Optics of ultracold neutrons and the neutron-microscope problem

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Some problems of the optics of ultracold neutrons (UCNs) are examined. The possibility is discussed of designing high-resolution neutron-optical systems, and in particular, a neutron microscope using UCNs. The problems are analyzed of the specific distortions involving the appreciable action on the motion of UCNs of the Earth's gravitational forces. Experiments on focusing of neutrons are reviewed, the methods of designing neutron-optical systems for UCNs are presented, and the existing neutron-microscope projects are described.

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1. INTRODUCTION

Neutron optics constitutes one of the essential components of neutron physics. Nearly a majority of the studies performed now in the neutron-research centers involve in some way the methods or problems of neutron optics. Correspondingly, the bibliography on neutron optics is very extensive.

The monographs of Refs. 1–9 have been devoted to the problems of the physics and optics of slow neutrons, as well as the neutron-optic methods of studying condensed media. A set of the fundamental problems of neutron optics is presented in Ref. 10.

The methods of neutron optics are widely employed in experiments in which the object of study is the neutron itself. References 11 and 12 were devoted to these problems; their publication coincided with the fiftieth anniversary of the discovery of the neutron. The review of Ref. 13 commemorated the same date. The features of modern neutron-optical instruments are examined in Ref. 14.

The reviews listed above involve the physics and optics of warm and cold neutrons. These are neutrons with velocities from several hundred to thousands of meters per second and wave lengths on a scale of several angstroms to tens of angstroms.

However, in the past ten-fifteen years a new situation has arisen in neutron optics[15–17] in connection with the discovery of ultracold neutrons (UCNs). UCNs are neutrons having velocities of the order of 10 m/s and less and wavelengths from several hundred to thousands of angstroms. The reviews of Refs. 16–21 have been devoted to the physics of UCNs.

One of the remarkable features of UCNs is their ability to undergo total reflection from the surface of condensed matter at all angles of incidence. Hence the potentiality has arisen from the discovery of UCNs of prolonged confinement of neutrons in closed vessels. This has substantially influenced the development of neutron optics.

In connection with the problem of storing UCNs, the problems have been studied in greater detail of reflection and refraction of neutron waves, the problem of validity of the description of the interaction of neutrons with matter has been analyzed in detail by introducing the optical potential,

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and the features have been studied of the dispersion law of neutron waves.

The phenomenon of total reflection of UCNs has been demonstrated in numerous elegant experiments, but it still has seemed remarkable for a long time, although the physics of the event has been fully understandable. Despite the lack of direct experiments, there has been no doubt that the reflection of UCNs is specular. Hence the idea has arisen of reflective optical elements for UCNs, followed by the proposal to design a neutron microscope.[22]

Another important distinctive feature of UCN optics involves the low value of the energy of these neutrons. Hence their motion becomes appreciably affected by such small forces as the force of gravity of the Earth, while the trajectory of a neutron is appreciably curved in the gravitational field. The same happens with a light beam in an optically *inhomogeneous medium. Therefore the analogy naturally* arose with the optics of inhomogeneous media.

Optical problems are mentioned to some extent in all studies on UCNs, including those cited above. Among the original studies pertaining to this problem we must cite Refs. 23 and 24. Reference 25 was devoted to problems of experimental study of the wave optics of UCNs. The problems that stand in the path of designing a neutron microscope have been briefly analyzed.[26,27]

The present article is devoted to some problems of the optics of UCNs, mainly involved with the problem of the neutron microscope. In the time elapsed since the appearance of the idea of the possibility of this instrument substantial changes have arisen in this field. While 10–12 years ago this possibility seemed a dream of the distant future, at present not only has the image of a simple object in neutron rays been obtained, but a stage has set in of actual construction of high-resolution neutron-optic instruments. Along this pathway specific difficulties have arisen and are being solved, mainly involving the need of building optical systems in an optically inhomogeneous medium.

Hence it seems quite opportune to review the current state of the instrumental optics of UCNs—a new field of neutron optics.

The theme of this article has dictated in a number of cases a pure *optical* approach *to* presenting a number of problems, which perhaps is not quite customary for specialists in neutron optics. Hence the article begins with a section devoted to the problem of the relationship between the ordinary quasiclassical solution of the problem of the motion of a particle in a potential field and the ray approach in optics.

Section 3 briefly presents the fundamental features of the optics of UCNs. Here we devote major attention to the problem of the reflection of a neutron wave.

Further, the specific problems involving the action of the gravitational force of the Earth on the motion of UCNs are treated.

The subsequent sections are directly devoted to problems of the instrumental optics of UCNs. Here the current state of the problem of the neutron microscope is reviewed. In particular, the experiments are described on the focusing of neutrons, in which it has been shown possible to obtain a neutron image. Some methods are presented of designing neutron-optical systems, and the existing projects for high-resolution instruments are described.

In closing, we examine the problems of the expected potential of future instruments, with the aim of making some prognoses on the possible application of a neutron microscope.

2. OPTICS OF A SLOW MASSIVE PARTICLE

As is known, when one treats the problem of the formation of an optical image, one must take into account essentially the wave aspects of the problem. On the other hand, a large number of the problems of instrumental optics can be solved by the methods of geometric optics. The same situation exists in the case of neutron optics. As is well known, the classical dynamics and quantum mechanics of a free particle enjoy the same relationship as geometric and wave optics. This analogy is especially pictorial whenever the quasiclassical approach is valid, i.e., in practically all the problems treated below. Since we are interested in the problem of *propagation and interference of neutron waves, of course,* we can carry out the entire treatment in pure wave language. However, the trajectory treatment, just like the ray approach of geometric optics, often allows one to elucidate the fundamental features of a phenomenon. Moreover, owing to the total analogy of ray (geometric) optics and classical dynamics, in optical terminology one can describe also the motion of a "classical" neutron when the wave features of the pattern are not of interest to us. This approach looks somewhat artificial, but it proves fruitful in treating optical problems, since it enables one to use a number of the propositions of geometric optics in the form of a ready result.

As we know, the stationary Schrödinger equation

$$\nabla^2\Psi + \frac{2m}{\hbar^2}[E - V(\mathbf{r})]\Psi = 0 \qquad (2.1)$$

is completely similar to the Helmholtz wave equation

$$\nabla^2 A + k^2 A = 0, \qquad (2.2)$$

This is especially clearly seen if we write it in the following form:

$$\nabla^2\Psi + k^2(\mathbf{r})\Psi = 0, \quad k^2 = \frac{2m}{\hbar^2}[E - V(\mathbf{r})], \qquad (2.3)$$

here $k(\mathbf{r})$ is the wave number. Adopting optical terminology, we introduce the definition of the refractive index

$$n^2(\mathbf{r}) = \frac{k^2(V = V(\mathbf{r}))}{k_0^2(V = V(0))}. \qquad (2.4)$$

Then we obtain the following expression from the definition (2.3) of the wave number k:

$$n^2(\mathbf{r}) = 1 - \frac{V(\mathbf{r})}{E} = 1 - \lambda^2\frac{2m}{h^2}V(\mathbf{r}), \quad \lambda = \frac{2\pi}{k_0}. \qquad (2.5)$$

We can draw a conclusion from (2.5) that will be essential below. In the case in which the propagation of the neutron wave is correctly described by introducing the potential $V(\mathbf{r})$ independent of k, the characteristic dispersion law holds:

$$|n^2 - 1| \propto \lambda^2. \qquad (2.6)$$

Upon substituting the quasiclassical solution

$$\Psi(\mathbf{r}) = \Psi_0(\mathbf{r})\exp(iS(\mathbf{r})), \qquad (2.7)$$

into the Schrödinger equation, we obtain the eikonal equation for the real part of (2.7):

$$(\nabla S)^2 = k^2, \tag{2.8}$$

provided that

$$\nabla^2\Psi \ll k^2\Psi_0. \tag{2.9}$$

The condition (2.9) is the condition for applicability of the quasiclassical approximation. The eikonal equation is the fundamental equation of geometric optics. In essence almost all of optics is contained in Eqs. (2.7) and (2.8). It is natural to identify the surface $S =$ const with the wavefront, the direction of the vector $\mathbf{k} = \nabla S$ determines the ray direction, while the following integral determines the phase:

$$\int \mathbf{k}\, \mathrm{d}\mathbf{l} = \frac{1}{\hbar} \int \mathbf{p}\, \mathrm{d}\mathbf{l}. \tag{2.10}$$

While in comparing the dynamics of a massive particle and geometric optics one can speak of an analogy (moreover, one established already by Hamilton), in the optical formulation of the Schrödinger equation the essence is not a simple analogy, but the wave nature of the quantum approach itself. The optical-mechanical analogy is one of the foundations of this approach. Indeed, the Schrödinger equation was constructed on the basis of two assumptions: the validity of the de Broglie hypothesis of the association of a particle with a certain wave defined by its momentum $\lambda = h/mv$, and the Hamiltonian principle

$$\int_{A,\, t_1}^{B,\, t_2} L(\dot{x}, \dot{y}, \dot{z}, x, y, z)\, \mathrm{d}t = \min, \tag{2.11}$$

here L is the Lagrangian, A and B are the initial and final points of the trajectory, and t_1 and t_2 are respectively the initial and final time. In the stationary case $E =$ const the Hamiltonian principle reduces to the principle of least action of Maupertuis[28]:

$$I = \int_{A,\, t_1}^{B,\, t_2} (L + E)\, \mathrm{d}t = \min, \tag{2.12}$$

here E is the toal energy, with $E = \Sigma \dot{x} p_x - L$. The time t enters into (2.12) only formally, since after substituting the expression for E into it, we obtain:

$$I = \int_{A,\, t_1}^{B,\, t_2} \sum p_x \dot{x}\, \mathrm{d}t = \int_A^B \sum p_x\, \mathrm{d}x = \int_A^B \mathbf{p}\, \mathrm{d}\mathbf{r} = \min, \tag{2.13}$$

here $d\mathbf{r}$ is an element of the path along the trajectory. Equation (2.13) is fully equivalent to the Fermat principle known in optics:

$$\delta \int k\, \mathrm{d}S = 0 \quad \text{or} \quad \delta \int n\, \mathrm{d}S = 0. \tag{2.14}$$

Here Eq. (2.13) contains more rigorous requirements than (2.14), since the variation equals zero not only for an extremum, but also when the optical path (the action) is stationary, in particular, when an optical image of one of the points A and B lies between these points.

Thus, to calculate the wave pattern of the field, one must proceed just as in optics, i.e., superpose the amplitudes of the waves corresponding to different rays (trajectories), taking into account the phases, which are defined as $\int \mathbf{k}\cdot d\mathbf{r}$ along the trajectories.[1)] Here the Kirchhoff diffraction theory is fully applicable.

Despite the extremely close analogy between ordinary optics and wave mechanics, there is, nevertheless, one rather essential difference. In optics the Fermat principle (2.14) can be written as follows:

$$I = \int_A^B n\, \mathrm{d}S = \int_A^B \frac{c}{v}\, \mathrm{d}S = c \int_A^B \mathrm{d}t, \quad \delta I = 0. \tag{2.15}$$

It expresses the minimality or stationarity of the time of propagation of light between the points A and B. But in the case of a massive nonrelativistic particle we have

$$k\, \mathrm{d}S = \frac{m}{\hbar} v \cdot v\, \mathrm{d}t.$$

Then we must write (2.14) in the form:

$$\delta \int_A^B v^2\, \mathrm{d}t = 0 \quad \text{or} \quad \delta \int_A^B n^2\, \mathrm{d}t = 0. \tag{2.16}$$

(This situation has been noted in Ref. 25). Therefore, although the surface of the wave front in the optics of a massive particle is, as before, the surface of equal phases and the phase itself is determined by the integral along the "classical trajectory," the time that a classical particle would take in reaching this surface by different trajectories generally differs.

Of course, in starting from the outset with the stationary Schrödinger equation, we rule the question of the time of propagation of the wave out of order. We are dealing with a situation in which only the trajectories of the classical particles along which we integrate are nonisochronic.

Previously the same situation had been analyzed in connection with the problem of the stability of the interference pattern in a neutron interferometer in the presence of forces acting on the neutron.[29–31] It was found that, if the interferometer is built to make the phase advance along two arms the same, but an external force makes the "classical" trajectories nonisochronic, then the first variation of the phase difference with respect to the momentum is very large. Here the interference pattern proves highly sensitive to the magnitude of the momentum. To observe it, even in the lowest interference order, one must ensure a very high degree of monochromatization. On the other hand, with an adjustment corresponding to complete isochronicity, the pattern in the first order is stationary, even in the presence of a phase difference in the two arms, and the first variation of the phase difference with respect to the momentum is zero. Owing to the considerable effect of the gravitational force on the motion of UCNs, one must take this phenomenon into account in designing interferometers for UCNs.

We note that, while in an interferometer the problem is the interference of two rays, in an optical instrument the image is formed by interference of an infinite number of rays. Therefore the isochronicity of rays is apparently a necessary condition for formation of a stable image in optical systems for UCNs.

In general the effect of the force of gravity and gravitational chromatic aberrations are one of the serious difficulties in instrumental UCN optics.

3. SOME FEATURES OF THE OPTICS OF ULTRACOLD NEUTRONS (UCNs)

3.1. The effective (optical) potential

As the definition of the refractive index (2.4) implies, fixing the potential $V(\mathbf{r})$ completely characterizes an optical medium for a particle. Let us examine the possible forms of potentials that can affect the propagation of a neutron wave.

The specific properties of ultracold neutrons are manifested primarily in the features of their interaction with the medium. This interaction can be described by introducing the corresponding effective potential, which is often called the optical potential. The simplest way to derive the expression for this potential consists in averaging the microscopic potential that correctly describes the scattering of a neutron by the elementary scatterers over the volume of the medium. The fundamental contribution to neutron scattering comes from the nuclei. Since the wavelength of slow neutrons is considerably larger than the dimensions of a nucleus, in calculating the scattering cross-section one can neglect the extension of the nuclear interaction and introduce into the treatment a potential that describes a point interaction. This is the so-called Fermi pseudopotential

$$V(\mathbf{r}) = \frac{h^2}{2\pi m} b\delta(\mathbf{r}-\mathbf{r}_0). \tag{3.1}$$

The quantity b is called the coherent nuclear scattering length and is determined from the scattering cross-sections for slow neutrons $\sigma_{\text{scatt}} = 4\pi b^2$. The scattering length b is connected to the amplitude of coherent nuclear forward scattering f_0 by the relationship

$$b = \lim_{k\to 0} f_0\,{}^{21}.$$

The scattering lengths for most nuclei are of the order of 10^{-12} cm and are much smaller than the distance between the nuclei. This is the second circumstance that justifies the use of the Fermi pseudopotential. For simplicity we shall consider the nuclei monoisotopic and spin-free. Then the scattering will be fully coherent. For a medium with a density N of nuclei the averaging of the pseudopotential over the volume yields the simple expression:

$$U = N\int V(\mathbf{r})\,d^3r = \frac{h^2}{2\pi m} Nb. \tag{3.2}$$

This formula is easily generalized to the case in which the material consists of several elements (isotopes), and the spins are arbitrary:

$$U_{\text{opt}} = \frac{h^2}{2\pi m}\sum_i N_i b_{i\,\text{coh}}. \tag{3.3}$$

Here N_i is the density of nuclei of type i, while $b_{i\,\text{coh}}$ is their corresponding scattering length averaged over the spin states of the neutron and the nucleus.

Upon substituting the expression (3.3) that we have derived for the optical potential into the formula (2.5) for the refractive index, we directly obtain:

$$n^2 = 1 - \lambda^2 \frac{\sum_i N_i b_i}{\;}. \tag{3.4}$$

In the case of monoisotopic composition and spin-free nuclei for which the summation drops out, one can write Eq. (3.4) in another way:

$$n^2 = 1 - \lambda^2 \frac{Nb}{\pi} = 1 - \frac{1}{k^2}\cdot 4\pi Nb. \tag{3.5}$$

It makes sense to compare this expression with the well-known formula of Lax, which holds for waves and scatterers of any type[32]:

$$k'^2 = k^2 + C\cdot 4\pi N f(0). \tag{3.6}$$

Here $f(0)$ is the amplitude of forward scattering, k' is the wave number in the medium, k is the wave number in vacuo, and C is a coefficient that allows for the difference between the effective (acting) field at the scatterer and the external coherent field:

$$C = \frac{\text{effective field}}{\text{coherent field}}.$$

As we see, Eqs. (3.5) and (3.6) coincide under the condition

$$C = 1, \quad f(0) = -b = \text{const}. \tag{3.7}$$

The change in the wave number in the medium leads to a phase shift of a wave passing through matter as compared with the phase of a wave propagating in a vacuum. This phase shift is determined by the relationship

$$\Delta\varphi = kd(n-1). \tag{3.8}$$

Here d is the thickness of the layer of material. At present this shift can be easily measured with neutron interferometers. The change in the wave number at the phase boundary leads to refraction of the wave and to appearance of a reflected wave.

The formula of Lax defines the relationship between the wave number of the incident wave and the wave propagating in the medium in the same direction, but it tells nothing about the amplitude of the reflected and refracted waves when the original wave is incident at an arbitrary angle. In this case the corresponding expressions must contain the scattering amplitude at the given angle $f(\vartheta)$ instead of the amplitude of forward scattering $f(0)$ (see, e.g., Ref. 33). If one uses the model of the potential of (3.3) with a sharp boundary to describe the interaction of the neutron with the medium, the needed relationships will be determined simply from the condition of continuity at the boundary, and are fully analogous to the Fresnel formulas.[24] Consequently the potential model is valid only when the conditions are satisfied that

$$f(\vartheta) \equiv f(0) = -b, \quad C = 1 \tag{3.9}$$

throughout the volume of the material.

Only the S-wave makes a substantial contribution to the scattering for slow neutrons in general and for ultracold neutrons in particular. Therefore the scattering is isotropic and one can replace $f(\vartheta)$ by $f(0)$ in all cases. As is known, at low neutron energies the scattering amplitude is constant and a replacement of $f(0)$ by $-b$ is also fully justified. As regards the magnitude of the Lax coefficient C, apparently its real part equals unity, while the imaginary part is much smaller than the real part. A set of problems associated with this has been discussed in Refs. 20, 23, and 33–35. In general the rather detailed analysis has shown that the degree of justification of using the potential of (3.3) is very great. The possible deviations from this model, if such exist, must lead to very subtle effects.[20,23,36,37] Hence the characteristic dispersion law of (2.6) and (3.4) is valid.

We see from (3.2) that the sign of the potential is determined by the sign of the scattering length. The scattering length is the distance between the position of the (point) nucleus and the point where the wave function of the scattered neutron vanishes. This quantity is directly associated with the phase shift of the scattered wave with respect to the incident wave. Nuclei exist in nature having both positive and negative scattering lengths (this fact was already noted by Fermi when experimental data on this question were lacking). However, for most nuclei we have $b > 0$, and hence $U > 0$, while $n < 1$. For thermal neutrons the deviation of the refractive index from unity is very small: 10^{-6}. However, it can be considerable for neutrons with energy $E \gtrsim U$. In the case in which $E < U$, the neutron wave cannot propagate in the material at all, and total (external) reflection occurs at all angles of incidence. Since here the wave function of the neutron rapidly decays in the material, the absorption upon reflection is very small. Precisely these neutrons are those now called ultracold. Ya. B. Zel'dovich was the first to call attention to the features of reflection of UCNs in 1959.[15] The characteristic magnitude of U for most substances is of the order of 10^{-7}eV, which is the order of magnitude of the energy of UCNs. Since the magnitude of U for a given substance determines the maximum kinetic energy of a neutron at which it can penetrate into the substance, it is also often called the boundary energy for the given substance. It corresponds to the limiting wavelength and the limiting velocity:

$$\lambda_0 = \left(\frac{\pi}{\sum_i N_i b_i}\right)^{1/2}, \quad v_0 = \left(\frac{2E_0}{m}\right)^{1/2} = \frac{h}{m}\left(\frac{\sum_i N_i b_i}{\pi}\right)^{1/2}. \tag{3.10}$$

One can derive these expressions directly from (3.5) by setting $n = 0$.

The magnitudes of the limiting energy, velocity, and wavelength for a number of materials are given in Table I.

3.2. Reflection and absorption of UCNs

As we have already mentioned, a number of the most essential problems of the reflection of UCNs were treated by Ya. B. Zel'dovich,[15] and then in greater detail by F. L. Shapiro.[16] The study of I. M. Frank[24] was devoted especially to the problems of reflection and absorption of ultracold neutrons. Following this study, we shall present only some final results of the theory.

In presenting the expressions for the optical potential (3.3) and for the refractive index (3.4), we omitted the problem of absorption of neutrons. One can easily take absorption into account by assuming the scattering amplitude b and concomitantly the potential and the square of the refractive index to be complex quantities. Here one can naturally associate the quantity n^2 with the magnitude of the dielectric permittivity.[23] Then we have

$$n^2 = \varepsilon = \varepsilon' + i\varepsilon'' = \left(1 - \frac{v_0^2}{v^2}\right) + i\frac{v_i^2}{v^2}, \qquad v_0^2 = \frac{h^2}{m^2}\frac{Nb'}{\pi}, \quad v_i^2 = \frac{h^2}{m^2}\frac{Nb''}{\pi}. \tag{3.11}$$

Here b' and b'' are the real and imaginary parts of the scattering amplitude for the medium. Here we assume in agreement with experiment that $b' = b_0$, where b_0' is the real part of the scattering amplitude for a single nucleus,[3)] while we have

$$b'' = \frac{k}{4\pi}\sigma = \frac{k}{4\pi}(\sigma_c + \sigma_n). \tag{3.12}$$

Here σ_c is the capture cross-section, and σ_n is the cross-section for inelastic scattering that leads to heating of the neutrons. The quantity v_i in (3.11) has been introduced by analogy with the limiting velocity v_0 for universality of notation. Here the refractive index is also complex, and we have

$$n = n' + in'', \quad n'^2 = \frac{\varepsilon'}{2} + \frac{1}{2}(\varepsilon'^2 + \varepsilon''^2)^{1/2}, \qquad n''^2 = -\frac{\varepsilon'}{2} + \frac{1}{2}(\varepsilon'^2 + \varepsilon''^2)^{1/2}. \tag{3.13}$$

In neutron optics ε' almost always exceeds ε'' in absolute magnitude. In the rare cases in which $b < 0$, we have $\varepsilon' > 0$, and an analogy exists with the optics of dielectrics. However, we usually have $b > 0$, and ε' is positive only when the neutron velocity exceeds the limiting velocity. Yet in the case of reflection of UCNs we have $v < v_0$, $\varepsilon' = n' - n'' < 0$. This situation, and specifically the complex character of ε and the negative value of ε' exceeding ε'' in modulus, is characteristic of the optics of metals.

Neutron optics differs in another substantial feature. One can show that when two conditions are satisfied, namely, when the angles of incidence and reflection are equal, and when the dispersion law (2.6) is valid, all the properties of reflection and refraction of a neutron wave are fully determined by the component of the momentum k_z normal to the surface.[24] Apparently both these conditions are satisfied to high accuracy.

We can easily carry out the transformation in all the formulas from n to n_z by replacing the value of the velocity with its component. In this case, upon substituting (3.11) into (3.13), we obtain

TABLE I. Effective potentials, limiting velocities, and wavelengths for certain materials.*

Material	U, meV	$V_{\lim}$ m/s	$\lambda_{\lim}$ Å	Material	U, meV	$V_{\lim}$ m/s	$\lambda_{\lim}$ Å
Nickel-58	380	8.52	462	Boron-free glass	90	4.15	953
	306	7.65	517				
Nickel	282	7.35	539	Lead	91	4.17	948
	208	6.31	627	Magnesium	58.4	3.34	1184
Iron	341	8.07	490	Aluminum	54.1	3.21	1232
	80	3.91	1011	Silicon	54	3.21	1231
Beryllium	249	6.91	573	Polyethylene	− 8.7	(1.29)	(3063)
Graphite	196	6.12	646	Vanadium	− 8.8	(1.30)	(3047)
Copper	168	5.66	699	Water	− 14.8	(1.68)	(2353)
Heavy water	166	5.63	703	Titanium	− 48.7	(3.05)	(1296)
Carbon monoxide	101	4.39	901				

*Table taken from: V. I. Lushchikov, Study of the Properties of Ultracold Neutrons (In Russian), Dissertation for the degree of doctor of physico-mathematical sciences in the form of a scientific report, Joint Institute for Nuclear Research 3-85-43, Dubna, 1985.

$$n_z'^2 = \frac{1}{2v_z^2}\{(v_z^2 - v_0^2) + [(v_z^2 - v_0^2) + v_i^4]^{1/2}\},$$
$$n_z''^2 = \frac{1}{2v_z^2}\{(v_0^2 - v_z^2) + [(v_z^2 - v_0^2) + v_i^4]^{1/2}\}, \quad (3.14)$$

here we take the positive value of the root. As a rule, v_i^2 is at least three orders of magnitude smaller than v_0^2, and hence $(v_z^2 - v_0^2)^2 \gg v_i^4$, apart from a narrow range of velocities directly near the threshold. In the reflection of UCNs we find $v_z < v_0$, and we have the following from (3.14)

$$n_z' = \frac{v_i^4}{4v_z^2(v_0^2 - v_z^2)}, \quad n_z'' = \frac{v_0^2 - v_z^2}{v_z^2}.$$

If the assumption holds that $b = \text{const}$, i.e., $\sigma = \sigma_c + \sigma_n$ obeys a $1/v$ law, then $\sigma(v)v = \text{const}$, and we have

$$v_i^2 = \frac{h^2}{m^2}\frac{Nb''}{\pi} = \frac{h}{m} N\sigma(v)\, v.$$

Then we have

$$n_z'^2 = \frac{h^2}{4m^2v_z^2}\frac{N^2[\sigma(v)v]^2}{v_0^2 - v_z^2}, \quad n_z''^2 = \frac{v_0^2 - v_z^2}{v_z^2}. \quad (3.15)$$

Here we have $n''^2 \gg n'^2$, i.e., the wave is strongly attenuated. This attenuation does not depend on N, and mainly involves reflection rather than absorption. The density of neutrons declines exponentially with increasing distance from the interface:

$$\rho = \rho_0 \exp(-2kn_z''z) = \rho_0 \exp\left[-\frac{2m}{\hbar}(v_0^2 - v_z^2)^{1/2} z\right]. \quad (3.16)$$

We can easily calculate the fraction of absorbed neutrons by starting with the condition of continuity of the wave function at the plane interface. One can write the solution simply by analogy with light. For the amplitude of the reflected wave the corresponding Fresnel coefficient r is

$$r = \frac{(1 - n_z') - in_z''}{(1 + n_z') + in_z''}. \quad (3.17)$$

Evidently the fraction of the neutrons absorbed in the medium is

$$\alpha = 1 - |r|^2 = \frac{4n_z'}{(1 + n_z')^2 + n_z''^2}. \quad (3.18)$$

Using (3.14), we can easily obtain the exact value of α. In the approximation with $(v_z^2 - v_0^2)^2 \gg v_i^4$ we have

$$\alpha = \frac{2v_i^2 v_z}{v_0^2(v_0^2 - v_z^2)^{1/2}} = \frac{2b''}{b'}\frac{v_z}{(v_0^2 - v_z^2)^{1/2}} = \frac{2v_z}{mv_0^2}\frac{\hbar N\sigma(v)v}{(v_0^2 - v_z^2)^{1/2}}. \quad (3.19)$$

And at the very threshold where $(v_z^2 - v_0^2)^2 \ll v_i^4$ we have

$$\alpha = \frac{2\sqrt{2}\, v_i}{v_0}. \quad (3.20)$$

Thus for UCNs the reflection coefficient $R = (1 - \alpha)$ differs from unity by an amount of the scale of $\eta = b''/b' \approx 10^{-3}$. At the very threshold we have $R \approx 1 - \sqrt{\eta}$, while it rapidly declines above the threshold. We see that the reflection of UCNs is actually almost total, while we can often neglect the quantity b'', of course, apart from those cases in which we are specially interested in the problem of absorption.

Let us take up some other features of the total reflection. We see from (3.16) that the amplitude of the wave inside the material declines exponentially with increasing distance from the surface, while the decay constant depends only on the normal component of the velocity:

$$\Psi = \Psi_0 e^{-\varkappa z}, \quad \varkappa = \frac{m}{\hbar}(v_0^2 - v_z^2)^{1/2}. \quad (3.20')$$

As is known, in the total reflection of bounded light beams when the wave is penetrating a second medium, the beam is displaced along the tangential component of the velocity.[38] This displacement becomes greater with greater depth of penetration of the wave. Of course, this phenomenon occurs also for waves of a different type. A. A. Seregin[39] has called attention to the fact that this longitudinal displacement must occur also for neutron reflection.

Perhaps this phenomenon must be considered in designing mirror optical systems of high resolution. The point is that, owing to the displacement of the beam along the surface, the image of a point source in a plane mirror ceases to coincide with the mirror-image point, but becomes a function of the angle of incidence. Thus the rays from a point source after reflection form a certain caustic.

One can easily estimate the magnitude of the additional displacement of the image in the direction of the normal to the mirror ζ by starting with the magnitude of the tangential displacement of the ray given in Ref. 38:

$$\zeta = -\frac{\lambda}{\pi}\frac{\sin^4\theta - n^2}{(\sin^2\theta - n^2)^{3/2}}. \quad (3.21)$$

Here θ is the angle of incidence, λ is the wavelength, and n is the refractive index. Upon substituting the value of n^2 from (3.11) and dropping the imaginary part involving absorption, we obtain:

$$\zeta = -\frac{\lambda}{\pi}\frac{v(v^2\sin^4\theta + v_0^2 - v^2)}{(v_0^2 - v^2\cos^2\theta)^{3/2}}. \quad (3.22)$$

We can estimate the order of magnitude of ζ. If the relative deviation of the normal component of the velocity $v_z = v\cos\theta$ from the limiting value v_0 amounts to 0.03, then for small θ we have $\zeta \approx 10\lambda$. We see that one must take this effect into account in a high-resolution optical system.

Another effect in total reflection was pointed out in Ref. 40. It was found that, if a wave is propagating in an inhomogeneous medium, then the reflection law can break down in certain cases. As applied to neutrons such a situation arises when the neutron moves with an acceleration (e.g., owing to the force of gravity) not directed normally to the plane of the mirror. The same phenomenon arises if the mirror itself moves with acceleration. The source of the effect is the fact that the magnitude of the phase change of the wave upon reflection involves the normal component of the velocity. Under the stated conditions a phase-shift gradient arises along the surface of the mirror. The effect increases as one approaches the total-reflection limit, but it is rather small. Thus, for a magnitude of the limiting velocity $v_0 = 320$ cm/s, which corresponds to reflection from aluminum, the angle of incidence $\theta = 45°$ and an acceleration equal to that of free fall, the difference between the angles of incidence and reflection amounts to 10^{-6} rad if the relative deviation of the normal velocity from the limiting value is 0.03, and 7×10^{-6} if this deviation amounts to 3×10^{-3}.

3.3. Gravitational and magnetic potentials

Since a neutron possesses mass and a magnetic moment, the gravitational field of the Earth and an external magnetic field can affect its motion. Here the gravitational potential is

$$V_{gr} = mgz. \quad (3.23)$$

Numerically we have $mg = 0.98\times10^{-7}$ eV/m. Thus the change in the potential energy of the neutron on rising 1 m is of the same order of magnitude as the value of the optical potential and the energy of the UCN. Therefore, while for thermal neutrons of energies of the order of 10^{-2} eV the effect of gravitation is practically negligibly small, the action of the field of the Earth on the motion of UCNs is quite substantial. From the standpoint of the dynamics of a massive particle, this action simply reduces to a curvature of the trajectory. To use optical terminology, one can speak of curving of the rays, as is usual in the optics of inhomogeneous media. Actually, if we substitute into the general expression (2.5) the value of the gravitational potential $V_{gr}(\mathbf{r}) = mgz$, we directly obtain:

$$n^2(z) = 1 - \lambda^2 \frac{2m^2 g}{h^2} z = 1 - \frac{2gz}{v^2(0)}. \tag{3.24}$$

Analogously we can take into account the effect of magnetic fields on the motion of the neutron. The potential of magnetic interaction is

$$V_{magn} = -(\mu \mathbf{B}). \tag{3.25}$$

The magnitude of the magnetic moment is $\mu = 6.02 \times 10^{-8}$ eV/T. Hence we see that magnetic fields can exert a substantial influence on the motion of a neutron only at field intensities of the scale of 1 T. Moreover, a homogeneous field does not perturb the motion. In contrast, inhomogeneous magnetic fields can be used in solving certain optical problems. In particular, V. V. Vladimirskiĭ proposed in 1960 using such fields to create magnetic mirrors and traps.[41] These ideas were developed experimentally in Refs. 42–46. There have been proposals to create magnetic lenses for neutrons.[47–49] However, owing to the smallness of the magnetic moment, also in those cases in which the field is not created intentionally, the magnetic action on a neutron is usually not large.[4)] But the gravitational action is always rather significant. Consequently we can assert that the low energy of UCNs has the result that the optics of UCNs is always the optics of inhomogeneous media.[51]

4. OPTICS OF ULTRACOLD NEUTRONS IN A GRAVITATIONAL FIELD AS THE OPTICS OF INHOMOGENEOUS MEDIA

4.1. Geometric optics of UCNs in the presence of gravitational force

Thus taking into account the effect of the gravitational field of the Earth on the motion of UCNs can lead to introducing a refractive index of the form of (3.24). Proceeding in this manner, we can at once use all the results already known from the optics of inhomogeneous media, as has been done, e.g., in Ref. 52 in analyzing problems of formation of a neutron image.

It is evident from the form of (3.24) that any optical system for UCNs is seemingly immersed in an optically inhomogeneous medium having dispersion. This form of the refractive index is characteristic of a rarefied plasma with a linearly varying density[5)]

$$n^2(\omega, z) = 1 - \frac{\omega_p^2(z)}{\omega^2}. \tag{4.1}$$

Here $\omega_p(z) = 2e(\pi N(z)/m)^{1/2}$ is the so-called plasma frequency, and $N(z) = N_0 z$. The optics of a linear plasma layer has been studied, although the problem of designing an optical system in such a medium has apparently not arisen.

An extensive literature has been devoted to the problems of optics of inhomogeneous media. We shall point out here only the monographs of L. M. Brekhovskikh[38] and Yu. A. Kravtsov and Yu. I. Orlov.[53]

Having attributed a refractive index to the vacuum, we have proceeded rather formally, although perhaps this procedure can be justified rigorously enough also in pure wave language by treating the interference of the secondary waves scattered by the varying potential.

However, in geometric optics the microscopic nature of the refractive index is not at all essential since, just as in classical mechanics, we can forget the wave nature of matter.

4.2. Field of applicability of the trajectory approach. Caustic of a point monochromatic UCN source

Let a point monochromatic UCN source lie at the point $z = 0$ of infinite space. All the neutrons leaving the source move in the gravitational field of the Earth along parabolas. Evidently the region of space occupied by the trajectories of the neutrons is bounded by a certain surface, or caustic.

Generally one terms the envelope of a family of rays as the caustic.[6)] The caustic is a surface on which a singularity exists. Near the caustic the rays become closer, and they intersect on the caustic itself. Hence the number of rays passing through each point of space varies jumpwise on the caustic. An increase in the amplitude of the wave arises from the intersection of the rays. If one calculates the amplitude by the methods of geometric optics, then on the caustic the amplitude will have an infinite value, which indicates the inapplicability of geometric optics in this region. Nevertheless an increase in the amplitude, i.e., focusing of the field on the caustic, actually does take place.

One usually attributes an extra phase shift to the wave upon reflection of the ray from a caustic. Upon contact with a nonsingular caustic a phase shift arises of $-\pi/2$. Upon passing through a three-dimensional focus the phase shift is $-\pi$.

The concentration of rays near a caustic leads to a considerable role of interference phenomena near the caustic surface, and this eliminates the divergence of the field amplitude. Near the caustic, in the so-called caustic region, an interference pattern arises in space. The characteristic scale Λ from the region of the caustic shadow to the first interference maximum should apparently be considered to be an estimate of the dimensions of the region of inapplicability of geometric optics.

Returning to the problem of the caustic of a point UCN source, or what is the same, of the region of applicability of the trajectory approach to the optics of UCNs, we shall formulate the problem in optical language. As before, we shall introduce the refractive index $n^2 = 1 - (2g/v^2)z$, where v is the initial velocity of the UCNs. Owing to the monochromaticity of the source we have $2g/v^2 = \text{const} = a$. Thus we have $n^2 = 1 - az$, which is a more compressed way of writing Eq. (4.1) for the refractive index in a linear plasma layer. The problem of the caustic in a linear layer is well known in optics.[53–57] In this case the equation of the caustic has the form

$$a\rho^2 = 4(1 - az), \tag{4.2}$$

while the caustic is a paraboloid of rotation with its vertex at $z = H = 1/a$, where $n^2 = 0$ (Fig. 1).

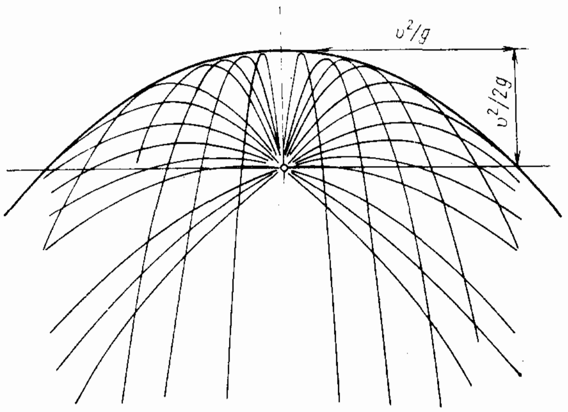

FIG. 1. Trajectories of UCNs in the gravitational field of the Earth. The rays from a source in a linear plasma layer have the same form.

The distance from the caustic to the first interference maximum can be calculated exactly for this case. Recently M. Berry has analyzed this problem in connection with the problem that we have examined[58] and found that

$$\Lambda = \left(\frac{\hbar^2}{m^2 g}\right)^{1/2} \approx 2.5\cdot 10^{-3}\ \text{cm}.$$

Here, as above, m is the mass of a neutron, and g is the acceleration of free fall.

One can solve this same problem quite qualitatively by identifying the concept of the region of applicability of the trajectory approach with the width of the caustic zone. At present a rather clear heuristic approach to the formulation of sufficient conditions for applicability of the method of geometric optics has been formulated.[53,59]

The fundamental idea of this approach is to introduce around the ray a certain volume or tube within which wave phenomena are already substantial. By analogy with the concept of Fresnel zones, the authors have called this the Fresnel volume. When an observation point lies far from the caustic, the Fresnel volumes of two rays, of which ray 1 has arrived directly from the source to the observation point, while ray 2 has been reflected from a caustic, are strongly separated in space (Fig. 2a). As the observation point approaches the caustic, a substantial overlap of the Fresnel volumes sets in. According to Ref. 59, this implies the inapplicability of the geometric-optic approach (Fig. 2b). Simple geometric constructions show[53,60,61] that the difference of eikonals along the rays 1 and 2 near a simple caustic is

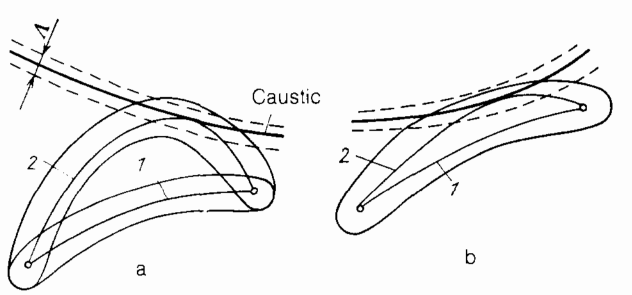

FIG. 2. Fresnel volumes of rays near a caustic. a—Point of observation sufficiently remote from the caustic. b—Point of observation close to the caustic.

$$\delta\psi = |\psi_1 - \psi_2| \approx \frac{4}{3}\beta^{1/2}|l_N|^{3/2}, \quad \beta = 2n_c|K_{ray}|. \qquad (4.3)$$

Here l_N is the distance along the normal from the caustic, n_c is the refractive index at the caustic, $K_{rel} = |K_{caust} - K_{ray} \cos\delta|$ is a quantity determining the relative curvature of the ray and the caustic. K_{caust} is the curvature of the normal cross-section of the caustic in the direction of the ray, K_{ray} is the curvature of the ray at the point of contact, and δ is the angle between the normal to the caustic and the principal normal to the ray.

We shall assume that a correct estimate of the width of the caustic zone is the dimension l_N, such that the difference in phase advances along the rays 1 and 2 (without allowing for the phase shift at the caustic) is π. Then we have $4/3k_0\beta^{1/2}l_N^{3/2} = \pi$, where k_0 is the wave number at the source, and

$$l_N = \left(\frac{3\pi}{4k_0}\right)^{3/2}\beta^{-1/3} = 1.77\Lambda, \quad \Lambda = k_0^{-2/3}\beta^{-1/3}. \qquad (4.4)$$

Since the refractive index depends only on Z, all the rays lie in planes passing through the origin of coordinates and $\delta = 0$. It remains to determine the relative curvature of the ray and the caustic. We can obtain the equation of the ray either from geometric-optical considerations or from elementary mechanics. Writing the equations of the ray and the caustic in the explicit forms

$$\begin{aligned} z &= \frac{1}{a} - \frac{a\rho^2}{4} && \text{for the caustic,} \\ z &= \operatorname{ctg}\theta\cdot\rho - \frac{a\rho^2}{4\sin^2\theta} && \text{for the ray} \end{aligned} \qquad (4.5)$$

and performing simple calculations, we obtain

$$\Lambda = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3} \approx 6\cdot 10^{-4}\ \text{cm}, \qquad (4.6)$$

while the width of the caustic zone is $l_N = 1.77\Lambda \approx 10.6\times 10^{-4}$ cm. This result is very close to the exact result obtained specifically for this case by M. Berry.

4.3. Limit of applicability of the geometric-optic approach at low UCN energies

Let us examine the problem of the applicability of geometric optics to the case of very small energies of UCNs existing in a gravitational field. In the upward motion of a neutron, its wavelength increases as it slows. Evidently the conditions for applicability of geometric optics break down when the wave number varies on its own scale in an altitude drop of the order of the wave length. Let this occur in an altitude drop $\Delta z = l$. Upon setting $l = v_2^2/2g \approx \lambda = \hbar/mv$, we obtain

$$l = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3}.$$

This is the same value as the one that characterizes the interference pattern near the caustic. We can easily estimate the order of magnitude of the corresponding energy:

$$\varepsilon = mgl = \left(\frac{m\hbar^2 g^2}{2}\right)^{1/3} \approx 0.6\cdot 10^{-12}\ \text{eV}. \qquad (4.7)$$

This problem has been treated in greater detail in Refs. 62 and 63, where the problem was analyzed of storing UCNs of extremely small energies in a plane. A more accurate quantum-mechanical calculation yields close-lying values for the linear constant l and the energy constant ε of the problem.

Thus the energy limit of applicability of the trajectory approach to optical calculations for UCNs in the gravitational field of the Earth amounts to about 10^{-12} eV.

In concluding hereby the treatment of the general problems of the optics of a very slow neutron wave, we shall turn more directly to problems of instrumental optics. However, the optical aspects of the effect of the gravitational force on a neutron will lie at the center of our attention also further on, since precisely this effect creates the greatest difficulties in the practical optics of UCNs.

5. INSTRUMENTAL OPTICS OF VERY SLOW NEUTRONS

5.1. The current state of studies on the optics of very slow neutrons

The possibility of focusing UCNs was already mentioned in the first study of F. L. Shapiro and his associates[79] reporting the discovery of UCNs. In Ref. 91 F. L. Shapiro noted that perhaps traps for UCNs will be useful as a source for neutron microscopy. The problem of designing a mirror neutron microscope (NM) using UCNs was discussed more concretely by I. M. Frank.[22] At that time the very phenomenon of total reflection of neutrons from a phase boundary at all angles of incidence still seemed quite unusual. It seemed highly attractive to use this remarkable property to construct a reflecting optical system—in the first stage simply a concave mirror—and thus obtain an image "in neutron rays." This study also pointed out a substantial difficulty on the path of creating such an optical instrument involving the appreciable effect on the motion of UCNs of the gravitational force of the Earth.

Several years later A. Steyerl and G. Schütz proposed to use a new optical element for this purpose—a zone mirror, i.e., a concave mirror with a zone phase structure.[64] Here, for a certain range of neutron wavelengths the gravitational chromatism is compensated by the intrinsic chromatism of the zone mirror. In 1979 N. T. Kashukeev and N. F. Chikov first obtained a neutron-optical image of a slit with two-dimensional focusing.[65] In this experiment the neutrons, propagating between two horizontal plane mirrors, were focused by a cylindrical mirror with a vertical generator. With this experimental arrangement the problem of gravitational distortions was naturally eliminated.

At the same time the theoretical study of the problem continued. Reference 52 discussed certain wave features of the optics of UCNs. In particular, the question was raised of the formation by a neutron wave of an optical image in the case in which the perturbing action of the gravitational field of the Earth exerts an appreciable influence. Attention was called to the fact that the propagation of a neutron wave in a potential field can be described by introducing a refractive index that depends on the velocity and coordinates. Thus a full analogy arose with the propagation of light in an optically inhomogeneous medium showing dispersion. The validity of the Fermi principle in an optically inhomogeneous medium implies the possibility of forming an image with neutron waves. The discussion of this problem was continued in Ref. 51, which noted that the result of Ref. 52 is to some extent trivial, since it is implied by the optical-mechanical analogy of Hamilton. However, the optical approach to the dynamics of a massive particle, usual in electron optics, was new for neutron optics and proved highly productive. The description of the motion of a neutron in the gravitational field of the Earth by introducing the refractive index enabled finding methods of calculating optical systems for UCNs. It is especially simple to do this in the paraxial approximation.[51]

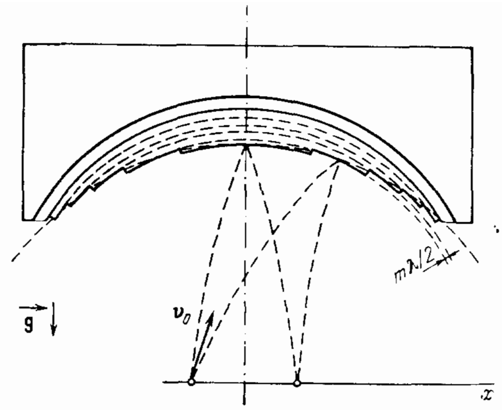

FIG. 3. Zone mirror for UCNs.[64,66]

In 1980 G. Schütz, A. Steyerl, and W. Mampe demonstrated the possibility of three-dimensional focusing of UCNs.[66] Using the zone mirror proposed in Ref. 64, they obtained the image of a slit with a magnification of 6× (Fig. 3). At the same time a definite progress was achieved in focusing of neutrons faster than ultracold—the so-called very cold neutrons. Experiments were performed at Grenoble with neutrons having velocities of the order of 200 m/s and a wavelength about 20 Å. Lenses[67] and a zone plate[68] have been used as the optical element (Fig. 4). The optical strength of the focusing elements for such neutrons is small. Hence it was necessary to use an instrument with a length of the order of 10 m. An analysis of the possibilities of using optical elements of different types in a neutron microscope has been presented in Refs. 26 and 27. Besides mirrors, lenses, and Fresnel plates, the possibility was discussed of using also magnetic lenses.[47–49] It was concluded that it was preferable to use mirror optics in the microscope, since a mirror is the only optical element possessing no intrinsic chromatism, although, of course, the difficulties involving gravitational distortions remain.

Two new ways were proposed for diminishing the gravitational aberrations.[26,27] In particular, it was proposed to compensate the gravitational force of the Earth with a magnetic field by placing the optical system in an inhomogeneous magnetic field. This problem was discussed in greater detail in Ref. 69. Below we shall describe a neutron micro-

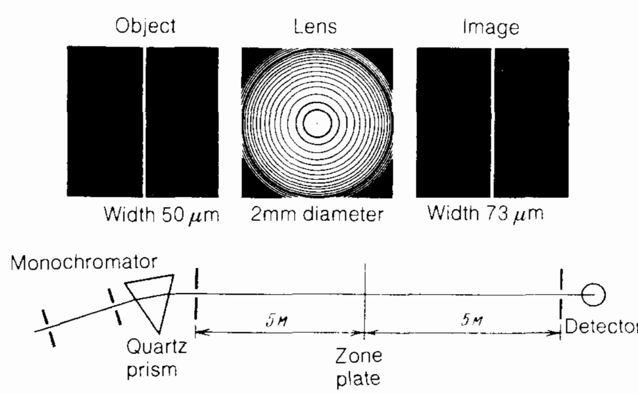

FIG. 4. Focusing of VCNs by a zone plate.[68]

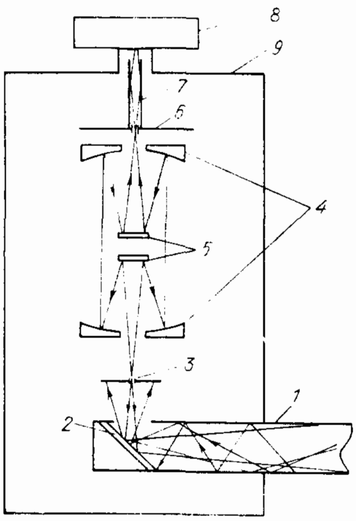

FIG. 5. Multimirror optical instrument for UCNs.[70] 1—neutron guide; 2—auxiliary mirror; 3—slit object; 4—concave mirror; 5—plane mirrors; 6—analyzing slit; 7—neutron guide of the detector; 8—detector; 9—vacuum chamber.

scope project based on this idea. The second way consisted in proposing the use of a relatively complex multimirror optical system in which the mirrors lie at different heights and are characterized by differing dispersions.

The idea of a multimirror optical system has been developed. An instrument based on this principle has been built, and the image of a slit with a low magnification has been obtained with it[70] (Fig. 5). A similar principle was used to build a neutron microscope with a magnification of 50× and tests of it have been reported in Refs. 25 and 71.

5.2. Optical elements for very slow neutrons

Let us examine the fundamental properties of optical elements that can be used in principle in a neutron microscope. It is natural to start this examination with a refractive lens. Let the material of the lens have a coherent scattering length b. Then the refractive index of the lens is

$$n=\left(1-\lambda^2\frac{Nb}{\pi}\right)^{1/2}=\left(1-\frac{v_0^2}{v^2}\right)^{1/2}. \qquad (5.1)$$

Here v_0 is the limiting velocity of neutrons for this material. As we know from optics, the focal length of the lens is

$$f=\frac{1}{(n-1)(R_1^{-1}-R_2^{-1})}. \qquad (5.2)$$

Here R_1 and R_2 are the radii of curvature of the two surfaces of the lens. Since usually we have $b>0$ and $n<1$, a converging lens having a minimal value of the focal length will be doubly concave. Upon assuming that $R_1<0$ and $R_1=-R_2$, we have

$$f=\frac{R}{2(n-1)}. \qquad (5.3)$$

For simplicity we shall expand (5.1) in a series omitting terms of the order of $(v_0/v)^4$, which is valid for $v>2v_0$ to an accuracy better than 0.01:

$$n=1-\frac{v_0^2}{2v^2}. \qquad (5.4)$$

Upon substituting (5.4) into (5.3), we obtain

$$f=R\frac{v^2}{v_0^2}. \qquad (5.5)$$

Hence we see that a neutron lens has very strong chromatism, with $f\propto v^2\propto 1/\lambda^2$.

One can also estimate the maximum relative aperture of a neutron lens. Evidently the maximum possible dimension of the aperture diaphragm is the diameter of the lens. Hence the relative aperture cannot exceed $2R/f=2v_0^2/v^2$. We see that not only does the focal length increase with increasing velocity of the neutron, but also, generally the relative aperture decreases. Thus the numerical aperture of a focusing lens in the instrument used for seeking the electric charge of the neutron[67] amounted to only 4×10^{-4}. Moreover, at low neutron velocities the use of lenses is restricted by the increased absorption.

Another optical element suitable in principle for focusing VCNs[71] is the zone plate. The question of using it, in particular, in connection with the problem of the neutron microscope, was studied in Ref. 72. The focal length is determined in this case by the expression

$$f=\frac{r_M^2}{M\lambda}. \qquad (5.6)$$

Here M is the number of zones of the plate and r_M is the radius of the largest zone. As we can easily see, a zone plate has chromatism, which is inherent in all diffraction instruments, with $f\propto 1/\lambda$. In practice the maximum dimension of the plate is determined by the smallest possible distance between neighboring zones: $\delta=r_M/2M$. Here the magnitude of the relative aperture is

$$\frac{1}{F}=\frac{2r_M}{f}=\frac{\lambda}{\delta}. \qquad (5.7)$$

The scale of the quantity δ attainable with modern technology is about 1 μm. For VCNs with velocities $v=15$ m/s ($\lambda=260$ Å), we obtain a relative aperture of 1:40.

Thus zone plates also possess appreciable dispersion, and at the current level of technology also a small relative aperture.

In contrast to lenses and zone plates, mirror optical elements *per se* are completely achromatic and have no fundamental restrictions on relative aperture. Chromatism of the instrument as a whole can arise only from gravitational distortions. However, one should speak in this case rather of the chromatism of the medium, which exists in any method of focusing, than of the chromatism of the optical element itself. The complicated aberrations that arise must be corrected for the system as a whole.

When one uses mirrors with total reflection at all angles of incidence, naturally restrictions arise on the magnitude of the velocity of the neutrons employed. This circumstance restricts both the possible resolution and the intensity. The largest value of the limiting velocity among the substances used as coatings for neutron mirrors is possessed by the isotope ^{58}Ni. For it we have $v_0=7.6$ m/s, $\lambda=520$ Å. The range of velocities can be extended by going to optics with inclined-incidence mirrors or using multilayer interference coatings, the so-called supermirrors.[73–75]

5.3. An ideal optical element for neutrons in the presence of gravitational force

Thus, apparently mirror optics is most suitable for designing high-quality optical systems for UCNs. The methods of calculation and the technology of production of mirror objectives are rather well developed for light. The difference of UCNs consists in the shorter wavelength and the substantial effect on them of gravitational forces, and this presents a fundamental difficulty.

The description of the motion of a neutron by introducing the refractive index simplifies the theoretical analysis of the problem. However, it yields no ready recipes, since, apparently, there was no necessity earlier to face the need of immersing an optical system in an inhomogeneous medium with dispersion.

Let us illustrate the optical properties of such a medium. We shall assume that an ideal optical element is removed from the medium into a vacuum. Let us assume the refractive index of the medium at the interface to be unity. Let our optical element with its optical axis along the z axis focus the rays from a point source (Fig. 6). The converging spherical wave through the boundary plane $z=0$ enters our layerwise-inhomogeneous medium. We shall denote the radius of curvature of the front at the coordinate origin as R. We shall assume that the value of R is much smaller than the maximum height of rise $v^2/2g$, i.e., that the curvature of the rays is small. The characteristic dimensionless parameter of the problem is the quantity $\xi = gR/v^2$, $\xi \ll 1$. We shall perform the calculations in the first order in ξ. We can easily determine that a ray that has entered the medium at the angle θ to the z axis intersects it at the height $z' = R[1-(\xi/2\cos^2\theta)]$. The essential point is that rays entering the medium even at an arbitrarily small angle intersect the optical axis below the initial center of curvature of the front. That is, the paraxial focus is shifted with respect to the point $z=R$. Its position is:

$$F = R\left(1-\frac{\xi}{2}\right) = R\left(1-\frac{gR}{2v^2}\right). \tag{5.8}$$

The magnitude of the gravitational displacement $\Delta F = gR^2/2v^2$ depends quadratically on the velocity. This is one of the manifestations of gravitational chromatism. An arbitrary ray entering the medium at the angle θ to the z axis intersects it below the paraxial focus, with the distance from the point of intersection to the focus being:

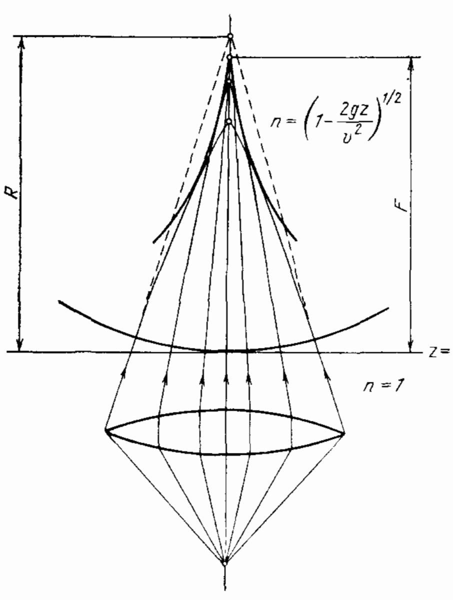

FIG. 6. Propagation of a convergent spherical wave front in a medium having a "gravitational" refractive index.

$$\zeta = F - z' = \frac{\xi R}{2}\,\mathrm{tg}^2\theta, \tag{5.9}$$

while the angle of the ray with the z axis at its point of intersection is determined from the condition:

$$\sin\alpha = \sin\theta\,(1+\xi). \tag{5.10}$$

We can easily derive from (5.9) and (5.10) the equation of the tangent to the ray at the point of intersection with the z axis·

$$\rho = (F-z)\,\tau - \frac{\xi R}{2}\,\tau^3, \quad \tau = \mathrm{tg}\,\alpha. \tag{5.11}$$

Solving (5.11) simultaneously with the equation $\partial\rho/\partial\tau = 0$, we obtain the equation of the caustic[8]:

$$\rho^2 = \frac{8}{27}\,\frac{v^2}{gF^2}\,(F-z)^3. \tag{5.12}$$

We note that our introduction of the interface at $z=0$ played a purely auxiliary role, since the geometry of the rays was determined by the refraction of the rays throughout the medium for $z>0$, while refraction at the phase boundary was absent. Thus, even for monochromatic neutrons the medium gives rise to aberrations, in particular spherical aberration. Moreover, owing to the disperison of the medium, both the position of the focus and the magnitude of the spherical aberration depend on the velocity (wavelength) of the neutron. Therefore, in the design of optical instruments using very slow neutrons, two fundamental and generally coupled problems arise. First, one must achromatize the system, and second, one must learn to operate with appreciable apertures, for which one has to correct the system for the fundamental aberrations by allowing for the properties of the medium.

5.4. Design of optical systems in an inhomogeneous medium in the paraxial approximation

The fundamental characteristics of an optical system, such as the location of the image, the focal length, the location of the principal planes, etc., can be determined by using the small-angle approximation, i.e., the paraxial approximation. Owing to the chromatism of the medium, all these quantities generally must depend on the velocity. In certain cases one can achromatize the system. Here one must distinguish achromatization of location, in which the location of the image plane is stationary, and achromatization of magnification, in which the ratio of angles is preserved for a ray near the object and near the image plane. We emphasize that as yet we are speaking only of small angles.

For paraxial calculation of neutron-optical systems, it is useful to employ the matrix method well known in optics.[76] Its generalization to the case of a layerwise-inhomogeneous medium offers no difficulties.[51] The essence of the method consists in the following. Any ray in an optical system having axial symmetry can be described by the coordinates Z, y, and V, where Z is the coordinate of an arbitrary point on the optical axis, which coincides with the z axis, y is the displacement of the ray from the optical axis in a plane passing through the point Z, and $V = n\vartheta$, where n is the refractive index and ϑ is the angle of the ray with the z axis. In the paraxial approximation all the angles entering the calculation are small enough that the odd trigonometric functions of these angles can be considered equal to the angles themselves. The regions of space having different refractive indices are separated by certain separation planes. The problem of the form of these surfaces is not treated, and the surface is characterized by its paraxial radius of curvature. In a homogeneous medium the ray propagates rectilinearly between the separation surfaces, and in refraction and reflection the relation is linear between the angles of incidence and refraction (or reflection). The linear relation of the coordinates is the fundamental distinctive feature of the paraxial approximation. Therefore every elementary change in the coordinates of the ray can be characterized by an elementary matrix of dimensionality 2×2, so that

$$\begin{pmatrix} y_2 \\ V_2 \end{pmatrix} = \hat{M} \begin{pmatrix} y_1 \\ V_1 \end{pmatrix}, \tag{5.13}$$

where the subscripts 1 and 2 pertain to certain planes in front of and behind the element of interest to us. Here the matrices for displacement $\hat{\tau}$, reflection $\hat{R}$, and refraction $\hat{F}$ are written as follows:

$$\hat{\tau} = \begin{pmatrix} 1 & \frac{t}{n} \\ 0 & 1 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} 1 & 0 \\ \pm\frac{2n}{r} & 1 \end{pmatrix}, \quad \hat{F} = \begin{pmatrix} 1 & 0 \\ \pm\frac{n_1-n_2}{r} & 1 \end{pmatrix}. \tag{5.14}$$

Here n is the refractive index, r is the radius of curvature of the reflective or refractive surface (the sign involves the direction of convexity), the subscripts 1 and 2 in the refractive matrix pertain to the refractive indices on the different sides of the surface, and $t = |Z_1 - Z_2|$ is the change in the coordinate upon displacement.

The transformation of the coordinates performed by the system as a whole also has the form of (5.13), but here the mattrix $\hat{M}$ of the system is the result of successive multiplication of all the elementary matrices, while the order of multiplication corresponds to the order of change of the coordinates y and v of the ray passing through the system. The matrix of the system fully defines its optical properties. When the optical system is immersed in an inhomogeneous medium, only the displacement matrix is subjected to modification. We can easily derive by direct calculation that the matrix for displacement through N thin layers having refractive indices n_i differs from $\hat{\tau}$ in (5.14) only in the replacement of the quantity t/n by the sum $\Sigma_i^N(t_i/n_i)$. For a medium having a smooth inhomogeneity we can naturally go from summation to integration. Hence the form of the displacement matrix is:

$$\hat{\tau} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \quad T = \int_{Z_1}^{Z_2} \frac{dt}{n(t)}. \tag{5.15}$$

In the case of a gravitational force acting on the neutron, the refractive index is determined from (3.24). Then we have

$$T = \frac{v^2}{g}\left|\left(1-\frac{2gZ_1}{v_2}\right)^{1/2} - \left(1-\frac{2gZ_2}{v^2}\right)^{1/2}\right|. \tag{5.16}$$

Thus, in this case also we can easily obtain the characteristic matrix of a system consisting of any number of optical elements.

We emphasize that the ideas presented above hold only for a layerwise inhomogeneous medium in which the refractive index depends only on the one coordinate Z, and the optical axis lies along it, in other words, for a vertical arrangement of the instrument.

Owing to the dependence of the refractive index on v^2, this same quantity entered into Eq. (5.16), simply replacing some linear dimension in the ordinary case of a homogeneous medium. Since linear dimensions always enter into all the relationships determining the properties of the system, replacing them with a quantity of the type of (5.16) causes all the parameters of the instrument to depend on the velocity of the neutron. Let us give two simple examples. First of all, let us calculate in the paraxial approximation the location of the focus for a concave mirror focusing UCNs. Let a parallel beam of UCNs be incident vertically downward on a concave mirror or radius R lying at the height $z = 0$. In defining the focal length for light, we would have to obtain the value $f = R/2$. Now we shall obtain this same value, but not for the linear dimension, but for the effective displacement T as defined by Eq. (5.16) with $Z_1 = 0$. Upon solving the equation

$$\frac{R}{2} = T = \frac{v^2}{g}\left[1-\left(1-\frac{2gz}{v^2}\right)^{1/2}\right]$$

for Z under the assumption that $z \ll v^2/2g$, we obtain

$$Z = f_n = \frac{R}{2} - \frac{gR^2}{8v^2}. \tag{5.17}$$

As it should, the focus for neutrons lies below the light focus, and its location depends on the velocity. We note that the distance that we have calculated from the mirror to the focal point generally does not coincide with the focal length. Defining the latter as the distance from the focus to the point of intersection of the primary parallel rays and the tangents to the rays at the focus, we obtain

$$F_n = \frac{R}{2} - \frac{gR^2}{4v^2}. \tag{5.18}$$

The reason for the difference is the curvilinearity of the rays. The change in the magnitude of the focal length leads to a gravitational variation in the optical magnification.

Now let us calculate the magnification of a concave mirror. As before, let the source lie above the mirror. We denote the distance from the source to the mirror by a. In the case of light the result is: $k_l = [1-(2a/R)]^{-1}$, where k_l is the optical magnification. Now, instead of the linear dimension a, the expression for the magnification again contains the value of the effective displacement. In the same approximations we obtain

$$k_n = \frac{1}{1-(2a/R)\left[1+(ga/2v^2)\right]}. \tag{5.19}$$

We see that both the difference of the result from the optical value and the magnitude of the chromatism of magnification sharply increase near the value $a = R/2$, i.e., at large magnifications. The chromatism of magnification of neutron-optical systems for UCNs apparently constitutes a considerable difficulty on the path of designing systems of large magnification and good resolution.

5.5. Achromatization of optical systems

In most cases chromatic aberrations are manifested even in systems with small apertures. The method of paraxial calculation presented above is quite suitable for analyzing them. In some cases this method enables one to some extent to achromatize the instrument. We shall illustrate this fact. Let the system consist of a certain number of reflective elements. To determine the characteristic matrix we must write down all the elementary matrices for displacement and reflection, and then multiply them. Here the first matrix will be the displacement matrix from the last reflecting surface before the image plane, and the next will be the displacement matrix from the object to the first reflecting surface:

$$\hat{M}=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\times\hat{M}_2\times\ldots\times\hat{M}_i\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}. \qquad (5.20)$$

Here a is the first segment of the system, and b is the last segment. Then we have

$$\hat{M}_0=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\times\hat{M}'.$$

Here $\hat{M}'$ is the matrix obtained by multiplying all the matrices, starting with the second one. Let us neglect for now the gravitational force, i.e., go over to the ordinary optics of homogeneous media. We shall denote the elements of the matrix $\hat{M}$ with unprimed letters, and those of the matrix $\hat{M}'$ with primed letters. We denote also the case of no gravitation with the subscript zero. Then we have

$$\hat{M}_0=\begin{pmatrix}A_0 & B_0\\ C_0 & D_0\end{pmatrix}=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\times\begin{pmatrix}A_0' & B_0'\\ C_0' & D_0'\end{pmatrix}. \qquad (5.21)$$

We recall that

$$\begin{pmatrix}y_2\\ V_2\end{pmatrix}=\begin{pmatrix}A_0 & B_0\\ C_0 & D_0\end{pmatrix}\times\begin{pmatrix}y_1\\ V_1\end{pmatrix}, \qquad (5.22)$$

where the subscripts 1 and 2 pertain to certain planes. Let us make plane 1 coincide with the plane of the source, and plane 2 with the image plane. If the source lies on the optical axis, then its image must also be on the axis. Here we have $y_1=y_2=0$. The condition $y_2=0$ implies that $B_0=0$. But we obtain from (5.21) that $B_0=B_0'+D_0'b$. Hence the location of the image is determined by the equation

$$b=-\frac{B_0'}{D_0'}. \qquad (5.23)$$

As is known, the optical magnification of the system is $k=n_1\sin\vartheta_1/n_2\sin\vartheta_2$, or in the paraxial approximation,

$$k=\frac{n_1\vartheta_1}{n_2\vartheta_2}=\frac{V_1}{V_2}. \qquad (5.24)$$

Thus we obtain from (5.22) and (5.24) and the condition $y_1=0$:

$$k=\frac{1}{D_0}=\frac{1}{D_0'}. \qquad (5.25)$$

Now we shall take into account the force of gravity by replacing all the linear dimensions by the effective displacements of (5.16) that correspond to them. Moreover, we should take into account the fact that the refractive index in the sites where the mirrors lie is not unity, while the corresponding reflection matrices contain quantities of the type $2n_i(z)/r_i$. The location of the image will be determined by an expression of the type of (5.23), which in this case is converted into an equation in b. Formally the expression for the magnification does not change. Thus we have

$$T_b=-\frac{B'}{D'},\quad k=\frac{1}{D'}. \qquad (5.26)$$

Now the elements of the matrix M' are functions of the velocity. Apparently, decreasing the chromatic distortions is favored by decreasing the linear dimensions of the system. If all the linear dimensions satisfy the condition $l_i\ll v^2/g$, then we can expand the refractive index and the corresponding quantities T_i in a series. We obtain

$$B'=B_0'+\frac{F_1(l_i)g}{v^2}+\frac{F_2(l_i)g^2}{v^4}+\ldots, \qquad (5.27)$$

$$D'=D_0'+\frac{\Phi_1(l_i)g}{v^2}+\frac{\Phi_2(l_i)g^2}{v^4}+\ldots \qquad (5.28)$$

($i=1,2,\ldots$), where the coefficients F_j and Φ_j have the dimensions of distance to the jth power. If we neglect terms of the order of g^2/v^4, then the condition for achromatization with respect to magnification reduces to the equation $\Phi_1=0$. The quantity T_b in the same approximation has the form $T_b=b\,[1+(\beta+b)(g/v_2^2)]$, where β also has the dimensions of distance and is determined by the location of the last mirror. Thus the location of the image is determined by the equation

$$T_b=-\frac{B'}{D'},\quad b\left[1+\frac{(\beta+b)g}{v^2}\right]=-\frac{B_0'+(F_1g/v^2)}{D_0'+(\Phi_1g/v^2)}. \qquad (5.29)$$

Solving (5.29) for b and equating the term with g/v^2 to zero, we obtain the condition for achromatization with respect to location. Of course, the validity of neglecting the higher terms in (5.27) and (5.28) must be analyzed sufficiently thoroughly. A multielement system contains a sufficient number of parameters, and sometimes one can achromatize the system by taking the higher orders into account. In particular, the telescopic system consisting of two concave mirrors and additional plane mirrors that was employed in Ref. 70 and is shown in Fig. 5, was achromatized with respect to location.

In some cases one can obtain conditions that enable simultaneous differential achromatization in the first order in g/v^2, both in location and in magnification. Thus, for the simple two-mirror system consisting of two concave mirrors with radii of curvature R_1 and R_2 (Fig. 7), achromatization with respect to location in the first order in g/v^2 is achieved in a telescopic geometry when the source lies at the focus of the first mirror $a=R_1/2$. In this same approximation achromatization of magnification will be achieved if simultaneously the following condition is satisfied:

$$R_1(1+\alpha)=6d. \qquad (5.30)$$

Here $\alpha=R_2/R_1$, and d is the distance between mirrors. Here the location of the image L and the optical magnification k coincide with the same quantities for light. We note that coincidence of the values of the magnification for neutrons and for light does not imply coincidence of the aperture angles in the image plane, since in the neutron case the gravitational force leads to a certain difference in the refractive index in the image plane from that in the plane of the source.

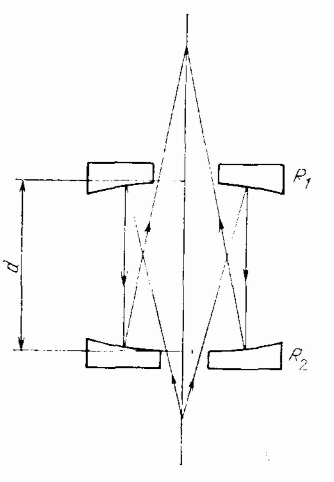

FIG. 7. An optical system for UCNs achromatic in the first order.

Another case of simultaneous achromatization in the first order in location and in magnitude has been presented in a recently published paper.[71] The fundamental difference in the instrument described in this study consists in employing the distinctive properties of the parabola of travel that are manifested when the neutrons pass through an apogee on their path.

Let us examine how the properties of a focusing concave mirror are altered in this case (Fig. 8). Let the neutron source lie on the axis of the mirror of radius R at the height a. We shall assume that the source is rather close to the focus, so that the neutrons after reflection from the mirror pass through the apogee of the parabolic trajectory before they cross the axis at the image point. Let us put the coordinate origin at the vertex of the mirror. We shall determine the location of the image by again using the matrix method. To do this, we shall introduce an auxiliary plane at the height c from the mirror. Let this plane lie above the image of the source, but below the apogee.

We must supplement our matrix apparatus with a matrix that describes the displacement of the ray with respect to the axis between the two crossings of the plane c. We can easily convince ourselves that this matrix is the matrix

$$\hat{\Delta} = \begin{pmatrix} 1 & \frac{2v^2}{g} n(z=c) \\ 0 & 1 \end{pmatrix}. \qquad (5.31)$$

The matrix $\hat{\Delta}$ is the matrix for displacement by the amount $\eta = (2v^2/g)n(c)$, and it does not alter the variable V. Actually the absolute value of V is determined by the magnitude of the angle made by the trajectory with the vertical axis and the value of the refractive index, and it changes only upon reflection. The sign of V is determined by whether the trajectory approaches the axis or departs from it.

Let us denote by b the segment from the plane C to the image. Then we can easily calculate the matrix of the system

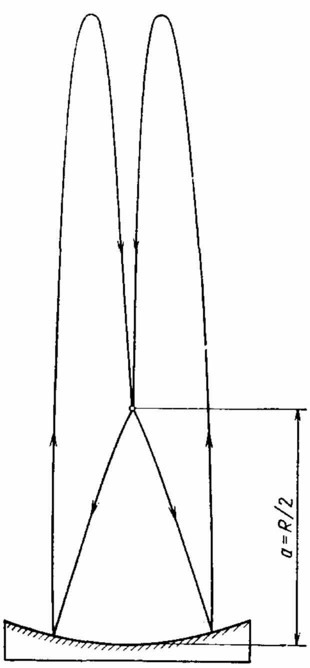

FIG. 8. Properties of a concave mirror in focusing UCNs. The location of the image coincides with the source when the point source is placed at the "light" focus.

$$\hat{M} = \begin{pmatrix} 1 & T_b \\ 0 & 1 \end{pmatrix} \times \hat{M}'; \quad \hat{M}' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}. \qquad (5.32)$$

Here the elements of the matrix $\hat{M}'$ are written as:

$$A' = 1 - \frac{2}{R}(\eta + T_c),$$
$$B' = T_a + (\eta + T_c)\left(1 - \frac{2T_a}{R}\right), \qquad (5.33)$$
$$C' = -\frac{2}{R}, \quad D' = 1 - \frac{T_a}{R}.$$

As before, the value of b is determined by an equation of the form of (5.29). Upon substituting into it the expressions for the displacements T and the refractive index $n(c)$, we obtain that, when $a = R/2$, the relationship holds that $b - c = a = R/2$.

Hence, in the case in which the source lies at the point of the light focus, its location exactly coincides with the location of the image without depending on the velocity. That is, achromatization in location occurs in all orders.

Chromatism of magnification remains considerable here, since the quantity D' that determines the magnification, as before, is determined by Eq. (5.19). One can easily see that when the value $a = R/2$ is substituted into (5.19), only terms depending on g and v^2 remain in the denominator.

In the neutron microscope described in Ref. 71, the chromatism of magnification is compensated in the first order by introducing a second, convex mirror. Here one can maintain the achromatization of location, but indeed now only in the first order. We shall return again to describing this instrument somewhat later in Sec. 6.1.

As a rule, it is difficult or impossible to achieve complete achromatization in all orders simultaneously in location and in magnification. In this regard it is useful to examine how the chromatism of location and that of magnification influence the resolution of the instrument.

Let the system have the magnification k, entrance aperture θ, and position of the image plane L, where L is measured, e.g., from the last mirror. The quantities k and L are given for a certain value of velocity of the neutrons. If the source is not monochromatic, then for the existing range of velocities, both the magnification and the location of the image lie in a certain range of values $k \pm \Delta k$, $L \pm \Delta L$.

Let us examine the role of the chromatism of location. The dimension of the image spot from a point source placed on the axis will be determined by the exit aperture angle ϑ and the chromatic spread of the image plane

$$\Delta R_L = 2\Delta L \cdot \mathrm{tg}\, \vartheta \simeq 2\Delta L \cdot \vartheta = \frac{2\Delta L \cdot \theta}{k}.$$

We have assumed that $\theta/\vartheta = k$, neglecting the small difference in the refractive indices in the plane of the object and the image. Then the resolution in the object plane will be k times smaller than the dimension of the spot in the image plane. It is defined as: $\delta_L = 2\Delta L \cdot \theta / k^2$. We can easily estimate the resolution at the edge of the field. In this case a ray proceeding to an edge point of the image lying at a distance d from the axis makes the angle with the axis $\alpha = d/L$. Then the dimension of the scattering spot in the image plane at the edge of the field amounts to $k'_L = d\Delta L/L$, while the resolution in the plane of the object is $\delta'_L = d\Delta L/kL$. If the dimension of the image is considered fixed, then, as we see, the resolution at the edge of the object is improved with increasing magnification as $1/k$, but as $1/k^2$ at the center of the object.

Now let us examine the distortions associated with the chromatic difference of magnification. Evidently these distortions are absent in the center of the field, where the displacement from the axis is zero. The central point is focused into a point on the axis, regardless of the value of the magnification. The greatest distortions will arise for a point most remote from the axis. The dimension of the scattering spot from this point amounts to $\Delta R_k = d'\Delta k$, while the resolution in the object plane is $\delta_k = d'\Delta k/k$, where d' is the dimension of the field in the object plane. Thus, for a fixed value of the dimension of the field in the object plane, the resolution increases with increasing magnification as $1/k$. Usually the dimension of the field must decrease with increasing magnification in inverse proportion to the magnification, since in practice the dimension of the image field is apparently restricted, e.g., by the design of the detector. Therefore the resolution improves in practice with increasing magnification as $1/k^2$.

The conclusion suggests itself from what we have said that, instead of achromatizing the system, one should simply increase as much as possible the magnitude of the optical magnification, which will lead to improving the resolution. This would actually be so if one could fix the values of Δk and ΔL with increasing magnification. Unfortunately this does not happen. Actually the value of the magnification is determined by Eq. (5.26). This means that large magnification corresponds to a small value of the quantity D'. Hence the first term D'_0 in the expansion (5.27) of D' in terms of g/v^2 is also small. This directly implies that the relative contribution of the higher terms in g/v^2 in the expression for D' increases with increasing k. Here the chromatism of magnification increases. We have already faced this situation in treating the chromatic aberrations of a single concave mirror [see (5.19)].

5.6. The nonparaxial case. An aplanat for UCNs[9]

Setting aside for now the problem of gravitational chromatism, let us discuss the problem of the possibility of using large entrance apertures in neutron-optical systems. The importance of this problem is evident. First, the limiting theoretical resolution of the instrument is associated with the numerical aperture. Second, the statistical potentialities of the instrument improve with increasing aperture, i.e., collecting power. It is precisely the statistics that can limit the resolution when one employs UCNs.

As is known, an optical system makes possible the sharp imaging of a small region near the axis of a plane perpendicular to the optical axis, i.e., is aplanatic, if the so-called sine condition is satisfied:

$$n_1 y_1 \sin\alpha_1 = n_2 y_2 \sin\alpha_2. \qquad (5.34)$$

Here n, α, and y are respectively the refractive indices, angles of the ray to the axis, and displacement with respect to the axis. The subscripts pertain to the object plane and the image plane. As is known from geometric optics, one can completely achieve axial stigmatism using one aspheric refractive or reflective surface, i.e., rid the system of spherical aberrations in all orders and achieve a sharp image of a point lying on the axis. One can obtain aplanatism by using two aspheric surfaces. Here the system becomes free from coma.[10] These conclusions are universal in character and are not restricted

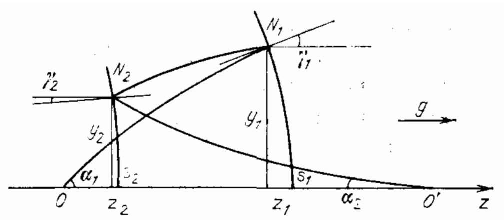

FIG. 9. On the design of an aplanatic mirror pair for UCNs.

to the case of homogeneous media.[77] This implies that one can design a special aplanat even in the case of UCNs in the presence of gravitational force.

The numerical methods of calculating an aplanatic mirror pair known in technical optics can be easily modified for the case of an inhomogeneous medium having a refractive index of the form of (3.24). For example, it is convenient to use the method given in Ref. 78. Let us explain the principle of calculation.

Let the optical system (the objective) contain two reflecting surfaces. We can easily determine from a paraxial calculation the mutual arrangement and the paraxial radii of the mirrors so that the system will have a given magnification. Moreover, in this calculation one must determine the location of the source and of the image plane. Our problem consists in finding surfaces such that a ray leaving the source at an arbitrary angle α_1 arrives at the image point O' while satisfying the sine condition. Let the refractive index be unity at the source point, with the acceleration of free fall g directed along the Oz axis. Let us assume that the ray leaves the axial point O of the source at the angle α_1. This ray (the neutron) intersects the surface of the first mirror at some point N_1 at the height y_1 from the axis (Fig. 9). Let us assume that the displacement of the point N_1 along the axis with respect to the vertex S_1 of the mirror is Z_1. Of course the quantities y_1 and Z_1 are not arbitrary, but must satisfy the equation of a parabola for the ray that we have chosen. For the initial ray that we have chosen in this way, there is only one ray which passes through the image point O' and satisfies the sine condition

$$\sin\alpha_1 = k\left(1 + \frac{2g\,|OO'|}{v^2}\right)^{1/2}\sin\alpha_2. \qquad (5.35)$$

This ray leaves the second mirror at the point N_2 having the coordinates y_2 and Z_2. The coordinates of this point must also satisfy the condition of a parabola for the exit ray. Thus, we have only two free parameters in the choice of the reflection points N_1 and N_2—the ray coordinates as calculated along the chosen rays. This coordinate can be, e.g., the displacement along the ray or the time of flight. Upon fixing the points N_1 and N_2, we have also determined the velocity of the neutron at these points, which depends only on the initial velocity and the height z. Between the points N_1 and N_2 the neutron also moves in a parabola with the known initial velocity

$$v_1 = [v^2 + 2g\,(|OS_1| - z_1)]^{1/2}. \qquad (5.36)$$

If we know the velocities of the neutron at the points of intersection with the mirror, we can easily calculate also the angle of the trajectory with respect to the optical axis at these points, since only the axial component of the velocity varies during the motion. Thus we now know the values of the angles α_{11} and α_{22}—the initial angle before reflection from the first mirror and the final angle after reflection from the second mirror, as well as the angles α_{01} and α_{02}—the initial and final angles of the intermediate rate. It remains to use the law of reflection to determine the angles of the normals to the surfaces of the mirrors at the points of reflection:

$$\gamma_1=\frac{\alpha_{02}+\alpha_{22}}{2},\quad \gamma_2=\frac{\alpha_{01}+\alpha_{11}}{2}. \tag{5.37}$$

Now we can calculate the equations of the surfaces of the mirrors. Let us specify them in the form

$$\begin{aligned} z^1&=A_1y_1^2+B_1y_1^4+C_1y_1^6+D_1y_1^8,\\ z^2&=A_2y_2^2+B_2y_2^4+C_2y_2^6+D_2y_2^8. \end{aligned} \tag{5.38}$$

The odd powers of y are lacking owing to the axial symmetry of the problem. Then the tangents of the angles of the normal are

$$\begin{aligned} \operatorname{tg}\gamma_1&=\frac{dz_1}{dy_1}=2A_1y_1+4B_1y_1^3+6C_1y_1^5+8D_1y_1^7,\\ \operatorname{tg}\gamma_2&=\frac{dz_2}{dy_2}=2A_2y_2+4B_2y_2^3+6C_2y_2^5+8D_2y_2^7. \end{aligned} \tag{5.39}$$

We note that the coefficients A_1 and A_2 in (5.38) and (5.39) determine the curvature of the mirrors near the z axis, while the paraxial radii of curvature were fixed by the conditions of the problem. We have

$$A^1=\frac{1}{2R_1},\quad A_2=\frac{1}{2R_2}. \tag{5.40}$$

Here R_1 and R_2 are the paraxial radii of the mirrors. Thus we have derived the two equations (5.39) for one ray and can determine the two coefficients in the expansion (5.38). Upon fixing N rays at N heights, we can determine N coefficients in addition to A_1 and A_2, which are known from the conditions of the problem.

One can easily perform the calculations by the iteration method, while assuming the surfaces of the mirrors to be plane in the initial approximation.

6. A NEUTRON MICROSCOPE USING ULTRACOLD NEUTRONS

6.1. A two-mirror achromatic microscope

Recently a neutron microscope with a magnification of 50× has been tested at the Laue-Langevin Institute at Grenoble.[71] The instrument was constructed so that the neutrons after reflection in the first parabolic mirror pass through the apogee of the parabola of flight before they reach the second spherical mirror (Fig. 10). Here the mirrors were arranged to achromatize the system in the first order. The problem of achromatizing such a system was discussed in Sec. 5.5.

The numerical aperture of the instrument amounted to $A = 0.32$. As the authors report, the dominating role in restricting the resolution is played by the higher-order gravitational chromatism, which amounts to several millimeters in the image plane for neutrons with velocities in the range from 5.5 to 6.7 m/s. The geometric aberrations, such as the spherical aberration and the coma, amount to values of the order of 1 mm. That is, they restrict the resolution to a value of the scale of 20 μm in the object plane.

The test experiment consisted in scanning the field of view with a slit moving in the object plane. Here the dimension of the analyzing slit (1.06 mm) exceeded the magnitude of the calculated field of view of 0.35×0.35 mm^2. All the

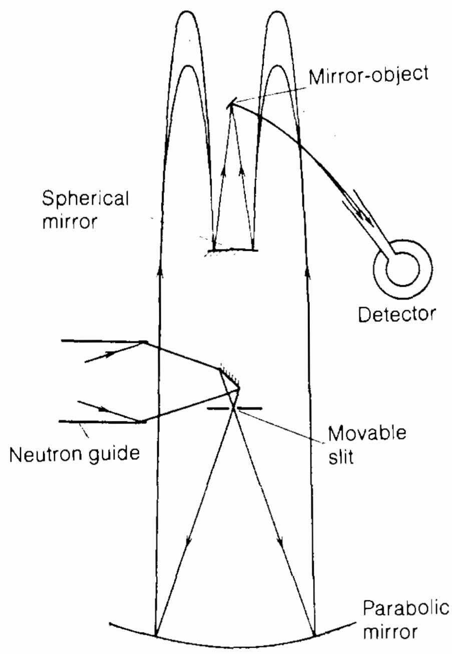

FIG. 10. Neutron microscope.[71]

neutrons that entered the image plane were recorded by the detector, whose dimensions limited the dimension of the field.

The experimentally obtained dependence of the counting rate on the position of the slit agrees with the calculated dependence. Apparently the studies with this instrument will be continued in the near future after installation in this same reactor of a UCN source having an intensity exceeding that previously existing by more than a hundredfold.

6.2. A neutron microscope with an aplanat

Although the problem of chromatism was partially solved in the apparatus described above, owing to the large dimensions of the instrument, it is precisely the chromatism that still presents the main problem. The geometric aberrations of the first mirror, including the gravitational aberrations, introduce smaller distortions, and their role is hardly discussed. On the other hand, in Sec. 5.6 we have treated the possibility of creating an aplanatic system for monochromatic waves, but without treating the problem of chromatism. Naturally the problem arises of the possibility of designing a microscope with an aplanatic objective without special measures for achromatization. One of the useful measures is miniaturization of the whole system. In particular, the dimension of the microscope, as in an ordinary optical instrument, can amount to a value of the order of 10–15 cm or even smaller as compared with the almost two-meter dimensions of the Laue-Langevin Institute microscope.

A two-mirror objective was designed by the method described in Sec. 5.6 for investigating this problem. The calculation was performed for four rays, and hence, five coefficients of the expansion (5.38) were determined. The focal length of the objective amounted to 2.8 mm, the magnification was 25×, and the numerical aperture $A = 0.5$. The objective was calculated for neutrons with a velocity of 6 m/s ($\lambda = 660$ Å). A very simple design of the microscope was chosen in which the objective projects the object into the detector plane.

A trajectory calculation was performed throughout the entire range of working angles for several points of an extended object with transverse dimensions of 0.6 mm. The resolution was taken to be the maximum dimension of the diffuse spot in the detector plane divided by the magnification. The calculation was performed for a certain range of wavelengths. In particular, when UCNs with wavelengths from 630 to 690 Å were used, the calculated resolutions at the center and at the edge of the field amounted respectively to 1.36 and 2.8 μm. When the spectral range was extended within the range 610–720 Å, the resolution deteriorated to the values of 2.2 and 3.9 μm.

It is interesting to note that the special form of the mirrors, which is optimal for parabolic rays, makes this objective more suitable for UCNs than for light. The resolution for light has a value that varies over the field in the range from 5.6 to 12 μm.

Apparently one can assert that the aplanatic objective is suitable for building a simple neutron microscope with a resolution of several micrometers using neutrons with wavelengths lying over a rather broad range. A defect of the instrument is the need for preparing aspheric mirrors with optical accuracy, which presents certain technological difficulties.

6.3. Microscope with magnetic compensation of the force of gravity. Fundamental problems

One could completely solve the problem of gravitational chromatism and higher-order gravitational aberrations if one could compensate the force of the Earth's gravity with a magnetic field acting on the magnetic moment of the neutron. This idea was first expressed in Refs. 26 and 27 and developed in Ref. 69. As will be shown below, it is impossible to neutralize completely the force action on the neutron in an extended region of space. However, if the optical part of the microscope consists simply of an objective that projects the object into the detector plane, then one can try to compensate the force of gravity in the region of the objective. Here the neutrons leaving the objective again enter the region of action of gravitation. It is important to analyze what are the consequences of this.

We see from the expression (5.26) for the optical magnification $1/D'$ in the presence of gravitational force that the magnitude of the optical magnification is completely determined by the matrix $\hat{M}'$, and the geometry of the last segment does not enter directly into the expression for the magnification. Conversely, the chromatism of location is determined by the properties of the entire system, including the last displacement. This implies that, if one has an ideal objective that forms a spherical converging wave from a point source, then the system as a whole will not have chromatic aberrations of magnification, but the circumstance that the gravitational force is acting on the neutrons on the path along the last segment gives rise to chromatism of location, and as a consequence, to gravitational spherical aberration. This hypothetical case has been treated above in Sec. 5.3. One can easily see that the radius of curvature of the initial front is the magnitude of the last segment L of our single-objective microscope in the absence of gravitational force. For neutrons of velocity v the location of the image plane is displaced downward by the amount $gL^2/2v^2$. For the range of velocities from v_1 to v_2 the scatter in location of image planes will amount to

$$2\Delta L=\frac{gL^2}{2}\left(\frac{1}{v_1^2}-\frac{1}{v_2^2}\right).$$

Now it is easy to estimate the limiting magnification, which arises from the distortions on the last segment. Let the quantity L amount to 15 cm, the magnification to 50×, the range of velocities from 4 to 7 m/s, and the entrance aperture angle to $\theta = 45°$. Upon using the results of Sec. 5.5, we find that the limiting resolution in the center of the field amounts to $\delta_l = 2\Delta L\theta/k^2 = 1\ \mu$m. To estimate the resolution at the edge of the field, let us fix the dimensions of the image detector. If the diameter of the detector is, e.g., 1.5 cm, then the resolution at the edge of the field will amount to $\sim 1.5\ \mu$m. As we see, the last segment does not introduce excessively large distortions, even when one uses a broad velocity spectrum. It remains to solve the problem of the "ideal" objective. Let us examine in greater detail the idea of magnetic compensation of the gravitational force.[26,27,69] In an inhomogeneous magnetic field the force acting on the neutron is $\mathbf{F} = \nabla(\boldsymbol{\mu}\cdot\mathbf{B})$, where $\boldsymbol{\mu}$ is the magnetic moment of a neutron. The condition for complete compensation of the gravitational force has the form

$$\nabla(\mu\mathbf{B}) + mg\mathbf{k} = 0. \tag{6.1}$$

Here $\mathbf{k}$ is a unit vector along the z axis. Of course, the condition (6.1) can be satisfied only for a definite polarization of the neutrons.

When very slow neutrons are passing through a region containing a magnetic field, usually the condition of adiabaticity is very well fulfilled. That is, the direction of the magnetic moment "tracks" the direction of the field, while the projection of the magnetic moment of the neutron on the direction of the induction vector is conserved. Here Eq. (6.1) is rewritten as: $\mu\nabla|B| + mg\mathbf{k} = 0$, or in the cylindrical system of coordinates:

$$\mu\frac{\partial|B|}{\partial z} = mg, \tag{6.2}$$

$$\frac{\partial|B|}{\partial\rho} = 0. \tag{6.3}$$

Substitution of numerical values into (6.2) yields the required value of the gradient $\partial|B|/\partial z = 1.7\times10^{-2}$ T/cm.

Of course, one cannot actually attain exact fulfillment of (6.2) and (6.3) throughout the entire region of space. However, these equations can be compatible at a certain point. Therefore the following approximate conditions can hold for a certain extended region of space:

$$\frac{\partial|B|}{\partial z}\approx\frac{m}{\mu}g,\quad \frac{\partial|B|}{\partial\rho}\ll\frac{\partial|B|}{\partial z}. \tag{6.4}$$

Reference 69 treated in this regard a field produced by a current-carrying loop. Figure 11 reproduces a diagram from this study showing the values of B, $\partial|B|/\partial z$, and $\partial|B|/\partial\rho$ from a loop of unit radius through which a unit current is flowing for different values of ρ and z. We see that the curves for $\partial|B|/\partial\rho$ intersect the axis of abscissas in the region $z\approx0.6$ and for different values of ρ, so that the values of the radial gradient for small values of ρ are small also at a certain finite distance from the point $z = 0.6$. At the same time we see that in the region $0.5 < z < 0.7$ and $\rho < 0.2$ the value of the vertical gradient varies weakly, while its absolute value substantially exceeds the value of the radial gradient. Thus the conditions

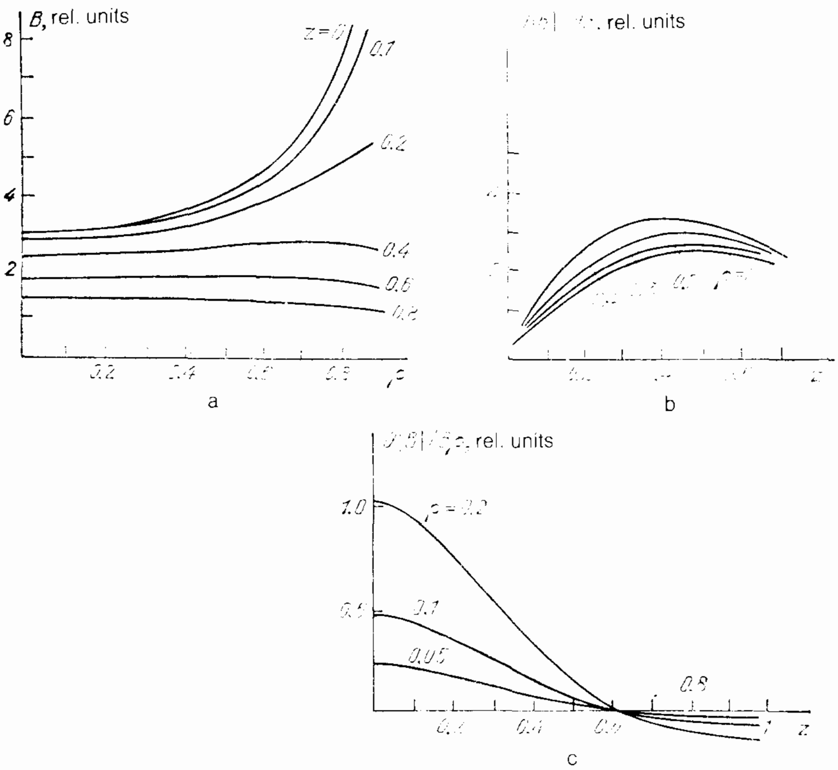

FIG. 11. The magnetic field and field gradient from a current-bearing loop of unit radius. a—Magnitude of the field for different heights z referred to the plane of the loop. b—Vertical field gradient as a function of the height z for different distances ρ from the axis. c—Radial field gradient as a function of the height z for different distances ρ from the axis.

(6.4) can be fulfilled to a certain degree of accuracy. Both the problem of the constancy of $\partial B/\partial z$ in space and the problem of the smallness of $\partial B/\partial \rho$ involve the ratio of the dimensions of the region of space where the stated conditions must be satisfied to the dimensions of the loop. We note that the magnitude of the current to create the needed field gradient at the point $z = 0.6$ increases in proportion to the square of the radius of the loop. If we use a current-bearing ring as the magnetic system, then to compensate the gravitational force by a factor of 40 in a region of space of volume $2\times2\times2$ cm^3, the radius of the winding must be 40 cm, and the value of the current 500 kA. Hence we see that it is difficult to compensate fully the gravitational force throughout the volume of an optical system. Conversely, it is quite possible to shield the objective effectively from the perturbing action of gravitation. Estimates of the distortions that arise if the gravitational force is not compensated on the last segment were given above. One must view these estimates as upper limits, since for a real system a considerable, though not total, compensation will exist also in this region.

Now let us turn to the problem of the spherical aberration of a wave from an initially spherical front propagating in the gravitational field. The source of this aberration was discussed in Sec. 5.3. As is known from optics, when a lens has spherical aberration the phase of the waves arriving at the focal point depends on the distance from the axis to the exit point of the ray from the lens. We can assume to a certain degree of approximation that a region of the lens corresponding to the first Fresnel zone participates in formation of the image. Here the Fresnel zone is determined by the condition that the eikonals of all the rays from this zone of the lens to the focus differ by no more than $\lambda/2$. We can determine the effective Fresnel zone for the case studied above. We shall assume that a small objective has been corrected for the ordinary aberrations, while the gravitational force in the region where the objective lies is fully compensated. If the source is a point source, then the objective forms a convergent spherical wave. As this wave propagates, the properties of the "medium" vary as the compensating action of the magnetic field weakens. We shall assume that the compensation disappears abruptly at a certain plane. Let us locate the origin of coordinates at the point of intersection of the optical axis with this plane. As we can easily see, this is the formulation of the conditions with which the problem of the caustic was treated in Sec. 5.3 (see Fig. 6).

Let us calculate the change in phase on the path from our separation plane $z = 0$ through which the spherical front passes to a certain plane $z = \chi$. In the quasiclassical approximation we have

$$\Phi = \frac{m}{\hbar}\int_0^t v^2\,dt. \qquad (6.5)$$

Going from integration with respect to t over to integration with respect to z, we obtain

$$\Phi = \frac{m}{\hbar}\int_0^{\chi} (v^2 - 2gz)\,\frac{dz}{(v^2\cos^2\theta - 2gz)^{1/2}}\,. \qquad (6.6)$$

Here θ is the initial polar angle of the trajectory and v is the velocity of the neutron. Taking into account the initial phase distribution at $z = 0$ and integrating, we obtain the following, apart from terms of the order of $\xi^2 = (gk/v^2)^2$:

$$\Phi = k\left(1 + \frac{3gR}{2v^2}\,\theta^4\right)\left\{\frac{\chi - R}{\cos\theta} + R + \frac{g\,[\chi^2 - R^2(1-\cos\theta)^2]}{2v^2\cos^3\theta} - \frac{g\,[\chi^2 - R^2(1-\cos^2\theta)]}{v^2\cos\theta}\right\}. \qquad (6.7)$$

Here R is the radius of curvature of the front at the origin of coordinates, and k is the wave number. We shall determine the location of the focus in the paraxial approximation from the condition of stationarity of the phase at the focus (the Fermat principle) in the limit of small angles θ, i.e.,

$$\lim_{\theta\to 0}\frac{\partial\Phi}{\partial\theta}\Big|_{\chi=F}=0.$$

Differentiating (6.7) at $\theta=0$ and setting $\chi=F$, we obtain

$$F-R+\frac{1}{2}\frac{gF^2}{v^2}=0. \qquad (6.8)$$

Solving (6.8) for F, we obtain with an accuracy up to ξ^2:

$$F=R-\frac{gR^2}{2v^2}. \qquad (6.9)$$

This is the same value as obtained earlier from pure geometric considerations in (5.8). Substituting $\chi=F$ into (6.7), we can easily calculate the difference in the phase changes along the trajectories for two rays: with an arbitrary initial angle and with $\theta=0$. In the small-θ approximation we obtain:

$$\Delta\Phi=[\Phi(\theta)-\Phi(0)]\,|_{\chi=F}=2kR\,\frac{gR}{v^2}\,\theta^4. \qquad (6.10)$$

This is the phase difference of the two rays entering the image plane. An essential point is that, strictly speaking, only one ray, which had been propagating exactly along the axis, passes through the focal point. Actually, as we saw above, the ray having an arbitrary initial angle crosses the z axis below the focus—this is spherical aberration. Hence this ray will cross the plane $z=F$ at a certain distance Δr from the focal point, which lies on the z axis and which is the point of observation in our case. Evidently, from simple considerations (Fig. 12), in order to take into account the phase at the observation point, one must subtract the quantity $k'\Delta l$ from the phase advance up to the plane $z=F$. Here we have $\Delta l=\Delta r\sin\alpha$, and α is the angle the trajectory makes with the z axis in the $z=F$ plane:

$$\Delta\varphi'=k'\,\Delta r\sin\alpha=(\mathbf{k}'\,\Delta\mathbf{r}). \qquad (6.11)$$

(The problem of the phase of the wave at an observation point lying near the classical trajectory has been discussed in Ref. 78.)

We obtain the following expression from Eq. (5.11) for the tangent to the ray near the focus:

$$\Delta r=\rho\,|_{z=F}=\frac{\xi R}{2}\,\mathrm{tg}^3\alpha=\frac{gR^2}{2v^2}\,\mathrm{tg}^3\alpha. \qquad (6.12)$$

Generally the quantities k' and α differ from k and θ, but

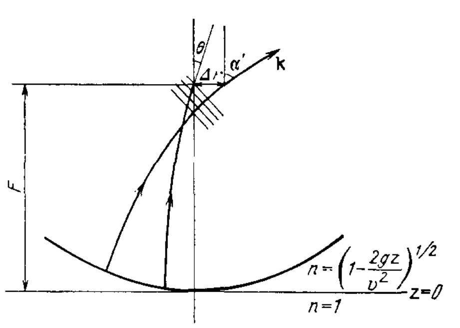

FIG. 12. On the determination of the phase of an arbitrary ray at the focal point.

these differences are on the scale of ξ. We can neglect them since the quantity Δr itself is of the order of ξR. Thus we obtain to the accuracy of ξ^2 that the additional phase shift is

$$\Delta\varphi'=k\Delta r\sin\theta=\frac{gR^2}{2v^2}\,\mathrm{tg}^3\theta\sin\theta\approx\frac{gR^2}{v^2}\,\frac{\theta^4}{2}. \qquad (6.13)$$

The total phase difference at the observation point (at the focus) between the two rays with initial angles of θ and $\theta=0$ will be determined from (6.10) and (6.13):

$$\Delta\varphi=\Phi'(\theta)-\Phi'(0)=kR\,\frac{gR}{v^2}\left(2\theta^4-\frac{\theta^4}{2}\right)=\frac{3}{2}\,kR\,\frac{gR}{v^2}\,\theta^4. \qquad (6.14)$$

Thus we see that the gravitational spherical aberration in principle imposes certain restrictions on the possible limits of resolution of the instrument. If the exit pupil of the objective is smaller than the dimension of the "gravitational" Fresnel zone, then the limiting resolution is determined, as usual, by the expression

$$\delta\approx\frac{0.61\lambda}{A}.$$

Here λ is the wavelength, A is the numerical aperture, $A=\sin\theta_{\max}$, where $\theta_{\max}$ is the aperture angle. In the opposite case the magnitude of the aperture angle is restricted by the dimension of the effective exit pupil—the dimension obtained above for the Fresnel zone. Exactly the same situation exists in ordinary optics in the presence of spherical aberration. Geometric estimates of the field at the focus of the lens in the presence of spherical aberration are treated, e.g., in Ref. 53.

Now let us estimate the dimensions of the Fresnel zone for a case that seems realistic from the standpoint of the possibility of designing a neutron microscope. As before, let the distance to the image plane be $R=15$ cm. Let the neutron velocity be $v=6$ m/s, which corresponds to a value of the wave vector $k=0.95\times10^6\ \mathrm{cm}^{-1}$. Let us determine from (6.14) the value of the angle for which $\Delta\varphi=\pi$. We obtain $\theta=0.137$ rad. Hence the radius of the first Fresnel zone is

$$\rho_F=R\ \mathrm{tg}\,\theta\approx 2\ \mathrm{cm}.$$

Thus we see that, for an objective with dimensions characteristic of an ordinary optical microscope, the gravitational distortions on the last segment of flight do not lead to fundamental restrictions of resolution, even in the total absence of magnetic compensation in this region. Simultaneously we can conclude that the trajectory calculation is fully valid when estimating the resolving power of a neutron microscope up to resolutions comparable with the wavelength.

6.4. Microscope with magnetic compensation of the force of gravity. Possible construction

To estimate the possible resolution of a neutron microscope we have performed a trajectory calculation of the resolving power of a simple microscope with magnetic compensation of the gravitational force in the region of the objective.

The optical system of the instrument consisted only of the objective, which projected the object into the image plane (the detector plane) (Fig. 13). Apparently it is not very essential which of the possible schemes of a mirror objective is chosen for this case. An objective was chosen very similar to that described in Sec. 6.2. The objective consisted of two aspheric mirrors with equal paraxial radii of 8.442

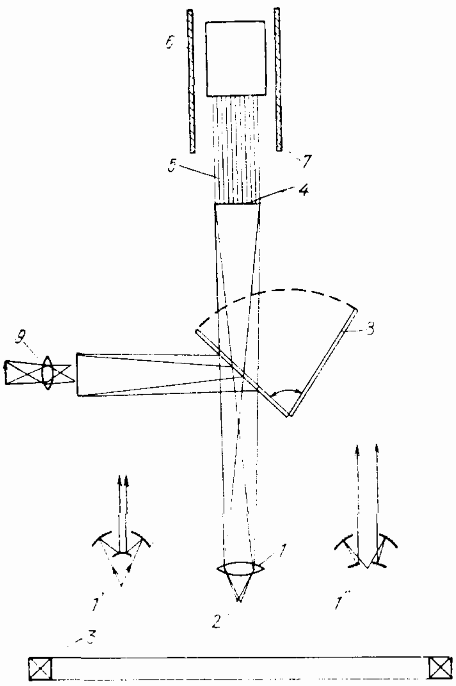

FIG. 13. Possible design of a neutron microscope with magnetic compensation of the gravitational field. *1*—objective (*1'* and *1''* are possible variants of mirror objectives); *2*—object; *3*—coil of superconductive magnet; *4*—image plane, in which the sensitive layer of the detector is placed; *5*—fiber light guide; *6*—light amplifier; *7*—magnetic shielding; *8*—hinged mirror; *9*—ocular.

mm. The objective was designed in such a way that in the absence of gravitation it yielded an aplanatic image of a plane object in a plane separated from it by the distance 15 cm with an optical magnification of 50×. The numerical aperture was $A = 0.5$. The maximum distance of the extreme ray from the axis inside the objective was about 3.5 mm, and 1.5 mm in the exit beam. A current-bearing loop of diameter 1 m was taken as the magnetic system in this model. To make the axial induction gradient at the height $z = 0.6\,R$ equal to the required value of 1.7×10^{-2} T/cm, the current in the loop must amount to 8.23×10^{5} A. If one uses a superconductive winding with a current density of 10^4 A/cm^2 to create the needed induction, then the winding must have a cross-section of the scale of 90–100 cm^2. We can expect, as was verified by calculation, that the pattern of induction gradients from the rectangular winding of cross-section about 10×10 cm^2 practically does not differ from that obtained for a thin coil.

The exact calculation of the trajectory of motion of a particle in a field inhomogeneous in two coordinates is a rather difficult problem. Therefore several assumptions simplifying the problem were made.

Since the magnitude of the maximal deviation of the trajectory of the neutron from the axis amounted to only 7×10^{-3} in units of the radius, the radial dependence of the quantity $\partial |B|/\partial z$ was neglected in calculating the vertical force (see Fig. 11b). The dependence of $\partial |B|/\partial z$ on z was described by a step function whose value was tabulated at an appropriate number of points. The radial gradient was approximated by the function

$$\frac{\partial\,|B|}{\partial\rho} = \rho\,(A_i + B_i\rho). \qquad (6.15)$$

Here the coefficients A_i and B_i were also tabulated.

Under the assumption that the velocities of the neutrons lie in the range 4.6–6 m/s (range of wavelengths 660–880 Å), while the dimension of the detector is 1.5×1.5 cm^2, it was found that the resolution at the center of the field is of the scale of the wavelength, while at the edge of the field it amounts to 0.3 μm. Of course, one must not attribute too much value to the absolute values of the obtained resolution. One must treat them as estimates of the maximum possible resolution. Nevertheless we can state with assurance that neutron-microscope designs with magnetic compensation in the region of the objective have great potentialities.

6.5. The neutron microscope and the problem of the intensity of UCNs

In 1973 in a now classic study, "Ultracold Neutrons,"[16] *F. L. Shapiro mentioned the neutron microscope* as one of the distinct potentialities of the use of UCNs. Actually, at that relatively recent time, the very statement of such a possibility in the future seemed rather bold. And even now, after the passage of more than a decade, the opinion that the neutron microscope belongs to the distant future is widespread. The basis for this is the very low counting rate in the optical experiments on UCNs performed up to now. The point is that instruments were used in these experiments with a small numerical aperture, and hence with a small collecting power, and further, they were installed at relatively weak UCN sources. As an example we point out an experiment in which a multimirror achromatic system was used[70] that had a collecting power of the order of 10^{-2} (in the sense of the fraction of the solid angle in units of 2π). As is now clear, it is quite practical to employ objectives with a numerical aperture of the order of 0.5–0.6 having a collecting power of 0.1–0.2.

An important second factor is the rather fast progress in the technology of UCN sources (Fig. 14). It is interesting to estimate the potentialities of today and the near future from the standpoint of realistic neutron-microscope projects.

The best UCN source now existing[86] has a flux density of UCNs of $f = 1.2\times 10^{3}$ n/cm^{2}s with a density of thermal flux at the bottom of the channel of $F_0 = 6\times 10^{13}$ n/cm^{2}s. A simple copy of such a source at a high-flux reactor such as

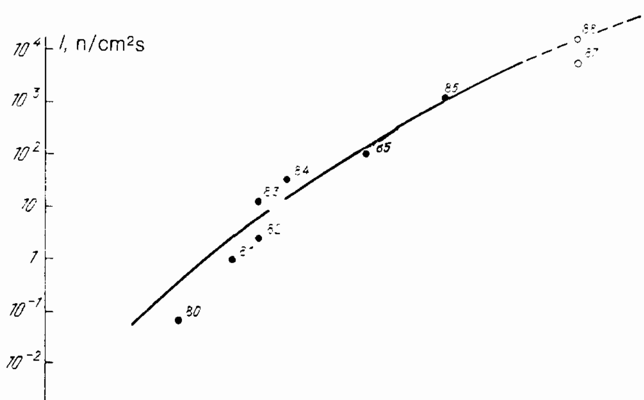

FIG. 14. Increase in the flux densities of exisitng UCN sources.

the HFR at the Laue-Langevin Institute at Grenoble or the PIK reactor being built at the Leningrad Institute of Nuclear Physics will yield an increase in flux by an order of magnitude. Perhaps by using a number of design features one can increase the yield again by a factor of two or three.

Let us adopt for estimates a magnitude of the UCN flux density of 2×10^4 n/cm^{2}s, which is close to the published data of the corresponding projects.[87,88] Let us estimate the "statistical" resolution at such a source. The count in a time T from an elementry region of the object will be

$$N = fT\Omega\mathscr{E}\delta^2.$$

Here δ is the linear dimension of the element, f is the flux, Ω is the solid angle employed, which is determined by the aperture, and $\mathscr{E}$ is the efficiency of the detector. Let $N = 100$. Such statistics suffices not only to record the count, i.e., distinguish "black" from "white," but also to attribute a certain gradation in brightness to all elements. (We are assuming as yet that the background over the time of measurement is rather small.) We shall assume that one can employ a solid angle $\Omega = 0.2$. A reasonable estimate of the efficiency of the UCN detector is $\mathscr{E} = 0.8$. Then with a counting time $T = 10^6$ s (10 days), we find that $\delta = 1.77\times 10^{-4}$ cm. If we employ magnetic compensation, then we must use polarized neutrons, which will increase the estimate to 2.5 μm. Undoubtedly we must treat this value as an estimate, but we should consider the result of this estimate to be the practicality of microscope projects with a resolution of the order of a micrometer, even in the very near future.

Turning to the longer view, we can state with a certain degree of assurance that even more intense UCN sources are possible, in particular, with pulsed neutron sources at modern accelerators.[89] Also the "inner" reserve in the construction of optical systems themselves is not bad. If it were possible to go from UCNs to faster neutrons, then the intensity would increase correspondingly. We recall that the intensity increases as the cube of the velocity of the neutrons employed. Therefore a shift, say, to multilayer mirrors with a limiting velocity of reflection of 15 m/s, which already seems now quite realistic, will yield a gain in intensity of almost an order of magnitude.

6.6. Proposed possibilities of the neutron microscope

It seems opportune to try to elucidate the most essential features of the neutron microscope based on very slow neutrons that determine its potentialities of application.

As is known, the neutron-optical properties of a medium depend substantially on its nuclear (chemical) composition. In fact, the refractive index for a neutron wave is determined by Eq. (3.4),

$$n^2 = 1 - \lambda^2 \frac{\sum_i N_i b_i}{\pi}.$$

Here b_i is the scattering length for the nuclei of different types, and N_i is the density of the corresponding nuclei. One of the features of neutron optics is the lack of a monotonic dependence of the scattering length on the atomic number or nuclear charge, as occurs, e.g., for x-rays. Consequently substantial differences exist in the refractive indices, not only for adjacent chemical elements, but also often for adjacent isotopes of a single element. The coefficients of reflection and transmission of neutrons are associated with the value of the refractive index. Therefore substances with differing but similar chemical composition can sometimes differ substantially in their neutron-optical properties. This difference can consist in the value of the limiting energy or the value of the refractive index. The value of the limiting energy can vary from element to element severalfold, while remaining, as a rule, of the order of 10^{-7} eV. This means that, for UCNs having an energy of this same order of magnitude, the magnitudes of the coefficient of reflection can differ substantially for substances of differing chemical composition. Therefore a neutron microscope with UCNs can show chemical contrast. This situation had been pointed out earlier.[26,27,89,71] In an ordinary microscope design the chemical contrast rapidly declines with increasing neutron energy, since in this case the material becomes almost completely transparent. Here the refractive index in (3.4) approaches unity, although its difference from unity is still quite substantial for very cold neutrons with energies exceeding the limiting energy severalfold. In this range of energies, it is not so much the coefficient of reflection, which is very small here, that depends on the chemical composition, as the phase of the neutron wave that has passed through the specimen [see (3.8)]. Apparently, when one uses very cold neutrons (VCNs), one can obtain chemical contrast only when one can apply to the neutron microscope one of the phase-contrast schemes known in ordinary optics. The difficulty consists in the fact that the application of the known phase-contrast schemes to a projecting neutron microscope substantially lowers the collecting power of the instrument. Nevertheless we can hope that a phase-contrast neutron microscope can be built. An essential feature of a future neutron microscope is isotope sensitivity, or isotope contrast. Apparently the circumstance that will have the greatest practical significance will be the fact that the two stable isotopes of hydrogen have strongly differing (even in sign) values of the coherent scattering length. For hydrogen $b = -3.741$, while for deuterium $b = 6.674$ in units of 10^{-13} cm. Therefore ordinary and deuterated materials strongly differ in their limiting energies, or what is the same thing, in their values of the mean scattering length. Thus, for ordinary and heavy water the values of the mean scattering length are: $b = 1.68$ and 19.14 Fm. Upon deuteration, also more complicated organic materials change appreciably in neutron-optical properties. In this regard we can hope that it will be possible in neutron-microscopic study to use the method of an optical deuterium marker.[27] The negative scattering length of the proton distinguishes hydrogen from most other elements entering into organic materials. This has the result that materials rather similar in composition can differ appreciably in their neutron-optical properties owing to differing hydrogen content. This situation is illustrated by Table II, for which the data are taken from Ref. 90. All of this gives grounds for assuming that a neutron microscope can be especially useful in biological studies.

As regards the resolution of the instrument, setting aside the problem of the statistical potentialities of the sources, we shall assume it to be limited by the wavelength. Then, when one uses UCNs the resolution can be of the order of 600–800 Å, while upon going to VCNs it is severalfold better. Thus, in resolution the neutron microscope can occupy an intermediate niche between the ordinary and the elec-

TABLE II. Refractive indices of certain amino-acid residues for neutrons with three velocities.*

Amino acid	Chemical composition	Number of exchangeable hydrogen atoms	Refractive index, $v = 5$ m/s, Ordinary	$v = 5$ m/s, Deuterated	$v = 20$ m/s, Ordinary	$v = 20$ m/s, Deuterated	$v = 50$ m/s, Ordinary	$v = 50$ m/s, Deuterated
Glycine	C_2NOH_3	1	0,692	0,408	0,9836	0,9736	0,99739	0,69958
Alanine	C_3NOH_5	1	0,801	0,642	0,9887	0,9815	0,99821	0,99706
Valine	C_5NOH_9	1	0,890	0,803	0,9934	0,9888	0,99895	0,99822
Leucine	C_6NOH_{11}	1	0,913	0,842	0,9948	0,9909	0,99916	0,99853
Phenylalanine	C_6NOH_{11}	1	0,770	0,701	0,9872	0,9840	0,99796	0,99745
Tyrosine	$C_9NO_2H_9$	2	0,733	0,577	0,9854	0,9789	0,99768	0,99666
Tryptophan	$C_{11}NO_2H_{10}$	2	0,702	0,563	0,9840	0,9784	0,99746	0,99658
	$C_4NO_3H_4$	1	0,570	0,375	0,9787	0,9728	0,99662	0,99569
Aspartic acid								
Glutamic acid	$C_5NO_3H_6$	1	0,738	0,564	0,9857	0,9785	0,99772	0,99659
Serine								
Threonine	$C_3NO_2H_5$	2	0,743	0,363	0,9859	0,9725	0,99776	0,99564
Asparagine	$C_4NO_2H_7$	2	0,806	0,557	0,9890	0,9782	0,99825	0,99654
Glutamine	$C_4N_2O_2H_6$	3	0,699	0,169	0,9839	0,9692	0,99744	0,99513
Lysine	$C_5N_2O_2H_8$	3	0,812	0,441	0,9893	0,9745	0,99829	0,99596
Arginine	$C_6N_2OH_{13}$	4	0,905	0,590	0,9943	0,9794	0,99910	0,99674
Histidine	$C_6N_4OH_{13}$	6	0,785	Imaginary	0,9879	0,9659	0,99808	0,99462
Methionine	$C_6N_3OH_{6,5}$	1,5	0,639	0,471	0,9813	0,9754	0,99703	0,99610
	C_5NOH_9	1	0,891	0,820	0,9935	0,9837	0,99897	0,99836

*The values of the refractive index in the deuterated case correspond to the case in which all the exchangeable hydrogen atoms have been exchanged for deuterium with the solvent.

tron microscope, at the same time possessing substantially new qualities.

The question of the choice of the range of neutron energies most suitable to use in microscopy will be solved in time, starting with practical considerations. Both UCNs and VCNs have their advantages and defects from this standpoint. When one goes to faster neutrons the chemical contrast diminishes and it becomes impossible to use mirrors with total reflection at normal incidence. However, as we have already pointed out, here the intensity increases as the cube of the velocity, while the wavelength decreases, which enables one to obtain better resolution.

7. CONCLUSION

In closing the discussion of the problem of the neutron microscope, we wish to present some results. Apparently as the chief result of recent time one should view the change in the psychology of physicists concerned with this and related problems. From being a matter for the distant future, the problem of the neutron microscope has now been brought into the rank of practical problems.

A certain progress exists in the theoretical and computational sphere. The ways have been outlined for correcting gravitational aberrations. More or less realistic microscope projects have been appearing, with which one will be able to obtain images of objects with a resolution characteristic of light microscopes.

However, on the experimental level the advances have been considerably more modest. The fundamental results here are: focusing of very cold neutrons has been carried out by using a lens and a zone plate. Here the resolution was of the order of 50 μm at a wavelength of 20 Å. There is an experiment on two-dimensional focusing of UCNs with a resolution on the scale of 1 mm. Three experiments have been performed on three-dimensional focusing of UCNs using a zone mirror, a multimirror instrument, and the neutron microscope at Grenoble. A magnified image of a slit was observed, with magnification ranging from $1.375\times$ to $50\times$. A physically more essential characteristic is the resolution of the instrument. In the stated experiments the resolution, strictly speaking, was not measured, although apparently it was no poorer than 100 μm at a wavelength of 600–800 Å. The image was analyzed by scanning the detector with a movable slit in the image plane or in the object plane.

Of course, it is substantially more advantageous to design optical experiments by using some type of position-sensitive detector, rather than in a scanning geometry.

The numerical aperture in the experiments with VCNs amounted to only 4×10^{-4}, so that the obtained resolution exceeds the wave limit of this instrument in total by an order of magnitude. In the experiments with UCNs the numerical aperture was of the order of 0.2–0.3. The existing or projected UCN sources already enable one to operate with resolutions of the order several micrometers under the condition of increasing the collecting power of the instrument, i.e., the numerical aperture, and using a position-sensitive detector. Thus the problem of building instruments with a resolution of the order of the light microscope is actually a problem for today.

A second problem to be solved consists in seeking practical methods of achieving chemical and isotope contrasts in neutron-optical instruments. At present this problem is of independent interest, and one can set up the appropriate experiments even with instruments of lower resolution, but possessing sufficient aperture. There are as yet no experiments directed toward solving this problem.

In closing I wish to thank my colleagues S. S. Arzumanov, A. N. Strepetov, and S. V. Masalovich, with whom I have had the satisfaction of collaborating in the field of instrumental neutron optics of UCNs, which is the topic of this article. Of course, all the fundamental problems treated in this paper have been discussed with them. I thank I. M. Frank for numerous and fruitful discussions, and also P. A. Krupchitskiĭ, who took on himself the labor of reading the manuscript and making a set of highly useful remarks.

I take the opportuity of acknowledging S. T. Belyaev

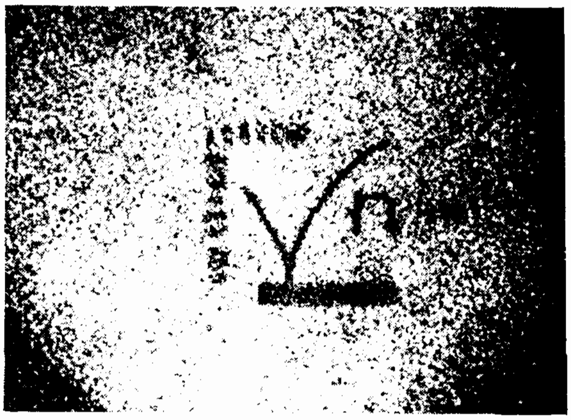

FIG. 15. Image of an object recorded in neutron "light." The dark regions correspond to a nickel coating.

and I. I. Gurevich for supporting the work.

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During the time of preparing this manuscript for publication several studies have appeared in print on the problem of the neutron microscope or directly bearing on it. A brief review of their content is given below.

An important event is the startup of two intense UCN sources at Grenoble[92] and Leningrad.[93] The creation of these sources enables performing studies on the practical optics of UCNs on a completely new level. In Sec. 6.1 the proposal was expressed of continuing the studies with the NM described in Ref. 71 at a new source. Reference 94 has reported the concrete plans of research with this instrument. The same study reported an experiment on focusing VCNs with a velocity of 10–13 m/s in a mirror optical instrument with a multilayer interference coating, or supermirror. A resolution of 230 μm was obtained in the image plane.

Reference 95 has reported on an experiment performed by a group from the I. V. Kurchatov Institute of Atomic Energy using the LIYaF source.[93] This experiment employed an image detector based on a scintillator sensitized to UCNs, with subsequent amplification of the light and recording on photofilm. Here for the first time all the elements of a neutron microscope were present: an object illuminated with neutrons, an optical system forming an image in the detector plane, and a detector visualizing the image. The achromatic instrument was used whose design was discussed in Sec. 5.5 and which is shown in Fig. 7. A two-dimensional image was obtained of a cross-shaped diaphragm and of a more complex object prepared by photolithography. A thin-layer (2000 Å) nickel pattern was deposited on a silicon substrate, and its image in neutron rays is shown in Fig. 15. Thus the existence of neutron contrast in UCN optics was demonstrated. The resolution of the instrument is no poorer than 100 μm.

A study[96] has also appeared that discusses the possibilities of neutron-microscopic investigations in biology.

[1)] This is true if the trajectory does not pass through a focus and is not reflected from a caustic. In the converse case one must take into account an additional caustic phase shift.

[2)] For more details on the definition and physical meaning of the quantities b and f_0 see Refs. 3 and 10.

[3)] In speaking of a single nucleus, we only assume the absence of an effect of the waves scattered by other nuclei. However, the question at all times is the amplitude of scattering by a fixed nucleus. We recall that the scattering amplitudes by free and bound nuclei differ by the amount of the effective mass $b = a(A + m)/A$, where A is the atomic number of the nucleus, and m is the mass of the neutron.

[4)] Cases can occur in which the influence of laboratory magnetic fields is significant, e.g., in optical instruments with a large trajectory length of the scale of several meters and with rather rigorous requirements on resolution. One must also consider stray magnetic fields in building a neutron interferometer using UCNs. In studies seeking the EDM of the neutron this is generally one of the most serious problems.[50]

[5)] The analogy of the refractive index for neutrons in a medium and the refractive index for short-wavelength electromagnetic radiation was pointed out in Ref. 23.

[6)] For more detail on caustics, see, e.g., Ref. 54.

[7)] VCNs: very cold neutrons. Neutrons are often called this whose velocities, in contrast to UCNs, somewhat exceed the limiting velocity of the material.

[8)] Strictly speaking, to find the equation of the caustic one must start not from (5.11), but from the equation of the parabolic trajectory. However, near the focus the equation found in this way reduces to (5.12) owing to the small curvature of the rays.

[9)] The results presented in this section and also in Secs. 6.2 and 6.4 were obtained jointly with A. N. Strepetov.

[10)] Coma: an aberration of a lens or mirror manifested in the fact that each annular zone of the lens forms its own extended image of an extraaxial point of the source, with this image possessing a complicated comet-shaped form.

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