to \infty$, we find the following result with the help of (2.56):

$$\mathscr{E}^f(m, a) = a\varepsilon^f(m, a) + R(m, a), \qquad (3.7)$$

$$R(m, a) = \lim_{a'\to\infty}\int_{-1/2}^{1/2} (a\Delta(m, a, \zeta) - a'\Delta(m, a', \zeta))\,d\zeta. \qquad (3.8)$$

In the integrals with Δ, the change of variables $x/a \to \zeta$, $x/a' \to \zeta$ has been made.

We now change the order of integration in R, making use of (3.4). We would then expect that for any value of y the integral over the variable ζ in (3.8) would vanish, so we must set $R(m, a) = 0$ and take $\mathscr{E}^{\rm f} = a\varepsilon^{\rm f}$ with $\varepsilon^{\rm f}$ from (3.3) for all values of m.

The integral of $\Delta(m, a, x)$ over the coordinate does not depend on the value of a. It determines the wall energy $\mathscr{E}_{\rm w}$ (Subsection 2.10). A direct calculation leads to an infinite value of $\mathscr{E}_{\rm w}$ because of the increase in $\Delta \sim -m^2[x \pm (a/2)]^2$ near the wall. In the case $\lambda \neq \infty$ (semitransparent walls), we asymptotically have[3)] $\Delta \sim -m^2[a(x \pm a/2]^{-1}$ (Ref.40).

3.2. Vacuum energy of bounded three-dimensional volumes. Research on the Casimir effect in the case of regions with a complicated multiparameter geometry required the development of some nontrivial analytic methods which make it possible to work effectively with multiple divergent sums and integrals.

Let us consider the vacuum energy of a massless scalar field in a parallelepiped with edges a, b, and c and with homogeneous conditions at the faces (the $I \times I \times I$ topology). A calculation of this energy, which is conveniently carried out through the repeated use of the Abel-Plana formula, yields the following result[11] (see Appendix I for the details of the calculations, including those associated with the presence of the faces and edges of the parallelepiped):

$$\mathscr{E} = \Big[-\frac{\pi^2}{1440 a^4} + \frac{\zeta(3)}{32\pi a^3}\left(\frac{1}{b} + \frac{1}{c}\right) - \frac{\pi}{96 a^2 bc} - \frac{\pi}{a^4}\alpha\left(\frac{b}{a}, \frac{c}{b}\right)\Big] abc. \qquad (3.9)$$

Here $\zeta(3) = 1.202$ is the value of the Riemann zeta function, and α represents certain integrosums which are exponentially small under the conditions $c \gg b \gg a$, proportional to $\exp(-2\pi c/b)$ and $\exp(-2\pi b/a)$. Without loss of generality, we can always choose to denote the sides in an order such that we can completely discard α (the contribution of α is less than 1%, even for a cube). In a cubic configuration, with $a = b = c$ (and with allowance for α), we have

$$\mathscr{E} = -\frac{0.015}{a}. \qquad (3.10)$$

Under the conditions $a \ll b$, c, the quantity $\mathscr{E}/bc$ from (3.9) is asymptotically equal to the energy per unit area in the case of infinite plates (Subsection 3.1).

Other versions of the topology, specified by replacing the homogeneous conditions along one or several variables by periodic conditions,[11] can be studied in a corresponding way (§5).

As in the one-dimensional case, the energy turns out to be exponentially small for a massive field under the condition $ma \gg 1$.

3.3. Electromagnetic field between plates. An electromagnetic field has the distinctive features that it is transverse and that so-called natural boundary conditions are imposed at the wall: the vanishing of the tangential components of the electric field,

$$\mathbf{E}_t = 0. \qquad (3.11)$$

The vacuum energy and the force per unit area were in fact derived by Casimir[1] for the very simple case of unbounded plane-parallel plates separated by a distance a under condition (3.11):

$$\mathscr{E}^{\rm f} = -\frac{\pi^2}{720 a^3}, \qquad F = -\frac{\pi^2}{240 a^4}. \qquad (3.12)$$

This result can be found easily by taking limits in the more general results for the electromagnetic field vacuum inside an ideally conducting parallelepiped (Subsection 3.4) and between plates of a real metal (Subsection 4.2). In accordance with (3.12), the plates tend to move toward each other, as in the case of a scalar field.

Here are all the components of the vacuum energy-momentum tensor of the electromagnetic field between parallel plates, with conditions (3.11) at these plates:

$$\langle T_{ik}\rangle = \frac{\pi^2}{720 a^4}\begin{pmatrix} -1 & & & \\ & -3 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The trace of the energy-momentum tensor, $\langle T_i^i\rangle$, vanishes, as it should for massless field.

The vacuum energy-momentum tensor of an electromagnetic field can also be calculated for other sufficiently symmetric configurations. For example, the energy density inside a dihedral angle, with conditions (3.11) at the faces, is[16]

$$\varepsilon = \langle T_{00}\rangle = -\frac{1}{720\pi^2\rho^4}\left(\frac{\pi^2}{\alpha^2} - 1\right)\left(\frac{\pi^2}{\alpha^2} + 11\right),$$

where the notation is explained by Fig. 3. When we take the limits $\alpha \to 0$ and $\rho \to \infty$ under the condition $\alpha\rho = a$, we return to the case of parallel plates, (3.12) (see also Ref. 21).

3.4. Electromagnetic field in bounded three-dimensional volumes. The eigenfrequencies of a parallelepiped with conditions (3.11) at the walls are the same as those for a scalar field:

$$\omega^2 = \pi^2\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right).$$

Condtion (3.11) does not require that all the components of the electric and magnetic fields vanish at the walls. Accordingly, and in contrast with the case of a scalar field, we can have oscillations with $n_1 = 0$, $n_2 \neq 0$, and $n_3 \neq 0$ and with interchanges of the n_i. The oscillations with the entire set $n_i \neq 0$, on the other hand, are doubly degenerate.[84]

Here is the result of a calculation of $\mathscr{E}$ for a square resonator under the assumption $a = b$ (Refs. 11 and 13):

$$\frac{\mathscr{E}}{a^2 c} = -\frac{1}{a^4}\left[\frac{\pi^2}{720} + \frac{\zeta(3)}{16\pi} - \frac{\pi}{24}\frac{a}{c}\right], \qquad a \leqslant c,$$
$$= -\frac{1}{c^4}\left[\frac{\pi^2}{720} - \frac{\pi}{48}\left(\frac{c}{a}\right)^2 - \left(\frac{\pi}{48} - \frac{\zeta(3)}{16\pi}\right)\left(\frac{c}{a}\right)^3\right], \quad a \geqslant c. \qquad (3.13)$$

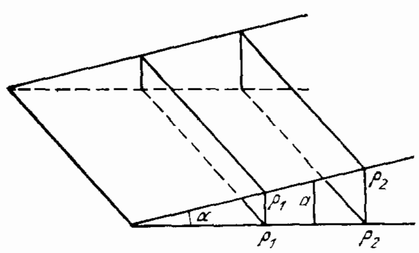

FIG. 3. The parameters characterizing the wedge-shaped region.

For a cubic volume (with corrections of the α type) the total vacuum energy of the electromagnetic field is

$$\mathscr{E}=\frac{0.0916}{a}. \qquad (3.14)$$

This result differs by a sign from the corresponding quantity for the scalar case, (3.10).

We turn now to the vacuum energy of the electromagnetic field in the presence of a sphere of radius R, at which boundary condition (3.11) holds. Casimir initially suggested[2] that in the case of a sphere the force caused by the vacuum energy would correspond to an attraction, as in the case of parallel plates, and might thus counterbalance the Coulomb repulsion. This suggestion raised the hope it would be possible to realize a model of an elementary particle in which the fine-structure constant α could be found unambiguously from the equilibrium condition (in literal accordance with Feynman's prediction that it would sometime become possible to express α in terms of the roots of Bessel functions).

The vacuum energy of a sphere was actually calculated in Refs. 25–27. Because of the irregularity of the distribution of eigenvalues of the Laplacian for a sphere, this problem was solved by numerical methods. The result is

$$\mathscr{E} = 0.09235/2R,$$

very close to the value given in (3.14) for a cube of side $a = 2R$ (one might say that the vacuum energy "does not get into" the corners of the cube).

The fact that the energy of a sphere is positive means that the sphere is tending to expand (repulsive forces); this circumstance is quite different from the case of two parallel plates. Accordingly, Casimir's suggestion was not confirmed (incidentally, see Subsection 6.1).

There is considerable interest in the behavior of the renormalized vacuum energy density ε as a function of the distance from the center of the sphere. It turns out that as we approach the surface of the sphere, $r = R$, the energy density ε increases without bound[28]:

$$\varepsilon \approx \frac{\text{const}}{R\rho^3}, \qquad (3.15)$$

where $\rho = R - r \ll R$.

How do we reconcile the presence of a nonintegrable singularity in ε near the boundary with the finite value given above for $\mathscr{E}$ for a sphere?

The distribution of eigenvalues of the Laplacian for a three-dimensional region of volume V which is bounded by a surface of area S with an average curvature K is given asymptotically in the limit $k \to \infty$ by[91]

$$n(k)=\frac{V}{2\pi^2}k^2-\frac{1}{8\pi}Sk+\frac{S}{6\pi^2}K+O(k^{-2}). \qquad (3.16)$$

A subtraction of the contribution from Minkowski space, which is carried out explicitly or implicitly in each regularization procedure, definitely removes the first term from (3.16), which is proportional to V. The term proportional to S stems from the "surface" energy of the boundaries and various possible definitions of $\mathscr{E}$ for a massive field (Subsection 2.10 and Appendix II).

The average surface curvature K has opposite signs for the inner and outer sides of a boundary. In calculating the energy density for the external space, we find a singularity near the surface of this sphere which is the same as in (3.15), except that it has the opposite sign [this circumstance is naturally incorporated in (3.15), when we note that we have $\rho < 0$ on the outside].

In summary, the value given above for the energy of a sphere agrees with $\mathscr{E}^{\text{tot}}$ in the terminology of Subsection 2.10: $\mathscr{E} = \mathscr{E}^{\text{tot}}$. In other words, $\mathscr{E}$ is the vacuum deformation energy over the entire unbounded space during an adiabatic decrease in the transparency of a spherical boundary from $\lambda = 0$ to $\lambda = \infty$. In evaluating the integral of ε over the entire space in the principal-value sense these singularities cancel out, so we find the finite value for the energy which is given above (see also Ref. 20).

3.5. Can the Casimir energy cross zero? It follows immediately from (3.13) that the vacuum energy of the electromagnetic field in a parallelepiped with sides $a \times b \times c$ is positive if $a = b = c$ (i.e., for a cubic volume), but it changes sign when the resonator is "stretched out." For example, in the particular case of a square cross section, $b = c$, the energy is positive in the interval

$$0.408 < \frac{c}{a} < 3.48,$$

crosses zero at the ends of this interval, and is negative outside it. A corresponding behavior of the energy is observed for certain other topological types, including the case without walls, i.e., with periodic conditions along all the coordinates (Subsection 5.2). A change in the sign of the vacuum energy should also be expected upon the deformation of a sphere into a fairly prolate ellipsoid.

As was mentioned earlier, in Subsection 2.10, in problems without walls the determination is unambiguous, and there is no arbitrariness associated with terms of the type in (2.55) or with the surface term in (3.16). Consequently, the assertion regarding the sign of $\mathscr{E}$ for such problems is absolute (different methods yield the same values for $\mathscr{E}$; see, for example, Refs. 11 and 17).

When there are boundaries, on the other hand, the energy is determined within a constant. The choice of this constant which was made above, however, is physically grounded since both the forces and the energy tend toward zero as the dimensions of the system increase without bound.

In discussing the possibility that the Casimir energy crosses zero it is also useful to note the case of a conducting cylindrical surface. In a sense, such a surface is an intermediate case between the configuration of two parallel plates and the configuration of a sphere, and the two latter configurations differ in the sign of the vacuum energy. Indeed, approximate methods did lead to a vanishing result for a cylinder in Refs. 29 and 30. A more recent paper,[31] however, found a negative energy per unit length of the cylinder, roughly twice the value for a resonator of square cross section. See Ref. 103 regarding another approximate method.

3.6. Spinor field. The formulation of an impenetrability condition at walls requires a separate study in the case of a spinor field. In order to derive a physically grounded boundary condition, we need to work from a model of an interaction with a specific field.[18] In the case of a discontinuous scalar potential $V = V_0\theta(x)$, which simulates a boundary (wall) at $x = 0$, Dirac's equation takes the following form, after we extract the time dependence:

$$(\omega\gamma^0 + i\gamma\nabla - m - V_0\theta(x))\psi = 0. \qquad (3.16')$$

Impenetrability of a wall corresponds to the limit $V_0 \to \infty$. The term with V_0 can obviously be canceled only by a derivative of ψ; i.e., at $x > 0$ we have $\psi \sim \exp(-V_0 x)$. Since ψ is continuous at $x = 0$, the following condition holds at boundary Γ:

$$(i\gamma \mathbf{n}\psi + \psi)|_\Gamma = 0, \tag{3.17}$$

where **n** is the normal to the boundary.

If, on the other hand, we determine the condition at the boundary with the help of a more realistic field—the 0-component of the electromagnetic vector potential—we find that it is not possible to arrange impenetrability of a barrier of a given height U for all values of the energy. The reason is that under the condition $|U - \omega| > m$ (ω is the frequency of the mode) we run into the well-known Klein paradox[92]; i.e., according to the present interpretation, particle and antiparticle currents appear and cancel each other out. For a configuration of two infinite parallel plates at $x = 0$ and $x = a$, we can seek particular solutions of (3.16) in the form

$$\psi = \begin{pmatrix} \varphi \\ -\dfrac{i\sigma\nabla\varphi}{m+\omega} \end{pmatrix}, \qquad \varphi = (ue^{ikx} + ve^{-ikx})\, e^{i\mathbf{q}\boldsymbol{\rho}}, \tag{3.18}$$

where u and v are constant spinors, $\boldsymbol{\sigma}$ is the vector of Pauli matrices, and **q** and **ρ** are 2-vectors in the plane of the plates. With the functions ψ from (3.18), conditions (3.17), imposed at the two plates, are compatible only if

$$f(ka) \equiv \mu \sin ka + ka \cos ka = 0. \tag{3.19}$$

If $\mu \equiv ma = 0$, we find the values $k_n = (\pi/a)[n + (1/2)]$ from (3.19). In calculating the energy we must therefore use a formula for summing over half-integer numbers, (2.39). We then find the following value for the vacuum energy per unit area of the plates[66]:

$$\mathscr{E} = -\frac{2\pi^2}{3a^3}\int_0^\infty \frac{y^3\, dy}{e^{2\pi y}+1} = -\frac{7\pi^2}{2880a^3}. \tag{3.20}$$

For completeness, we also write all the components of the energy-momentum tensor of a neutrino field:

$$\langle T_{ik}\rangle = \frac{7\pi^2}{5760a^4}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix}.$$

In the case $m \neq 0$, we use an argument principle for the summation over the roots of Eq. (3.19). This principle is based on the equality[93]

$$\frac{1}{2\pi i}\oint_C \varphi(z)\frac{d}{dz}\ln f(z)\, dz = \sum \varphi(z_0) - \sum \varphi(z_\infty), \tag{3.21}$$

where the summation is over the zeros (the points z_0) and poles (the points z_∞) of the function $f(z)$. The contour C is chosen in such a way that all the roots of the function $f(z)$ from (3.19) are enclosed by the contour.

As a result we find an expression for the vacuum energy $\mathscr{E}^f$ defined in (2.56) (Ref. 18):

$$\mathscr{E}^f(a,\mu) = -\frac{1}{\pi^2 a^3}\int_\mu^\infty (x^2-\mu^2)^{1/2}\ln\left(1+\frac{x-\mu}{x+\mu}e^{-2x}\right)dx.$$

In the case $\mu = 0$, this expression is the same as (3.20), while at $\mu \gg 1$ it gives us $\mathscr{E}^f \sim e^{-2\mu}$.

To conclude this subsection of the paper we note that for a spinor field the simplest boundary condition, $\psi = 0$, cannot be imposed at the boundary, since it would contradict Dirac's equation. Condition (3.17), on the other hand, not only is compatible with the Dirac equation but also satisfies the natural requirement that the current of particles across the boundary must vanish.

4. INCORPORATING THE REAL PROPERTIES OF THE MEDIUM BOUNDING THE QUANTIZATION VOLUME

4.1. Fluctuation van der Waals interaction: relationship with the Casimir effect. Since in certain problems the Casimir force may be thought of as a particular or limiting case of molecular (van der Waals) forces, we will take a brief look at the reasons for the appearance of, and methods for calculating, the latter forces.[36,37]

Let us assume that two macroscopic objects are separated by a characteristic distance a. Most of the energy of the objects is determined by the interactions at atomic distances d, i.e., by the short-range interaction, so the energy is proportional to the volume of the objects. There is, however, a small component of the energy which depends on the shape and relative positions of the objects. For two isolated atoms this energy, $\mathscr{E} \sim (d/a)^6$, corresponds to the interaction of two mutually polarizing dipoles which arise as a result of quantum fluctuations of the charge distribution in the atoms. In solids, a macroscopically large number of atoms participate in the interaction because of the long-range effects. It is thus convenient to calculate forces in terms of the fluctuation electromagnetic field, which is determined at equilibrium by the permittivity ε and the magnetic permeability μ of the objects (although the average values of the fluctuation fields are zero, the energies and forces are determined by quadratic combinations of these fields, i.e., by their correlation functions).

The force was originally defined in the literature as a component of the stress tensor due to fluctuation fields.[32]

Later, a linear relationship between the correlation functions of an equilibrium field and the temperature Green's functions $D_{ik}(\mathbf{r},\mathbf{r}')$ of a photon in the medium was used. The stress tensor was thus found from D_{ik} through the use of some linear differential operator.[33,36] The divergence $D_{ik}(\mathbf{r},\mathbf{r}') \sim |\mathbf{r}-\mathbf{r}'|^{-1}$ was eliminated through the replacement

$$D_{ik}(\mathbf{r},\mathbf{r}') \to D_{ik}(\mathbf{r},\mathbf{r}') - \bar{D}_{ik}(\mathbf{r}-\mathbf{r}'), \tag{4.1}$$

where $\bar{D}_{ik}$ is the Green's function of a homogeneous and unbounded auxiliary medium which has the same permittivity as the real medium at the point under consideration. This procedure was justified as the removal of the contribution from short-wave fluctuations of length scale d, which are not pertinent to the given problem.

The Casimir force for a gap between metals is found from these calculations by taking the limit $|\varepsilon| \to \infty$, which does not depend on whether ε is real or imaginary.

Finally, a simpler and more constructive method for calculating the force was proposed.[35,37] That method is to calculate the force as the derivative of the free energy $\mathscr{F}$ of a system of oscillators whose resonant frequencies $\omega_\alpha(a)$ are determined by the geometry and also by ε and μ, i.e., by the solution of the classical electrodynamic problem. The equilibrium occupation numbers of the oscillators,

$$n(\omega_\alpha)=\frac{1}{2}+\frac{1}{\exp(\omega_\alpha/T)-1}, \qquad (4.2)$$

become the characteristic halves of zero-point vibrations in the temperature limit $T\to 0$ (as $T\to 0$, the free energy becomes equal to the energy). Since the sum over the frequencies which determines $\mathcal{F}$ diverges, a regularization procedure having the same meaning as those used above is employed in order to find a finite result.

Barash and Ginzburg[35] found proof that this approach is valid even when there is absorption, in which case simple expressions such as (4.2) are meaningless. Speaking somewhat loosely, we could say that since $\mathcal{F}$ is ultimately determined by the behavior of ε at the imaginary axis in the complex ω plane, where the relation $\operatorname{Im}\varepsilon(i\zeta)=0$ always holds, the presence of damping, i.e., $\operatorname{Im}\varepsilon(\omega)\neq 0$, on the real axis, is unimportant to the applicability of the method.

A calculation of the forces in terms of the eigenfrequency spectrum is similar in terms of the ideas involved to the theory of the Casimir effect, but it brings into consideration the entire medium in the unbounded volume, with properties which depend on the coordinates.

It is thus useful to have, to supplement the methods outlined above, a simpler method which generalizes the approach presented in §2 and §3 to the case of nonideal boundaries through a modification of the boundary condition which would incorporate all the properties of the medium to the minimum extent necessary. There would then be no need to consider explicitly the region occupied by the medium; the result would be to simplify the calculations, especially for curvilinear and complicated regions.

4.2. An impedance boundary condition reflects the properties of the medium. We know[84] that the penetration of an electromagnetic field into a real metal can be dealt with effectively by imposing an "impedance" condition at the boundary:

$$\mathbf{E}_t=Z(\omega)\,[\mathbf{H}_t\,\mathbf{n}], \qquad (4.3)$$

where $\mathbf{E}_t$ and $\mathbf{H}_t$ are the tangential components of the electric and magnetic fields, and $\mathbf{n}$ is the inward normal (into the medium) to the boundary (Fig. 4). The impedance $Z(\omega)$ can be expressed in a simple way in terms of the permittivity $\varepsilon(\omega)$ and the magnetic permeability $\mu(\omega)$ of the medium: $Z(\omega)=(\mu(\omega)/\varepsilon(\omega))^{1/2}$. The relation $|Z|\ll 1$ usually holds. Boundary condition (4.3) also holds in cases in which $\varepsilon(\omega)$ and $\mu(\omega)$ do not have a direct meaning, e.g., in the region of an anomalous skin effect or for low-frequency superconductors.

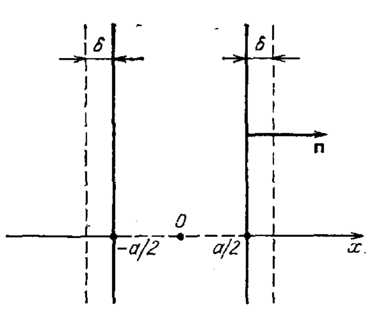

FIG. 4. Penetration of a field into a real medium.

The impedance $Z(\omega)$ is frequently replaced by the function $\delta(\omega)=iZ(\omega)/\omega$, which has the meaning, in the case $\delta=\operatorname{Re}\delta$, of the depth to which the electric field penetrates into the plate material. If there is an $\operatorname{Im}\delta\neq 0$, the attenuation of the field in the medium is taken into account. An ideal metal evidently corresponds to the case $Z=\delta=0$, and condition (4.3) becomes (3.11).

Let us examine the vacuum energy and Casimir forces between two parallel plates made of a real metal and modeled by half-spaces separated by an empty gap $-a/2<x<a/2$ (Fig. 4).

Finding the complete system of solutions of Maxwell's equations which satisfy boundary conditions (4.3) at $x=\pm a/2$, we find dispersion relations for the spectrum ω_n (Ref. 38):

$$\sin\frac{\varkappa a}{2}+\frac{1}{\varkappa}\,\delta(\mathbf{Q}^2+\varkappa^2)\cos\frac{\varkappa a}{2}=0,$$
$$\cos\frac{\varkappa a}{2}-\frac{1}{\varkappa}\,\delta(\mathbf{Q}^2+\varkappa^2)\sin\frac{\varkappa a}{2}=0, \qquad (4.4)$$

and also

$$\cos\frac{\varkappa a}{2}-\delta\varkappa\sin\frac{\varkappa a}{2}=0,\qquad \sin\frac{\varkappa a}{2}+\delta\varkappa\cos\frac{\varkappa a}{2}=0. \qquad (4.5)$$

Here $\mathbf{Q}$ is a two-dimensional vector in the plane of the plates (y,z), and $\varkappa^2\equiv\omega^2-\mathbf{Q}^2$.

The eigenfrequency spectrum in the Casimir effect can also be defined in terms of the reflection coefficients for electromagnetic waves, which can be expressed approximately in terms of a surface impedance.[39]

If the values of δ and, along with them, the eigenfrequencies ω_n are real, it is convenient to determine the vacuum energy $E=(1/2)\Sigma_n\omega_n$ by means of the argument principle (3.21). Taking this approach, we find

$$E(a,\delta)=\frac{1}{2\pi}\int_0^\infty d\zeta\int_0^\infty dQ\,\frac{Q}{2\pi}\ln D(Q,i\zeta), \qquad (4.6)$$

where $Q=|\mathbf{Q}|$, and where we have made the substitutions $\omega\to i\zeta$, $\varkappa\to i(\zeta^2+Q^2)^{1/2}\equiv iR$. The function D vanishes on each solution of Eqs. (4.4) and (4.5). We could take D to be the product of the left-hand sides of these equations.

For arbitrary complex δ, the eigenfrequencies $\omega=(\varkappa^2+Q^2)^{1/2}$ found as the solutions of (4.4), (4.5) have an imaginary increment. As was mentioned in the preceding section, the results calculated for the vacuum energy depend only on the behavior of ε (and δ) on the imaginary frequency axis, where they are definitely real. Accordingly, expression (4.6) is also valid in the case of complex ω_n after a rotation of the contour, $\omega\to i\zeta$, the integral in (4.6) is obviously real].

In order to give E a finite value in this case, we use procedure (2.56). As a result, a finite value of $\mathscr{E}^f$ is found from (4.6) by replacing the function D by

$$D_{\text{reg}}=D_0^2\left[1+\frac{4\sigma}{(1+\sigma)^2}\,\frac{1}{e^{2Ra}-1}\right]\left[1+\frac{4\delta R}{(1+\delta R)^2}\,\frac{1}{e^{2Ra}-1}\right], \qquad (4.7)$$

where

$$D_0=1-\exp(-2Ra).\qquad \sigma=\frac{\delta\zeta^2}{R}.$$

The value of the attractive force acting on a unit area of the plates can be determined from the relation $F = -\partial\mathscr{E}^{\rm f}/\partial a$, as usual.

For ideal plates with $\delta = 0$ we have $D_{\rm reg} = D_0^2$, and from (4.6) and (4.7) we find results which have already been established, (3.12).

In the case $\delta \neq 0$, it is most convenient to carry out specific calculations of the corrections to the energies and the Casimir forces for the nonideal nature of the metal making up the plates by using a perturbation theory in powers of the small parameter δ/a. At the distances (a) between the plates which are ordinarily used in the experiments, in the micron range, we can assume $\delta(\omega) = \delta_0 = \Omega^{-1}$ for metals, where Ω is the effective plasma frequency.[85] For the force acting between the plates we then find, in the first two orders in δ_0/a, the following expression[38]:

$$F = -\frac{\pi^2}{240a^4}\left[1 - \frac{16}{3}\frac{\delta_0}{a} + 24\left(\frac{\delta_0}{a}\right)^2\right]. \tag{4.8}$$

The first-order correction from the right side of (4.8) was found in Ref. 33 with a coefficient which differs by a factor of about five from the actual value, which was first derived in Ref. 41.

Since at $a \sim 1\ \mu$m we have $\delta_0 \sim 0.1a$, the corrections to the Casimir forces for the nonideal nature of the plate material are fairly important. Corrections for the roughness of the plates were found in Ref. 104.

This method was applied to various frequency dependences of the impedance Z in Refs. 38 (corresponding to the cases of the anomalous skin effect, the normal skin effect, superconducting plates, metal-plates coated with an insulator, and plates made of an anisotropic material). It was found that in all cases this method yields simple and graphic expressions for the corresponding corrections to the Casimir forces (see also Refs. 43 and 44).

In discussing the corrections to Casimir's classical result, (3.12), we should also mention the deviations from it which arise if the plates are not strictly parallel and also because of the finite areas of the plates. For example, if plates of width $l = \rho_2 - \rho_1$ (Fig. 3) make an angle α with each other, we find the following expression[24] for the force acting on a unit area of the plates from the result for the dihedral angle (Subsection 3.3):

$$F = -\frac{\pi^2}{240a^4}\left[1 + \frac{10}{3}\left(\frac{\alpha l}{2a}\right)^2\right], \tag{4.9}$$

where a is the average distance between the plates. With $l = 1$ cm, $a = 1\ \mu$m, and $\alpha \lesssim 2''$, the correction to the force for the deviation from a parallel arrangement is less than 1%.

Comparing (3.13) for the Casimir energy of a parallelepiped with (3.12), we can estimate the corrections to (3.12) for plates of finite area S. The minimum deviations from force (3.12) occur when lateral metal screens are placed right up against the plates; the relative magnitude of these deviations is of the order of $10a^2/S$ for square plates. With $S = 1\ {\rm cm}^2$ and $a = 1\ \mu$m, for example, this quantity is equal to 10^{-5}%.

4.3. Temperature corrections. There are two ways by which the temperature can affect Casimir forces. First, the temperature can have a direct effect through the temperature dependences $\varepsilon(T)$ and $\mu(T)$. These dependences obviously disappear in the limit of the ideal case, $|\varepsilon| \to \infty$. Second, at $T \neq 0$ the force is determined by the derivative not of the energy but of the free energy $\mathscr{F}$, which is a function of the temperature (the definitions of the energy and the free energy are the same only at $T = 0$). There is also a dependence $\mathscr{F} = \mathscr{F}(T)$ in the case of ideal, impenetrable boundaries. It is this case which we will discuss below in the example of the electromagnetic field between ideally conducting plates. For convenience in the discussion below, we will explicitly introduce $\hbar$ and c in the equations in this subsection.

A characteristic parameter is $T_{\rm e} = \hbar c/2a$ (we are taking the Boltzmann constant to be equal to unity). From the general results[36] we easily find the following expression in the case with $T/T_{\rm e} \ll 1$ and $|\varepsilon| \to \infty$:

$$F = -\frac{\pi^2\hbar c}{240a^4}\left[1 + \frac{1}{3}\left(\frac{T}{T_{\rm e}}\right)^4\right] \tag{4.10}$$

Expression (4.10) has been derived previously by a slightly different method.[34]

The opposite limit, $T/T_{\rm e} \gg 1$, corresponds to $aT \gg \hbar c$. Substituting in the characteristic frequency $\omega \sim c/a$, we find the condition for a semiclassical situation, $\hbar\omega \ll T$. We would therefore expect $\hbar$ to drop out of the expression for the force.

This is what we indeed find, and in the limit $|\varepsilon| \to \infty$ we have[36]

$$F = -\frac{T\zeta(3)}{8\pi a^3}. \tag{4.11}$$

Temperature corrections to the Casimir forces have also been studied for other configurations of the boundaries, e.g., for a sphere.[29]

4.4. What is measured in an experiment? The Casimir forces between two parallel metal plates were first measured by Sparnaay,[3] who found qualitative confirmation of theoretical result (3.12) for the attractive force. The plates in that study were made of chromium or of chromium-plated iron and were separated by a distance a ranging from 0.5 to 2 μm. The plates were first brought into contact and then pulled apart by a special mechanical system. A spring counterbalancing the plates was stretched by the Casimir forces, with the result that the capacitance of capacitor C changed (see the experimental layout in Fig. 5). Measurements of C made it possible in turn to determine the attractive force (F) between the plates beginning at values $\sim 10^{-4}$ dyn.

The relative error in the measurements of F in Ref. 3 was of the order of 100%, since the hysteresis in the mechanical system used there introduced an error $\Delta a \approx 0.3\ \mu$m in the value of a. This circumstance prevented a rigorous quantitative confirmation of result (3.12).

Later experiments were carried out to measure the attractive forces between dielectric objects, since it was possi-

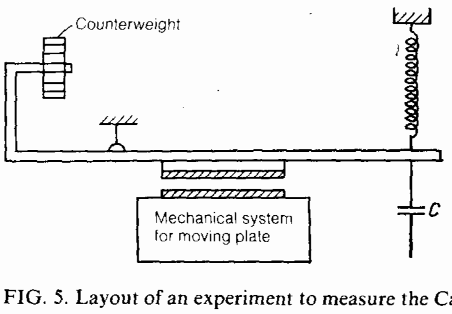

FIG. 5. Layout of an experiment to measure the Casimir force.

ble to measure distances more accurately for such objects. For example, Derjaguin *et al.*[4] used an ampere-balance apparatus with feedback to study the attraction between a spherical lens and a flat plate, both made of quartz. The error in the measurement of the distance between the objects was $\Delta a \approx 0.01\ \mu$m. The results of the measurement of the force (obtained with a relative error $\approx 20\%$) turned out to agree well with the theoretical predictions (Subsection 4.1).

Some even more precise experiments were described in Refs. 5, where the forces between the lateral surfaces of two mica-clad glass cylinders, arranged perpendicular to each other, were measured. Multibeam interferometry and the method of equal-chromatic-order fringes made it possible to measure the distance between the cylinders within an error $\Delta a \approx (2\text{–}3)\cdot 10^{-4}\mu$m. The relative error in the measurement of the forces between the cylinders was 5–10%. Within these errors, the results agreed with the theoretical predictions (the literature on measurements of van der Waals forces is reviewed in Ref. 6).

In all these experiments, what was actually observed was a force caused by the existence of vacuum oscillations of the electromagnetic field. The zero-point oscillations of the electron-positron field are also observed experimentally, indirectly, through their contribution to the Lamb shift and to the anomalous magnetic moment of an electron in the Coulomb field of a nucleus.

In summary, although the fluctuation forces between solid objects have now been studied in some detail experimentally, the actual result derived by Casimir regarding the forces between plane plates of a good metal has received only qualitative confirmation. Since this is the most characteristic case, from the standpoint that the force is independent of the microstructure of the plates, it would be interesting to see some new and precise experiments to measure the Casimir forces between metals.

How experiments carried out to measure fluctuation forces might be useful at the frontiers of elementary particle physics is discussed in Subsection 6.4 of this review.

5. NONTRIVIAL TOPOLOGY OF SPACE-TIME AND COSMOLOGICAL APPLICATIONS

5.1. The Casimir effect on a Möbius sheet. A Möbius sheet is the simplest two-dimensional nonorientable manifold; i.e., it has a nontrivial topology. A graphic model of such a sheet can be devised by cutting a cylinder (Fig. 6a) parallel to its axis, twisting it, and cementing it back together (Fig. 6b). A Möbius sheet has no inside and no outside. When we draw a line at the center of the strip and make one revolution we come back to our starting point, but on the opposite side of the strip; after two revolutions we close the line.

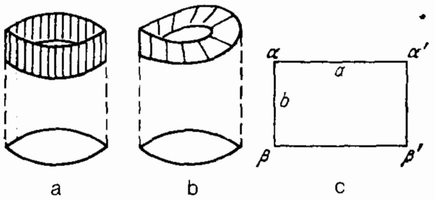

FIG. 6. Möbius sheet and a cylinder.

This graphic representation requires going into a third dimension. If we remain in a two-dimensional manifold, we must formally specify a Möbius sheet through an identification of the boundary points of the "strip" in such a way that we have $\alpha \equiv \beta'$ and $\beta \equiv \alpha'$ (for a cylinder, we would obviously have $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$; Fig. 6c). The corresponding conditions for the field on a Möbius sheet, which replace the periodicity condition along the length on a cylinder, take the form

$$\varphi(0, y) = \varphi(a, b-y), \quad \varphi(x, 0) = \varphi(x, b) = 0, \\ \partial_x \varphi(0, y) = \partial_x \varphi(a, b-y). \tag{5.1}$$

Satisfying conditions (5.1) by means of a product of trigonometric functions, we find two families of frequencies:

$$\omega_{mk}^{(1)} = 2\pi \left\{ \left[\frac{m+(1/2)}{a} \right]^2 + \left(\frac{k}{b} \right)^2 \right\}^{1/2}, \quad k \geqslant 1, \\ \omega_{mk}^{(2)} = 2\pi \left\{ \left(\frac{m}{a} \right)^2 + \left[\frac{k+(1/2)}{b} \right]^2 \right\}^{1/2}, \quad k \geqslant 0, \tag{5.2}$$

where we have $-\infty < m < \infty$ everywhere.

The renormalized energy found by applying formulas of the Abel-Plana type, (2.35), (2.39), is (at exponential accuracy)[12]

$$\mathscr{E} = \frac{\pi}{24a}\left(-\frac{3\zeta(3)}{2\pi^2}l - 1 + l^{-1} \right), \quad l > 1, \\ = -\frac{\zeta(3)}{16\pi a} l^{-2}, \quad l < 1, \tag{5.3}$$

where $l \equiv b/a$.

Corresponding calculations for the surface of a cylinder yield[12]

$$\mathscr{E}_{\text{cyl}} = \frac{\pi}{24a}\left(-\frac{12\zeta(3)}{\pi^2}l + 2 - l^{-1} \right), \quad l > \frac{1}{2}, \\ = -\frac{\zeta(3)}{16\pi a} l^{-2}, \quad l < \frac{1}{2}.$$

Consequently, the difference in energies disappears at a small value of l (in practice, at $l < 1/2$); i.e., the topological distinctions cease to play a role when the topologically nonequivalent boundaries are removed.

A projection of a cylinder or a Möbius sheet onto a plane, a circle S^1, is called a "stratification base," and the segments I which make up the width of the strip are "layers." The cylinder in Fig. 6a is the direct product $S^1 \times I$, and the Möbius sheet is not; i.e., there is a nontrivial stratification (cf. Subsection 2.6).

5.2. The 3-torus and other topologies. Three-dimensional problems provide several versions of a change in topology through the formal replacement of one or more segments I in a parallelepiped $I \times I \times I$ by a circle S^1. Detailed results for various types of fields and topologies are given in Refs. 9 and 11.

As an example we consider the average vacuum energy density for a massless spinor field in the 3-torus configuration $S^1 \times S^1 \times S^1$, obtained through an identification of opposite faces of the parallelepiped. Here there are no walls. The periodicity conditions lead to the following result for the vacuum energy density under the conditions $a \leqslant b \leqslant c$:

$$\varepsilon = \frac{2\pi^2}{45a^4} + \frac{2\zeta(3)}{\pi a b^3} + \frac{2\pi}{3abc^2}$$

(as in Section 3, exponentially small terms have been discarded).

We obviously have $\varepsilon > 0$ here for all relations among the sides (we have $\varepsilon \approx 3.23a^{-4}$ for the case $a = b = c$). For an electromagnetic field in the same configuration, however, the sign of ε changes upon a change in a relation between sides. For example, with $a = b = c$ we have $\varepsilon = 0.0932/a^4$, while with $a = b \neq c$ the energy retains its sign over the interval $0.478 < c/a < 3.26$.

For a massless scalar field in a 3-torus configuration with $a = b = c$ we have the result[19] $\varepsilon \approx -0.8375a^4$ (see Appendix II regarding the cases of the $S^1 \times S^1$ and the $S^1 \times S^1 \times R^1$ configurations). Interestingly, for a spinor field with antiperiodic conditions in the identification of opposite faces of the parallelepiped [such conditions are more natural for a spinor field; see (2.37) and the corresponding discussion in terms of the variables y, z] we again find a negative density for the Casimir energy. For a two-component neutrino field,[56] for example, we have $\varepsilon \approx -0.3914a^{-4}$. In other words, a twisted configuration of a spinor field is preferred from the energy standpoint.

Other results, in particular, for the $S^1 \times I \times I$ configuration, which is equivalent to an ordinary torus, are given in Refs. 9, 11, and 19.

5.3. Casimir effect on a sphere. We now consider the vacuum energy on a sphere S^2, i.e., a two-dimensional spherical surface. This example is interesting not only because it presents a new type of topology but also because it has a curvature. This example is useful as a simple analog of cosmological problems.

The metric on a sphere of radius a is

$$ds^2 = a^2 (d\eta^2 - d\theta^2 - \sin^2\theta \, d\varphi^2), \tag{5.4}$$

where we have introduced the time variable $\eta = t/a$. The scalar curvature is $R = 2a^{-2}$.

To find the frequency spectrum we first need to generalize the Klein-Fok equation to the case of an arbitrary metric g_{ik}. The simplest method is to replace the ordinary derivatives ∂_k by covariant derivatives ∇_k:

$$(\nabla_k \nabla^k + m^2) \varphi = 0. \tag{5.5}$$

This method for incorporating the curvature of a space in the equation for a scalar field is called "minimal coupling." In this case the equation does not satisfy the requirement of "conformal invariance"; i.e., there is no field transformation $\varphi \to \tilde{\varphi}$ of such a nature that the wave equation retains its form under the mapping ($m = 0$)

$$g_{ik} \to \tilde{g}_{ik} = e^{-2\sigma(x)} g_{ik}. \tag{5.6}$$

Physically, conformal invariance means that the behavior must be identical in Riemann spaces (5.6) of massless particles which do not introduce into the problem the scale m^{-1}. This invariance holds for the "conformal-coupling equation" ($m = 0$)[94]

$$(\nabla_k \nabla^k + \xi R + m^2) \varphi = 0, \tag{5.7}$$

where $\xi = \xi_c = (D-2)/4(D-1)$ is the same as the constant which was introduced in §3. A variation over the metric of the Lagrangian corresponding to (5.7) yields a metric energy-momentum tensor (see Subsection 3.1 for the case of Minkowski space).

For a real scalar field we have

$$T_{ik} = \frac{1}{2} (\nabla_i \varphi \nabla_k \varphi + \nabla_k \varphi \nabla_i \varphi) - \frac{1}{2} g_{ik} [\nabla_l \varphi \nabla^l \varphi - (m^2 + \xi R) \varphi^2] - \xi (R_{ik} + \nabla_i \nabla_k - g_{ik} \nabla_l \nabla^l) \varphi^2. \tag{5.8}$$

With $\xi = 1/8$ (since $D = 3$), Eq. (5.7) becomes

$$\partial_\eta^2 \varphi - \Delta\varphi - \left(m^2 a^2 + \frac{1}{4} \right) \varphi = 0$$

and has the following eigenfunctions and eigenfrequencies:

$$\varphi_{lm}(\eta, \theta, \varphi) = \frac{1}{(2\omega_l)^{1/2}} e^{i\omega_l \eta} Y_{lm}(\theta, \varphi),$$

$$\omega_l^2 = m^2 a^2 + \left(l + \frac{1}{2} \right)^2, \tag{5.9}$$

where Y_{lm} are the spherical harmonics.

In this problem the energy-momentum tensor is clearly independent of the coordinates. Working from (5.8) and (5.9), and renormalizing, we find[12]

$$\langle T_i^k \rangle = \frac{m^3}{4\pi} \left[S_1(ma) \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} + S_{-1}(ma) \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right],$$

$$S_n(\mu) = \int_0^1 (1 - p^2)^{n/2} p \, [\exp(2\pi p\mu) + 1]^{-1} dp. \tag{5.10}$$

It follows from (5.10) that for a massless field the Casimir energy on a sphere vanishes. In limiting cases we have

$$\varepsilon = \frac{m^3}{12\pi} \quad (\mu \ll 1), \qquad \varepsilon = \frac{m}{96\pi a^2} \quad (\mu \gg 1). \tag{5.11}$$

The behavior in (5.10) and (5.11) is different from the behavior examined above and also different from the behavior in the corresponding problem with $D = 4$ (Subsection 5.4).

5.4. The Casimir effect in cosmology. Cosmology offers us perhaps the most grandiose example of the use of a space with a non-Euclidean topology in physics. In inflationary models of the universe, the 3-space is known to be closed[46] (or bounded by the wall of a bubble), so the vacuum energy-momentum tensor contains a contribution of topological nature, which becomes important in the early stages of the evolution.

In this subsection we present the results calculated on the Casimir effect in the most important case for cosmology: that of homogeneous and isotropic models of a closed type (cross sections $t =$ const are 3-spheres with topology S^3). We will discuss a possible role of the Casimir energy density in research on the universe.

In comoving coordinates, the metric of a homogeneous and isotropic closed space-time is[95]

$$ds^2 = g_{ik} dx^i dx^k = a^2(\eta) (d\eta^2 - dl^2), \tag{5.12}$$

where $\eta = \int dt/a(\eta)$, and dl^2 is the metric of the 3-space with a curvature of $+1 (0 \leqslant \chi \leqslant \pi)$, given by

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta = d\chi^2 + \sin^2\chi \, (d\theta^2 + \sin^2\theta \, d\varphi^2). \tag{5.13}$$

For power-law scale factors $a(\eta)$, (5.12) and (5.13) constitute the metric of a closed Friedmann model, while in the case $a = a_0/\cos\eta = a_0 \mathrm{ch}(t/a_0)$ it is that of a de Sitter model, which corresponds to an exponential "inflation" of the 3-space [we recall that $a(t)$ has the meaning of the radius of curvature of the 3-space at time t].

In contrast with all of the problems discussed above, which were steady-state problems, here the quantity a, found through a solution of Einstein's equations, depends on the time. As a result, in addition to the Casimir component of the vacuum energy-momentum tensor (associated with the deviation of the topology of the 3-space from a Euclidean topology) there are some additional polarization terms, which depend on derivatives of the scale factor. There are also terms which describe the creation of particles from the vacuum by the gravitational field.

The fundamental questions associated with the creation of particles from vacuum by external fields have been studied in Refs. 47–49, 61, and 62. For our purposes it is important to note[50] that the very possibility of such an effect in a gravitational field stems from the violation of energy dominance conditions in quantum theory:

$$\varepsilon \geqslant 0,\ |P| \leqslant \varepsilon,\ \varepsilon + 3P \geqslant 0$$

(ε is the energy density, and P is the pressure of the quantized fields in the vacuum state). These conditions start to hold only after the "final" conversion of the virtual particles into real particles.

The polarization of the vacuum determined by the derivatives of a is related to terms in the effective Lagrangian of the gravitational field which are quadratic in the curvature.[96,97] The energy dominance conditions may also be violated for this polarization. A complete theory of the effects of particle creation and vacuum polarization in a gravitational field is given in Refs. 7 and 8; here we will consider only the Casimir effect itself, which also occurs in the case $da/d\eta = 0$.

Separating variables in the Klein-Fok equation (5.7) in metric (5.12), (5.13), and calculating the vacuum expectation value of the energy-momentum tensor (5.8), we finally find

$$\langle 0 \mid T_{00} \mid 0 \rangle = \frac{1}{4\pi^2 a^2} \sum_{n=1}^{\infty} n^2 \omega_n, \quad \omega_n = (m^2 a^2 + n^2)^{1/2},$$
$$\langle 0 \mid T_{\alpha\beta} \mid 0 \rangle = \frac{1}{12\pi^2 a^2} \sum_{n=1}^{\infty} \frac{n^4}{\omega_n} \gamma_{\alpha\beta}. \tag{5.14}$$

The same expressions are found for a plane Minkowski space (the sums are replaced by integrals from 0 to ∞).

A regularization of results (5.14) is carried out by procedure (2.33), i.e., by discarding the contribution of the Minkowski space tangent to the Riemann space under consideration at the given point. Through the method of dimensional regularization one can demonstrate[51-53] that such a procedure is precisely equivalent to a renormalization of the cosmological constant Λ in the effective Lagrangian for the gravitational field (the physical value of Λ_{ren} is discussed in Ref. 45). In practice, procedure (2.33) corresponds to the use of Abel-Plana formula (2.35). As a result we find[54]

$$\langle T_{00} \rangle = \frac{1}{2\pi^2 a^2} \int_{ma}^{\infty} \frac{\lambda^2 (\lambda^2 - m^2 a^2)^{1/2}\, d\lambda}{\exp(2\pi\lambda) - 1},$$
$$\langle T_{\alpha\beta} \rangle = \frac{1}{6\pi^2 a^2} \int_{ma}^{\infty} \frac{\lambda^4 (\lambda^2 - m^2 a^2)^{-1/2}\, d\lambda}{\exp(2\pi\lambda) - 1} \gamma_{\alpha\beta}. \tag{5.15}$$

The quantities in (5.15) are defined unambiguously, since the space-time under consideration here has no boundaries.

For massless particles, the integrals in (5.15) can be evaluated analytically[19,55]:

$$\langle T_{00} \rangle = \frac{1}{480\pi^2 a^2}, \quad \langle T_{\alpha\beta} \rangle = \frac{1}{1440\pi^2 a^2} \gamma_{\alpha\beta}. \tag{5.16}$$

The distribution with respect to λ in (5.15) with $m = 0$ is the same as a Bose spectrum with an effective temperature $T_{eff} = 1/2\pi a$ (Refs. 54 and 63).

For $m \neq 0$, the values found for $\langle T_{ik} \rangle$ in Ref. 55 by numerical methods lay in the interval $0 \leqslant ma \leqslant 1.5$. It turned out that at $ma \gtrsim 0.5$ the Casimir energy-momentum tensor of a massive field ceases to satisfy the energy dominance conditions. At $ma \gg 1$ the following can be found with the help of Watson's lemma:

$$\langle T_{00} \rangle \approx \frac{(ma)^{5/2}}{8\pi^3 a^2} e^{-2\pi ma},$$
$$\langle T_{\alpha\beta} \rangle = \approx \frac{(ma)^{7/2}}{12\pi^2 a^2} \gamma_{\alpha\beta} e^{-2\pi ma}.$$

In the case of a massless spinor field, the results corresponding to (5.16) are[19,55]

$$\langle T_{00} \rangle = \frac{17}{960\pi^2 a^2}, \quad \langle T_{\alpha\beta} \rangle = \frac{17}{2880\pi^2 a^2} \gamma_{\alpha\beta}. \tag{5.17}$$

The spectrum is a Fermi spectrum, with the same effective temperature T_{eff} as for a scalar field.

The existence of Casimir contributions of the type in (5.16) and (5.17) to the vacuum energy-momentum tensor demonstrates once again that the global structure of the space-time is reflected in the local properties of the physical vacuum. This fact poses the intriguing possibility in principle of reconstructing the topological structure of the universe as a whole, in particular, solving the finiteness-infiniteness problem of 3-space on the basis of the results of purely local measurements.

Casimir terms (5.16) and (5.17) in the overall vacuum energy-momentum tensor play an important role in the construction of so-called self-consistent models of the universe, in which the gravitational field is produced by the vacuum energy-momentum tensor as a source in accordance with Einstein's equations, while the vacuum, conversely, is polarized by this gravitational field itself.[57-59] The discovery of such models has provided a serious scientific basis for analyzing the hypothesis that the entire universe around us originated from a physical vacuum.[60]

Another reason why the Casimir effect is pertinent to cosmology is the possible topological nontriviality of the 3-space of the universe. For example, Zel'dovich and Starobinskiĭ[56] have studied the quantum creation of a universe with a plane 3-space having the topology of a 3-torus. In this case, substitution of the Casimir energy density summed over all the fields, $\varepsilon = -Aa^{-4}$ ($A > 0$; Subsection 5.2), into the right side of Einstein's equations makes it possible to find a nonsingular cosmological model of the inflationary type. The Casimir effect in supersymmetry and supergravity theories was also studied in Ref. 64 in connection with the problem of quantum creation of the universe with a nontrivial topology.

6. THE CASIMIR EFFECT IN ELEMENTARY PARTICLE PHYSICS

6.1. The vacuum energy in the bag model. A simplified and phenomenological description of the structure of hadrons in quantum chromodynamics (QCD) gives us the so-called bag model,[65] according to which the hadrons are bubbles in a QCD vacuum which enclose quarks and gluons, whose currents through the walls of the bubble are zero (in other words, there is confinement). Since the QCD vacuum has a negative energy density, its absence from a bubble is equivalent to a positive volume energy. The relation between the latter and an energy of a surface-tension type ($\mathscr{E} \sim a^{-1}$), which also has a Casimir component, determines the bubble radius a, the hadron mass, and other observable characteristics.

In the absence of real quarks, the Casimir energy of a bag would be caused by quantum fluctuations of the quark and gluon fields. Two contributions to this energy come from the quark field described by the Dirac equation and the color gluon field which approximately satisfies equations analogous to Maxwell's equations in which the roles of the electric and magnetic fields are interchanged.[10]

For a spherical cavity of radius a, under impenetrability conditions of the type in (3.17) at a boundary, and with quark masses ignored, the following result has been found[67]:

$$E_q \approx -\frac{1}{144\pi a} + \frac{1}{3\pi a \delta^2}. \tag{6.1}$$

The term in (6.1) which diverges as the cutoff parameter is allowed to go to zero is canceled by a term of the opposite sign for external modes[68] (i.e., $\mathscr{E}^{\text{tot}}$ remains finite; cf. Subsection 3.4). An analysis of the contributions of both the inner and outer regions in Ref. 68 led to a positive final result:

$$\mathscr{E}_q^{\text{tot}} \approx \frac{0.02}{a}. \tag{6.2}$$

In the case of gluons, when the conditions for the color electric field **E** and the color magnetic field **B** at the boundary

$$\mathbf{nE}\,|_\Gamma = 0, \quad [\mathbf{nB}]\,|_\Gamma = 0, \tag{6.3}$$

are taken into account (**n** is the outward normal), the following expression was found[22,67,68] in place of (6.1):

$$E_g \approx \frac{11}{72\pi a} - \frac{4}{3\pi a \delta^2}. \tag{6.4}$$

This expression depends on the cutoff parameter δ (see also Ref. 70).

The finite energy of the gluon field, $\mathscr{E}_g^{\text{tot}}$, can be found from the known result for the Casimir energy of a sphere with a permittivity ε_1 and a magnetic permeability μ_1 situated in an infinite medium with ε_2 and μ_2 under the condition $\varepsilon_1\mu_1 = \varepsilon\mu_2 = 1$ (Ref. 69):

$$\mathscr{E}_\mu^{\text{tot}}(a) = \mathscr{E}^{\text{tot}}(a)\left(\frac{\mu_{12}-1}{\mu_{12}+1}\right)^2\left[1+0.311\,\frac{\mu_{12}}{(\mu_{12}+1)^2}\right], \tag{6.5}$$

where $\mu_{12} \equiv \mu_1/\mu_2$, $\mathscr{E}^{\text{tot}}(a) \approx 0.092/2a$ is the same as the energy of an ideally conducting sphere, from Subsection 3.4.

It can be seen from conditions (6.3) that the Casimir energy of a gluon field is found from (6.5) under the conditions $\mu_1 = 1, \mu_2 \to \infty$, i.e., $\mu_{12} \to 0$; this energy is the same as $\mathscr{E}^{\text{tot}}(a)$. The result incorporating the contributions of all components of the gluon field is $8\mathscr{E}^{\text{tot}}(a)$.

Dividing the energy

$$8\mathscr{E}^{\text{tot}}(a) + 3\mathscr{E}_q^{\text{tot}} \approx \frac{0.43}{a}$$

by the mass of a gluon with a radius $a \sim 1$ fm, we find an estimate of the relative role played by the Casimir energy in the bag energy: $\approx 9\%$. We note, however, that in practice the values of the parameters which determine the bag energy are treated as adjustable parameters and are used to fit the spectrum and the magnetic moments of the hadrons.

6.2. Multidimensional field theories of the Kaluza-Klein type. Interest in the development of field theory models of the Kaluza-Klein type has revived and strengthened in recent years. Despite the diversity of such models, all are based on one leading idea: The actual dimensionality of the space-time is $D = 4 + N > 4$, but $N > 0$ of the dimensions "spontaneously compactify," i.e., form an N-dimensional compact space whose geometric dimensions are of the order of Planck dimensions, $a \approx l_{pe} = G^{1/2} \approx 10^{-33}$ cm (G is the gravitational constant). In the simplest models, the limit $a \to 0$ is in fact taken. These additional N dimensions are not directly observable, but their presence has an implicit effect on the form of the equations of motion found for the 4-space from the original equations of the $(4 + N)$-dimensional theory. In this connection one would say that ordinary four-dimensional physics is a low-energy approximation of the more general $(4 + N)$-dimensional theory.[74,87]

The first theory of this type was proposed by Kaluza[71] and pursued by Klein.[72] That theory dealt with a unification of gravitation and electromagnetism in a five-dimensional theory of gravitation.

We would like to discuss a very simple but nontrivial example which illustrates the possibilities of a theory of this type.[73,74] The action for the gravitational field in a five-dimensional space-time is

$$S^{(5)} = \frac{1}{16\pi G_5}\int |g^{(5)}|^{1/2} R^{(5)} d^5x, \tag{6.6}$$

where $R^{(D)}$ is the scalar curvature, and g_{AB} is a five-dimensional metric, which we choose in the form

$$g_{AB} = \begin{pmatrix} g_{nm} + A_n A_m & A_m \\ A_n & 1 \end{pmatrix}, \qquad g^{(5)} \equiv \det g_{AB}; \tag{6.7}$$

here $n, m = 0, 1, 2, 3$; and the functions A_n are certain 4-vector functions. The topology of a given manifold is $M^4 \times S^1$, where S^1 denotes as before a circle of radius a (along the fifth coordinate, x^4). Assuming that g_{AB} is independent of x^4, as is customarily done for models of this sort, substituting (6.7) into (6.6), and integrating over x^4, we find the effective action

$$S^{(4)} = -\frac{1}{16\pi G}\int |g^{(4)}|^{1/2}\left(R^{(4)} + \frac{1}{4}F_{mn}F^{mn}\right) d^4x, \tag{6.8}$$

in which a $U(1)$-gauge term with $F_{mn} = \partial_m A_n - \partial_n A_m$, which is characteristic of the electromagnetic field, has arisen "spontaneously."[74]

In the usual notation, the role of the gauge field in (6.8) should be played by the quantity $\tilde{A} = (16\pi G)^{-1/2}A$. It thus becomes possible to relate the electric charge of a particle to G and to generate qualitative estimates of probable values of a (Ref. 74).

Compactification ideas are presently used widely in various versions of supersymmetry theories, including supergravity theories.[87] One possible mechanism for spontaneous compactification is related to a self-consistent incorporation of a Casimir energy-momentum tensor on the right side of Einstein's equations (see the following subsection).

6.3. Compactification due to the Casimir energy. In a $(4+N)$-dimensional space-time, Einstein's equations take the form

$$R_{AB}-\frac{1}{2}g_{AB}R=G^{(D)}(\langle T_{AB}\rangle+\Lambda g_{AB}), \qquad (6.9)$$

where R_{AB} is the Ricci tensor, $G^{(D)}$ is the gravitational constant, which is related to the usual quantity G by $G^{(D)} = GV(S^N)$, where V is the volume of the N-dimensional sphere S^N, and $\langle T_{AB}\rangle$ is the renormalized Casimir energy-momentum tensor.

We seek a solution of (6.9) which corresponds to the manifold $M^4 \times S^N$. Correspondingly, the metric and the Ricci tensor become

$$g_{AB}=\begin{pmatrix}\eta_{mn} & 0\\ 0 & g_{\mu\nu}\end{pmatrix}, \qquad R_{AB}=\begin{pmatrix}0 & 0\\ 0 & \frac{R}{N}g_{\mu\nu}\end{pmatrix}, \qquad (6.10)$$

where η_{mn} is the metric of Minkowski space M^4, $g_{\mu\nu}$ is the metric on an N-dimensional sphere of radius a, and R is the scalar curvature on it.[75–77]

Expressions (6.10) are compatible in form with Eqs. (6.9) if the energy-momentum tensor is also of block form, and we have

$$\langle T_{mn}\rangle = T_1\eta_{mn}, \qquad \langle T_{\mu\nu}\rangle = T_2 g_{\mu\nu}. \qquad (6.11)$$

As usual, we assume that the trace of the energy-momentum tensor vanishes: $g_{AB}\langle T_{AB}\rangle = 0$. We set T_1 equal to the density of the Casimir energy of quantized fields: $T_1 = \langle T_{00}\rangle$. From dimensionality considerations for massless fields we have $T_1 = C/a^D$. Substituting (6.10) and (6.11) into (6.9), we find the following conditions which must be satisfied by the adjustable parameters a and Λ:

$$a^{2+N}=-\frac{CDG^{(D)}}{N(N-1)}, \qquad \Lambda=-\frac{N(N-1)(N+2)}{2a^2D}.$$

The condition $C<0$ must obviously hold. Values of C for boson fields, $C_1 \approx -1\cdot 10^{-4}$, and fermion fields, $C_2 \approx 9\cdot 10^{-4}$, are given for the case $N=2$ in Ref. 77 ($G^{(6)} = 4\pi a^2 G$). Consequently, within a constant determined by the sum of C_i over all possible fields, we find $a \sim G^{1/2}$. In other words, in this model, with a sufficiently large number of boson fields, we do indeed have a spontaneous compactification with a length scale of the order of the Planck length. The case of odd values of N was studied in Ref. 76.

A compactification in cosmological theories of the Kaluza-Klein type due to the Casimir effect was studied in Refs. 82b in connection with the possible onset of an inflationary regime in the evolution of the universe.

6.4. Refinement of the constants of particle physics on the basis of the Casimir effect. In modern elementary particle theory, a large number of hypothetical light or massless particles have been introduced (the axion, the scalar axion, the arion, the axino, etc.; see the review by Ansel'm *et al.*[79]). In particular, light scalar particles should arise because of a breaking of any global symmetry in a supersymmetry theory.

The presence of light or massless particles gives rise to some new (i.e., not electromagnetic or gravitational) long-range forces between macroscopic objects.[82a] These forces can be detected in experiments carried out to measure Casimir forces, if one observes differences between the experimental value of the force and the corresponding theoretical prediction. In any case, measurement of the Casimir forces make if possible to impose limitations on those additional (non-Casimir) forces between macroscopic objects which would arise as a result of the exchange of hypothetical light particles. It thus becomes possible to find limitations on the constants of such particles.

Analyzing the results of some old studies carried out to measure Casimir forces between a plane and a spherical lens,[5,6] Kuz'min *et al.*[78] found a limitation on the mass m and on the Yukawa coupling constant of a light scalar particle with fermions, f:

$$\frac{f}{m^3}<10^{-13}\ \mathrm{eV}^{-3}. \qquad (6.12)$$

If we apply this limitation to a supersymmetry theory of grand unification with a unification scale $\sim 10^{17}$ GeV, which contains the Peccei-Quinn $U(1)$ symmetry, which is broken at the same scale, then by taking the light scalar particle to be a scalar axion we find the estimate $M > 10^3$ GeV for the scale over which the supersymmetry is broken[78] (in models with a supersymmetry which is not broken, there is no Casimir effect[80]). This is the best estimate which has been obtained by any method (including those based on astrophysical data); see also Ref. 81.

As was shown in Ref. 78, the Casimir effect is in general the best tool for seeking long-range forces caused by scalar particles over broad ranges of the coupling constants, $10^{-18} < f < 10^{-9}$, and masses, 10^{-6} eV $< m < 10$ eV. The Casimir effect was used in Refs. 82c to refine the most stringent known upper limits on the constants of hypothetical long-range forces which fall off in accordance with a power law $F \sim r^{-n}$ (the refinements were by an order of magnitude in the $n=3$ case and by a factor of 200 in the $n=4$ case). Accordingly, new and precise experiments to measure Casimir forces could be decisive in tests of the predictions of supersymmetry theories.

7. CONCLUSION

As was shown above, the long list of phenomena in which a vacuum of quantized fields acquires a certain energy density as a result of the boundedness of the quantization volume or a deviation from a Euclidean topology of the manifold can be interpreted in a common way and described as the Casimir effect. The appearance of a Casimir force between macroscopic objects is essentially a macroscopic manifestation of the zero-point oscillations of the electromagnetic field. Although it is observed in local measurements, this effect turns out to be a unique source of information about the topological structure of the universe as a whole. It is thus not surprising to note the diversity and benefits of the various applications of this effect: in macroscopic physics, cosmology, hadron physics, supersymmetry and supergravitation, and the refinement of the constants in elementary particle physics.

Great things are expected of the Casimir effect: It may become a new testing ground for the predictions of fundamental physical theories.

Not all the problems which involve calculations of Casimir energies and forces have been solved satisfactorily at this point. For example, we need a further refinement of the problem of surface divergences. Much more will undoubtedly be accomplished in the development of approximate methods for calculating the Casimir effect for more-complex configurations of boundaries.

Our purposes in this review have been to present the physical and mathematical foundations of the Casimir effect and to point out its most important applications. In the section of the bibliography on each of these effects one can find voluminous additional material containing model-dependent results and details.

In conclusion I wish to thank the late Ya. B. Zel'dovich for support of the idea of writing this review and for several useful comments which he offered regarding the contents of this review in the spring of 1986.

APPENDIX I

We wish to illustrate the method for calculating the vacuum energy through the repeated use of the Abel-Plana formulas. We use as an example the double sum

$$S=\sum_{n=1}^{\infty} s_n,\qquad s_n=\sum_{m=1}^{\infty}(n^2+m^2)^{1/2}, \qquad (A.1)$$

where the coefficients of n and m have been set equal to unity for brevity.

First holding n fixed, we evaluate the sum over m with the help of (2.35):

$$s_n=\int_0^{\infty}(n^2+x^2)^{1/2}\,\mathrm{d}x-\frac{n}{2}+2I(n),$$
$$I(z)=-\int_z^{\infty}\frac{(t^2-z^2)^{1/2}\,\mathrm{d}t}{\exp(2\pi t)-1}. \qquad (A.2)$$

We now carry out the summation over n. We apply the Abel-Plana formula to the second term on the right side of (A.2), and we do the same to the first term, changing the order of the integration and the summation. As a result, using I as defined in (A.2), we find

$$S=\int_0^{\infty}\mathrm{d}x\left[\int_0^{\infty}\mathrm{d}y\,(x^2+y^2)^{1/2}-\frac{x}{2}+2I(x)\right]$$
$$-\frac{1}{2}\int_0^{\infty}\mathrm{d}y\,y-I(0)+2\sum_{n=1}^{\infty}I(n). \qquad (A.3)$$

Through a regularization, we should discard the infinite integrals in (A.3), so that the final contribution, reg S, comes from all the terms in (A.3) which contain $I(x)$, $I(n)$, and $I(0)$.

A regularization of the sum (A.1) over frequencies in a region (a rectangle) with boundaries requires discarding the contributions of unbounded spaces,

$$J_2=\int_0^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,(x^2+y^2)^{1/2},\qquad J_1=\int_0^{\infty}\mathrm{d}x\,x, \qquad (A.4)$$

of both the same dimensionality as the region under consideration (J_2) and lower dimensionality (J_1).

The discarding of J_1 may be thought of as the elimination of a certain infinite energy which is associated with the presence of a boundary on the region (cf. Subsection 3.4).

Other functions of many variables are regularized in a corresponding way.

Sums of the type in (A.1) can be expressed in terms of Einstein's zeta-function, which is defined for a p-dimensional sum of squares by

$$Z_p\left|\begin{matrix}\mathbf{g}\\ \mathbf{h}\end{matrix}\right|(s)\equiv\sum_{\mathbf{m}}{}'\,[(\mathbf{m}+\mathbf{g})^2]^{-ps/2}\exp(2\pi i(\mathbf{m},\mathbf{h})). \qquad (A.5)$$

Here we have introduced p-dimensional vectors $\mathbf{g}$, $\mathbf{h}$, and $\mathbf{m}$; the vector $\mathbf{m}$ has integer components ($m_i=0,\ \pm1,\ \pm2,\ldots$); and in the summation we omit the term with $\mathbf{m}+\mathbf{g}=0$ if such exists. In the case pertinent here, $\mathbf{h}=0$, the functional equation[98]

$$Z_p\left|\begin{matrix}\mathbf{g}\\ 0\end{matrix}\right|(s)\,\pi^{-ps/2}\Gamma\left(\frac{ps}{2}\right)=Z_p\left|\begin{matrix}0\\ -\mathbf{g}\end{matrix}\right|(\sigma)\,\pi^{-p\sigma/2}\Gamma\left(\frac{p\sigma}{2}\right) \qquad (A.6)$$

holds, where $\sigma=1-s$.

APPENDIX II

The sums of certain multiple converging series of the type in (A.1) can be calculated efficiently with the help of the Jacobi θ-function.[99,100] As an example we follow Ref. 100, considering the series

$$S(r)=\sum_{m,\,n=-\infty}^{\infty}{}'\,(m^2+n^2)^{-r} \qquad (A.7)$$

at $r>1$ [a term with $m=n=0$ has been omitted from (A.7)]. Using the obvious identity [a Mellin transformation of $\exp(-bt)$]

$$\int_0^{\infty}t^{r-1}e^{-bt}\,\mathrm{d}t=b^{-r}\Gamma(r),$$

we can put (A.7) in the form

$$S(r)=\frac{1}{\Gamma(r)}\int_0^{\infty}t^{r-1}\sum_{m,\,n=-\infty}^{\infty}{}'\,\exp[-(m^2+n^2)t]\,\mathrm{d}t. \qquad (A.8)$$

Now using the definition

$$\theta_3(0,q)\equiv\sum_{m=-\infty}^{\infty}q^{m^2}$$

and setting $q=\exp(-t)$, we find from (A.8)

$$S(r)=\frac{1}{\Gamma(r)}\int_0^{\infty}t^{r-1}(\theta_3^2(0,e^{-t})-1), \qquad (A.9)$$

where the subtrahend in the outer set of parentheses stems from the absence of a term with $m=n=0$ from (A.7). The known representation[98]

$$\frac{1}{4}(\theta_3^2(0,q)-1)=\sum_{i=1}^{\infty}\frac{q^i}{1+q^{2i}}=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(-1)^j q^{i(2j+1)}$$

leads, after substitution into (A.9) and an integration, to

$$S(r)=4\zeta(r)\beta(r),\qquad \beta(r)\equiv\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^r}, \qquad (A.10)$$

where $\zeta(r)$ is the Riemann zeta-function.

We thus see that the double series (A.7) has been transformed into a product of known one-dimensional series.

Let us consider, for example, a scalar field on the 2-

torus $S^1 \times S^1$, i.e., on a square of side a, with an identification of opposite sides. The unregularized energy density

$$\frac{\pi}{a^3} \sum_{n,\, m=-\infty}^{\infty}{}' (n^2+m^2)^{1/2} = \frac{\pi}{a^3} Z_2 \begin{vmatrix} 0 \\ 0 \end{vmatrix} \left(-\frac{1}{2}\right) \qquad (A.11)$$

diverges, of course. However, if we use (A.6) as a definition of the regularized value of the sum, we can express (A.11) in terms of $Z_2(3/2)$; then making use of (A.10) with $r = 3/2$, we find

$$\varepsilon = -\frac{1}{4\pi a^3} Z_2 \begin{vmatrix} 0 \\ 0 \end{vmatrix} \left(\frac{3}{2}\right) = -\frac{\zeta(3/2)\,\beta(3/2)}{\pi a^3}. \qquad (A.12)$$

Using $\zeta(3/2) = 2.612$ and $\beta(3/2) = 0.8645$, we find $\varepsilon = -0.718/a^3$. The same numerical value can be found by taking Fourier transforms of (A.11); the regularization reduces to discarding the term with zero momentum.[11]

The energy density in the topology $S^1 \times S^1 \times R^1$ is also expressed in terms of the sum (A.7) with $r = 2$ after a dimensional regularization of the integral with respect to a continuous variable. The result is[101] $\varepsilon = -G/3a^4$, where $G = \beta(2) \approx 0.915$ is the Catalan constant.

[1)]Zero-point vibrations in a solid are manifested as a consequence of the fluctuations in the positions of the atoms which make up the string: $\overline{y}^2 \neq 0$. Light is scattered by these fluctuations: if the fluctuations are large, $\overline{y}^2 \approx d$, where d is the distance between atoms, a crystal cannot form (this is the typical situation for light atoms, e.g., He), etc.

[2)]This redefinition of the energies of course does not eliminate the actual effects of the zero-point oscillations of vacuum, e.g., the Lamb shift. Interestingly, Feynman[96] has given a qualitative explanation of the Lamb shift involving a redefinition of the vacuum energy of a closed region.

[3)]In the case $\xi \neq \xi_c$, even for a massless field, a singularity $\Delta \sim (\xi - \xi_c)[x \pm (a/2)]^{-4}$ appears,[15] and it should also be assigned to the wall energy.

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