G. M. Zaslavskiĭ, R. Z. Sagdeev, O. A. Usikov, and A. A. Chernikov. *Stochastic web and structure generation.* In Hamiltonian systems arbitrarily small perturbations destroy singular trajectories (separatrices), passing through saddle points. For periodic perturbations the destruction is of a stochastic character, generating in the vicinity of a separatrix a region of stochastic dynamics—a stochastic layer.[1] In phase space narrow stochastic layers can intersect, forming a complicated network of channels (stochastic web) along which the particle can wander. The existence of such a web in the general case of Hamiltonian systems with $N \geqslant 3$ degrees of freedom (six-dimensional phase space) was predicted by Arnol'd[2] (Arnol'd diffusion).

The boundary of existence of a stochastic web with $N = 3$ can be lowered to the minimum $N = 3/2$ (i.e., the phase-space is three-dimensional), if some additional resonance conditions leading to strong degeneracy of the system are satisfied.[3] An example of such a system is a linear oscillator, perturbed by a periodic sequence of δ pulses. Its Hamiltonian has the form

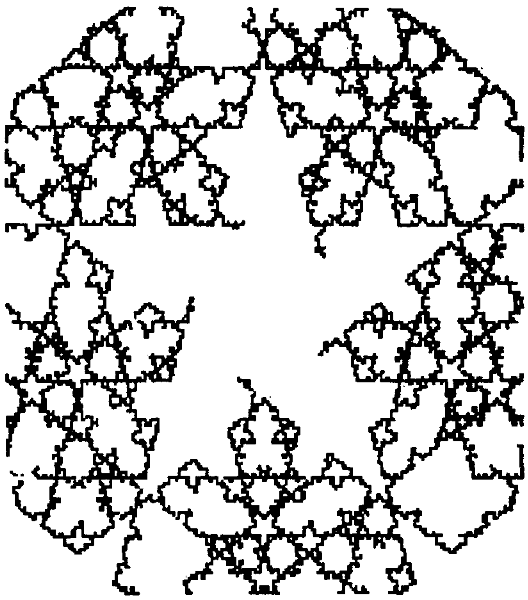

FIG. 1. Example of a stochastic web with a symmetry of fifth order. Inner "window"—an example of the well-known fractal tree. The "window" appears owing to the existence of cantorotors and is also filled by elements of the web after a sufficiently long time.

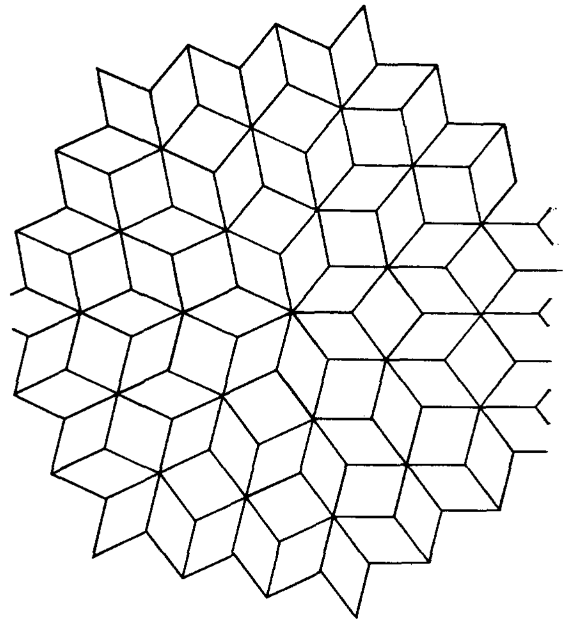

FIG. 2. Example of nonperiodic covering of the plane with a symmetry of seventh order with the help of two elements—thick and thin rhombi.

$$H=\frac{1}{2}(\dot{x}^2+\omega_H^2x^2)-\frac{K}{T}\cos x\sum\delta(t-nT), \qquad (1)$$

where K is a dimensionless perturbation parameter and T is the period of the perturbation. The variable t appears as a third component of the phase space, in addition to x and $\dot{x}$. The problem of the motion of a particle in a constant magnetic field and in the field of a wide wave packet, propagating perpendicularly to a magnetic field, leads to the same expression (1). Here ω_H is the Larmor frequency.

The Hamiltonian (1) has the special property that the interaction of the rotational symmetry of the trajectories of the motion in a magnetic field and the translational symmetry along the x axis, determined by the wave packet owing to the term with cos x, has a quite simple form. The interaction of these symmetries is strongest under resonance conditions:

$$\alpha\equiv\omega_H T=\frac{2\pi p}{q}\quad(p<q), \qquad (2)$$

where p and q are integers (in what follows $p=1$ can be used everywhere).

The equations of motion for (1) can be reduced exactly to "a mapping with twisting"

$$\hat{M}_\alpha:\begin{cases}\bar{u}=\left[u+\frac{K}{\alpha}\sin\vartheta\right]\cos\alpha+v\sin\alpha,\\ \bar{v}=-\left[u+\frac{K}{\alpha}\sin\vartheta\right]\sin\alpha+v\cos\alpha,\end{cases} \qquad (3)$$

where $u=\dot{x}/\omega_H$, $v=-x$, and the pitch of the mapping equals T. The mapping (3) under the condition (2) generates in the (u,v) plane a stochastic web with arbitrarily small values of K with a thickness $\sim\exp(-\mathrm{const}/K)$. The web exhibits approximate rotational symmetry of order q and is fractal. As the interaction parameter K increases the thickness of the web increases, and for large K the phase plane is covered by a stochastic sea, in which separate small islands remain at the locations of the cells of the web.

Under the resonance condition (2) it is possible to separate from the Hamiltonian (1) the part which creates the "bare" separatrices, and the part which dresses them in a stochastic layer.[4] The first part has the following form:

$$H_q=-\frac{K}{\alpha^2 q}\sum_{j=1}^{q}\cos\left(u\cos\frac{2\pi j}{q}+v\sin\frac{2\pi j}{q}\right), \qquad (4)$$

which brings us immediately to the assertion that we are dealing with structures of the "quasicrystal" type with $q\neq 2$, 3, 4, and 6. The corresponding structures in the (u,v) plane are formed by lines of constant energy $E=H_q$. In particular, for $q=5$ Penrose's mosaic can be placed with the help of a simple algorithm on the contour lines $H_q=E$.[4] An important relationship is thereby established between the Penrose covering and the properties of the "covering generator" (tiler mapping) $\hat{M}_{\alpha=2\pi/q}$ with $q=5$ (Fig. 1).

The existence of a Hamiltonian H_q in an explicit form and the tiler mapping $\hat{M}_{\alpha=2\pi/q}$ (q is an integer) enables the study of many important properties of similar structures with a symmetry of the "quasicrystal" type. They are determined as structures with the symmetry of the contour lines of H_q (q is an integer). The properties of the density of states for $q\geqslant 7$ are similar to those of the density of states of a liquid. Structures with $q=5, 7, \ldots$ can appear as intermediate structures at the order-chaos transition (Fig. 2).

The family of contour lines for H_q are states from a large number of singly-coupled regions with different areas and geometries. This family is also fractal and exhibits scaling properties. The distribution of elliptical and hyperbolic points is concentrated primarily in some energy bands and apparently likewise retains a symmetry of the "quasicrystal" type.[5]

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