From the Editorial Board

The article "Fluctuation-dissipation relations. Role of the finiteness of the correlation time. Quantum generalization of Nyquist's formula" by Yu. L. Klimontovich was submitted for publication in Uspekhi Fizicheskikh Nauk. In this article it is asserted that the well-known Callen-Welton formula holds in the quantum region only under conditions of weak dissipation, while the quantum generalization of Nyquist's formula in the general case differs substantially from the generally accepted derivation in the literature. The reviewers to whom we sent the article did not agree with Yu. L. Klimontovich's conclusions. Nevertheless, discussions (in particular, at seminars) indicate that the questions Yu. L. Klimontovich addresses are by no means clear to everyone. Moreover, Yu. L. Klimontovich is not only a well-known specialist in statistical physics, but he is also the author of the book "Statistical Physics" (Nauka, Moscow (1985)), a textbook for physics students. In this book (Chapter 11, Sec. 5) the quantum generalization of Nyquist's formula is also derived in an unusual manner.

In view of this situation we thought it would be appropriate to publish Yu. L. Klimontovich's article simultaneously with the article by V. L. Ginzburg and L. P. Pitaevskiĭ, in which Yu. L. Klimontovich's article is criticized from the standpoint of the position adopted in the literature. In addition, this issure of Uspekhi contains an article by V. I. Tatarskiĭ, which is directly relevant to the problems discussed. In addition, V. I. Tatarskiĭ, like V. L. Ginzburg and L. P. Pitaevskiĭ, regards the standard expression for Nyquist's quantum formula to be correct.

We believe that the three indicated articles taken together elucidate quite fully the questions under discussion and will enable the readers to draw their own conclusions. The Editorial Board does not propose to have these questions discussed further in Uspekhi.

Fluctuation-dissipation relations. Role of the finiteness of the correlation time. Quantum generalization of Nyquist's formula

Yu. L. Klimontovich

M. V. Lomonosov Moscow State University

The fluctuation-dissipation relations (FDR) in physical systems are studied at all levels of the statistical description. The most general FDR are the relations for the fluctuations of many-body distribution functions. It is pointed out the problem of formulation of FDR is related to the problem of deriving irreversible equations based on the reversible equations of classical and quantum mechanics. The FDR are divided into two classes: 1) FDR for fluctuations with infinite correlation times ("collisionless approximation"), which correspond to infinitely narrow resonances, and 2) FDR for fluctuations with finite correlation times ("collisional approximation"). The corresponding spectral densities have finite widths, determined by the "collision integrals." The fundamental questions about which different viewpoints have been published in the literature are critically analyzed: 1) the limits of applicability of the Callen-Welton formula and 2) the quantum generalization of Nyquist's formula for the intensity of a Langevin source of oscillatory systems. It is shown that the traditional form of the quantum Nyquist formula is not well-founded and leads to unphysical consequences. A different expression, used in the literature, for the quantum Nyquist formula is examined. It is not universal, but holds in many important cases. Its region of applicability is determined by the corresponding quantum kinetic equations. The consequences of the two forms of the quantum Nyquist formula, which can be checked experimentally, are studied. The question of the formulation of the quantum Nyquist formula is studied as a part of the general problem of determining the intensity of a Langevin source and the corresponding diffusion coefficient in quantum systems. It arises, in particular, also in quantum electrodynamics in the calculation of the Lamb shift (Sec. 12). In this connection two derivations of Bethe's formula for the Lamb shift are analyzed. It is established that the "subtraction formalism" of quantum electrodynamics corresponds to the nontraditional form of the quantum Nyquist formula. The exposition is illustrated with many specific examples.

CONTENTS

1. INTRODUCTION

The first fluctuation-dissipation relation (FDR) was established by Einstein in 1905 in his work on the theory of Brownian motion. It has the form $D = m\gamma kT$, and relates the diffusion coefficient (in momentum space) characterizing the molecular motion of the medium, the coefficient of friction γ, and the temperature T. A short time later Langevin proposed a different method for describing Brownian motion based on the equations of mechanics with a random force—Langevin's force, characterizing the role of the molecular motion. For a free particle Langevin's equation has the form

$$\frac{dr}{dt} = v, \quad \frac{dv}{dt} + \gamma v = \frac{1}{m} f(t). \tag{1.1}$$

The correlation function of the random force is given by the expression

$$\langle f_i(t) f_j(t') \rangle = 2D\delta_{ij}\,\delta(t-t'), \quad D = m\gamma kT, \qquad (1.2)$$

which corresponds to the zeroth-order approximation in the small parameter τ_{cor}/τ_{rel} —the ratio of the correlation time of molecular collisions to the relaxation time $\tau_{rel} = 1/\gamma$. In this approximation the spectral density of the force

$$\frac{1}{3}(f^2)_\omega = 2D \qquad D = m\gamma kT \qquad (1.3)$$

is independent of the frequency. This means that the noise attributable to the molecular motion is *white noise*.

The second relation in (1.3) shows that D plays a dual role: not only that of a diffusion coefficient, but also that of the noise intensity in Langevin's equation. Here it is important that the definition of both the diffusion coefficient and the noise intensity is based on Maxwell's distribution for the velocities of Brownian particles.

In 1928 Nyquist used Langevin's equation

$$\frac{dq}{dt} = I \qquad l\,\frac{dI}{dt} + RI + \frac{1}{C}\,q = \mathscr{E}(t) \qquad (1.4)$$

to describe thermal oscillations (Brownian motion) in an electric circuit. The role of Langevin's force is played by a random emf, whose spectral density is given by the expression

$$(\mathscr{E}^2)_\omega = 2RkT \qquad (1.5)$$

—*Nyquist's formula*. In (1.4) and (1.5) L, C, and R are the inductance, capacitance, and ohmic resistance. The latter corresponds to $m\gamma$ in formulas (1.2) and (1.3). The spectral densities of the current and charge are expressed in terms of the spectral density of the emf:

$$(I^2)_\omega = \frac{(\mathscr{E}^2)_\omega}{|Z(\omega)|^2} = \frac{2RkT}{|Z(\omega)|^2} \qquad (q^2)_\omega = \frac{(I^2)_\omega}{\omega^2}$$

$$Z(\omega) = R - i\left(L\omega - \frac{1}{C\omega}\right) \qquad (1.6)$$

The one-time moments (variances) are determined by integrating over ω:

$$\langle I^2 \rangle = \frac{kT}{L} \qquad \langle q^2 \rangle = CkT \cdot \qquad (1.7)$$

The following integrals are employed in the derivation of these formulas:

$$L\int_{-\infty}^{\infty} \frac{2R}{|Z(\omega)|^2}\frac{d\omega}{2\pi} = 1, \quad \frac{1}{C}\int_{-\infty}^{\infty} \frac{2R}{\omega^2|Z(\omega)|^2}\frac{d\omega}{2\pi} = 1 \cdot \qquad (1.8)$$

To calculate the thermal noise in an electric circuit Nyquist could have used Einstein's method also. For this one must write the corresponding Fokker-Planck equation for the distribution function $f(q,I,t)$:

$$\frac{\partial f}{\partial t} + I\frac{\partial f}{\partial q} - \omega_0^2 q\frac{\partial f}{\partial I} = \frac{D}{L^2}\frac{\partial^2 f}{\partial I^2} + \frac{\partial}{\partial I}\left(\frac{R}{L}If\right), \qquad \omega_0^2 = \frac{1}{LC}\cdot \qquad (1.9)$$

The equilibrium solution of this equation

$$f = C\exp\left(-\frac{H(q,I)}{kT}\right),$$

$$\int f(q,I)\,dq\,dI = 1, \qquad H = \frac{LI^2}{2} + \frac{q^2}{2C} \qquad (1.10)$$

is an example of the Gibbs distribution for an oscillator in a thermostat. The expressions for the one-time moments

found with the help of this distribution are identical to (1.7).

The spectral densities of the current and charge are calculated by the standard procedure. Namely, together with (1.9) an equation of the same form for the two-time distribution function is employed:

$$f(q, I, t, q', I', t'). \qquad (1.11)$$

With its help the system of equations of the two-time moments is obtained:

$$\frac{d}{d\tau}\langle qI\rangle_\tau = \langle II\rangle_\tau, \quad L\frac{d\langle II\rangle_\tau}{d\tau} + R\langle II\rangle_\tau + \frac{1}{C}\langle qI\rangle_\tau = 0,$$

$$\tau = t - t' > 0 \qquad (1.12)$$

The equation is supplemented with the "initial conditions" (1.7), which are found with the help of the Gibbs distribution (1.10). The solution of Eqs. (1.12) yields the formulas (1.16) and, as a consequence, Nyquist's formula (1.5).

We have presented the two simplest examples of classical FDR. Many FDR are studied in statistical physics (see, for example, Refs. 1-7) in order to describe kinetic, hydrodynamic, and diffusion processes in the most diverse systems.

Additional questions arise in the formulation of FDR for quantum systems. Indeed, how, for example, should the classical Nyquist formula (1.5) be formulated in the quantum case? This question was first answered in Nyquist's paper.[8] Taking into account the zero-point oscillations Nyquist's result consists of the following: one must make the substitution

$$kT \rightarrow \frac{1}{2}\hbar\omega \coth\frac{\hbar\omega}{2kT} \equiv kT_\omega \qquad (1.13)$$

in the classical formula. Then we arrive at the *quantum Nyquist formula*

$$(\mathscr{E}^2)_\omega = 2RkT_\omega. \qquad (1.14)$$

The expression (1.7) for the spectral density of the current assumes the form

$$(I^2)_\omega = \frac{2RkT_\omega}{|Z(\omega)|^2}, \quad \text{and} \quad \langle I^2\rangle = \int_{-\infty}^{\infty}\frac{2RkT_\omega}{|Z(\omega)|^2}\frac{d\omega}{2\pi}. \qquad (1.15)$$

The operation (1.13)—substitution of the average energy of a quantum oscillator for the average energy of the classical oscillator (circuit)—appears to be obvious. There is, however, another possibility for generalizing the formula (1.5): the substitution (1.13) is made not at the running frequency of the spectrum ω, but rather at the characteristic frequency ω_0,[6,7,32] i.e.,

$$kT \rightarrow \frac{1}{2}\hbar\omega_0 \coth\frac{\hbar\omega_0}{2kT} \equiv kT_{\omega_0}. \qquad (1.16)$$

Then the classical Nyquist formula (1.15) assumes the form

$$(\mathscr{E}^2)_\omega = 2RkT_{\omega_0}. \qquad (1.17)$$

Thus the spectrum, like in the classical theory, is white noise. There appears, however, a dependence on the characteristic frequency ω_0—a parameter in the Hamiltonian in the Gibbs distribution (1.10).

Let us compare the two quantum generalizations of Nyquist's formula presented above.

1. The formula (1.14) contains two parameters ("widths") of the line or two corresponding relaxation times

$$\tau^0 \sim \frac{1}{\omega}, \quad \tau^{(\tau)} \sim \frac{\hbar}{kT}. \tag{1.18}$$

The first characterizes the spectrum of zero-point oscillations, and the second characterizes the thermal motion. These times are independent of the parameters of the system—the parameters in the Hamiltonian, and are in this sense *universal* relaxation characteristics. In contrast to this formula, the formula (1.17) is not universal, since the concept of white noise itself is relative.

2. The transition to the classical approximation (the condition $kT \gg \hbar\omega$) in (1.14) is not completely determined, since ω here is the running frequency of the spectrum, while the contribution of zero-point oscillations is largest at high frequencies.

3. Since the zero-point oscillations lead in (1.15) to a diverging integral, it is necessary to introduce the parameter ω_{max}. In so doing, the corresponding contributions to $L \langle I^2 \rangle$ for the classical oscillator and for the LR circuit equal

$$L \langle I^2 \rangle = \frac{1}{2\pi} \hbar \frac{R}{L} \ln \frac{\omega_{max}}{\omega_0},$$

$$L \langle I^2 \rangle = \frac{1}{4\pi} \hbar \frac{R}{L} \ln \left[1 + \frac{\omega_{max}^2}{(R/L)^2} \right]. \tag{1.19}$$

We can see that the mean kinetic energy of the classical oscillator *in the equilibrium state is a linear function* of the dissipative parameter R. The other part of the integral in (1.15), determined by thermal fluctuations, also depends on R.

This result is inconsistent with the statistical theory, according to which in the equilibrium state the one-time correlation functions (here $\langle I^2 \rangle$ and $\langle q^2 \rangle$) depend only on the temperature and the parameters in the Hamiltonian. Otherwise the thermodynamic functions (here $\langle I^2 \rangle$ and $\langle q^2 \rangle$) would depend on the nature of the process by which the equilibrium state is established. A contradiction with the second law of thermodynamics would then arise.

The use of the quantum Nyquist formula in the form (1.17) instead of (1.14) does not lead to the indicated difficulties. Indeed, for the classical oscillator and the LR circuit ($\omega_0 = 0$) the average kinetic energy is determined by the formulas

$$L \langle I^2 \rangle_R = kT_{\omega_0}, \quad L \langle I^2 \rangle_{LR} = kT, \tag{1.20}$$

and therefore the system is in equilibrium with the thermostat.

4. The formulas (1.14) and (1.17) lead to the fact that the behavior of the function $(I^2)_\omega$ is substantially different at high and low frequencies.

The difference between the formulas (1.14) and (1.17) in the calculation of integral characteristics vanishes only under the condition of an exact resonance ($R \to 0$), when

$$\text{Re} \frac{1}{Z(\omega)} = \frac{R}{R^2 + [L\omega - (1/C\omega)]^2} \to \pi\delta \left(L\omega - \frac{1}{C\omega} \right). \tag{1.21}$$

Thus the main difficulties associated with the use of the quantum Nyquist formula (1.14) occur only for a finite width of the resonance R/L (or finite correlation time $\tau_{cor} = L/R$).

The problem of the formulation of the quantum Nyquist formula is part of the general problem of determining the intensity of Langevin sources and the corresponding diffusion coefficients in quantum systems. In particular, as we shall see in Sec. 12, it also arises in quantum electrodynamics in the calculation of the Lamb shift.

In this work we shall analyze the FDR for physical systems at different levels of the statistical description. The most general FDR are those for fluctuations of many-body distribution functions (see Secs. 5 and 6). Their formulation reveals the relationship between the problem of the formulation of FDR and the problem of the derivation of the irreversible equations of the statistical theory based on the reversible equations of classical and quantum mechanics.[43,44]

It is useful to divide the FDR into two classes: 1) FDR for fluctuations with infinite correlation times—"collisionless approximation," which correspond to infinitely narrow resonances, and 2) FDR for fluctuations with finite correlation times—"collisional approximation." The corresponding spectral densities have finite widths, determined by the "collision integrals" in the corresponding kinetic equations.

Among FDR of the first class the Callen-Welton formula is the best known. In the quantum theory it relates the spectral densities of the internal thermodynamic parameters with the dissipative parts of the corresponding susceptibilities. The second class includes, in particular, the quantum Nyquist formula (1.17). The collision integral in it determines the "width" of the white noise.

In the course of the exposition questions about which there are differing viewpoints in the literature are examined in detail. They include: 1) the question of the limits of applicability of the Callen-Welton formula and 2) the question of the quantum generalization of the classical Nyquist formula (1.5).

In connection with what was said above, we shall trace the manner in which the quantum Nyquist formula (1.14) is established. Two methods are known. The first method is based on the Callen-Welton formula and will be examined in Sec. 4. The other method is based on the second law of thermodynamics and some additional considerations. It is precisely this approach that was employed in Nyquist's work.[8] We shall examine here a simpler variant, presented in Refs. 9 and 10.

2. THERMODYNAMIC DERIVATION OF THE QUANTUM NYQUIST FORMULA

Following Refs. 9 and 10, we shall replace the classical oscillator under study by two complex resistances connected in series, such that $Z(\omega) = Z_1(\omega) + Z_2(\omega)$ and, in particular, $R = R_1 + R_2$. The total power evolved in the circuit at the frequency ω due to thermal fluctuations is given by

$$P = R (I^2)_\omega = R \frac{(\mathscr{E}^2)_\omega}{|Z(\omega)|^2}. \tag{2.1}$$

Since the spectral density of the emf according to both the formula (1.14) and the formula (1.17) is proportional to $R = R_1 + R_2$, the power P is proportional to $(R_1 + R_2)^2$ and can therefore be represented as a sum of four contributions $P_1 + P_{12} + P_{21} + P_2$. The second law of thermodynamics implies the equality $P_{12} = P_{21}$. We write it in the form

$$\frac{R_2 (\mathscr{E}_1^2)_\omega}{|Z(\omega)|^2} = \frac{R_1 (\mathscr{E}_2^2)_\omega}{|Z(\omega)|^2}. \tag{2.2}$$

This implies that

$$\frac{(\mathscr{E}_1^2)_\omega}{R_1} = \frac{(\mathscr{E}_2^2)_\omega}{R_2}. \tag{2.3}$$

and therefore the ratio of the spectral density of the emf to the corresponding resistance is independent of the number of the subsystem (of the numbers 1 and 2). On this basis it is concluded in Refs. 9 and 10 that the ratio (2.3) is a universal function of the frequency and the temperature $f(\omega,T)$. In our notation $f(\omega,T) \equiv 2kT_\omega$. How, then, should the form of the function $f(\omega,T)$ (or kT_ω) be determined?

For this, in Refs. 9 and 10 the equality (1.15), relating the average kinetic energy $L\langle I^2\rangle$ with the as yet unknown function kT_ω, is employed. Since according to (2.3) $kT_\omega \equiv f(\omega,T)/2$ is a universal function of the frequency and temperature, to determine it, it is sufficient (?) to study the particular case of infinitely small damping ($R \to 0$), when the substitution (1.21) can be made in the integrand in (1.15). At the same time the function kT_ω can be removed from the integrand at the frequency $\omega = \omega_0$. Finally, employing the formula (1.20) for the average kinetic energy of an undamped quantum oscillator, we arrive at the conclusion that the function kT_ω sought at the frequency ω_0 is determined by the expression

$$\frac{1}{2} f(\omega_0, T) \equiv kT_{\omega_0} = \frac{1}{2}\hbar\omega_0 \operatorname{cth}\frac{\hbar\omega_0}{2kT} . \tag{2.4}$$

To find the form of the function $f(\omega,T)$ at an arbitrary frequency, the following step is used in Refs. 9 and 10: since $f(\omega,T)$ is a universal function, it can be reconstructed from the particular result (2.4), obtained in the approximation of an infinitely narrow resonance, by making the substitution $\omega \to \omega_0$. As a result we arrive at the following expression for the function sought:

$$\frac{1}{2} f(\omega, t) \equiv kT_\omega = \frac{1}{2}\hbar\omega \operatorname{cth}\frac{\hbar\omega}{2kT} , \tag{2.5}$$

and therefore at the quantum formula (1.14).

The final step made above, however, qualitatively alters the entire picture. Indeed, the function (2.4) depends only on the temperature and the parameters in the Hamiltonian L, C (or ω_0), while the function (2.5) depends on the running frequency of the spectrum ω and, therefore, $\langle I^2\rangle$ in (1.15) depends on R.

There is no basis for such a radical change. Moreover, the function (2.4) also satisfies the equality (2.3), since the frequency ω_0 refers to the entire circuit with the total impedance $Z(\omega)$. Thus preference cannot be given based on the equality (2.3) to the quantum generalization of Nyquist's formula (1.14) over the formula (1.17).

Another "derivation" of the quantum Nyquist formula in the form (1.14) is based on the Callen-Welton formula, which relates the spectral density of an arbitrary internal parameter X to the imaginary part of the corresponding susceptibility tensor (see, for example, Sec. 124 in Ref. 1):

$$(X_i X_j)_\omega = \hbar \operatorname{Im} \alpha_{ij}(\omega) \operatorname{cth}\frac{\hbar\omega}{2kT}. \tag{2.6}$$

We shall see in Sec. 4 that the derivation of Nyquist's formula (1.14) from the quantum Callen-Welton formula is not well-founded. Here we merely point out that the integral of (2.6) over the frequency determining the one-time correlation function $\langle x_i x_j\rangle$ in the state of thermal equilibrium, depends only on the temperature and the parameters in the Hamiltonian, but not on the dissipative parameters, for example, the resistance R in the calculation of the correlation functions $\langle I^2\rangle$ and $\langle q^2\rangle$.

This discrepancy between the consequences of the quantum Nyquist formula and the derivations of the statistical theory of the equilibrium state have long attracted investigators. Thus G. S. Gorelik in his paper[11] "Some applications of the second law of thermodynamics to electric fluctuations" showed for the example of an RC circuit that the mean-square fluctuations of the charge, i.e., the quantity $\langle q^2\rangle$, cannot depend in the equilibrium state on the resistance R. A year later V. L. Ginzburg published a paper[9] entitled "Some questions in the theory of electrical fluctuations." In it the author wrote: "The results presented (the fact that $\langle q^2\rangle$ is independent of R), obtained in Ref. 3 (in our case Ref. 11), are completely *paradoxical*, since they are purely classical and contradict the quantum Nyquist formula [here (1.14)], while it appears that in their derivation no classical assumptions were made." This viewpoint has not changed over the years (see Ref. 10).

Thus far attention has been devoted primarily to the oscillatory circuit. Let us examine some consequences to which the two quantum generalizations of Nyquist's formulas lead for dissipative, but not oscillatory systems. We shall begin with the LR circuit, when $C = \infty$ and, therefore, $\omega_0 = 0$. The expression for $\langle I^2\rangle$ is identical in this case with the classical expression [see (1.20)]. The situation changes completely when the quantum Nyquist formula (1.14) is used. Indeed, in this case

$$L(I^2)_\omega = \frac{2R/L}{\omega^2+(R/L)^2} kT_\omega, \quad L\langle I^2\rangle = L\int_{-\infty}^{\infty} (I^2)_\omega \frac{d\omega}{2\pi} . \tag{2.7}$$

The last formula implies that the average kinetic energy is not equal to kT. Moreover, it depends on the resistance R.

To emphasize even more strongly the difference between the results (2.7) and (1.20), we shall write down the corresponding formulas for separate charged particles. Consider a rectilinear cylindrical conductor of length l and with a transverse cross section S. Then the quantities L and R are defined as follows:

$$L = \frac{ml}{e^2 n S} , \quad R = \frac{ml}{e^2 n S}\nu, \text{ and } \frac{R}{L} = \nu; \tag{2.8}$$

where e and m are the electron charge and mass, n is the average electron density, and ν is the electron-phonon collision frequency. Let us go over from $L\langle I^2\rangle$ to the average kinetic energy of an electron. From (1.20) we have

$$\frac{m\langle v^2\rangle}{2} = \frac{1}{N}\frac{3}{2}kT; \tag{2.9}$$

N is the total number of particles in the sample. From (2.7) we obtain a different result:

$$\frac{m\langle v^2\rangle}{2} = \frac{1}{N}\frac{3}{2}\int_{-\infty}^{\infty}\frac{2\nu}{\omega^2+\nu^2} kT_\omega \frac{d\omega}{2\pi} \neq \frac{3}{2N}kT. \tag{2.10}$$

Here, as in (2.3), the contribution of zero-point oscillations can be separated out. The corresponding integral diverges logarithmically [compare with (1.19)]. Moreover, the average kinetic energy of the electron in the equilibrium state depends on the collision frequency ν.

To enlarge the class of "quantum Nyquist formulas" we shall study the motion of an electron in a thermal electromagnetic field. For this, it would seem natural to proceed as follows.

We define the spectral density of Langevin's force, equivalent to the action of a thermal electromagnetic field (taking into account the isotropy of the motion), as follows:

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$$\frac{1}{3} (f^2)_\omega = \frac{1}{3} e^2 (\delta E)^2_\omega \equiv \frac{4\pi^2}{3} e^2 \rho_\omega = \frac{4}{3} e^2 \frac{\omega^3}{c^3} kT_\omega. \quad (2.11)$$

Planck's formula is used here for the function ρ_ω.

We denote by $\gamma(\omega) = 2e^2\omega^2/3mc^3$ the coefficient of radiation friction. Then the expression for the spectral density of Langevin's force (per component) can be represented in the form of the quantum Nyquist formula (1.14) taking into account the frequency dependence of the coefficient of friction:

$$\frac{1}{3} (f^2)_\omega := 2m\gamma(\omega) kT_\omega. \quad (2.12)$$

In the dipole approximation (the condition $c/\omega \gg e^2/mc^2$, equivalent to the inequality $\gamma(\omega) \ll \omega$) Langevin's equation assumes the form

$$m \frac{dv}{dt} + m\hat{\gamma}v = f(t), \quad \text{отсюда} \quad (-i\omega + \gamma(\omega)) v_\omega = \frac{f_\omega}{m}. \quad (2.13)$$

The operator $\hat{\gamma}$ is the radiation friction operator. With its help we find an expression for the spectral density of the particle velocity

$$m (v^2)_\omega = \frac{1}{m} \frac{(f^2)_\omega}{\omega^2 + \gamma^2(\omega)} = \frac{2\gamma(\omega)}{\omega^2 + \gamma^2(\omega)} \cdot 3kT_\omega. \quad (2.14)$$

This result is unsatisfactory for the same reason as the formulas (2.7) and (2.10). Indeed, Eq. (2.14) gives for the average kinetic energy of the electron the expression [unlike (2.10) the factor $1/N$ does not occur here since there is only one particle]

$$\frac{m\langle v^2 \rangle}{2} = \frac{3}{2} \int_{-\infty}^{\infty} \frac{2\gamma(\omega)}{\omega^2 + \gamma^2(\omega)} kT_\omega \frac{d\omega}{2\pi}. \quad (2.15)$$

This implies that the equality

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT, \quad (2.16)$$

which is the condition for equilibrium between the noninteracting particles and the field, holds only in the zeroth-order approximation in the parameter $\hbar\gamma/kT$. We shall return to a discussion of this equation.

For the RC circuit the quantum Nyquist formula (1.14) leads to the equality

$$\frac{1}{C} \langle q^2 \rangle = \int_{-\infty}^{\infty} \frac{1/RC}{\omega^2 + (1/RC)^2} kT_\omega \frac{d\omega}{2\pi}, \quad (2.17)$$

and therefore the average potential energy depends not only on the temperature, but also on the resistance R.

Let us summarize the discussion up to this point.

The quantum Nyquist formula in the form (1.14) is attractive due to its "universality"—the existence of characteristic correlation times (1.18) which are independent of the parameters of the system. However, first of all, the "thermodynamic derivation" of this formula is not convincing and, second, the consequences of the formula (1.14) are inconsistent with the statistical theory of the equilibrium state.

The quantum Nyquist formula in the form (1.17) is not, of course, universal. The determination of the conditions under which it is applicable is part of the general problem of substantiating Langevin's equations for quantum systems. The foundation for this, as will be demonstrated using concrete examples, are the kinetic equations. We point out only that the formula (1.17) agrees with the equality (2.3),

expressing the second law of thermodynamics, and does not lead to unphysical consequences.

The relation between the quantum Nyquist formula and the Callen-Welton formula will be examined in Secs. 4–6. We shall see that there are no foundations for regarding the Nyquist formula as following from the Callen-Welton formula. Moreover, these formulas refer to two limiting cases: the Callen-Welton formula corresponds to the case of infinitely narrow resonances, when the "collision frequencies" approach zero, whereas the Nyquist formula (1.17) refers to white noise. This means that the correlation time of the noise is much shorter than all characteristic times in the problem. However, we shall first define more clearly the concepts of dissipative and nondissipative characteristics.

3. NONDISSIPATIVE AND DISSIPATIVE CHARACTERISTICS

It was asserted above that in the equilibrium state the one-time correlation functions are independent of the dissipative characteristics. Let us try to define more clearly the concepts of nondissipative and dissipative characteristics of an arbitrary equilibrium system. The possibility of introducing two different types of characteristics is a result of the dual role of the interaction of particles and fields in macroscopic systems.

A wide class of systems of particles is described by a Hamiltonian in which the potential energy is determined by the interaction potential for pairs of particles $\Phi(r)$. Then the thermodynamic functions—the average energy, the pressure, the one-time moments of additive functions of dynamic variables (particle density, charge, current, etc.)—are determined by the one-particle f_1 and two-particle f_2 distribution functions.

In the equilibrium state, in the absence of external fields, the function $f_1(r) = 1$, while f_2 and the corresponding correlation function g_2 depend only on the modulus of the distance between the particles. Correspondingly the spatial Fourier components are even functions of the wave vector, i.e., $f_2(k) = f_2(-k)$ and $g_2(k) = g_2(-k)$.

The functions $f_2(|r|)$ and $g_2(|r|)$ and the thermodynamic functions determined by them are nondissipative characteristics of the system. The contribution of interactions to them can be arbitrarily large, i.e., there is no general requirement that the interaction be small.

As an example we shall present the well-known expression for the average kinetic energy of a particle, interacting arbitrarily strongly with its environment. In the classical theory the average kinetic energy is independent of the interaction and is completely determined by the temperature. In the quantum theory, however, the average kinetic energy is determined by the equality (see Sec. 33 in Ref. 1 and Sec. 67 in Ref. 12)

$$\frac{m\langle v^2 \rangle}{2} = \frac{3}{2} kT + \frac{\hbar^2 n}{24mkT} \int \frac{\partial^2 \Phi}{\partial r^2} g_2(|r|) \, dr; \quad (3.1)$$

where n is the average density of atoms.

Thus in the quantum theory the interaction affects the magnitude of the average kinetic energy. This contribution, however, unlike the quantum contribution to the formula (2.15), which is a consequence of the quantum Nyquist formula (1.14), is determined by the *nondissipative* function $g_2(|r|)$.

In the nonequilibrium state the function $g_2(k,p,p',t)$,

which depends now on the particle momenta also, is complex. Its real part determines the nondissipative characteristics, while the imaginary part determines the dissipative characteristics (Sec. 58 in Ref. 12). The latter include the collision integrals, in terms of which the kinetic coefficients, in particular, the electrical resistance R, are expressed. The function $\mathrm{Im} g_2(k,p,p',t)$ determines the relaxation to equilibrium. In the equilibrium state, however, $\mathrm{Im} g_2 = 0$. As a consequence the collision integrals also vanish.

Naturally the two-time characteristics and the corresponding spectral densities depend on the function $\mathrm{Im} g_2$ in the equilibrium state also. In the transition to the one-time moments, however, the dependence on the dissipative characteristics is lost.

As another example we shall examine the FDR for a completely ionized Coulomb plasma [see, for example, (74.3) in Ref. 12]:

$$(\delta E \delta E)_{\omega, k} = \frac{8\pi}{\omega} \frac{\mathrm{Im}\, \varepsilon(\omega, k)}{|\varepsilon(\omega, k)|^2} \cdot \frac{1}{2} \hbar\omega \coth \frac{\hbar\omega}{2kT}. \quad (3.2)$$

The dielectric permittivity of the plasma

$$\varepsilon(\omega, k) = 1 + \sum_a \frac{4\pi e_a^2 n_a}{k^2} \frac{V}{(2\pi\hbar)^3}$$

$$\times \int dp \frac{f_a\left(p + \frac{1}{2}\hbar k\right) - f_a\left(p - \frac{1}{2}\hbar k\right)}{\hbar(\omega + i\Delta - kv)},$$

$$\Delta \to 0, \quad (3.3)$$

can be studied for complex frequencies also.

With the help of (3.2) and (3.3) we find by integrating over ω the spatial spectral density of fluctuations of the field [see (75.5) in Ref. 12]:

$$(\delta E \delta E)_k = 4\pi n k T \left[\left(1 - \frac{1}{\varepsilon(0, k)}\right) \right.$$

$$\left. + 2 \sum_{l=1}^{\infty} \mathrm{Re}\left(1 - \frac{1}{\varepsilon(i \cdot 2\pi l k T/\hbar, k)}\right)\right]. \quad (3.4)$$

We can see that the spectral density (3.4) is determined by the function $\varepsilon(0,k)$, which is identical to the real part of the function $\varepsilon(\omega,k)$ at zero frequency, and by the dielectric permittivity at the imaginary frequency $\varepsilon(i \cdot 2\pi l k T/k,k)$. The formula (3.3) implies that in the equilibrium state, when $f(p)$ is the Maxwell distribution, the function $\mathrm{Im}\,\varepsilon(i2\pi kT/\hbar,k) = 0$. As a result the spectral density (3.4) is expressed entirely in terms of the real part of the dielectric permittivity, i.e., in terms of the *nondissipative characteristic*.

4. THERMODYNAMIC FORM OF THE FDR. CALLEN-WELTON FORMULA

The Callen-Welton formula relates the spectral density of an arbitrary internal parameter x_i with the imaginary part of the susceptibility tensor α_{ij}. We impose below the restriction $\alpha_{ij} = \alpha_{ji}$. The spectral density is given by the formula [see Sec. 124 in Ref. 1 and (10.4.21) in Ref. 6]

$$(X_i X_j)_\omega = \pi \sum_{nm} (X_i)_{nm} (X_j)_{mn} \delta(\omega - \omega_{nm})(f_m + f_n). \quad (4.1)$$

The second expression, which is necessary in order to establish the Callen-Welton formula, determines the imaginary part of the tensor $\alpha_{ij}(\omega)$ [see Sec. 124 in Ref. 1 and (10.4.11) in Ref. 6]:

$$\mathrm{Im}\, \alpha_{ij}(\omega) = \frac{\pi}{\hbar} \sum_{nm} (X_i)_{nm} (X_j)_{mn} \delta(\omega - \omega_{nm})(f_m - f_n). \quad (4.2)$$

We shall employ an equality which holds for the Gibb's distribution:

$$\delta(\omega - \omega_{nm}) \frac{f_n + f_m}{f_m - f_n} = \delta(\omega - \omega_{nm}) \coth \frac{\hbar\omega}{2kT}. \quad (4.3)$$

The Callen-Welton formula (2.6) follows from (4.1)–(4.3).

Thus the possibility of going over from (4.1) and (4.2) to the Callen-Welton formula is linked with the presence of the functions $\delta(\omega - \omega_{nm})$, corresponding to the condition of an infinitely narrow resonance for every transition $n \to m$, in (4.1) and (4.2). We shall regard δ functions as limits of distributions with a resonance width Δ (for example, the function $(1/\pi)\Delta/[(\omega - \omega_{nm})^2 + \Delta^2]$ in the limit $\Delta \to 0$).

The transition to δ functions is justified in the presence of "broad" functions $(1/2)\hbar\omega_{nm} \coth(\hbar\omega_{nm}/kT)$, whose widths are characterized by two parameters $kT/\hbar \equiv \tau_\hbar^{-1}$ and ω_{nm}. Let us return to the question of the derivation of the quantum Nyquist formula (1.14). It is based on the following (see, for example Sec. 78 in Ref. 4). In the formula (2.6), which is regarded as universal, it is assumed that the function $\mathrm{Im}\,\alpha(\omega)$ is proportional to $R/|Z(\omega)|^2$ for arbitrary values of R. But, in this case, the equality (4.3) breaks down and therefore the expressions (4.1) and (4.2) no longer reduce to the Callen-Welton formula (2.6). We also note that the integral over the frequencies ω, calculated with the help of the Callen-Welton formula (2.6), yields the following expression for the one-time correlation function of the fluctuations of an arbitrary internal parameter x:

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} (X_i X_i)_\omega \frac{d\omega}{2\pi} = \frac{1}{2} \sum_{nm} |X_{nm}|^2 (f_m + f_n). \quad (4.4)$$

We can see that the one-time correlation function depends only on the temperature and the parameters in the Hamiltonian.

Thus the formula (2.6) is obtained by comparing the expressions (4.1) and (4.2), which *separately* determine the spectral density of the fluctuations, and the imaginary part of the corresponding susceptibility. It is obvious that dissipation already appears in the formula (4.1), since it cannot arise merely as a result of the algebraic transformation of the expression (4.1) (with the help of (4.2)) into the Callen-Welton formula (2.6).

The formula (4.1), leading to the Callen-Welton formula, is thereby itself an example of a fluctuation-dissipation relation. There then arises the question: what is the dissipative factor in the formula (4.1)? The answer to this question is given in the next section.

5. KINETIC FORM OF THE FLUCTUATION-DISSIPATION RELATIONS

To calculate the kinetic fluctuations (fluctuations in the distribution functions) it is necessary to have more general FDR, relating the spectral densities of the fluctuations of distribution functions with the imaginary parts of the corresponding susceptibilities. The most general relations of this type are FDR for many-body distribution functions, whose evolution to the equilibrium state is described by the corresponding kinetic equations.[13,14,15,7] These FDR, of course, are extremely complicated for direct application, but they are very convenient as a point of departure in going over

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to the description employing simpler distribution functions.

We note first of all that the spectral density of an arbitrary internal parameter, appearing, in particular, in the Callen-Welton formula, can be expressed in terms of the spectral density of the fluctuations of the many-body distribution function (density matrix in the quantum theory):

$$(X_i X_j)_\omega = \sum_{nm} \sum_{n_1 m_1} (X_i)_{nm} (X_j^*)_{n_1 m_1} (\delta f_{nm} \delta f_{n_1 m_1})_\omega. \quad (5.1)$$

Thus the problem can be reduced to the calculation of the spectral density of the fluctuations δf_N and δf_{nm}. This can be done as follows.[7,15]

In the classical theory for a system of N particles one can introduce two distribution functions in a $6N$-dimensional phase space X:

$$f_N^M (x, t) = \delta (x - X(t)), \quad f_N (x, t) = \overline{f_N^M (x, t)}; \quad (5.2)$$

$X(t)$ is a $6N$-dimensional vector of the coordinates and momenta of the constituent particles of the system.

In an incomplete description the first of the functions (5.2) is random. The second determines the average distribution with respect to the Gibbs ensemble. The one-time correlation function of the fluctions $\delta f_N = f_N^M - f_N$, i.e., the quantity

$$\langle \delta f_N (x, t) \delta f_N (x' t) \rangle = \delta (x - x') f_N (x, t) - f_N (x, t) f_N (x', t), \quad (5.3)$$

is a measure of the incompleteness of the statistical description.

In the quantum theory two corresponding density matrices are introduced. In the equilibrium state the spectral density of the many-body density matrix is defined by the expression[6,7,15]

$$(\delta f_{nm} \delta f_{n_1 m_1})_\omega = \pi \delta (\omega - \omega_{nm}) \delta_{nn_1} \delta_{mm_1} (f_m + f_n). \quad (5.4)$$

It can be rewritten in the form of an FDR:

$$(\delta f_{nm} \delta f_{n_1 m_1})_\omega = \hbar \operatorname{Im} A_{nmn_1 m_1} (\omega) \operatorname{cth} \frac{\hbar \omega}{2kT}, \quad (5.5)$$

if the following susceptibility is introduced:

$$A_{nmn_1 m_1}^1 (\omega) = -\frac{\delta_{nn_1} \delta_{mm_1} (f_m - f_n)}{\hbar (\omega - \omega_{nm} + i\Delta)}, \quad \Delta \to 0. \quad (5.6)$$

To go over from the FDR (5.5) to the Callen-Welton formula (2.6) the expression (5.5) must be substituted into (5.1) and summed over n_1, m_1. In so doing the imaginary part of the tensor $\operatorname{Im} \alpha_{ij} (\omega)$ is expressed in terms of (5.6):

$$\operatorname{Im} \alpha_{ij} (\omega) = \sum_{nm} \sum_{u_1 m_1} (X_i)_{nm} (X_j^*)_{n_1 m_1} \operatorname{Im} A_{nmn_1 m_1} (\omega). \quad (5.7)$$

We note two more consequences of the FDR (5.4). The integral over the frequencies in (5.4) [or (5.5)] leads to the following expression for the one-time correlation function:

$$\langle \delta f_{nm} (t) \delta f_{n_1 m_1}^* (t) \rangle = \frac{1}{2} \delta_{nn_1} \delta_{mm_1} (f_m + f_n) - \delta_{nm} \delta_{n_1 m_1} f_n f_{m_1}. \quad (5.8)$$

The second term on the right side ensures that the equalities $\Sigma_n \delta f_{nn} = 0$ and $\Sigma_{n_1} \delta f_{n_1 n_1} = 0$. The one-time correlation function (5.8) is independent of the dissipative characteristics.

The second result follows directly from (5.6): the real part of the susceptibility at zero frequency

$$\operatorname{Re} A_{nmn_1 m_1} (0) = \frac{\delta_{nn_1} \delta_{mm_1}}{\hbar \omega_{nm}} (f_m - f_n) \quad (5.9)$$

in the equilibrium state, when $f_m > f_n$ for $E_n > E_m$, is positive.

In the formulas (5.4)–(5.9) n and m are complete sets of quantum numbers of a system of N particles. Based on them it is possible to go over to the single-particle description. Then, for example, the form of the expressions (5.4) and (5.5) remains the same, but the expression for the susceptibility (5.6) is replaced by the following expression:

$$A_{nmn_1 m_1} (\omega) = -\frac{1}{N} \frac{\delta_{nn_1} \delta_{mm_1} (f_m - f_n)}{\hbar (\omega - \omega_{nm} + i\Delta)}, \quad \Delta \to 0. \quad (5.10)$$

Compared with the expression (5.6), here there is a factor of $1/N$, since the single-particle distribution functions are collective variables of additive type. In going over to the single-particle description n, m are sets of quantum numbers for separate particles.

An example is the FDR for a rarefied gas in the Boltzmann approximation

$$(\delta f_{p_1 p_1'} \delta f_{p_2 r_2'})_\omega = \hbar \operatorname{Im} A (\omega, p_1, p_1', p_2, p_2') \operatorname{cth} \frac{\hbar \omega}{2kT}. \quad (5.11)$$

The imaginary part of the corresponding susceptibility is determined by the formula

$$\operatorname{Im} A (\omega, p_1, p_1', p_2, p_2') = \frac{\pi}{N \hbar} \frac{(2\pi \hbar)^6}{V^2} \delta (p_1 - p_2)$$
$$\times \delta (p_1' - p_2') \delta \left(\omega - \frac{1}{\hbar} (E_{p_1} - E_{p_1'}) \right)$$
$$\times (f_1 (p_1') - f_1 (p_1)). \quad (5.12)$$

We shall return, finally, to the question of the conditions of admissibility of the approximation of infinitely narrow resonances ($\Delta \to 0$), which is employed not only in the derivation of the Callen-Welton formula (2.6), but also in the derivation of the FDR (5.5) for fluctuations of many-body distribution functions (5.5).

The canonical Gibbs distribution f_n is established owing to the interaction of the system under study with a thermostat. We denote the corresponding relaxation time as $\tau_{\rm rel}$. The Liouville equation does not describe this of establishing the Gibbs distribution process. It is reversible, since its solution is determined by the solution of the equations of dynamics, and the spread in the initial conditions—the initial distribution—merely moves about the phase space with the volume remaining constant (Liouville's theorem). For the same reason the entropy of the system also remains constant as a function of time.

The description of the relaxation process at the level of many-body distribution functions requires the corresponding kinetic equations, which take into account the incompleteness of the description of the system under study. An equation of this type was first derived by M. A. Leontovich for a Boltzmann gas.[13] The incompleteness of the description in this case is actually attributable to the use (explicit or implicit) of the condition that the initial small-scale (with correlation times much shorter than the mean free-flight time) correlations become weaker.[7,12,14,18] Other forms of the kinetic equations for many-body distribution functions are also available.[6,7,14] Here the canonical Gibbs distribution is determined from the condition that the corresponding collision integral I_n vanishes.

The time $\tau_{\rm rel}$ characterizes the relaxation to the equilibrium state. Once equilibrium is achieved the canonical distribution, naturally, is independent of $\tau_{\rm rel}$. For the fluctu-

ations δf_N and δf_{nm}, however, this is no longer so.

Indeed, it is possible to separate small-scale fluctuations, for which $\tau_{cor} \ll \tau_{rel}$, and large-scale fluctuations, for which $\tau_{cor} \gtrsim \tau_{rel}$. The collision integrals in the kinetic equations are determined by the small-scale fluctuations, since the condition that the starting correlations become weaker can be employed only for them. Large-scale (kinetic) fluctuations, however, are calculated based on the kinetic equations themselves.[5-7,12,16]

The finiteness of the relaxation time τ_{rel} determines a lower limit for Δ. Employing the inequalities introduced in Sec. 4 we arrive at the double inequality

$$\max \left(\frac{kT}{\hbar}, \omega_{nm} \right) \gg \Delta \gg \frac{1}{\tau_{rel}}. \tag{5.13}$$

The inequality on the right indicates that only the small-scale fluctuations are included, while the inequality on the left makes it possible to go over to the "infinitely narrow resonance approximation," i.e., to introduce the function $\delta(\omega - \omega_{nm})$. Since the condition $\tau_{cor} \ll \tau_{rel}$ separates a region of fluctuations for whose calculation "collisions" do not play an important role, in this sense the fluctuation-dissipation relations presented in Secs. 4 and 5 refer to the "collisionless region." The term "collisionless approximation" is widely employed in plasma theory.[7,12,19] In the case of a gas this corresponds to the approximation of free-molecular flow (see Ref. 7, Sec. 7 of Chap. 9).

The fluctuation-dissipation relations for the one- and two-particle distribution functions are of greatest practical value. In particular, according to the formula (5.1) the spectral density of the additive internal parameters (the current and charge in the case of an electrical circuit) are related to the spectral density of the fluctuations by the one-particle distribution function.

The inequalities (5.13) remain in force in going over to the description based on the one-particle distribution functions. Here ω_{nm} are the transition frequencies for an individual atom taking into account the renormalization owing to the effect of the medium, and τ_{rel} is the relaxation time determined by the collision integrals in the corresponding kinetic equations for the one-particle distribution functions.

In the classical description the lefthand inequality of (5.13) always holds. The Callen-Welton formula in this case has the form

$$(X_i X_j)_\omega = 2 \frac{\operatorname{Im} \alpha_{ij}(\omega)}{\omega} kT \tag{5.14}$$

and holds for an arbitrary width of the resonance.

Let us return to the question posed at the end of the preceding section. We can now give an answer to it. Indeed, the dissipative factor in the FDR (5.14) for the spectral density of an N-particle distribution function and therefore in the formulas (4.1) and (2.6) also is the function $\delta(\omega - \omega_{nm})$. Its "width" is characterized by the times of the small-scale—"collisionless"—fluctuations, which, in turn, determine the "collision integrals" in the corresponding kinetic equations.

If the function $\delta(\omega - \omega_{nm})$ in (5.4) and (4.1) is replaced, for example, by a Lorentz line of width Δ, then the result of the calculation will depend substantially on the stage at which the passage to the limit $\Delta \to 0$ is performed. In deriving the FDR the limit was taken in the final expressions. If, however, the quantity Δ is set equal identically to

zero in the starting equations, then they will become reversible and it will no longer be possible to derive the FDR.

The parameter Δ separates the region of small-scale—"collisionless"—fluctuations. One of the conditions for deriving the FDR is the *condition of partial weakening of the starting correlations*. Another factor is also important: *the one-time correlation function, which is a measure of the incompleteness of the statistical description, must differ from zero.*[7,12,15] Thus the question of the formulation of FDR is closely linked with the problem of irreversibility.[43,44]

6. FDR FOR FLUCTUATIONS WITH FINITE CORRELATION TIMES

The separation of fluctuations into small-scale—"collisionless"—and large-scale [the choice of the parameter Δ satisfying the inequalities (5.13)] is linked with the introduction of infinitely small scales of time and length.[7,17,12]

We begin with the formulation of FDR for fluctuations of many-body distribution functions (density matrices), but this time for collision region. We denote by δI_{nm} the operator corresponding to the linearized collision integral. Then the expression for the spectral density of the fluctuations will be determined by the expression

$$(\delta f_{nm} \delta f_{n_1 m_1})_\omega = \operatorname{Re} \frac{1}{\omega - \omega_{nm} + i\delta I_{nm}} \delta_{nn_1} \delta_{mm_1} (f_m + f_n). \tag{6.1}$$

It goes over into (5.4) under the substitutions $\delta I_{nm} \to \Delta$ and $\Delta \to 0$.

The spectral density (6.1) can be determined by two methods: by solving the equation for the two-time correlation function with the "initial" (at $t = t'$) condition (5.8) [in the classical theory—with the condition (5.3)] or by solving the corresponding Langevin equation

$$\left(\frac{\partial}{\partial t} + i\omega_{nm} + \delta I_{nm} \right) f_{nm} = y_{nm}, \quad \langle y_{nm} \rangle = 0. \tag{6.2}$$

The spectral density of the Langevin source equals

$$(y_{nm} y_{n_1 m_1})_\omega = \frac{1}{2} (\delta I_{nm} + \delta I_{n_1 m_1}) \delta_{nn_1} \delta_{mm_1} (f_m + f_n). \tag{6.3}$$

The expression (6.1) can be rewritten in the form of an FDR:

$$(\delta f_{nm} \delta f_{n_1 m_1})_\omega = \hbar \operatorname{Im} A_{nmn_1 m_1}(\omega) \frac{f_m + f_n}{f_m - f_n}, \tag{6.4}$$

by introducing the corresponding susceptibility [compare with (5.10)]

$$A_{nmn_1 m_1}(\omega) = - \frac{1}{\hbar(\omega - \omega_{nm} + i\delta I_{nm})} \delta_{nn_1} \delta_{mm_1} (f_m - f_n). \tag{6.5}$$

Because of the finiteness of the width of the resonance there is now no basis for employing the equality (5.3) in order to reduce the FDR (6.4) to the form (5.5) and correspondingly to the formula (2.6).

It has already been pointed out that the classical expression (5.14) remains valid also for a finite correlation time of the fluctuations (for a finite width of the resonance). The proof for a plasma is given, for example, in Sec. 64 of Ref. 12. It can also be taken over to the case of arbitrary internal parameters.

We shall examine some examples of FDR for fluctuations with a finite correlation time.

6.1. FDR for a Boltzmann gas[16,5,11]

In calculating the fluctuations of the one-particle density matrix in a Boltzmann gas the operator δI_{nm} is replaced by the operator δI_p, which is determined by the structure of the linearized quantum Boltzmann collision integral. The spectral density of the fluctuations $\delta f_{pp'}$ is determined by the expression

$$(\delta f_{p_1 p_1'}\delta f_{p_2 p_2'})_\omega = \frac{1}{N}\, \mathrm{Re}\; \frac{t}{\omega - (E_{p_1} - E_{p_1'})/\hbar + i\delta I_p}$$

$$\times \frac{(2\pi\hbar)^6}{V^2}\, \delta\,(p_1 - p_2)\, \delta\,(p_1' - p_2')\,(f\,(p_1) + f\,(p_1')). \tag{6.6}$$

It can be represented in the form of an FDR:

$$(\delta f_{p_1 p_1'}\delta f_{p_2 p_2})_\omega = \hbar\, \mathrm{Im}\, A\,(p_1,\ p_1',\ p_2,\ p_2',\ \omega)\frac{f\,(p_1') + f\,(p_1)}{f\,(p_1') - f\,(p_1)}, \tag{6.7}$$

by introducing the susceptibility [compare with (5.12)]

$$A\,(p_1,\ p_1,\ p_.,\ p_2',\ \omega) = -\frac{1}{N\hbar}\;\frac{1}{\omega - [(E_{p_1} - E_{p_1'})/\hbar] + i\delta T_p}$$

$$\times \frac{(2\pi\hbar)^6}{V^2}\, \delta\,(p_1 - p_2)\, \delta\,(p_1' - p_2')$$

$$\times (f\,(p_1') - f\,(p_1)). \tag{6.8}$$

It is now no longer possible to write the FDR (6.7) in the form (5.11) because of the finiteness of the width of the resonance.

The FDR (6.7), unlike (5.11), can also be employed for nonequilibrium states. Based on it, a fluctuation representation of the Boltzmann collision integral can be constructed.[7,17] The fluctuation representation of the collision integrals is usually employed only within the framework of perturbation theory or the polarization approximation.[6,7,19]

6.2. Planck's formula

In calculating the distribution of thermal radiation in a cavity the finiteness of the correlation time of the fluctuations is determined by the dissipative boundary conditions. We shall take this into account by introducing the effective conductivity. $\gamma = 4\pi\sigma$ is the corresponding decay constant. We shall ignore here the dependence of γ on the frequency and wave number. The classical expression for the spatial-temporal spectral density of the field fluctuations has the form [(3.12), Chap. 14 of Ref. 7]

$$\frac{1}{4\pi}\,(\delta E\delta E)_{\omega,\,k} = \frac{4\gamma\omega^2}{(\omega^2 - \omega_k^2)^2 + \gamma^2\omega^2}\, kT,\quad \omega_k = ck. \tag{6.9}$$

In the quantum case, following Planck, we replace kT by the average energy of the k th oscillator with frequency ω_k [compare with (1.16)]:

$$kT \to kT_{\omega_k} = \frac{\hbar\omega_k}{2}\, \mathrm{cth}\, \frac{\hbar\omega_k}{2kT}. \tag{6.10}$$

The formulas (6.9) and (6.10) imply that in the quantum case the one-time correlation function is also independent of γ. This is natural, since the correlation function $(\delta E)_k^2$ is a nondissipative characteristic (see Sec. 3).

The transition from (6.9) and (6.10) to Planck's formula can be made only in the zeroth-order approximation in the parameter $\hbar\gamma/kT$ (exact resonance), when the substitution

$$\frac{\gamma\omega^2}{(\omega^2 - \omega_k^2)^2 + \gamma^2\omega^2} \to \pi\,|\omega|\,\delta\,(\omega^2 - \omega_k^2) \tag{6.11}$$

can be made [compare with (1.21)]. Under this condition integration over k yields Planck's formula.

If we do not follow Planck and instead of the substitution (6.10) take, like in (1.13), the limit $kT \to kT_\omega$, then the one-time correlation function $(\delta E)_k^2$ will depend on γ and we shall arrive at the same difficulties as for the Nyquist formula (1.14).

So, there exist two types of fluctuation-dissipation relations. The first type includes the Callen-Welton formula (2.6), the FDR (5.5), and the more specific FDR for fluctuations of one-particle distribution functions, for example, (5.11) for the Boltzmann gas. In all these cases the inequalities (5.13) hold. They make it possible to take into account only the small-scale fluctuations and to use the resonance conditions for which the inequality (4.3) holds. It is specified in concrete cases.

The fluctuation-dissipation relations of the second kind occur for long-lived correlations, for which the correlation times τ_{cor} are comparable to τ_{rel}. These could be fluctuations of the distribution functions in the kinetic theory, fluctuations of local hydrodynamic and thermodynamic functions in hydrodynamics, and fluctuations of the current and charge in electrical circuits. These FDR also include Nyquist's formula. In addition, the quantum variants of Nyquist's formula (1.14) and (1.17) are not equivalent. We shall see which of the two expressions does the statistical theory yield. We shall do this using the example of a system of atoms interacting with an electromagnetic field. The latter will play here the role of a thermostat.

7. FDR FOR A SYSTEM OF TWO-LEVEL ATOMS

The kinetic equation—the equation of balance of the populations of the energy levels of molecules with two quantum states—was first proposed in 1916 in A. Einstein's famous paper.

Let us examine the collision integral taking into account all transitions between discrete levels [see (2.1), Chap. 19 in Ref. 7]:

$$I_n = \sum_m \left[B_m^n \frac{1}{4\pi^2}\,(\delta E\delta E)_{\omega_{nm}}\,(f_m - f_n) - \frac{1}{2}\,A_m^n\,(f_m + f_n)\right]; \tag{7.1}$$

where B_m^n and A_m^n are Einstein coefficients. The spectral density of the field fluctuations in the equilibrium state

$$(\delta E\delta E)_{\omega_{nm}} = 4\pi^2\,\frac{\omega_{nm}^3}{\pi^2 c^3}\, kT_{\omega_{nm}} \tag{7.2}$$

differs only by a factor of $4\pi^2$ from Planck's function $\rho_{\omega_{nm}}$. The derivation of the kinetic equation for the function f_n employs the inequalities (5.13). This gives a basis for separating in the collision integral (7.1) the resonance contributions at the frequencies $\omega = \omega_{nm}$.

The calculation of the kinetic fluctuations for a system of two-level atoms[6,7] yields the following expression for the spatial-temporal spectral density of the current fluctuations:

$$(\delta I\delta I)_{\omega,\,k} = \frac{2\gamma_{nm}}{\gamma_{nm}^2 + [\omega - (\omega_{nm}^2/\omega)]^2}\,\frac{N}{V}\,\omega_{nm}^2\,|d_{nm}|^2\,(f_m + f_n). \tag{7.3}$$

It can be written in the form of an FDR:

$$(\delta I\delta I)_{\omega,k} = 3\cdot 2\,\mathrm{Re}\,\sigma\,(\omega)\,kT_{\omega_{nm}}, \tag{7.4}$$

by introducing the conductivity $\sigma(\omega)$.

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Let us compare the FDR (7.4) with the quantum generalizations (1.14) and (1.17) of the classical Nyquist formula. The quantities δI and σ in (7.4) correspond to the emf $\mathscr{E}$ and the resistance R, while the transition frequency ω_{nm} corresponds to the characteristic frequency of the circuit ω_0. We can see that the FDR (7.4) has the structure of (1.17) and not that of (1.14).

The two-level atom approximation corresponds to one of the two limiting cases. The other corresponds to the model of quantum atomic oscillators. It corresponds more directly to the model of an oscillatory circuit.

8. SYSTEM OF QUANTUM ATOMIC OSCILLATORS. DIFFUSION COEFFICIENT. QUANTUM NYQUIST FORMULA

Let us return to the kinetic equation with the collision integral (7.1). We shall employ it for the system of one-dimensional quantum atomic oscillators with the characteristic frequency ω_0. With the help of (7.1) we find

$$I_n = \gamma \left\{ \operatorname{cth}\left(\frac{\hbar\omega_0}{2kT}\right) \left[\frac{n+1}{2}(f_{n+1}-f_n) + \frac{n}{2}(f_{n-1}-f_n) \right] \right.$$
$$\left. - \left[-\frac{n+1}{2}(f_{n+1}+f_n) + \frac{n}{2}(f_{n-1}+f_n) \right] \right\} ; \quad (8.1)$$

here we introduce the following notation for the coefficient of radiation friction

$$\gamma = \frac{2e^2\omega_0^2}{3mc^3} . \quad (8.2)$$

Using the expression (8.1), we shall write the equation for the average energy as follows

$$\langle E \rangle = \sum_n \left(n + \frac{1}{2} \right) \hbar\omega_0 f_n(t). \quad (8.3)$$

Here there are two convenient forms of this equation:

$$\frac{d\langle E \rangle}{dt} = \gamma(\omega_0)(kT_{\omega_0} - \langle E \rangle) \equiv D - \gamma(\omega_0)\langle E \rangle. \quad (8.4)$$

In the second case the expression for the quantum diffusion coefficient

$$D = \gamma(\omega_0)kT_{\omega_0} \quad (8.5)$$

is used. It can be regarded as the quantum generalization of Einstein's formula for the diffusion coefficient in the classical Fokker-Planck equation (1.9). We shall study the relationship of this equation to the quantum equation for the function f_n with the collision integral (8.1).

Namely, we shall replace the equation for the function $f_n(t)$ by the effective Fokker-Planck equation for the energy distribution function $f(E,t)$. We write this equation in the form

$$\frac{\partial f(E,t)}{\partial t} = D\frac{\partial}{\partial E}\left(E\frac{\partial f}{\partial E} \right) + \frac{\partial}{\partial E}(\gamma F f), \quad \int_0^\infty f(E,t)\,dE = 1, \quad (8.6)$$

and we determine the diffusion coefficient from the condition that the equation for the average energy

$$\langle E \rangle = \int_0^\infty Ef\,dE$$

coincide with the exact equation (8.4). As a result we obtain the expression (8.5) for the diffusion coefficient. Finally, we can go over from (8.6) to the Fokker-Planck equation for the current and charge distribution function $f(q,I,t)$. Making the substitution

$$E \to \frac{LI^2}{2} + \frac{q^2}{2C} , \qquad \gamma \to \frac{R}{L} , \quad D \to \frac{D}{L} , \quad (8.7)$$

we arrive at the Fokker-Planck equation (1.9) with the quantum diffusion coefficient

$$D = RkT_{\omega_0} \quad (8.8)$$

and, as a consequence, to the quantum Nyquist formula (1.17) for the spectral density of the emf in the corresponding Langevin equation. The conditions for the applicability of the Fokker-Planck equation (1.9) with the quantum diffusion coefficient are discussed, for example, in Ref. 30 in an analysis of the quantum properties of high-Q macroscopic resonators.

9. QUANTUM NYQUIST FORMULA FOR A PLASMA CLASSICAL OSCILLATOR

In Secs. 7 and 8 the characteristic frequencies ω_{ab} and ω_0 are characteristic of individual atoms. Let us examine an example in which the characteristic frequency is a characteristic of the system as a whole. The plasma classical oscillator with a Langmuir frequency for the plasma electrons ω_e will serve as an example.

Langevin's equations for the charged particles of the plasma are very complicated owing to the nonlinearity of the kinetic equations. It is possible, however, to derive a model Fokker-Planck equation for the distribution function of collective variables also taking into account the quantum effects. We proceed as follows.

We define the coefficient of friction in terms of the effective collision frequency ν, and the diffusion coefficient by the formula

$$D = m\nu kT_{\text{eff}} . \quad (9.1)$$

Here T_{eff} is the effective temperature,

$$kT_{\text{eff}} = \frac{\int d\omega\,dk\,\delta(\omega-kv)\,((kv)^2/k^2)\,(\delta E\delta E)_{\omega,k}}{\int d\omega\,dk\,\delta(\omega-kv)\,(kv/k^2)\,(\operatorname{Im}\varepsilon(\omega,k)/|\varepsilon(\omega,k)|^2)} . \quad (9.2)$$

The spectral density of the fluctuations of the field and the dielectric permittivity taking quantum effects into account, are determined by the well-known formulas (3.2) and (3.3). The expressions (3.2) and (9.2) imply that in the classical limit, when $\hbar \to 0$, the effective (quantum) temperature is $T_{\text{eff}} = T$. The formula (9.2) can be regarded as a consequence of Einstein's relation $D_{ij}v_j = A_i$ for a plasma.[6,7,19]

Langevin's equation for a plasma oscillator with a characteristic frequency $\omega_e = (4\pi e^2 n/m)^{1/2}$ is identical in form to (1.4). Here the spectral density of the random emf is determined by the expression ($T_{\text{eff}} = T_{\omega_e}$)

$$(\mathscr{E}^2)_\omega = 2RkT_{\omega_e} \equiv 2R\frac{1}{2}\hbar\omega_e\operatorname{cth}\frac{\hbar\omega_e}{2kT} . \quad (9.3)$$

It corresponds to the quantum Nyquist formula (1.17), in which the characteristic frequency ω_0 is determined by the characteristic frequency of the electron oscillations in the plasma.

10. NYQUIST FORMULA FOR A CIRCUIT WITH NONLINEAR FRICTION

The calculation of the fluctuations in nonlinear systems has been of interest for a long time.[21-28] For a circuit with a nonlinear resistance the problem reduces to the solution of the Fokker-Planck equation with higher order derivatives.[22] Thus, for example, when the resistance is a quadratic func-

tion of the current, terms with third and fourth order derivatives appear in the equation for the distribution function $f(I,q,t)$. At the same time, however, the theory does not determine uniquely the coefficients in front of the higher-order derivatives. In these coefficients a constant which is unrelated to the dissipative characteristics of the circuit remains undetermined. For a quadratic nonlinearity, by suitably choosing this constant, the coefficients in front of the third and fourth derivatives can be made to vanish and the equation for the function $f(I,q,t)$ can thereby be reduced to the standard Fokker-Planck equation.

So, the Fokker-Planck equation can be employed for calculating the fluctuations in a circuit with a nonlinear resistance under the condition that the function $R(I)$ is even (Sec. 9 of Chap. 4 in Ref. 6). We shall employ this possibility to extend Einstein's and Nyquist's formulas to the case of a nonlinear circuit.

We shall write Langevin's equations for the current and charge in the form [compare with (1.3)]

$$\frac{dq}{dt} = I, \quad L\frac{dI}{dt} + R(I)I + \frac{1}{c}q - \frac{1}{2}\frac{dD(I)}{dI}$$
$$= D^{1/2}(I)\,y(t) \equiv \mathscr{E}(I,\,t); \quad (10.1)$$

$R(I)$ is an even function of the current. For a quadratic nonlinearity

$$R(I) = R + \frac{R_1 LI^2}{kT}. \quad (10.2)$$

R_1 is the nonlinear resistance in units of LI^2/kT. The random emf, equivalent to the action on the circuit of randomly moving charges, now depends not only on the time, but also on the current strength. It is represented in the form $D^{1/2}(I)y(t)$, where $D(I)$ is an as yet unknown intensity of the noise. $y(t)$ is a random source, determined by the two first moments:

$$\langle y(t)\rangle = 0, \quad \langle y(t)\,y(t')\rangle = 2\delta\,(t-t'). \quad (10.3)$$

The additional term $-(1/2)dD/dI$ on the left side of Eq. (10.1) for I is introduced in order to simplify the Fokker-Planck equation and put it into a form corresponding to the general structure of kinetic equations. It has the following form:

$$\frac{\partial f}{\partial t} + I\frac{\partial f}{\partial q} - \omega_0^2 q\frac{\partial f}{\partial q} = \frac{1}{L^2}\frac{\partial}{\partial I}\left(D(I)\frac{\partial f}{\partial I}\right) + \frac{\partial}{\partial I}\left(\frac{R(I)}{L}If\right). \quad (10.4)$$

The function $D(I)$ is found from the condition that the solution of Eq. (9.4) in the equilibrium state should also have the form (1.10). As a result we arrive at the following expression for the diffusion coefficient, taking into account both the quantum effects and the nonlinear dependence of the resistance on the current:

$$D(I) = R(I)\frac{1}{2}\hbar\omega_0\coth\frac{\hbar\omega_0}{2kT} \equiv R(I)\,kT_{\omega_0}. \quad (10.5)$$

As for linear circuits, the one-time correlation functions of the fluctuations of the current and charge $\langle I^2\rangle$ and $\langle q^2\rangle$ in the equilibrium state depend only on the temperature and nondissipative parameters L and C.

In the particular case of an LR circuit ($1/C\to0$, $\omega_0\to0$), which corresponds to the motion of a free Brownian particle in a medium with nonlinear friction, the quantum effects become insignificant and the formula (9.4) assumes the form

$$D(I) = R(I)\,kT. \quad (10.6)$$

The transition from (10.5) to (10.6) corresponds to the zeroth-order approximation in the parameter $\hbar\omega_0/kT$. Another limiting case, when the parameter $\hbar\omega_0/kT$ is large and therefore the contribution of zero-point oscillations predominates is interesting. This situation occurs in lasers.

11. POWER DEPENDENCE OF THE SPECTRAL WIDTH OF OSCILLATIONS OF CLASSICAL AND QUANTUM OSCILLATORS

In a classical oscillator of the Thompson type the source of noise is not only the thermal motion of charges in the circuit, but also the shot noise of the anodic current.[26-28] In quantum oscillators shot noise is caused by the discreteness (atomic structure) of the working medium.[6,7] It can be taken into account by introducing random sources of polarization into the field equations. Here we shall examine the effect of the thermal noise (taking into account quantum effects and dissipative nonlinearity) on the spectrum of oscillations in classical and quantum oscillators.

The spectral line of an oscillator has a structure which is characterized by at least two parameters: the width of the spectrum of fluctuations of the amplitude and the diffusion coefficient of the phase. So as not to complicate the analysis and reveal as simply as possible the role of quantum effects and dissipative nonlinearity, we shall employ an approximate description of the spectrum by introducing an effective Lorentz line. The line width in this case is determined by the equations for the energy in the first-moment approximation (Chaps. 4 and 11 in Ref. 6, Chaps. 12 and 22 in Ref. 7).

The Fokker-Planck equation for the distribution function of the energy of the oscillations in the oscillator $f(E,t)$, taking into account the energy dependence of the diffusion coefficient, can be written in the form [compare with (4.10.3) in Ref. 6 and (1.8), Chap. 12 in Ref. 7]

$$\frac{\partial f(E,\,t)}{\partial t} = \frac{\partial}{\partial E}\left(D(E)E\frac{\partial f}{\partial E}\right) + \frac{\partial}{\partial E}[(a+bE)Ef]. \quad (11.1)$$

The diffusion coefficient is defined by the expression

$$D(E) = (\gamma + bE)\cdot\frac{1}{2}\hbar\omega_0\coth\frac{\hbar\omega_0}{2kT}. \quad (11.2)$$

Compared with (8.5) here we introduce the nonlinear friction ($\gamma \to \gamma + bE$) with the nonlinearity coefficient b, which corresponds to R_1 in (10.2). The quantity a is given by

$$a = \gamma - a_0; \quad (11.3)$$

a_0 characterizes the feedback. At the generation threshold $a_0 = \gamma$ and $a = 0$. In the absence of feedback ($a_0 = 0$, $a = \gamma$) Eqs. (11.1) and (11.2) imply that the equilibrium solution of Eq. (11.1) is identical to the equilibrium energy distribution for a quantum oscillator with the characteristic frequency ω_0.

With the help of Eq. (11.1) the following expression is obtained for the distribution of the average energy over the spectrum[6,7]

$$E(\omega) = \frac{2\lambda}{\omega^2+\lambda^2}\langle E\rangle, \quad \int_{-\infty}^{\infty}E(\omega)\frac{d\omega}{2\pi} = \langle E\rangle. \quad (11.4)$$

Here the half-width of the spectrum λ is determined by the formula

$$\lambda = \frac{D[1+(b/\gamma)\langle E\rangle]}{2\langle E\rangle}, \quad (11.5)$$

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where

$$D = \gamma \cdot \frac{1}{2} \hbar\omega_0 \coth\frac{\hbar\omega_0}{2kT} \equiv \gamma kT_{\omega_0} \qquad (11.6)$$

is the quantum diffusion coefficient of a linear system [identical to (8.5)].

In the equilibrium state the average energy $\langle E \rangle = kT_{\omega_0}$ and

$$\lambda = \frac{1}{2}(\gamma + bkT_{\omega_0}). \qquad (11.7)$$

In the regime with developed oscillations, but with a quite small nonlinearity $(b\langle E\rangle/\gamma \ll 1)$ the quantity λ is determined by the expression

$$\lambda = \frac{D}{2\langle E\rangle} \equiv \frac{\gamma kT_{\omega_0}}{2\langle E\rangle} \qquad (11.8)$$

and decreases as the intensity of the oscillations increases. In particular, in the quantum limit $(\hbar\omega_0 \gg kT)$ Eq. (11.8) implies the well-known formula of Schawlow-Townes for the width of the spectrum (see Refs. 23 and 32, Chap. 11 in Ref. 6). (A comparison with the formulas of Haken and Lax is also given there.) The formula (11.5) implies that as the energy of the oscillations increases further saturation sets in (the line width $\lambda \to Db/2\gamma$).

12. QUANTUM NYQUIST FORMULA AND THE LAMB SHIFT

Interest in the theory of Brownian motion in quantum systems has increased strongly in recent years (see, for example, Refs. 29–31). In Ref. 31 the quantum Nyquist formula (1.14) is used to explain the results of experimental studies of voltage fluctuations in Josephson junctions. These data are apparently still inadequate to make an unequivocal choice between the two expressions (1.14) and (1.17).

We shall show that the Lamb shift—the shift in the levels of the hydrogen atom owing to the effect of vacuum (in the approximation $T = 0$) fluctuations of the electromagnetic field—is a "judge." The Lamb shift was first discovered in Ref. 33.

There are two methods for calculating the Lamb shift. One method is based on the "subtraction formalism" of quantum electrodynamics. The first calculation was carried out by Bethe[34] (see Refs. 35–38). The second method, which at first glance appears to be less formal, was proposed by Welton—one of the authors of the Callen-Welton formula—in Ref. 39. This method is presented in textbooks.[40,41]

We shall show that these two methods of calculation are actually based on two different quantum Nyquist formulas: the formula (1.14) in Welton's method and the formula (1.17) in the "subtraction formalism" of quantum electrodynamics. Both methods, however, yield the same Bethe formula for the shift of the levels. What is going on? We shall see that Welton's method is not systematic and preference must be given to the "subtraction formalism" and therefore the quantum Nyquist formula in the form (1.17).

Welton's method is based on the use of Langevin's equation (2.13) to calculate the fluctuations in the motion of the electron in the ground state of the hydrogen atom. The agreement of the result with Bethe's formula turns out to be accidental here, since in the determination of the energy shift only the change in the potential energy owing to the fluctuation displacement of the electron is taken into account. Inclusion of the velocity fluctuations, determining the change

in the kinetic energy of the electron, however, substantially changes the entire picture.

Indeed, consider the formula (2.15), determining the fluctuation contribution of the kinetic energy. At $T = 0$ the integral over ω diverges as ω^2. Since the existence of fluctuations of the electron-positron vacuum is not taken into account, we cut-off the region of the integration by the condition $\omega < \omega_{max} = mc^2/\hbar$. Employing the notation $\gamma(\omega)$ for the coefficient of radiation friction at the frequency ω, $\alpha = e^2/\hbar c$ for the fine-structure constant, and $E_0 = me^4/2\hbar^2$ for the ground state energy of the hydrogen atom (or the ionization energy), we arrive at the result

$$\frac{1}{3}m\langle v^2\rangle = \frac{1}{2\pi}\hbar\gamma(\omega_{max}) \sim \frac{1}{\alpha}|E_0| \gg |E_0|. \qquad (12.1)$$

This result is unsatisfactory in two respects. First, the average kinetic energy at $T = 0$ does not equal zero and depends on the dissipative characteristic $\gamma(\omega_{max})$. Second, there is an enormous difference between the temperature of the particle $(1/3)m\langle v^2\rangle$ and the temperature of the field (here $T = 0$). This is obviously inconsistent with the experimental data.

Both deficiencies of the calculation by Welton's method vanish when the appropriate kinetic equation for the electron velocity distribution function $f(p,t)$ is used to find the average kinetic energy of the electron in the presence of interactions with the fluctuation field.[6,42] In the state of equilibrium $f(p)$ is the Maxwell distribution, so that the equality (2.16) also holds in quantum theory, i.e., there is no difference between the temperatures of the electrons and of the field. It is important that the components of the spectral density $(\delta E\delta E)_{\omega,k}$ for only those values of ω and k for which the laws of conservation of energy and momentum of the particles and of the field quanta play a role in the collision integral in establishing the equilibrium distribution. These conditions do not hold for the Langevin equations (2.13) and (2.12). This is one of the reasons that Welton's method is unsystematic. The second reason is that the classical model of the motion of an electron in an atom, which forms the foundation of the calculation, is inadequate.

The width and shift of atomic levels due to the fluctuation of the electromagnetic field can be calculated by using the corresponding system of kinetic equations for the diagonal and nondiagonal density matrices $f_n(t)$ and $f_{nm}(t)$.[6,7,42] The collision integral $I_n(t)$ in the equation for the function $f_n(t)$ in the approximation of stationary atoms is determined by the expression (7.1).

To calculate the shift in the level we shall employ the expression (9.2.8) from Ref. 6

$$\Delta E_n = \frac{1}{3\hbar}\sum_{n_1}|d_{nn_1}|^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{(\delta E\delta E)_\omega}{\omega - \omega_{n,n}}. \qquad (12.2)$$

It defines not the complete, but rather only the induced contribution of fluctuations of the field (the full expression will be presented below). The induced contribution is determining only in the calculation of the quadratic Stark effect in an external field.

If Planck's distribution is employed in the expression (12.2) in the calculation of the level shift, which corresponds to the use of the quantum Nyquist formula (1.14) [see Eqs. (2.13) and (2.12)], then a power-law divergence appears in the integration over ω. To derive Bethe's formula the expression (7.2), i.e., the spectral density only at the

transition frequencies, must be used in (12.2) for the spectral density of the field. Indeed, it is this function that enters into the collision integral (7.1), determining the relaxation to the equilibrium distribution in the presence of interactions with the fluctuation field.

Thus, based on the structure of the collision integral (7.1), in calculating the level shift we shall employ the spectral density $(\delta E\delta E)_\omega$ only at the transition frequencies $\omega_{n_1 n}$. This is equivalent to the introduction of noise (here—fluctuations of the electromagnetic field) based on the quantum Nyquist formula (1.17), and not (1.14).

We recall that the formula (12.2) determines only the induced contribution to the shift in the energy level. With the help of the kinetic equation for the nondiagonal density matrix $f_{nm}(t)$, in the calculation of the level shift not only the induced but also the spontaneous transitions can be taken into account (see Chap. 9 in Ref. 6). Taking into account spontaneous transitions leads to the substitution (we assume at the outset $\omega = \omega_{n_1 n}$) [see (9.4.7) in Ref. 6]

$$(\delta E\delta E)_{\omega_{n_1 n}} \to (\delta E\delta E)_{\omega_{n_1 n}} + \frac{2\hbar\omega_{n_1 n}^3}{c^3} \bullet \qquad (12.3)$$

At $T = 0$ the right side equals $4\hbar\omega_{n_1 n}^3/c^3$, if $\omega_{n_1 n} > 0$, and zero if $\omega_{n_1 n} < 0$. As a result, after substituting (12.3) into (12.2) and integrating over ω we arrive at Bethe's formula

$$\Delta E_n = \frac{2}{3\pi c^3} \sum_{n_1} |d_{n_1 n}|^2 \omega_{n_1 n}^3 \ln \frac{\omega_{\max}}{\omega_{n_1 n}}$$

$$\approx \frac{4e^4}{3m^2c^3} \hbar |\psi_n(0)|^2 \ln \frac{\omega_{\max}}{\langle\omega_{n_1 n}\rangle}. \qquad (12.4)$$

In summing over n_1, in the argument of the logarithm we made the substitution $\omega_{n_1 n} \to \langle\omega_{n_1 n}\rangle$. Since the quantity $|\psi_n(\theta)|^2$ differs from zero only for the S state, (taking into account the fact that $\langle\omega_{n_1 n}\rangle \sim E_n/\hbar$) Bethe's formula (12.4) can be represented in the form[41]

$$\Delta E_n = \frac{8}{3\pi} |E_0| \frac{\alpha^3}{n^3} \ln \frac{2n^2}{\alpha^2} ; \qquad (12.5)$$

$|E_0|$ is the energy of the ground state of the hydrogen atom, α is the fine-structure constant, and n is the principal quantum number. Taking into account some additional factors leads to a refinement of Bethe's formula.[35-38]

13. CONCLUSIONS

Thus we have given a brief review of fluctuation-dissipation relations for different physical systems at all levels of the statistical description. The most general relations, as we saw, are FDR for many-body distribution functions. They are too complicated to be used directly, but they are convenient starting points in the formulation of simpler FDR.

It was shown using many examples that of the two methods, based respectively on the kinetic equations and on Langevin's equations, the first method must be given preference. In addition, based on the kinetic equations it is possible to establish the applicability of Langevin's equations and to determine, in particular, the spectral densities of the Langevin sources. In this method the fundamental difficulties to which, for example, the quantum Nyquist formula (1.14) leads, do not arise.

Not all questions, of course, were included in this review. Thus the kinetic equations with retardation—equations of the non-Markovian type—were practically not examined. They include equations with radiation friction. An example of such equations is the kinetic equation for the nondiagonal density matrix of the system atoms + electromagnetic field. The effects of retardation appear in the spectral-frequency dependence of both the dissipative (for example, line widths) and nondissipative (for example, frequency or energy shifts) characteristics.

Fluctuation dissipation relations are widely employed also in nonequilibrium statistical theory, in particular, in the construction of the kinetic equations themselves. In so doing the conditions for the validity of the corresponding FDR are also established. The validity of the FDR obtained is confirmed indirectly by the "efficiency" of the equations so obtained.[1)]

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