Types of transformations used in physics, and particle "exchange"

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Usp. Fiz. Nauk **152**, 521–529 (July 1987)

An active transformation of a physical system implies its motion. A passive transformation of the system is a change in the method of describing it. An analogy transformation means transition to another physical system, similar in some respect to the original one. In some cases transformations of one type may imitate those of another. The operation of particle permutation in quantum mechanics implies a passive transition to describing the same state of the system by a different method of introducing particle numbering. The problems of transitions from describing identical particles to describing different ones and that of "explaining" the probabilistic meaning of wave functions statistically are touched upon.

CONTENTS

1. INTRODUCTION

Various types of transformations are considered or used both in physics discussions and in calculations. In many cases the content and object of these transformations are not sufficiently clearly defined, thus becoming a source of confusion. And yet the transformations, just as any mathematical procedure, can be a reliable instrument in the hands of the researcher, but only when it is totally clear to what purpose and how they are applied.

The term "transformation" is used in physics in a substantially wider context than a transformation of some set into itself. Therefore, the theorem that mappings of any set onto itself form a group is, generally speaking, not applicable to all transformations considered in physics. Consequently, it is useful to mention the mathematical nature of the corresponding transformations.

Almost thirty years ago Bargman[1] introduced the terms "active transformation" and "passive transformation." Since then these terms have been used by many authors. It seems, however, that they are used in various contexts, some of them differing from the original definition. It is advisable to classify the transformations used in physics, and on this basis, in particular, to explain terms such as "particle exchange" or "particle permutation," and, in this context, using the example of the probabilistic meaning of wave functions, touch upon the problem of explaining some concepts by others.

2. ACTIVE TRANSFORMATIONS

According to the original definition,[1] an active transformation of a physical system is its *motion*, i.e., a variation in its characteristics under the effect of some internal or external interactions. We are dealing with motion of the same physical system, evolving in time, and considered either as approximately isolated, or coupled with other systems and treated from the point of view of the same reference system and by the same method of description. Strictly speaking, there exists only one example of active motion—the evolution of the universe. Most of the time we treat or consider active motions of systems of bodies or particles, weakly coupled with other systems.

Active motions are continuous. No jumps or reflections are included. The result of active motion of a body, occurring between some initial and final times, can lead to a configuration which could have been obtained by a jump at the initial moment of time if such a jump were feasible, but this configuration refers in reality to another, subsequent moment of time, so that in the evolution space of the system,[2] parameterized by independent system coordinates and time, these two positions correspond to points shifted not only in coordinates, but also in time.

It is important to note that the active motions of arbitrary physical systems do not form groups, but only partial semigroupoids.[1)] For time displacements ($t_1 \to t_2$) (simultaneous variation is implied of all time-dependent characteris-

 0038-5670/87/070644-05$01.80

tics) the products $(t_1 \to t_2) \times (t_3 \to t_4)$ are defined only for $t_3 = t_2$, and in this case provide the transition $(t_1 \to t_4)$, and no single displacement $(t_1 \to t_2)$ has an inverse [besides the identity $(t \to t)$]. This is so because macroscopic systems (at any rate from multimolecular to planetary level) vary irreversibly, while the isolation of microscopic systems from macroscopic ones is an approximate idealization, whose meaning consists of the fact that in considering some interactions occurring during a quite short time interval, such as collisions, one can ignore the interaction of particles with macroscopic systems.

3. PASSIVE TRANSFORMATIONS

All kinds of *variations of methods of describing* physical systems refer to passive transformations. It is well known that each physical state, motion, or effect admits many different descriptions. Examples of variations in descriptions are canonical transformations of phase space in mechanics, including transitions to different reference systems, both inertial and noninertial, transitions to different representations in quantum mechanics, or changes of gauge in electrodynamics. Sometimes the same concepts are associated with different terms, or the description of effects is translated into a different language (in the terminological or linguistic sense, in the sense of notation or machine language). All these changes are examples of passive transformations.

Each passive transformation has an inverse, but they also do not always form a group, only a partial groupoid. Only in special cases can the transformations be combined in an arbitrary order. Thus transitions to describing physical systems from other inertial reference systems, obtained from an original one by a shift in coordinates or time, by spatial rotation or a Galileo or Lorentz transformation, form a group, since products of these transformations are defined in an arbitrary sequence. Passive transformations can be either continuous or discrete, including reflections. Several descriptions of the same effect can be realized simultaneously, for example by different observers, a comparison of which is a passive transformation of descriptions.

The aforementioned variety of descriptions of each phenomenon precludes the identification of our knowledge concerning a phenomenon with any of its descriptions. More accurately: the image of a phenomenon (an objective, although perhaps also a relative one) is the class of equivalence of its descriptions. Differently stated, our knowledge concerning each phenomenon is relative not only because no measurement can be carried out with absolute accuracy and completeness, but also because its description always contains both elements reflecting the effect, and elements related only to the method of description. In particular, the quantum-mechanical state vector is the class of equivalence of wave functions (projections of this vector) in all possible representations.

The variety of methods of describing physical systems makes it possible to select for each problem the representation most suitable for its solution. The clarity of terminology and the appropriateness of notation are an important pedagogical tool, but the essence of the phenomena described is independent of the passive transformations.

4. ANALOGY TRANSFORMATIONS

The active and passive transformations of physical systems do not exhaust all transformations used in physics. There exists no generally accepted term for transformations not included in one of the types considered above, and we call them analogy transformations. They consist of a transition from considering some physical system to considering a different system, similar in some respect to the original one, i.e., possessing some similar features with it, but at the same time also containing different characteristics, less essential in the context under consideration.

An analogy transformation for some physical transformation, in particular, is a qualitative, or, if possible, a quantitative mental construction of a model of the original system, reflecting, on the one hand, all the essential properties of the system in the aspect considered and of the interactions in it, and sufficiently simplified, and on the other hand, capable of undergoing a mathematical analysis. If these two requirements can be combined, the model constructed serves for analysis of experiments (finding of parameters), prediction of results of new experiments, and use of properties of the system for technological applications.

An example of an analogy transformation is the transition from considering a right-handed screw to considering a left-handed one, or a transition from a crystal rotating polarized light to the right to studying a crystal of similar structure, but rotating the light to the left. In this case we exclude actively affecting the system, i.e., preparing a left-handed screw or a left-handed crystal from material of their right-handed counterparts, not only because this is possible only in principle and is not done at all in comparing corresponding objects, but also primarily because this is what is needed for making the indicated comparison at the same moment of time. The transition from a right-handed object to a similar left-handed one is not a passive transformation, since the transition to a different reference system or a method of description for all the coupled systems cannot remove the objective difference between them, so that a left-handed nut can not be screwed onto a right-handed screw. A reflection of the reference system is equivalent to a transition from a right-handed to a left-handed object only when one ignores the variations in the description, occurring during reflection for objects not subjected to an analogy transformation.

Another example of an analogy transformation is the so-called time reversal. E. Wigner, who first investigated in detail the application of this transformation to quantum mechanics, wrote[4]: "The term 'reversal of the direction of motion' is, apparently, more accurate, though longer, than the term 'time reversal'." Indeed, in this operation the state of motion of some physical system is contrasted with the motion of a different system, differing from the first by reversal of the directions of motion of all the particles of the system and the directions of all the angular momenta at the same moment of time. The case in which the reversed system is obtained from the original one by means of actively influencing it is excluded from consideration, since the system thus reversed would be formed in a successive moment of time, while the states of several other systems would also change. The transformation of reversal of direction of motion is equivalent to the passive transformation of time reversal for isolated microsystems only.

The analogy transformations do not always form groups, since there is no inverse for the mapping of a complex object on a simpler one, and the products of transformations are not always defined in an arbitrary sequence.

5. COMPARISON OF TRANSFORMATIONS

The definitions of the three types of transformations mentioned are not always mutually exclusive. Firstly, the identity transformation can refer to any of these types. Secondly, passive transformations are a special case of analogy transformations, in which a system similar to the original coincides with the original, but is treated from a different point of view.

If some characteristics of the transformed systems are not taken into account, or the transformation is applied to a bounded object, considered as isolated, it is possible that transformations of one type imitate transformations of another type. Thus, a body moving as a result of some active influence of other bodies into a new position changes its coordinates (ignoring a shift in time and motion of other bodies) in the same way, as if it were to be considered from the point of view of a new coordinate system, undergoing a displacement oppositely to the active spatial shift of the body, provided one disregards the fact that other bodies, not affected by the shift, do not change their positions in the former case, and are shifted all together at once in the latter case. Exactly the same can be said about active rotation of a body, conditionally equivalent to opposite rotation of the coordinate system, as well as about corresponding measurement pairs of body velocities, and Galileo or Lorentz transformations. The Poincaré group forms a set of passive transformations, but not of active (in the full sense of the word) motions of bodies or particles. Therefore it is untrue, as is sometimes stated, that the active and passive points of view on each transformation of the Poincaré group are completely equivalent.

Charge conjugation transformations and other internal symmetry operations of isolated microsystems are in essence analogy transformations. However, in the corresponding abstract space they have the appearance of either the result of an active rotation, or of a passive transformation, and therefore can be included in symmetry groups.

6. PARTICLE PERMUTATION

In the presentation of quantum mechanics the section on systems of identical particles is particularly difficult to explain. R. Mirman[5] investigated the reasons for this difficulty in detail, and found that none of the available textbooks contained a flawless discussion of this point. The various authors do not define the meaning of "particle permutation" or "particle exchange," and attempt to explain one of the terms "identity," "equality," and "indistinguishability" by another, while the corresponding concepts do not have noncoinciding definitions.

Considering the original and translated literature on quantum mechanics in the Russian language,[6–26] it is easily noted that, although all authors lead their readers to identical and correct equations for wave functions of systems of identical particles, the presentation of this point is lacking everywhere in completeness and consistency. All authors of the books mentioned and of the books quoted by Mirman do not provide a clear response to at least one of the two questions: 1) What kind of transformation is the "particle permutation" operation (or of what does it consist); and: 2) Does the wave function obtained by this operation describe the same state as the original one, or a different one? Many authors of the books mentioned totally avoid both problems, and more directly to the equations; the meaning of the transformations is left outside the boundaries of the text.

Using the example of a system of an electron on Earth, and an electron generated by decay of an unstable particle in another galaxy, Mirman shows that the active interpretation of particle permutation is already eliminated by the requirement of causality. In none of the books considered do the authors insist explictly on explaining particle permutation as an experimental intrusion into the system investigated, but the words "let us permute the particles" lead an insufficiently sophisticated reader specifically to this incorrect interpretation. As a result he visualizes particle permutation roughly as a permutation of identical nuts within one machine.

The transformation called particle permutation consists of comparing wave function values at points of the representation space differing by the replacement of permuted particles, and, consequently, is an *analogy transformation*, relating wave functions of systems, in which the particles play permuted roles.

On the other hand, the same transformation can be considered as a *passive* transformation, consisting of a transition to describing the same state of the same system using a different method of numbering the particles[27] or, what is the same, by a different order of axis sequence in the representation space, such as configuration space.[24]

Figure 1 shows these two points of view on the particle permutation transformation for the wave function $\psi(x_1,x_2)$ of the simplest case of one-dimensional motion of two identical fermions, with $\psi(x_2,x_1) = -\psi(x_1,x_2)$. The arrow pointing to the left and upwards denotes comparison of the values of the same wave function at different points of configuration space, while the arrow pointing to the right denotes comparison of wave function values, selected at the same point, but with different methods of introducing coordinates.

7. IDENTICAL AND DISTINCT PARTICLES

Changing coordinates in the representation space is also possible when the particles differ in some respect. In this case, however, the operators expressing the system characteristics in one set of coordinates in terms of those in another,

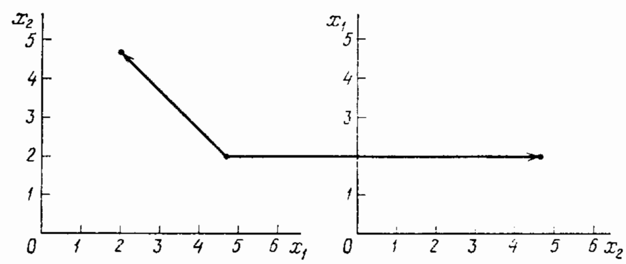

FIG. 1.

as well as the matirces transforming the wave functions, do not form a representation of the finite permutation group, but do form a representation of a finite partial groupoid,[28] where the products are defined not in an arbitrary order. Thus, for two particles the groupoid of rearrangements consists of three elements: the identity, the direct transition, and the inverse transition; the squares of each of the two transitions are undefined. The unitary irreducible representation of this groupoid is one-dimensional, contains a single arbitrary phase, and consists of unity and direct and inverse changes of the wave function phase by the indicated fixed phase. However, the change in the phase of the wave function can always be attributed to a gauge transformation [the groupoid is embedded in the U(1) group], while the value of the arbitrary phase does not affect any observable. Therefore, for a two-particle system the deviation of the groupoid of rearrangements from the permutation group is not essential. For a three-particle system the groupoid of rearrangements is of order 28, contains two arbitrary parameters, and the effect of the corresponding unitary representations of this groupoid does not reduce to the effect of unitary representations of the permutation group of three particles. The groupoid is embedded in one of the subgroups of the SU(3) group. The Racah coefficients for addition of three angular momenta also form a representation of this groupoid (the parameters, arbitrary in the general case, are expressed in terms of the angular momenta).

As to the state of a system of several noninteracting particles, a verbal paradox is indeed generated. On the one hand, if the particles do not interact (or even cannot interact due to the nonsatisfaction of the causality conditions), their joint wave function is the product of particle wave functions, as is the case for any system of noninteracting clusters. On the other hand, if this state of the system is assigned a wave function symmetrized in the representation variables, this does not result in observable consequences, since these particles interact with any third system independently of each other.

In the example of particles in different galaxies, provided by Mirman, the wave function cannot, naturally, be symmetrized until particles begin to interact, but the use in place of it of a symmetrized function does not lead to a change in any observable effects. Similarly one resolves apparent paradoxes with symmetrized systems of neutral K-mesons, provided by the authors of Ref. 29. A fully relativistic quantum theory of systems of particles, which allows for the description of systems both of particles interacting for a long time (exchanging infinite series of field quanta), and of noninteracting particles, will have to trace out the conversion, during the evolution process of the system, of the nonsymmetrized wave function into a symmetrized one as a result of particle interactions. It is essential that the theory must be relativistic, because under the assumption of infinite propagation velocity of the interactions systems of identical particles are symmetrized instantaneously.

The authors of Ref. 29 categorically reject the possibility of theoretical assertions not verifiable experimentally in principle. However, we have already discussed above passive transformations as transitions between different descriptions of physical systems. Clearly, an assertion of superiority of one description over another cannot be verified (though the rate of convergence of the approximations in an approximate description of the physical system can be quite different for them). Similarly, the presence or absence of certain wave function symmetry of a system of noninteracting particles is an assertion which does not have observable consequences, and therefore cannot be verified.

8. CHANGE IN TERMINOLOGY AND DEFINITION OF CONCEPTS

A change in terminology is a passive transformation. Naturally, by itself it cannot serve as a means of defining or explaining a concept the term for which is being changed. An example is the problem of "explaining" the probabilistic meaning of a wave function. One concept can explain another, either if they are defined independently, and then it is proved that they reflect one and the same entity, or that the concept being explained is a special case of a more general concept, or if the concept being explained is directly defined as a consequence of the concept providing the explanation on imposition of certain conditions.[27] Nothing like this exists in attempts of explaining the probabilistic meaning of a wave function in terms of the quantum ensemble concept,[18,30] since there exist no distinct definitions of these two concepts.

The fact that to find *experimentally* a probability distribution, in particular to measure the absolute value of a wave function, one must carry out many observations, does not provide any justification for defining the concept of probability in terms of statistics. The latter is essentially the opinion regarding the foundations of the theory of probability, which at one time was expounded by R. von Mises. This point of view has never been able to be carried through consistently, and such attempts became only a chapter in the history of probability theory, when in 1933 A. N. Kolmogorov formulated an axiomatic definition of the concept of the probability field (see Ref. 31). Later he wrote[32]: "The existence of an axiomatized probability theory rids us of the temptation to "define" probability by methods, claiming to combine their direct scientific persuasiveness with accommodation to constructing a formal rigorous mathematical theory on their basis . . . To this sort of definitions belongs the definition of probability as a frequency limit with unbounded increase in the number of trials. The assumption of probabilistic nature of trials, i.e., of a tendency of frequencies to group around a constant value, by itself is true (as is the assumption of randomness of any phenomenon) only upon the satisfaction of certain conditions, which cannot hold indefinitely and with unrestricted accuracy. Therefore, the exact transition to the limit as the frequency tends to the probable value cannot have any real significance. The formulation of the principle of frequency stability in turning to such a limiting transition requires determination of admissible methods of finding infinite sequences of trials, which can also be a mathematical fiction." [2)] The opposite opinion concerning the possibility of defining probability as a frequency limit, in particular, is reflected in definition (1.1) of Landau and Lifshitz[33]; however, the subsequent exposition in that book is not based on this definition.

A source of confusion of the concepts here is the use of the word "statistics" in two different meanings. Many authors, particularly nonmathematicians, refer as statistical to phenomena in which some random factors participate, as well as the theory of these effects (statistics in the wide sense of the word). Thus arose the term "statistical physics." Statistics in the wide sense of the word (for example, the theory of ensembles) is a synonym of the concept of fields of random events and their corresponding probabilities. In the case of quantum effects the latter *are defined* by the wave functions.

Statistics in the narrower sense of this concept is a set of methods (and their theory), making it possible to obtain information concerning some probability distribution on the basis of a finite number of appropriate experiments. From the necessity of employing statistics (in the narrow sense) for estimating parameters of a probability distribution (see, for example, Ref. 34) it does not at all follow that probability itself must or can be "explained" by statistics. Therefore, one should speak not about the statistical, but about the *probabilistic* meaning of a wave function and about *statistical measurements* required to establish its absolute value from experiment. If the number of experiments or their accuracy are not high, the predominance of elements of relative truth over elements of absolute truth does not justify regarding the experimental results as nonobjective (unlike the statement in Ref. 18, p. 59).

The author is grateful to I. A. Yakovlev and G. G. Samburova for useful discussions on the topic of this paper.

[1)]As is well known (see, for example, Ref. 3), a set G of elements $e, a, b, c, \ldots$, in which only the one composition operation $a\cdot b$ of two elements a and b, leading to an element of the same set is defined, is a group if: 1) the composition operation is defined for any pair of elements of the set in an arbitrary order, $a\cdot b\in G$, and $b\cdot a\in G$; 2) the composition operation is associative, i.e., $(a\cdot b)\cdot c = a\cdot(b\cdot c)$; 3) there exists a left identity e in G, i.e., $e\cdot a = a$ for all a in G; 4) each element $a\in G$ can be inverted, i.e., there exists an element $a^{-1}\in G$, such that $a^{-1}\cdot a = e$. Various generalizations of the concept of a group are obtained by weakening one or more of these conditions. In particular, if condition 4) is satisfied not for all $a\cdot G$, then G is a semigroup. If condition 1) is weakened, i.e., the composition operation of elements of G is defined not for all pairs, then G is called a partial groupoid; at the same time condition 2) can be violated, in particular, $(a\cdot b)\cdot c$ may be defined, but not $a\cdot(b\cdot c)$. If these two types of weakened conditions are encountered simultaneously, a partial semigroupoid is obtained.

[2)]To reduce this thought ad absurdum, it is said that in this approach it is allowed, for example, to define a geometric point on a plane as the limit of spots of chalk, imprinted on a blackboard as the pressure on the piece of chalk is reduced.

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Translated by Nathan Jacobi