Fractional charge in quantum field theory and solid-state physics

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Topological solitons that polarize the Dirac vacuum can assume fractional fermion (and electric) charge. The effect is investigated in detail in low-dimensional [(1 + 1)- and (2 + 1)-dimensional space-time] quantum field theory models. A review is given of phenomena associated with fractional charges found experimentally in quasi-one-dimensional (spin-charge anomaly in lightly doped polyacetylene) and two-dimensional systems (quantum Hall effect). Topics covered include: (1) fractional charge in one-dimensional quantum field theory models, (2) anomalous quantum numbers in the Peierls-Fröhlich system, and (3) two-dimensional models of quantum field theory and the quantum Hall effect.

CONTENTS

1. INTRODUCTION

Advances in quantum field theory were significantly influenced during the last decade by topological ideas and methods. In addition to traditional studies of local field dynamics, the global characteristics of this theory have attracted particular attention. The topological charge occupies an important place among the new concepts that have emerged in this approach to quantum field theory models. The conservation of this charge is assured not by the dynamics of the interaction, but by nontrivial boundary conditions. The topological charge is essentially a global characteristic of the theory, and is carried by objects with spatial structure, i.e., solitons.

In 1976, Jackiw and Rebbi investigated the effect of fermion fields on the quantum dynamics of solitons and discovered a striking fact: in one-dimensional models, a topologically stable soliton (kink) polarizes the Dirac vacuum so that it acquires a half-integer fermion charge (electric charge, if the fermions are electrically charged).[1]

This result is widely known among specialists in the quantum theory of solitons, but was far removed from possible experimental consequences. Indeed, one-dimensional models cannot claim to be realistic theories in high-energy physics, and are used only as a testing ground for the theoretical verification of different hypotheses that are subsequently transferred to the real three-dimensional world.

But then, in 1979, Su, Schrieffer, and Heeger[36] published their paper, in which they investigated the linear polymer trans-polyacetylene [trans-$(CH)_x$], an object which at first sight seemed far removed from any possible application of quantum field theory. Pure polyacetylene is a good dielectric, but it acquires relatively high conductivity already when lightly doped. Moreover, the increase in conductivity (by 10–12 orders of magnitude) is accompanied by a reduction in magnetic susceptibility, practically down to zero. This experimetnal fact suggests that free charge carriers in trans-$(CH)_x$ have zero spins. On the contrary, in undoped polyacetylene, there are mobile neutral objects with spin 1/2 (see, for example, the review given in Ref. 39). To explain the anomalous spin-charge relationship and the unusual optical properties of weaky-doped polyacetylene, Su *et al.*[36] and Rice[37] suggest that the free charge (spin) carriers in trans-$(CH)_x$ are topological structure defects, namely, solitons of the order parameter of the Peierls dielectric, whose existence was first predicted by Brazovskiĭ.[35]

The work of Brazovskiĭ,[35] Su *et al.*[36] and Rice and Timonen[37] stimulated unusual activity in theoretical and experimental studies of Peierls dielectrics (PD), including po-

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lyacetylene. It was soon recognized that the above anomalies of polyacetylene were manifestations of the Jackiw-Rebbi charge fragmentation effect, masked by the spin of the real electron.[3,47] It was found that the functional model of Peierls dielectrics was largely the same as the previously studied one-dimensional quantum field theory models.[43–50,40]

Polyacetylene belongs to a special class of Peierls dielectrics (dimerized structures) with real order parameter. In general, a Peierls structural phase transition is accompanied by the emergence of a complex order parameter whose phase describes collective subgap motion of electrons in a filled band (Fröhlich charge-density wave). Charge-density solitons were first discussed within the framework of the phenomenological approach by Rice, Bishop, and Krumhansl[57] and can have an arbitrary fractional charge. The electrical production of such soliton-antisoliton pairs is one of the possible mechanisms of nonlinear conductivity[58,65] of a number of quasi-one-dimensional compounds (TaS_3, $NbSe_3$, etc.).

A situation will therefore seem to have emerged in which formal one-dimensional quantum field theory models are being successfully applied to quasi-one-dimensional conducting media, whereas experiments in this branch of physics, which is remote from quantum field theory, partially confirm a prediction that is unusual for a modern high-energy physics model, namely, the fragmentation of the fermion charge. On the other hand, the existence of fractional charge in quasi-one-dimensional physics has not as yet been directly verified by experiment.

Fractional quantum numbers were first observed by Tsui *et al.*[92] who measured the Hall conductivity of a two-dimensional electron gas (quantum Hall effect) in heterojunctions based on gallium arsenide (GaAs-AlGaAs). Integer quantization of the transverse component of magnetoresistance and the simultaneous vanishing of the longitudinal component in a gas of two-dimensional electrons in a strong magnetic field were discovered in 1980 by von Klitzing *et al.*[90] and were interpreted in terms of the coherent motion of delocalized electronic states corresponding to completely filled Landau levels. The high precision with which the integer quantization of Hall conductivity was observed has been explained in terms of the topological nature of the effect.[101,105,106]

The existence of the Hall-type topological current was predicted soon after in gauge models of quantum field theory in $(2+1)$-dimensional space-time.[86–88] In particular, it was shown that, in two-dimensional quantum electrodynamics, the electric fields $\mathbf{E}$ give rise to a nondissipative vacuum current with conductivity $\sigma = e^2/4\pi\hbar$ at right-angles to the direction of $\mathbf{E}$ (e is the electron charge). However, while, for the Peierls-Fröhlich system, field-theoretical models can be obtained from microscopic Hamiltonians and therefore claim to provide a description of the real situation, the relationship between the topological vacuum current in $2D$-electrodynamics and the Hall current of nonrelativistic two-dimensional electrons in a quantizing magnetic field is purely formal and does not involve the different physical factors that explain both effects.

The integer quantum Hall effect (QHE) can be explained in terms of the single-particle model of $2D$-electrons interacting with an external electromagnetic field and with impurities. The experimentally observed fractional quantization of Hall conductivity is a consequence of many-particle effects, e.g., the Coulomb interaction between electrons. The model of an incompressible electron quantum liquid[107] was proposed by Laughlin to explain the fractional QHE. The ground state of this model corresponds to a condensate of electrons rotating about a common center of mass, and quasiparticles (quasielectrons and quasiholes) have a finite excitation energy and a fractional electric charge. But possibly the most unusual property of the Laughlin quasiparticles is their anomalous statistics.[121,122] Here again, there is a close conjunction of ideas and methods developed in quantum field theory and in solid-state physics.

In 1982, Wilczek studied the motion of a particle of charge q in the field of an infinitesimally thin solenoid producing a flux Φ, and pointed out that the quantity $\Delta_\Phi = q\Phi/2\pi$ could be regarded as the spin of this two-dimensional bound system (the composite consisting of a particle and magnetic field flux has been given the name, "anyon").[116] The quantum-mechanical ensemble of indentical anyons is then found to obey anomalous statistics which, in general, is intermediate between the traditional Fermi and Bose statistics of ordinary particles.[116,120] The anomalous statistics of quasiparticles in fractional quantum Hall effect is a further example of the interdependence of fundamental ideas that arise in different branches of modern physics.

The topological charge (and the associated anomalies in the polarization of the fermion vacuum) is not a specific property of low-dimensional models. It is present in spaces of any dimension. However, in this review, we shall confine our attention to one- and two-dimensional theories because it is precisely in low-dimensional physics that current research has shown an intimate connection between the predictions of quantum field theory models and many unusual effects found experimentally in the physics of condensed media.

Our aim is to provide a systematic account of theoretical facts on the problem of fractional charges. We have tried to present our material in a form that will be interesting to specialists working either in quantum field theory or solid-state physics. However, the account is based on ideas developed in quantum field theory (this also applies to the terminology that we shall use). The "solid-state" aspect of the problem will not be fully dealt with and most attention will be devoted to the field-theoretical point of view. To a lesser extent, this also applies to the section devoted to the quantum Hall effect but, here again, we shall say very little about the part played by disorder, which is very important in this context, but is not in our direct line of sight here. Unfortunately, the experimental situation in this field cannot yet be regarded as fully settled. Moreover, the experimental facts relating to the quantum Hall effect will also be presented in a very brief form. All this additional information can be gleaned, for example, from published reviews (Refs. 39–42, 61, 94, 135, 136.

2. FRACTIONAL CHARGE IN ONE-DIMENSIONAL QUANTUM FIELD THEORY MODELS

2.1. Solitons in the Jackiw-Rebbi model

We shall begin our study of the phenomenon of fractional charges with the simplest one-dimensional model describing the interaction between fermion ψ and scalar φ fields:

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\varphi)^2-U(\varphi)+\bar{\chi}(i\gamma_\mu\partial_\mu-g\varphi)\psi; \qquad (2.1)$$

where $\bar{\psi}\equiv\psi^+\gamma_0$, γ_μ are the Dirac matrices $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$ and g is the Yukawa coupling constant. The condition imposed on the potential $U(\varphi)$ [in addition to the requirement of renormalization of the model (2.1)] is that the boson sector of the model must contain static topologically stable solitons. Suppose that $U(\varphi)$ has, for example, the form shown in Fig. 1 [with the correct choice of the potential, $U(\varphi)=\lambda(\varphi^2-\varphi_0^2)^2$, the Lagrangian (2.1) is sometimes referred to as the Jackiw-Rebbi model]. The solutions of (2.1) for $g\to 0$ that have the asymptotic behavior

$$\varphi_{\bar{S}}(x)=\begin{cases}\varphi_0, & x\to\infty,\\ -\varphi_0, & x\to-\infty,\end{cases}\quad \varphi_S(x)=\begin{cases}-\varphi_0, & x\to\infty,\\ \varphi_0, & x\to-\infty,\end{cases} \qquad (2.2)$$

have finite energy and are absolutely stable. It is also clear that such solitons occur for the Yukawa interaction between the scalar fields φ and fermions (at least for weak coupling constants). Once $U(\varphi)$ has been chosen, we can readily find the solutions of the classical equations of motion for the fields φ and ψ, which correspond to the boundary conditions (2.2) (see, for example, Ref. 1). However, for our purposes, the explicit form of the soliton solutions is unimportant. We must examine what influence the topologically stable soliton has on the fermion vacuum.

2.2. Supersymmetry, zero modes, and fractional charges

We shall assume that the static soliton $\varphi_S(x)$ is an external field, in which case, (2.1) reduces to the Lagrangian for Dirac electrons with position-dependent "mass" $m(x)=g\varphi_S(x)$. We recall that, in the $(1+1)$-dimensional space, ψ is a two-component Dirac spinor and describes two degrees of freedom (particle-antiparticle). Since one-dimensional electrons do not have kinematic spin (there is no rotation group), the Dirac matrices degenerate to the Pauli matrices $(\gamma_0,\gamma_1,\gamma_5=\gamma_0\gamma_1)\to\sigma_i$. For example, if we take $\gamma_0=-\sigma_1,\gamma_1=-i\sigma_3$ and assume that $\psi(x,t)=e^{i\omega t}\psi(x)$, we have

$$H_D\psi(x)=\omega\psi(x),\quad H_D=\sigma_2\hat{p}-\sigma_1 m(x), \qquad (2.3)$$

where ω is the energy of the fermions in the static scalar field $m(x)$, $\hat{p}=-i\partial_x$. Since $\{H_D,\sigma_3\}=0$, we can readily construct two anticommuting Hermitian operators $Q_{1,2}$ whose squares are equal:

$$Q_1\equiv H_D,\quad Q_2=i\sigma_3H_D,\quad \{Q_1,Q_2\}=0, \qquad (2.4)$$

$$H_S=H_D^2=Q_1^2=Q_2^2=\hat{p}^2+m^2(x)+\sigma_3m'(x). \qquad (2.5)$$

The operator H_S is the Hamiltonian of the supersymmetric Witten quantum mechanics. According to the general properties of supersymmetric theories, the Hamiltonian (2.5) has the following properties (see, for example, the review in Ref. 32): (1) all the "energy" levels $E_S>0$ of the Hamiltonian H_S are at least double degenerate, which is a consequence of the charge symmetry $\psi_{-\omega}=\sigma_3\psi_\omega$, and (2) the zero mode $H_S\psi_0=0$ is not degenerate and the very existence of this normalized solution depends on the "topology" of the potential $m(x)$. In particular, when $m(x)$ assumes different signs as $x\to\pm\infty$, the zero energy level exists independently of the specific local behavior of the function $m(x)$. Conversely, when the potential $m(x)$ has the same sign at spatial infinities, the zero mode is absent.

It is readily seen that the zero modes of the Hamiltonians H_S and H_D are identical. Since the requirement of topological stability of the soliton $\varphi_S(x)$ satisfies the condition for the existence of the zero mode of the supersymmetric Hamiltonian H_S, we must conclude that a bound state with zero energy appears on the kink.

For spinless fermions, the zero mode can be either empty or singly filled. Since the energy of the fermion in the zero energy level is, of course, zero, the states of the soliton are doubly degenerate in energy. However, these states differ by the sign of the fermion charge. Let us suppose that F_+ is the fermion charge of the soliton with filled zero mode and F_- the charge in the unfilled zero mode. We then have $F_+-F_-=1$. Since considerations involving charge symmetry suggest that the fermion charges of these states are numerically equal, we find that $F_+=-F_-=1/2$. All this can be readily translated into rigorous algebraic language and the fractional charge of the kink can be proved[1,9] (see also Ref. 4, where the connection with supersymmetry is employed). We shall not reproduce the formal algebraic proof here. Instead, we will consider analytic methods that yield the same final results, but are more convenient in the study of the behavior of a fractional charge in an external field.

2.3. Fractionally-charged solitons and bosonization

A simple proof of the existence of fractionally charged solitons in the model defined by (2.1) can be obtained[2] by the bosonization method (see also Refs. 3 and 5).

We recall that, in $(1+1)$-dimensional space-time, quantum-mechanical systems can be described in the equivalent languages of boson or fermion variables. This equivalence was noted as far back as the 1960s for massless, free theories of fermions and boson fields (see, for example, the review given in Ref. 24). In particular, there was the simple fact that the canonical commutation relation for Bose operators

$$[\sigma(x),\dot{\sigma}(y)]=i\delta(x-y) \qquad (2.6)$$

transformed into the single-time fermion current commutator

$$[j^\mu(x),j^\nu(y)]=-\frac{i}{\pi}\varepsilon^{\mu\nu}\delta'(x-y),\quad \varepsilon^{01}=-\varepsilon^{10}=1 \qquad (2.7)$$

under the formal replacement

$$j^\mu(x)=\frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu\sigma. \qquad (2.8)$$

The canonical energy-momentum tensor of the free scalar

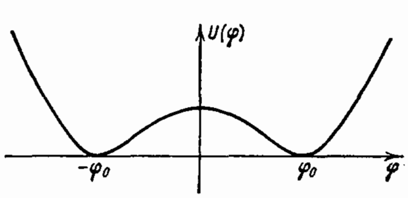

FIG. 1.

field transforms into the energy-momentum tensor of massless fermions, written in terms of the fermion current density.

In 1975, Coleman[22] proved a theorem on the equivalence of two nontrivial models of one-dimensional quantum field theory, namely, the massive Thirring model

$$\mathcal{L}_T = \bar{\psi} i\gamma_\mu \partial_\mu \psi - m\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2 \tag{2.9}$$

and the quantum sine-Gordon model

$$\mathcal{L}_G = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{\alpha_0}{\beta^2}(1-\cos\beta\sigma); \tag{2.10}$$

where ψ is the two-component Dirac spinor, σ is a real scalar field, m is the fermion mass, and β, g are the coupling constants of the scalar and fermion fields.

When the Fermi fields are expressed directly in terms of the Bose operators, the result takes a very unwieldy nonlocal form.[23,25] Structurally, the Fermi fields assume the form $F \sim e^B$, where B are the Bose operators, and the anticommutation relations for F appear because the commutator of the Bose operators is equal to the c-number $i\pi$:

$$F_1F_2 = e^{B_1}e^{B_2} = e^{B_2}e^{B_1}e^{[B_1,\,B_2]} = e^{B_2}e^{B_1}e^{i\pi} = -F_2F_1.$$

For bilinear combinations of Fermi fields, the transformation formulas have a simple local form and are convenient in practical calculations. In particular, the transformation from the massive Thirring model to the sine-Gordon model is accomplished by replacing the normal operator products as follows:

$$:\bar{\psi} i\gamma_\mu\partial_\mu\psi: \Longleftrightarrow \frac{1}{2} :(\partial_\mu\sigma)^2:, \tag{2.11}$$

$$:\bar{\psi}\gamma_\mu\psi: \Longleftrightarrow \frac{\beta}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\sigma, \tag{2.12}$$

$$:\bar{\psi}\psi: \Longleftrightarrow \mu : \cos(\beta\sigma): \tag{2.13}$$

The field coupling constants are then related by

$$1+\frac{g}{\pi} = \frac{4\pi}{\beta^2}. \tag{2.14}$$

(We note that the constant μ in (2.10) is proportional to the mass for which the normal ordering procedure is carried out.)

It follows from (2.14) that, when $\beta^2 = 4\pi$ ($g = 0$), the Thirring model degenerates to the free fermion model which corresponds in the boson formalism to the topologically stable solitons of the sine-Gordon model (for $\beta^2 = 4\pi$). This property enables us to use the bosonization procedure in numerous models of one-dimensional quantum field theory by considering the fields interacting with fermions as external (adiabatic approximation).

After bosonization, the Lagrangian (2.1) assumes the form

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - V(\sigma,\varphi), \tag{2.15}$$
$$V(\sigma,\varphi) = U(\varphi) + g\mu\varphi\cos(2\sqrt{\pi}\sigma).$$

According to (2.8) and (2.12), the fermion charge

$$Q_F = \int dx\, \bar{\psi}\gamma_0\psi \tag{2.16}$$

is identical with the topological charge of solitons in the boson formulation:

$$Q_F = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, \partial_x\sigma = \frac{\Delta\sigma}{\sqrt{\pi}}, \quad \Delta\sigma = \sigma(\infty) - \sigma(-\infty). \tag{2.17}$$

To determine Q_F, it is therefore sufficient to determine the vacuum values of the scalar fields φ and σ. In the quasiclassical approximation, this problem reduces to the study of the minima of the potential $V(\sigma,\varphi)$ (2.15):

$$U'(\varphi) + g\mu\cos(2\sqrt{\pi}\sigma) = 0, \quad \sin(2\sqrt{\pi}\sigma) = 0. \tag{2.18}$$

For a weak Yukawa coupling constant g (when the entire calculation scheme can be justified), the solutions of (2.18) are as shown in Fig. 2 ($\pm\bar{\varphi}$ are the vacuum values of the field φ). Stable static inhomogeneous solutions of the model (2.12) that reach the same equilibrium value of the field φ($+\bar{\varphi}$ or $-\bar{\varphi}$) as $x \to \pm\infty$ have $|\Delta\sigma| = \sqrt{\pi}$, so that according to (2.17) they have unit fermion charge. This is a fermion of the model defined by (2.1) (more precisely, a polaron, i.e., a locally perturbed scalar field vacuum with a bound single-fermion state). For solitons that assume asymptotic field values of different sign for $x \to \pm\infty$, we have $|\Delta\sigma| = \sqrt{\pi}/2$, and their fermion charge is half-integral. These states correspond to kinks. Thus, solitons with unfilled zero-energy level have the charge $Q_F = -1/2$ and those with a filled level have $Q_F = +1/2$. We note that the kink will also change the density of states of fermions in the continuum, but this perturbation reduces merely to a redefinition of the energy of a quantum soliton.

2.4. Chiral anomaly and fractional charge

In this Section, we shall generalize the problem to some extent, i.e., we shall consider vacuum polarization by a complex scalar field $\varphi = \rho e^{i\theta}$ [$\rho(x) > 0$]. The Lagrangian for fermions in the external field φ, which satisfies the general symmetry requirements (in relation to hermiticity, Lorentz invariance, and P-parity) is

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial_\mu - g\rho(x)e^{i\gamma_5\theta(x)})\psi, \quad \gamma_5 \equiv \gamma_0\gamma_1, \tag{2.19}$$

where g is the Yukawa coupling constant. It is convenient to transform the Lagrangian (2.19) by separating out the term that is the mass of the Dirac electron $m = g\rho$ for $\rho = \text{const}$. The form of this unitary transformation (chiral rotation) is uniquely dictated by the form of the fermion-boson coupling in (2.19):

$$U_\theta\psi U_\theta^{-1} = \exp\left(-\frac{i}{2}\gamma_5\theta(x)\right)\psi,$$
$$U_\theta\bar{\psi}U_\theta^{-1} = \bar{\psi}\exp\left(-\frac{i}{2}\gamma_5\theta(x)\right). \tag{2.20}$$

If we use the canonical anticommutation relations for the Fermi fields, we can readily verify that the chiral rotation operator U_θ can be written in the form

$$U_\theta = \exp\left(\frac{i}{2}\int j_0^5(x)\,\theta(x)\,dx\right), \quad j_\mu^5(x) = \bar{\psi}\gamma_\mu\gamma_5\psi, \tag{2.21}$$

where j_μ^5 is the axial-vector current.

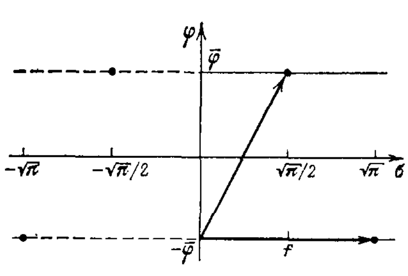

FIG. 2.

Transformations such as (2.20) can be readily carried out using the formal operator identity

$$e^{iA}Be^{-iA}=B+i[A, B]+\frac{i^2}{2}[A, [A, B]]+\cdots, \quad (2.22)$$

if we know the algebra of the operators A, B. In our case, we must evaluate the commutator of the vector and axial-vector currents. We note that the Dirac matrices satisfy the simple relation $\gamma^\mu\gamma^5=\varepsilon^{\mu\nu}\gamma_\nu$ in $(1+1)$-dimensional space, and the only nontrivial commutator (of the densities of vector and axial-vector charges) is readily found using, for example, the bosonization procedure (2.6)–(2.8):

$$[j_0^5(x), j_0(y)]=-\frac{i}{\pi}\delta'(x-y). \quad (2.23)$$

The relations given by (2.20)–(2.23) are sufficient to enable us to derive the chiral-transformed Lagrangian (2.19)[5,6]:

$$U_\theta\mathscr{L}U_\theta^{-1}\equiv\mathscr{L}_\theta=\bar{\psi}i\gamma_\mu\partial_\mu\psi-g\rho\bar{\psi}\psi+\frac{1}{2}\bar{\psi}\gamma^\mu\psi\varepsilon_{\mu\nu}\partial^\nu\theta+\frac{1}{8\pi}(\partial_\mu\theta)^2. \quad (2.24)$$

We draw attention to the last term in (2.24), which is due to the appearance of Schwinger terms in the charge-density algebra (2.23). In the functional approach, this addition to the classical Lagrangian is due to the transformation Jacobian for the change of variables $\psi=\exp[-(i/2)\gamma_5\theta(x)]\chi$ in the functional integral over the Fermi fields.[29,30]

The transformation formulas for the vector and axial-vector currents under local chiral rotations can be readily found using (2.20)–(2.23):

$$j_\mu^\theta(x)=j_\mu(x)-\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\nu\theta, \quad j_\mu^{5\theta}(x)=j_\mu^5(x)-\frac{1}{2\pi}\partial_\mu\theta. \quad (2.25)$$

According to the first of these, the fermion charge of vacuum

$$Q_F=\int_{-\infty}^{\infty}dx\,\langle 0|\bar{\psi}\gamma_0\psi|0\rangle-\frac{\Delta\theta}{2\pi}, \quad \Delta\theta=\theta(\infty)-\theta(-\infty) \quad (2.26)$$

consists of two parts, namely, the topological charge due to the nonzero phase difference $\Delta\theta$ and the polarization charge induced by the interaction between the vacuum fermions and the real scalar (ρ) and vector ($A^5\equiv\varepsilon_{\mu\nu}\partial^\nu\theta/2$) fields [see (2.24)].

Without imposing restrictions on the form of $\rho(x)$ and $\theta(x)$, we can say very little about the integral part of the total vacuum charge determined by filling all the fermion states of negative energy in an inhomogeneous external field. On the other hand, the fractional part is connected only with the global (topological) characteristics of the external field and can be calculated for a sufficiently general situation.

We now turn to the first term in (2.26). This is the ground-state charge $Q_F^{\rm pol}$, induced by the positive-definite scalar field $\rho(x)$ and vector field A_μ^5:

$$Q_F^{\rm pol}=\int_{-\infty}^{\infty}dx\,\frac{\delta S_{\rm eff}}{\delta A_0^5}; \quad (2.27)$$

where $S_{\rm eff}$ is the effective action, deduced from the Lagrangian (2.24) without the last term and integrated over the Fermi fields:

$$\exp(iS_{\rm eff})=\int D\bar{\psi}D\psi\exp\left(i\int d^2x\mathscr{L}\right).$$

For soliton-type external sources, $\rho(x)\to\rho_0$ for $|x|\to\infty$ and, according to the general theorems discussed earlier, the Dirac $1D$-Hamiltonian does not have zero modes in this scalar field. The interaction between the fermions and the field $\rho(x)$ does not, therefore, affect the fractional part of the vacuum charge.

The vector field A_μ^5 polarizes the fermion vacuum. It is clear that the effective action in (2.27) is a functional of only $\partial_\mu A_\nu^5(\mu\neq\nu)$. For weakly inhomogeneous quasistationary fields, $|\dot{A}_1|$, $|A_0'|\ll g\rho_0$, this coupling is effectively local and, in the lowest order in the external field, it provides a contribution to the vacuum energy density that is proportional to $(A_0')^2\sim(\theta'')^2$, $(\dot{A}_1)^2\sim(\ddot{\theta})^2$. This is much less than the contribution of the chiral anomaly, $(\partial_\mu\theta)^2$, which plays the part of the mass of the vector field A_μ^5. The contribution $Q_F^{\rm pol}$ can therefore be neglected, and the fractional part of the vacuum charge in the external complex scalar field is proportional to the topological charge of the phase solitons

$$Q_F=-\frac{\Delta\theta}{2\pi}. \quad (2.28)$$

The question is: what does this formula give us for the model of a real scalar field that we examined above? In this case, the phase θ can assume the values 0 or π (mod 2π). For topologically stable solitons ($\varphi\to\pm\varphi_0$ as $x\to\pm\infty$), the phase difference is $|\Delta\theta|=\pi$ and (2.28) reproduces the results obtained in Sections 2.2 and 2.3: the charge of the kink with unfilled zero-energy level is $Q_F=-1/2$, and that with the filled level is $Q_F=-1/2+1=+1/2$.

In the general case of complex scalar fields, the relation given by (2.28) does not forbid the existence in the vacuum of one-dimensional fermions with arbitrary fractional part of the charge.[5–14]

2.5. Interaction between fractional charge and electric field. Topological nature of fractional charge

When fermion models are electrically charged, the fractional fermion charge of solitons transforms into the fractional electric charge $Q=eQ_F$, and we have to consider the important problem of its interaction with the electromagnetic field.

In the model defined by (2.19), the external electric field is introduced by the standard device of extending the derivative: $\partial_\mu\to D_\mu=\partial_\mu-ieA_\mu$. After chiral rotation of the fermion fields, the Lagrangian for the model assumes the form

$$\mathscr{L}^\theta(A)=\bar{\psi}(i\gamma_\mu D_\mu-g\rho)\psi+\frac{1}{2}\bar{\psi}\gamma^\mu\psi\varepsilon_{\mu\nu}\partial^\nu\theta+\frac{1}{8\pi}(\partial_\mu\theta)^2-\frac{e}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu}\theta, \quad (2.29)$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and the last two terms in (2.29) appear because of the presence of anomalous Schwinger terms in the commutator of the charge densities (2.23) (chiral anomaly). We shall now show that the last term in the Lagrangian (2.29) describes the interaction between the external field and the fractional charge. Integrating this expression by parts, we obtain

$$\mathscr{L}_{\rm int}=-\frac{e}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu}\theta\Rightarrow eA_\mu j_t^\mu, \quad j_t^\mu=-\frac{1}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\theta; \quad (2.30)$$

where j_t^μ is the topological current whose conservation for a smooth function $\theta(x,t)$ is assured by the trivial identity

$\partial_\mu j_t^\mu \sim \partial_\mu \partial_\nu \theta \varepsilon^{\mu\nu} \equiv 0$. The zero component of this current integrated with respect to the space coordinate is equal to the fractional vacuum charge (i.e., the charge of the phase soliton). According to (2.30), the interaction between the electric field and the topological current has the standard electrodynamic form: it is local despite the collective character of the fractional charge.

The different methods that we have used to investigate the reaction of vacuum to an external topologically nontrivial field relate the fractional fermion charge and the global characteristics of the field.

We may therefore expect that, in the most general case, the fractional fermion vacuum charge can be expressed in terms of the topological invariants of the theory.

It is convenient to relate the vacuum charge to the spectral parameters of the problem. We are essentially investigating the asymmetry of the spectrum of the Dirac equation in flat space in an external field. The fermion charge operator $\hat{Q}_F$ has the required properties under the transformation of C-conjugation $C\hat{Q}_F C^{-1} = -\hat{Q}_F$, and it is well-known that it can be expressed in terms of the fermion field commutator

$$\hat{Q}_F = \frac{1}{2} \int dx\, [\psi^+, \psi], \quad Q_F = \langle 0|\hat{Q}_F|0\rangle. \tag{2.31}$$

Expanding ψ and ψ^+ into a series in terms of the eigenfunctions $\psi_k(x)$ of the Dirac equation with positive and negative energies ω_k, we can readily write the charge operator in the form (see, for example, Ref. 130):

$$\hat{Q}_F = \sum_k (\hat{n}_k - \hat{\bar{n}}_k) - \frac{1}{2}\Big(\sum_{k,\ \omega_k \geqslant 0} 1 - \sum_{k,\ \omega_k < 0} 1\Big), \tag{2.32}$$

where $\hat{n}_k$ ($\hat{\bar{n}}_k$) is the particle number (antiparticle number) operator. Standard texts on quantum field theory usually assume that the last term in (2.32) is zero. The vacuum charge is then also found to be zero, $\langle 0|\hat{Q}_F|0\rangle = 0$. This procedure is undoubtedly correct for free fermions, but it is not always valid when external fields are present. Actually, the expression

$$\eta(0) = \sum_{k,\ \omega_k \geqslant 0} 1 - \sum_{k,\ \omega_k < 0} 1 \equiv \sum_k \operatorname{sgn} \omega_k \tag{2.33}$$

in (2.32) is a formal series. To achieve a correct description of the sum, (2.33) must be regularized. It is convenient to use the following regularization:

$$\eta(0) = \lim_{s\to 0} \eta(s), \quad \eta(s) = \sum_{k,\ \omega_k \neq 0} (\operatorname{sgn} \omega_k)|\omega_k|^{-s} + n_0, \tag{2.34}$$

where n_0 is the number of zero modes. The quantity $\eta(\theta)$ is called the Atiyah-Palodi-Singer η-invariant.[79] Thus, the topological fermion vacuum charge can be expressed in terms of the η-invariant:[7,8]

$$Q_F = -\frac{1}{2}\eta(0). \tag{2.35}$$

For Dirac fermions in the external scalar field $\varphi(x)$ (2.3), the charge symmetry of the problem ensures that the energy spectrum ω_k is symmetric with respect to zero. This means that only the zero modes n_0 contribute to the sum in (2.34). For an external field that has asymptotic values of different sign as $x \to \pm\infty$, there is only one zero mode, $\eta(0) = 1$, and we again find that the fermion charge of a kink is half-integral.

When there is no charge symmetry, all states (including the continuum) contribute to the sum in (2.34), and the calculation of spectral asymmetry is less elementary (a general discussion and particular cases can be found, for example, in Refs. 8 and 11).

3. ANOMALOUS QUANTUM NUMBERS IN THE PEIERLS-FRÖHLICH SYSTEM

The existence of extended objects with unusual quantum numbers, i.e., fractional fermion and electric charges, was first predicted in quantum field theory.[1] A similar conclusion was reached independently in Refs. 35–37, 38, and 57, where a theoretical study was made of the formation of topological structure defects in a number of quasi-one-dimensional conducting compounds. These compounds belong to a class of models that are called Peierls dielectrics. We shall now use the language of quantum field theory to formulate a microscopic model of a Peierls dielectric, and will show how the phenomenon of fractional charges that has been examined in detail in quantum field theory has its parallels in the physics of quasi-one-dimensional condensed media.

3.1. Peierls dielectric: a model of dynamic mass generation

The Peierls dielectric belongs to a class of media in which the gap Δ at the Fermi level in the spectrum of conduction electrons is an order parameter. The appearance of the gap is in itself energetically favorable because it reduces the energy of filled electron states. In the Peierls transition, this splitting of the band occurs as a result of the electron-phonon interaction.

In the language of quantum field theory, the appearance of the gap at the Fermi level signifies that the fermions acquire mass; the Yukawa fermion-boson coupling corresponds to the electron-phonon interaction. The nonzero vacuum expectation value of the scalar field for the electron-lattice system, which is necessary for mass generation, is assured by the phonon condensate, i.e., a macroscopic shift of ions from their position of equilibrium. The system therefore undergoes a structural phase transformation.

The above analogy can be made more precise (see the review papers in Refs. 33 and 34). As an example, consider a one-dimensional metal with half-filled conduction band. Suppose that a Peierls phase transition has taken place in the system and a gap has appeared in the electron spectrum (Fig. 3) at the Fermi level (which we shall take as the origin of energy from now on). In the quasiclassical approximation, the energy density can then be written as the sum of two contributions, namely, the negative energy of electrons in the filled band (vacuum) and the positive energy of the phonon condensate:

$$\varepsilon(\Delta) = -2\int_{-k_F}^{k_F} \frac{dk}{2\pi}(\varepsilon_k^2 + \Delta^2)^{1/2} + \frac{\Delta^2}{g^2} + \frac{\dot{\Delta}^2}{g^2\tilde{\omega}^2}; \tag{3.1}$$

where $\varepsilon_k = -W\cos(ka)$ is the spectrum of conduction electrons in the strong-coupling model (W is the half-width of the conduction band, a is the interatomic separation), k_F is the Fermi momentum ($k_F = \pi/2a$, in our case), and g is the electron-phonon coupling constant. The last two terms in (3.1) correspond to the elastic and kinetic energies of the

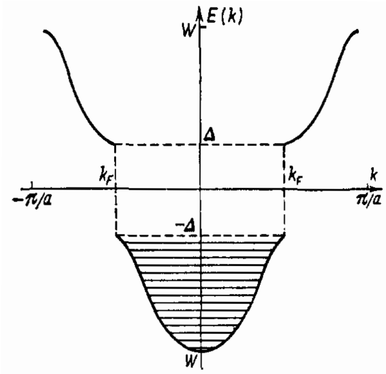

FIG. 3. $k_F = \pi/2a$ is the Fermi momentum.

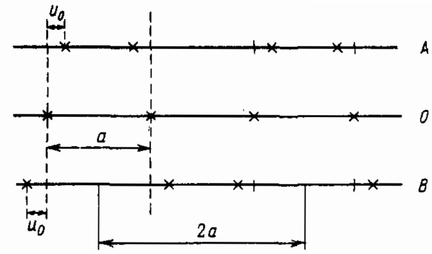

FIG. 4.

lattice in the harmonic approximation (because of the small ion shift, $|\Delta| \sim |u_0| \ll a$; Fig. 4), and $\bar{\omega}$ is the frequency of condensed phonons with momentum $Q = 2k_F$. The quantities W, $\bar{\omega}$ and g can be regarded as the bare constants of Peierls theory.

The equilibrium gap Δ_0 is determined from the condition that the energy $\varepsilon(\Delta)$ be a minimum:

$$\frac{\partial\varepsilon}{\partial\Delta} = 0 \Rightarrow \Delta = \pm\Delta_0, \quad \Delta_0 = 4W\exp\left(-\frac{\pi\hbar v_F}{g^2}\right). \quad (3.2)$$

Thus, at low temperatures, the one-dimensional metal with one electron per atom must be in the dielectric phase, which is characterized by twice the lattice period (dimerization) and the order parameter $\pm\Delta_0$.

For weak electron-phonon coupling constants, the resulting gap is $\Delta_0 \ll W$. In this approximation,

$$\varepsilon(\Delta) - \varepsilon(0) = \frac{\Delta^2}{2\pi}\left[\ln\left(\frac{\Delta}{\Delta_0}\right)^2 - 1\right] \quad (3.3)$$

(we are using a system of units in which $\hbar = v_F = 1$) and (3.3) remains unaltered if we linearize the electron dispersion relation near $k \simeq k_F$:

$$E(k) \Rightarrow E(p) = \pm(p^2 + \Delta^2)^{1/2}, \quad p = k - k_F. \quad (3.4)$$

After linearization, the electron sector of the Peierls dielectric model assumes a form typical for the models of one-dimensional relativistic quantum-field theory:

$$\mathcal{L} = \bar{\psi}_\sigma(i\gamma_\mu\partial_\mu - \Delta)\psi_\sigma - \frac{\Delta^2}{g^2} + \frac{\dot{\Delta}^2}{g^2\bar{\omega}^2}, \quad (3.5)$$

where ψ is a two-component relativistic spinor describing two degrees of freedom of the Dirac field (particle-antiparticle) in (1 + 1)-dimensional space. The Dirac matrices γ_μ then degenerate to the Pauli matrices. The spin of the real electron is taken into account by the "isotopic" doubling of the components, $\sigma = 1, 2$.

3.2. Microscopic model of a Peierls dielectric

In the microscopic description of the Peierls phase transition,[33,34] the starting point is usually the Hamiltonian for the electron-phonon Fröhlich interaction:

$$H = \sum_{k,\sigma}\varepsilon_k a^+_{k,\sigma}a_{k,\sigma} + \sum_q \omega_q b^+_q b_q + N^{-1/2}\sum_{k,q,\sigma} g a^+_{k+q,\sigma}a_{k,\sigma}(b_q + b^+_{-q}); \quad (3.6)$$

where $a^+_{k,\sigma}(a_{k,\sigma})$, $b^+_q(b_q)$ are the electron and phonon creation (annihilation) operators, ε_k, ω_q are the electron and phonon spectra, g is the electron-phonon coupling constant, and N is the number of atoms in a chain of length L ($N = L/a$). The functional model of the Peierls dielectric is derived from the Hamiltonian (3.6) by averaging over $|k| \sim k_F$ and linearizing the electron spectrum. A rigorous derivation of this model, that includes effects due to the commensurability of k_F and the Brillouin momentum π/a, is given, for example, in Refs. 42 and 56. Without concentrating our attention on the specificity of solid-state aspects of the derivation of the model, we shall now extend the results of the preceding section to the general case of arbitrary filling of the conduction band.

In our case, the Peierls dielectric is characterized by the complex order parameter $\Delta = |\Delta|e^{i\varphi} \sim N^{-1/2}\langle b_Q\rangle$, whose modulus is the gap in the conduction-electron spectrum at the Fermi level and whose phase φ describes large-scale variations in the chemical potential. The resulting modulation of the charge is called the Fröhlich charge-density wave (CDW).

The interaction between electrons and the complex order parameter results, as in the example considered earlier, in the appearance of the mass of Dirac electrons. In the field-theoretic approach, we can readily write down the general form of the terms describing the interaction between Dirac electrons ψ and the order parameter Δ that is consistent with symmetry considerations:

$$\mathcal{L}_{\psi\Delta} = -\Delta_1\bar{\psi}_\sigma\psi_\sigma - i\Delta_2\bar{\psi}_\sigma\gamma_5\psi_\sigma, \quad (3.7)$$

where $\Delta_1 = |\Delta|\cos\varphi$ and $\Delta_2 = |\Delta|\sin\varphi$ are the real and imaginary parts of the order parameter, respectively. The same result follows when the functional model is derived from the Fröhlich Hamiltonian.

The complete Lagrangian for the functional model of the Peierls dielectric is

$$\mathcal{L} = \frac{|\dot{\Delta}|^2}{g^2\bar{\omega}^2} - \frac{|\Delta|^2}{g^2} + \bar{\psi}_\sigma(i\gamma_\mu\partial_\mu - \Delta_1 - i\gamma_5\Delta_2)\psi_\sigma. \quad (3.8)$$

In reality, (3.8) describes the dynamics of the Peierls dielectric with the so-called incommensurable charge-density wave, for which the degree to which the band is filled cannot be represented by a ratio of integers. Commensurability effects will be taken into account below but, for the moment, let us compare (3.8) with known models of one-dimensional quantum field theory.

Without the first term (kinetic energy of the lattice), the Lagrangian (3.8) is identical with the quasiclassical Lagrangian $U(1) \otimes U(1)$ of the chiral-invariant Gross-Neveu model.[43] The authors of Ref. 43 have examined the mechanism of dynamic generation of mass in relativistic models with four-fermion interaction.

In the $U(1)\otimes U(1)$ chiral-invariant one-dimensional model

$$\mathcal{L}_{GN}=i\bar{\psi}_\sigma\gamma_\mu\partial_\mu\psi_\sigma+\frac{g^2}{2}[(\bar{\psi}_\sigma\psi_\sigma)^2-(\bar{\psi}_\sigma\gamma_5\psi_\sigma)^2] \tag{3.9}$$

($\sigma=1,\ldots,N_m$), if we substitute

$$\Delta_1=-g^2\bar{\psi}_\sigma\psi_\sigma,\quad \Delta_2=ig^2\bar{\psi}_\sigma\gamma_5\psi_\sigma \tag{3.10}$$

the Lagrangian (3.9) becomes formally identical with (3.8) (without the first term). Hence, it follows that, if the $\Delta_{1,2}$ in (3.10) are weakly-fluctuating fields, the four-fermion interaction effectively reduces to the model describing the dynamics of free fermions in the self-consistent complex scalar field. For the Gross-Neveu model, the condition that the fluctuations of Δ are small is satisfied for $N_m\gg 1$. For the Peierls dielectric (3.8), the classical behavior of the field Δ is assured not by the multicomponent nature of the electron wave function but by the existence of a slow subsystem, i.e., the lattice. In the model defined by (3.8), there are two internal time scales, namely, the electron scale $t_e\sim\Delta_0^{-1}$ (Δ_0 is the equilibrium gap) and the small phonon scale $t_L\sim(g\bar{\omega})^{-1}$. Their ratio $\alpha=t_e/t_L$ is always small in the system we are considering (for polyacetylene, $\alpha\sim0.1$), so that the adiabatic approximation is valid and electronic processes are investigated against the background of a nonfluctuating lattice.[56]

The model (3.5) is a special case of (3.8) for a real order parameter ($\Delta_2=0$) and describes a one-dimensional Peierls dielectric with twice the period of the trans-polyacetylene-type lattice. When the field Δ can be considered to be classical [the last term in (3.5) is omitted], the Lagrangian (3.5) becomes identical with the quasiclassical Lagrangian of the Gross-Neveu model.[48] The remarkable property of this model is that it contains solitons.[44] As far as the Peierls dielectric is concerned, we are interested only in static soliton solutions of the Gross-Neveu model because the evolution of the order parameter Δ in time is determined by the kinetic lattice components in (3.5).

The quasiclassical approximation is used in Ref. 44 to show that there are two types of static soliton solutions, namely, topological solitons (kinks) of the order parameter

$$\Delta(x)=\Delta_0\,\mathrm{th}\,\Delta_0x,\quad E_s=N_m\frac{\Delta_0}{\pi} \tag{3.11}$$

(E_S) is the soliton energy), where, on each kink, there can be localized $n_0=0,1,\ldots,N_m$ spinless fermions with energy $\omega_F=0$; and bound kink-antikink pairs (polarons)

$$\Delta=\Delta_0\Big[1+\gamma\,\mathrm{th}\left(\Delta_0\gamma x-\frac{1}{4}\ln\frac{1+\gamma}{1-\gamma}\right)$$
$$-\gamma\,\mathrm{th}\left(\Delta_0\gamma x+\frac{1}{4}\ln\frac{1+\gamma}{1-\gamma}\right)\Big], \tag{3.12}$$

$$E_p=\frac{2}{\pi}N_m\gamma\Delta_0,\quad \gamma=\sin\left(\frac{n_0}{N_m}\cdot\frac{\pi}{2}\right); \tag{3.13}$$

where E_p is the polaron energy and $n_0=1,2,\ldots<N_m$ is the number of fermions localized on the polaron. For a Peierls dielectric such as trans-$(CH)_x$, the index σ corresponds exclusively to spin degeneracy, and $N_m=2$. The phenomenological model of a lightly doped trans-polyacetylene, that is convenient for the description of soliton-type excitation, is proposed in Refs. 37 and 45.

3.3. Chiral anomaly and charge-density wave

Let us rewrite the Lagrangian (2.8) in terms of the modulus $|\Delta|$ and phase φ of the order parameter, and let us introduce the electromagnetic field $\partial_\mu\to D_\mu=\partial_\mu-ieA_\mu$:

$$\mathcal{L}=\frac{|\dot{\Delta}|^2}{g^2\bar{\omega}^2}-\frac{|\Delta|^2}{g^2}+\frac{|\Delta|^2\dot{\varphi}^2}{g^2\bar{\omega}^2}+\bar{\psi}_\sigma i\gamma_\mu D_\mu\psi_\sigma$$
$$-|\Delta|\bar{\psi}_\sigma\exp(i\gamma_5\varphi)\psi_\sigma-\frac{1}{4}F_{\mu\nu}^2. \tag{3.14}$$

The electronic sector of the model (3.14) is analogous to that examined in Section 2.4, and we shall therefore use the results obtained previously to derive the effective Lagrangian. Integrating (3.14) with respect to the fermion field (see Sec. 2.4), we find that

$$\mathcal{L}_{eff}=-\frac{1}{4}F_{\mu\nu}^2+\mathcal{L}_{eff}^{(\Delta)}+\mathcal{L}_{eff}^{(\varphi)}, \tag{3.15}$$

$$\int\mathcal{L}_{eff}^{(\Delta)}d^2x=\int d^2x\left(\frac{|\dot{\Delta}|^2}{g^2\bar{\omega}^2}-\frac{|\Delta|^2}{g^2}\right)$$
$$-i\ln\mathrm{Det}(i\gamma_\mu D_\mu-|\Delta|), \tag{3.16}$$

$$\mathcal{L}_{eff}^{(\varphi)}=|\Delta|^2\frac{\dot{\varphi}^2}{g^2\bar{\omega}^2}-\frac{1}{4\pi}(\varphi')^2-\frac{e}{\pi}E\varphi, \tag{3.17}$$

where E is the electric field. In deriving (3.17), the phase φ was assumed to be a slightly nonuniform field $|\varphi'|\ll\Delta_0$ and time derivatives of the phase "of electronic origin" were omitted because the analogous lattice terms were greater by the factor $(\Delta_0/\bar{\omega})^2\gg1$. The real part of $\mathcal{L}_{eff}^{(\Delta)}$ is determined by the polarization of the vacuum by the external electric field, whereas the imaginary part is the rate of creation of electron-hole pairs via the tunnel effect (see, for example, Ref. 131). It is clear that the latter process becomes significant only in fields $E_c\sim\Delta_0^2/e$, so that, when $E\ll E_c$, we shall assume that the modulus of the order parameter is homogeneous. For a constant external field, the problem of polarization of the ground state of the Peierls dielectric can be solved exactly.[46] When $E\ll E_c$, we have $\mathrm{Re}\,\mathcal{L}_{eff}(\Delta_0,E)\sim E^2$ and the first two terms in (3.15) reduce to the standard expression for the energy of the electric field in the medium:

$$\mathcal{L}_E=\frac{\varepsilon_\Delta E^2}{2},\quad \varepsilon_\Delta=1+\frac{1}{6}\left(\frac{\omega_p}{\Delta_0}\right)^2; \tag{3.18}$$

where ε_Δ is the permittivity of the Peierls dielectric due to virtual electron-hole transitions,[55] and $\omega_p^2=8e^2n_f$ (n_f is the two-dimensional density of chains in the sample).

The expression given by (3.17) is the Lagrangian of the incommensurate charge-density wave in the electric field

$$\mathcal{L}^{(\varphi)}=\frac{\Delta_0^2}{g^2\bar{\omega}^2}[\dot{\varphi}^2-c_0^2(\varphi')^2]-\frac{e}{\pi}E\varphi, \tag{3.19}$$

where $c_0=g\bar{\omega}/2\sqrt{\pi}\Delta_0$. We note that both the phase gradient and the interaction of the phase with the external field appear in this approach because of the chiral anomaly.[66] Comparison with (2.29) shows that this expression contains the additional factor 2 in both terms. This is due to the fact that we have taken into account the spin of the real electron [the isospin σ of the model (3.8)]. In accordance with (2.30) and (3.16), the topological current is given by

$$j_{CDW}^\mu=-\frac{e}{\pi}\varepsilon^{\mu\nu}\partial_\nu\varphi. \tag{3.20}$$

This shows that the charge density in the charge-density wave is related to the phase gradient, and the CDW current is related to the time derivative of the phase:

$$\rho_{\text{CDW}} = -\frac{e}{\pi}\varphi', \quad j_{\text{CDW}} = \frac{e}{\pi}\dot{\varphi}. \tag{3.21}$$

It then follows that the total charge of the charge-density wave is

$$Q_{\text{CDW}} = \int\limits_{-\infty}^{\infty} dx\, \rho_{\text{CDW}} = -\frac{e}{\pi}\Delta\varphi \tag{3.22}$$

and is the "topological characteristic" that depends only on the global phase change $\Delta\varphi = \varphi(\infty) - \varphi(-\infty)$. In the incommensurable charge-density wave, the phase difference is not fixed. The action corresponding to the Lagrangian (3.19) is invariant under a phase change by an arbitrary angle ($\varphi \to \varphi + \alpha$) and, when Coulomb effects can be neglected, the incommensurable charge-density wave is a goldstone under spontaneous $U(1)$-symmetry breaking. The topological fragmentation of charge appears when we take into account the commensurability energy that preserves the degeneracy of the ground state of the Peierls dielectric but only under discrete phase transformations.

3.4. The spin-charge anomaly for solitons in polyacetylene

A very important special case is that of the structural Peierls transition with doubling of the lattice period (twofold commensurability). The microscopic functional model of the Peierls dielectric (3.1)–(3.5) contains one collective degree of freedom, namely, the real order parameter Δ. The phase φ is absent as an independent dynamic variable. The linear polymer trans-polyacetylene [trans-$(CH)_x$] is an example of this.[39]

As already noted, functional models of the Peierls dielectric, such as trans-polyacetylene (3.5), are identical with the $N_m = 2$ model of Gross and Neveu in the adiabatic approximation. Since the energy density in this model has two energy-degenerate minima ($\pm\Delta_0$), topologically stable solitons can exist in this system. These solitons are the domain walls (structure defects) separating phases that are degenerate in energy in the dimerized chain of carbon atoms (Fig. 5).

The explicit form of the soliton solutions is not important for the purposes of the present review. We merely note that, according to (3.11)–(3.13), two types of inhomogeneous quantum objects can exist in the Peierls dielectric with real order parameter [trans-$(CH)_x$, $N_m = 2$]: topologically stable solitons [kinks $\Delta_S(x) \to \pm\Delta_0$ for $x \to \pm\infty$] with energy $E_S = 2\Delta_0/\pi$, and polarons [bound $S\bar{S}$-pairs: $\Delta_p(x) \to \Delta_0$ for $|x| \to \infty$] with energy $E_p = 2\sqrt{2}\Delta_0/\pi$. The polaron carries one localized electron (hole) state $|\omega_p| = \Delta_0/\sqrt{2}$. Since $E_p < \Delta_0$, the free electron is unstable in the conduction band (hole in the valence band) against self-localization with the formation of a polaron[40] in a time $t > \omega^{-1}$. The polaron is therefore always charged ($Q_p + \pm e$) and has a half-integral spin. A pair of polarons is, however, unstable against decay into an $S\bar{S}$ pair.[41]

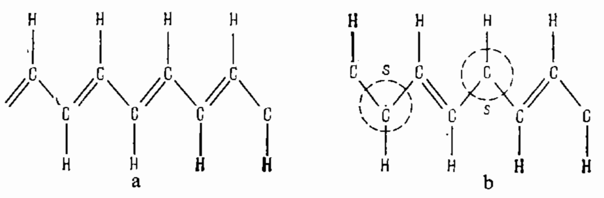

FIG. 5. a—Defect-free chain of the *trans* phase of polyacetylene; the attenuation of single and double bonds results in a structure with a doubled period. b—Chain of *trans* $(CH)_x$, containing a soliton-antisoliton pair ($S\bar{S}$).

Solitons in polyacetylene have unusual quantum numbers. The expression given by (3.22) for the CDW charge is also valid for a model with a real order parameter Δ. Although, in this case, the phase is rigidly fixed ($\varphi = 0, \pi$; mod 2π), nevertheless, for the domain wall separating the energy-degenerate PD vacua, $\Delta = \pm\Delta_0$, the phase difference is $|\Delta\varphi| = \pi$. According to (3.22), the kink $\Delta\varphi = \pi$ must therefore acquire the topological charge $Q_S = -e$ (the antikink $\Delta\varphi = -\pi$ acquires $Q_{\bar{S}} = e$. The spin of the charge solitons is then zero because, in deriving (3.20)–(3.22), we carried out a summation over the spin index σ. As we have seen, the topologically stable soliton produces an electronic state strictly at the center of the forbidden band (zero mode). When the electron spin is taken into account, this level can be empty, singly filled, or doubly filled. For the electrically neutral system, it necessarily follows that the collective (topological) charge must be compensated by the electron charge (hole charge) arriving from the filled band and forming a zero-energy bound state. This compensation over distances $l \gg \xi_0$ ($\xi_0 = \hbar v_F/\Delta_0$ is the coherence length of the Peierls dielectric, which is equal to the characteristic size of the soliton) is possible because of the local relationship between the topological charge density and the external field, given by (3.19) and (3.20). The kink becomes neutral but assumes a spin of 1/2. An additional electron (hole) with opposite spin projection can be inserted into the zero-energy level by doping the transpolyacetylene. The kink then loses its spin but acquires charge. Uncharged topological structure defects are thus seen to have paramagnetic properties in polyacetylene, whereas solitons are diamagnetic.[39]

The following simple observation provides an intuitive explanation of the nontrivial spin-charge coupling. The structural formula for the trans-modification of polyacetylene is shown in Fig. 5a). Dimerization corresponds to the alternation of single and double bonds between the carbon atoms. It is clear that a local change in structure contains at least one soliton-antisoliton ($S\bar{S}$) pair (see Fig. 5b). It is readily seen that the addition of a single bond to the $S\bar{S}$-pair converts this structure into a defect-free chain (A- or B-phase in Fig. 4, depending on the boundary conditions). For spinless electrons, a single bond corresponds to one electron, so that, by virtue of symmetry, each kink must have a half-integral charge. The presence of spin, which doubles the number of electrons corresponding to a single chemical bond, masks the fractional charge but leads to the anomalous spin-charge coupling for solitons in polyacetylene (a detailed review of experimental data confirming the presence of solitons in polyacetylene can be found in Refs. 39 and 42).

For arbitrary isospin $\sigma = 1, 2, \ldots, N_m$ (number of components in the electron function), the topological charge of solitons can be readily found by generalizing (3.14), (3.17), and (3.22):

$$Q_S = -N_m \frac{e}{2\pi}\Delta\varphi \Rightarrow -N_m \frac{e}{2}. \tag{3.23}$$

The case $N_m = 1$ corresponds to effectively spinless electrons[54] and $N_m = 4$ can occur in carbyne chains.[53]

3.5. Solitons in a commensurable density wave

So far, we have investigated the Peierls dielectric by reducing it to a one-dimensional quantum field theory model. The specificity of the solid-state aspect of the problem was reflected only in the choice of a particular form for the kinetic energy associated with the order parameter: $\dot{\Delta}^2/g^2\bar{\omega}^2$. Below, we shall take into account a further two factors that are significant in condensed media.

Real quasi-one-dimensional conductors are periodic structures. The charge-density wave in the functional model (3.20), in which the periodicity of the lattice has not been taken into account, is a goldstone. This approach can be justified if the period of the CDW is commensurable with the lattice period, and other effects giving rise to pinning (chaining) of the CDW are negligible. The propagation of the CDW is then nonactivational. In practice, pinning always occurs in the case of charge-density waves, and small oscillations in the phase of the order parameter have a nonzero threshold frequency.

In many quasi-one-dimensional conductors, the basic reason for pinning is the correlation between the CDW period and the lattice period. When the conduction band is full, which is indicated by the ratio of mutually simple integers $\nu = N/M$ ($N<M$, $M>2$ is the commensurability index $M=\pi N/k_{\rm F}a$ and a is the lattice period, the CDW Lagrangian acquires an additional term, the commensurability energy[55]

$$\mathcal{L}_{\rm com}=\frac{\lambda}{g^2}\,\Delta^2\left(\frac{\Delta}{\varepsilon_{\rm F}}\right)^{M-2}\cos M\varphi,\quad \lambda\sim 1\ , \tag{3.24}$$

where $\varepsilon_{\rm F}$ is the Fermi energy. When this is taken into account, the Lagrangian for the commensurable charge-density wave becomes identical with the Lagrangian for the sine-Gordon model. Although the ratio $\Delta/\varepsilon_{\rm F}$ in (3.24) is small, the inclusion of commensurability pinning has a number of important consequences. First, small oscillations in phase acquire a mass given by $\omega_0 = \lambda^{1/2}(\Delta/\varepsilon_{\rm F})^{(M/2)-1}M\bar{\omega}$ and, second, solitons with fractional charge appear in the system.

Actually, the correlation energy (3.24) lifts the degeneracy of the PD Lagrangian under continuous transformations of phase, and our model has only a discrete set of vacua $\varphi_K = K\cdot 2\pi/M$. The inhomogeneous solutions that "couple" the vacua φ_K and $\varphi_{K\pm1}$ are topologically stable solitons (phase solitons) with fractional charge[57]

$$Q_{\rm S}=-eN_{\rm m}\,\frac{\Delta\varphi}{2\pi}\overset{N_{\rm m}=2}{\Longrightarrow}\pm\frac{2e}{M}. \tag{3.25}$$

The next modification of the CDW Lagrangian involves the inclusion of three-dimensional effects.[68,69] These can give rise to lateral stiffness in the system, i.e., there is a region of size $\xi_\perp \gg n_{\rm f}^{-1/2}$, in which $\varphi(x_\perp)=$ const (Ref. 42) (the lateral coherence of the CDW has been confirmed experimentally[61,62a,64]).

For coherent phase fluctuations in a cluster containing a large number of chains, $N_K\sim\xi_\perp^2 n_{\rm f}$, it is important to take into account their Coulomb interaction. Actually, the electrostatic interaction between collective charges in a cluster of size $\xi_\perp \gg n_{\rm f}^{-1/2}$ is effectively one-dimensional and is therefore significant for the development of CDW electrodynamics. The solution of the self-consistent problem leads to the following structure of the CDW Lagrangian in a constant external electric field E[65]:

$$\mathcal{L}(\varphi)=N_0\Big[\frac{1}{2}\dot{\varphi}^2-\frac{c_0^2}{2}(\varphi')^2-\frac{1}{2}\omega_\varphi^2\left(\varphi+\frac{E}{e^*}\right)^2+\frac{\omega_0^2}{M^2}\cos(M\varphi)\Big], \tag{3.26}$$

where $e^* = en_{\rm f}/\pi\varepsilon_\parallel$ ($\varepsilon_\parallel$ is the static permittivity) and

$$N_0=\frac{2N_k}{g^2}\frac{\Delta_0^2}{\bar{\omega}^2},\quad c_0=\frac{g\bar{\omega}}{2\sqrt{\pi}\,\Delta_0},\quad \omega_\varphi^2=ee^*\frac{g^2\bar{\omega}^2}{\Delta_0^2},$$
$$\omega_0^2=\lambda\left(\frac{\Delta_0}{\varepsilon_{\rm F}}\right)^{M-2}M^2\bar{\varphi}^2. \tag{3.27}$$

It is readily seen that inclusion of the electrostatic interaction of coherent CDW fluctuations has led to the appearance of the plasma frequency ω_φ in the phase mode. Substituting the permittivity of the Peierls dielectric $\varepsilon_\parallel = \varepsilon_\Delta \gg 1$ (3.18) into the definition of ω_φ, we have[60] $\omega_\varphi^2 = 3g^2\bar{\omega}^2/\pi$.

We note that there are a number of approaches in the literature that describe the CDW dynamics in a field (see, for example, Refs. 60, 61, and 67). We draw particular attention to the model (3.26) because the corresponding Lagrangian has parallels in quantum field theory (see below). In all probability, the CDW description using (3.26) is valid at low temperatures and when the effect of impurities is small.

It is interesting that a slight modification of (3.26) that does not alter the structure of the model leads to a graphic picture[65] that explains the nonlinear electrical conductivity of the CDW described by the empirical formula[61]

$$\sigma(E)=\sigma_0+\sigma_1\left(1-\frac{E_{\rm T}}{E}\right)\exp\left(-\frac{E_0}{E-E_{\rm T}}\right),\quad E>E_{\rm T}; \tag{3.28}$$

where $\sigma_{0,1}$ are the conductivity coefficients and $E_{\rm T}$ is the threshold field at which the non-ohmic contribution to $\sigma(E)$ appears.

The nonlinear conductivity (3.28) used in this approach is due to the instability of the ground state of the model (3.26) in an external field, and is intimately related to the idea of confinement of fractionally-charged phase solitons.

3.6. Soliton confinement

The Lagrangian (3.26) resembles the boson form of the Lagrangian in the massive Schwinger model (MSM). This model describes the one-dimensional quantum electrodynamics (see, for example, Refs. 7–73):

$$\mathcal{L}^{\rm F}=\bar{\psi}\{i\gamma_\mu(\partial_\mu-ieA_\mu)-m\}\psi-\frac{1}{4}F_{\mu\nu}^2. \tag{3.29}$$

where ψ represents the spinless Dirac fermions of mass m, $\bar{\psi}\equiv\psi^+\gamma_0$, $F_{\mu\nu}=\partial_\mu A_\partial-\partial_\nu A_\mu$, and A_μ is a vector field.

In (1 + 1)-dimensional space-time, the electromagnetic interaction reduces to the Coulomb interaction between the charges and to the interaction between the charges and the external field E. In gauge theories, the field A_0 is not an independent dynamic variable (the corresponding generalized momentum $\pi_\mu = F_{0\mu}$ is zero, $\pi_0\equiv0$) and, in the Hamiltonian formalism, it must be eliminated from the description using the coupling equations. In the $A_1=0$ gauge, the MSM Hamiltonian and the coupling equations have the form

$$H^{\rm F}=\bar{\psi}i\gamma_1\partial_x\psi+m\bar{\psi}\psi+\frac{1}{2}F_{01}^2,\quad \partial_x^2A_0=-ej_0\ , \tag{3.30}$$

where $j_\mu = \bar{\psi}\gamma_\mu\psi$. The specificity of one-dimensional electrodynamics (the presence of only the Coulomb interaction between the charges) enables us to apply to (3.29) the fermion-boson equivalence formulas (2.11)–(2.14) in the case of free fermions, $\beta^2 = 4\pi$. When this transformation has been made, the MSM Lagrangian assumes the boson form[72]

$$\mathcal{L}^B = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{e^2}{2\pi}\left(\sigma + \frac{\theta}{2\sqrt{\pi}}\right)^2 + \mu m \cos(2\sqrt{\pi}\,\sigma). \tag{3.31}$$

The parameter $\theta (0 \leqslant \theta \leqslant 2\pi)$ formally appears in (3.31) as an integration constant during the bosonization of the coupling equation (3.30)

$$F_{01}^2 = (\partial_x A_0)^2 = e^2\left(\partial_x^{-1} j_0 + \frac{\theta}{2\pi}\right)^2 \Rightarrow \frac{e^2}{\pi}\left(\sigma + \frac{\theta}{2\sqrt{\pi}}\right)^2 \tag{3.32}$$

and has a simple physical interpretation. It is proportional to the strength E of the uniform electric field characterizing the vacuum of one-dimensional quantum electrodynamics ($\theta = E\,2\pi/e$). Actually, in one-dimensional space, the Coulomb interaction force between charges does not depend on the separation between them, so that the existence of the vacuum electric field with amplitude $|E| \leqslant e/2$ is not inconsistent with the equilibrium condition for the system (there is no creation of charge pairs).

According to (3.31), massless ($m = 0$) one-dimensional electrodynamics is equivalent to the theory of the free neutral scalar field with mass[70] $m_\sigma = e/\sqrt{\pi}$. Since neutral particles do not interact with a uniform electric field, the massless Schwinger model does not depend on the angle θ ($\mathcal{L}_\theta = \mathcal{L}_{\theta=0}$ when $\sigma \to \sigma - \theta/2\sqrt{\pi}$). The question is: what is the effect of introducing the mass?

It is readily seen that, under strong coupling conditions ($m_\sigma \gg m$), and whatever the dependence on θ, the MSM potential (3.31)

$$V(\sigma) = \frac{1}{2} m_\sigma^2 \left(\sigma + \frac{\theta}{2\sqrt{\pi}}\right)^2 - m\mu \cos(2\sqrt{\pi}\,\sigma) \tag{3.33}$$

has a unique vacuum, so that the qualitative predictions of the model are, in this case, the same as the predictions of the massless Schwinger model. In particular, the spectrum does not contain fermions which, according to the Coleman fermion-boson equivalence, correspond to topological solitons of the boson Lagrangian (3.31). Under weak coupling conditions ($m_\sigma \ll m$), the potential $V(\sigma)$ has two local minima $\sigma \simeq 0$ and $\sigma \simeq \sqrt{\pi}$, one of which is metastable for $|\theta| \neq \pi$. This potential does not forbid the existence of solitons bound (for $|\theta| \neq \pi$) by a linearly increasing potential into soliton-antisoliton pairs (confinement phase). This phase is conveniently described in terms of fermion fields forming the fermion-antifermion pairs due to the Coulomb potential that increases linearly with distance (one-dimensional Coulomb potential).

The basic difference between the Lagrangian of the coherent CDW (3.26) and the MSM Lagrangian (3.31) is the arbitrary coefficient (not equal to $2\sqrt{\pi}$) in the argument of the cosine in (3.26). This means [see (2.13)–(2.14)] that there is an interaction (in addition to the Coulomb interaction) of the soliton-antisoliton pairs that results in the formation of bions. In a bion, the interaction between solitons is exponentially small at distances $x \gg d$, where d is the size of the soliton.

The decisive assumption in the derivation of (3.26) was that there was inter-chain phase coherence, which enabled us to reduce the static interaction between effective charges for distances $x \gg d_\parallel \gg \xi_\perp$ to the one-dimensional interaction ($d_\parallel$ is the characteristic size of the phase perturbation along a chain). It is precisely for this type of phase fluctuation that we can justify the appearance of the term proportional to φ^2 in the CDW Lagrangian (3.26). In this case, the exponentially small "bion potential" can be neglected, and the electrodynamics of coherent CDW in weak fields is equivalent to one-dimensional electrodynamics of spinless massive fermions.

In accordance with the description of the MSM given above, for $\omega_0 \gg \omega_\varphi$, the solitons of coherent CDW in (3.26) in fields

$$E < E_T = \frac{\pi e^*}{M} = \frac{e}{M}\frac{n_f}{\varepsilon_\parallel} \tag{3.34}$$

are in the confinement phase. It is readily seen that the deconfinement field E_T is identical with the field due to a uniformly charged plane, namely, the "condenser" plate formed by coherent solitons (kink-antikink pair) with mean charge density $n_f Q_S$:

$$E_T = \frac{1}{2}\frac{n_f Q_S}{\varepsilon_\parallel} = \frac{e}{M}\frac{n_f}{\varepsilon_\parallel}. \tag{3.35}$$

We have given a possible mechanism for the confinement of CDW solitons that depends on the electrostatic interaction between phase solitons that are strictly correlated on different chains. However, even if we abandon the assumption of the cluster nature of the motion of the CDW, the solitons on a particular chain will be in the confinement phase,[35] characterized by the absence of free solitons and the fact that the energy of interaction of an $S\bar{S}$-pair increases with increasing separation between solitons.

3.7. Tunneling as a means of creating soliton-antisoliton pairs and nonlinear conductivity

We have noted in Section 3.5 that the model defined by (3.26) was based on the assumption that there were no free electrons. Topological solitons carrying an electric charge can then contribute to CDW conductivity. When the dielectrization of the Fermi surface is not complete (there are impurity levels or temperatures sufficient for appreciable concentrations of excitations lying above the gap), the soliton charge is screened.[40] We note, however, that, in real one-dimensional chains at low temperatures, the mobility of free electrons is suppressed by localization effects. When free electrons are subject to the strong localization effect, they produce high static permittivity[132] $\varepsilon(0)$, but do not lead to the Debye screening of soliton charges. Phenomenological allowance for free carriers then reduces to the replacement of $\varepsilon_\parallel$ with[65] $\varepsilon(0)$. This two-fluid model (Peierls subsystem plus free electrons in the localization regime) can be used to ensure agreement between theoretical predictions and experimental data[61] without significant modification of our ideas on collective excitations in the Peierls dielectric.

We shall therefore assume that the topological solitons in the charge density wave have an electric charge given by (3.25). Since small oscillations in the phase of the order parameter for arbitrarily small pinning ($\omega_0 \neq 0$) do not con-

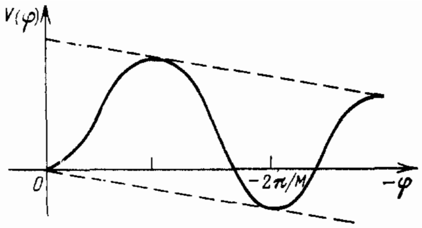

FIG. 6.

tribute to the dc conductivity [$\sigma(\omega)\to 0$ for $\omega\to 0$ (Ref. 55)], the nonlinear static conductivity of the Peierls-Fröhlich system is due to CDW solitons and is proportional to the density of the $S\bar{S}$-pairs. At low temperatures, $T\ll\omega_0$, the principal mechanism for the formation of the $S\bar{S}$-pairs is their creation by the electric field as a result of the tunnel effect.

The rate of creation of $S\bar{S}$-pairs by a constant electric field in the Peierls-Fröhlich system was first calculated by Maki[58] without taking the confinement effect into account (see also Refs. 5 and 62b), who used the then traditional instanton procedure. In complete agreement with analogous calculations performed for one-dimensional models of quantum field theory,[75–78] the $S\bar{S}$-pair production process was found to have no threshold.

Confinement effects lead to the appearance of a threshold field E_T, at which the formation of soliton-antisoliton pairs becomes energetically more convenient. Let us calculate the rate of formation of $S\bar{S}$-pairs near the threshold in this case, using the model (3.26) and the fermionization procedure.[76]

The potential for the model (3.26) is

$$V(\varphi)=N_0\left[\frac{1}{2}\omega_\varphi^2\left(\varphi+\frac{E}{e^*}\right)^2-\frac{\omega_0^2}{M^2}\cos(M\varphi)\right] \tag{3.36}$$

and, for a realistic ratio of the activations of the phase mode, i.e., $\omega\ll\omega_0$ [$\varepsilon_\parallel\sim\varepsilon(0)\sim 10^7$ for trichalcogenides], it has an absolute minimum $\varphi=0$ only in fields $E<E_T=en_f/M\varepsilon(0)$ (3.34)–(3.35). It is readily verified that, when $E>E_T$, the state $\varphi=-2\pi/M$ corresponds to a lower energy. The tunneling decay $|0\rangle\to|-2\pi/M\rangle$ is accompanied by the creation of an $S\bar{S}$-pair (Fig. 6) with charge density n_fQ_S. In weak fields (near the pair-production threshold) or, equivalently, for a weak "warping" of the vacua, i.e., $\Delta V\ll E_S/d$ (E_S is the energy and d the size of the soliton), such that

$$\Delta V=V(0)-V\left(-\frac{2\pi}{M}\right)=2N_K|Q_S|(E-E_T) \tag{3.37}$$

the solitons can be regarded as free and the fermion language is convenient in calculations of the tunneling decay probability. The Lagrangian for the coherent CDW (3.26) is then equivalent to the Lagrangian for free spinless (one-dimensional) Dirac fermions in a uniform external electric field ($qE^*\equiv\Delta V$). The rate of creation of $S\bar{S}$ pairs is therefore identical with the well-known expression for the rate of creation of fermion-antifermion pairs of charge q in a constant field $E^*(qE^*=\Delta V)$:

$$I_{S\bar{S}}=\frac{\Delta V}{2\pi\hbar N_K}\exp\left(-\pi\frac{E_S^2}{\Delta V\cdot c_0\hbar}\right). \tag{3.38}$$

If we express the quantities in (3.38) in terms of the parameters of the problem, we find[65] that the nonlinear CDW conductivity $\hat{\sigma}(E)\sim n_fI_{S\bar{S}}/E$ is given by (3.28). The experiments reported in Ref. 61 have also established that the threshold field satisfies the universal relation $E_T\varepsilon(0)=\text{const}$, which is readily explained by the above model of electrostatic confinement [see (3.34) and (3.35)]. Using the same assumptions as when we took into account the contribution of nondielectrized electrons to the permittivity $\varepsilon(0)$, we can qualitatively explain the temperature and concentration dependence of the threshold field.[65]

4. TWO-DIMENSIONAL FIELD THEORY MODELS AND THE QUANTUM HALL EFFECT

So far, we have investigated quantum-field theory models in (1 + 1)-dimensional space-time and have related their predictions to experimentally established phenomena in the physics of quasi-one-dimensional compounds whenever the physical situation could be reduced to an effectively one-dimensional one. In this Section, we examine anomalies in gauge models of two-dimensional quantum-field theory and discuss their connection with the quantization of the Hall conductivity of two-dimensional electrons in a strong magnetic field (quantum Hall effect[90]).

We recall that, in odd-dimensional spaces (space + time), we cannot construct matrices (γ_{D+1}) that anticommute with all the Dirac matrices γ_μ. This means that there are no chiral transformations in such spaces. However, in odd-dimensional space-time, there are in gauge theories anomalous terms in the effective action that are of "topological origin." These terms are commonly referred to as topological action (or the Chern-Simons action).

Studies of two-dimensional quantum electrodynamics (2D-QED) are particularly intersting for our purposes here. Therefore, without pausing to consider the hierarchy of anomalies in gauge theories and their nontrivial mathematical structure (connection with characteristic classes in differential geometry and cocycles in cohomology[80–83]), we shall use simple physical considerations to write out the anomalous extra term in the 2D-QED action.

4.1. Topological action in two-dimensional quantum electrodynamics

Consider Dirac fermions ψ in (2 + 1)-dimensional space in an external electromagnetic field A_μ:

$$\mathcal{L}=\bar{\psi}\{i\gamma_\mu(\partial_\mu-ieA_\mu)-m\}\psi; \tag{4.1}$$

where m is the fermion mass, γ_μ are the Dirac matrices that in this case ($\mu=0,1,2$) reduce to the three Pauli matrices σ. The external field will be considered to be the magnetic field $B=\varepsilon^{ij}F_{ij}/2$ ($i,j=1,2$) and, following the scheme expounded in Section 2.5, we shall determine the "charge" of the vacuum of two-dimensional fermions, induced by the external gauge field. According to (2.31)–(2.34), this charge is due to the asymmetry of the spectrum of the Dirac equation:

$$Q_F=-\frac{1}{2}\sum_{\{k\}}\operatorname{sgn}\omega_{\{k\}}, \tag{4.2}$$

where $\{k\}$ is the set of quantum numbers. In this particular case, the required equation is

$$H_D\psi=\omega\psi,\quad H_D=\boldsymbol{\alpha}(\mathbf{p}-e\mathbf{A})+\beta m, \tag{4.3}$$

where $\boldsymbol{\alpha}_{1,2}$, $\boldsymbol{\beta}$ is the set of Pauli matrices, and it is readily shown that all the eigenvalues of (4.3) with $|\omega|>m$ are symmetric relative to zero and only the states $\omega_- = -m$ (or $\omega_+ = +m$) produce an asymmetry of the spectrum. Actually, since the operators H_D and $[H_D, \beta]$ anticommute

$$\{H_D, [H_D, \beta]\} = [H_D^2, \beta] = 0, \tag{4.4}$$

the eigenfunctions ψ_ω and $\psi_{-\omega} = [H_D, \beta]\psi_\omega$ have energies that are equal in magnitude and opposite in sign. Accordingly, their contribution to (4.2) is zero. Next, the Hamiltonian of the 2D-electrons (4.3) has the following matrix structure:

$$H_D = \begin{pmatrix} m & D^- \\ D^+ & -m \end{pmatrix}, \quad D^\pm = D_1 \pm iD_2, \quad D_j = p_j - eA_j, \tag{4.5}$$

and the operators $D^\pm$ do not commute in the magnetic field B, i.e., $[D^-, D^+] = -2eB(x,y)$. The "zero modes" ψ_m ($[H_D, \beta]\psi_m = 0$) satisfy the equation

$$H_D\psi_\pm = \omega_\pm\psi_\pm, \quad \psi_\pm = \frac{1}{2}(1 \pm \beta)\psi_m. \tag{4.6}$$

It is readily verfied that, because the operators $D^\pm$ do not commute, only one of the two equations $D^\pm \psi_\mp = 0$ can be satisfied (depending on the sign of eB). According to (4.5) and (4.6), this means that the zero modes are always asymmetric, i.e., $eB > 0$, $\omega_+ = +m$ (the state $\omega_- = -m$ is not present in the spectrum) and vice versa. The spectrum of (4.3) is therefore asymmetric, and the external magnetic field induces a nonzero "charge" [86] (4.2) in the vacuum.

To evaluate (4.2), we must know the degree of degeneracy of the zero modes. This can be done by transforming from H_D to the "Hamiltonian"

$$H_S = H_D^2 - m^2 = Q_1^2, \quad Q_1 = \alpha \mathbf{D}. \tag{4.7}$$

The operator H_S is the Hamiltonian of supersymmetric quantum mechanics with supercharges Q_1 and $Q_2 = i\beta Q_1$; $\{Q_1, Q_2\} = 0$ (see Ref. 32 for further details). The general properties of supersymmetric theories then lead to the following: (1) all states of H_S with energy $E_S^2 = \omega^2 - m^2 > 0$ are doubly degenerate even in a nonuniform magnetic field $B(x,y)$ (for the nonrelativistic problem, this is the well-known degeneracy in the spin of the electron) and (2) the zero modes $E_S = 0$ ($\omega_+ = +m$ or $\omega_- = -m$) do not have superpartners (they are spin-polarized). Their degree of degeneracy is a topological invariant that depends only on the magnetic-field flux[84] $N_t = |\Phi|/\Phi_0$ ($\Phi_0 = 2\pi/e$ is the quantum of the magnetic flux). For a magnetic field with infinite flux, the degree of degeneracy is infinite, and the density of states is

$$\tilde{\nu} = \frac{|\Phi|}{\Phi_0 \int dS}, \tag{4.8}$$

where dS is an element of area and $\tilde{\nu} = |eB|/2\pi$ for $B = \text{const}$. In view of (4.8), we can readily show that the density of the "charge" induced in the vacuum is given by

$$J_0 = -\frac{e}{2}\tilde{\nu}\,\text{sgn}(eBm) = -\text{sgn}\, m \cdot \frac{e^2}{8\pi}\varepsilon^{ij}F_{ij}. \tag{4.9}$$

The question is: why bring in the induced charge even though we are considering the reaction of the vacuum to an external magnetic field? We shall show below that, when the sign of the mass is fixed, two-dimensional Dirac fermions can assume only one (out of the two possible) spin value. States with positive and negative energies have opposite spins. A magnetic field will polarize the spins and give rise to partial "flipover." However, in the two-dimensional case, the sign of the spin is strictly correlated with the sign of the energy, so that spin polarization automatically involves an asymmetric spectrum and the magnetic-field induced charge appears in the vacuum.

The relativistically covariant generalization of (4.9) leads to the following simple expression for the vacuum currents:[86-88]

$$J^\mu = -\text{sgn}\, m \cdot \frac{e^2}{8\pi}\varepsilon^{\mu\rho\sigma}F_{\rho\sigma}. \tag{4.10}$$

The current J^μ is topological, $\partial_\mu J^\mu \equiv 0$, and gauge invariant. However, the vacuum current is an axial vector and therefore has unusual P-parity.

It is well-known that any nondissipative current can be obtained by varying the action with respect to the potential. The 2D-QED action then acquires the additional topological term

$$S_{CS} = (\text{sgn}\, m)\frac{e^2}{16\pi}\int d^2x\, dt\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} = i\theta_0 \int \omega_3, \tag{4.11}$$

where $\theta_0 = 2\pi e^2$ sgn m and

$$\int \omega_3 \equiv \frac{1}{32\pi^2}\int d^3x\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \tag{4.12}$$

is the $U(1)$-topological invariant of Chern and Simons (see, for example, Refs. 28 and 81) in three-dimensional (Euclidean) space. It is readily verified that the Abelian Chern-Simons action (4.12) is gauge-invariant, so that the constant θ_0 with which (4.12) appears in the effective action of the theory is not quantized and, in general, can depend on the parameters of the medium such as temperature, chemical potential, and so on.[20]

4.2. Topological current in relativistic and nonrelativistic systems of 2D-electrons

Equation (4.10) leads to an unexpected conclusion: in the relativistic theory of two-dimensional electrons, the electric field $\mathbf{E}$ gives rise to a vacuum (nondissipative) Hall-type current (at right-angles to $\mathbf{E}$), given by

$$J^i = (\text{sgn}\, m)\,\sigma\varepsilon^{ij}E_j \tag{4.13}$$

with quantum conductivity $\sigma = e^2/2h$ ($h = 2\pi\hbar$).

The standard expression for the Hall current of real particles is

$$j_i = j_0 v_i^D = j_0 \frac{1}{B}\varepsilon_{ij}E_j; \tag{4.14}$$

where $j_0 = en$ is the charge density, B is the uniform magnetic field, v_i^D is the drift velocity of electrons in the crossed electric and magnetic fields $E < B$, and $V^D = E/B$. Although the formula for the current density given by (4.14) in the form of a product of charge density and charge velocity has the classical nonrelativistic form, the substitution $j_0 = e\tilde{\nu}$ makes it exact for the relativistic single-particle quantum-mechanical problem ($e\tilde{\nu} = e|eB|/2\pi$ is the charge density of a filled Landau level). This can be verified by direct calculation[100] and has a simple explanation. It is clear that the degree of degeneracy $\tilde{\nu}$ of the Landau levels in the uniform magnetic field is the same for both relativistic and nonrelativistic particles. The drift velocity of Landau orbitals is also unaltered. Actually, since the electric and magnetic fields E_y and B are orthogonal in the laboratory frame of coordinates, and $E_y < B$, there is an inertial reference frame[126] (moving with the velocity $v_x = E_y/B$) in which the electric field is zero, so that there is no charged current. Transforming back to the laboratory system, we can readily find the mean current due to the drift of Landau orbitals:

$$j_i^K = \sum_{K=1}^{K} j_i = K\frac{e^2}{h}(\text{sgn}\, eB)\,\varepsilon_{ij}E_j. \quad (4.15)$$

The total current of K filled levels (without taking the spin degeneracy into account) is

$$j_i = (\text{sgn}\, eB)\frac{e^2}{h}\varepsilon_{ij}E_j. \quad (4.16)$$

We recall that, for Pauli electrons with normal magnetic moment in a magnetic field, all the Landau levels (except for the lowest) are twofold degenerate in the spin direction. However, for conduction electrons in condensed media, the effective mass m^* of an electron that appears in the diamagnetic part of the Hamiltonian and the mass m that determines the paramagnetic response of the system, are generally different ($m^* \neq m$). Degeneracy in the spin is thus lifted, all electrons in each energy level are polarized, and the level separation is of the order of $w_c = eB/m^*$.

The Hall current of the first fully occupied Landau levels (4.16) has a structure similar to that of (4.13), which has led a number of authors[97–99] to relate the topological current in 2D-QED to the observed whole-number quantization of Hall conductivity. However, we shall see later that the physical picture underlying (4.10) and (4.13) and the reason for the quantization of the Hall conductivity of real electrons are different, and the connection between 2D-QED and the quantum Hall effect discussed in these papers is purely formal.

We shall now attempt to explain, first, the "physical" reason for the anomalous reaction of the 2D-electron vacuum to the external electromagnetic field. The most surprising fact is that the current vector **J** is perpendicular to the electric field although, in contrast to (4.16), the magnetic field is not explicitly present in (4.13) (instead of sgn(eB), we have sgn m).

As already noted, in three-dimensional space-time, the algebra of the Dirac matrices is isomorphic to the algebra of Pauli matrices. The wave function of relativistic 2D-electrons is therefore a two-component spinor that describes only two degrees of freedom (particle-antiparticle). Kinematics is thus seen to forbid the existence of the spin degree of freedom in the case of the 2D-fermions. On the other hand, the existence of the rotation group in the O(2) plane formally allows the Dirac fermion to have spin $s = \pm 1/2$ (in two-dimensional space, the spin is, of course, a pseudoscalar and not an axial vector). Consequently, there is a kinematic reason that forces the fermion to occupy one of the two possible polarization states. This reason is the mass of the fermion.[1)] Actually, the mass term $m\bar\psi\psi = m\psi^+\beta\psi$ is invariant in the two-dimensional case under P-reflections of the coordinates, $x \to -x$, $y \to y$ (simultaneous reflection of the two coordinates is equivalent to rotation) because $P_\psi(x,y,t)P^{-1} \sim \alpha_2\psi(-x,y,t)$, and only the combined operation of P-reflection and the replacement $m \to -m$ leaves the Lagrangian unaltered[85] (see also Jackiw's lectures[28]). It is therefore natural to introduce the spin matrix for the two-dimensional fermions $\Sigma^{(2)} = (\text{sgn}\, m)\beta/2$ and, since β is also the matrix of the sign of energy, states with positive and negative energy have spin polarizations of opposite sign. This is most clearly seen in the Foldy-Wouthuysen representation in which the free-electron Hamiltonian is $H_D^{F-W} = \beta(\mathbf{p}^2 + m^2)^{1/2}$ and $[\Sigma, H_D^{F-W}] = 0$.

Thus, the Dirac vacuum of 2D-fermions has not only an electric charge, but also nonzero magnetization. Naturally, the total "charge" of vacuum is not physically meaningful, but the part of it that is induced by the external field is, i.e., $\Delta Q = Q\{F_{\mu\nu}\} - Q\{F_{\mu\nu} = 0\}$.

We now turn directly to our problem. Let B be the external magnetic field, so that, depending on the sign of eB, the spectrum of the Dirac equation (4.3) does not contain states with energy $\omega = +m$ or $\omega = -m$. Next, all the vacuum states of 2D-Dirac electrons are completely polarized whether or not the magnetic field is present, but the number of vacuum states with energy $E < -m$ is lower by an amount equal to half the states that have "condensed" on the levels $\omega_- = -m$ or $\omega_+ = m$. Hence, the induced vacuum magnetization, is determined by the zero modes of the operator (4.7), in complete agreement with the quantum-mechanical calculation given in the last Section. If, in addition to the magnetic field, there is also an electric field **E** in the x, y plane, the latter will give rise to the drift of charges filling the Landau levels at right-angles to **E**. The "charge" density induced by the magnetic field is $J_0 = -2\, \text{sgn}\, m \cdot S$ [$S = (\tilde\nu/4)\text{sgn}(eB)$ is the induced spin], so that, substituting J_0 in (4.14), we obtain exactly the same expression as the formula given by (4.13) for the vacuum current of 2D-electrons.

In the standard quantum electrodynamic approach, the anomalous (topological) addition to the effective Heisenberg-Euler Lagrangian of 2D-electrons was first obtained in Ref. 87.

In its physical significance (vacuum polarization), the current (4.13) is specific precisely to relativistic 2D-fermions. This property ensures that the magnetic field B does not appear in the expression for the current density (4.13), and the direction of J for given e is determined by the sign of the mass [and not the sign of eB, as it is in standard formulas for the Hall current (4.15)]. The topological nature of the vacuum current manifests itself also in the fact that (4.13) remains valid even in a nonuniform magnetic field $B(x,y)$ (it is important to ensure that the flux remains constant; see last Section), whereas (4.15) and (4.16) are meaningful only for $B = \text{const}$.

Thus, the topological addition to the 2D-QED action cannot determine the Hall current of real 2D-electrons in solids (see also Ref. 100) and we must seek other physical factors to explain the quantization of Hall conductivity (see next Section).

To conclude this Section, we must make one further remark about the transition from the relativistic to the nonrelativistic description of 2D-fermion systems. It is well-known that, in contrast to relativistic fermions, nonrelativistic two-dimensional fermions can have the two polarization states $s = \pm 1/2$. Since nonrelativistic physics is a limiting case of relativity, is this not a contradiction? The paradox can be resolved as follows. In the $(2+1)$-dimensional space, the sign of mass is an additional new quantum number of Dirac fermions. In nonrelativistic physics, mass can only be positive. Since the sign of mass in nonrelativistic physics is fundamentally unobservable, the transition from the Dirac picture to the Pauli picture must, in addition, involve averaging over the quantum number "sign of mass." It can be readily appreciated that this "mixture" of Dirac electrons (after averaging, the theory becomes P-even) gives the spin degree of freedom to the Pauli 2D-electrons.

4.3. Integral quantum Hall effect

In 1980, von Klitzing *et al.* discovered the following unusual phenomenon. In the system of 2D-electrons that exists in the conducting layer (channel) of MDS structures (metal-dielectric-semiconductor) or heterojunctions placed in a strong magnetic field ($B \sim 100$ KG) perpendicular to the conducting plane, the Hall component of magnetoresistance ρ_{xy} at low temperatures ($T \sim 1$ K) has a piecewise constant form as a function of the magnetic field (or as a function of the charge density in the channel, which is set by the voltage across the MDS structures; see Fig. 7):

$$\rho_{xy}^{\text{exp}} = \frac{1}{K(B)} \frac{h}{e^2}, \tag{4.17}$$

where $K(B)$ is a step-function which assumes integer values on each "step" with a precision that is unusual for solid-state effects ($\Delta K/K \sim 10^{-7}$; see the reviews given in Refs. 94 and 95). In the same range of parameters, the diagonal component of the magnetoresistance ρ_{xx} is practically zero ($\rho_{xx} \lesssim 10^{-10}$) when the quantization of the Hall resistance takes place. The observed picture is stable against changes in temperature, concentration of impurities, and type of structure containing the conducting 2D-electron layer, over a wide range of variation of these parameters. The high precision of the quantization of Hall resistance, and the universality of the observed picture, have led to a search for general laws of quantum physics underlying this effect.

The components σ_{xx}, σ_{xy} of the conductivity matrix and the components ρ_{xx}, ρ_{xy} of the magnetoresistance matrix are related by

$$\sigma_{xy} = -\frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2}, \qquad \sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2}. \tag{4.18}$$

It follows that, if ρ_{xx} vanishes for $\rho_{xy} \neq 0$, this implies the simultaneous vanishing of the longitudinal component of conductivity.

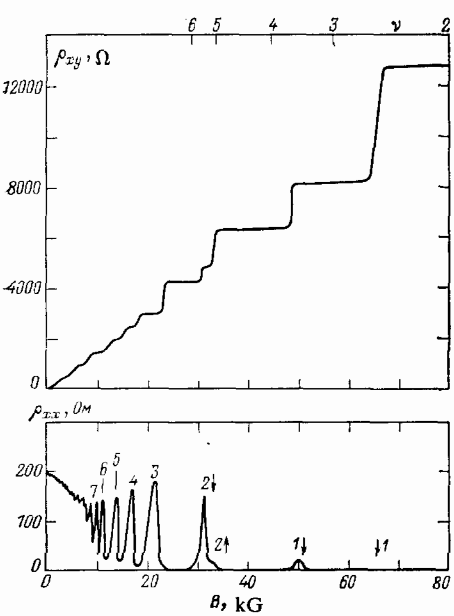

FIG. 7. Experimental curves taken from Ref. 95.

In the ideal gas of 2D-electrons, the vanishing of σ_{xx} and the whole-number values of $\sigma_{xy}/\sigma_{xy}^{(0)}$ (where $\sigma_{xy}^{(0)} \equiv e^2/h$ is the quantum-mechanical unit of conductivity) are achieved only at individual points when the density of electrons (or, for a given density, the strength of the magnetic field) is such that the first K Landau levels are completely filled (4.16). The variation in longitudinal magnetoresistance ρ_{xx} that is periodic in $1/B$ is therefore an expected effect (Shubnikov-de Haas oscillations). The unusual feature of von Klitzing's result, which has given rise to a steady flow of theoretical papers, is that, for real 2D-electrons, these properties are observed with high precision over a finite range of variation of electron density (magnetic field). The reaction of 2D-electrons interacting with one another and with impurities to the external electric field is always the same (excluding very narrow regions, for which $\rho_{xx} \neq 0$), as it is in the ideal Fermi gas with completely filled Landau levels.

A qualitative explanation of this behavior of real 2D-electron systems appears to be as follows. Because of the interaction between conduction electrons and impurities, the infinitely degenerate δ-sharp Landau levels spread out into bands (recent measurements[96] actually indicate an extended Gaussian distribution of the electron density of states). At the edges of the bands, all the states are localized and do not contribute to conductivity. For high impurity concentrations (low electron mobility in the channel of a field transistor, or the conducting layer of a heterojunction), an increase in the density of 2D-electrons is therefore accompanied by a slow displacement of the Fermi level ε_F over the localized states (in the mobility gap) between the two continuous-spectrum regions. In this situation, only the filled regions ($\varepsilon < \varepsilon_F$) of delocalized states (near the centers of the Landau levels) contribute to the Hall conductivity. The fact that it is precisely the existence of localized states that is responsible for the steps on the function $\sigma_{xy}(B)$ is confirmed by comparisons of the width of the steps for specimens of low and high charge mobility. It has been demonstrated experimentally that the step width decreases with increasing mobility.

Of particular importance for this range of questions is the paper by Laughlin,[101] who first gave the general proof (without using a specific form of the interaction Hamiltonian or perturbation theory) of the integral quantization of Hall conductivity, based on gauge invariance. We follow Ref. 101 and consider a "thought experiment" involving the "measurement" of Hall conductivity in the geometry of Fig. 8. Here, the system of 2D-electrons producing a current I in a magnetic field **B** is arranged in the form of a cylinder with

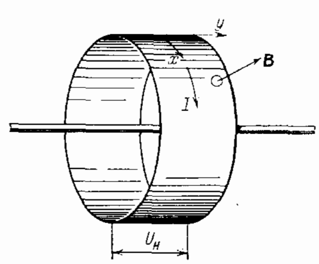

FIG. 8. Laughlin's "thought experiment": I—azimuthal current of electrons, U_H—Hall voltage, **B**—constant magnetic field, perpendicular to the surface.

an infinitely long solenoid running along its axis. Since the magnetic field **B** is perpendicular to the surface of the cylinder, the moving charges experience the Lorentz force which pushes the electrons to one of the ends of the cylinder. A potential difference U_H (Hall voltage) is applied between the ends of the cylinder to compensate this force.

Let Φ be the flux produced by the solenoid, so that the potential A_φ on the surface of the cylinder is $A_\varphi = \Phi/2\pi R$, where R is the radius of the cylinder. The current I can now be found from the general thermodynamic relationship (see, for example, Ref. 129, p. 176) in terms of the change δF in the free energy of the system and the change $\delta\Phi$ in the magnetic flux through the circuit:

$$I_H = \frac{\partial F}{\partial \Phi}. \qquad (4.19)$$

Our problem is to evaluate (4.19) for an increase in the flux for which the initial and final states of the system are identical. A similar device (thermodynamic cycle) usually yields the quantization of the system parameters. It was precisely for this reason that Laughlin chose his original geometry of the experiment with a fictitious solenoid.

When the flux in the solenoid is equal to a whole number of quanta, $\Phi = K\Phi_0$, an infinitely long solenoid is physically unobservable in the external region. Actually, in the gauge-invariant theory, the Hamiltonian for charged particles in an electromagnetic field depends only on the kinematic (gauge-invariant) momenta

$$H = H(\mathbf{p}^{(j)} - e\mathbf{A}^{(j)}), \qquad (4.20)$$

where j is the particle number. The gauge transformation

$$A_\varphi^{(j)} \to A_\varphi^{(j)} - \frac{1}{R}\frac{\partial}{\partial \varphi^{(j)}}\Lambda^{(j)} \qquad \Lambda^{(j)} = \frac{\Phi}{2\pi}\varphi^{(j)} \qquad (4.21)$$

completely removes the potential from the region outside the solenoid. At the same time, the charged-particle wave functions are multiplied by the phase factor

$$\exp\left(ie\sum_j \Lambda^{(j)}\right) = \prod_j \exp\left(i\frac{\Phi}{\Phi_0}\varphi^{(j)}\right). \qquad (4.22)$$

For an arbitrary flux Φ, this gauge transformation is singular, and (4.22) shows that the corresponding wave functions are not single-valued. The requirement of single-valuedness leads to the flux quantization $\Phi = K\Phi_0$, where K ia an integer.

Suppose now that the chemical potential of the system of 2D-electrons lies in the mobility gap (this is equivalent to the requirement of $\sigma_{xx} = 0$). The excited states of the system are then separated by a finite energy gap from the ground state, and cannot be reached as a result of an adiabatic change in the parameters. An adiabatic change in the flux in the solenoid will therefore leave the system in the ground state. When this state is not degenerate, the initial ($|\Phi = 0\rangle$) and final ($|\Phi = \Phi_0\rangle$) states of the many-body Hamiltonian must be identical by virtue of gauge invariance. However, the wave function of the system in the final state has acquired the additional phase factor (4.22). This factor can, nevertheless, be removed by performing a translation by a distance $a = \Phi_0/2\pi RB$ along the axis of the cylinder (we recall that translations in a magnetic field are accompanied by the phase rotation of the wave functions of charged particles; see, for example, Refs. 128, Section 60). By virtue of the translational invariance of the system (on average, in the presence of impurities) and the nondegenerate character of the ground state, the resulting wave function leads to the same average characteristics of the system as $|\Phi = 0\rangle$. Physically, this means that, when there are delocalized electron states, a change in the flux by Φ_0 is accompanied by the transport of a whole number of electrons N_0 from one end of the cylinder to the other (localized electrons do not, of course, participate in this transport). Hence, the change in the free energy is $\Delta F = N_0 e U_H$ and, according to the definition given by (4.19),

$$I_H = \langle I_H\rangle = \frac{\Delta F}{\Phi_0} = N_0\frac{e^2}{h}U_H, \quad \bar{\sigma}_{xy} = N_0\frac{e^2}{h}, \qquad (4.23)$$

where $\langle I_H\rangle$ is the mean Hall current as the flux in the solenoid varies from 0 to Φ_0 or, in other words, it is the average of (4.19) over the flux:

$$\langle I_H\rangle = \Phi_0^{-1}\int_0^{\Phi_0} d\Phi\, \frac{\partial F}{\partial \Phi}.$$

It is only because of the presence of the gap in the excitation spectrum that the actual (I_H) and average ($\langle I_H\rangle$) Hall currents are equal with exponential precision. In other words, the wave function of delocalized electrons contributing to the Hall conductivity is coherent.

Thus, (1) gauge invariance, (2) the existence of the mobility gap, and (3) the nondegenerate character of the ground state reduce the problem of quantization of Hall conductivity to the quantization of the electron charge. Laughlin's qualitative treatment has been generalized,[105,106] and a rigorous theorem on the topological nature of the average Hall conductivity has been established, subject to assumptions (1)–(3).

In the above, essentially thermodynamic, discussion, we have deliberately avoided specifying the form of the Hamiltonian or the integer N_0 because the proof of the quantization of $\bar{\sigma}_{xy}$ is valid under conditions (1)–(3) irrespectively of the form of the many-body Hamiltonian of the 2D-electrons or the type of energy-level classification. In particular, it is valid for the perfect gas of charged particles for the same isolated values of the magnetic field B for which the complete occupation of the Landau levels takes place. Comparison of (4.23) with (4.16) shows that the number N_0 is then equal to the number of filled Landau levels, i.e., in the Laughlin thought experiment, a change in the flux by Φ_0 is accompanied by the transfer of exactly one electron from one edge of the cylinder to the other for each filled Landau level. This conclusion also follows from elementary microscopic analysis (see, for example, Section 3.2.3 in Ref. 134). It is natural to suppose that Landau systematics will not be affected by the presence of weak disorder. However, the important questions that are raised by the role of disorder are outside the framework of our review.[102,104]

4.4. Quasiparticles with fractional charge

Two years after the discovery of the whole-number quantization of Hall conductivity, Tsui *et al.*[92] measured the magnetoresistance of selectively doped heterojunctions GaAs-$Al_xGa_{1-x}As$ with maximum mobility μ for these specimens ($\mu \sim 10^6$ cm^2/V·s) at temperatures $T \lesssim 0.1$ K, and discovered a plateau on the function $\rho_{xy}(B)$ and the simultaneous vanishing of $\rho_{xx}(B)$ for fractional ($\nu = 1/3$) filling of the lowest Landau level. This phenomenon is now called the fractional quantum Hall effect. Subsequent studies of FQHE in heterojunctions (see Ref. 95) and in silicon MDS structures[93] have shown that fractional quantization of the Hall conductivity occurs for rational values of $\nu = q/p$, where p is always an odd number ($\nu^{\exp} = 1/3, 2/3, 4/3, 5/3, 1/5, 2/5, 3/5, 4/5, 6/5, 2/7, 3/7, 4/7, 4/9, 5/9$). As in FQHE, fractional quantization occurs with high precision [for the steps, $\nu = 1/3, 2/3$ and $\Delta\rho_{xy}/\rho_{xy} \sim 3\times 10^{-5}$ (Ref. 95)], but FQHE is observed at lower temperatures and only for specimens with low carrier density and high charge mobility.

Laughlin's formal arguments that result in whole-number quantization of $\bar{\sigma}_{xy}$ are readily generalized to the case of fractional quantization.[111] Since, in experiments performed under the conditions of quantization of transverse conductivity, the longitudinal conductivity is zero, the system must have a mobility gap. Whole-number quantization is a consequence of the additional assumption that the wave function of the ground state is nondegenerate (see last Section). Conversely, the hypothesis that the ground state has a finite degree of degeneracy provides an explanation of fractional quantization in the spirit of Laughlin's thermodynamic arguments.

Actually, in Laughlin's "thought experiment," the adiabatic increase in the solenoid flux per flux quantum Φ_0 led only to the transfer of a whole number of electrons because of the gauge invariance and nondegeneracy of the ground state. In the case of a p-fold degeneracy of the ground state, the flux of the solenoid must be increased by $p\Phi_0$ flux quanta in order to return the system to the initial configuration. Suppose that the net result is that q electrons are transported from one end of the cylinder to the other. According to (4.23), the Hall conductivity then assumes the fractional value $\bar{\sigma}_{xy} = (q/p)(e^2/h)$ (see also Ref. 105b, which cites arguments in favor of the topological character of the quantization of Hall conductivity under FQHE conditions). We shall see below that certain anomalous properties of excitations in FQHE (their anomalous statistics) indicate that odd values of p are more likely. This provides partial justification for the empirical rule demanding an odd denominator.

The first question that arises in the study of FQHE is: what is responsible for the gap in the excitation spectrum? We know that the Fermi energy is equal to the energy of the lowest Landau level in this case, and there is no gap in the excitation spectrum in the single-particle theory of 2D-electrons. Many-body effects, i.e., the Coulomb interaction between the charges, must therefore be taken into account if FQHE is to be explained.

For a constant magnetic field, two physically equivalent ways of describing the electrons are usually employed. In the Landau gauge, $\mathbf{A} = (0, Bx)$, the separation of variables in the Schrödinger equation can be performed in terms of the Cartesian coordinates, and the conserved quantum number with which energy degeneracy is associated is the momentum k along the y-axis. Accordingly, the wave function of the electron in the Landau ground state in a rectangular region of size L is

$$\Psi_{n=0,\,k}(x,\ y) = \pi^{-1/4}(a_0L)^{-1/2}\,e^{iky}\,\exp\left[-\frac{1}{2a_0^2}(x-x_0)^2\right], \tag{4.24}$$

where $a_0 = (eB)^{-1/2}$ is the magnetic length ($\hbar = c = 1$) and $x_0 = ka_0^2$ is the center of the Landau orbit. The degeneracy of energy with respect to the momentum k is thus seen to signify a clear independence of the electron energy of the position of the center of the orbit on the x, y plane. The number of states is

$$N_t = \tilde{\nu}\int \mathrm{d}S = \frac{\Phi}{\Phi_0} = \frac{eB}{2\pi}L^2 = \frac{L^2}{2\pi a_0^2} \equiv \frac{S}{S_0}. \tag{4.25}$$

In the symmetric gauge, $\mathbf{A} = (-yB/2, xB/2)$, the Schrödinger equation allows the separation of variables in the case of polar coordinates. The conserved quantum number that does not influence the energy is now the angular momentum m of the electron. In terms of the complex variables $z = x - iy$, $\bar{z} = x + iy$, the electron wave function for the lowest energy level is[84,107]

$$\Psi_{n=0,\,m}(z,\ \bar{z}) = (2^{m+1}\pi m!)^{-1/2}\,z^m\exp\left[-\frac{1}{4a_0^2}|z|^2\right]. \tag{4.26}$$

The mathematical convenience of (4.26), as compared with (4.24), is that, with the exception of the exponential factor that is common to all the electrons filling the Landau level, (4.26) is a holomorphic function of the complex coordinates [the representation (4.26) is therefore a holomorphic representation[84]; for the QHE context, see Ref. 110a). Since the Coulomb interaction conserves the total angular momentum

$$M = \sum_{i=1}^{N} m_i$$

the representation given by (4.26) is the natural basis for the wave functions of the many-body Hamiltonian.

The first theoretical explanation of FQHE was given by Laughlin.[107] His model of the incompressible quantum electron fluid makes use of the gauge (4.26). Tao and Thouless[113] continued the approach, using the gauge (4.24) and the idea of the charge density wave on Landau orbitals. The Laughlin scheme seems to be the most successful attempt to describe the symmetry properties of a many-body system of electrons in a strong magnetic field. However, since both approaches yield many qualitatively similar predictions, including the prediction of quasiparticles with fractional charges, which is important in our context here, we must now summarize the Tao-Thouless approach.

4.4.1. Laughlin's approach

In FQHE, the magnetic fields are so strong (and the disorder is so weak) that the energy gap Δ between regions of delocalized states is greater than the characteristic (Coulomb) energy of electron-electron correlations: $e^2/a_0 \lesssim \Delta$ ($\Delta \sim \omega_c \sim eB/m^*$). This means that a particular combination of single-particle wave functions such as (4.26) may provide an adequate approximation for the description of the ground and lowest-lying excited states of the N-particle Hamiltonian H of electrons in the Landau ground state with Coulomb interaction

$$H = H_0 + \sum_j V(\mathbf{r}_j) + \sum_{i<j}\frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|},$$
$$H_0 = \sum_{j=1}^{N}\frac{1}{2m_j^*}(\mathbf{p}_j - e\mathbf{A}_j)^2; \tag{4.27}$$

where H_0 is the Hamiltonian for free 2D-electrons in a transverse magnetic field, m^* is the effective mass of electrons (j is the particle label), and $V(\mathbf{r}_j)$ is the potential of the lattice that ensures the system is electrically neutral.

For the simplest type of fractional occupation number $\nu = 1/p$ (p is odd), Laughlin proposed the following trial function[107] for the ground state of the Hamiltonian (4.27):

$$\Psi_L(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^p \exp\left(-\frac{1}{4a_0^2} \sum_{j=1}^{N} |z_j|^2\right). \tag{4.28}$$

The exponential factor in this expression is obvious and needs no comment. The specific form of the preexponential function $f(z_1, z_2, \ldots, z_N)$ can be justified as follows. The function f must be (1) completely antisymmetric in the coordinates z_j (Pauli's principle) and (2) it must be a homogeneous polynomial [see the representation (4.26)] of degree

$$M = \sum_{i=1}^{N} m_i$$

(conservation of total angular momentum M). Finally, f must minimize the Coulomb energy. This requirement is strictly taken into account only for systems with a finite (small) number N of electrons, and enables us to write it in the form of (4.28).

Actually, the Coulomb energy of electrons decreases with increasing orbital angular momentum m_{ij} of each pair of particles. This suggests that the function f should be taken in the form of the product of factors $(z_i - z_j)^{m_{ij}}$ (describing the orbital motion of the pair ij) with equal and maximum possible (for the given total angular momentum M) values of the exponents $m_{ij} = m_{\max}$. Requirement (2) immediately leads to $M = C_2^N m_{\max}$ [$C_2^N = \frac{1}{2}N(N-1)$ is the number of different pairs among the N electrons].

To relate $m_{\max}$ to the occupation number ν, Laughlin used the formal but precise analogy between the model of 2D-electrons in the Landau ground state and the theory of classical one-component plasmas.[107] We shall now consider qualitative arguments, involving a quasiclassical quantity such as the mean area "occupied" by a rotating quantum particle, to obtain the same results in a graphic manner. It is readily seen that, in a strong magnetic field, an electron rotating around a common center of inertia with angular momentum m sweeps out an area S_m that is greater by a factor of m (for $m \gg 1$) than the area $S_0 = 2\pi a_0^2$ occupied by the flux quantum Φ_0($S_m = \pi\langle m|r^2|m\rangle \simeq 2\pi a_0^2 m$). On the other hand, in the case of a fractional occupation number $\nu \equiv N/N_t = NS_0/S = 1/p$, the maximum area occupied by a uniform distribution of charge per particle is $S/N = pS_0$. Hence, $m_{ij} = m_{\max} = p$, and we obtain (4.28).

A more "rigorous" justification of the preexponential function can be obtained in spherical geometry when the plane of the 2D-electrons is compactified into a sphere at the center of which there is a monopole producing a magnetic field B that is perpendicular to the surface of the sphere.[108a] In this case, the magnetic field flux is finite, the degree of degeneracy of the Landau levels is finite, and the problem of N-particle level occupation is correctly posed. This approach [108a] has been used to show that the Laughlin function (4.28), written in terms of variables on a sphere, is the exact solution for the three-particle problem. For $N > 3$, the function Ψ_L is always the approximate wave function for the ground state of electrons experiencing the Coulomb interaction, and its particular variational property is that it suppresses states in which pairs of particles are at the shortest distances from one another (see also Ref. 108b).

Although one of the requirements that initially enabled a determination to be made of the form of Ψ_L was the conservation of total angular momentum, the important properties of the Laughlin function are entirely due to behavior at short distances. The Coulomb potential was replaced by the short-range repulsive potential in Ref. 112, and simple analysis showed that, in this case, the Laughlin function is the exact wave function for the ground state of 2D-electrons in the partially filled lowest Landau level[112] with $\nu = 1/p$ (p–odd).

Within the framework of the picture described above, in which the 2D-electrons rotate around a common center of mass, it seems likely that the change in the area occupied by the Laughlin quantum liquid, which is due to centrifugal barriers, can occur only in quanta (incompressibility). Thus, the removal of one electron (production of a hole) for $\nu = 1/p$ is equivalent to a reduction in the area by $S_p = p \cdot 2a_0^2 = pS_0$. We are therefore entitled to ask: what would happen if we changed the area by the amount S_0 occupied by the flux quantum Φ_0?

It is readily seen that this excitation is equivalent to placing at some point ξ in the liquid an infinitely thin solenoid in which the flux changes adiabatically by Φ_0. In this case, the single-particle wave functions (5.31) have their orbital angular momentum increased by unity, so that $z^m \to z^{m+1}$ [see (4.22) and (4.26)] and the many-particle state becomes

$$\Psi_+ = \prod_{i=1}^{N} (z_i - \xi)\, \Psi_L(z_1, z_2, \ldots, z_N) \equiv \hat{A}_\xi \Psi_L. \tag{4.29}$$

According to Laughlin,[107] this is the wave function of a quasihole $\tilde{h}$ with complex coordinate ξ. The wave function of the quasielectron $\tilde{e}$ can be readily constructed by analogy with (4.29). This is done by applying the adjoint $\hat{A}_\xi^+$ of the operator $\hat{A}_\xi$ to the preexponential factor in Ψ_L (the exponential factor in the holomorphic representation can be removed by introducing it into the integration measure[133,110a]):

$$\Psi_- = \hat{A}_\xi^+ \Psi_L, \quad \hat{A}_\xi^+ = \prod_{i=1}^{N} \left(\frac{\partial}{\partial z_i} - \frac{\xi}{a_0^2}\right). \tag{4.30}$$

Quasiholes (quasielectrons) have finite dimensions $R \sim a_0$ and a finite excitation energy $\tilde{\Delta} \sim (1/\bar{p}^2) e^2/a_0$ (see Ref. 107 for further details). This means that, in FQHE, quasiparticles constitute a local change in the electron density. Their charge is $\pm e^*$ and they carry a quantum of flux Φ_0. Physically, the flux is due to a circular current flowing around the quasiparticles.

The unusual property of the Laughlin quasiparticles is their fractional charge $e^* = \pm e/p$, where e is the charge of the electron.[107] We are led to this relation even in the intuitive picture because, when $\nu = 1/p$, there is one electron for p quanta of flux. However, like the fractional charge of one-dimensional solitons, the fractional charge of two-dimensional quasiparticles is a topological parameter. The fraction appears not as a result of averaging, but as a precise quantum number. We shall follow Ref. 123 and give a simple "topological" derivation of the fractional charge of quasiparticles for an arbitrary rational occupation factor $\nu = q/p$.

As we have seen, under the conditions of FQHE, a change in flux by one quantum, $\Delta\Phi = \pm\Phi_0$, is always due to a local change in the electron density (formation of quasiparticles). Let $\nu = q/p$ and suppose that our problem is to find the effective charge e_ν^* of the quasiparticles. By taking an electron around a closed contour containing one flux quantum, we change the phase of its wave function by $\Delta\varphi_e = \pm 2\pi$. Correspondingly, one circuit of a localized flux quantum around a resting electron produces a change in the phase of the quasiparticle by $\Delta\varphi_{\tilde{e}} = \mp 2\pi$. If we place N electrons in the interior of the contour, we have $\Delta\varphi_{\tilde{e}} = \mp 2\pi N$. On the other hand, a change in the phase of the wave function is always due to the total flux Φ through the contour (4.22): $\Delta\varphi_{\tilde{e}} = e_\nu^*\Phi$. Recalling that $\Phi = N\Phi_0/\nu$, and equating the change in the quasiparticle phase found by the two different methods, we obtain

$$e_\nu^* = \mp\nu e. \qquad (4.31)$$

It is readily seen that, when $\nu = 1/p$, the creation of p quasiparticles at a given point is equivalent to the creation of one real electron (hole).[109,110c] Actually, p-fold application of the quasihole creation operator $\hat{A}_\xi$ to the Laughlin wave functions (4.28) produces the same state, but with one "removed" electron, since

$$\Psi_h = (\hat{A}_\xi)^p\,\Psi_L = \prod_{i=1}^{N}(z_i - \xi)^p\,\Psi_L(z_1, z_2, \ldots, z_N) \qquad (4.32)$$

describes the wave function of the Laughlin quantum liquid with one unoccupied (at ξ) electron state.[110b]

4.4.2. The Tao-Thouless approach

In the Landau gauge, the single-particle states (4.24) are plane waves in one of the coordinates (for example, y) and Gaussian wave packets (of size a_0) in the other coordinate. The problem of placing the particles in the Landau orbitals, and of minimizing their Coulomb energy, is therefore effectively one-dimensional in this case because it reduces to the problem of finding the centers x_0 of the occupied orbitals. In the Tao-Thouless scheme,[113] the partially filled ($\nu = 1/p$) lowest Landau level of the ground state corresponds to the formation of a regular superlattice (centers x_0) with period $a_p = pa$, where $a = L/N_t$ is the distance between the centers of the orbitals (see Fig. 9a, where, for convenience, we have taken $\nu = 1/2$). By construction, this state is p-fold degenerate.

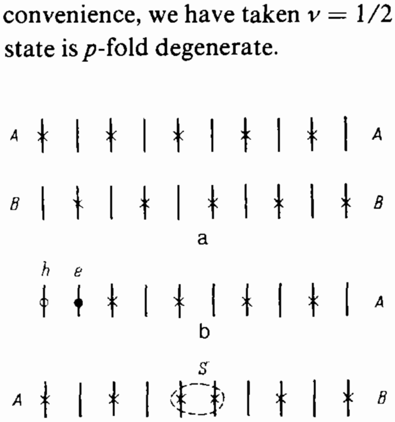

FIG. 9. a—Energy-degenerate configurations (A and B) superlattices for $\nu = 1/2$. b—Superlattice with electron-hole excitation. c—Topological defect—a soliton in Landau orbitals.

The elementary perturbations are now the states in which the electron is in one of the unoccupied ground-state orbitals (particle) or is absent from an occupied orbital (hole) (see Fig. 9b). Qualitatively, there is no doubt that, because of the regular disposition of the Coulomb-interacting electrons in the orbitals, the formation of a particle-hole pair requires a finite amount of excitation energy (this is, of course, literally the characteristic Coulomb energy $\tilde{A} \sim e^2/a_0$ and calculations based on perturbation theory of the Coulomb potential enable us to determine the small numerical factor[113]).

Apart from single-particle excitations, the system can have collective states, i.e., topologically stable solitons and antisolitons[114,115] (see Fig. 9c). Their existence is the unavoidable consequence of the finite degeneracy of the ground state. Solitons are then the "domain walls" separating energy-degenerate phases and, from this point of view, the Tao-Thouless solitons are completely analogous to solitons of the commensurate charge-density wave (see Section 3). For the fractional occupation number $\nu = 1/p$, the commensurability index of the superlattice is p and the topological excitations in this model carry fractional charge $e^* = \pm e/p$, just like the CDW solitons.[2)] At this point, we have complete agreement between the Laughlin and Tao-Thouless theories.

4.5. Hierarchies of quasiparticles and anomalous statistics

Both the Laughlin and Tao-Thouless theories in their original form were capable of explaining the quantum Hall conductivity, but only for the simplest fractional values of the occupation factor of the Landau ground level, $\nu = 1/p$ (p–odd). Subsequent experiments showed that there was a plateau on the function $\sigma_{xy}(B)$, which corresponded to a rational level occupation $\nu = q/p$, and attempts were made to modify the theoretical FQHE models accordingly.[108,109,121] The CDW model is readily generalized to rational occupation factors $\nu = q/p$ by having regular clusters of q particles in a superlattice of period $a_p = pL/N_t$, where L is the size of the system and N_t the number of Landau orbitals. The idea of structural hierarchy of quasiparticles[108a,121] was introduced to explain the quantization of σ_{xy} for $\nu = q/p$ within the Laughlin model.

Let $\nu = 1/p$, in which case, the 2D electrons form a condensate (incompressible Laughlin quantum liquid) in a strong magnetic field in which excitations (quasiparticles) have finite activation energy. Since quasiparticles are local objects carrying an electric charge, they can be due to impurities. When the density of 2D-electrons is close to $\nu = 1/p$, the density of quasiparticles is low (an increase in the number of electrons by one is equivalent to the excitation of p quasielectrons). They are all pinned by impurities and do not contribute to conductivity at low temperatures. Hence, $\sigma_{xx} = 0$ and σ_{xy} corresponds exactly to the Hall conductivity of the partially filled ($\nu = 1/p$) Landau ground state. Further change in the electron density produces an increase in the density of quasiparticles. Delocalized quasiparticle states appear, and, by analogy with the Laughlin picture, they themselves form a condensate, so that there is a new stable occupation factor $\nu_1 = q_1/p_1$. The preceding discussion remains valid for the new structure, and we thus have a hierarchial structure of quasiparticles and their condensates.

In accordance with the Laughlin arguments, the trial wave function for a condensate of quasiparticles with effective charge $e^+ = \pm qe$ is[121]

$$\Psi_{\mathrm{L-H}} = Q\{Z_j\}P\{Z_j\}\exp\left(-\frac{|q|}{4a_0^2}\sum_{j=1}^{N_S}|Z_j|^2\right), \quad (4.33)$$

where Z_j is the complex coordinate of quasiparticle j. The splitting of the preexponential function into the product of two factors has the following simple interpretation. The polynomial $P\{Z_j\}$ is always chosen to be symmetric

$$P\{Z_j\} = \prod_{i<j}(Z_i - Z_j)^{2l}, \quad (4.34)$$

where l is a positive integer. Its variational properties were discussed in detail in Sec. 4.4.1. The function $Q\{Z_j\}$ determines the symmetry properties of the quasiparticle wave function under quasiparticle interchange. For electrons, requirements of Fermi statistics unambiguously lead to the polynomial form

$$Q(z_j) = \prod_{i<j}(z_i - z_j),$$

and, according to Laughlin (see Sec. 4.4.1), we have $2l+1 = \nu^{-1}$. For quasiparticles that carry not only a charge but also the magnetic flux Φ_0, the symmetry properties of the many-particle wave function turn out to be much more complicated.

In contrast to three-dimensional space (and spaces of higher dimensionality), indistinguishable quantum particles in two-dimensional space can, in general, have anomalous statistics.[116,120]

In the functional approach, the particle statistics in a space of dimensionality d is determined by the one-dimensional irreducible representations of the fundamental homotopy group of the configuration space of N indistinguishable particles, $\pi_1(M_{dN})$. It is readily shown that, for $d \geqslant 3$, $\pi_1(M_{3N}) = S_N$ (S_N is the permutation group). Since there are only two types of irreducible representation of S_N, i.e., $\chi_+ = 1$ and $\chi_- = \pm 1$ (even and odd permutations), there are only two types of statistics in spaces with $d \geqslant 3$, namely, Bose-Einstein (χ_+) and Fermi-Dirac (χ_-). For $d = 2$, $\pi_1(M_{2N})$ is an infinite non-Abelian group. Its one-dimensional unitary representations have the form

$$\chi_\theta = \exp(-i\theta) \quad (0 \leqslant \theta < 2\pi)$$

so that two-dimensional indistinguishable quantum objects can obey anomalous (theta) statistics that is intermediate between boson ($\theta = 0$) and fermion ($\theta = \pi$) statistics.

As a simple example, consider the bound system of a spinless particle of charge q and an infinitesimally thin solenoid carrying a flux Φ and cutting through through the plane of motion of the particle (Wilczek called this system "anyon"[116]). The solenoid potential A_φ acting on a particle at a point r is $A_\varphi = \Phi/2\pi r$ and, in the singular gauge (4.21), can be completely removed from the region outside the solenoid. According to the general requirement of gauge invariance, the transformation of the potentials is accompanied by phase rotation of the wave function. Hence, in the gauge $A_\varphi = 0$ (4.21), the wave function of a charged particle is, in general, multivalued:

$$\Psi_\Phi(\varphi) = e^{iq\Lambda}\Psi(\varphi) = \exp\left(i\frac{q\Phi}{2\pi}\varphi\right)\Psi(\varphi), \quad (4.35)$$

$$\Psi_\Phi(\varphi + 2\pi) = e^{i\theta}\Psi_\Phi(\varphi), \quad \theta \equiv q\Phi, \quad (4.36)$$

where $\Psi(\varphi)$ is the single-valued particle wave function.

A system of two identical anyons is equivalent (this can be readily shown by transforming to relative coordinates) to a single anyon, but with a flux $\Phi \to 2\Phi$. If we use this fact, we can readily generalize (4.35) to a many-particle wave function of N identical anyons

$$\Psi_\Phi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \prod_{i<j}\exp\left(i\frac{\theta}{\pi}\varphi_{ij}\right)\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N); \quad (4.37)$$

where $\mathbf{r}_j$ is a two-dimensional vector representing the position of the charged particle, φ_{ij} is the azimuthal angle of the pair ij, and $\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ is a single-valued wave function. According to (4.37), a permutation of any two anyons ($\Delta\varphi_{ij} = \pi$) multiplies the multiparticle function Ψ_Φ by the phase factor $e^{i\theta}$. Hence, for $\Delta_\Phi \equiv \theta/2\pi = q\Phi/2\pi$ equal to an integer, the anyons are bosons, whereas for Δ_Φ equal to half an integer they are fermions. In the general case, we have anomalous statistics (θ-statistics). To reconcile spin with statistics, it is convenient to consider Δ_Φ as the anyon spin.[3] In terms of the complex coordinates $z_j, \bar{z}_j$ of the particles, the multivalued wave function (4.37) assumes the form[120]

$$\Psi_\Phi = \prod_{i<j}(z_i - z_j)^{\theta/\pi} f\{z_j, \bar{z}_j\}, \quad (4.38)$$

where f is a single-valued function of the coordinates.

Let us now determine the phase θ for quasiparticles in FQHE in the simplest case, where the anyon quasiparticles are "constructed" from electrons (the first structure level). The change $\Delta\varphi$ in the phase of the wave function of a quasiparticle in a closed adiabatic path around another quasiparticle is $\Delta\varphi = e^*\Phi_0$ and, for $\nu = 1/p$, we find from (4.31) that $|\Delta\varphi| = 2\pi/p$. Hence, the phase change that accompanies the interchange of quasiparticles is $\Delta\varphi/2 = \pi\nu = \theta$. According to the general structure of (4.38), the factor $Q\{Z_j\}$ in the pseudowave function of the Laughlin-Halperin quasiparticles (4.33) is now

$$Q\{Z_j\} = \prod_{i<j}(Z_i - Z_j)^{\pm 1/p}, \quad (4.39)$$

where the signs $+$ ($-$) correspond to hole and electron quasiparticles, respectively.

In conclusion of this Section, we shall show that the anomalous statistics of quasiparticles provides at least a partial theoretical justification of the empirical odd denominator rule.[125] To show this, consider two clusters, each of which consists of p quasielectrons. Since each quasiparticle obeys anomalous statistics with $\theta = \pi\nu$, and the interchange of two clusters is equivalent to p^2 interchanges of pairs of quasiparticles, the parameter θ_p that represents the cluster statistics is $\theta_p = p^2\theta$. Accordingly, when $\nu = q/p$ ($q < p$; q, p are mutually primitive numbers), the interchange of two clusters multiplies the quasiparticle wave function by $\exp(i\theta_p) = \exp(i\pi pq)$. According to the equivalence theorem, the excitation of p quasielectrons in FQHE with $\nu = q/p$ is equivalent to the formation of q electrons. Since electrons obey Fermi statistics, the same phase factor is given by

$$e^{i\pi pq} = e^{i\pi q^2}. \quad (4.40)$$

When p is even, this necessarily means that q must also be even and the numbers, q, p cannot be mutually primitive. In other words, for even denominators p, anomalous statistics forbids the simultaneous formation in the system of a large number of quasiparticles, and the condensation of quasiparticles into the incompressible quantum liquid is impossible.

5. CONCLUSION

It is now more than ten years since the publication of the first paper on fractionally charged solitons. The fractional charge problem has since been carefully examined, both in field theory and in solid-state physics. It will be particularly important to find real quasi-one-dimensional and quasi-two-dimensional systems exhibiting phenomena closely related to the fractional charge effect. In this new research area the ideas and methods developed in parallel in quantum field theory and in solid-state physics have unexpectedly found an extensive region of contact. One hopes that further advances in theoretical and experimental work will "convert" the as yet unusual phenomenon of fractional charge into a generally accepted concept of modern fundamental physics.

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[1)]The two-dimensional massless fermion cannot have kinematic spin because relativistic invariance demands that the spin be either parallel or antiparallel to momentum, which is absurd for particles that "live" on a plane.

[2)]We recall that the electrons occupying Landau levels are polarized so that they are effectively spinless. Hence, the topological charge of solitons in Landau orbitals ($N_m = 1$) is smaller by a factor of two than the topological charge of solitons of commensurate CDW (3.25).

[3)]See Refs. 116–120 for a detailed discussion of fractional spin.

[4)]This review deals with numerous problems in quantum field theory and solid-state physics. The following list of references does not therefore claim to be exhaustive. In some cases, we have confined ourselves to listing only the first original papers and reviews. (After this review was submitted for publication, two important review papers appeared,[135,136] in which fractional charges in quantum field theory and the current state of QHE were discussed.)

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