METHODOLOGICAL NOTES

Stimulated radiation from high-current relativistic electron beams

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The review, which is methodological in nature, discusses the most common mechanisms of stimulated radiation of electron beams such as single-particle and collective Cherenkov effects, undulator and synchrotron radiation, the anomalous Doppler effect, and Thomson and Raman scattering and radiation. The relation between the mechanisms of spontaneous radiation of an individual electron and stimulated radiation in electron beams is made clear, the basic principles of linear electrodynamics of radiative beam instabilities are stated, and the principal mechanisms of their nonlinear stabilization are investigated. The discussion of widely different stimulated processes in electron beams is carried out from a unified point of view with use of a simple mathematical apparatus and such generally accepted physical laws as the conservation laws and Newton's laws. Therefore the review, which is intended for specialists in the fields of plasma electrodynamics and high-frequency electronics, is written also for persons familiar with only the fundamentals of physics taught in a general university course.

CONTENTS

1. SPONTANEOUS RADIATION AND THE CONDITIONS OF ITS EXISTENCE

The basis of the transformation of the energy of directed motion of electron beams into electromagnetic radiation is the phenomenon of resonance interaction of an individual electron with the field of a monochromatic wave. In the first approximation in analysis of this interaction usually the following two equivalent approaches are used: either one calculates the work done by the electromagnetic field on an electron executing a specified motion unperturbed by this field,[1,3] or one considers the excitation by an unperturbed electron of a field oscillator.[2] The effect known as spontaneous radiation is described in just this way in classical electrodynamics. Let us see what are the conditions under which spontaneous radiation occurs, using here the first of the approaches mentioned above.

We shall calculate the work of the field of a monochromatic wave on an electron moving uniformly in a straight line, assuming that the direction of propagation of the wave and the direction of motion of the electron coincide. Representing the field of the wave in the form $\mathbf{E}(z,t) = \mathbf{E}\sin(\omega t - k_{\parallel} z + \varphi)$, we obtain for the amount of work the following expression:

$$A_T = e\int_{-T/2}^{T/2} dt\,\mathbf{u}\mathbf{E}(z,t)\,|_{z=u_{\parallel}t} = eu_{\parallel}E_{\parallel}\sin\varphi\,\frac{2}{\omega-k_{\parallel}u_{\parallel}}\sin\left(\frac{\omega-k_{\parallel}u_{\parallel}}{2}T\right)\xrightarrow[T\to\infty]{}\pi eu_{\parallel}E_{\parallel}\times\sin\varphi\,\delta(\omega-k_{\parallel}u_{\parallel}); \qquad (1.1)$$

here $u_{\parallel}$ is the velocity of the electron, which is directed along the Oz axis, $E_{\parallel}$ is the component of the electric field of the wave in the direction of motion, T is the time of interaction of the electron with the wave, which is assumed large in comparison with $1/\omega$, and φ is the initial phase of the wave.

From Eq. (1.1) it follows that, first, the sign of the work is determined by the phase of the field φ; second, the work is nonzero only if the component of the electric field of the wave in the direction of motion is nonzero; third, $A_\infty \neq 0$ only under the conditions of Cherenkov resonance, i.e.,

$$\omega(k_{\parallel}) = k_{\parallel}u_{\parallel}. \qquad (1.2)$$

Here we have taken into account that the wave has a completely definite dispersion law $\omega(k_{\parallel})$ (or $\omega(\mathbf{k})$ if the direction of the radiation does not coincide with the direction of motion of the electron). The spontaneous radiation which occurs under the conditions (1.2) is called the Cherenkov effect.[2-5] This spontaneous effect must not be confused with the stimulated Cherenkov radiation effect which will be discussed below.

If the electron, under the action of some external forces, executes in addition to its longitudinal motion also some oscillatory motion, then new conditions of radiation arise. For example, suppose that the electron is moving in a longitudinal constant magnetic field with a nonzero transverse (with respect to the magnetic field) velocity. Here $z = u_{\parallel}t$, $v_x + iv_y = u_{\perp}\exp(i\omega_H t/\gamma)$, where $u_{\perp}$ is the transverse velocity, ω_H is the electron cyclotron frequency, and $\gamma = [1-(u_{\parallel}^2/c^2)-(u_{\perp}^2/c^2)]^{-1/2}$. Therefore the electron

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in addition to its uniform longitudinal motion executes a rotational transverse motion with a rotation frequency ω_H/γ. Calculation of the work of the field of a transverse circularly polarized electro-magnetic wave $(E_x + iE_y = iE_\perp \times \exp[\pm i(\omega t - k_\parallel z + \varphi)]$ on such an electron gives a result analogous to (1.1):

$$A_\infty = \pi e u_\perp E_\perp \sin\varphi\,\delta\left(\omega - k_\parallel u_\parallel \pm \frac{\omega_H}{\gamma}\right). \tag{1.3}$$

However, the condition of radiation in this case turns out to be substantially different:

$$\omega(k_\parallel) = k_\parallel u_\parallel \pm \frac{\omega_H}{\gamma}. \tag{1.4}$$

In the more general case with arbitrary polarization and direction of wave propagation, the conditions of radiation are written in the form

$$\omega(k_\parallel) = k_\parallel u_\parallel \pm n\frac{\omega_H}{\gamma} \qquad (n = 1, 2, 3, \ldots). \tag{1.5}$$

The spontaneous radiation which arises when the conditions (1.5) are satisfied is called magnetic bremsstrahlung or cyclotron radiation[6] (synchrotron radiation for large n; Ref. 2). It must not be confused with the stimulated cyclotron radiation to be discussed below.

In general, if the frequency of oscillation of an electron in external fields is ω_0, then one has the following conditions of spontaneous radiation:

$$\omega(k_\parallel) = k_\parallel u_\parallel \pm n\omega_0, \qquad n = 1, 2, 3, \ldots, \tag{1.6}$$

which is called undulator radiation.[2,7] Undulator radiation is most frequently dipole in nature (magnetic bremsstrahlung can be considered to be one of the various forms of dipole radiation). Spontaneous undulator radiation must not be confused with the stimulated radiation which arises in motion of a beam of electrons in an undulator.

Let us return now to an electron moving uniformly and in a straight line but in a system which is periodically nonuniform in the direction of its motion. It is well known that the fields in such systems can be represented in the form of a sum of Brillouin harmonics[8]:

$$\mathbf{E}(z, t) = \mathbf{E}\exp(-i\omega t + ik_\parallel z + i\varphi)\sum_{n=-\infty}^{+\infty} a_n \exp(in\chi z), \tag{1.7}$$

where $2\pi/\chi$ is the spatial period of nonuniformity of the system. Here if h is a small parameter which determines the depth of modulation of the periodic system, then $a_n \sim h^n$. Calculation of the work of the field (1.7) on a rectilinearly moving electron gives the following result:

$$A_\infty = \pi e u_\parallel E_\parallel \sin\varphi \sum_{n=-\infty}^{+\infty} a_n \delta(\omega - k_\parallel u_\parallel - n\chi u_\parallel). \tag{1.8}$$

From this we obtain the condition of spontaneous radiation of an electron in periodically nonuniform systems[9]:

$$\omega(k_\parallel) = k_\parallel u_\parallel \pm n\chi u_\parallel, \qquad n = 1, 2, 3, \ldots. \tag{1.9}$$

This radiation also must not be confused with the corresponding stimulated radiation of a beam of electrons.

The expressions for the work done (1.1), (1.3), and (1.8) are linear in the amplitude of the field of the electromagnetic wave, since in calculating them the field was taken at points on a specified unperturbed trajectory of the electron, that is, on the trajectory of the zero-order approximation. We can say that these expressions characterize stimulated radiation of first order. However, it is possible to have spontaneous radiation also of a higher order in the field, for example, second order.

Suppose that an electron is moving in a straight line in the field of two waves with a longitudinal component of the form $E_\parallel = E_{\parallel 1}\sin(\omega_1 t - k_{\parallel 1} z + \varphi_1) + E_{\parallel 2} \times \sin(\omega_2 t - k_{\parallel 2} z + \varphi_2)$ and let $\Omega_{1,2} \equiv \omega_{1,2} - k_{\parallel 1,2} u \neq 0$, i.e., there is no spontaneous (Cherenkov) radiation of first order. We shall calculate the work of such a field on the electron. In the first-order approximation the averaged work calculated along the zero-order approximation trajectory is zero. In the same first-order approximation the electron simply oscillates in the field of two waves with frequencies $\Omega_{1,2}$. We shall take into account these oscillations under the integral in Eq. (1.1) (that is, we shall make in that equation the substitution $z = u_\parallel t + \tilde{z}$, where $\tilde{z}$ is the first-order approximation correction; after simple calculations we find that

$$A_\infty = \pi\frac{e^2 u_\parallel E_{\parallel 1} E_{\parallel 2}}{2m|\Omega_1\Omega_2|}[(k_{\parallel 1} - k_{\parallel 2})\sin(\varphi_1 - \varphi_2)\,\delta(\Omega_1 - \Omega_2) + (k_{\parallel 1} + k_{\parallel 2})\sin(\varphi_1 + \varphi_2)\,\delta(\Omega_1 + \Omega_2)]. \tag{1.10}$$

Therefore the work is nonzero if one of the conditions of combination (Raman) resonance is satisfied:

$$|\omega_1(k_{\parallel 1}) - k_{\parallel 2} u_\parallel| \mp |\omega_2(k_{\parallel 2}) - k_{\parallel 2} u_\parallel| = 0. \tag{1.11}$$

This process should be regarded as spontaneous radiation of a wave with frequency ω_1 by an electron executing a specified motion in the field of a wave with frequency ω_2, or vice versa. Spontaneous radiation of second order occurring under the conditions (1.11) is ordinary Thomson scattering by a moving electron (Ref. 6).[1)] It must not be confused with the stimulated scattering which is discussed below.

Spontaneous effects of higher than second order appear relatively rarely, and therefore we shall not dwell on them. We mention only that they are all contained in the conditions (1.6) if by ω_0 we understand the frequency of oscillations of an electron (or the sum of such frequencies) in corresponding fields.

Therefore the conditions of spontaneous radiation reduce to coincidence of the frequency of the radiation field in the coordinate system moving with velocity $u_\parallel$ with the frequency of oscillations of the electron or one of its harmonics. If there are many electrons, then the radiation of one of them can stimulate the same radiation of other electrons, acting on their motion. In this way we have stimulated (induced) radiation which is stimulated by spontaneous radiation. It occurs practically under the same resonance conditions as spontaneous radiation, that is, on coincidence of the frequency $\omega - k_\parallel u_\parallel$ with the electron's own frequency. Therefore the importance of the conditions of spontaneous radiation lies in the fact that they determine the radiation spectra both of individual electrons and of electron beams as a whole.

2. SINGLE-PARTICLE STIMULATED PROCESSES. THE CHERENKOV EFFECT, UNDULATOR RADIATION, THOMSON SCATTERING, AND THOMSON RADIATION

Turning to presentation of the theory of stimulated radiation of electron beams, we shall attempt initially to proceed in the same way as in Section 1 in discussion of spontaneous radiation. Namely, in calculation of the work of the field of a wave of radiation on a beam we shall calculate the work of this field on each unperturbed electron, and then sum the results. In the case of a monochromatic field it is sufficient to carry out the summation only over the electrons which are present in a wavelength of the radiation. In this way, using Eq. (1.1), we obtain for the amount of work

$$A_\infty = \pi e u_\parallel E_\parallel \delta(\omega - k_\parallel u_\parallel) \sum_j \sin \varphi_j, \qquad (2.1)$$

where φ_j is the phase of the field relative to the jth electron. However, if the beam is not modulated, the phases φ_j are uniformly distributed in the interval $(0, 2\pi)$. Therefore the sum in (2.1) vanishes.

Therefore the summation over the unperturbed individual electrons gives no radiation. This also is understandable since the coherent waves from each electron cancel each other as the result of interference. In order to obtain nonvanishing coherent radiation it is necessary to give up the assumption that the motions of the individual electrons are independent (unperturbed), i.e., it is necessary to take into account the reaction of the field of the radiation wave on each electron of the beam and in this way to consider the phases φ_j as changing under the action of this field. In just this self-consistent approach we have a phasing (bunching) of the electrons in the field of the wave and stimulated coherent radiation appears. As far as spontaneous radiation is concerned, it is not present in unmodulated beams.

We shall start with a discussion of stimulated radiation in the case of a rectilinear beam of electrons executing one-dimensional motion along a very strong external magnetic field. The frequency of the field of the radiated wave $E_\parallel(z,t) = (1/2)[E_\parallel \exp(-i\omega t + ik_\parallel z) + \text{c.c.}]$ we shall assume to be close to one of the eigenfrequencies of the system in which the beam is propagating, that is, $\omega \approx \omega(k_\parallel)$. Strictly speaking, the frequency ω contains an imaginary correction $i\delta\omega$ (increment) due to the exponential rise of the field in the linear stage of the radiation process, i.e., we shall understand ω to mean $\omega + i\delta\omega$.[2)] Under the conditions considered, the perturbation of the electron trajectories is determined by the equations of motion

$$\frac{dz}{dt} = v_\parallel, \quad \frac{dv_\parallel}{dt} = \frac{e}{m} \beta^2 \left(1 - \frac{v_\parallel^2}{c^2}\right)^{3/2} E_\parallel(z, t), \qquad (2.2)$$

where β is a parameter which characterizes the efficiency of coupling of the beam with the radiation field. This parameter can have various natures. As an illustration we have shown in Fig. 1 a picture of the interaction of a beam which is "thin" in its transverse cross section with the retarded surface wave of a dielectric. In this case $\beta \sim \Psi(x_0)$, where $\Psi(x)$ is the normalized transverse distribution of the field component $E_\parallel(z,t)$.

However, writing the equations of motion (2.2) taking into account the radiation field only on the right-hand side is not in general sufficient. The point is that the beam itself is an oscillatory system which contains an entire set of collective eigenfrequencies—plasma oscillations. If these oscillations are excited in radiation, then in the right-hand side of the second equation of (2.2) it will be necessary to add the corresponding beam fields. Processes in which collective modes of the beam are not excited are called single-particle processes. In the present section only such processes are considered; therefore the equations of motion are written just in the form (2.2). Collective processes and the criteria for their appearance will be investigated below.

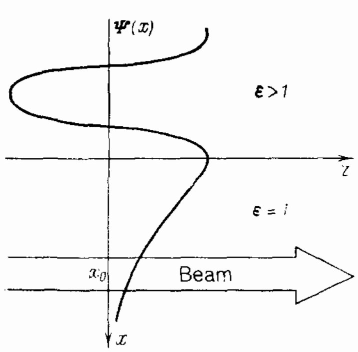

FIG. 1. Clarification of the meaning of the coupling parameter between the beam and the radiation.

Linearizing the equation of motion (2.2), we obtain in first order in the field a correction $\tilde{z}_j(t)$ to the unperturbed trajectory of the jth electron $z_j = u_\parallel(t - t_0) + z_{0j}$, emerging at time t_0 from a point z_{0j}. In finding this correction it is convenient to assume that at $t = t_0 \to -\infty$ there is no radiation, which corresponds to the assumption of adiabatic turning on of the radiation field in the past. Then substituting $\tilde{z}_j$ into the second equation (2.2), we obtain after averaging over all electrons per wavelength of the radiation field the law of variation of the average density of the beam momentum P_e:

$$\frac{dP_e}{dt} = k_\parallel \frac{|E_\parallel|^2}{8\pi} \frac{\omega_b^2 \gamma^{-3} (\omega - k_\parallel u_\parallel) \delta\omega}{[(\omega - k_\parallel u_\parallel)^2 + \delta\omega^2]^2} \beta^2, \qquad (2.3)$$

in which $\gamma = [1 - (u_\parallel^2/c^2)]^{-1/2}$ is the relativistic factor of the energy of the beam electrons, $\omega_b = (4\pi e^2 n_b/m)^{1/2}$ is the Langmuir frequency of the electrons, and n_b is their density. We note that the averaging in derivation of (2.3) was carried out in the usual way, namely,

$$\langle x \rangle = \frac{1}{N} \sum_{j=1}^{N} x_j,$$

where N is the number of electrons in a wavelength and X is the electron characteristic being averaged. For example, $P_e = n_b m \langle v_\parallel [1 - (v_\parallel^2/c^2)]^{-1/2} \rangle$. It is obvious also that small variations of P_e and of the average density of the beam energy W_e are connected by the relation

$$\delta W_e = u_\parallel \delta P_e. \qquad (2.4)$$

In what follows we shall need the conservation of energy and momentum in the radiation process:

$$\frac{d}{dt}(W + W_e) = 0, \quad \frac{d}{dt}(P + P_e) = 0, \qquad (2.5)$$

in which W and P are the densities of the energy and momentum of the wave, and also the expression for P in terms of the square of the amplitude of the radiation field,

$$P = \alpha \frac{k_\parallel}{\omega} \frac{|E_\parallel|^2}{8\pi}. \qquad (2.6)$$

Here α is a parameter determined by the specific nature of the wave excited by the beam. It will be defined more completely below.

In using Eqs. (2.4)–(2.6), we shall transform (2.3) to an equation for the densities of the energy and momentum (W, P) of the radiation:

$$\frac{d}{dt}(W, P) = -\frac{\beta^2}{\alpha} \frac{\omega\omega_b^2 \gamma^{-3} (\omega - k_\parallel u_\parallel) \delta\omega}{[(\omega - k_\parallel u_\parallel)^2 + \delta\omega^2]^2} (W, P). \qquad (2.7)$$

From (2.7) we can draw a conclusion of a general nature: if the energy of the wave increases with time, that is, if $\delta\omega > 0$, then $\omega - k_\parallel u_\parallel < 0$, and the radiating beam necessarily overtakes the wave. We shall maximize the right-hand side in (2.7) by adjustment of $\omega - k_\parallel u_\parallel$, that is, we shall find the optimal condition of radiation. We obtain

$$\omega - k_{\parallel} u_{\parallel} = -\frac{1}{\sqrt{3}}\,\delta\omega. \tag{2.8}$$

Let us now determine the increment $\delta\omega$. If we take into account that $\dot W = 2\delta\omega W$ [the density of the energy W is quadratic in the field, and therefore it rises as $\exp(2\delta\omega t)$], we find from Eqs. (2.7) and (2.8)

$$\delta\omega = \frac{\sqrt{3}}{2}\left(\frac{1}{4}\,\frac{\beta^2}{\alpha}\,\omega\omega_b^2\gamma^{-3}\right)^{1/3}. \tag{2.9}$$

In addition, it can be seen from Eqs. (2.8) and (2.9) that in the case of low-density beams in which $\delta\omega \to 0$ (more precisely, $\delta\omega \ll \omega$), the optimal condition of stimulated radiation (2.8) practically coincides with the condition of spontaneous Cherenkov radiation (1.2). Incidentally, from Eqs. (1.2), (2.4), and (2.5) we obtain the well known relation between the energy and momentum of a wave of any nature[3)]

$$P = \frac{k_{\parallel}}{\omega} W. \tag{2.10}$$

Equations (2.7)–(2.9) are extremely general, since the individual features of a system in which the beam radiates are contained only in the factor α (for an unbounded electron plasma $\alpha = 1/2$, and from (2.9) we obtain for $\beta = \sqrt{2\alpha}$ $= \sqrt{3}/2(\omega_b^2\omega/2\gamma^3)^{1/3}$, which is the known increment of beam-plasma instability[10,11)]). The processes described by Eqs. (2.7)–(2.9) are called single-particle stimulated Cherenkov effects. These processes are characterized by a common feature—longitudinal grouping of the electrons in the wave field and formation in it of retarding phases of the radiating electron bunches. This feature is also responsible for the overtaking of the radiated wave by the beam as reflected in (2.8).[10–14]

If the beam radiates in a periodically nonuniform system, then the wave field is given by the real part of the expression (1.7) with a complex frequency. Substituting this field into the right-hand side of the second equation (2.2) and essentially repeating the derivation of relation (2.7), we obtain the following equation for the radiation energy density:

$$\frac{dW}{dt} = -\frac{1}{\alpha}\left\{\sum_{n=-\infty}^{\infty} u_{\parallel} k_n \frac{\beta^2 |a_n|^2 \omega_b^2\gamma^{-3}(\omega - k_{\parallel}u_{\parallel})\,\delta\omega}{[(\omega - k_{\parallel}u_{\parallel})^2 + \delta\omega^2]^2}\right\} W, \tag{2.11}$$

where $k_n = k_{\parallel} - n\chi$. It follows from this that in radiation in a periodic system Eqs. (2.8) and (2.9) are preserved with replacement of $k_{\parallel}$ by k_n and of ω_b^2 by $|a_n|^2\omega_b^2$ respectively, and the optimal condition of radiation reduces to one of the conditions (1.9). In addition, in the sum which enters into (2.12) we need retain only the single term corresponding to that n value for which (1.9) is satisfied. Equation (2.11) determines the single-particle stimulated Cherenkov effect in a periodically nonuniform system.[14,15]

Now let each beam electron, in addition to uniform motion, execute in the external fields also a specified oscillatory motion with a fixed amplitude. For example, $z = u_{\parallel}t + z_0 + \hat z$, where $\hat z(t) - (2\pi/\omega_0)$ is a periodic function which does not have a constant component. In this case instead of (2.11) we obtain

$$\frac{dW}{dt} = -\frac{\omega}{\alpha}\sum_{n=-\infty}^{\infty}\left\{\beta^2 |A_n|^2 \frac{\omega_b^2\gamma^{-3}(\omega_n - k_{\parallel}u_{\parallel})\,\delta\omega}{[(\omega_n - k_{\parallel}u_{\parallel})^2 + \delta\omega^2]^2}\right\} W, \tag{2.12}$$

where $\omega_n = \omega - n\omega_0$ and $A_n \sim [\max_t (k_{\parallel}\hat z)^n]$. The process defined by Eq. (2.12) we shall call single-particle stimulated undulator radiation.[16,17] For stimulated undulator radiation, Eqs. (2.8) and (2.9) are valid with replacement of ω by ω_n and of ω_b^2 by $|A_n|^2\omega_b^2$ respectively, and the condition of radiation reduces to (1.6). In addition, as in the case of a periodic system, in Eq. (2.12) it is sufficient to retain only a single term in the infinite sum.

The wave radiated in an undulator undergoes a Doppler shift of the frequency. For example, from the condition $\omega = k_{\parallel}u_{\parallel} + \omega_0$ for the very simple spectrum $\omega = \omega(k_{\parallel}) = \mathrm{sign}(k_{\parallel})k_{\parallel}c$ we have

$$\omega = \omega_0\left(1 - \mathrm{sign}\,k_{\parallel}\cdot\frac{u_{\parallel}}{c}\right)^{-1}. \tag{2.13}$$

A similar shift of frequency occurs also in radiation in a periodic system.

Equation (2.12) is valid if the amplitude of the electron oscillations with frequency ω_0 is maintained at a constant level. In just this case one speaks of undulator radiation. However, if the oscillation amplitude is variable, a separate treatment is necessary. We shall carry out this treatment, using the conservation laws. Suppose that the oscillations of an electron are due to the action on it of an electromagnetic wave with frequency ω_2 and wave number $k_{\parallel 2}$. Thomson scattering of this wave by the beam electrons occurs. Obviously in this case $\omega_0 = \omega_2 - k_{\parallel 2}u_{\parallel}$. Denoting the frequency and wave number of the scattered wave by ω_1 and $k_{\parallel 1}$ respectively and taking into account that in Thomson scattering the radiation condition has the form (1.11), we obtain for the frequencies $\omega_{1,2}$ and the wave numbers $k_{\parallel 1,2}$ the relations of combination (Raman) resonance. We shall write them in the form

$$(\omega_1 - k_{\parallel 1}u_{\parallel}) - (\omega_2 - k_{\parallel 2}u_{\parallel}) = 0, \tag{2.14}$$

assuming that $\omega_1 > 0$ and that the sign of ω_2 is arbitrary.[4)] Equation (2.14) must now be considered as the condition of stimulated Thomson scattering or radiation, while (1.11) gives the conditions of spontaneous radiation of second order.

We shall express the energies and momenta of the incident and scattered waves in terms of the numbers of quanta N_1 and N_2:

$$W_1 = \omega_1 N_1, \quad W_2 = |\omega_2| N_2, \quad P_1 = k_{\parallel 1}N_1, \quad P_2 = \mathrm{sign}\,\omega_2\cdot k_{\parallel 2}N_2. \tag{2.15}$$

Then, writing the conservation laws for small variations of the energy and momentum in the form

$$\omega_1\,\delta N_1 + |\omega_2|\,\delta N_2 + \delta W_e = 0, \quad k_{\parallel 1}\,\delta N_1 + \mathrm{sign}\,\omega_2\cdot k_{\parallel 2}\,\delta N_2 + \delta P_e = 0 \tag{2.16}$$

and taking into account Eqs. (2.14) and (2.4), we obtain from (2.16) the well known Manley-Rowe relations[18]:

$$\delta N_1 + \mathrm{sign}\,\omega_2\cdot\delta N_2 = 0, \quad (\omega_1 - \omega_2)\,\delta N_1 + \delta W_e = 0. \tag{2.17}$$

Let $\omega_1 > \omega_2 > 0$. Then $N_1 + N_2 = \mathrm{const}$, that is, the total number of quanta is conserved. If in addition $\delta W_e < 0$ (the beam gives up energy to the scattered wave), then $\delta N_1 > 0$ and $\delta N_2 < 0$. Consequently stimulated scattering occurs with an increase of the frequency. The opposite process is also possible—scattering with decrease of the frequency, and it is accompanied by an increase of the average energy of the electron beam, that is, by acceleration of the electrons.

Now let $\omega_2 < 0$. In this case $\delta N_1 > 0$ and $\delta N_2 > 0$, that is, the total number of quanta increases. Consequently stimulated radiation of two waves occurs. The inverse process is also possible—absorption of two waves, and here the beam is accelerated.

It should be mentioned that the requirement $\omega_2 < 0$ in itself has no physical meaning, just as the sign of the frequency has in itself no meaning. Simply by setting $\omega_2 < 0$ we turn on in (2.14) also the condition (1.11), taken with the lower sign (plus).[5)]

It is obvious that if in the processes of Thomson scattering and radiation the amplitude of the wave with frequency ω_2 is assumed to be constant (the case of high-power pumping), then for the energy density of the wave with frequency ω_1 it is easy to obtain an equation of the type (2.12) with $\omega_0 = \omega_2 - k_{\parallel 2} u_\parallel$, that is, it is easy to go over to the undulator case. Nevertheless, undulator radiation must be distinguished from Thomon processes since the pumping in an undulator is created and maintained by external sources and is not an eigenwave of the system in which the beam is propagated. For this same reason it is necessary to distinguish the spontaneous processes (1.6) and (1.10).

We shall make an additional important remark on the behavior of Eq. (2.7) (and equations similar to it). In derivation of this equation it was assumed that all electrons at $t = t_0$ have an identical velocity $u_\parallel$, and therefore under the resonance conditions (2.8) they all interact strongly with the radiated wave. However, in real beams the electrons have a spread in velocity which can be taken into account by introducing at $t = t_0$ an unperturbed electron distribution function $f_0(v_\parallel)$ normalized to unity. The generalization of Eq. (2.7) to the case of a beam with a spread in velocities is obvious and has the form

$$\frac{d}{dt}(W,\ P) = -\frac{\beta^2\omega\omega_b^2}{\alpha} J\cdot(W,\ P),$$

$$J = \int_{-\infty}^{+\infty} f_0(v_\parallel)\frac{\delta\omega\,(\omega - k_\parallel v_\parallel)\,dv_\parallel}{[(\omega - k_\parallel v_\parallel)^2 + \delta\omega^2]^2}. \qquad (2.18)$$

In the case of a single-velocity beam in which $f_0(v_\parallel) = \delta(v_\parallel - u_\parallel)$, from (2.18) we obtain (2.7) and the results (2.8) and (2.9). The same results are obtained if $f_0(v_\parallel)$ has a finite but small width, namely,

$$\frac{\Delta v_\parallel}{u_\parallel} \ll \frac{\delta\omega}{\omega}. \qquad (2.19)$$

where $\Delta v_\parallel$ is the half-width of the distribution function and $\delta\omega$ is the increment (2.9). Actually, if (2.19) is satisfied, the function $x(x^2 + \delta\omega^2)^{-2}$, where $x = \omega - k_\parallel v_\parallel$, can be taken outside the integral sign in the expression for J also. The condition (2.19) in this case means that all beam electrons interact strongly with the radiated wave, or, as one can say, the interaction has a hydrodynamical nature (it is integrated over the distribution function).

Let us consider now the opposite limit, i.e., the inverse of (2.19). In this case we can take out from under the integral sign in the formula for J the quantity $f_0(v_\parallel)$; however, the subsequent integration here gives zero. To make the result more accurate we shall expand $f_0(v_\parallel)$ in the vicinity of the phase velocity of the wave $\omega/k_\parallel$ with accuracy to terms of first order:

$$f_0(v_\parallel) = f_0\left(\frac{\omega}{k_\parallel}\right) + \left(v_\parallel - \frac{\omega}{k_\parallel}\right)\frac{\partial f_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega/k_\parallel}.$$

Following this, carrying out the elementary integration, we obtain the following result:

$$\frac{d}{dt}(W,\ P) = \frac{\pi}{2}\beta^2\frac{\omega\omega_b^2}{\alpha}\frac{1}{k_\parallel^2}\frac{\partial f_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega/k_\parallel}(W,\ P). \qquad (2.20)$$

From this we find the expression which determines the direction and rate of change of the energy density of the radiation:

$$\delta\omega = \frac{\pi}{2}\beta^2\frac{\omega\omega_b^2}{\alpha k_\parallel^2}\frac{\partial f_0}{\partial v_\parallel}\bigg|_{v_\parallel = \omega/k_\parallel}, \qquad (2.21)$$

which differs substantially from (2.9).

From (2.21) it can be seen that for $\partial f_0/\partial v_\parallel < 0$, when the number of electrons lagging the wave is greater than the number of electrons leading it, $\delta\omega < 0$, i.e., the intensity of the radiation decreases. This is the well known Landau damping effect.[1,18] However, if $\partial f_0/\partial v_\parallel > 0$, then the intensity of the radiation increases. In this case one speaks of the inverse Landau damping effect.[45,47]

The Landau effects are due to resonance interaction with the wave of a small part of the electron beam—only those electrons for which the Cherenkov resonance condition (1.2) is satisfied. The quantity (2.21) is determined by the detailed structure of the distribution function, and therefore the Landau effects are kinetic. The strictly resonance nature of kinetic effects, and also the proportionality of the increment $\delta\omega$ to the density of radiators to the first power, makes them similar to spontaneous-radiation effects. In what follows we shall not discuss kinetic effects. We mention them here only for completeness of the presentation.

3. COLLECTIVE STIMULATED PROCESSES. THE COLLECTIVE CHERENKOV EFFECT OR THE RAMAN EFFECT. RAMAN SCATTERING AND RAMAN RADIATION

In derivation of Eq. (2.7), we made use of Eq. (2.2), in the right-hand side of which there is only the field of the radiated wave. However, this is not always valid. Indeed, the radiation modulates the electron flux in density, and additional beam fields arise in it, which is schematically shown in Fig. 2. The beam fields result in the existence of a whole group of collective stimulated processes.

Beam fields in the electron equation of motion are rather simple to take into account qualitatively. Indeed, on displacement of an electron from its equilibrium position, it is acted on by a restoring force which is proportional to the displacement. The proportionality coefficient is the square of the frequency of plasma oscillations of the beam,[19] which we shall designate as Ω_b^2.[6)] Taking this into account, we can write the equations of motion of the electron in the form [cf. Eq. (2.2)]

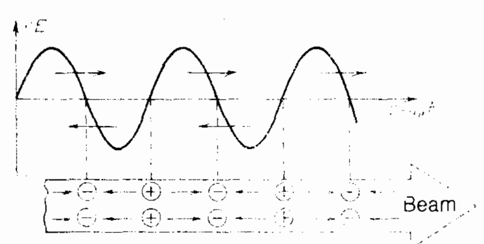

FIG. 2. Modulation of a beam and the appearance of beam fields.

$$\frac{dz}{dt} = v_\parallel,$$

$$\frac{dv_\parallel}{dt} + \Omega_b^2 (z - z_n) = \frac{e}{m} \beta^2 \left(1 - \frac{v_\parallel^2}{c^2} \right)^{3/2} E_\parallel (z, t), \quad (3.1)$$

where $z_n = z_0 + u_\parallel (t - t_0)$ is the unperturbed trajectory emerging from the point z_0.

From Eq. (3.1), proceeding exactly in the same way as in derivation of Eq. (2.3), we can easily obtain the following law of variation of the average density of the beam momentum P_e:

$$\frac{dP_e}{dt} = k_\parallel \frac{|E_\parallel|^2}{8\pi} \frac{\beta^2 \omega_b^2 \gamma^{-3} (\omega - k_\parallel u_\parallel) \delta\omega}{[(\omega - k_\parallel u_\parallel)^2 - \Omega_b^2 - \delta^2\omega]^2 + 4 (\omega - k_\parallel u_\parallel)^2 \delta^2\omega}. \quad (3.2)$$

The further manipulation of Eq. (3.2) involves use of the conservation of energy and momentum. In writing down these conservation laws it is necessary to take into account the electromagnetic energy and momentum of the plasma oscillations of the beam. In regard to the electromagnetic momentum, in the case of potential plasma oscillations it is equal to zero. Limiting the discussion to just this case,[7)] we shall write the momentum-conservation law in the form (2.5). However, the energy-conservation law must be written as follows:

$$\frac{d}{dt} (W + W_e + W_\parallel) = 0, \quad (3.3)$$

including in the total balance in addition to the densities of the radiation energy and of the average kinetic energy of the electrons also the density of the electrostatic energy of the plasma oscillations $W_\parallel$.

Using the momentum conservation law and Eq. (2.10), we transform Eq. (3.2) to the form

$$\frac{d}{dt} (W, P) = -\frac{\omega}{\alpha} \frac{\beta^2 \omega_b^2 \gamma^{-3} (\omega - k_\parallel u_\parallel) \delta\omega}{[(\omega - k_\parallel u_\parallel)^2 - \Omega_b^2 - \delta\omega^2]^2 + 4 (\omega - k_\parallel u_\parallel)^2 \delta\omega^2} (W, P). \quad (3.4)$$

As $\Omega_b^2 \to 0$ Eq. (3.4) goes over into Eq. (2.7). However, this is also understandable—in a rectilinear completely magnetized beam with a vanishingly small density of electrons, radiation can be due only to the single-particle spontaneous Cherenkov effect.

Let us discuss now the case of high densities of electrons, in which

$$\Omega_b^2 \gg \delta\omega^2, \quad (3.5)$$

and Eq. (3.4) is written in the following form:

$$\frac{d}{dt} (W, P) = -\frac{\omega}{\alpha} \frac{\beta^2 \omega_b^2 \gamma^{-3} (\omega - k_\parallel u_\parallel) \delta\omega}{[(\omega - k_\parallel u_\parallel)^2 - \Omega_b^2]^2 + 4 (\omega - k_\parallel u_\parallel)^2 \delta^2\omega} (W, P). \quad (3.6)$$

It follows from (3.6) that, as in the case of (2.7), the radiation energy density W increases with time, provided that $\omega - k_\parallel u_\parallel < 0$. The maximum of the increment is realized for the condition

$$\omega (k_\parallel) = k_\parallel u_\parallel - \Omega_b, \quad (3.7)$$

which is different from (2.8). The increment itself also has a form different from (2.9):

$$\delta\omega = \frac{1}{2} \beta \left(\frac{1}{2\alpha} \frac{\omega_b^2 \gamma^{-3} \omega}{\Omega_b} \right)^{1/2}. \quad (3.8)$$

Using now the latter expression, we shall give an explicit form to the inequality (3.5):

$$\Omega_b \gg \frac{1}{8} \frac{\beta^2}{\alpha} \omega \frac{\omega_b^2 \gamma^{-3}}{\Omega_b^2}, \quad (3.5')$$

thereby expressing explicitly the criterion of high density of the beam electrons.

Radiation under the conditions (3.7) and (3.5) is called stimulated Raman effect.[20] This is a collective effect, into which the single-particle Cherenkov effect is transformed with increase of the density of electrons in the beam. The meaning of the name "Raman effect" will be made clear below. We note here that the term "collective Cherenkov effect" would probably be more appropriate in the present case, and we shall use this terminology in what follows.

To make clear the physical essence of the collective Cherenkov effect we recall that the spectra of beam plasma waves are determined by the expressions

$$\omega = k_\parallel u_\parallel \pm \Omega_b. \quad (3.9)$$

The spectrum with the minus sign corresponds to the so-called slow wave with negative energy. In excitation of this wave, energy is removed from the beam.[21] However, as can be seen from Eq. (3.9), the condition of resonance (3.7) is the condition of excitation of just the slow wave. The energy liberated here goes into radiation.[8)]

It is possible, by means of the equations of "cold" hydrodynamics and the Poisson equation, to show independently that the electrostatic energy of the plasma oscillations of a beam is given by the formulas

$$W_\parallel = \pm \frac{\Omega_b}{k_\parallel u_\parallel} \delta W_e. \quad (3.10)$$

The upper sign here refers to the fast wave, and the lower—to the slow wave. Since electrostatic energy is always positive, then from Eq. (3.10) it is again evident that on excitation in the beam of a slow wave it slows down, i.e., $\delta W_e < 0$.

Taking Eq. (3.10) into account, we rewrite the conservation law (3.3) in the form

$$\frac{d}{dt} (W + W_b^{(-)}) = 0, \quad (3.11)$$

where

$$W_b^{(-)} = \left(1 - \frac{\Omega_b}{k_\parallel u_\parallel} \right) \delta W_e \quad (3.12)$$

is the total energy of the slow wave. Now using (3.7) and (2.4) and the general relation (2.10), we obtain from (3.12) an expression for the momentum of the slow beam wave:

$$P_b^{(-)} = \frac{k_\parallel}{\omega} W_b^{(-)} = \delta P_e, \quad (3.13)$$

which in the potential (electrostatic) case considered is obvious.

The process just discussed, like all stimulated processes investigated in the previous section, leads to instability corresponding to an exponential rise of the field with time. Indeed, both (2.7) and (3.6) for $\omega - k_\parallel u_\parallel < 0$, and also (2.11) and (2.12), reduce to the form

$$\frac{dW}{dt} = 2\delta\omega \cdot W \quad (3.14)$$

with positive $\delta\omega$. Therefore the energy density of the radiation in the linear approximation grows without limit at the expense of the beam energy with arbitrarily small initial perturbations in the system.[9)]

We shall now give an example of a completely different process. We shall set in (3.6) $\omega - k_{\parallel} u_{\parallel} = \Omega_b$, which, as can be seen from (3.9), is the condition of resonance interaction of the radiation with the fast beam wave. Here Eq. (3.6) reduces to a form different from (3.14):

$$\frac{dW}{dt} = i \cdot 2\delta\omega \cdot W, \tag{3.15}$$

where $\delta\omega$ coincides with (3.8). This is one of the examples of interaction of two waves with positive energy. In such an interaction, energy from the fast beam wave is periodically pumped over into radiation and vice versa.

The energy and momentum of the fast beam wave are given by the expressions

$$W_b^{(+)} = \left(1 + \frac{\Omega_b}{k_{\parallel} u_{\parallel}}\right) \delta W_e, \quad P_b^{(+)} = \delta P_e, \tag{3.16}$$

in which, as can be seen from (3.10), δW_e (and δP_e) are greater than zero.[10)] However, the conservation law (3.3) reduces to the form

$$\frac{d}{dt}(W + W_b^{(+)}) = 0. \tag{3.17}$$

Since the signs of W and W_b^+ coincide, an increase of one of them is accompanied by a decrease of the other. Consequently, for development of a process involving a fast wave it is necessary to have in the system a finite initial perturbation, which also is a source of energy. The characteristic behavior of the energy densities in the processes (3.14) and (3.15) is shown in Fig. 4. A radiation process of the type (3.15) is not an instability; it is a simple decay of the initial perturbation. Note that if the inequality (3.5) is not satisfied, such decay is impossible: it is suppressed by the instability based on the stimulated single-particle Cherenkov effect.

The stimulated collective Cherenkov effect is realized also in periodic structures. Here the resonance condition is written in the form

$$\omega(k_{\parallel}) = k_{\parallel} u_{\parallel} + n\chi u_{\parallel} \pm \Omega_b, \tag{3.18}$$

and Eqs. (3.8) and (3.5) are retained with replacement of ω_b^2 by $|a_n|^2 \omega_b^2$.

With increase of the beam density, there is also a change in form of the stimulated undulator radiation defined by Eq. (2.12). For example, the resonance condition is written in the form

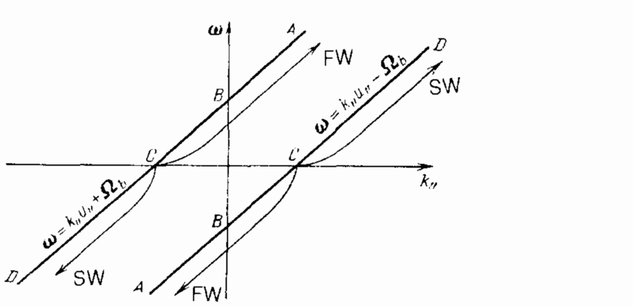

FIG. 3. Dispersion and signs of the energy and momentum of beam waves. AC—fast wave (FW); CD—slow wave (SW); AB: $\delta W_e > 0$, $P_b > 0$, $W_b > 0$; BC: $\delta W_e > 0$, $P_b < 0$, $W_b > 0$; CD: $\delta W_e < 0$, $P_b < 0$, $W_b < 0$.

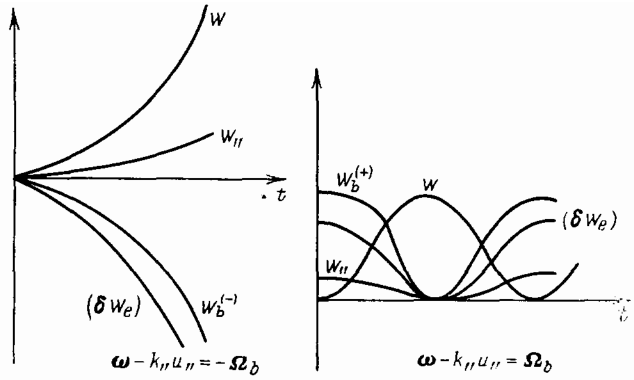

FIG. 4. Change of the energies of waves in a radiative instability and with simple decay of the initial perturbation.

$$\omega(k_{\parallel}) = k_{\parallel} u_{\parallel} + n\omega_0 \pm \Omega_b, \tag{3.19}$$

and Eqs. (3.8) and (3.5) are retained with the substitution $\omega_b^2 \to |A_n|^2 \omega_b^2$. Radiation occurring under the conditions (3.19) we shall call collective undulator radiation.

In conclusion we shall discuss the interaction of two waves under conditions in which the amplitude of the scattered wave is not fixed. The resonance condition (3.19) in this case we shall write in the form

$$(\omega_1 - k_{\parallel 1} u_{\parallel}) - (\omega_2 - k_{\parallel 2} u_{\parallel}) = \pm\Omega_b, \tag{3.20}$$

having generalized the condition of Thomson scattering and radiation (2.14) to the case of dense beams. If we use the notation

$$\omega = \omega_1 - \omega_2, \quad k_{\parallel} = k_{\parallel 1} - k_{\parallel 2}, \tag{3.21}$$

then Eq. (3.20) coincides with the conditions (3.9) of resonance excitation of beam waves. On the other hand, the relations (3.21) are the conditions of nonlinear interaction of the waves.[18,31] The processes (3.20) are just such interactions of two electromagnetic waves and one beam wave, and they must be assigned to processes of the Raman scattering and radiation type.

Let us consider the conservation laws in the interaction of waves. For this purpose we shall express the energy and momentum of the plasma oscillations of the beam in terms of the corresponding number of quanta (plasmons) N_b:

$$W_b^{(\pm)} = \pm\omega N_b, \quad P_b^{(\pm)} = \pm k_{\parallel} N_b \quad (\omega > 0). \tag{3.22}$$

We shall take first in condition (3.20) the minus sign, i.e., we shall consider the interaction of electromagnetic waves with a slow beam wave. In this case the conservation of energy and momentum are written in the form

$$\begin{gathered} \omega_1 \delta N_1 + |\omega_2| \delta N_2 - \omega \delta N_b = 0, \\ k_{\parallel 1} \delta N_1 + \text{sign}\, \omega_2 \cdot k_{\parallel 2} \delta N_2 - k_{\parallel} \delta N_b = 0. \end{gathered} \tag{3.23}$$

From this, with inclusion of (3.21), we obtain the Manley-Rowe relations[18]

$$\begin{gathered} \delta N_1 - \delta N_b = 0, \quad \delta N_2 + \text{sign}\, \omega_2 \cdot \delta N_b = 0, \\ \delta N_1 + \text{sign}\, \omega_2 \cdot \delta N_2 = 0, \end{gathered} \tag{3.24}$$

which are analyzed below on the assumption that in the initial state the beam is not perturbed, that is, $\delta N_b > 0$.

If $\omega_1 > \omega_2 > 0$, then $\delta N_1 > 0$, and $\delta N_2 < 0$. This is Raman scattering with increase of the frequency, that is, the process into which Thomson scattering with increase of the frequency is transformed on increase of the beam density. If $\omega_1 > 0$ and $\omega_2 < 0$, then, as can be seen from (3.24), we have $\delta N_1 > 0$ and $\delta N_2 > 0$, that is, the amplitudes of the two electromagnetic waves grow simultaneously. In this case Raman radiation of two waves is realized. The Thomson radiation of two waves discussed previously goes over at once into Raman radiation of two waves with increase of the beam density.

In interaction of electromagnetic waves with the fast beam wave [the plus sign in (3.20) and (3.22)] the conservation laws and the Manley-Rowe relations have the form

$$\begin{gathered}\omega_1\delta N_1 + |\omega_2|\delta N_2 + \omega\delta N_b = 0,\\ k_{\parallel 1}\delta N_1 + \text{sign}\,\omega_2\cdot k_{\parallel 2}\cdot\delta N_2 + k_\parallel\delta N_b = 0,\\ \delta N_1 + \delta N_b = 0, \quad \delta N_2 - \text{sign}\,\omega_2\cdot\delta N_b = 0,\\ \delta N_1 + \text{sign}\,\omega_2\cdot\delta N_2 = 0.\end{gathered} \tag{3.25}$$

If $\omega_1 > \omega_2 > 0$ and the beam in its initial state is not perturbed ($\delta N_b > 0$), then $\delta N_1 < 0$ and $\delta N_2 > 0$, that is, there is scattering with reduction of the frequency with simultaneous excitation of the fast wave of the beam. The source of energy is the energy of the high-frequency electromagnetic wave. In the beam system the frequency of the scattered wave is less than the frequency of the incident wave,

$$\Omega_2 = \Omega_1 - \Omega_b. \tag{3.26}$$

However, this relation is satisfied in the Raman scattering of light (the Raman-Mandel'shtam-Landsberg effect[22]) and corresponds to the normal Stokes line in the scattering spectrum. For this reason all processes which take into account the correction Ω_b in the conditions for radiation and scattering are called Raman. However, this terminology does not take into account singularities associated with motion of the beam and with the negative energy of the slow beam wave. Therefore its application to all these processes, in our opinion, is not completely justified.[11]

If $\delta N_b < 0$ (the beam is initially excited) and $\omega_2 > 0$, then $\delta N_1 > 0$ and $\delta N_2 < 0$. This is scattering with increase of the frequency, not at the expense of the energy of directed motion of the electrons, but rather at the expense of the energy of the low-frequency wave and the fast beam wave. This case completely corresponds to classical Raman scattering with an anti-Stokes spectrum.

If $\delta N_b < 0$ and $\omega_2 < 0$, then $\delta N_1 > 0$ and $\delta N_2 > 0$. This means that the two waves radiate at the expense of the energy of the plasma oscillations of the electrons of the beam.

All processes with participation of the fast wave are not properly assigned to radiative beam instabilities, since for them the presence in the beam of energy of directed motion is not fundamental. They are based on the simple decay of the initial wave perturbation into wave perturbations of another type. On decrease of the beam density, all processes involving the fast wave (like processes involving the slow wave) are modified into the stimulated Cherenkov and Thomson effects, that is, into single-particle radiative instabilities.

4. ANOMALOUS AND NORMAL DOPPLER EFFECTS

In all the single-particle processes discussed in Section 2, the average energy of the electron flux was determined by the energy of its rectilinear quasiuniform motion (quasiuniform because during radiation this energy slowly decreases). A completely different case is that of an electron oscillator, that is, an electron in the field of gyroscopic or conservative external forces. Let us consider how a beam of such oscillators radiates. Here it is necessary to take into account that, in addition to the density of the "longitudinal" momentum P_e and the density of the "longitudinal" energy W_e, in the beam there is a density of "transverse" energy of the oscillator motion $W_\perp$.[12] Writing down the laws of conservation of energy and momentum for the radiation

$$\frac{d}{dt}(W + W_e + W_\perp) = 0, \qquad \frac{d}{dt}(P + P_e) = 0. \tag{4.1}$$

we obtain for the variations of the energy of the electromagnetic field and W_e the following relations:

$$\delta W = -\frac{\omega}{\omega - k_\parallel u_\parallel}\delta W_\perp, \qquad \delta W_e = \frac{k_\parallel u_\parallel}{\omega - k_\parallel u_\parallel}\delta W_\perp. \tag{4.2}$$

In obtaining Eq. (4.2) we have taken into account the relations (2.10) and (2.4). It is evident from this that in the Cherenkov effect ($\omega = k_\parallel u_\parallel$) the energy of oscillatory motion of the electron does not change ($\delta W_\perp = 0$). Different situations are realized for $\omega \neq k_\parallel u_\parallel$.

We shall assume for the moment that $\omega > 0$ and the beam is radiating, that is, $\delta W > 0$. If $\omega - k_\parallel u_\parallel < 0$ (here automatically $k_\parallel > 0$), then $\delta W_\perp > 0$ and $\delta W_e < 0$. Consequently, in radiation the transverse energy of the oscillatory motion of the electrons increases. Energy of the longitudinal motion is expended on this, and it also goes into radiation. This radiation is called the anomalous Doppler effect.[2,12,23] Therefore in the anomalous Doppler effect even a beam electron moving initially in a straight line, on radiating, begins to oscillate.

If $\omega - k_\parallel u_\parallel > 0$ and $k_\parallel > 0$, then $\delta W_\perp < 0$ and $\delta W_e < 0$. Consequently, an electron beam radiates both as the result of its own transverse oscillatory energy, and as the result of the energy of longitudinal translational motion. However, if at the beginning of the process $W_\perp = 0$, then there is no radiation. Thus, for radiation an initial reserve of oscillatory energy is necessary. This radiation is called the normal Doppler effect.[2,12,23]

Finally, for $\omega - k_\parallel u_\parallel > 0$ and $k_\parallel < 0$ we have $\delta W_\perp < 0$ and $\delta W_e > 0$. That is, in radiating, the beam is accelerated in the longitudinal direction. Again the energy of the transverse oscillatory motion is expended in this process. This process in the literature is also called the normal Doppler effect.

Let us mention one additional circumstance which follows from the equations (4.2). For example, suppose that the radiation spectrum $\omega(k_\parallel)$ has the very simple form $\omega = k_\parallel c_0$, where c_0 is the phase velocity of the electromagnetic waves. Then, if $c_0 - u_\parallel \ll u_\parallel$, in the normal effect we have

$$|\delta W| \approx |\delta W_e| = \frac{u_\parallel}{c_0 - u_\parallel}|\delta W_\perp| \gg |\delta W_\perp|. \tag{4.3}$$

That is, an electron beam radiates at the expense of the energy of longitudinal motion, while the oscillatory (transverse) energy changes very little, being, as it were, a seed. A similar situation exists in the anomalous Doppler effect, in which $c_0 < u_\parallel$ and $u_\parallel - c_0 \ll u_\parallel$. Therefore at $c_0 \sim u_\parallel$ the Doppler effects are energetically equivalent to the stimulated Cherenkov effect.

In Fig. 5 we have shown the regions of frequencies and

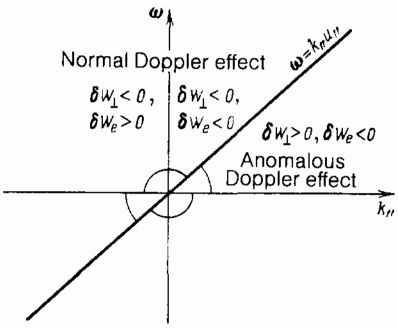

FIG. 5. The regions of the anomalous and normal Doppler effects.

wave numbers in which the various effects are possible. The figure generalizes the discussion carried out above for $\omega > 0$ to negative frequencies ($\omega < 0$).

We shall consider as an illustration of the anomalous effect the radiation of an electron initially traveling in a straight line in a constant magnetic field.[24] We shall obtain a formula of the type (2.7) for the intensity of radiation. For this purpose we shall write down the equations of motion of the electrons for the velocity components $v_\parallel$ and $v_\perp = v_x + iv_y$:

$$\frac{dv_\perp}{dt} + i\omega_H v_\perp = -\frac{e}{mc}\left(\frac{\partial A_\perp}{\partial t} + v_\parallel \frac{\partial A_\perp}{\partial z}\right), \qquad (4.4)$$

$$\frac{dv_\parallel}{dt} = \frac{1}{2}\frac{e}{mc}\left(v_\perp \frac{\partial A_\perp^*}{\partial z} + \text{c.c.}\right),$$

where $A_\perp = A_x + iA_y$ is the vector potential. We shall take the direction of rotation of the plane of polarization of the field of radiation in accordance with the formula $A_\perp = A_0 \exp(i\omega t - k_\parallel z)$, where ω is a complex frequency (here $\omega \to \omega - i\delta\omega$). Taking $v_\perp$ from the linearized first equation of (4.4) and substituting it into the second equation, we obtain with inclusion of (4.1) and (2.10) the following equation for the intensity of radiation:

$$\frac{d}{dt}(W, P) = \frac{1}{\alpha}\,\omega\,\frac{\omega_b^2\omega_H\delta\omega}{\omega^2\left[(\omega - k_\parallel u_\parallel + \omega_H)^2 + \delta\omega^2\right]}(W, P), \qquad (4.5)$$

where α is a parameter determined by the nature of the wave in accordance with the formula

$$W = \alpha\,\frac{\omega^2}{c^2}\,\frac{|A_0|^2}{4\pi}. \qquad (4.6)$$

Incidentally, in derivation of Eq. (4.5) it can be shown that the right-hand side of the second equation of (4.4) does not depend on the coordinate z. Consequently, in this case there is no bunching of the electrons in the wave field, that is, all electrons are slowed down identically, regardless of their phase. This distinguishes the considered process substantially from the stimulated Cherenkov effect.

From Eq. (4.5) we determine in an obvious way the condition of optimal radiation in the anomalous Doppler effect:

$$\omega(k_\parallel) = k_\parallel u_\parallel - \omega_H, \qquad (4.7)$$

which coincides exactly with one of the conditions of spontaneous radiation (1.4) (for $\gamma = 1$, since $u_\parallel \ll c$), and the increment[13)] is

$$\delta\omega = \left(\frac{1}{\alpha}\,\frac{\omega_b^2\omega_H}{2\omega}\right)^{1/2}. \qquad (4.8)$$

It is easy to show also that the electron energy densities W_e and $W_\perp$ and the radiation density W are connected with each other by relations of the type (4.2).

It is appropriate to mention also the following circumstances. Equations (4.5), (4.7), and (4.8) differ substantially from those obtained in discussion of stimulated Cherenkov effects (in a uniform or periodic system), since they qualitatively define a different effect. Here radiation arises even in an initially rectilinear beam ($u_\perp = 0$), since in the anomalous Doppler effect the electron "untwists" and acquires a transverse velocity. In regard to spontaneous radiation, as can be seen from (1.3), for $u_\perp = 0$ it is impossible. Finally, the effect considered is a single-particle effect since in the right-hand sides of the equations (4.4) there is no contribution from collective beam fields; there is no contribution from the energy and momentum of such fields also in the conservation laws (4.1).

We shall make one further important remark. The conditions of stimulated undulator and Cherenkov radiation (1.6) and (1.9) fall in the regions of the anomalous and normal Doppler effects. However, it is erroneous to identify them for this reason with the latter. We can speak of Doppler effects if the oscillatory properties of the electrons are due to gyroscopic or conservative external forces, that is, if we can introduce a conserved quantity $W_\perp$.

There is a deep analogy[25] between the collective Cherenkov effect and the anomalous Doppler effect. In the first place, the condition (3.7), since $\omega - k_\parallel u_\parallel < 0$, falls just into the region of the anomalous effect. Second, in the anomalous Doppler effect internal degrees of freedom of the beam are excited, which are determined by the individual properties of the electron oscillators. Internal degrees of freedom—plasma oscillations—are excited also in the collective Cherenkov effect, but they are determined already not by the individual properties of the electrons, but by the collective behavior of the beam as a whole. This analogy can be seen also from the similarity of the equations (3.6) and (4.5), especially, under the conditions (3.5), in the simplified form of writing of Eq. (3.6);

$$\frac{d}{dt}(W, P) = \frac{\omega}{\alpha}\,\frac{\beta^2}{4\Omega_b^2}\,\frac{\omega_b^2\gamma^{-3}\Omega_b\delta\omega}{(\omega - k_\parallel u_\parallel + \Omega_b)^2 + \delta\omega^2}(W, P). \qquad (4.9)$$

The analogy can be carried still further if with inclusion of (4.7) we rewrite the second relation of (4.2) in the form

$$W_\perp = -\frac{\omega_H}{k_\parallel u_\parallel}\,\delta W_e \qquad (4.10)$$

and compare it with the second expression of (3.10). Formally this reduces to the replacement $\omega_H \to \Omega_b$. From the physical point of view it is much more important that the electrostatic energy of plasma oscillations $W_\parallel$ in the collective Cherenkov effect is equivalent to the energy $W_\perp$ in the anomalous Doppler effect. It can also be said that the relation (4.7) determines the spectrum of the wave with negative energy, but in the absence of a slow beam wave, the wave of (4.7) is single-particle.

Since in the collective Cherenkov effect under the conditions (3.7), as in the anomalous Doppler effect, a faster-than-light beam is radiating (i.e., the phase velocity of the wave of radiation is less than the velocity of the beam), collective Cherenkov radiation under the conditions (3.7) is frequently called anomalous Doppler radiation.

In the case of the normal Doppler effect the velocity of the electrons is less than the phase velocity of the wave, that is, a slower-than-light beam is radiating. Here an initial reserve of energy of the transverse motion $W_\perp$ is necessary for radiation. However, the same thing occurs in interaction of a beam and radiation under the conditions $\omega - k_\parallel u_\parallel = \Omega_b$. Therefore the normal Doppler effect is not assigned to radiative instabilities, i.e., under the conditions $\omega - k_\parallel u_\parallel = \omega_H > 0$ an electron beam is stable against excitation of electromagnetic waves. However, in the case of relativistic beams the latter statement can become invalid since there is an additional mechanism of stimulated radiation which we shall consider in the next section.

5. STIMULATED CYCLOTRON RADIATION

Spontaneous radiation, which we were discussing in Section 1, is due to an actual displacement of electric charges in space—an accelerated displacement in dipole radiation and a uniform displacement in Cherenkov radiation in a retarding medium or in a periodic structure. In unperturbed spatially homogeneous beams there is no actual displacement of charges, since at any moment of time at any point of space the charge density is $\rho = \text{const}$. Therefore there is also no spontaneous radiation.[14)] One can say, however, that spontaneous radiation initiates stimulated radiation. For example, any initial regular modulation of a beam, even one which arises at random, creates spatially coherent spontaneous radiation, which in turn produces further modulation of the beam. As a result the beam is broken up into a sequence of radiating electron bunches or, in other words, it is bunched by the radiation field. This bunching is the principal factor in the transition of spontaneous (bare) radiation into stimulated radiation. The seed itself, in the form of an initial modulation of the beam, can be arbitrarily small. In addition, it is understandable that the principal cause producing radiation in the case of bunched electrons is the polarization current of the beam.

What we have said applies to all systems considered in Sections 2 and 3. For example, the spontaneous Cherenkov effect initiates the following things: in the case of beams of low density—single-particle stimulated Cherenkov radiation, and in beams of high density in faster-than light motion—collective Cherenkov radiation. Spontaneous undulator or dipole radiation initiates stimulated undulator radiation—single-particle and collective. The spontaneous effect of second order (Thomson scattering) initiates in beams of low density the effects of Thomson scattering and radiation, and in dense beams—it initiates stimulated Raman scattering or radiation. In all these cases only the uniform motion of the electron flux has been considered, and consequently only polarization currents have been taken into account, which is quite adequate.

There is another, opposite case in which the charge density of the beam is not perturbed ($\rho = \text{const}$) and a rotational variable current ($\text{div}\,\mathbf{j} = 0$) induced in the beam radiates. This radiation is initiated by other spontaneous effects and can be analogous to the radiation from a current-carrying frame.[26] A corresponding example was discussed in Section 4 in derivation of Eq. (4.5), in which the radiation of an initially rectilinear beam in a constant magnetic field was investigated. Stimulated radiation of such a beam arises in the following way. A bare circularly polarized current produces a spontaneous field of the same polarization. In the general case $\mathbf{jE} = 0$, and there is no radiation. However, if the condition (4.7) is satisfied, the situation changes considerably: the vectors **E** and **j** become parallel, and therefore the transverse velocity of the electrons begins to increase and the longitudinal component of the Lorentz force slows down the entire flux as a whole without modulating it. Of course, in the most general case the sources of the radiation from beams are both rotational currents and polarization currents.

There is an additional spontaneous effect which initiates stimulated radiation, namely, stimulated cyclotron radiation (or magnetic bremsstrahlung). Here also $\mathbf{jE} \neq 0$, but **j** is now not the current of an individual electron but the current of an electron bunch formed as the result of bunching in a Larmor orbit. Stimulated cyclotron radiation is a purely relativistic effect. It is initiated by spontaneous magnetic bremsstrahlung. Let us turn to a more detailed discussion of it.

Let the electrons be uniformly distributed on the Larmor circle and let their velocities be directed as shown in Fig. 6. The frequency of rotation of such electrons in an external magnetic field B_0 is ω_H/γ, that is, it depends on their energy. Also let the electric-field-strength vector of a circularly polarized wave rotate at each point of space with a frequency ω close to ω_H/γ. If $\omega \approx \omega_H/\gamma$, then in the portion of the orbit ABC the electrons receive energy from the field; therefore the value of γ increases and the rotation frequency of the electrons drops. As a result the electrons from the portion ABC attempt to shift toward the point A. It is easy to see that the electrons from the portion ADC also shift toward the same point. As the result an electron bunch arises near point A on the Larmor orbit. However, at $\omega = \omega_H/\gamma$, as a result of the symmetry of the picture with respect to the direction of **E**, the combined work of the field on the electrons is equal to zero. There is no symmetry for $\omega \neq \omega_H/\gamma$, when the work is nonzero and stimulated cyclotron radiation can arise, that is, radiation of rotating electron bunches produced by the radiation.[27] It is clear that the key aspect here is the dependence of the electron rotation frequency on γ, that is, relativistic behavior.

For a quantitative investigation of the effect we shall write the equation for the longitudinal momentum of a beam electron:

$$\frac{dP_\parallel}{dt} = \frac{e}{2c}\left(v_\perp \frac{\partial A^*_\perp}{\partial z} + v^*_\perp \frac{\partial A_\perp}{\partial z}\right), \tag{5.1}$$

and represent the longitudinal coordinate of the electron and the components of the velocity and the radiation field in the form

$$\begin{aligned} &z = u_\parallel (t - t_0) + z_{0j} + \tilde{z}_j, \qquad v_\parallel = u_\parallel + \tilde{v}_\parallel, \\ &v_\perp = (u_\perp + \tilde{v}_\perp)\exp\left[-i\frac{\omega_H}{\gamma}(t - t_0) - i\varphi_{0j} - i\tilde{\varphi}\right], \\ &A = A_0 \exp(i\omega t - ik_\parallel z) \qquad (\omega \to \omega - i\delta\omega). \end{aligned} \tag{5.2}$$

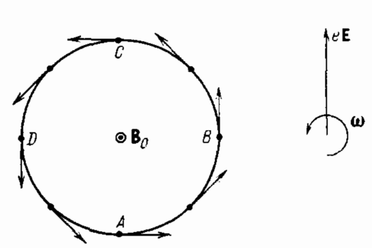

FIG. 6. Magnetic-cyclotron bunching of electrons in the Larmor orbit.

where the quantities with a tilde (˜) are small perturbations proportional to A_0, and z_{0j} and φ_{0j} are the initial coordinate and initial phase of rotation of the jth electron. We shall average Eq. (5.1), taking into account in it terms up to quadratic in the perturbations. As a result we obtain the relation

$$\frac{d\langle P_\parallel\rangle}{dt} = -ik_\parallel \frac{en_b}{c}\Big\langle \frac{1}{2}(u_\perp A_0 + \tilde{v}_\perp A_0 + iu_\perp \tilde{\theta} A_0) \times \exp(i\Delta t + i\theta_{0j}) - \text{c.c.}\Big\rangle, \quad (5.3)$$

in which $\Delta = \omega - k_\parallel u_\parallel + (\omega_H/\gamma), \theta_{0j} = \varphi_{0j} - k_\parallel z_{0j}$, and $\tilde{\theta} = \tilde{\varphi} - k_\parallel \tilde{z}$ is the perturbation of the helical phase. As before, the averaging in (5.3) is understood in the same sense as in the derivation of Eq. (2.3).

Let us discuss the meaning of Eq. (5.3). The term proportional to $u_\perp A_0$ determines the action of the field on an unperturbed electron, that is, it is responsible for the spontaneous radiation. In the averaging it disappears. The term $\sim \tilde{v}_\perp A_0$ determines the stimulated anomalous Doppler radiation studied previously in the example of (4.5). The new element is the term $\sim u_\perp \tilde{\theta} A_0$ which describes the phase bunching of the particles. It is specifically this term that is responsible for stimulated magnetic bremsstrahlung.

The further manipulation of Eq. (5.3) reduces to calculation in the linear approximation of the perturbations $\tilde{v}_\perp$ and $\tilde{\theta}$ and to application of momentum conservation and of Eq. (2.10). Omitting the simple intermediate steps, we shall give the final equation for the density of energy and momentum of the radiation:

$$\frac{d}{dt}(W, P) = \frac{1}{\alpha}\omega(S_1 + S_2)(W, P); \quad (5.4)$$

here

$$S_1 = \frac{\omega_b^2 \gamma^{-2} \omega_H \omega^{-2} \delta\omega}{[\omega - k_\parallel u_\parallel + (\omega_H/\gamma)]^2 + \delta\omega^2}, \quad (5.5)$$

$$S_2 = \frac{u_\perp^2}{c^2}\left(1 - \frac{k_\parallel^2 c^2}{\omega^2}\right) \frac{\omega_b^2 \gamma^{-1}[\omega - k_\parallel u_\parallel + (\omega_H/\gamma)]\,\delta\omega}{\{[\omega - k_\parallel u_\parallel + (\omega_H/\gamma)]^2 + \delta\omega^2\}^2}. \quad (5.6)$$

As $u_\perp^2/c^2 \to 0$, when $S_2 \to 0$, Eq. (5.4) goes over into (4.5), or more precisely into its relativistic analog. In the opposite case for

$$|S_2| \gg |S_1| \quad (5.7)$$

Eq. (5.4) turns out to be substantially different and in its structure reminds us of Eq. (2.7). This also is to be expected, since the effect of stimulated cyclotron radiation, like the single-particle stimulated Cherenkov effect, is due to bunching of electrons, in the present case with a perturbation of the helical phase $\tilde{\theta}$. Note that the perturbation $\tilde{\theta}$ leads to polarization of the flux: a rotating dipole moment arises, which is specifically the source of radiation.

Taking into account that in the case (5.7) Eq. (5.4) is similar to Eq. (2.8), we shall write down without difficulty the increment of the instability in stimulated cyclotron radiation:

$$\delta\omega = \frac{\sqrt{3}}{2}\left[\frac{1}{4\alpha}\frac{u_\perp^2}{c^2}\omega_b^2\gamma^{-1}\left|\omega\left(1 - \frac{k_\parallel^2 c^2}{\omega^2}\right)\right|\right]^{1/3}. \quad (5.8)$$

In addition the condition of radiation is written in the form

$$\omega - k_\parallel u_\parallel + \frac{\omega_H}{\gamma} = \pm\frac{1}{\sqrt{3}}\delta\omega. \quad (5.9)$$

In the latter expression the + sign is taken only for radiation of waves with a phase velocity greater than $u_\parallel$, but less than the velocity of light c. In the opposite case the sign − is chosen.

Using Eqs. (5.5), (5.6), and (5.8), we shall give an explicit form to the inequality (5.7), namely

$$\omega_b^2 \ll \left(\frac{u_\perp^2}{c^2}\gamma^2\right)^2 \left(\frac{\omega}{\omega_H}\right)^3 \left[\omega\left(1 - \frac{k_\parallel^2 c^2}{\omega^2}\right)\right]^2. \quad (5.10)$$

On increase of the density of the flux, the inequality (5.10) is violated and stimulated cyclotron radiation becomes impossible. It goes over either into the anomalous Doppler effect or into the normal Doppler effect. The latter process, as was shown previously, is not an instability.

We note in addition that in obtaining the formulas (4.2) we made important use of the relation $\delta W_e = u_\parallel \delta P_e$ and of the possibility of separating the mechanical energy of the electron into longitudinal and transverse parts. In the case of an electron in a magnetic field this can be done with accuracy to $u_\perp^2/c^2$. However, stimulated cyclotron radiation is an effect of the same order. Therefore the formulas (4.2) are not applicable, and it is difficult to say how the longitudinal and transverse energies of the electrons change (even if this separation can be approximately carried out) in stimulated cyclotron radiation without a special analysis.

6. THE MECHANISMS OF NONLINEAR STABILIZATION OF RADIATIVE BEAM INSTABILITIES

The results of Sections 2–5 permit us to determine the principal mechanisms of nonlinear stabilization of radiative beam instabilities and to estimate the maximum energy density of stimulated radiation of electron beams. We shall carry out an appropriate qualitative discussion, beginning with the single-particle stimulated Cherenkov effect.

Making use of energy conservation (2.5) and expressing the change of the density of the average energy of the beam electrons W_e in terms of their average velocity $v_\parallel$, we obtain the following expression for the energy density of the radiation:

$$W = \alpha\frac{|E_\parallel|^2}{8\pi} = mc^2\gamma^3\frac{u_\parallel^2}{c^2}n_b\left(1 - \frac{v_\parallel}{u_\parallel}\right). \quad (6.1)$$

Therefore in evaluating the energy density of the radiation it is necessary to estimate the change of the average velocity of the electrons $u_\parallel - v_\parallel$, which is easily done by means of Eq. (2.7). Indeed, if we substitute into Eq. (2.7) instead of $u_\parallel$ the running value of the average velocity $v_\parallel$, then it will be seen that the increase of the energy density of the radiation is cut short at $\omega = k_\parallel v_\parallel$, that is, on decrease of the average velocity of the beam down to the phase velocity of the wave $\omega/k_\parallel$. Consequently $v_{\parallel\min} = \omega/k_\parallel$. Now taking into account Eq. (2.8), we can write

$$\left(1 - \frac{v_\parallel}{u_\parallel}\right)_{\max} = \frac{1}{\sqrt{3}}\frac{\delta\omega}{\omega}. \quad (6.2)$$

From this and Eq. (6.1) we obtain the desired estimate for the maximum energy density of the radiation:

$$\left(\alpha\frac{\|E_\parallel|^2}{8\pi}\right)_{\max} = n_b mc^2\gamma\cdot\gamma^2\frac{u_\parallel^2}{c^2}\frac{\delta\omega}{\omega}, \quad (6.3)$$

where $\delta\omega$ is the increment defined by Eq. (2.9).

The estimate (6.3) can be obtained also by another means. In accordance with Eq. (2.8) the beam overtakes the wave and thereby transfers to it its energy. For each beam electron the wave is equivalent to a sequence of potential barriers, over which it passes with a velocity $\sim\delta\omega/k_{\parallel}$. If in the rest system of the wave the height of the potential barriers exceeds the kinetic energy of the electron, then capture of the electrons by the field of the wave will occur. The captured electrons on the average are stationary with respect to the wave and therefore on the average energy is not exchanged with them. The condition of capture is easily obtained from the equations of motion (2.2). It reduces to the following (Refs. 28 and 29)[15]:

$$\beta\frac{eE_{\parallel}}{k_{\parallel}}\gamma^{-3}\approx m\left(\frac{\omega}{k_{\parallel}}-u_{\parallel}\right)^2. \tag{6.4}$$

From this and Eq. (2.8) the estimate (6.3) also follows. Therefore the stabilization of the instability due to the single-particle stimulated Cherenkov effect involves capture of electrons of the beam by the field of the radiation.

An important characteristic of a radiative instability is the electron efficiency η, which is defined as the ratio of the change of the kinetic energy density of the beam to the initial energy density. Using Eqs. (6.1) and (6.3), we find that in the induced Cherenkov effect

$$\eta_{\max}\approx\frac{1}{2}\frac{u_{\parallel}^2}{c^2}\frac{\gamma}{\gamma-1}\cdot 2\gamma^2\frac{\delta\omega}{\omega}. \tag{6.5}$$

It should be mentioned that in obtaining (6.1) we assumed fulfillment of the inequalities

$$\left|1-\frac{v_{\parallel}}{u_{\parallel}}\right|\ll 1,\qquad \gamma^2\left|1-\frac{v_{\parallel}}{u_{\parallel}}\right|\ll 1, \tag{6.6}$$

the first of which, as can be seen from (6.2), is unconditionally satisfied. The second inequality of (6.6) reduces to smallness of the efficiency (6.5). In Fig. 7 we have shown the maximum efficiency in the single-particle Cherenkov effect (without taking into account the factor $(1/2)(u_{\parallel}^2/c^2)\gamma/(\gamma-1)$ as a function of the parameter $\mu_1=2\gamma^2\delta\omega/\omega$, obtained in numerical solution of the nonlinear problem.[30] The drop of the efficiency at large μ_1 is due to the difficulty of bunching relativistic electrons in the retarding phases of the field.

The estimates (6.3) and (6.5) and the dependence of the efficiency on μ_1 are valid also for the stimulated Cherenkov effect in a periodic structure and for stimulated undulator radiation. It is necessary only to interpret as $\delta\omega$ and ω in (6.3), (6.5), and μ_1 in terms of appropriate quantities. Similar results hold also for Thomson scattering and radiation if the amplitude of one of the electromagnetic waves changes slightly (for example, in the case of an intense pumping wave).

Let us turn now to discussion of collective processes which occur in dense beams when the condition (3.5) or

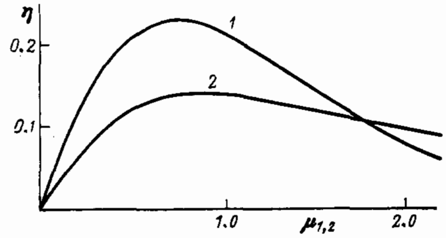

FIG. 7. Electron efficiencies in stimulated radiation. 1—Dependence of the efficiency on μ_1 in the single-particle Cherenkov effect, 2—dependence of the efficiency on μ_2 in the collective Cherenkov effect.

$$\nu\equiv\frac{\delta\omega}{\Omega_b}\ll 1 \tag{6.7}$$

holds. We recall that single-particle processes occur under conditions opposite to (6.7). We shall begin with the collective Cherenkov effect.

Taking into account that in the collective Cherenkov effect instead of (2.7) we have (3.6), we obtain the following estimate for the maximum slowing down of the beam:

$$\left(1-\frac{v_{\parallel}}{u_{\parallel}}\right)_{\max}\approx\frac{\Omega_b}{k_{\parallel}u_{\parallel}}. \tag{6.8}$$

From this and from (6.1) we obtain

$$\left(\alpha\frac{|E_{\parallel}|^2}{8\pi}\right)_{\max}\approx n_b mc^2\gamma\cdot\gamma^2\frac{u_{\parallel}^2}{c^2}\frac{\Omega_b}{k_{\parallel}u_{\parallel}}, \tag{6.9}$$

$$\eta_{\max}=\frac{1}{2}\frac{u_{\parallel}^2}{c^2}\frac{\gamma}{\gamma-1}\cdot 2\gamma^2\frac{\Omega_b}{k_{\parallel}u_{\parallel}}, \tag{6.10}$$

and the inequalities (6.6) reduce to the condition $\Omega_b\ll k_{\parallel}u_{\parallel}$ and to the smallness of the efficiency (6.11).[16]

The estimates (6.8)–(6.10), like (6.4), are estimates of the capture of electrons, not by the field of the radiation, but by the field of the slow beam wave (self-capture[17]). (This is natural, since under the conditions (6.7) the field of the beam wave is considerably greater than the field of the radiation.)

In Fig. 7 we have shown the efficiency as a function of the parameter $\mu_2=2\gamma^2\Omega_b/k_{\parallel}u_{\parallel}$, obtained in numerical solution of the nonlinear problem (for $\nu=0.3$).

It turns out, however, that the values of the energy density (6.9) and of the efficiency (6.10) are not always achieved, and that the dependence of the efficiency on μ_2 shown in Fig. 7 is not always realized. A detailed numerical analysis shows that the estimates (6.10) and (6.11) are valid if the coefficient ν is not very small (from 0.1 up to 1). For smaller ν other nonlinear processes not related to capture set in. We shall discuss them.

Let us return to Eq. (4.9), which is valid if the inequality (3.5) is very strong. It can be seen that the right-hand side of (4.9) has a well expressed resonance nature and is large if

$$\frac{(\omega-k_{\parallel}u_{\parallel}+\Omega_b)^2}{\Omega_b^2}<\nu^2. \tag{6.11}$$

It is obvious that on violation of the inequality (6.11) the instability is stabilized. This inequality is violated either as the result of change of the beam velocity or as the result of a dependence of the frequency of the plasma oscillations of the beam on their amplitude. These factors can be combined under the general designation "nonlinear shift of frequency".[31]

If the nonlinear shift of frequency is due only to slowing down of the beam, then from (6.11) we obtain the estimate

$$\left(1-\frac{v_{\parallel}}{u_{\parallel}}\right)_{\max}=\nu\frac{\Omega_b}{k_{\parallel}u_{\parallel}}=\frac{\delta\omega}{k_{\parallel}u_{\parallel}}, \tag{6.12}$$

in which the increment $\delta\omega$ is defined in (3.8). From this and from (6.1) we obtain

$$\left(\alpha\frac{|E_{\parallel}|^2}{8\pi}\right)_{\max}=n_b mc^2\gamma\cdot\gamma^2\frac{u_{\parallel}^2}{c^2}\nu\frac{\Omega_b}{k_{\parallel}u_{\parallel}}, \tag{6.13}$$

$$\eta_{\max}\approx\frac{1}{2}\frac{u_{\parallel}^2}{c^2}\frac{\gamma}{\gamma-1}\mu_2\nu. \tag{6.14}$$

We note that in the case of stabilization of an instability by a nonlinear shift of the frequencies, the dynamics of the stabilization can be investigated analytically.[32] For example, it is possible to obtain an exact expression for the maximum efficiency[33]:

$$\eta_{\max}=\frac{u_{\parallel}^2}{c^2}\frac{\gamma}{\gamma-1}\nu\mu_2\left(1+\frac{3}{2}\mu_2+\frac{3}{8}\mu_2^2\right)^{-1}. \qquad (6.15)$$

Equation (6.15) takes into account both the nonlinear slowing down of the beam and the relativistic nonlinear change of the frequency of the plasma oscillations of the beam. For $\mu_2 \ll 1$, when relativistic effects are unimportant, Eq. (6.15) reduces (with accuracy to 1/2) to the estimate (6.14).

Let us consider the fact that for $\nu \sim 1$ the estimates (6.13) and (6.14) go over into (6.9) and (6.10) respectively. However, from the point of view of nonlinear processes these estimates correspond to completely different cases. For example, at not very small $\nu(\gtrsim 0.1)$ the stabilization of the instability is due to capture of beam electrons by the field of slow waves and subsequent development of strong turbulence. At very small ν turbulence does not arise and there is no capture, since as the result of the nonlinear frequency shift the amplitude of the beam wave does not manage to rise up to the capture value.[33]

The results (6.8)–(6.15) are valid also for the collective Cherenkov effect in a periodic structure and for collective undulator radiation. It is necessary only to interpret $\delta\omega$ in the formulas obtained as the corresponding increment. Similar results exist also for Raman scattering and radiation provided that the amplitude of one of the electromagnetic waves changes only slightly (the case of an intense pumping wave).

Let us now consider the anomalous Doppler effect in the case of an initially rectilinear nonrelativistic beam of electrons moving in a constant external magnetic field. Using Eqs. (4.1) and (4.10), we express the energy density of the radiation in terms of the average longitudinal velocity of the beam electrons

$$W=\alpha\frac{\omega^2}{c^2}\frac{|A_0|^2}{4\pi}=n_b m u_{\parallel}^2\frac{\omega}{k_{\parallel}u_{\parallel}}\left(1-\frac{v_{\parallel}}{u_{\parallel}}\right). \qquad (6.16)$$

For estimation of the difference $(u_{\parallel}-v_{\parallel})$ we shall consider Eq. (4.5), which in its structure is equivalent to Eq. (4.9). Utilizing the resonance nature of the right-hand side of Eq. (4.5), we obtain in analogy with (6.11) and (6.12) the following estimate:

$$\left(1-\frac{v_{\parallel}}{u_{\parallel}}\right)_{\max}\approx\frac{\omega_H}{k_{\parallel}u_{\parallel}}\frac{\delta\omega}{\omega_H}, \qquad (6.17)$$

where $\delta\omega$ is the increment determined by the formula (4.8). From (6.17) and (6.16) we obtain the desired estimates for the maximum energy density of the radiation and the maximum electron efficiency[34]:

$$\left(\alpha\frac{\omega^2}{c^2}\frac{|A_0|^2}{4\pi}\right)_{\max}\approx n_b m u_{\parallel}^2\frac{\omega\omega_H}{(k_{\parallel}u_{\parallel})^2}\frac{\delta\omega}{\omega_H}, \qquad (6.18)$$

$$\eta_{\max}\approx 2\frac{\omega\omega_H}{(k_{\parallel}u_{\parallel})^2}\frac{\delta\omega}{\omega_H}. \qquad (6.19)$$

From the derivation of the estimates (6.19) and (6.18) it is clear that the stabilization of the instability is due to violation of the resonance condition (4.7), that is, to the nonlinear frequency shift. In turn the nonlinear frequency shift is due to the slowing down of the electron beam.

In conclusion we shall consider stimulated magnetic bremsstrahlung and shall give the corresponding nonlinear estimates. For this purpose we shall use Eq. (5.4), in which when the inequality (5.10) is satisfied it is possible to neglect the term which contains S_1. Without the term proportional to S_1, Eq. (5.4) coincides in structure with Eq. (2.7), and the condition of stimulated cyclotron radiation (5.8) is similar to the condition (2.8). Therefore it is clear that in the case of magnetic bremsstrahlung stabilization of an instability sets in at $\omega - k_{\parallel}v_{\parallel} + (\omega_H/\tilde{\gamma}) = 0$, where $v_{\parallel}$ and $\tilde{\gamma}$ are the running average values of the longitudinal velocity and relativistic factor of the electrons. Then, using (5.9), we obtain the relation

$$\left|k_{\parallel}u_{\parallel}\left(1-\frac{v_{\parallel}}{u_{\parallel}}\right)+\frac{\omega_H}{\gamma}\left(1-\frac{\tilde{\gamma}}{\gamma}\right)\right|_{\max}\approx\delta\omega, \qquad (6.20)$$

where $\delta\omega$ is the increment (5.8). From the conservation of energy and momentum we have

$$W_{\max}=\left(\alpha\frac{\omega^2}{c^2}\frac{|A_0|^2}{4\pi}\right)_{\max}=n_b mc^2\gamma\left(1-\frac{\tilde{\gamma}}{\gamma}\right)_{\max},$$

$$P_{\max}=\frac{k_{\parallel}}{\omega}W_{\max}$$

$$=n_b m u_{\parallel}\gamma\left[\left(1-\frac{v_{\parallel}}{u_{\parallel}}\right)_{\max}+\left(1-\frac{\tilde{\gamma}}{\gamma}\right)_{\max}\right]. \qquad (6.21)$$

From this and from (6.20) we obtain the desired estimate

$$\left(\alpha\frac{\omega^2}{c^2}\frac{A_0^2}{4\pi}\right)_{\max}=n_b mc^2\gamma\left|\frac{\delta\omega}{\omega}\frac{\omega}{(k_{\parallel}^2c^2/\omega)-k_{\parallel}u_{\parallel}+(\omega_H/\gamma)}\right| \qquad (6.22)$$

and expressions for the maximum electron efficiency

$$\eta_{\max}=\left|\frac{\delta\omega}{\omega}\frac{\omega}{(k_{\parallel}^2c^2/\omega)-k_{\parallel}u_{\parallel}+(\omega_H/\gamma)}\right|. \qquad (6.23)$$

The expressions (6.22) and (6.23) have an interesting property. If $|\omega^2 - k_{\parallel}^2c^2| \gg \delta\omega$, then, as we can easily see,

$$\eta_{\max}\approx\frac{\delta\omega}{\omega}\left|\frac{k_{\parallel}^2c^2}{\omega^2}-1\right|^{-1}\ll 1.$$

However, if the phase velocity of the radiated wave is equal to the velocity of light c, then with inclusion of (5.8) we find that $\eta_{\max} \sim 1$. This is so-called autoresonance: at $\omega = k_{\parallel}c$ the changes of $v_{\parallel}$ and $\tilde{\gamma}$ are such that the resonance conditions (5.8) are kept unchanged during the entire process. Here it is necessary to mention that in the autoresonance regime, or, more precisely, near it, the increment (5.7) is extremely small, and the density of the beam, in view of the inequality (5.9), must be small. We can state that the autoresonance regime of stimulated magnetic bremsstrahlung can be realized only in beams of very low density.

7. THE LINEAR DISPERSION EQUATION

The dispersion-equation method is an effective means of investigation of the initial stage of radiative beam instabilities. The dispersion equation is a condition of solubility of field equations with a beam current linear in the field on the right-hand side. This current is expressed in terms of the beam-conductivity tensor σ_{ij} by the formula

$$j_i=\sigma_{ij}E_j \qquad (i,\ j=x,\ y,\ z). \qquad (7.1)$$

For calculation of the conductivity tensor it is possible, for example, to resort to integration of the linearized equations of motion of the individual electrons, which in essence was done previously in derivation of formulas of the type (2.7). Here we shall set forth the basis of the dispersion-equation method, which permits not only illustration of the preceding results but also obtaining of new results.

In the linear approximation the conditions of spontaneous radiation are manifested as poles in the components of the beam-conductivity tensor. In the general case it is difficult to say anything about the structure of these components, and each specific beam requires individual discussion. However, it is possible to learn a great deal in the example of beams moving along a very strong external magnetic field (completely magnetized beams). We shall begin with them.

If a completely magnetized beam is moving along an axis Oz along which the external magnetic field is also directed, then the only nonzero component of the beam-conductivity tensor will have the form[19]

$$\sigma_{zz}=i\frac{\omega}{4\pi}\frac{\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel})^2}. \quad (7.2)$$

We shall assume for the moment that in solution of the field equations the electron beam can be considered as a small perturbation. This means that the polarization (structure) of the wave radiated by the beam can be considered the same as if there were no beam. Taking into account the current (7.1) and (7.2) in the field equations on the basis of perturbation theory leads to a dispersion equation of the following form[14]:

$$D(\omega,\ k_{\parallel})=\frac{\beta^2\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel})^2}, \quad (7.3)$$

where $D(\omega,k_{\parallel})$ is a function whose zeros determine the dispersion of the waves radiated by the beam $\omega=\omega(k_{\parallel})$, and β is a coupling parameter introduced previously [see Eq. (2.2)]. On fulfillment of the resonance condition $\omega(k_{\parallel})=k_{\parallel}u_{\parallel}$ we obtain from (7.3) the following expression for the complex frequency:

$$\omega=k_{\parallel}u_{\parallel}+\frac{-1+i\sqrt{3}}{2}\left|\beta^2\omega_b^2\gamma^{-3}\left(\frac{\partial D}{\partial\omega}\right)^{-1}\right|^{1/3}. \quad (7.4)$$

The imaginary part of (7.4) is the increment of the instability due to the single-particle stimulated Cherenkov effect. It is easy to see that the result (7.4) coincides with (2.8) and (2.9) for

$$\alpha=\frac{1}{4}\frac{\partial}{\partial\omega}(\omega D). \quad (7.5)$$

The latter expression and Eq. (2.6) agree with the determination of the momentum and energy of the waves in terms of the dispersion function $D(\omega,k_{\parallel})$.[46]

With increase of the beam density, the dispersion equation (7.3) becomes invalid. Actually a dense beam, being modulated, appreciably distorts the polarization of the field. It is possible to take into account the distortion of the field polarization by generalizing the concept of beam conductivity (7.2). We shall define the conductivity not as the response of the beam to the total electromagnetic field, but only as the response to the radiation field. Here instead of (7.2) we obtain the following expression:

$$\sigma_{zz}=i\frac{\omega}{4\pi}\frac{\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel})^2-\Omega_b^2}. \quad (7.6)$$

It is easy to see that the generalized conductivity actually has the form (7.6) if we proceed from the electron equations of motion written in the form (3.1).

Using (7.6), we obtain instead of (7.3) a dispersion equation which determines the radiative instabilities of dense beams[35]:

$$D(\omega,\ k_{\parallel})=\frac{\beta^2\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel})^2-\Omega_b^2}. \quad (7.7)$$

On fulfillment of the resonance condition $\omega(k_{\parallel})=k_{\parallel}u_{\parallel}-\Omega_b$ we obtain from (7.7) an equation for the complex frequency:

$$\omega=\omega(k_{\parallel})+i\beta\left|\frac{\omega_b^2\gamma^{-3}}{\Omega_b}\left(\frac{\partial D}{\partial\omega}\right)^{-1}\right|^{1/2}. \quad (7.8)$$

The imaginary part of (7.8) is the increment of the instability due to the collective Cherenkov effect. It is easy to see that the result (7.8) coincides with (3.6) and (3.7). In addition, the condition of its applicability reduces to the inequality (3.8).

The structure of the dispersion equations for radiative instabilities of beams in periodic structures and undulators turns out to be the same as (7.3) and (7.7). The main difference reduces to the replacement of $(\omega-k_{\parallel}u_{\parallel})$ by $(\omega-k_{\parallel}u_{\parallel}-\chi u_{\parallel})$ or by $(\omega-k_{\parallel}u_{\parallel}-\omega_0)$.

Let us discuss a number of examples.

In the case of an unbounded beam and an unbounded plasma the dispersion equation has the form (7.7) with the parameter[12] $\beta=1$ (for $\omega\approx\omega_p$)

$$1-\frac{\omega_p^2}{\omega^2}=\frac{\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel})^2-\Omega_b^2}, \quad (7.9)$$

where ω_p is the Langmuir frequency of the electrons of the plasma and $\Omega_b^2=\omega_b^2\gamma^{-3}$. It is easy to show that for $\Omega_b\ll\omega_p$ Eq. (7.9) has a solution only of the type (7.4), since the inequality (3.8) cannot be satisfied. Therefore Eq. (7.9) actually reduces to the form (7.3). Thus the only mechanism of radiative instability of an unbounded beam of low density in an unbounded plasma is the single-particle stimulated Cherenkov effect. Here purely longitudinal plasma waves are radiated.

In the case of a transversely inhomogeneous beam and plasma the dispersion equation retains the structure of (7.9):

$$1-\frac{\Omega_p^2}{\omega^2}=\frac{\beta^2\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel})^2-\Omega_b^2}, \quad (7.10)$$

where Ω_p, Ω_b and β are rather complicated functions of the geometry.[32] It is most important that $\beta<1$. Furthermore, if the beam and plasma are spread out in space, then this parameter is extremely small. Therefore both the single-particle and the collective Cherenkov effects can be realized.

Now let the electron beam be propagated in a waveguide filled with a periodically modulated dielectric with a permittivity $\varepsilon=1+\tilde{\varepsilon}\cos\chi z$. The dispersion equation which describes the eigenoscillations of such a system has the form

$$1-\frac{k_\perp^2}{(\omega^2/c^2)-k_{\parallel}^2}=\frac{|\tilde{\varepsilon}|^2\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel}-\chi u_{\parallel})^2-\Omega_b^2}, \quad (7.11)$$

where $k_\perp$ is the transverse eigennumber of the waveguide. Equation (7.11) does not differ in structure from (7.7). The dispersion equation has the same form also in the case of a beam radiating in an electrostatic undulator[17]:

$$1-\frac{k_\perp^2}{(\omega^2/c^2)-k_{\parallel}^2}=\frac{|k_{\parallel}z_E|^2\omega_b^2\gamma^{-3}}{(\omega-k_{\parallel}u_{\parallel}-\omega_0)^2-\Omega_b^2}, \quad (7.12)$$

where $\omega_0=\chi u_{\parallel}$, and Z_E is the amplitude of the oscillations of a beam electron in the pumping field. For $\omega_0=\omega_2-k_{\parallel 2}u_{\parallel}$ Eq. (7.12) describes the beam instability in the field of the electromagnetic pumping wave (scattering) (Ref. 36).[17)]

It is easy to see that as $\Omega_b\to 0$, that is, for a vanishingly small beam density, Eq. (7.7) goes over into (7.3). In other words, Eq. (7.7) is valid for any beam density. This very simple situation is realized if the radiation field and the field of the plasma oscillations of the beam have different space-time scales, that is, if these fields are easy to distinguish from each other. For example, in radiation of a beam in a periodic structure the wavelength of the radiation is $2\pi/k_\parallel$, while the length of the beam wave is $2\pi/(k_\parallel + \chi)$; in the case of the example (7.10) for $\beta \ll 1$ the radiation field (plasma oscillations) and the beam fields are separated in space (for $\beta \sim 1$ the oscillations of the plasma and the beam are indistinguishable but, as was mentioned in analysis of (7.9), the fields of the beam for $\beta \sim 1$ are negligibly small), and so forth. However, there are also more complicated systems, two of which we shall now discuss.

If an electron is propagated in a waveguide with an isotropic dielectric filling, then the dispersion equation has the form

$$k_\perp^2 + \left(k_\parallel^2 - \frac{\omega^2}{c_0^2}\right)\left[1 - \frac{\omega_b^2\gamma^{-3}}{\varepsilon_0(\omega - k_\parallel u_\parallel)^2}\right] = 0,$$

$$c_0 = \frac{c}{\varepsilon_0^{1/2}} < u_\parallel, \qquad (7.13)$$

where ε_0 is the permittivity of the dielectric. It would appear that, dividing (7.13) by $k_\parallel^2 - (\omega^2/c_0^2)$, it is possible to obtain an equation of the type (7.3). However, it turns out that (7.13) has solutions also which are completely different from the single-particle solutions (7.4). By means of the substitution

$$\omega = \omega(k_\parallel) + \delta\omega = k_\parallel u_\parallel + \delta\omega, \qquad (7.14)$$

where $\omega(k_\parallel) = [(k_\perp^2 + k_\parallel^2)c_0^2]^{1/2}$, we reduce Eq. (7.13) to the following form:

$$\delta^3\omega = \frac{1}{2}\frac{\omega_b^2\gamma^{-3}k_\perp^2c_0^2}{\varepsilon_0\omega}\left(1 + 2\frac{\omega\delta\omega}{k_\perp^2c_0^2}\right). \qquad (7.15)$$

From this in the case of beams of low density when

$$\omega_b^2\gamma^{-3} \ll \varepsilon_0 k_\perp^2 c_0^2\left(\frac{k_\perp c_0}{\omega}\right)^2, \qquad (7.16)$$

we find the increment

$$\delta\omega = \frac{-1 + i\sqrt{3}}{2}\left(\frac{1}{2}\frac{\omega_b^2\gamma^{-3}k_\perp^2c_0^2}{\varepsilon_0\omega}\right)^{1/3}, \qquad (7.17)$$

which is obviously due to the single-particle Cherenkov effect. If the inequality inverse to (7.16) is satisfied, then all roots of Eq. (7.15) turn out to be real, which means that the instability at the point of the single-particle Cherenkov resonance disappears. One can show that in the case inverse to (7.16) Eq. (7.13) has the following solution (for $\omega \gg k_\perp c_0$):

$$\omega = \omega(k_\parallel) + \frac{1}{2}i\left(\frac{k_\perp^2c_0^2}{\omega}\Omega_b\right)^{1/2}, \qquad (7.18)$$

where $\Omega_b^2 = (1/\varepsilon_0)\omega_b^2\gamma^{-3}$. The solution (7.18) is realized if the resonance condition $\omega(k_\parallel) = k_\parallel u_\parallel - \Omega_b$ is satisfied. The regime (7.18) obviously corresponds to the collective Cherenkov effect. Therefore the disappearance of the instability at the point of the single-particle Cherenkov resonance is due to the fact that on increase of the beam density the instability becomes collective.

In order to discover how collective beam fields arise in a dielectric waveguide, we shall express the electric field components in terms of the Hertz electric potential[37] Ψ:

$$|E_\perp| = |k_\perp k_\parallel \Psi|, \quad E_\parallel = -\left(k_\parallel^2 - \frac{\omega^2}{c_0^2}\right)\Psi. \qquad (7.19)$$

If in (7.19) we substitute $\omega = \omega(k_\parallel)$ without taking into account $\delta\omega$ (and restrict the discussion to this case), then we find that $E_\perp/E_\parallel$ is the same for any beam density. In this way it is possible to arrive at once at the solutions (7.17). However, in dense beams in calculation of $E_\parallel$ it is necessary to take into account the correction to the frequency $\delta\omega$. Here the ratio $E_\perp/E_\parallel$ becomes a function of the beam density, that is, on increase of the density the polarization of the waveguide field is distorted. This distortion is due to the appearance of beam fields, which are shown schematically in Fig. 2. Taking into account these fields in the case of dense beams also leads to the solutions (7.18).[18]

As a further example we shall discuss the nonpotential instability of a uniform electron beam penetrating a uniform magnetized plasma waveguide. The dispersion equation in such a system has the form[14]

$$k_\perp^2 + \left(k_\parallel^2 - \frac{\omega^2}{c^2}\right)\left(1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2\gamma^{-3}}{(\omega - k_\parallel u_\parallel)^2}\right) = 0. \qquad (7.20)$$

Using the substitution (7.14), in which $\omega(k_\parallel)$ is the dispersion dependence of the plasma waveguide without a beam, we obtain from (7.20) the following relation[35] (for $u_\parallel \sim c$):

$$\frac{\delta\omega}{\omega}\left[\left(\frac{\delta\omega}{\omega}\right)^2 + \gamma^2\frac{\omega_b^2\gamma^{-7}}{k_\perp^2u_\parallel^2}\right] = \frac{1}{2}\frac{\omega_b^2\gamma^{-7}}{k_\perp^2u_\parallel^2}. \qquad (7.21)$$

From this in the case of beams of low density in which

$$\omega_b^2\gamma^{-2} \ll k_\perp^2 u_\parallel^2, \qquad (7.22)$$

we find an increment

$$\delta\omega = \frac{-1 + i\sqrt{3}}{2}\left(\frac{1}{2}\frac{\omega_b^2\gamma^{-7}}{k_\perp^2u^2}\right)^{1/3}\omega, \qquad (7.23)$$

which is obviously due to the single-particle Cherenkov effect. On the other hand, if the inequality inverse to (7.22) is satisfied, then the increment turns out to be

$$\delta\omega = i\left(\frac{\omega_b^2\gamma^{-5}}{k_\perp^2u^2}\right)^{1/2}\omega. \qquad (7.24)$$

The instability with the increment (7.24) is realized at the same point of the single-particle Cherenkov resonance as (7.23). Since $\delta\omega$ in (7.24) is pure imaginary, there is no relative motion between the beam and the field (in the single-particle and collective Cherenkov effects such motion is essential), and therefore the instability with the increment (7.24) is aperiodic, or an instability of the negative-mass type. This is not a radiative instability but is due to the self-modulation of the beam in a medium with negative permittivity, that is, with excitation of beam fields which cannot be radiated.[38,39] We note also that in this instability there is a significant rearrangement of the structure of the waveguide field. Instabilities of the negative-mass type naturally cannot be discussed by the methods of Sections 2–6.

Up to this time we have considered the electron beam to be completely magnetized, which in fact is not very important. Actually, if the beam current which excites the radiation is related primarily to modulation, then the dispersion equations have the structure (7.13), (7.7), or of the type (7.13), (7.20). This can be seen from the following considerations. The modulation of the beam is determined by the perturbation of the electron trajectories. In order to find these perturbations, it is necessary to integrate the equations of motion twice. As a result of this integration, in the denominators of the components of the beam-conductivity tensor the quantities $(\omega - k_\parallel u_\parallel)^2$ or $(\omega - k_\parallel u_\parallel - \omega_0)^2$ appear, that is, the structure (7.2) or (7.6) is preserved.

However, if the beam current involves only velocity modulation, then in the conductivity tensor there will be poles not of second order, but of first order. Actually this modulation is determined by perturbations of the electron velocity, and in calculating them the equations of motion need to be integrated only once. As an illustration we shall discuss the instability of a rectilinear electron beam moving along a finite magnetic field in an isotropic dielectric with permittivity ε_0. For radiation strictly along the direction of motion of the beam, the dispersion equation has the form[24]

$$\omega^2-k_\parallel^2 c_0^2=\frac{\omega_b^2\gamma^{-1}(\omega-k_\parallel u_\parallel)}{\varepsilon_0[\omega-k_\parallel u_\parallel+(\omega_H/\gamma)]}. \quad (7.25)$$

From this it is easy to obtain the results given in Section 4 in discussion of the anomalous Doppler effect. From Eq. (7.25) it also can be seen that in the region of the normal Doppler effect $[\omega=-k_\parallel c_0=k_\parallel u_\parallel-(\omega_H/\gamma)]$ the frequency turns out to be real, that is, the beam is stable.

In the case of the magnetic-bremsstrahlung mechanism of radiation from an electron beam there is a bunching of the electrons in the Larmor orbits, that is, a polarization of the beam arises. As a result in Eq. (7.25) a beam term with a second-order pole appears:

$$\omega^2-k_\parallel^2 c_0^2=\frac{\omega_b^2\gamma^{-1}}{\varepsilon_0}\left\{\frac{\omega-k_\parallel u_\parallel}{\omega-k_\parallel u_\parallel+(\omega_H/\gamma)}-\frac{1}{2}\frac{u_\perp^2}{c^2}\frac{(\omega^2-k_\parallel^2c^2)}{[\omega-k_\parallel u_\parallel+(\omega_H/\gamma)]^2}\right\}. \quad (7.26)$$

From Eq. (7.26) it is easy to obtain the increment of the instability based on stimulated magnetic bremsstrahlung, which was given in Section 5.

Relativistic bunching of electrons in Larmor orbits is related to still another instability which we shall discuss in the case of a beam of electron oscillators propagating along an external magnetic field in an evacuated metallic waveguide. In this case to obtain the dispersion equation it is necessary to use the rather cumbersome apparatus of the tensor operator of the permittivity of the oscillator flux.[40] Since we do not have the possibility of discussing the details of the derivation here, we shall present the final form of the dispersion equation[35]:

$$\left(k_\perp^2+k_\parallel^2-\frac{\omega^2}{c^2}\right)(1-\varepsilon_\perp)=-\frac{\omega^2}{c^2}\beta^2\varepsilon_\perp; \quad (7.27)$$

here $\beta^2\leqslant 1$ is the coupling parameter determined by the geometry of the waveguide and the transverse geometry of the beam, while $\varepsilon_\perp$ coincides with the right-hand side of Eqs. (7.26) multiplied by $\varepsilon_0/2\omega^2$. If the beam density is small and the inequality

$$\frac{u_\perp}{c}\frac{\omega_b}{\omega}\ll\beta^2\gamma^{1/2} \quad (7.28)$$

is satisfied, then in the resonance $\omega(k_\parallel)=-[(k_\perp^2+k_\parallel^2)c^2]^{1/2}=k_\parallel u_H-(\omega_H/\gamma)$ from (7.27) we obtain an increment of the type (5.8) of the radiative instability based on stimulated magnetic bremsstrahlung. However, in the limit inverse to (7.28) and again at the point of resonance the increment turns out to be completely different:

$$\delta\omega=i\left(\frac{1}{2}\frac{u_\perp^2}{c^2}\omega_b^2\gamma^{-1}\frac{k_\perp^2c^2}{\omega^2}\right)^{1/2}. \quad (7.29)$$

This increment is of the same type as (7.24), that is, it describes an aperiodic nonradiative instability. This instability is well known in the theory of cyclic accelerators, where it has been called the negative-mass instability.[41] It is very likely that the negative-mass instability affects the efficiency of operation of gyrotrons[42] when dense beams are used in them. At least there is a theoretical confirmation of this.[43]

We note that the dispersion-equation method as a result of its high efficiency is frequently used in various applied calculations. For example, the dispersion equations given by us in Section 7 form the basis of the linear theory of various microwave electronic devices: Eqs. (7.11) and (7.12) are used in the theory of radiators in periodic structures and in the theory of undulators; (7.13) and (7.20) are used in the theory of dielectric and plasma accelerators and high-voltage generators; Eq. (7.26) is used in the theory of gyrotrons. One of the important problems which can be solved by the dispersion-equation method is the determination of the conditions of self-excitation of various radiators in electron beams. More details regarding this problem and other problems of microwave electronics can be found, for example, in the review by Bogdankevich *et al.*[14]

In conclusion we shall discuss an additional process which, at first glance, is not completely justified to mention together with radiative beam instabilities. However, in reality this is not the case. We are speaking of the Buneman instability or of the instability of a plasma with current.[47] The well known dispersion equation of this instability

$$\omega^2-\omega_i^2=\frac{\omega_e^2\omega_i^2}{(\omega-k_\parallel u_\parallel)^2-\omega_e^2} \quad (7.30)$$

has the form of (7.7). Here ω_e is the electron Langmuir frequency, ω_i is the ion Langmuir frequency ($\omega_i\ll\omega_e$), and $u_\parallel$ is the velocity of motion of the plasma electrons relative to the ions. The condition (3.7) of resonance between slow electron and ion Langmuir waves is written in the form

$$\omega_i=k_\parallel u_\parallel-\omega_e. \quad (7.31)$$

Since $\omega_i\ll\omega_e$, the slow wave of the electrons in (7.31) has an energy close to zero but negative. This is responsible for the instability. In the resonance (7.31) we determine from (7.30) the increment (more precisely, the complex frequency)

$$\omega=\delta\omega=\frac{1+i\sqrt{3}}{2}\left(\frac{\omega_i^2\omega_e}{2}\right)^{1/3}, \quad (7.32)$$

in which, as is easily seen,

$$\omega_i\ll|\delta\omega|\ll\omega_e. \quad (7.33)$$

The right-hand inequality in (7.33) is equivalent to (3.5), and therefore the Buneman resonance instability is of the class of the stimulated collective Cherenkov effect or of the anomalous Doppler effect. However, the dependence of the increment (7.32) on ω_e does not agree in its structure with the general expression (7.8). The entire question lies in the left-hand inequality of (7.33). Indeed, if we go over to the rest system of the electrons, then the source of the instability must be considered to be the ion flux. Here the left-hand inequality of (7.33) becomes equivalent to the inequality inverse to (3.5). Consequently, with respect to the ions the Buneman instability is the single-particle stimulated Cherenkov effect—radiation by an ion flux of electron Langmuir waves. Therefore the increment (7.32) also recalls in its form (7.4). Thus the use of this or that terminology is not always unique, although in any case it correctly reflects the essence of the matter. For example, adopting the point of view that the Buneman instability is an anomalous Doppler effect, we shall apply the estimate (6.8) for determination of the minimum velocity of the electrons relative to the ions. Since Ω_b in our case is ω_e and $\omega_e \approx k_{\parallel} u_{\parallel}$, we have $v_{\parallel \min} \approx 0$, that is, the current in a plasma in the case of a Buneman instability is interrupted completely. It is possible to arrive at the same conclusion also from the point of view that the Buneman instability is single-particle stimulated radiation of the ions. This is confirmed also by exact nonlinear calculations.[48]

Thus, we have considered the principal radiative beam instabilities which develop near the radiation line. In the case of low-density beams all radiative instabilities are single-particle, and with increase of the beam density they become collective. There are systems in which an increase of the beam density leads to the suppression of a radiative instability by a nonradiative aperiodic process. We note that beams can be aperiodically unstable also far from the radiation line. These instabilities are due either to relativistic effects of the negative-mass type[44] or to a negative value of the effective dielectric permittivity of the system in which the beam is propagated.

[1)]The not very accurate name Compton scattering is widely used in the literature. We shall stay with the more correct terminology Thomson scattering. Compton scattering (the Compton effect) is a quantum process due to the corpuscular properties of light. The Compton shift of frequency in Compton scattering $\Delta\omega/\omega \approx \hbar\omega/mc^2$ must be considered only in the region of very high frequencies (in scattering by an electron in the frequency region above $10^{20}\,s^{-1}$). The departure from coincidence of the frequencies of the scattered and incident waves in (1.11) is due only to the motion of the electron and is related to the Doppler effect. In the rest system of the electron we would have $\omega_1 = \omega_2$, as also should occur in classical Thomson scattering.

[2)]To avoid misunderstandings we note that we are not discussing radiation from a "bare" beam. In fact, a beam which is neutral in charge and rectilinear is stable. It begins to radiate when it is placed in a medium or system in which there are eigenwaves with dispersion laws $\omega(k_{\parallel})$ satisfying definite resonance conditions; see Eq. (2.8) and the following discussion. These eigenwaves we shall call radiated waves or radiation. Therefore only under certain conditions (in motion in a plasma, retarding medium, periodic structure, and the like) will a beam begin to radiate and will the increment $\delta\omega$ appear.

[3)]The vector relation $\mathbf{P} = \mathbf{k}/\omega W$ applies. However, having in mind that in applications one deals with waves in waveguide structures with a single direction of propagation, we do not introduce the vectors **P** and **k** here. In waveguides there is only a longitudinal propagation constant $k_{\parallel}$, and in the transverse direction the waves are standing waves. This fact has already been taken into account by us in writing down the conditions for spontaneous radiation.

[4)]Equation (2.14) with $\omega_0 = \omega_2 - k_{\parallel 2} u_{\parallel}$ follows from the conditions of undulator radiation (1.6).

[5)]The actual form of writing the wave perturbation (1/2) $[\exp(-i\omega + ik_{\parallel} z) + \text{c.c.}]$ contains both $+\omega$ and $-\omega$, and therefore the sign of only the frequency means nothing; it is necessary to take into account also the sign of the wave number $k_{\parallel}$. For example, waves with $(\omega, k_{\parallel})$ and $(-\omega, -k_{\parallel})$ are completely indistinguishable, while waves with $(\omega, k_{\parallel})$ and $(-\omega, k_{\parallel})$ are propagated in different directions. Since we have decided to write the resonance condition in the form (2.14) with $\omega_1 > 0$ and since we wish to take into account in this condition waves with all possible directions of propagation, it is necessary for us not to impose any restrictions on the sign of ω_2. Otherwise the resonance condition would be written in two forms (1.11) and the unified form of the presentation would be destroyed.

[6)]In the general case $\Omega_b^2 \neq \omega_b^2$, but always $\Omega_b^2 \sim \omega_b^2 \sim n_b$. For example, in a nonrelativistic magnetized beam $\Omega_b^2 = \omega_b^2 k_{\parallel}^2/(k_{\perp}^2 + k_{\parallel}^2)$, where $k_{\perp}$ is the component of the wave vector transverse to the external magnetic field.[19]

[7)]The assumption that the plasma oscillations of the beam are potential has been made already in Eq. (3.1).

[8)]The presence in a system of waves with negative energy means that it is not in equilibrium, or that there is present in the unperturbed system a certain reserve of energy (in a beam this is the kinetic energy of motion). It is clear that such a system can be perturbed in such a way that its total energy decreases. Such a perturbation is a wave with negative energy.

[9)]More precisely, the condition of instability has the form $\omega - k_{\parallel} u_{\parallel} < 0$, and $\omega > 0$ or $\omega - k_{\parallel} u_{\parallel} > 0$ and $\omega < 0$. It is obvious that these two forms are equivalent. Unless specifically stated otherwise, we will assume everywhere that $\omega > 0$.

[10)]The beam waves have singularities for $\omega/k_{\parallel} < 0$, as can be seen from the diagram in Fig. 3. Since the singularities appear in the low-frequency region $|\omega| < \Omega_b$, we do not consider them.

[11)]The Raman-Mandel'shtam-Landsberg effect is the name given to the scattering of light by an atom (or molecule) accompanied by excitation of the latter. For excitation of an atom it is necessary to communicate to it energy, which is removed from the incident light quantum, as a result of which the frequency of the secondary (scattered) quantum turns out to be less than that of the incident quantum (the normal Stokes line of scattering). To remove excitation in an atom, energy must be taken away—it goes into increase of the frequency of the scattered quantum (the anti-Stokes line of scattering). In these cases an atomic system is equivalent to a beam with a wave of positive energy excited in it. The analog of a beam with a wave of negative energy excited in it would be an atomic system, for excitation of which it would be necessary to remove energy from it, and for removal of the excitation, on the other hand, it would be necessary to communicate energy to it. If now the theory of the Raman-Mandel'shtam-Landsberg effect is generalized to include such atomic systems, its analogy with the theory of scattering of waves on a beam will be complete.

[12)]In this section for the sake of clarity we discuss nonrelativistic electron beams. However, everything that is said remains in force also for the relativistic case. For a flux of nonrelativistic electrons in a constant magnetic field $W_{\perp} = (1/2) m n_b \omega_H^2 R_L^2$, where n_b is the density of electrons and R_L is the Larmor radius. In the general case $W_{\perp} = (1/2) m n_b \omega_0^2 A_{\perp}^2$, where ω_0 is the frequency and $A_{\perp}$ is the amplitude of the oscillations.

[13)]In the case of radiation of circular waves in an isotropic dielectric with permittivity ε_0, we have $\alpha = \varepsilon_0$ and Eq. (4.8) reduces to the form $\delta\omega = (\omega_b^2 \omega_H / 2\varepsilon_0 \omega)^{1/2}$. The latter result is well known.[24]

[14)]The static electric and magnetic fields of a beam of electrons are not considered, since they have no relation to radiation.

[15)]If $\tilde{z}$ is the perturbation of the electron trajectory, then Eq. (6.4) is equivalent to the inequality $|k_{\parallel} \tilde{z}| \gtrsim 1$, which denotes a strong perturbation of the trajectory. For estimation of $\tilde{z}$ the linearized equations (2.2) are sufficient.

[16)]Strictly speaking, instead of (6.1) in the collective effect, as follows from (3.10) and (3.11), the following relation exists: $W = n_b mc^2 \gamma^3 [1 - (v_{\parallel}/u_{\parallel})][1 - (\Omega_b / k_{\parallel} u_{\parallel})] u_{\parallel}^2/c^2$. However, for $\Omega_b \ll k_{\parallel} u_{\parallel}$ it does not differ from (6.1).

[17)]In Eqs. (7.11) and (7.12), as follows from their derivation (see Refs. 17 and 36), the quantities $|\tilde{\varepsilon}|$ and $|k_{\parallel z_E}|$ are small parameters.

[18)]The distortion of the polarization of the radiation field can be taken into account also in systems of the type (7.11) and (7.12). However, in these systems as a result of the different spatial scales of the fields of the beam and of the radiation, taking into account the distortion leads to small corrections. These corrections, furthermore, have no relation to the eigenoscillations of the beam.

[1]V. P. Silin and A. A. Rukhadze, Electromagnetic Properties of Plasma and Plasma-Like Media (in Russian), Atomizdat, M., 1961.
[2]V. L. Ginzburg, Theoretical Physics and Astrophysics (in Russian), Nauka, M., 1981 [In English: Pergamon Press, Oxford, 1979].
[3]I. E. Tamm and I. M. Frank, Dokl. Akad. Nauk SSSR **14**, 107 (1937).
[4]V. L. Ginzburg, Dokl. Akad. Nauk SSSR **23**, 773 (1939).
[5]B. M. Bolotovskiĭ, Usp. Fiz. Nauk **75**, 295 (1961) [Sov. Phys. Usp. **4**, 781 (1962)].
[6]L. D. Landau and E. M. Lifshitz, Field Theory (in Russian), Nauka, M., 1967 [Engl. transl., The Classical Theory of Fields, 3rd ed., Pergamon Press, Oxford, 1971].
[7]D. F. Alferov, Yu. A. Bashmakov, and E. G. Bessonov, Proc. (Tr.) P.N. Lebedev Phys. Inst. Acad. Sci. USSR **80**, 100 (1975).
[8]C. Kittel, Introduction to Solid State Physics, Wiley, 1976, Appendix A [Russ. transl., Nauka, M., 1978].
[9]B. M. Bolotovskiĭ and G. V. Voskresenskiĭ, Usp. Fiz. Nauk **94**, 377 (1968) [Sov. Phys. Usp. **11**, 143 (1968)].
[10]A. I. Akhiezer and Ya. B. Faĭnberg, Dokl. Akad. Nauk SSSR **69**, 555 (1949).
[11]D. Bohm and E. P. Gross, Phys. Rev. **75**, 1864 (1949).
[12]Ya. B. Fainberg, Atomnaya Énergiya **11**, 313 (1961) [Sov. Atomic Energy].
[13]V. I. Kurilko, Dokl. Akad. Nauk SSSR **208**, 1059 (1973) [Sov. Phys. Dokl. **18**, 132 (1973)].

[14]L. S. Bogdankevich, M. V Kuzelev, and A. A. Rukhadze, Usp. Fiz. Nauk **133**, 3 (1981) [Sov. Phys. Usp. **24**, 1 (1981)].
[15]L. S. Bogdankevich, M. V. Kuzelev, and A. A. Rukhadze, Zh. Tekh. Fiz. **50**, 233 (1980) [Sov. Phys. Tech. Phys. **25**, 143 (1980)].
[16]A. A. Varfolomeev, Free Electron Lasers (in Russian), Preprint of the Kurchatov Atomic Energy Institute, No. 4626, M., 1981.
[17]M. V. Kuzelev, Zh. Tekh. Fiz. **53**, 1029 (1983) [Sov. Phys. Tech. Phys. **28**, 623 (1983)].
[18]B. B. Kadomtsev, Collective Phenomena in Plasma (in Russian), Nauka, M., 1976.
[19]Electrodynamics of Plasma (in Russian), Edited by A. I. Akhiezer, Nauka, M., 1974.
[20]P. Sprangle and V. L. Granatstein, Appl. Phys. Lett. **25**, 377 (1974).
[21]B. B. Kadomtsev, A. B. Mikhaĭlovskiĭ, and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **47**, 2266 (1964) [Sov. Phys. JETP **20**, 1517(1965)].
[22]M. Born, Atomic Physics, Hafner, New York, 1957.
[23]I. M. Frank, Nobel Lecture II Dec. 1958. Russian version in Usp. Fiz. Nauk **68**, 397 (1959).
[24]V. V. Zheleznyakov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **2**, 14 (1959); **3**, 57 (1959).
[25]M. V. Nezlin, Usp. Fiz. Nauk **120**, 481 (1976) [Sov. Phys. Usp. **19**, 946 (1976)].
[26]A. N. Matveev, Electrodynamics and the Theory of Relativity (in Russian), Vysshaya shkola, M., 1964.
[27]A. V. Gaponov, Zh. Eksp. Teor. Fiz. **39**, 326 (1960) [Sov. Phys. JETP **12**, 232 (1961)].
[28]R. I. Kovtun and A. A. Rukhadze, Zh. Eksp. Teor. Fiz. **58**, 1709 (1970) [Sov. Phys. JETP **31**, 915 (1970)].
[29]V. D. Shapiro and V. I. Shevchenko, Zh. Eksp. Teor. Fiz. **60**, 1023 (1971) [Sov. Phys. JETP **33**, 555 (1971)].
[30]M. V. Kuzelev and A. A. Rukhadze, Kratk. Soobshch. Fiz. No. 11 10, (1985) [Sov. Phys. Lebedev Inst. Rep. No. 11 (1985)].
[31]H. Wilhelmsson and J. Weiland, Coherent Nonlinear Interaction of Waves in Plasmas, Pergamon, N.Y., 1976 [Russ. transl., Energoizdat, M., 1981].
[32]M. V. Kuzelev and V. A. Panin, Izv. Vyssh. Uchebn. Zaved. Fiz. **27** (No. 1), 31 (1984) [Sov. Phys. J. **27** (No. 1), **25** (1984)].
[33]M. V. Kuselev, A. A. Rukhadze, and G. V. Sanadze, Zh. Eksp. Teor. Fiz. **89**, 1591 (1985) [Sov. Phys. JETP **62**, 921 (1985)].
[34]N. S. Ginzburg, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **22**, 470 (1979) [Radiophys. Quantum Electron. (1979)].
[35]V. V. Bogdanov, M. V. Kuzelev, and A. A. Rukhadze, Fiz. Plazmy **10**, 548 (1984) [Sov. J. Plasma Phys. **10**, 319 (1984)].
[36]M. V. Kuzelev, and V. A. Panin, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **27**, 426 (1984) [Radiophys. Quantum Electron. (1984)].
[37]M. Born and E. Wolf, Principles of Optics, Pergamon Press, N.Y., 1959 [Russ. transl., Nauka, M., 1970].
[38]M. V. Kuzelev, A. A. Rukhadze, and D. S. Filippychev, Fiz. Plazmy **8**, 537 (1982) [Sov. J. Plasma Phys. **8**, 302 (1982)].
[39]N. I. Aĭzatskiĭ Fiz. Plazmy **6**, 597 (1980) [Sov. J. Plasma Phys. **6**, 327 (1980)].
[40]Nonlinear and Nonequilibrium Processes in Plasma Radiophysics (in Russian), edited by V. N. Oraevskiĭ, Nauka, M., 1982.
[41]A. A. Kolomenskiĭ, Physical Basis of Methods of Charged Particle Acceleration (in Russian), Moscow State University Publishing House, M., 1980.
[42]V. L. Bratman, N. S. Ginzburg *et al.*, Intern. J. Electron. **51**, 541 (1981).
[43]V. V. Bogdanov, M. V. Kuzelev, and A. A. Rukhadze, Kratk. Soobshch. Fiz. No. 2, 17 (1986) [Sov. Phys. Lebedev Inst. Rep. No. 2, 20 (1986)].
[44]V. V. Bogdanov and M. V. Kuzelev, Kratk. Soobshch. Fiz. No. 4 3 (1985) [Sov. Phys. Lebedev Inst. Rep. No. 4 (1985)].
[45]A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Usp. Fiz. Nauk **73**, 701 (1961) [Sov. Phys. Usp. **4**, 332 (1961)].
[46]V. M. Agranovich and V. L. Ginzburg, Spatial Dispersion in Crystal Optics and the Theory of Excitons (in Russian), Nauka, M., 1965 and 1979 [Eng. transl., Wiley, New York, 1967 and Springer-Verlag, Berlin, 1984].
[47]A. A. Ivanov, Physics of Highly Nonequilibrium Plasma (in Russian), Atomizdat, M., 1977.
[48]M. V. Kuzelev, A. A. Rukhadze, and D. S. Filippychev, Kratk. Soobshch. Fiz. No. 5, 25 (1981) [Sov. Phys. Lebedev Inst. Rep. No. 5, 21 (1981)].

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