General properties of electromagnetic response functions

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The present state of the general theory of response functions of material media, describing the universal properties of these functions that characterize all types of medium, is surveyed. Topics covered include recent results on the range of admissible values of static permeability, the form of the Landau functional for the electrodynamic ordering of a medium, and the electrodynamics of a medium interacting with a magnetic monopole. Applications to high-temperature superconductivity, anomalous diamagnetism, and monopole detection are discussed.

CONTENTS

1. INTRODUCTION

Response functions describing the reaction of a material medium to an external electromagnetic field play a fundamental part in the formalism of macroscopic electrodynamics. They are intimately related to the basic electrodynamic parameters of a medium, namely, its permittivity and magnetic permeability (see Refs. 1 and 2), and serve as a concentrated source of information on the effect of interactions with the medium and on the structure and properties of the medium itself. It is precisely in the character of the response functions (above all, their dependence on frequency ω and wave vector $\mathbf{k}$ of the external field[1)]) that the individual properties of the medium are found to manifest themselves.

On the other hand, response functions have a number of general properties, common to all media. They follow from universal relationships (dispersion relations, sum rules, and several inequalities) that are derived directly from the general requirements of causality, stability of the medium, its symmetry properties, and so on, without the use of specific models of the medium. Such relationships are important if only because they are among the relatively small number of exact results of many-body theory. Their importance has increased particularly in the last few years since it has become clear that the solution of many problems that are of current interest relies precisely on these general properties of response functions. At the same time, there is now little doubt that many of the propositions referring to these properties, that are widely scattered throughout the literature, are actually inexact or definitely incorrect.

The examples given below should illustrate the problems themselves as well as their connection with the general properties of response functions and the inconsistencies and contradictions that ensue from statements made in the literature.

(1) The problem of high-temperature superconductivity is how to produce a radical increase in the critical temperature of superconductors. The essential point here is that the effective static interaction between charges of the same sign in a medium must be attractive and not repulsive, as it is in vacuum. The terms of the longitudinal permittivity $\varepsilon(\omega, \mathbf{k})$, this means that subject to certain restrictions, we must have $\varepsilon(0, \mathbf{k}) < 0$ (for large k).[7–9] However, this inequality is not consistent with the widely held view that $\varepsilon(0, \mathbf{k}) \geqslant 1$ for media in equilibrium. Were the latter condition found to be true, it would mean that the threshold for the longitudinal instability of the medium, $\varepsilon(0, \mathbf{k}) = 0$, could not be reached, i.e., the medium would not crystallize, and so on.

(2) The question of anomalous diamagnetism has to do with the existence of nonsuperconducting media with an anomalously high negative magnetic susceptibility (a possible example is an ordered medium with a spontaneous current in the ground state; see Refs. 10 and 11 in this connection). However, standard analysis, based on the Landau theory[12] of phase transitions, obviates this possibility: the corresponding energy functional

$$\mathscr{E} = M^2 [2 (\mu (0, \mathbf{k}) - 1)]^{-1} + \beta M^4 - \mathbf{M} \cdot \mathbf{H}$$

leads to a response that is paramagnetic in character [$\mathbf{M}$ is the magnetization, related directly to the spontaneous current, $\mathbf{H}$ is the external magnetic field, and $\mu(\omega, \mathbf{k})$ is the magnetic permeability of the medium]. Moreover, the functional itself must describe not only the ordering process, but also small magnetization fluctuations in the unordered medium, which is in clear conflict with the actual existence of

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diamagnetic media, e.g., diamagnetic atomic gases, because the condition for the stability of a medium against the growth of such fluctuations is $\mu(0, \mathbf{k}) > 1$.

(3) The detection of the magnetic monopole is attracting considerable attention because the discovery of this particle, with the required properties, would confirm the validity of the strategy adopted in the unified theory of fundamental interactions. The detection of the jump in the magnetic flux that occurs as a result of the Meissner effect when the monopole crosses a superconducting ring is particularly relevant in this context. Direct dynamic analysis leads to the conclusion that this jump should occur. However, the Meissner effect does not appear within the framework of the generally accepted scheme for the macroscopic electrodynamics of the monopole. The point is that, according to the equation div $\mathbf{B} = 4\pi\tilde{\rho}$, where $\mathbf{B}$ is the magnetic induction and $\tilde{\rho}$ the magnetic charge density, the longitudinal field of the monopole does not "feel" the medium (whereas the transverse field vanishes, as in ordinary electrodynamics, together with the source velocity).

The above three examples (to be examined in greater detail in Sections 6, 8, and 10, respectively) already illustrate the necessity for a systematic and consistent analysis of the general properties of response functions that would answer questions relating to the range of admissible values of the static permeability of the medium, the form of the functional of the Landau theory in the language of response functions, and the response functions in monopole electrodynamics. This analysis is given below.[2)] In relation to the above problems, it removes all the above contradictions and answers the associated questions.

In the analysis given below, we shall confine our attention to the important special case of unordered, homogeneous, isotropic, and nongyrotropic equilibrium media, whose properties are invariant under translations and reflections of space and time, and also under rotations (in the rest frame of the medium as a whole[3)]). Perturbations of the medium are assumed small (linear electrodynamics). The medium is taken to be nonrelativistic to the extent to which this is consistent with the existence of magnetism.

The above restrictions are particularly convenient because they enable us to transform to the Fourier components of physical quantities. For a real quantity $A(t, \mathbf{x})$,

$$A^*(\omega, \mathbf{k}) = A(-\omega, -\mathbf{k}). \qquad (1.1)$$

We now introduce the longitudinal and transverse components (subscripts l and t, respectively) of the vector quantity $\mathbf{V}$:

$$\mathbf{V}_l = \mathbf{k}(\mathbf{k}\cdot\mathbf{V})k^{-2}, \qquad \mathbf{V}_t = \mathbf{V} - \mathbf{V}_l. \qquad (1.2)$$

The longitudinal and transverse components of physical quantities, and the equations considered below, will often be described as "electric" or "magnetic" with all their obvious imprecision. Operators corresponding to physical quantities will be taken in the Heisenberg representation. Accordingly, the expectation value of an operator $\hat{A}$ will be

$$A \equiv \langle\hat{A}\rangle = \mathrm{Sp}\,(\hat{R}\hat{A}), \qquad (1.3)$$

where $\hat{R}$ is the density matrix of the medium with the interaction turned off. The expectation value (1.3) is the result of statistical and quantum-mechanical averaging without smoothing over a small volume (see footnote 1).

Since we shall not mention the laws of Coulomb and Ampere, we shall use Heaviside units, which means that the factor 4π will disappear from the Maxwell equations (to transfer back to the usual units, the square of the charge in the expressions for observed quantities must be multiplied by this factor). The velocity of light, the Boltzmann and Planck constants, and the normalizing volume are all assumed equal to unity. The symbol d^3k means $d\mathbf{k}/(2\pi)^3$.

2. MAXWELL EQUATIONS AND CONSTITUTIVE RELATIONS

The electromagnetic field in a medium is described by the electric field $\mathbf{E}$ and magnetic induction $\mathbf{B}$, i.e., by the average values of the microscopic electric and magnetic fields. The vectors $\mathbf{E}$ and $\mathbf{B}$ have a direct physical meaning: they appear in the Lorentz force $e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ acting on a classical test particle. They also satisfy the Maxwell equations

$$\begin{aligned}
&\mathrm{curl}\,\mathbf{B} - \dot{\mathbf{E}} = \mathbf{j} = \mathbf{j}^e + \mathbf{j}^i \quad &(a),\\
&\mathrm{div}\,\mathbf{E} = \rho = \rho^e + \rho^i \quad &(b),\\
&\mathrm{curl}\,\mathbf{E} + \dot{\mathbf{B}} = 0 \quad &(c),\\
&\mathrm{div}\,\mathbf{B} = 0 \quad &(d),
\end{aligned} \qquad (2.1)$$

where ρ, $\mathbf{j}$ are the total charge and current densities (the indices e and i label external quantities and those induced in the medium by the external fields, respectively). They are all related by the continuity equations $\dot{\rho} + \mathrm{div}\,\mathbf{j} = 0, \ldots$, which enable us to ignore the longitudinal current components [see (1.2)].

Instead of ρ^i, $\mathbf{j}^i$, it is common to introduce the electric induction $\mathbf{D}$ and magnetic field $\mathbf{H}$, so that (2.1a) and (2.1b) can be rewritten in the form

$$\mathrm{curl}\,\mathbf{H} - \dot{\mathbf{D}} = \mathbf{j}^e \quad (a), \qquad \mathrm{div}\,\mathbf{D} = \rho^e \quad (b). \qquad (2.2)$$

In contrast to $\mathbf{E}$ and $\mathbf{B}$, the quantities $\mathbf{D}$ and $\mathbf{H}$ do not have direct physical meaning (this does not apply to $\mathbf{D}_l$ and the static $\mathbf{H}$) because the transformation $\mathbf{D}\to\mathbf{D} + \mathrm{curl}\,\mathbf{N}$, $\mathbf{H}\to\mathbf{H} + \dot{\mathbf{N}}$ with arbitrary $\mathbf{N}$ does not alter the form of (2.2).[13]

The Maxwell equations contain a number of redundant unknowns and must be complemented with the constitutive relations that contain information about individual properties of the medium. The latter usually relate ρ^i, $\mathbf{j}^i$ (or $\mathbf{D}$, $\mathbf{H}$) and the fields $\mathbf{E}$, $\mathbf{B}$. The structure of these constitutive relations is determined by the symmetry properties of the medium (see Section 1). In linear electrodynamics,

$$\rho^i = (1-\varepsilon)\,\mathrm{div}\,\mathbf{E}, \qquad \mathbf{j}^i_t = (\tilde{\varepsilon} - 1)\,\dot{\mathbf{E}}_t + \left(1 - \frac{1}{\tilde{\mu}}\right)\mathrm{curl}\,\mathbf{B} \qquad (2.3)$$

or

$$\mathbf{D}_l = \varepsilon\mathbf{E}_l, \qquad \mathbf{D}_t = \tilde{\varepsilon}\mathbf{E}_t, \qquad \mathbf{H} = \frac{1}{\tilde{\mu}}\mathbf{B}. \qquad (2.4)$$

The quantities ε, $\tilde{\varepsilon}$, and $\tilde{\mu}$ that parametrize these equations are, in general, integral operators acting in space and time (in the Fourier components, they are functions of ω and $\mathbf{k}$). Only two of them are independent. They correspond to the two types of field in the medium, i.e., the longitudinal field $\mathbf{E}_l$ and one [because of the strict relation (2.1c)] transverse field $\mathbf{E}_t$ or $\mathbf{B}$. The quantities $\tilde{\varepsilon}$ and $\tilde{\mu}$ have no independent meaning and can be varied arbitrarily provided the following quantity (normalized to unity in the absence of the medium) remains constant:

$$\eta = \left(\frac{k^2}{\tilde{\mu}} - \omega^2 \tilde{\varepsilon}\right)(k^2 - \omega^2)^{-1}. \tag{2.5}$$

This follows from the fact that it is possible to regroup the terms in the second relation in (2.3) [see (2.1c)], or from the above-mentioned abiguity of **H** and **D**.

Accordingly, there is a number of equivalent forms of constitutive relations. The two most widely used correspond to the choice

$$\tilde{\varepsilon} = \varepsilon, \quad \tilde{\mu} = \mu \tag{2.6}$$

and

$$\tilde{\varepsilon} = \varepsilon_t = \varepsilon + \left(1 - \frac{1}{\mu}\right)\frac{k^2}{\omega^2}, \quad \tilde{\mu} = 1, \tag{2.7}$$

where ε is the usual (longitudinal) permittivity, ε_t is the transverse permittivity, and μ the magnetic permeability. All such constitutive relations contain the single longitudinal parameter of the medium ε and differ by the form of the transverse parameter [for (2.6), this is μ and, for (2.7), ε_t].

The most natural and convenient transverse parameter of the medium is the quantity given by (2.5). It is related to μ and ε_t by

$$(k^2 - \omega^2)\,\eta = \frac{k^2}{\mu} - \omega^2 \varepsilon = k^2 - \omega^2 \varepsilon_t, \tag{2.8}$$

and is equal to $1/\mu$ for $\omega = 0$ and to ε_t for $\omega \to \infty$. In contrast to ε_t, the quantity η does not have a nonphysical singularity at $\omega = 0$ (see Ref. 5) and, in contrast to μ, it does have a direct physical meaning at all frequencies.[14]

The parameters ε and η of the medium have associated with them a particular form of constitutive equations, relating ρ^i, $\mathbf{j}^i$ not to the fields **E**, **B**, as before, but to the external sources:

$$\rho^i = \left(\frac{1}{\varepsilon} - 1\right)\rho^e, \quad \mathbf{j}^i_t = \left(\frac{1}{\eta} - 1\right)\mathbf{j}^e_t. \tag{2.9}$$

From the point of view of subsequent applications, it is convenient to reduce these equations to a different form by introducing the potential φ, **A**, defined by

$$\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}, \quad \mathbf{B} = \operatorname{curl}\mathbf{A},$$

which ensure that (2.1c) and (2.1d) become identities. If we adopt the gauge $\operatorname{div}\mathbf{A} = 0$, we can replace (2.1a) and (2.1b) with the following equations for the potentials:

$$\Delta\varphi = -\rho, \quad \Box\mathbf{A} = \mathbf{j}_t. \tag{2.10}$$

The analogous equations

$$\Delta\varphi^e = -\rho^e, \quad \Box\mathbf{A}^e = \mathbf{j}^e_t \tag{2.10'}$$

determine the external potentials φ^e, $\mathbf{A}^e$, produced by the same external sources in vacuum. The constitutive relations can be expressed in terms of these potentials [see (2.9)]

$$\rho = \frac{k^2 \varphi^e}{\varepsilon(\omega, \mathbf{k})}, \quad \mathbf{j}_t = \frac{(k^2 - \omega^2)\,\mathbf{A}^e}{\eta(\omega, \mathbf{k})}, \tag{2.11}$$

which are distinguished by simplicity, lack of ambiguity, and clear physical meaning. Another important advantage of these expressions is discussed in Section 3.[4]

3. ELECTROMAGNETIC RESPONSE FUNCTIONS

The constitutive relations were introduced above as equations complementing the Maxwell equations, so that together they form a closed system. Their physical meaning, on the other hand, is that they describe the reaction of the medium to an external electromagnetic field. This fact enables us to expose many general properties of the quantities that appear in the constitutive relations.

Suppose that the medium has been subjected to a weak external influence $\delta\mathfrak{J}(t, \mathbf{x})$ with the result that a parameter $\mathfrak{A}$ of the medium acquires the increment

$$\delta\mathfrak{A}(t, \mathbf{x}) = \int dt'\, d\mathbf{x}'\, \mathfrak{R}(t - t', \mathbf{x} - \mathbf{x}')\, \delta\mathfrak{J}(t', \mathbf{x}'), \tag{3.1}$$

or, in terms of the Fourier components,

$$\delta\mathfrak{A}(\omega, \mathbf{k}) = \mathfrak{R}(\omega, \mathbf{k})\, \delta\mathfrak{J}(\omega, \mathbf{k}). \tag{3.2}$$

In words:

(result of influence) = (response function) × (influence)

The constitutive relations introduced above have the same form (for a weak influence, they can be written as relationships between variations[5]).

For reasons that will become clear later, we subject the quantities in (3.1) and (3.2) to the following conditions:

A. The influence $\mathfrak{J}$ must be completely arbitrary; it must be capable of assuming any predetermined value, and its source must not experience the reaction of the medium.

B. The interaction between the external influence and the medium must be described by

$$\hat{H}_{\text{int}} = -\int d\mathbf{x}\, \hat{\mathfrak{A}}\, \delta\mathfrak{J}, \tag{3.3}$$

where $\hat{\mathfrak{A}}$ is an operator that depends on the dynamic variables of the medium [the quantity $\mathfrak{A}$ in (3.1) and (3.2) is equal to $\langle\hat{\mathfrak{A}}\rangle$], and $\mathfrak{J}$ is a given quantity.

Condition A does not predetermine the dynamic meaning of the quantities $\mathfrak{A}$ and $\mathfrak{J}$, but the stronger condition B gives them the meaning of generalized coordinate and force, respectively. From now on, we shall use the phrase "response function" to describe the quantity $\mathfrak{R}$ in (3.1) and (3.2) that satisfies at least condition A. A response function satisfying condition B can be referred to as the generalized susceptibility, in the absence of a better name (usually, the word "susceptibility" is used for the part of the response function that vanishes as $\omega \to \infty$).

Let us now explain which particular parameters of the medium that appear in the constitutive relations can be regarded as the response functions and generalized susceptibilities. In any case, condition A is satisfied by influences due to external sources ρ^e, $\mathbf{j}^e$, whose magnitudes are quite arbitrary. This enables us to identify the constitutive relations (2.9) and (2.11) with (3.1) and (3.2).[6] If, on the other hand, (3.3) is taken to be the well-known Hamiltonian for the interaction between the medium and external fields

$$\hat{H}_{\text{int}} = \int d\mathbf{x}\,(\hat{\rho}\,\delta\varphi^e - \hat{\mathbf{j}}_t \cdot \delta\mathbf{A}^e) \tag{3.4}$$

($\hat{\rho}$ and $\hat{\mathbf{j}}$ are, respectively, the total charge and total current density operators), the quantities that appear in the constitutive relation (2.11) must also satisfy condition B. This enables us to write

$$\mathfrak{J} = -\varphi^e,\ A^e, \quad \hat{\mathfrak{A}} = \hat{\rho},\ \hat{j}_t, \quad \mathfrak{R} = -\frac{k^2}{\varepsilon},\ \frac{k^2 - \omega^2}{\eta}. \tag{3.5}$$

where the quantities $\mathfrak{R}$ may be looked upon as the response functions and generalized susceptibilities.

It is significant that, in contrast to the external sources, the total current and charge densities ρ, $\mathbf{j}$ cannot be regarded as arbitrary and given in advance because they contain a contribution due to the medium itself. This prevents us from inverting the constitutive relations (2.11), reading them from right to left, or considering ε and η themselves as the response functions. However, it is important to note that this conclusion is not valid in the special case of a spatially homogeneous influence with $\mathbf{k}=0$ (or, more precisely, with $k\sim 1/L$, where L is a macroscopically large dimension of a specimen of the medium).

Let us then turn to the question of the physical implementation of the influences acting on the medium. The influences defined by (3.5) can be readily produced by placing a point charge, or a circuit carrying a given current, in the interior of the medium, and selecting Fourier components with arbitrary $\mathbf{k}$ (Fig. 1). On the other hand, when $\mathbf{k}=0$, we can insert the specimen of the medium into a capacitor with given charge on the plates, or inside a solenoid with given current flowing through it (Fig. 2). The same devices can be used to produce influences due to total charge or current. The first of these influences is produced when the capacitor is connected across a battery that maintains a given potential difference or, according to (2.10), a given total charge density. The second requires the use of a superconducting solenoid (more precisely, a superconducting cylinder with azimuthal currents) that produces a quantized magnetic flux, equal to the line integral of the potential $\mathbf{A}$ over the perimeter of the aperture in the superconductor. In this device, the potential $\mathbf{A}$ has a given value or, according to (2.10), the total current density has a given value. External charges flowing into the plates or away from them, or external currents flowing around the aperture in the semiconductor (Fig. 3) then play the part of $\mathfrak{A}$ (the result of the influence).

We emphasize that it is only for $\mathbf{k}=0$ that the devices stabilizing the values of ρ, $\mathbf{j}$ (battery or superconductor) can be taken outside the medium without distorting its structure and properties. However, when $\mathbf{k}\neq 0$, such devices must be introduced into the medium, and the necessary number of them increases with increasing $\mathbf{k}$, so that the new medium is rather different from the original. However, the properties of the medium will be distorted even if we use a specimen with small linear dimensions and large enough $k\sim 1/L$ (the distortion being due to surface effects whose contribution increases with $\mathbf{k}$).

Of course, the influences due to ρ, $\mathbf{j}$ that we have discussed must satisfy only condition A: external sources playing the part of $\mathfrak{A}$ cannot be described by an operator [see (3.3)] and their dependence on the state of the medium appears only at the level of expectation values due to boundary

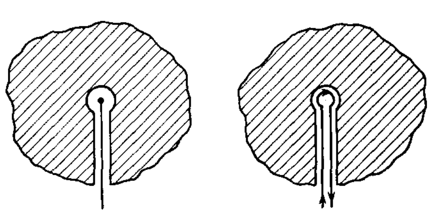

FIG. 1.

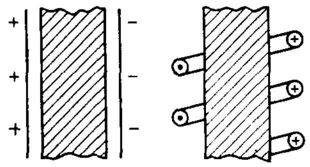

FIG. 2.

conditions on the surface of the specimen. It follows that, for the purposes of identification with (3.1) and (3.2), we can use the constitutive relations (2.9) in the form $\rho^e=\varepsilon\rho$, $\mathbf{j}_t^e=\eta\mathbf{j}_t$. This yields

$$\mathfrak{I}=\rho,\ \mathbf{j}_t,\quad \mathfrak{A}=\rho^e,\ \mathbf{j}_t^e,\quad \mathfrak{R}=\varepsilon(\omega,\ 0),\ \eta(\omega,\ 0). \tag{3.6}$$

Here, the quantities $\mathfrak{R}$ have the meaning of only the response functions, but not of the generalized susceptibilities.

The response functions in macroscopic electrodynamics are exhausted by the quantities $\mathfrak{R}$ in (3.5) and (3.6). The quantities ε and η (for arbitrary k), μ, and ε_t are not found among the response functions and, even less so, among the generalized susceptibilities. To conclude this Section, we note that the questions we have examined here are taken up in the book by Pines and Nozieres[16] and are developed in Refs. 14 and 17–20.

4. CAUSALITY AND DISPERSION RELATIONS

Response functions must satisfy the condition of causality, which specifies their dependence on frequency. In its simplest formulation, this condition states: "Cause always precedes in time its consequence" or, more concisely: "The future has no effect on the past." The condition is in agreement with general human experience as well as with the results of experiments and observations involving scales ranging from subnuclear to cosmic.[7)]

Let us now apply the causality condition to (3.1), taking $\mathfrak{I}$ as the cause and its result $\mathfrak{A}$ as the consequence. In accordance with condition A, the quantity $\delta\mathfrak{I}$ is arbitrary and can assume any predetermined value. Suppose that it is equal to zero for $t<t_0$ (t_0 is an arbitrary instant of time), but is otherwise arbitrary. The causality condition then demands that $\delta\mathfrak{A}$ must vanish for $t<t_0$ and, since $\delta\mathfrak{I}$ is arbitrary, this is possible only provided

$$\mathfrak{R}(t,\ \mathbf{x})=0,\quad t<0, \tag{4.1}$$

which can serve as a quantitative expression of the causality principle.

When (4.1) is satisfied, the Fourier component of the response function

$$\mathfrak{R}\ (\omega,\ \mathbf{k})=\int dt\, d\mathbf{x}\, \mathfrak{R}(t,\ \mathbf{x})\exp\left[-i\ (\mathbf{k}\cdot\mathbf{x}-\omega t)\right]$$

is given by an integral with respect to time that converges in the upper half-plane of the frequency ω, regarded as a complex variable. By virtue of the well-known theorem on converging Fourier integrals, this means that $\mathfrak{R}(\omega,\mathbf{k})$ is analytic in this region. Next, (1.1) leads to the following relations for an isotropic medium:

$$\mathrm{Re}\ \mathfrak{R}\ (\omega,\ \mathbf{k})=\mathrm{Re}\ \mathfrak{R}\ (-\omega,\ \mathbf{k}),$$

$$\mathrm{Im}\ \mathfrak{R}\ (\omega,\ \mathbf{k})=-\mathrm{Im}\ \mathfrak{R}\ (-\omega,\ \mathbf{k}). \tag{4.2}$$

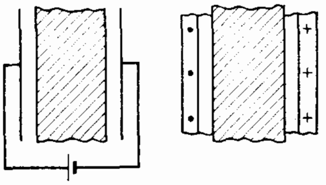

FIG. 3.

The Kramers-Kronig-type dispersion relations follow from this in the usual way (see Refs. 16 and 23):

$$\mathfrak{R}(\omega,\ \mathbf{k})=\mathfrak{R}(\infty,\ \mathbf{k})+\frac{1}{\pi}\int_0^\infty d\zeta^2\,\mathrm{Im}\,\mathfrak{R}(\zeta,\ \mathbf{k})\,[\zeta^2-(\omega+i\delta)^2]^{-1}, \tag{4.3}$$

where $\mathfrak{R}(\infty,\mathbf{k})$ is the limit of $\mathfrak{R}(\omega,\mathbf{k})$ as $\omega\to\infty$ and is a real number[8)] by virtue of (4.2).

The fact that condition A is necessary for the derivation of the analyticity of the response function can be illustrated by the following simple example. Suppose that the quantity $\delta\mathfrak{J}(\omega,\mathbf{k})$ is not arbitrary and has a zero in the upper half-plane of ω. It is then clear from (3.2) that the response function can have a pole at the same point, which does not prevent the vanishing of $\delta\mathfrak{A}(t,\mathbf{x})$ for $t<t_0$.

We emphasize that the derivation of the dispersion relations relies not only on the causality condition, but also on the condition that the medium is stable. The latter was implicitly assumed above and forbids an exponential increase in $\delta\mathfrak{A}$ and $\mathfrak{R}$ with time, which enables us to transform to the Fourier component. On the other hand, these conditions alone do not lead to restrictions on the analytic properties of the response function. For example, if it has a pole at $\omega=i\Gamma$, $\Gamma>0$, integration with respect to ω in the Fourier integral for $\mathfrak{R}(t,\ \mathbf{x})$ gives a stable but noncausal behavior of $\theta(-t)\exp(\Gamma t)$ when the contour is taken along the zero axis of ω. On the other hand, it produces an unstable but causal behavior of $-\theta(t)\exp(\Gamma t)$ when contour C of Fig. 4 is chosen.[21,25] Here, $\theta(t)$ is respectively equal to zero and unity for t smaller and greater than zero.

We now turn to the formulation of dispersion relations for the electrodynamic response functions (3.5) and (3.6), and begin with the term $\mathfrak{R}(\infty,\mathbf{k})$ in (4.3). As $\omega\to\infty$, we have the following asymptotic expressions [see Refs. 2, 16 and (2.5)]:

$$\varepsilon\to 1-\frac{\omega_p^2}{\omega^2}+\ldots,\quad \mu\to 1+\ldots,\quad \eta\to 1-\frac{\omega_p^2}{\omega^2}+\ldots, \tag{4.4}$$

where ω_p^2 is the square of the plasma frequency (see below). The significance of (4.4) is that the medium does not succeed in reacting to rapid external influences.

When (4.4) is taken into account, the dispersion relations have the following form in the electric case:

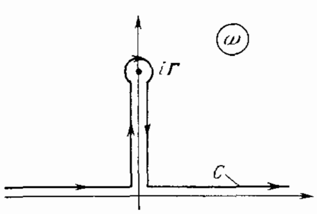

FIG. 4.

$$\varepsilon^{-1}(\omega,\ \mathbf{k})=1+\frac{1}{\pi}\int_0^\infty d\zeta^2\,\mathrm{Im}\,\varepsilon^{-1}(\zeta,\ \mathbf{k})\,[\zeta^2-(\omega+i\delta)^2]^{-1}, \tag{4.5}$$

$$\varepsilon(\omega,\ 0)=1+\frac{1}{\pi}\int_0^\infty d\zeta^2\,\mathrm{Im}\,\varepsilon(\zeta,\ 0)\,[\zeta^2-(\omega+i\delta)^2]^{-1}. \tag{4.5'}$$

In the static limit,

$$\varepsilon^{-1}(0,\ \mathbf{k})=1+\frac{2}{\pi}\int_0^\infty \frac{d\zeta}{\zeta}\,\mathrm{Im}\,\varepsilon^{-1}(\zeta,\ \mathbf{k}), \tag{4.6}$$

$$\varepsilon(0,\ 0)=1+\frac{2}{\pi}\int_0^\infty \frac{d\zeta}{\zeta}\,\mathrm{Im}\,\varepsilon(\zeta,\ 0). \tag{4.6'}$$

In accordance with (4.2), the static values of the response functions are real. The quantity $\varepsilon(0,0)$ must be understood as the limit corresponding to $\omega\to 0$, $k\to 0$, $\omega/k\to 0$ (the well-known Drude singularity $\varepsilon\sim i/\omega$ of a conducting medium corresponds to the opposite limit $k/\omega\to 0$).

On the other hand, for $\mathbf{k}\neq 0$, the permittivity itself need not satisfy the dispersion relations. It can have a simple pole on the positive imaginary frequency semi-axis:

$$\varepsilon(\omega,\ \mathbf{k})=1+\frac{1}{\pi}\times\int_0^\infty d\zeta^2\,\mathrm{Im}\,\varepsilon(\zeta,\ \mathbf{k})\,[\zeta^2-(\omega+i\delta)^2]^{-1}-\alpha(\mathbf{k})\times[\omega^2+\Gamma^2(\mathbf{k})]^{-1} \tag{4.7}$$

with the residue

$$\omega_p^2\geqslant\alpha\geqslant\Gamma^2\left(1+\frac{2}{\pi}\int_0^\infty\frac{d\zeta}{\zeta}\,\mathrm{Im}\,\varepsilon\right)>0$$

[this inequality means that $\varepsilon(0,\mathbf{k})\leqslant 0$]. Singularities of a wider class are not possible in (4.7) because they would lead to the appearance of zeros in $\varepsilon(\omega,\mathbf{k})$ that are forbidden by (4.5).[26] The expression given by (4.7) corresponds to the violation of condition A (see Section 3) for an influence due to the total charge: the quantity $\delta\rho=\delta\rho^e/\varepsilon$ has a zero at $\omega=i\Gamma$, and the ineffectiveness of the causality condition in this case has already been mentioned above.

In the magnetic case, it is more convenient to deal not with the response function $(k^2-\omega^2)/\eta$ [see (3.5)] itself, which grows on a large circle in the complex plane, but with the quantity $1/(k^2-\omega^2)\eta$, which has the same analytic properties and is also found among response function: $\delta A=\delta j_i^e/[(k^2-\omega^2)\eta]$ (see Section 2). This leads to the dispersion relation [see (4.4)]

$$[(k^2-\omega^2)\,\eta(\omega,\ \mathbf{k})]^{-1}=\frac{1}{\pi}\times\int_0^\infty d\zeta^2\,\mathrm{Im}\,[(k^2-\zeta^2)\,\eta(\zeta,\ \mathbf{k})]^{-1}\times[\zeta^2-(\omega+i\delta)^2]^{-1}. \tag{4.8}$$

In the static limit,

$$\mu(0,\ \mathbf{k})\,k^{-2}=\frac{2}{\pi}\int_0^\infty\frac{d\zeta}{\zeta}\,\mathrm{Im}\,[(k^2-\zeta^2)\,\eta(\zeta,\ \mathbf{k})]^{-1}. \tag{4.9}$$

As far as the response function (3.6) is concerned, the fact that $\eta(\omega,\ 0)=\varepsilon(\omega,0)$ [see (2.5)] means that the corresponding dispersion relation is identical with (4.5).

5. IMAGINARY PART OF RESPONSE FUNCTIONS

The dispersion relations can be derived using condition A alone. When condition B is satisfied as well, it is possible to perform a dynamic analysis of the interaction with the medium, which yields additional information about the properties of response functions. In this Section, we shall examine dynamic questions relating to the dissipative characteristics of the medium, i.e., the imaginary parts of its response functions. Dynamic aspects of the theory of response functions will be examined further in Section 7.

When an external influence is applied to a medium (which, because of its large volume, absorbs all the radiation that can penetrate it during the interaction), the energy transferred to the medium is eventually converted into heat. The energy Q dissipated in the medium per unit time is determined by the Hamiltonian (3.3), and is equal to $\int d\mathbf{x}\mathfrak{A}\dot{\mathfrak{J}}$. Transforming to Fourier components, and using (1.1), (3.2), and (4.2), we obtain the following expression for the total dissipated energy:[9]

$$\int_{-\infty}^{\infty} dt\, Q = \frac{1}{\pi}\int_0^{\infty} d\omega\omega \int d^3k\, \mathrm{Im}\, \mathfrak{R}(\omega, \mathbf{k})\, |\mathfrak{J}(\omega, \mathbf{k})|^2. \quad (5.1)$$

In an equilibrium medium $Q \geqslant 0$ and, because $\mathfrak{J}$ is arbitrary, we have

$$\mathrm{Im}\, \mathfrak{R}(\omega, \mathbf{k}) \geqslant 0 \quad (5.2)$$

for $\omega > 0$ [and the reverse inequality for $\omega < 0$; see (4.2)].

In particular, (5.1) describes the energy lost by a charged particle traveling through the medium. Bearing in mind subsequent applications (see Section 10, below), we now give another derivation of the formula, based not on the Hamiltonian (3.3) but directly on Maxwell equations. The energy lost by a particle producing fields **E** and **B** in the medium consists of the change in the field energy and the work done on particles of the medium. This work is determined by the current $\mathbf{j}^i$ induced by the particle itself. The result is

$$Q = \int d\mathbf{x}\, \mathbf{j}^i \cdot \mathbf{E} + \frac{1}{2}\int d\mathbf{x}\, (E^2 + B^2). \quad (5.3)$$

Using (2.1a) and omitting integrals over the surface of the medium, we can readily obtain the well-known expression

$$Q = -\int d\mathbf{x}\mathbf{j}^e \cdot \mathbf{E}, \quad (5.4)$$

which agrees with the energy balance $Q = W$ that follows from (2.1) and (2.2), where the change in the total energy of the medium per unit time is

$$W = \int d\mathbf{x}\ (\mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{H}\cdot\dot{\mathbf{B}}). \quad (5.5)$$

We can readily verify, using (2.1) and (3.5), that (5.1) and (5.4) lead to identical expressions for the longitudinal and transverse losses.[10]

We now return to inequality (5.2) and apply it to the generalized susceptibilities (3.5). In the electric case, this gives

$$\mathrm{Im}\, \varepsilon(\omega, \mathbf{k}) \geqslant 0, \quad (5.6)$$

and, in the magnetic case,

$$(k^2 - \omega^2)\, \mathrm{Im}\, \eta^{-1}(\omega, \mathbf{k}) \geqslant 0. \quad (5.7)$$

We can now use (2.8) to write the last inequality in the form

$$\omega^2\, \mathrm{Im}\, \varepsilon \geqslant k^2\, \mathrm{Im}\, \mu^{-1}, \qquad \mathrm{Im}\, \varepsilon_t(\omega, \mathbf{k}) \geqslant 0. \quad (5.8)$$

All these inequalities apply to positive frequencies.

We shall use them in the next Section to find the range of admissible values of dielectric and magnetic permeabilities. In the remainder of the present Section, however, we shall use these inequalities to derive the dispersion relations for a number of quantities that are not found among the response functions. The analyticity of these quantities in the upper half-plane of frequency follows not from the physical causality condition, but from the mathematical fact that some of the response functions are not only free from singularities (poles, cuts, and essential singular points), but also have no zeros in this region. It follows that the reciprocal of this kind of response function is also analytic and satisfies the dispersion relations.[17,27]

The fact that a function satisfying the dispersion relation such as (4.3) and having an imaginary part that does not change sign has no zeros necessarily means that the sign of this imaginary part and of the term outside the integral in the dispersion relations must be the same when the terms on the right-hand side of (4.3) cannot cancel out. In point of fact, the integral in the dispersion relations is real only on the imaginary frequency axis where its sign is the same as the sign of the imaginary part of the function under investigation.

In the electric case, the sign of the imaginary part and the term outside the integral in (4.5) are different, and the quantity $1/\varepsilon$ can have a zero and the dielectric permeability a pole in the upper frequency half-plane. On the other hand, in (4.5′), these signs are the same, and the fact that $\varepsilon(\omega, 0)$ cannot have a zero follows from (4.5). In the magnetic case, the term outside the integral is generally absent from (4.8), so that the quantity $(k^2 - \omega^2)\eta$ and, at the same time, $\eta(\omega, \mathbf{k})$ are analytic in the upper frequency half-plane. If we recall the asymptotic expressions given by (4.4), we see that this leads to a new dispersion relation, namely,

$$(k^2 - \omega^2)\,\eta(\omega, \mathbf{k}) = k^2 - \omega^2 + \omega_p^2 + \frac{1}{\pi}\int_0^{\infty} d\zeta^2 (k^2 - \zeta^2)\, \mathrm{Im}\, \eta(\zeta, \mathbf{k})\, [\zeta^2 - (\omega + i\delta)^2]^{-1} \quad (5.9)$$

with the static limit

$$k^2\mu^{-1}(0, \mathbf{k}) = k^2 + \omega_p^2 + \frac{2}{\pi}\int_0^{\infty} \frac{d\zeta}{\zeta}(k^2 - \zeta^2)\, \mathrm{Im}\, \eta(\zeta, \mathbf{k}). \quad (5.10)$$

According to (2.8), this also leads to the analyticity of ε_t and to the dispersion relation

$$\varepsilon_t(\omega, \mathbf{k}) = 1 + \frac{1}{\pi}\int_0^{\infty} d\zeta^2\, \mathrm{Im}\, \varepsilon_t(\zeta, \mathbf{k})\, [\zeta^2 - (\omega + i\delta)^2]^{-1}. \quad (5.11)$$

It is also readily seen that the quantity $1/\mu$ has a pole at the same point as ε, with a residue consistent with the quantity α [see (2.8) and (4.7)]. Since the sign of the term containing the integral and of the imaginary part in (5.11) is the same [see (5.8)], the quantity $1/\varepsilon_t$, for which we can also write down the corresponding dispersion relation, is analytic.[11]

There is particular interest in the question of the validity of the dispersion relation for the complex refractive index of the medium (the Kramers-Kronig relations)

$$n(\omega) = [\varepsilon(\omega, \mathbf{k})\,\mu(\omega, \mathbf{k})]^{1/2} = \varepsilon_t^{1/2}(\omega, \mathbf{k}), \quad k = \omega n(\omega)$$

[the propagation of a photon in the medium means that (2.8) must be equal to zero]. Since ε_t is analytic and has no zeros in the upper frequency half-plane (see above), the refractive index would be an analytic function, despite the presence of the square root in its definition, were it not for the fact that the actual situation gives rise to a considerable complication of the analytic properties of $n(\omega)$. This means that a weak-enough spatial dispersion of the medium is necessary for the validity of the Kramers-Kronig relations.

Summarizing, we give a list of the parameters of the medium that are analytic in the upper frequency half-plane and satisfy the dispersion relations:

$$\varepsilon^{-1}(\omega, \mathbf{k}),\ \varepsilon(\omega, 0), \eta^{-1}(\omega, \mathbf{k}), \eta(\omega, \mathbf{k}),\ \varepsilon_l(\omega, \mathbf{k}), \varepsilon_t^{-1}(\omega, \mathbf{k}).$$

Correspondingly, the following quantities can have a pole in the same region:

$$\varepsilon(\omega, \mathbf{k})\ (\mathbf{k} \neq 0), \quad \mu^{-1}(\omega, \mathbf{k}), \quad \mu(\omega, \mathbf{k}).$$

6. LIMITS OF ADMISSIBLE VALUES OF THE STATIC PERMEABILITIES OF A MEDIUM

By combining the static dispersion relations with the inequalities for the imaginary parts of the response functions, we can obtain inequalities for the static values of these functions, and hence determine the range of their admissible values. In the electric case, combining (4.6) and (5.6), we obtain the following inequalities for arbitrary values of $\mathbf{k}$:

$$\varepsilon^{-1}(0, \mathbf{k}) \leqslant 1, \tag{6.1}$$

i.e., either $\varepsilon(0, \mathbf{k}) \geqslant 1$ or $\varepsilon(0, \mathbf{k}) \leqslant 0$ [the second possibility occurs only when the dispersion relation for $\varepsilon(\omega, \mathbf{k})$ is violated; see Section 4]. When $\mathbf{k} = 0$, the relation given by (4.6′) is more restricted and yields

$$\varepsilon(0, 0) \geqslant 1 \tag{6.2}$$

[the meaning of $\varepsilon(0,0)$ is discussed in Section 4]. Condition (6.2) is identical with the condition that the compressibility of the medium be positive.[16]

Figure 5 shows the range of admissible values of $\varepsilon(0, \mathbf{k})$ (shaded). The boundaries of this region have the following meaning (see also the discussion given below): the point $\mathbf{k} = 0, \varepsilon \to \infty$ is the limit of stability against ferroelectric ordering (spontaneous appearance of $\mathbf{D}$ for $\mathbf{E} = 0$); the interval $0 \leqslant k < \infty$, $\varepsilon = 0$ represents the same situation for the transition to the state with a charge density wave (spontaneous appearance of $\mathbf{E}$ for $\mathbf{D} = 0$); and the segment $0 \leqslant k < \infty$, $\varepsilon = 1$ corresponds to the limiting case of total vacuum. In view of the foregoing, we cannot have ferroelectrics with $\mathbf{k} \neq 0$, or media with spontaneous uniform electric field (the external charge or a pair created by it would then be accelerated and take energy away from the medium, thus contradicting the condition of initial equilibrium of the medium[28]), or media with $0 \leqslant \varepsilon(0, \mathbf{k}) \leqslant 1$, i.e., the "diaelectrics."[12]

Several examples of real media with negative $\varepsilon(0, \mathbf{k})$ [and the simultaneous violation of the dispersion relations for $\varepsilon(\omega, \mathbf{k})$] are now known. They include nonideal plasmas, strong electrolytes, certain simple metals, and so on.[10,29] They are all characterized by a strong coupling between the constituent particles (strong local-field effects). Figure 6 shows schematically the form of the function $\varepsilon(0, \mathbf{k})$ for a classical nonideal single-component plasma for different values of the interaction parameter $\alpha = e^2 n^{1/3}/T$, which is equal to the ratio of the Coulomb to thermal (kinetic) energies.[30] When $\alpha \lesssim 1$, the dielectric permittivity is greater than one. For $1 \lesssim \alpha \lesssim 170$, it is negative, reaching the limits of the admissible region $\varepsilon = 0$ for $\alpha \simeq 170$, at which point a charge-density wave is produced (crystallization of plasma) with lattice parameter $1/k_0$. The negative sign of ε in the precrystallization state of the medium follows simply from the impossibility of reaching the crystallization point $\varepsilon = 0$ other than from the admissible region $\varepsilon < 0$.

In the violated dispersion relations (4.7), the quantity $\Gamma^2(\mathbf{k})$ can be qualitatively simulated by the expression $(k^2 - k_1^2)(k_2^2 - k^2)$, which corresponds to negative $\varepsilon(0, \mathbf{k})$ in the range $k_1 < k < k_2$. In nonconducting media, k_1 has a finite value, determined by the parameter of the medium, and, as this value is passed, the quantity $1/\varepsilon$ changes sign and vanishes at $k = k_1$. However, for conductors (in particular, plasmas; see Fig. 6), the quantity k_1 is zero, which corresponds to the Debye pole $1/k^2$ in $\varepsilon(0, \mathbf{k})$ with an "incorrect" sign of the residue. Correspondingly, the positive sign of $\varepsilon(0, \mathbf{k})$ is then restored only at the point $\mathbf{k} = 0$ itself.[29]

The above analysis completely removes the contradictions mentioned in example (1) of Section 1 and, in principle, offers the possibility of a radical increase in the critical temperature for superconductivity.

Turning now to the magnetic case, we shall consider a combination of static dispersion relations (4.9), (5.10) and the inequality (5.7), which will lead us to the inequality[13]

$$\mu(0, \mathbf{k}) \geqslant \left(1 + \frac{\omega_p^2}{k^2}\right)^{-1}. \tag{6.3}$$

Negative values of magnetic permeability are forbidden, and the fact that the right-hand side of (6.3) differs from unity means that diamagnetism is possible. Figure 7 shows the range of admissible values of $\mu(0, \mathbf{k})$ (shown shaded). The

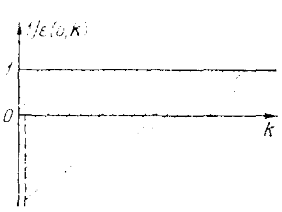

FIG. 5.

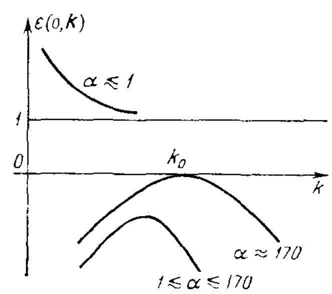

FIG. 6.

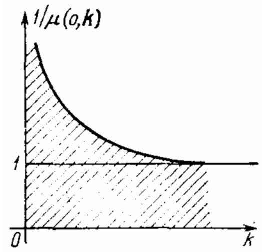

FIG. 7.

meaning of these limits is as follows. The point $\mathbf{k} = 0, \mu \to \infty$ is the limit of stability against ferromagnetic ordering (spontaneous appearance of $\mathbf{B}$ for $\mathbf{H} = 0$); the interval $0 < k < \infty$, $\mu \to \infty$ describes the same case for antiferromagnetic ordering; and the boundary $\mu = [1 + (\omega_p^2/k^2)]^{-1}$ represents the limiting state of the ideal diamagnetic substance that can be realized in a London superconductor at zero temperature.

In the system of units in which the velocity of light is c, the formula given by (6.3) contains the grouping ω_p^2/k^2c^2. Hence, in a nonrelativistic medium, although anomalous diamagnetism (low values of μ; see example 2 of Section 1) is actually possible in principle, it is confined to the relatively narrow region $k < \omega_p/c$ (see also Section 8, below[14)]). On the other hand, when k is not small, the right-hand side of (6.3) is indistinguishable from unity.

The quantity ω_p^2 in (6.3) is the square of the plasma frequency (see Section 4), i.e., it is equal to the sum of e^2n/m over all the charged-particle species (e is the particle charge, m its mass, and n the concentration). Charged particles are understood here to include bound complexes if, under the conditions under investigation (temperature, pressure, and so on), they participate in the response as a whole. Actually, in the nonrelativistic situation, we should be concerned with electrons and nuclei and, in some cases, valence electrons and ions. By referring the quantity ω_p^2 to much smaller structural components of the medium (for example, protons in nuclei), we simply introduce a more or less stringent limitation on the magnetic permeability because of the higher value of ω_p^2.

Returning now to the derivation of (6.3) [see (4.4), (4.9), and (5.10)], we note that the nonsinglevaluedness of ω_p^2 that we have been discussing corresponds to the existence of permittivities with intermediate asymptotic behavior (4.4) and different values of ω_p^2 reached successively as the frequency increases. This corresponds to the structure of $\mathrm{Im}\, \varepsilon^{-1}$, which takes the form of a set of peaks representing the successive excitation of increasingly finer structural components of the medium. At low pressures and temperatures, and relatively low values of ω and $\mathbf{k}$, the contribution of the higher peaks is negligible.

The questions examined above are discussed in Refs. 14, 17–20, 29, and 30.

7. PROPERTIES OF GENERALIZED SUSCEPTIBILITIES

The inequalities given by (6.1) and (6.3) were obtained above from the dispersion relations for the generalized susceptibilities (3.5), but they can also be derived in a purely dynamic manner, starting with general expressions, such as the Kubo formula. The generalized susceptibility is given by the variational derivative

$$\mathfrak{R} = \frac{\delta \langle \hat{\mathfrak{A}} \rangle}{\delta \mathfrak{J}} = \left\langle \frac{\delta \hat{\mathfrak{A}}}{\delta \mathfrak{J}} \right\rangle$$

for $\mathfrak{J} = 0$ [see (3.1) and (3.3)], where the operator is taken in the Heisenberg representation for the total interaction that includes (3.3), and the differentiation can be carried out under the symbol representing averaging [see (1.3)]. This operator depends on $\mathfrak{J}$ in two ways: explicitly and through the dynamic variables of the medium, whose evolution is determined by the total Hamiltonian $\hat{H}$ for the medium, which includes (3.3). Hence, $\mathfrak{R} = \mathfrak{R}_0 + \tilde{\mathfrak{R}}$ where the first term

$$\mathfrak{R}_0 (t - t', \mathbf{x} - \mathbf{x}') = \left\langle \frac{\partial \hat{\mathfrak{A}} (t, \mathbf{x})}{\partial \mathfrak{J} (t', \mathbf{x}')} \right\rangle \qquad (7.1)$$

describes the direct response of the medium ($\partial / \partial \mathfrak{J}$ is the symbol representing partial variational differentiation). On the other hand, the second term (Kubo formula)

$$\tilde{\mathfrak{R}} (t - t', \mathbf{x} - \mathbf{x}') = i\theta (t - t') \langle [\hat{\mathfrak{A}} (t, \mathbf{x}), \hat{\mathfrak{A}} (t', \mathbf{x}')] \rangle \qquad (7.2)$$

corresponds to the part of the response that is related to the change in the dynamic variables under the external influence.

The above formulas follow from the general expression for the derivative of the Heisenberg operator $\hat{O}(t)$ with respect to g (see Ref. 33)

$$\frac{d\tilde{O}(t)}{dg} = \frac{\partial \hat{O}(t)}{\partial g} + i \int_{-\infty}^{t} dt' \left[\frac{\partial \hat{H}(t')}{\partial g}, \hat{O}(t) \right], \qquad (7.3)$$

the validity of which can be readily verified by checking that theequation$d^2\hat{O}/dtdg = d^2\hat{O}/dgdt$ holds,usingtheHeisenberg equation

$$d\hat{O}(t)/dt = \partial \hat{O}(t)/\partial t + i [\hat{H}(t), \hat{O}(t)] \qquad (7.4)$$

and the Jacobi identity for the commutators. In deriving (7.2), we have also used the expression $\partial \hat{H} / \partial \mathfrak{J} = -\hat{\mathfrak{A}}$ that follows from (3.3).

The quantity $\tilde{\mathfrak{R}}$ can be written as the derivative with respect to $\mathfrak{J}$ of the part of the operator cf14A that does not explicitly depend on $\mathfrak{J}$ (this part is denoted below by $\tilde{\hat{\mathfrak{A}}}$). In the linear theory considered here, we can have

$$\tilde{\hat{\mathfrak{A}}} = \hat{\mathfrak{A}} - \frac{\mathfrak{J} \partial \hat{\mathfrak{A}}}{\partial \mathfrak{J}}, \quad \tilde{\mathfrak{A}} = \langle \tilde{\hat{\mathfrak{A}}} \rangle = \mathfrak{A} - \mathfrak{R}_0 \mathfrak{J}. \qquad (7.5)$$

In expressions that do not contain derivatives with respect to $\mathfrak{J}$, which is set equal to zero, we can replace $\hat{\mathfrak{A}}$, $\mathfrak{A}$ with $\hat{\mathfrak{A}}$, $\tilde{\mathfrak{A}}$.

Transforming in (7.2) to the Fourier components, and introducing the intermediate set of eigenfunctions of the Hamiltonian $\hat{H}$, we obtain

$$\tilde{\mathfrak{R}} (\omega, \mathbf{k}) = 2 \sum_{mn} \alpha_{mn} |\mathfrak{A}_{mn} (\mathbf{k})|^2 [\omega_{mn}^2 - (\omega + i\delta)^2]^{-1},$$

where $\omega_{mn} = E_m - E_n$, $\mathfrak{A}_{mn}$ is the Fourier component of the matrix element of the operator $\hat{\mathfrak{A}}$ in the Schrödinger representation. The quantity $\alpha_{mn} = (w_n - w_m)\omega_{mn}$, where w_n is the probability that the level of energy E_n is filled, is positive for the equilibrium medium (the probability of filling the level decreases with its energy). Hence, it follows that the quantity $\tilde{\mathfrak{R}}$ (like $\mathfrak{R}$) satisfies the dispersion relation and has an imaginary part whose sign is the same as that of the frequency. The above representation for $\tilde{\mathfrak{R}}$ leads to the following important inequality:

$$\tilde{\mathfrak{R}} (0, \mathbf{k}) \geqslant 0, \qquad (7.6)$$

which, in turn, eventually leads to (6.1) and (6.3). There is one further relation that expresses the fluctuation-dissipation theorem for an equilibrium medium at temperature T:

$$\langle\{\delta\hat{\mathfrak{A}}(\omega,\ \mathbf{k}),\ \delta\hat{\mathfrak{A}}(\omega',\ \mathbf{k}')\}\rangle = 2(2\pi)^4 \coth\frac{\omega}{2T}\,\mathrm{Im}\,\hat{\mathfrak{R}}(\omega,\ \mathbf{k})\,\delta(\omega+\omega')\,\delta(\mathbf{k}+\mathbf{k}'), \tag{7.7}$$

where $\delta\hat{\mathfrak{A}} = \hat{\mathfrak{A}} - \mathfrak{A}$ is the fluctuation in $\mathfrak{A}$, $\{\hat{a}, \hat{b}\} = \hat{a}\hat{b} + \hat{b}\hat{a}$. The derivation of this can be found, for example, in Refs. 4 and 12.

To derive the inequalities (6.1) and (6.3) from (7.6), we start with (3.5), then introduce the corresponding operators $\hat{\mathfrak{A}}$, and finally use (2.10′). In the electric case,

$$\hat{\mathfrak{A}} = \hat{\rho} = \rho^e + \hat{\rho}^i,\quad \mathfrak{J} = -\varphi^e = -\frac{\rho^e}{k^2},\quad \mathfrak{R} = -k^2/\varepsilon,$$

where $\hat{\rho}^i = \Sigma e\hat{n}$ and $\hat{n}$ is the particle number-density operator. Here and below, the sum is evaluated over all types of charged particles. From (7.1) and (7.3), we have

$$\mathfrak{R}_0 = -k^2,\quad \tilde{\mathfrak{R}} = k^2\left(1-\frac{1}{\varepsilon}\right),\quad \tilde{\hat{\mathfrak{A}}} = \hat{\rho}^i, \tag{7.8}$$

from which (6.1) follows.

In the magnetic case, in which

$$\hat{\mathfrak{A}} = \hat{\mathbf{j}}_t = \mathbf{j}_t^e + \hat{\mathbf{j}}_t^i,\quad \mathfrak{J} = \mathbf{A}^e,\quad \mathfrak{R} = \frac{k^2-\omega^2}{\eta},$$

the situation is more complicated: the explicit dependence of $\hat{\mathbf{j}}$ on $\mathbf{A}^e = \mathbf{j}_t^e/(k^2-\omega^2)$ is determined not only by $\mathbf{j}^e$, but also by the diamagnetic component of the current $\mathbf{j}^i$, which is linear in the total potential (including $\mathbf{A}^e$). When the structural components of the medium are chosen in accordance with the discussion of Section 6, this component is $\Sigma(e^2\hat{n}/m)\hat{\mathbf{A}}$. This complication manifests itself in the appearance of the factor $\varkappa(\omega, \mathbf{k})$, which is not equal to unity, in the expressions

$$\mathfrak{R}_0 = (k^2-\omega^2)\,\varkappa,$$

$$\tilde{\mathfrak{R}} = (k^2-\omega^2)(\eta^{-1}-\varkappa),\quad \tilde{\hat{\mathfrak{A}}} = \hat{\mathbf{j}}_t^i - (\varkappa-1)\,\mathbf{j}_t^e. \tag{7.9}$$

In the static case in which we are interested here, the operator $\hat{\mathbf{A}} = -\Delta^{-1}\hat{\mathbf{j}}_t$ can be replaced with its average value $\mathbf{j}_t/k^2$, which provides the dominant contribution to the significant range of low values of k that do not exceed ω_p/c by a large factor (see Section 6): the contribution of the fluctuating part of $\mathbf{A}$ is determined by the replacement of k^2 in the denominator with l^{-2}, where l is the correlation length (small in comparison with c/ω_p). Simple calculation shows that

$$\varkappa(0,\ \mathbf{k}) = \left(1+\frac{\omega_p^2}{k^2}\right)^{-1}, \tag{7.10}$$

and hence, using (7.6) and (7.9), we obtain the inequality given by (6.3).

The difference between the magnetic and electric cases in the present context is due to the existence of diamagnetism (current depending on potential) in the absence of diaelectricity (see Section 6). However, the more fundamental reason for this difference is that Faraday's induction law, which is responsible for the diamagnetic current, has no electric analog. It is therefore not surprising that relativistic media, described by the Dirac equation with a potential-independent current, can also be diamagnetic. The diamagnetism arises formally in the latter case because of the violation of the inequality given by (7.6) [although $\varkappa$ in (7.9) is actually equal to unity]. In the single-particle theory, this violation is due to negative energy levels (virtual transitions to which are responsible for diamagnetism) and in field theory with renormalizations.

8. STABILITY OF A MEDIUM AND LANDAU FUNCTIONAL

The dispersion relations that lead to the inequalities given by (6.1)–(6.3) are based, as emphasized in Section 4, not only on the principle of causality, but also on the requirement that the medium be stable. It is therefore not surprising that these inequalities are also the criteria for the stability of the medium against spontaneous rearrangement (ordering) with the appearance of nonzero $\mathfrak{A}$ (order parameter). More precisely, spontaneous ordering that is unrelated to the external influence corresponds to the order parameter $\tilde{\mathfrak{A}}$, which is the part of $\mathfrak{A}$ that does not explicitly depend on the external influence [see (7.5)].

The case $\tilde{\mathfrak{A}} \neq 0$ corresponds to a set of nonequilibrium states that differ by the magnitude of other, i.e., different from $\tilde{\mathfrak{A}}$, dynamic variables ξ of the medium. The energy $\mathscr{E}(\tilde{\mathfrak{A}}, \xi)$ of these states is reckoned from the energy of the original unordered state with $\tilde{\mathfrak{A}} = 0$.[15] From among these states, we select that corresponding to minimum $\mathscr{E}$ as a function of ξ for given $\tilde{\mathfrak{A}}$, i.e.,

$$\delta\mathscr{E} + \lambda\delta\tilde{\mathfrak{A}} = 0 \tag{8.1}$$

where λ is the Lagrange multiplier. This minimum is denoted $\mathscr{E}(\tilde{\mathfrak{A}})$ and is called the Landau functional. It lies at the basis of the Landau theory of phase transitions (see Section 1), which starts with the assumption that the relaxation of the order parameter is significantly slower than that of the variable ξ (incomplete equilibrium), while the medium is considered to be stable against variations in these variables.[12,34] On the other hand, the stability of the medium with respect to the order parameter itself is determined by the functional $\mathscr{E}(\tilde{\mathfrak{A}})$. In particular, the above-mentioned criterion for the stability of the unordered state is identical with the condition that $\mathscr{E}(\tilde{\mathfrak{A}})$ be a minimum for $\tilde{\mathfrak{A}} = 0$.

Since the relaxation of the order parameter is slow, the quantity $\mathscr{E}(\tilde{\mathfrak{A}})$ can be looked upon as the energy of the equilibrium state of the medium whose Hamiltonian differs from the true Hamiltonian $\hat{H}$ by the additional term $\hat{H}'$ that depends on the magnitude of the order parameter, assumed sufficiently small. The form of $\hat{H}'$ can be found from (8.1), for which we need to have at our disposal the expression for the change δO in the expectation value of an operator $\hat{O}$ under the influence of $\hat{H}'$ itself. This change corresponds to the rearrangement of the medium, i.e., to a change in its dynamic variables in the absence of any external influences, and is therefore described by a Kubo-type formula (see Section 7), but without the term containing the partial derivative[16] $\delta O = -\langle\hat{O}, \delta\hat{H}'\rangle$, where

$$\langle\hat{A},\ \hat{B}\rangle = i\int_{-\infty}^{t} dt'\,\langle[\hat{A}(t),\ \hat{B}(t')]\rangle.$$

Hence, by considering an individual Fourier component of the static order parameter, we obtain the expression for the change in this parameter ($\hat{O} = (\tilde{\mathfrak{A}})$

$$\delta\tilde{\mathfrak{A}} = -\langle\tilde{\hat{\mathfrak{A}}},\ \delta\hat{H}'\rangle, \tag{8.2}$$

and for the change in the energy ($\hat{O} = \hat{H}$), using (7.4),

$$\delta\mathscr{E} = \delta\langle\hat{H}\rangle = -\delta H' = \langle\hat{H}',\ \delta\hat{H}'\rangle. \tag{8.3}$$

Substitution of these expressions in (8.1) yields $\hat{H}' = \lambda\tilde{\mathfrak{A}}$. Using (8.2) and the equation $\langle\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}\rangle = \tilde{\mathfrak{R}}$ that follows from (7.2), we find that $\delta\lambda = -\delta\tilde{\mathfrak{A}}/\tilde{\mathfrak{R}}$ and, if we use (8.3), we obtain the expression $\mathscr{E} = \lambda^2\tilde{\mathfrak{R}}/2$. Assuming that the order parameter is small, we find that the effective Hamiltonian is

$$\hat{H}' = -\frac{\hat{\tilde{\mathfrak{A}}}(\mathbf{k})\,\delta\tilde{\mathfrak{A}}(\mathbf{k})}{\tilde{\mathfrak{R}}(0,\ \mathbf{k})} \tag{8.4}$$

and the first (quadratic) term in the Landau functional is

$$\mathscr{E} = \frac{(\delta\tilde{\mathfrak{A}}(\mathbf{k}))^2}{2\tilde{\mathfrak{R}}(0,\ \mathbf{k})}. \tag{8.5}$$

The condition for the stability of the medium against small fluctuations in $\tilde{\mathfrak{A}}$ is actually the same as (7.6) and this leads to inequalities (6.1) and (6.3) [see the next Section in connection with (6.2)].

When the medium is subjected to forced ordering by an external influence, we find from (7.1), (7.2), and (7.5) that

$$\delta\tilde{\mathfrak{A}}\ (0,\ \mathbf{k}) = \tilde{\mathfrak{R}}\ (0,\ \mathbf{k})\ \delta\mathfrak{I}\ (0,\ \mathbf{k}). \tag{8.6}$$

Comparing this with (8.4) and transforming to the coordinate representation, we have

$$\hat{H}' = -\int d\mathbf{x}\ \hat{\tilde{\mathfrak{A}}}\delta\mathfrak{I}. \tag{8.7}$$

If we close our eyes to the difference between the quantities $\hat{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}$, we see that the expression given by (8.7) is identical with the Hamiltonian for the external influence on the medium, given by (3.3). This corresponds to the well-known proposition that the ordered state of a medium can be looked upon as an equilibrium state in a certain specially chosen external field whose strength is determined by (8.6). However, it is also clear that this proposition is not literally valid although, as it turns out, it can be used in a somewhat modified form in the thermodynamic analysis of ordered media (see below, Section 9).

We now proceed to the derivation of the specific form of the Landau functional in the electrodynamics of material media, omitting the symbol δ in the order parameter for the sake of brevity. In the absence of the external field, we need not distinguish between $\tilde{\mathfrak{A}}$ and $\mathfrak{A}$ and take the latter as the order parameter, i.e., ρ and $\mathbf{j}$ (or $\mathbf{E}$ and $\mathbf{B}$). Using (8.5), (7.8), and (7.9), we find that, in the electric and magnetic cases,

$$\mathscr{E} = \frac{E^2}{2(1-\varepsilon^{-1})}, \qquad \mathscr{E} = \frac{B^2}{2(\mu - \varkappa)},$$

respectively, with ε, μ, and $\varkappa$ given in terms of the arguments $\omega = 0$ and $\mathbf{k}$ [see (7.10)]. These expressions must not be confused with the quantities $\varepsilon E^2/2$ and $B^2/2\mu$ that determine the energy of the medium in the external field. It is precisely this kind of confusion that has led, in the past, to the incorrect conclusion that ε must be positive.

When an external field is present, the Landau functional consists of (8.5) plus the energy of interaction between the order parameter and the external field and the energy of the external field itself. According to (3.3), this additional term is equal to $-\tilde{\mathfrak{A}}\mathfrak{I} - (\mathfrak{A} - \tilde{\mathfrak{A}})\mathfrak{I}/2$, so that, using (7.5), we finally obtain the Landau functional in the form

$$\mathscr{E} = \frac{(\tilde{\mathfrak{A}} - \tilde{\mathfrak{R}}\mathfrak{I})^2}{2\tilde{\mathfrak{R}}} - \frac{\mathfrak{R}\mathfrak{I}^2}{2}. \tag{8.8}$$

Taking the variation of this with respect to $\tilde{\mathfrak{A}}$ at fixed $\mathfrak{I}$, we obtain the correct equilibrium relation (8.6) and, at the equilibrium point itself, the correct result for the energy of the medium in the external field, $D^2/2\varepsilon$ and $-\mu H^2/2$, due to external charges and currents (the expression $B^2/2\mu$, given above, on the other hand, corresponds to the effect of the total current in the superconducting solenoid[2]).

To evaluate (8.8), it is convenient to introduce the polarization $\mathbf{P} = \mathbf{D} - \mathbf{E}$ and magnetization $\mathbf{M} = \mathbf{B} - \mathbf{H}$. According to (7.8) and (7.9), we then have

$$\tilde{\mathfrak{A}} = \frac{i\mathbf{k}\cdot\mathbf{P}}{k^2}, \qquad \tilde{\mathfrak{A}} = i\,[\mathbf{k},\ \mathbf{M} - (\varkappa - 1)\,\mathbf{H}]. \tag{8.9}$$

The significant point is that, in the magnetic case, the order parameter does not reduce to the magnetization, but contains the additional diamagnetic term that removes from the magnetization its rapidly relaxing part due to the precession of orbits in the external field, but is unrelated to the order parameter as such. Finally, in the electric case,

$$\mathscr{E} = \frac{[\mathbf{P} - (1 - \varepsilon^{-1})\,\mathbf{D}]^2}{2(1 - \varepsilon^{-1})} + \frac{D^2}{2\varepsilon}, \tag{8.10}$$

and, in the magnetic case,

$$\mathscr{E} = \frac{[\mathbf{M} - (\mu - 1)\,\mathbf{H}]^2}{2(\mu - \varkappa)} - \mu\frac{H^2}{2}. \tag{8.11}$$

The last expression gives

$$\mathscr{E} = \frac{M^2}{2(\mu - \varkappa)} - \frac{\mathbf{M}\cdot\mathbf{H}(\mu - 1)}{\mu - \varkappa} + \ldots,$$

which is very different from the expression given in Section 1, but is entirely consistent with the diamagnetism of an unordered system. This removes the contradiction mentioned in example 2 of Section 1. Moreover, in the ordered state with spontaneous current, the lower limit of the magnetic permeability is equal not to unity (as would be expected from the expression given in Section 1), but to $\varkappa$. It follows that the widely held belief that the vector nature of the order parameter presupposes a paramagnetic response is actually unjustified, so that there can be no doubt about the basic possibility of the existence of anomalous diamagnetic substances as media with spontaneous currents in the ground state (for small nonzero values of k; see Section 6).

9. STABILITY OF A MEDIUM AND LANDAU FUNCTIONAL (THERMODYNAMIC APPROACH)

The above dynamic approach, based on the Hamiltonian (3.3) and the existence of the operator $\tilde{\mathfrak{A}}$, is inadequate if only because it does not extend to the Landau functional for ferroelectric ordering and the inequality given by (6.2): the response function ε is not the generalized susceptibility. The more general thermodynamic approach, usually adopted in the derivation of the Landau functional and the stability conditions, is free from this defect. However, the standard thermodynamic scheme leads to difficulties with the description of diamagnetism. The results of Section 8 show how the scheme has to be modified to become universal.

Let us first translate the results of Section 8 to the thermodynamic language. Suppose that the medium has been subjected to an external influence $\mathfrak{J}$, which results in the appearance of the order parameter $\tilde{\mathfrak{A}}$, whose magnitude is given by (8.6). The corresponding change in the energy $\mathscr{E}(\mathfrak{J})$, which is a functional of $\mathfrak{J}$, is determined by the expectation value (3.3):

$$\delta\mathscr{E}(\mathfrak{J}) = -\mathfrak{A}\delta\mathfrak{J} = -\tilde{\mathfrak{A}}\delta\mathfrak{J} - (\mathfrak{A} - \tilde{\mathfrak{A}})\,\delta\mathfrak{J}. \qquad (9.1)$$

The first term in this expression is the work expended directly in ordering the medium and the second contains the change in the energy of the external influence in vacuum and (in the magnetic case) the work associated with precession in the external field. The second term is unrelated to ordering (the change in the dynamic variables of the medium) and should be omitted. The transformation to the functional $\mathscr{E}(\tilde{\mathfrak{A}})$ in which we are interested, which corresponds to a change of the argument, is accomplished in thermodynamics by the Legendre transformation

$$\delta\mathscr{E}(\mathfrak{A}) = \delta\mathscr{E}(\mathfrak{J}) + \delta(\tilde{\mathfrak{A}}\mathfrak{J}) = \mathfrak{J}\delta\tilde{\mathfrak{A}}, \qquad (9.2)$$

where $\mathfrak{J}(\tilde{\mathfrak{A}})$ is determined by (8.6). It is readily seen that (9.2) and (8.5) are exactly equivalent, as are their consequences relating to the form of the Landau functional and the stability conditions.

Physically, the above procedure corresponds (subject to the reservation relating to the difference between $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$; see Section 8) to the so-called Leontovich principle[35]: to find the energy of a nonequilibrium state with a given order parameter, the state is transformed into an equilibrium state by introducing the appropriate external field and then subtracting from the energy of the medium in this field the "surplus" work expended in turning on the field. This subtraction corresponds to the Legendre transformation. It is important to emphasize that, in the electric case, the replacement of $\mathfrak{A}$ with $\tilde{\mathfrak{A}}$ is relatively "innocent" and corresponds to the replacement of $\mathbf{E}$ with $-\mathbf{P}$, where the first term in (9.1) has the direct interpretation as the energy of a dipole in an external field, $-\mathbf{P}\cdot\delta\mathbf{D}$. However, in the magnetic case, the difference between the order parameter and the magnetization [see (8.9)] means that the simple picture is no longer valid, in the final analysis, because of the presence of diamagnetic effects.

Passing now to the case where ordering occurs in a given total (and not merely external, as above) field, so that the dynamic approach is inappropriate, we emphasize that (9.2) is still valid. Here, we must start with (9.1) as the primary thermodynamic relation and perform the transformation from $\mathfrak{A}$ to $\tilde{\mathfrak{A}}$ by subtracting terms directly related to the (constant) total field. In the simplest electric case (ordering inside a capacitor with given potential difference across it, i.e., given field $\mathbf{E}$: see Section 3), the change in the energy of the medium is given by the well-known expression $-\rho^e\delta\varphi$, which leads to $\mathfrak{J} = \varphi$ and $\mathfrak{A} = \rho^e = \rho - \rho^i$, and hence to $\tilde{\mathfrak{A}} = -\rho^i$.[17] We then find that (9.1) assumes the form $-\mathbf{P}\cdot\delta\mathbf{E}$, which is simply the energy of the dipole in a given field $\mathbf{E}$.

Accordingly, the Landau functional for ferroelectric ordering has the following form: for $\mathbf{E} = 0$ (order parameter $\mathbf{D}$)

$$\mathscr{E} = \frac{D^2}{2(\varepsilon - 1)}, \qquad (9.3)$$

and, for arbitrary $\mathbf{E}$,

$$\mathscr{E} = \frac{[\mathbf{P} - (\varepsilon - 1)\,\mathbf{E}]^2}{2(\varepsilon - 1)} - \frac{\varepsilon E^2}{2}. \qquad (9.4)$$

The stability condition that follows from this is exactly the same as (6.2), and it is immediately obvious from (9.3) why this inequality applies only to the $\mathbf{k} = 0$ case (see Section 3). The point is that, in a medium that is free from external influences, and in which there are no external charges, so that div $\mathbf{D} = 0$ is valid, the order parameter $\mathbf{D}$ can appear only for $\mathbf{k} = 0$.

Comparison of (8.10) and (9.5) in the absence of external influences will show that the difference between these expressions, which correspond, respectively, to ordering inside a shorted and uncharged capacitor, is positive (and equal to $P^2/2$). This is a further indication of the instability of the medium with a uniform spontaneous electric field (see Section 6), which finds it advantageous to convert into uniform induction.

The results found in the last two Sections are in agreement with the statement made in Section 6 about the physical significance of the admissible range of values of the permeabilities of the medium. The questions examined in these Sections were partly discussed in Refs. 14 and 18–20.

10. ELECTRODYNAMICS OF THE MAGNETIC MONOPOLE

In this concluding Section, we consider the electrodynamics of a medium subjected to the influence of a magnetic charge (monopole). The medium itself is assumed to be free from monopoles. Apart from their independent interest, the questions examined below should result in a better understanding of ordinary electrodynamics, providing a "lateral view" of it.

The Maxwell equations with monopole forces that restore the symmetry between electric and magnetic fields differ from (2.1c) and (2.1d) as follows:

$$\text{curl}\,\mathbf{E} + \dot{\mathbf{B}} = -\tilde{\mathbf{j}}^e \ \ (\text{c}), \qquad \text{div}\,\mathbf{B} = \tilde{\rho}^e \ \ (\text{d}), \qquad (10.1)$$

where $\tilde{\rho}^e$, $\tilde{\mathbf{j}}^e$ are the external monopole charge and current densities that are also related by the continuity equation. These two equations do not include the induced charge and current of monopoles because the medium itself is free from monopoles. On the other hand, the induced charge and current of ordinary particles that appear in (2.1a) and (2.1b) are given by (2.3), as before.

However, in these formulas, we can no longer perform the regrouping of terms mentioned in Section 2 that modifies the quantities $\tilde{\varepsilon}$ and $\tilde{\mu}$: the fields $\mathbf{E}_t$ and $\mathbf{B}_t$ are no longer related by the stringent relation (2.1c) [see (10.1c)]. It follows that, in monopole electrodynamics, the quantities $\tilde{\varepsilon}$ and $\tilde{\mu}$ are not arbitrary [within the framework of (2.5) and (2.8)] as they are in ordinary electrodynamics, but acquire independent meaning as quantities describing the response of the medium to the independent influences $\mathbf{E}_t$ and $\mathbf{B}_t$.[18] The number of independent influences acting on the medium is now no longer two, as before, but three, namely, $\mathbf{E}_l$, $\mathbf{E}_t$, and $\mathbf{B}_t$, and this is represented by the three independent parameters of the medium ε, η, and one of the quantities $\tilde{\varepsilon}$, $\tilde{\mu}$, which, as before, are related by (2.5). All this is clear from the following expressions that generalize (2.9):

$$\rho^i = (\varepsilon^{-1} - 1)\,\rho^e, \quad \mathbf{j}_t^i = (\eta^{-1} - 1)\,\mathbf{j}_t^e + (\tilde{\varepsilon} - \tilde{\mu}^{-1})\,\omega\,[\mathbf{k}\cdot\tilde{\mathbf{j}}^e]\,(k^2 - \omega^2)\,\eta]^{-1}. \qquad (10.2)$$

They are the response to ordinary current, determined by η alone, whereas the response to the monopole current is determined by a different combination of $\tilde{\varepsilon}, \tilde{\mu}$, namely, $\tilde{\varepsilon} - (1/\tilde{\mu})$.

We note that the fourth type of field, namely, the longitudinal field $\mathbf{B}_l$, is unrelated to any kind of influence on the medium, since its sources do not appear in (10.2). Conversely, this field does not "feel" the presence of the medium whose parameters do not appear in (10.1d).

It is clear from the foregoing that a special dynamic calculation of $\tilde{\varepsilon}$ and $\tilde{\mu}$ separately is necessary in monopole electrodynamics. Usually, however, $\tilde{\varepsilon}$ is simply identified with ε and $\tilde{\mu}$ with μ [see (2.6)], for which there is no more justification than for the choice $\tilde{\varepsilon} = \varepsilon_t, \tilde{\mu} = 1$ [see (2.7)]. It is precisely this choice that is found to be justified for classical (nonquantum) media. In this case, the distribution function that determines the induced current satisfies the transport equation with the force term $e(\mathbf{E} + \mathbf{v}\times\mathbf{B})\nabla_{\mathbf{p}} f_0$, where $\mathbf{v}$ is the velocity and $\mathbf{p}$ the momentum. In an equilibrium medium, f_0 depends only on energy, the gradient of which with respect to $\mathbf{p}$ is equal to $\mathbf{v}$. Hence, the magnetic field does not have a direct effect on the medium and, according to (2.3), this yields $\tilde{\mu} = 1$.

It is precisely this fact that removes the difficulties discussed in connection with example[3)] of Section 1. When $\omega \ll k \ll \lambda^{-1}$, i.e., in the special case in which we are interested here (well away from a slow monopole, where λ is the London penetration depth), the superconductor may be described as an ideal classical conducting fluid. Accordingly, $\mu = k^2\lambda^2$ (ideal diamagnetism), ε is finite, and $(k^2 - \omega^2)\eta = \lambda^{-2}$, $\tilde{\varepsilon} = -\omega^2\lambda^{-2}$, $\tilde{\mu} = 1$. On the other hand,

$$\mathbf{B}_t = i\omega\tilde{\varepsilon}\tilde{\mathbf{j}}_t^e [(k^2 - \omega^2)\eta]^{-1} \tag{10.3}$$

[cf. (2.1a), (2.1b), and (10.1)]. When $\tilde{\varepsilon} = \varepsilon$, the quantity $\mathbf{B}_t$ vanishes together with ω (with the velocity of the monopole) and the Meissner effect is truly absent (see Section 1). However, when $\tilde{\varepsilon} = \varepsilon_t$, the quantity given by (10.3) assumes the form $\mathbf{B}_0 - \mathbf{B}_l$, where

$$\mathbf{B}_0 = -\int_{-\infty}^{t} dt\, \tilde{\mathbf{j}}^e$$

is a "string" along the trajectory of the monopole. This means that the total field $\mathbf{B}_l + \mathbf{B}_t$ exhibits the Meissner effect, and the jump in the magnetic flux does actually occur.

It is important to note that, if the usual identification of $\tilde{\varepsilon}$ with ε and $\tilde{\mu}$ with μ were correct for media for which $\varepsilon(0, \mathbf{k}) < 0$ and the dispersion relations for ε were violated (see Section 6), we would have the extremely unusual phenomenon of a rapid and, in fact, "explosive" stopping of the monopole with the transfer of its kinetic energy to the energy of the field in the medium, i.e., in the final analysis, radiation and heat. This is already signald by the expression given by (10.3): the pole of $\tilde{\varepsilon}$ at $\omega = i\Gamma$ [cf. (4.7)] must be bypassed in accordance with the causality condition, which leads to an exponential increase of the form $\mathbf{B}_t \sim \exp(\Gamma t)$ (see Section 4 and Fig. 4). The significant point is that this increase does not indicate at all an instability of the medium as such: by hypothesis, the medium does not contain monopoles and therefore its natural fluctuations cannot simulate the influence of a monopole, as was the case for the influence due to a charge. It is precisely this result that ensures that the response to a monopole current [see the second relation in (10.2)] need not be analytic in the upper frequency half-plane, and the function $\tilde{\varepsilon}(\omega, \mathbf{k})$ can have a positive region. All this is a reflection of the simple fact that the particles of the medium are different in character from the sources of the external influence acting upon it.

A consistent analysis of the above effect must be based on a generalization of the theory of energy losses, given in Section 5, in which the monopole sources are included. As before, if we start with (5.3), but use the new Maxwell equations (10.1) and Eqs. (2.1a) and (2.1b), we can readily find the generalization of (5.4):

$$Q = -\int d\mathbf{x}\, (\mathbf{j}^e \cdot \mathbf{E} + \tilde{\mathbf{j}}^e \cdot \mathbf{B}). \tag{10.4}$$

When $\tilde{\varepsilon}$ is analytic in the upper frequency half-plane, the substitution of (10.3) into this expression leads to a formula that is analogous to (5.1) and gives the monopole energy loss:

$$\int_{-\infty}^{\infty} dt\, Q = \frac{1}{\pi} \int_0^{\infty} d\omega\omega \int d^3k\, \mathrm{Im}\, \{\tilde{\varepsilon}\, [(k^2 - \omega^2)\,\eta]^{-1}\}\, |\tilde{\mathbf{j}}_t^e|^2. \tag{10.5}$$

On the other hand, when $\tilde{\varepsilon}$ has a pole (see above), it follows from (10.4) and from the equation of motion $M\mathbf{v}\cdot\dot{\mathbf{v}} = -Q$ (M and $\mathbf{v}$ are the mass and velocity of the monopole) that there is actually a rapid slowing down of the motion even when M is very large.

It is important to note that the widely discussed expression for the monopole energy loss (see, for example, Ref. 36) differs from (10.4) by the replacement of $\mathbf{B}$ with $\mathbf{H}$. The derivation of this unnatural result (the force acting on the monopole would seem to be given by the mean field $\mathbf{B}$ and not the field in the vacuum $\mathbf{H}$) is based on the energy balance and the previous expression, given by (5.5), for the change in the energy of the medium. However, this expression is itself based on Faraday's law of induction (2.1c) (see Ref. 2), which is not valid in monopole electrodynamics. Equation (10.1c), on the other hand, leads to

$$W = \int d\mathbf{x}\, (\mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{H}\cdot\dot{\mathbf{B}} - \tilde{\mathbf{j}}^e \cdot (\mathbf{B} - \mathbf{H})),$$

which is in complete agreement with (10.4). We must add that all this, together with the foregoing discussion of the longitudinal field $\mathbf{B}_l$, is in agreement with the total absence of longitudinal monopole losses, whereas the expression containing the field $\mathbf{H}$ predicts the existence of such losses.

Turning now to the monopole "explosion effect," we note that this effect is not possible in classical media with $\tilde{\varepsilon} = \varepsilon_t$. This is due to the analyticity of ε_t, mentioned in Section 5. The existence of quantum media in which this effect may occur is still an open question.

We emphasize, in conclusion, that the entire foregoing discussion is concerned with linear monopole electrodynamics, which deals with relatively weak fields. On the other hand, the fields produced by the monopole, whose charge is greater by a factor of at least 137/2 than the charge of the electron, can hardly be regarded as weak, even in the most favorable case of a monopole moving slowly relative to the medium. Accordingly, it will be necessary to take into account the nonlinearity of the response functions in the field, the presence of specific terms of the form $\mathbf{B}\times\mathbf{E}$ (Faraday effect) in the expression for the induced current (2.3), and so on. Studies of nonlinear monopole electrodynamics have only just begun (see, for example, Ref. 36). In any case, there are real situations for which linear electrodynamics is valid, e.g., Cherenkov losses by a monopole in a classical medium are determined by distances from it that exceed the wavelength of the radiation, where the field of the monopole is low enough and linear electrodynamics is valid.

Details relating to the questions examined in this Section can be found in Ref. 37.

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1)This dependence (frequency and spatial dispersion) reflects the delay of the response and its nonlocal character. It is important to emphasize that inclusion of spatial dispersion makes averaging of physical quantities over a small volume unnecessary. Effects associated with the unsmoothed microstructure of the medium then manifest themselves in the complex and irregular dependence of the response functions on **k** for large values of this quantity.[3-6]

2)An expanded version of this paper will appear as a separate chapter in a monograph entitled, "Dielectric permittivity of condensed media," due to be published by the North-Holland Publishing Company.

3)See Ref. 3 in connection with anisotropic and gyrotropic media.

4)It is readily seen from (2.11) that $1/\varepsilon$ and $1/\eta$ are the renormalization factors reflecting the influence of the medium in expressions for the longitudinal and transverse components of the photon Green function in the medium.[15]

5)We shall use the Gothic fonts $\mathfrak{A}$ (answer), $\mathfrak{R}$ (response), and $\mathfrak{I}$ (influence) in order to emphasize quantities that are fundamental to our discussion in this paper.

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