

Interacting magnons

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The present state of the theory of spin waves in magnetic insulators is reviewed. Special emphasis is placed on systems with an arbitrary site spin. Some illustrative calculations are carried out on the magnetic characteristics of several systems with a low density of particles (magnons) which interact strongly with each other. The distinctive features of reduced-dimensionality systems which stem from the governing role of spin fluctuations are examined. A separate chapter is devoted to magnetic phase transitions, primarily orientational. Several effects which stem from the dipole interaction of spins are also discussed.

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1. INTRODUCTION

This is a review of magnons: the quanta of elementary excitations of the spin subsystem of a magnetically ordered crystal. We will be discussing only magnetic insulators, in which the electronic (orbital) excitations are separated from the ground state by energies of the order of the atomic energy ε_a . The excitations of interest here are those with energies significantly smaller than ε_a .

A spin wave (magnon) is one branch of the low-energy excitations of a crystal. If we restrict the discussion to long-wave oscillations, with $ak \ll 1$ (a is the interatomic distance, and \mathbf{k} is the wave vector), we can describe the low-energy excitations phenomenologically on the basis of the continuum approximation.¹⁻³ In this approach the general system of equations describing long-wavelength oscillations consists of the linearized equations of electrodynamics and the theory of elasticity (including the equations of optical oscillations) and also the Landau-Lifshitz equations for the densities of the magnetic moments. A natural way to derive the dispersion laws for the elementary excitations is to eliminate the magnetization and displacement vectors from the system of linearized equations. We are then left with only the equations of macroscopic electrodynamics, but with a spatial and temporal dispersion in the permittivity and the magnetic permeability. As a result, the dispersion laws for the elementary excitations are various solutions of the dispersion relation

$$D(\mathbf{k}, \omega) = 0. \quad (1.1)$$

Taking this approach, we find expressions which relate the frequencies ω of the excitation branches—polaritons and optical magnons (magnetic polaritons)—with the wave vector

\mathbf{k} (Ref. 4). An understanding of the crystal structure and the symmetry elements of the crystal enables one without solving the Landau-Lifshitz equations and the equations of elasticity theory to make several statements regarding the spatial and temporal dispersion of the tensors $\varepsilon_{ik}(\omega, \mathbf{k})$ and $\mu_{ik}(\omega, \mathbf{k})$ (Ref. 5).

Strictly speaking, for any arbitrary wave propagation direction in a crystal each of the wave branches is a coupled oscillation of all the quantities involved in the problem (e.g., the propagation of an optical magnon is accompanied by oscillations in the electric as well as the magnetic fields). Over a broad interval of wave vectors, however, it is sufficient to focus on one or a few most important oscillating quantities and to ignore the oscillations of other quantities. This simplification becomes possible either because of differences in the propagation velocities of different processes or because of small values of the constants of the interactions between them. In the case of an optical magnon (the branch in which the oscillations of the magnetic subsystem are manifested most strongly) the small value of the velocity of this magnon in comparison with the velocity of light, c , frequently allows us to ignore the oscillations of the electric field, while the weakness of the magnon-phonon interaction allows us to ignore elastic vibrations of lattice atoms. We then end up with purely magnetic oscillations, whose quanta are magnons.¹⁾

The equations describing long-wavelength magnetic oscillations are the magnetostatic equations

$$\text{rot } \mathbf{H} = 0, \quad \text{div } \mathbf{B} = 0, \quad B_i = \mu_{ik}(\omega, \mathbf{k}) H_k, \quad (1.2)$$

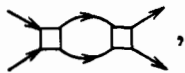
in which the spatial-temporal dispersion $\mu_{ik}(\omega, \mathbf{k})$ is determined exclusively by the magnetic structure of the crystal:

by the number of sublattices, the sign and magnitude of the exchange interaction between magnetic moments, the anisotropy energy, and the magnitude and direction of the external magnetic field. All these quantities can be introduced phenomenologically, and dispersion laws can accordingly be calculated for magnons.^{1,6} By comparing the formulas found by this approach with experimental results one can determine the constants introduced phenomenologically.

The phenomenological formulas for the magnon energies are sufficient for describing a long list of phenomena: resonance effects, the temperature dependence of the magnetization and of the magnon heat capacity at temperatures T low in comparison with the transition temperature T_c (and far from points of orientational transitions), etc. On the other hand, many effects (relaxation phenomena, the temperature dependence of the spectrum and of the points of orientational transitions along the magnetic-field scale, etc.) cannot be explained by a simple phenomenological model. For each phenomenon one could of course introduce new phenomenological constants (e.g., one could introduce a spin-wave lifetime τ for relaxation phenomena), but that approach is not very informative. A systematic theory for the nonlinear effects listed above will clearly require consideration of the interaction between magnons and also the interaction of magnons with other elementary excitations. The interaction between magnons is primarily of an exchange origin and thus not a minor one. It must therefore be incorporated completely (the first few terms of a perturbation theory are not sufficient). It is true that for small wave vectors the interaction is effectively weak because the amplitude (A) of the scattering of magnons by each other is proportional to the wave vectors. In an isotropic ferromagnetic, for example, the amplitude for scattering through a zero angle is

$$A = \begin{array}{c} k_1 \\ \swarrow \quad \searrow \\ \square \\ \swarrow \quad \searrow \\ k_2 \end{array} \propto \lambda_{ij} k_1^i k_2^j + \lambda_{ijkl} k_1^i k_2^j k_1^l k_2^m.$$

This amplitude is renormalized by the exchange interaction. Even in first-order perturbation theory the correction to A ,



without changing its dependence on the wave vectors, does change the important coefficient λ_{ijkl} , by an amount proportional to

$$\int \frac{d^3k}{e_k} \propto \int \frac{d^3k}{k^2}.$$

This integral "lands on" its upper limit, demonstrating that the continuum approximation cannot be used.

A systematic derivation of magnetic properties should begin with a microscopic analysis based on a consideration of the interactions of real atoms with each other. This "first-principles" approach is a very complicated one.⁷ On the other hand, the exchange interaction between atomic spins, which plays a very important role in magnetic phenomena, has little effect on the structure of a crystal lattice. We are thus justified in taking a semiphenomenological approach: We consider a definite crystal lattice at whose sites there are spins S_i (i is the radius vector of the lattice site) of fixed

magnitude.²⁾ The interaction between spins is specified by a model spin Hamiltonian which contains a set of constants. It is assumed that these constants can be calculated from the quantum theory of atoms.^{7,8}

The problem of determining the magnetic properties of a substance essentially reduces to one of studying the properties of a lattice gas of spin arrows.

2. SPIN HAMILTONIAN: TYPES OF MAGNETIC SUBSTANCES

Nature displays a rich variety of types of magnetic substances. They differ in particular in the number of magnetic atoms (the number of spin arrows) in the unit cell of the crystal; this number can range up to several tens. As the crystal cell becomes more complicated in structure, however, the spectrum of spin waves does not acquire any fundamentally new properties. For example, the number of low-lying branches in the magnon spectrum does not exceed three, regardless of the total number of magnetic atoms in a cell.⁹ Some fairly general ideas regarding the properties of waves of various types can be extracted from a consideration of some simple spin models: crystals with one or two magnetic atoms per cell. We will restrict the present discussion to such models.

Ordered magnetic substances divide in a natural way into groups of ferromagnetic and antiferromagnetic materials. In this simplest case they are described by the same exchange Hamiltonian, but the exchange integral J has different signs:

$$\mathcal{H}_{\text{ex}} = -\frac{1}{2} \sum_{i, \Delta} J(\Delta) S_i S_{i+\Delta}, \quad (2.1)$$

where $J > 0$ for a ferromagnetic and $J < 0$ for an antiferromagnetic material. The vector Δ joins nearest neighbors in a lattice. Since we are not talking about metals, in which free electrons mediate the long-range effect between spins (the RKKY interaction), the discussion can be restricted to the interaction of only nearest neighbors. Hamiltonian (2.1) is the simplest but by no means the only possible exchange Hamiltonian of a ferromagnetic material (the assertions which follow apply equally well to antiferromagnetic substances). In the first place, the interaction between a selected atom and its next-nearest neighbors may be important. Second, there may be situations in which an interaction involves more than a pair of spins [this is the case, for example, in solid ^3He (Ref. 10)]. Finally, in the case $S \neq 1/2$ the exchange Hamiltonian may contain, along with a term which is bilinear in the spins, terms of the type $(S_i S_{i+\Delta})^n$, where $2 \leq n \leq 2S$. Any added complexity in the Hamiltonian of course leads to a further complication of the corresponding phase diagram (for example, if there is a strong antiferromagnetic exchange between next-nearest neighbors, the original ferromagnetic structure becomes helimagnetic¹¹). The description of possible types of complex magnetic structures has been the subject of many studies.^{11,12} It is not our purpose in the present review to examine those studies. We see a different goal: to use the examples of very simple models to travel the path from a microscopic Hamiltonian to the macroscopic characteristics without introducing any phenomenological parameters and without resorting to order-of-magnitude calculations. For this reason we will consider only the standard Heisenberg Hamiltonian³⁾ (2.1).

We choose the anisotropy energy in the form

$$\mathcal{H}_a = -\frac{1}{2} \sum_{1, \Delta} g(\Delta) S_1^z S_{1+\Delta}^z. \quad (2.2)$$

Again, this is not the only possible form of the anisotropy energy; it might in general also include terms with other spin projections (not necessarily identical). The reason we have chosen the anisotropy energy in the form in (2.2) is that we can examine several fundamental properties of magnetic substances on the basis of a comparatively simple example. For the same reasons as for the exchange energy, the anisotropy energy is determined only by the interaction with the nearest surroundings of the magnetic atom. Admittedly, we must bear in mind that in general we would have $g(0) \neq 0$ but $J(0) \equiv 0$ (there is of course no exchange interaction of the spin of an atom with itself). The term containing $g(0)$ describes a single-ion anisotropy. It is physically meaningful for $S \neq 1/2$. The remainder, which contains $g(\Delta \neq 0)$, describes the unlike-ion anisotropy, which is frequently interpreted as an anisotropy of the exchange interaction.

The anisotropy described by (2.2) arises as a result of the spin-orbit interaction. The anisotropic part of the spin energy also includes the energy of the dipole-dipole interaction of spins (μ is the Bohr magneton):

$$\mathcal{H}_{dd} = 2\mu^2 \sum_{1 \neq 1'} [(S_1 S_{1'}) R_{11'}^2 - 3(S_1 R_{11'}) (S_{1'} R_{11'})] R_{11'}^{-5}. \quad (2.3)$$

In contrast with (2.2), the dipole-dipole interaction cannot be limited to nearest neighbors. In it we can single out a spin-isotropic term and a term which has the structure of an anisotropy energy. These terms make long-range corrections to $J(\Delta)$ and $g(\Delta)$, which are manifested in the dependence of the energy of the spin wave on the direction of the quasiwave vector \mathbf{k} in the limit $k \rightarrow 0$. Furthermore, \mathcal{H}_{dd} contains terms whose role does not reduce to one of renormalizing $J(\Delta)$ and $g(\Delta)$. These terms not only contribute to the angular dependence of the spectrum in the limit $k \rightarrow 0$ but also give rise to several "fine" effects, which we will be discussing below (§8). Until we take up these fine effects, we will ignore the dipole-dipole interaction for simplicity. As we will see below, this simplification is legitimate if the dimensionless anisotropy constant is greater than 4π .

The complete Hamiltonian, which contains Zeeman energy $2\mu H S_1^z$ along with \mathcal{H}_{ex} and \mathcal{H}_a , is

$$\mathcal{H} = -\frac{1}{2} \sum_{1, \Delta} (J(\Delta) S_1 S_{1+\Delta} + \beta(\Delta) S_1^z S_{1+\Delta}^z) - \sum_1 [\alpha (S_1^z)^2 - 2\mu H S_1^z]; \quad (2.4)$$

here $\beta(\Delta) \equiv g(\Delta \neq 0)$, and $\alpha \equiv g(\Delta = 0)$. We know from quantum mechanics that an anisotropy always arises as a result of relativistic interactions, so it must be small in comparison with the exchange to the extent that the ratio $(v/c)^2$ is small, where v is the velocity of the orbital motion of electrons, and c is the velocity of light. This is indeed the situation in a large number of real magnetic substances (see Table VII below). In recent years, however, research has revealed many substances in which the energy of the unlike-ion anisotropy is of the order of the exchange energy, so these substances can be classified quite accurately as Ising or planar magnetic substances (some corresponding examples are given in Table VII). As a rule, a situation of this sort arises because the exchange interaction (which is isotropic in spin

space) between magnetic ions turns out to be small for some reason or other.

To simplify the geometric side of the analysis, and to avoid some complicated equations, we restrict the discussion to a simple cubic lattice with a nearest-neighbor interaction ($J(\Delta) \equiv J, \beta(\Delta) \equiv \beta$). This is of course an idealization: writing the anisotropy energy as in (2) presupposes that the unit cell is at least tetragonal. However, an increase in the structural complexity of a unit cell seldom causes any qualitative changes.⁴⁾

The reason why a Hamiltonian as simple as that in (2.3) is capable of describing a variety of magnetic structures is that the signs of the anisotropy and exchange constants can be arbitrary. Before we list the various cases we note that the roles played by the single-ion and unlike-ion anisotropies are essentially identical (as long as they are small). The only exception is the case $S = 1/2$, in which the term containing the single-ion anisotropy reduces to a constant. For simplicity, we will not consider the unlike-ion anisotropy in this review. The interested reader can find the corresponding results in the original papers.^{13,14}

We first describe the various versions of the ground states and the phase diagrams, treating the spins as classical vectors. By taking this approach we are able in particular to associate definite signs of the constants in Hamiltonian (2.3) with types of magnetic structures.

We begin with ferromagnetic materials. If $\alpha > 0$, the magnetic moments are directed primarily along or opposite the z axis in the absence of a magnetic field at $T < T_c \sim JS$. This is an easy-axis ferromagnetic substance. Any arbitrarily weak field directed along the z axis will eliminate the phase transition from the paramagnetic phase to the ferromagnetic phase. Furthermore, no transition occurs in any arbitrary oblique direction of the magnetic field H (if $H_z \neq 0$). In this case, however, an increase in the field is accompanied by a smooth alignment of the magnetic moments in the field direction. The average magnetic moment M becomes parallel to H only in the limit of an infinitely strong field. This situation is different if the field H is directed strictly perpendicular to the z axis. In this case there exists a certain critical field $H_c \propto \alpha$ below which there is a nonzero order parameter M_z , while at $H > H_c$ the average magnetic moment M is directed along the field; i.e., we have $M_z = 0$ (Fig. 1). The reason why the case $H_z = 0$ is a special one is that in this geometry the field appears in the Ginzburg-Landau functional in combination with even powers of the order parameter.³ The critical field H_c depends on the temperature, vanishing at the Curie point of an easy-axis magnetic substance. The phase diagram of an easy-axis magnetic sub-

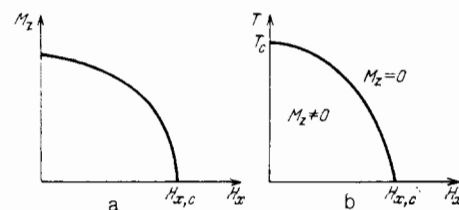


FIG. 1. a—Field dependence of the z component of the magnetic moment for an easy-axis ferromagnetic material in a transverse field H_x ; b—phase diagram of an easy-axis ferromagnetic material in a transverse field.

stance in a transverse field is shown in Fig. 1b. The points of an orientational transition at low temperatures ($T \ll T_c$) in a field $H \approx H_c$ and the Curie points of the transition in a zero field ($T = T_c$) lie on the same line in the phase diagram. In this sense, the state at $H > H_c(T)$ belongs to the paramagnetic phase even in the limit $T \rightarrow 0$.

If $\alpha < 0$, then the spins at $T < T_c$ will lie in the x, y plane. This case is an easy-plane ferromagnet. In this case, H_z and H_1 trade roles (in comparison with an easy-axis ferromagnet): A line of second-order phase transitions (of the same type as in Fig. 1 exists only if $H_1 \neq 0$). In a field H_1 , the symmetry group Z_2 , which reflects invariance under the substitution $M_z \leftrightarrow -M_z$, is preserved.

We turn now to antiferromagnetic substances.⁵⁾ Working on the basis of a classical description, we can say that in the case $\alpha > 0$ (an easy-axis antiferromagnetic material) and $H = 0$ the spins lie in a checkerboard order parallel to the z axis. In a field $\mathbf{H} \parallel \mathbf{Z}$, the antiparallel arrangement of neighboring spins persists up to a field $H_1 \propto (J\alpha)^{1/2}$, at which point a first-order phase transition (a spin-flop transition) occurs. The spins of the sublattices are established nearly perpendicular to the field. As H is increased further, the angle between the sublattices decreases, and in a field $H_2 \propto J$ the magnetic moments of the sublattices "collapse." This is a second-order phase transition (a spin-flip transition). Both of the transition points (H_1 and H_2) of course depend on the temperature.^{15,16} Figure 2a shows the phase diagram in terms of the variables H_z, T . The arrows show the directions of the magnetic moments of the sublattices in the different phases.

The spin-flip transition always occurs, regardless of the direction of the magnetic field, but the spin-flop transition occurs only if H is directed nearly parallel to the z axis.¹⁷ Figure 2b shows the phase diagram in the H_1, H_2 plane at $T < T_N$. We wish to stress that in contrast with a ferromagnetic material the transition line $H = H_2(T)$ [$H_2(T_N) = 0$] exists for an arbitrary direction of H . Correspondingly, the state at $H > H_2(T)$ is paramagnetic.

In an easy-plane antiferromagnet ($\alpha < 0$) at $H = 0$, the spins are ordered in an antiparallel fashion in the xy plane. In a magnetic field of arbitrary direction, only a spin-flip transition to a paramagnetic state occurs, at $H = H_2(T) \propto J$. For an easy-plane antiferromagnetic material, it is of fundamen-

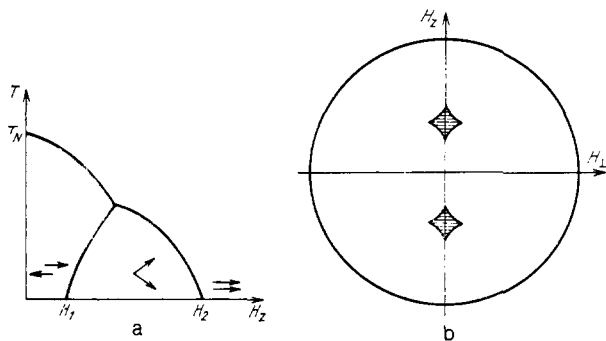


FIG. 2. a— H_z, T phase diagram of an easy-axis antiferromagnetic material (the arrows are the directions of the magnetic moments of the sublattices in the different phases); b— H_1, H_2 phase diagram of an easy-axis antiferromagnetic material (a spin-flop transition occurs in the hatched regions).

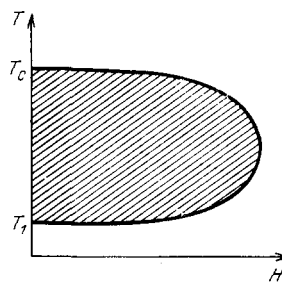


FIG. 3. Phase diagram of a magnetic substance whose anisotropy constant changes sign with decreasing temperature: $\alpha > 0$ at $T > T_1$ and $\alpha < 0$ at $T < T_1$. The hatched region is the region with $M_z \neq 0$.

tal importance to consider an additional term which has been omitted from (2.3) but which exists for crystals with a certain symmetry (e.g., orthorhombic). This additional term describes the so-called Dzyaloshinskii interaction and takes the form¹⁸

$$\frac{1}{2} \sum_{\mathbf{l}, \Delta} d(\Delta) (n_z |S_{\mathbf{l}} S_{\mathbf{l}+\Delta}), \quad |n_z| = 1.$$

The Dzyaloshinskii interaction is of relativistic origin, so the coefficient d is of the same order of magnitude as α . This interaction leads to a slight ferromagnetism in the case of an easy-plane antiferromagnetic material, i.e., to a skew of the sublattices in a zero magnetic field.^{15,18}

This list of possible types of ordering and, correspondingly, phase diagrams of course does not exhaust the list of magnetic structures which exist in nature, even those which are described by a model like (3). In particular, the anisotropy constant may depend on the temperature and may change sign, resulting in the conversion of an easy-plane magnetic substance into an easy-axis magnetic substance or vice versa. Figure 3 shows an example of a phase diagram of a magnetic substance for which the relation $\alpha > 0$ holds at $T > T_1$, while $\alpha < 0$ holds at $T < T_1$. The paramagnetic phase spans the entire unhatched region except for the segment $0 < T < T_1$ along the ordinate axis. We will not go into complications of this sort here, since we are interested in the properties of magnetic substances near $T = 0$.

In winding up this discussion of the properties of various types of magnetic substances we would like to point out that each type of course has representatives among real magnetic substances (some corresponding examples are gathered in Table VII, at the beginning of §6). In this review we will be concerned for the most part with theoretical work on the behavior of magnetic substances with an arbitrary spin at low temperatures. Although we will reproduce several well-known equations here, we will be focusing on some subtle effects. We will thus not dwell on the experimental confirmation of the basic properties of ferromagnetic and antiferromagnetic materials (such as the Bloch law for ferromagnetic substances or the dependence $C \propto T^3$ for the heat capacity of antiferromagnetic materials). Where we do know of experiments on the subtle effects, we will cite the corresponding papers.

3. TRANSITION TO BOSONS

In principle, a microscopic calculation of the properties of magnetic substances on the basis of Hamiltonian (2.3)

could be carried out directly in terms of spin operators.¹⁹⁻²¹ That approach, which uses the single-particle Hamiltonian as a zeroth order approximation, is a good one and also quite graphic when the nonlocal exchange interaction is small in comparison with the single-particle interaction.²²⁻²⁴ In the opposite limit—in which we are interested here—that approach is no longer graphic, and we find it more natural to introduce some usual Bose quasiparticles: magnons. The problem of constructing Bose analogs of spin Hamiltonians has been posed and solved in many places, dating back to the pioneering paper by Holstein and Primakoff.²⁵⁻³⁴ The primary difficulty in taking this path is that the finite-dimensional spin space, with $2S + 1$ dimensions (S is the lattice spin), must be related to an infinite-dimensional boson space. This linkup is usually made by introducing a projection operator which cuts off extraneous states.^{27,34} Admittedly, this approach requires an analysis of the role played by this operator in each particular problem. One could in principle also avoid using a projection operator by associating with spin operators some periodic functions of Bose operators:

$$S^z = f(\hat{N}), \quad S^- = a^+ F(\hat{N}), \quad S^+ = (S^-)^+,$$

where

$$F(x) = \{[S(S+1) - f(x)(f(x)-1)](x+1)^{-1}\}^{1/2},$$

$$\hat{N} = a^+ a,$$

$$f(x) = S - (2S+1)[x(2S+1)^{-1}].$$

The square brackets in the last equation mean the fractional part of the argument. This path (a Goldhirsch transformation^{31,32}) allows one to write out explicitly an exact Bose analog of a spin Hamiltonian, although in the form of an infinite series whose terms are products of an arbitrary number of creation and annihilation operators (analytic expressions have been derived for the corresponding coefficients only for the cases³² $S = 1/2$ and $S = 1$). Clearly, one cannot work with such a Hamiltonian in the general case. There is, however, a wide class of problems for which the condition of a slight deviation from ideality of a Bose gas of magnons is satisfied. We will be discussing some examples below; at this point we would like to point out that, as always, a slight deviation from ideality means that we can restrict the calculations to the lowest anharmonicities. In other words, we can work with simply the first few terms of the expansion of the spin-projection operators in series in normal products of Bose operators. For all the effects which we will be discussing below, which result from a nonideality of a Bose gas, for example, it is sufficient to work with the first three terms in the exact Goldhirsch expansions:

$$S_1^+ + S = a_1^+ a_1, \quad S > 1,$$

$$= a_1^+ a_1 - \frac{1}{2} a_1^+ a_1^+ a_1 a_1, \quad S = 1,$$

$$= a_1^+ a_1 - a_1^+ a_1^+ a_1 a_1 + \frac{2}{4} a_1^+ a_1^+ a_1 a_1 a_1, \quad S = \frac{1}{2},$$

$$S_1^- = (S_1^+)^+ = (2S)^{1/2} \left\{ a_1 - \left[1 - \left(1 - \frac{1}{2S} \right)^{1/2} \right] a_1^+ a_1 a_1 \right. \\ \left. - \frac{1}{2} \left[2 \left(1 - \frac{1}{2S} \right)^{1/2} - \left(1 - \frac{1}{S} \right)^{1/2} - 1 \right] \right. \\ \left. \times a_1^+ a_1^+ a_1 a_1 a_1 \right\}, \quad S > \frac{1}{2},$$

$$= a_1 - a_1^+ a_1 a_1 + \frac{1 + \sqrt{3}}{2\sqrt{3}} a_1^+ a_1^+ a_1 a_1 a_1, \quad S = \frac{1}{2}.$$

For $S > 1$ these equations can be derived from a compact Holstein-Primakoff transformation:

$$S_1^z = -S + a_1^+ a_1,$$

$$S_1^+ = (2S)^{1/2} a_1^+ \left(1 - \frac{a_1^+ a_1}{2S} \right)^{1/2}, \quad (3.2)$$

$$S_1^- = (2S)^{1/2} \left(1 - \frac{a_1^+ a_1}{2S} \right)^{1/2} a_1.$$

A modification of this transformation is the exact Goldhirsch transformation. Actually, however, the situation is simpler: Since the Holstein-Primakoff transformation does not by itself send the system out of the physical subspace (the matrix element between the states $|2S\rangle$ and $|2S+1\rangle$ is zero), the use of the Holstein-Primakoff transformation at temperatures low in comparison with T_c leads to correct results even in the cases $S = 1$ and $1/2$. This assertion is verified by direct calculations^{35,36}: The divergent series in $1/S$ which appear at the stage of the transition to bosons, even with a normal ordering of Bose operators, do not appear in the final results. We will accordingly be using transformation (3.2) everywhere, regarding it as exact (for—we wish to stress this point—an arbitrary value of S).⁶⁾

Let us formulate the conditions under which a gas of magnons can be regarded as slightly nonideal. As we mentioned earlier (§1), the exchange interaction between magnons is not small, so a slight deviation of a Bose gas from ideality could result only from a low density of quasiparticles. In all cases, a necessary condition for a low magnon density is that the temperature T must be low in comparison with the transition temperature T_c (or T_N) at $H = 0$. This condition is not always a sufficient condition, however, since in most magnetic structures describable by Hamiltonian (2.3) there are zero-point vibrations, which lead to a nonvanishing boson density $\langle a_1^+ a_1 \rangle$ in the ground state. This density must also be small. In ferromagnetic materials, the value of $\langle a_1^+ a_1 \rangle$ is made small at $T = 0$ by the small value of the ratio α/J . In antiferromagnetic materials this situation is more complicated, since even in the absence of anisotropy a state with a checkerboard arrangement of quantum spins is not an eigenstate of the spin Hamiltonian. In other words, the zero-point vibrations in an antiferromagnetic material result from exchange, so the boson density at $T = 0$ will generally not be parametrically small.⁷⁾

In this review we will focus on problems in which perturbation theory can be used correctly. We accordingly restrict the discussion to ferromagnetic materials with a slight anisotropy, while in the case of antiferromagnetic materials we will be concerned primarily with the field region near the spin-flip transition, where the structure is approximately ferromagnetic. We wish to emphasize the following point: The discussion is carried out in terms of an arbitrary site spin S (i.e., in terms of $S \sim 1$), so we will not make use of the possibility of obtaining a low magnon density as a result of the semiclassical nature of the spin. In other words, we will not use the expansion in $1/S$ which has been used frequently elsewhere.

The Bose analogs of spin Hamiltonians differ substantially, depending on whether there are zero-point vibrations in the system.⁸⁾ If there are no zero-point vibrations, the ground state of the magnetic substance is the same as its classical ground state. In this case we need write in the Bose Hamiltonian only those terms which contain equal numbers

TABLE I. Values of the magnon energy and of the seed amplitude for two-particle scattering for systems without zero-point vibrations. In the cases of an easy-plane ferromagnetic and an easy-plane antiferromagnetic material it is assumed that the magnetic field H_z is greater in magnitude than the critical field for the transition to the "paramagnetic" phase.

Model	Spectrum $\varepsilon_{\mathbf{k}}$	Amplitude Φ_{12}^{34}
1. Easy-axis (+) and easy-plane (-) ferromagnetic materials in a longitudinal field	$J^{(0)} S (1 - v_{\mathbf{k}}) \pm \pm 2\tilde{\alpha} S + 2\mu H_z$	$-\frac{J^{(0)}}{4} \left\{ v_{2-3} + v_{1-3} - 2S \left[1 - \left(1 - \frac{1}{2S} \right)^{1/2} \right] (v_1 + v_2 + v_3 + v_4) \right\} \pm \alpha$
2. Isotropic antiferromagnetic material above the spin-flip transition	$2(\mu H_z - J^{(0)} S) + J^{(1)} S (1 - v_{\mathbf{k}})$	$\frac{J^{(1)}}{4} \left\{ v_{2-3} + v_{1-3} + 2S \left[1 - \left(1 - \frac{1}{2S} \right)^{1/2} \right] \times (v_1 + v_2 + v_3 + v_4) \right\}$

of magnon creation and annihilation operators:

$$\mathcal{H} = E_{cl} + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \frac{1}{N} \sum_{1, 2, 3, 4} \Phi_{12}^{34} a_1^{\dagger} a_2^{\dagger} a_3 a_4 \Delta (1 + 2 - 3 - 4) + \dots \quad (3.3)$$

The indices 1, 2, 3, 4 correspond to $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$. In Hamiltonian (2.4), this situation corresponds to an easy-axis ferromagnetic material in a longitudinal field and also an easy-plane ferromagnetic or antiferromagnetic material in a strong field directed along the selected axis. The values of the magnon energy $\varepsilon_{\mathbf{k}}$ and of the amplitude Φ_{12}^{34} are given in Table I. We will use the notation

$$\tilde{\alpha} = \alpha \left(1 - \frac{1}{2S} \right), \quad v_{\mathbf{k}} = \frac{J(\mathbf{k})}{J^{(0)}},$$

where $J(\mathbf{k})$ is the \mathbf{k} th Fourier component of the exchange interaction $J(\Delta) = J$, given by

$$J(\mathbf{k}) = J \sum_{\Delta} e^{i\mathbf{k}\Delta}.$$

In the case in which there are zero-point vibrations, the structure of the Bose Hamiltonian is quite different. Specifically, terms with unequal numbers of magnon creation and annihilation operators appear^{25,35,39}:

$$\mathcal{H} = E_{cl} + \sum_{\mathbf{k}} \left[A_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \frac{1}{2} B_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}^{\dagger} + a_{\mathbf{k}} a_{-\mathbf{k}}) \right] + N^{1/2} \Phi_0 a_0^{\dagger} + \frac{1}{N^{1/2}} \sum_{1, 2, 3} \Phi_{12}^3 a_1^{\dagger} a_2^{\dagger} a_3 + \frac{1}{N} \sum_{1, 2, 3, 4} (\Phi_{123}^4 a_1^{\dagger} a_2^{\dagger} a_3^{\dagger} a_4 + \Phi_{12}^{34} a_1^{\dagger} a_2^{\dagger} a_3 a_4) + \frac{1}{N^2} \sum_{\substack{1, 2, 3, \\ 4, 5, 6}} \Phi_{123}^{456} a_1^{\dagger} a_2^{\dagger} a_3^{\dagger} a_4 a_5 a_6 + \text{H.a.} \quad (3.4)$$

In Hamiltonian (2.4), this situation corresponds to an easy-axis ferromagnetic material in a field which is not parallel to the selected crystal axis or to an easy-plane ferromagnetic or antiferromagnetic substance in an arbitrary field, except in the cases discussed above. The coefficients $\Phi_0, A_{\mathbf{k}}$, and $B_{\mathbf{k}}$ in (3.4) for certain models are listed in Table II. We will not reproduce here the lengthy expressions for the seed energy of

TABLE II. Values of the coefficients in the forms which are linear and quadratic in the Bose operators for the Hamiltonians of several systems containing zero-point vibrations (θ is the angle between the quantization axis and the selected axis of the crystal; for an antiferromagnetic material, the direction of the selected axis is set by the external magnetic field).

Model	Φ_0	$A_{\mathbf{k}}$	$B_{\mathbf{k}}$
1. Easy-plane ferromagnetic material in a longitudinal field	$-i(2NS)^{1/2} \sin \theta \times (\mu H_z - \tilde{\alpha} S \cos \theta)$	$J^{(0)} S (1 - v_{\mathbf{k}}) + 2\mu H_z \cos \theta + \tilde{\alpha} S (1 - 3 \cos^2 \theta)$	$\alpha S [1 - (2S)^{-1}]^{1/2} \sin^2 \theta$
2. Easy-axis ferromagnetic material in a transverse field	$-i(2NS)^{1/2} \cos \theta \times (\mu H_x - \tilde{\alpha} S \sin \theta)$	$J^{(1)} S (1 - v_{\mathbf{k}}) + 2\mu H_x \sin \theta + 2\tilde{\alpha} S \left(1 - \frac{3}{2} \sin^2 \theta \right)$	$\alpha S [1 - (2S)^{-1}]^{1/2} \sin^2 \theta$
3. Isotropic antiferromagnetic material in a field	$-i(2NS)^{1/2} \sin \theta \times (\mu H_z - J^{(1)} S \cos \theta)$	$J^{(0)} S (1 - v_{\mathbf{k}}) + 2\mu H_z \cos \theta + J^{(1)} S (v_{\mathbf{k}} \sin^2 \theta - 2 \cos^2 \theta)$	$J^{(0)} S v_{\mathbf{k}} \sin^2 \theta$

the ground state, E_{c1} , and for the coefficients in the anharmonic terms. We simply note that while terms of the type $\Phi_{123}^4 a_1^+ a_2^+ a_3^+ a_4$ with an even number of Bose operators exist whenever there are zero-point vibrations, the terms containing an odd number of operators arise only if the quantization axis makes an angle θ , where $0 < \theta < \pi/2$, with the selected axis of the magnetic substance. In the classical case ($S \rightarrow \infty$) the angle θ is found from the condition $\Phi_0 = 0$. In the general case (at a nonzero S) the equation for determining the angle θ is more complicated (more on this below), and Φ_0 is not necessarily zero.

We wish to stress that the values given for the coefficients of both Hamiltonian (3.3) and Hamiltonian (3.4) in Tables I and II fail to satisfy many natural requirements. For example, the exchange part Φ_{12}^{34} does not satisfy Adler's principle⁹; not all of the anisotropic terms vanish at $S = 1/2$; and the spectrum found through a direct diagonalization of the quadratic form in (3.4) is unstable at small values of k for $S \neq 1/2$. The existence of these parasitic properties demonstrates the need for anharmonic renormalizations; a description of these renormalizations is the basic content of the following sections of this review.

For systems describable by Hamiltonian (3.3) all the effects which stem from anharmonicities are manifested only at a nonzero temperature. Some problems which arise here are those of calculating the temperature renormalizations of the spin-wave energy (including the renormalization of the gap in the anisotropic case) and the magnetization and that of calculating the relaxation parameters. These questions will be taken up in §4.

In systems describable by Hamiltonian (3.4) the situation is more complicated since now the temperature effects are accompanied by anharmonic effects at $T = 0$ which result from quantum fluctuations (§5).

For the most part, the problems formulated above have been solved in the semiclassical approximation ($S \gg 1$), in which anharmonicities can be dealt with in second-order perturbation theory. We recall that we are interested in arbitrary values of S . In this case, magnetic systems with a predominant exchange interaction have an obvious distinguishing feature: The anharmonicities to which they lead are not small in the general case of arbitrary wave vectors, and all the terms of the corresponding perturbation-theory series must be summed.¹⁰

The strong interaction between quasiparticles is what distinguishes magnons from phonons. In the case of phonons *there are always zero-point vibrations*, but the coefficients of the anharmonicity and the density of quasiparticles depend on the same parameter Θ/ε_0 (Θ is the Debye temperature, and ε_0 is a characteristic atomic energy, given by $\varepsilon_0 \sim Ms^2$, where M is the mass of the ion, and s is the sound velocity). If $\Theta/\varepsilon_0 \ll 1$ (and this condition usually holds, by virtue of the small value of the ratio m/M , where m is the electron mass), the interaction and also the density of quasiparticles are simultaneously small at $T \ll \Theta$. In this case the phonons are similar to magnons in systems with $S \gg 1$. Admittedly, there is the distinction from phonons that in the case of magnons there can be situations in which there are no zero-point vibrations, so single-particle excitations are eigenstates of the spin Hamiltonian. In such cases a magnon is more reminiscent of a particle in a periodic field (a band particle) than of a phonon. For both magnons and phonons

there can be situations in which decay processes are forbidden by conservation laws (e.g., the cases of transverse phonons and magnons in an isotropic antiferromagnetic material), so the quasiparticle lifetime is infinite, despite the presence of zero-point vibrations.

4. SYSTEMS WITHOUT ZERO-POINT VIBRATIONS

It is clear from the discussion above that we are talking about systems in which the ground state is characterized by the maximum possible magnetization and is thus the same as the classical ground state. As we have pointed out, this requirement is satisfied by isotropic ferromagnetic substances, easy-axis ferromagnetic substances in a longitudinal field, and also uniaxial easy-plane ferromagnetic and antiferromagnetic substances in a sufficiently strong magnetic field directed along the selected axis. In all these systems the spectrum of long-wave single-particle states at $T = 0$ is of the form

$$\varepsilon_{\mathbf{k}} = \varepsilon_0 + JS(\mathbf{k}\mathbf{a})^2, \quad (4.1)$$

where the values of ε_0 are given in Table I for two specific cases (we are setting Planck's constant equal to one). For an arbitrary geometry of the exchange interaction, the quantity $JS(\mathbf{k}\mathbf{a})^2$ in (4.1) is replaced by $(1/2)A_{ij}k_i k_j$, where

$$A_{ij} = 2S \sum_{\Delta} J(\Delta) \Delta_i \Delta_j$$

is the tensor of inverse effective masses.

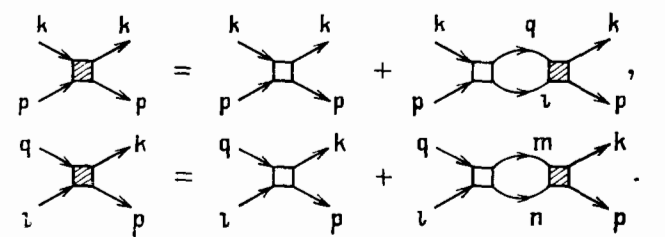
At nonzero temperatures, spectrum (4.1) is not exact, since this spin wave interacts with thermal magnons, whose characteristics (the gap ε_0 and the effective-mass tensor A_{ij}) are renormalized as a result; in addition, its lifetime $\tau(\mathbf{k})$ becomes finite.

The energy renormalization and the lifetime of the magnon are determined by the real and imaginary parts of the diagram



$$, \quad (4.2)$$

which contains the complete amplitude for scattering through a zero angle, $r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}}$. A calculation of this amplitude (for arbitrary S) requires summing the "exchange ladder," i.e., solving the integral equations



$$. \quad (4.3)$$

An analytic solution of these equations is possible thanks to a specific feature of the exchange interaction: The kernels of the integral equations can be factorized. To illustrate the solution of the integral equations in an application based on the Holstein-Primakoff formalism [see (3.2)], we have calculated the complete amplitude for scattering through a zero angle in an isotropic ferromagnetic material.

In the general case of an anisotropic magnetic substance

TABLE III. Temperature renormalizations of the gap, temperature renormalizations of the effective mass, and magnon lifetimes for systems which do not have zero-point vibrations. The factor $Q(S)$ is defined in the text proper [expressions (4.5)–(4.7)]. Explicit expressions for the coefficients A and A' are not reproduced here, since the corresponding temperature corrections to the spectrum are smaller than the lifetime of the quasiparticles with the same wave vectors. $Z_p(x) = \sum_{n=1}^{\infty} (e^{-nx}/n^p)$; in limiting cases: $Z_p(x) \approx \zeta(p)$ at $x \ll 1$ and $Z_p(x) \approx e^{-x}$ at $x \gg 1$ ($\tau = T/4\pi JS$).

Model	Gap in spectrum $\varepsilon_0(T)$	Inverse effective mass	Inverse magnon lifetime	
			ν_0	ν_h
1. Easy-axis (+) and easy-plane (–) ferromagnetic materials in a longitudinal field	$2\mu H_z \pm 2\tilde{\alpha}S$ $\times \left[1 - \frac{2}{S} \tau^{3/2} Z_{3/2} \left(\frac{\varepsilon_0}{T} \right) \right]$	$JS \left[1 - \frac{\pi}{S} Q(S) \tau^{5/2} Z_{5/2} \left(\frac{\varepsilon_0}{T} \right) \right],$ $\tilde{\alpha}S \ll T;$ $JS \left[1 - \frac{A}{S} \tau^{3/2} Z_{3/2} \left(\frac{\varepsilon_0}{T} \right) \right],$ $\tilde{\alpha}S \gg T$	$\frac{12}{\pi S} \tilde{\alpha} \left(\frac{\tilde{\alpha}}{J(0)} \right) \tau^2$ $\times Z_2 \left(\frac{\varepsilon_0}{T} \right)$	$\frac{3}{\pi S} \tilde{\alpha} \left(\frac{\tilde{\alpha}}{J(0)} \right) (ak) \tau^{3/2}$ $\times Z_{3/2} \left(\frac{\varepsilon_0}{T} \right),$ $\tilde{\alpha}S \gg JS(ka)^2;$ $\frac{J}{6\pi S} \tau^2 (ak)^4 \ln^2 \frac{T}{JS(ak)^2},$ $\tilde{\alpha}S \ll JS(ka)^2 \ll T;$ $\frac{J}{4S} (ak)^3 \tau^{5/2} Z_{5/2} \left(\frac{\varepsilon_0}{T} \right),$ $JS(ka)^2 \gg T, \tilde{\alpha}S$
2. Isotropic anti-ferromagnetic material above the spin-flip transition	$2\mu H_z - 2J(0)S$ $\times \left[1 - \frac{2}{S} \right]$ $\times \left(1 + \frac{W-1}{S} \right)^{-1}$ $\times \tau^{3/2} Z_{3/2} \left(\frac{\varepsilon_0}{T} \right) \Big]$	$JS \left[1 - \frac{A'}{S} \tau^{3/2} Z_{3/2} \left(\frac{\varepsilon_0}{T} \right) \right]$	$\frac{12}{\pi S} J(0)$ $\times \left(1 + \frac{W-1}{S} \right)^{-2}$ $\times \tau^2 Z_2 \left(\frac{\varepsilon_0}{T} \right)$	$\frac{3J(0)}{\pi S} \left(1 + \frac{W-1}{S} \right)^{-2}$ $\times (ka) \tau^{3/2} Z_{3/2} \left(\frac{\varepsilon_0}{T} \right)$

without zero-point vibrations, the complete scattering amplitude can be written in a natural way as the sum

$$r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}} = (r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}})_{\text{ex}} + (r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}})_{\text{rel}}. \quad (4.4)$$

It is the first term, of exchange origin, which determines the renormalization of the effective-mass tensor at temperatures which are not too low. The second term is of a relativistic nature. It determines the renormalization of the gap and—at very low temperatures—the effective-mass tensor. Both terms contribute to the damping of a spin wave with $k \neq 0$. At $k = 0$, the damping is determined exclusively by $(r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}})_{\text{rel}}$. The exchange and relativistic parts of the amplitude $r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}}$ are susceptible in different ways to exchange renormalizations. For the relativistic part, the primary renormalization reduces to the replacement of the single-ion anisotropy constant α by $\alpha^{41,42} \tilde{\alpha} = \alpha [1 - (2S)^{-1}]$; the unlike-ion anisotropy constant does not change at all.¹¹⁾ Nontrivial renormalizations of the relativistic part of the complete amplitude arise in higher orders in α/J and β/J . For the exchange part of $r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}}$, the renormalization is more important: It leads to a change in the structure of the amplitude. Specifically, the “parasitic” finite term at zero values of the quasiparticle momentum vanishes, so that Adler’s principle is restored.

Table III shows expressions for the renormalizations of the gap and of the effective mass and also for the magnon lifetime in various cases.^{1,14,19,24,27,41–44} In calculating the renormalizations of the spectrum in leading order in the temperature it is sufficient to consider only the value of $r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}}$ at $T = 0$, since the characteristic wave vectors of the virtual magnons are close to their maximum values. In the calculation of $\tau(\mathbf{k})$, the temperature dependence of the amplitude must not be ignored, since all the characteristic wave vectors of the problem are of the order of $\alpha^{-1}(T/JS)^{1/2}$, and the distribution functions are of the order of unity.

It can be seen from the expressions given in Table III that the renormalization of the effective mass due to $(r_{\mathbf{k}\mathbf{p}}^{\mathbf{k}\mathbf{p}})_{\text{ex}}$ contains a nontrivial spin factor

$$Q(S) = 1 + \frac{4\Gamma}{3(2S - \Gamma)} + \frac{W - 1}{3S}, \quad (4.5)$$

where

$$W = \frac{3}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos x - \cos y - \cos z} \approx 1.52 \quad (4.6)$$

is the Watson integral,⁴⁵ and

$$\Gamma = \frac{3}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{(1 - \cos x) \cos y dx dy dz}{3 - \cos x - \cos y - \cos z} \approx 0.2. \quad (4.7)$$

For a simple cubic lattice with nearest-neighbor interaction, an expression for $Q(S)$ was first derived by Dyson²⁷ by a formalism developed especially for the purpose.¹²⁾ The lifetime of a magnon, in contrast, does not have a factor of this sort, so the correct result for the damping is the same as the result derived in second-order perturbation theory.

The expressions in Table III are important for refining dispersion relations as such. They are less important for calculating the free energy and its derivatives (aside from the replacement of α by $\tilde{\alpha}$), since the temperature renormalization of the energy plays a lesser role than the incorporation

of the deviation of the magnon dispersion law at $T = 0$ from a quadratic law in the calculation of the macroscopic characteristics.

Table III shows equations for nearly exchange magnetic substances (with an anisotropy much less important than the exchange).

Strictly speaking, a small anisotropy is not a necessary condition for obtaining rigorous results for an arbitrary spin: The only important points are that the ground state be “classical” and that the temperature be low in comparison with T_c . Not only the calculations but also the final results are extremely complicated for an arbitrary value of the ratio $|\alpha|/J$. We restrict the present discussion to the expressions for the gap renormalization, $\Delta\varepsilon_0(T)$. For an arbitrary $|\alpha|/J$ the temperature dependence $\Delta\varepsilon_0(T)$ remains the same as at $|\alpha|/J \ll 1$, but an additional factor which depends explicitly on the spin and on the ratio $|\alpha|/J$ appears. The most interesting case is that of an easy-axis ferromagnetic substance: For a certain value of the anisotropy constant,

$$\alpha = \alpha_c = J(0) S [W - (2S)^{-1} (W - 1)]^{-1}$$

[see (4.6)], the scattering amplitude r_{00}^{00} diverges, forcing us to consider the finite size of the momentum of a virtual magnon in calculating the gap renormalization by means of (4.3).⁴² Calculations show that with $\alpha = \alpha_c$ the temperature correction to ε_0 changes sign, and the damping of γ_0 has a maximum (Fig. 4). The nature of the resonance effect is a change at $\alpha = \alpha_c$ in the nature of the low-lying two-particle excited states. At this value of α , the energy of a bound state of two magnons with $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = 0$ becomes lower than $2\varepsilon_0$, which is the lower energy boundary of the continuum of two-particle excitations.^{46,47}

We have been discussing the bound states which arise from the relativistic interaction. Exchange bound states in three-dimensional space exist only at large quasiparticle wave vectors.^{48–51} We will not pursue that question since it goes beyond the scope of the problems in which we are interested—problems which concern the behavior of long-wave

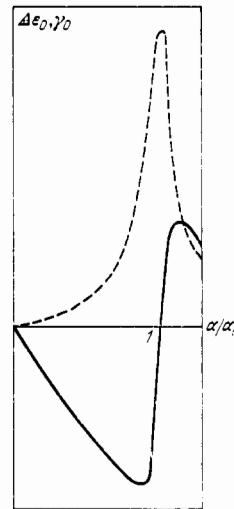


FIG. 4. The attenuation factor γ_0 and the temperature correction to the gap in the spectrum of an easy-axis ferromagnetic material, $\Delta\varepsilon_0$, as functions of the single-ion anisotropy constant.⁴² Solid line— $\Delta\varepsilon_0$; dashed line— γ_0 .

magnons. We simply note that in calculating the exchange bound states one needs to know the value of $|r_{12}^{34}|$ for arbitrary values of the momenta.^{46,51}

5. SYSTEMS WITH ZERO-POINT VIBRATIONS

As we have already mentioned, the quantum ground state does not coincide with the classical ground state in many systems because of zero-point vibrations. For such systems, the problem is not simply one of calculating the temperature renormalizations but also of correctly determining the magnetic characteristics at $T=0$. Among the models described by Hamiltonian (3.2), those which have zero-point vibrations are those in which the quantization axis does not coincide with the selected crystal axis: an easy-axis ferromagnetic material in an oblique field and easy-plane ferromagnetic and antiferromagnetic materials—the latter in nearly all cases, except in a situation in which a strong field is applied along the selected axis.

In terms of Bose operators, these models are described by a Hamiltonian like (3.4) (see Table II for the values of the coefficients). The structure of this Hamiltonian is the same as that for a nonideal Bose gas of particles with a condensate.^{11,52,53} The formal analogy between the Hamiltonians (in both cases there are nondiagonal terms $a_{\mathbf{k}}^+ a_{-\mathbf{k}}$ and $a_{\mathbf{k}} a_{-\mathbf{k}}$ in the quadratic form) means that we can use in the calculations the Belyaev technique for Bose systems with a condensate; this technique has been developed quite well for gases with low densities of particles which are interacting strongly with each other.⁵³ This situation literally refers to slightly anisotropic magnetic substances, in which the interaction between magnons is strong (an exchange interaction), and the density is low because the temperature and the anisotropy are slight in comparison with the exchange.

Calculations based on the standard Belyaev diagram technique imply the introduction of three Green's functions: the normal Green's function G^{+-} and two anomalous ones (G^{++} and G^{--}), which are equal to each other by virtue of the Hermitian nature of the Hamiltonian. For these functions we can write a system of Dyson equations, from which we find a dispersion relation for the spectrum of spin waves:

$$D(\mathbf{k}, \omega) \equiv (\omega - i\Sigma_A(\mathbf{k}, \omega))^2 + (A_{\mathbf{k}} - \Sigma_s(\mathbf{k}, \omega))^2 - (B_{\mathbf{k}} - \Sigma^{++}(\mathbf{k}, \omega))^2 = 0, \quad (5.1)$$

where

$$\Sigma_{s,A}(\mathbf{k}, \omega) = \frac{1}{2} (\Sigma^{+-}(\mathbf{k}, \omega) \pm \Sigma^{+-}(-\mathbf{k}, -\omega)), \quad (5.2)$$

and $\Sigma^{+-}(\mathbf{k}, \omega)$ and $\Sigma^{++}(\mathbf{k}, \omega)$ are the eigenenergy parts of the normal and of the anomalous Green's functions. The normal Green's function is related in the usual way to the quantity $\kappa_{\mathbf{k}} = \langle a_{\mathbf{k}}^+ a_{\mathbf{k}} \rangle$:

$$\kappa_{\mathbf{k}} = \frac{1}{2\pi} \int d\omega G^{+-}(\mathbf{k}, \omega) e^{i\omega t} |_{t \rightarrow +0}. \quad (5.3)$$

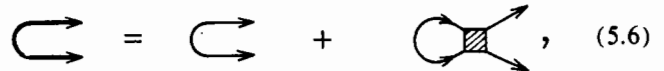
Calculations show that in the lowest order in the anisotropy the value of $\kappa_{\mathbf{k}}$ differs from the classical value [which can be found through a trivial diagonalization of the quadratic form in (3.4)] only by a natural quantum renormalization of the anisotropy constant,^{29,35,54-58} $\alpha \rightarrow \alpha[1 - (2S)^{-1}]$:

$$\kappa_{\mathbf{k}} = (A_{\mathbf{k}} - \varepsilon_{\mathbf{k}}) (2\varepsilon)^{-1} + A_{\mathbf{k}} \varepsilon_{\mathbf{k}}^{-1} n_{\mathbf{k}}, \quad (5.4)$$

where $n_{\mathbf{k}}$ is a Bose function, and

$$\varepsilon_{\mathbf{k}} = \{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2 [1 - (2S)^{-1}]\}^{1/2} \quad (5.5)$$

is the spectrum of spin waves (again, in lowest order in α/J). We wish to stress that the simplicity of this result does not mean that its derivation was simple: The calculation of the renormalization of the coefficient $B_{\mathbf{k}}$ ($B_{\mathbf{k}} \rightarrow B_{\mathbf{k}} [1 - (2S)^{-1}]^{1/2}$) requires solving the integral equation



$$\text{Diagrammatic equation (5.6)}$$

whose kernel factorizes by virtue of the structure of the exchange interaction.

At the same accuracy level, the angle θ , which specifies the magnetization direction, is found by equating the coefficient of the linear term in Hamiltonian (3.4) to zero (we denote the angle determined in this fashion by θ_0).

In the lowest order in the anisotropy, the expressions for the magnetic characteristics at $T=0$ thus differ from the classical expressions only by the renormalization of the anisotropy constant, $\alpha \rightarrow \alpha[1 - (2S)^{-1}]$. These expressions are given in Table IV. Also shown in this table are the results for an antiferromagnetic substance near the spin-flip transition.^{13,14,59} The particle density is low here by virtue of the condition $\theta \ll 1$. We recall that for an antiferromagnetic material the angle θ is the angle between the quantization axis of one of the sublattices and the direction of the external field.

In an easy-axis ferromagnetic material the order parameter is the z component of the magnetization. Since the order parameter has only a single component, i.e., since there is no axial symmetry, the spectrum has a finite gap everywhere except at the transition point. In an easy-plane ferromagnetic or antiferromagnetic material, in contrast, the order parameter near the spin flip is the transverse magnetization, i.e., a two-component vector. In such cases the system is invariant under rotations around the selected axis, and as a consequence the spin-wave spectrum is linear in the limit $k \rightarrow 0$, i.e., is of a Goldstone nature.

Before we discuss the anharmonic effects, we would like to go back to the formal analogy between the Hamiltonians of magnetic substances and Bose gases of particles. We would like to determine just how far we can pursue this analogy.

We know that in a Bose gas of particles with a condensate the long-wavelength excitations are sound waves.^{52,53} In magnons, as we have just stated, this assertion is correct only if the order parameter is of a vector nature (easy-plane ferromagnetic and antiferromagnetic materials; isotropic antiferromagnetic substances), and only in these cases is it meaningful to stress the analogy with a Bose gas of particles.

In the strict meaning of the word, magnons are quasi-particles which determine elementary excitations in a magnetic substance. The number of magnons is not conserved, so there is no condensate of magnons in an equilibrium state. On the other hand, the low-temperature properties of magnetic substances can be and frequently are conveniently described in terms of Bose operators a^+ and a , which are directly related to the spin-projection operators [see (3.2) in §3]. This is the approach which we take. These operators are customarily called "magnon creation and annihilation operators," and the concept of a condensate has a completely definite meaning for a description in terms of these operators. Specifically, in a Bose gas of particles a condensate—

TABLE IV. Magnetic characteristics at $T = 0$ of several systems containing zero-point vibrations. The expressions are written in lowest order in the small particle density: the small parameter $\tilde{\alpha}/J$ for easy-axis and easy-plane ferromagnetic materials and $\sin^2 \theta_0$ for an antiferromagnetic material (v_0 is the volume of the unit cell).

Model	Angle θ_0	Critical field $H_c^{(0)}$	Magnon energy ϵ_k	Longitudinal magnetization	Transverse magnetization
1. Easy-plane ferromagnetic material in a longitudinal field	$\cos \theta_0 = \frac{H_z}{H_{z,c}^{(0)}},$ $H_z < H_{z,c}^{(0)};$ $\theta_0 = 0, \quad H_z > H_{z,c}^{(0)}$	$\mu H_{z,c}^{(0)} = \tilde{\alpha} S$	$\{J^{(0)} S (1 - v_k) \times [2\tilde{\alpha} S \sin^2 \theta_0 + J^{(0)} S (1 - v_k)]\}^{1/2}$	$2\mu S v_0^{-1} \cos \theta_0$	$2\mu S v_0^{-1} \sin \theta_0$
2. Easy-axis ferromagnetic material in a transverse field	$\sin \theta_0 = \frac{H_x}{H_{x,c}^{(0)}},$ $H_x < H_{x,c}^{(0)};$ $\theta_0 = \frac{\pi}{2}, \quad H_x > H_{x,c}^{(0)}$	$\mu H_{x,c}^{(0)} = \tilde{\alpha} S$	$[J^{(0)} S (1 - v_k) + 2 (\mu H_x \sin \theta_0 + \tilde{\alpha} S \cos 2\theta_0)]^{1/2} [J^{(0)} S (1 - v_k) + 2 (\mu H_x \sin \theta_0 + \tilde{\alpha} S \cos^2 \theta_0)]^{1/2}$	$2\mu S v_0^{-1} \sin \theta_0$	$-2\mu S v_0^{-1} \cos \theta_0$
3. Isotropic antiferromagnetic material near a spin-flip transition	$\cos \theta_0 = \frac{H_z}{H_{z,c}^{(0)}},$ $H_z < H_{z,c}^{(0)};$ $\theta_0 = 0, \quad H_z > H_{z,c}^{(0)}$	$\mu H_{z,c}^{(0)} = J^{(0)} S$	$J^{(0)} S [(1 - v_k) (2 \sin^2 \theta_0 + 1 - v_k)]^{1/2}$	$2\mu S v_0^{-1} \cos \theta_0$	$2\mu S v_0^{-1} \sin \theta_0$

i.e., a cluster of n_0 particles in a unit volume in a state with $\mathbf{p} = 0$ —reflects the appearance of a long-range order in the system: The density-density correlation function tends toward n_0 at long range.¹¹ Correspondingly, the appearance of a long-range order in magnetic substances is characterized by the appearance of a nonzero value of $\langle \mathbf{S}^\perp \rangle$, whose square is the large-scale limit of the spin-spin correlation function [$\langle \mathbf{S}^\perp \rangle = \langle S^x, S^y \rangle$ is the projection of the spin onto the easy plane; for definiteness, we will speak in terms of an easy-plane ferromagnetic material].

According to (3.2), we have

$$\langle S^\perp \rangle = \left(S - \frac{1}{N} \sum_{\mathbf{k}} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle \right) \sin \theta,$$

or

$$\frac{N}{V} \frac{\langle S^\perp \rangle^2}{S^2} = \frac{N \sin^2 \theta}{V} \frac{2}{NS} \sum_{\mathbf{k}} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle \approx \frac{N}{V} \sin^2 \theta. \quad (5.7)$$

By analogy with the known equation for a Bose gas of particles,¹¹

$$n_0 + \frac{N}{V} \frac{1}{N} \sum_{\mathbf{k}} \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rangle = \frac{N}{V} \quad (5.8)$$

($b_{\mathbf{k}}^+$ and $b_{\mathbf{k}}^-$ are the Bose operators of the particles), we can thus identify the quantities

$$\frac{N \sin^2 \theta}{V}, \quad \frac{N}{V} \frac{\langle S^\perp \rangle^2}{S^2}, \quad \text{and} \quad \frac{2}{NS} \sum_{\mathbf{k}} \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle$$

with respectively the total density of particles, the density of particles in the condensate, and the "site" density of particles above the condensate. The first two of these correspondences are a bit on the formal side, although they do make it possible to deal with phase transitions in an easy-plane ferromagnetic or antiferromagnetic material as the disappearance of a Bose condensate.¹³ The last correspondence is a more natural one since the distribution function of the magnons above the condensate has the same form as that in a Bose gas [see (5.3) and (5.4)]. This correspondence makes the structure of the ground state clear: At $T = 0$, a magnetic substance contains (quasi) particles above the condensate, in an equilibrium number which is determined by the first term in (5.4) and which is small [$\sim (\alpha/J)^{3/2}$] to the extent to which the ratio of the anisotropy to the exchange is small. The deviation of the distribution function of the magnons above the condensate from $\kappa_{\mathbf{k}}|_{T=0}$ causes a relaxation of the spin system, which leads under certain conditions to a finite magnon lifetime at $T = 0$. The standard uv transformation in magnetic substances is also clarified: It may be regarded as a transition from particles to quasiparticles. In this connection we would like to point out that zero-point vibrations are important not only at $T = 0$ but also at non-zero temperatures: The temperature part of the particle distribution function $\kappa_{\mathbf{k}}$ is not a Bose function, in contrast with the quasiparticle distribution function, which is equal to¹⁴ $n_{\mathbf{k}}$.

There are several effects (e.g., the shift of the point of the orientational transition in an easy-axis ferromagnetic material), however, for which calculations require equations more accurate than those in Table IV—equations which incorporate the following terms in α/J . The calculations are carried out in the following way^{14,35}: The angle θ is

found from the vanishing of the sum of diagrams having a single external end,

$$\text{Diagram with one external end} = 0, \quad (5.9)$$

and the eigenenergy parts Σ^{+-} and Σ^{++} are given by the diagram series

$$\begin{aligned} \Sigma^{+-} &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\ &+ \text{Diagram 4} + \text{Diagram 5} \\ &+ \text{Diagram 6} + \text{Diagram 7}, \quad (5.10) \\ \Sigma^{++} &= \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} \\ &+ \text{Diagram 11} + \text{Diagram 12} + \text{Diagram 13} \\ &+ \text{Diagram 14} + \dots \quad (5.11) \end{aligned}$$

At the accuracy of the treatment below (to within the square of the small parameter) it is sufficient to consider only these diagrams. The complete vertices Φ which appear in the series are themselves solutions of integral equations which arise because of their strong exchange renormalization. These equations are shown in diagram form in Fig. 5.

The results of the quantum renormalizations are collected in Table V. There is a nontrivial dependence of the

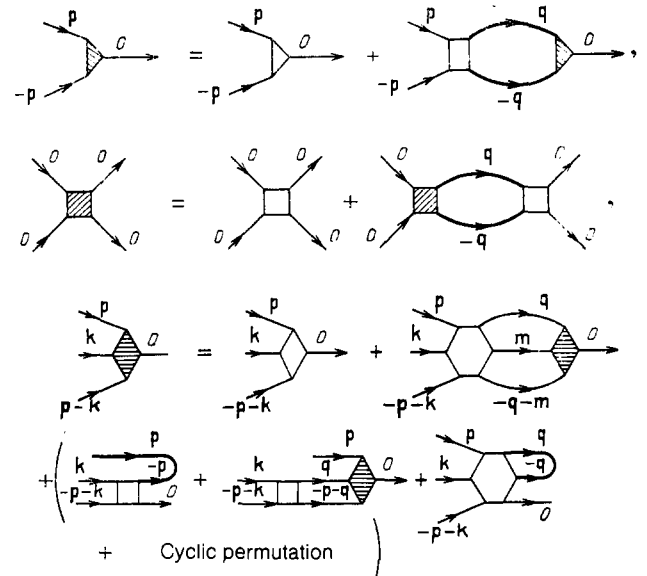


FIG. 5. Diagram equations for the complete vertices which appear in diagrams (5.10) and (5.11).

TABLE V. Quantum renormalizations ($T = 0$) of the magnetic characteristics of several systems containing zero-point vibrations $\lambda = W + \{2S[1 - (2S)^{-1}]\}^{-1}$.

Model	Angle θ	Critical field H_c	Magnon energy $\epsilon_k, ka \ll 1$	Longitudinal magnetization	Transverse magnetization
1. Easy-plane ferromagnetic material in a longitudinal field	$\cos \theta = \cos \theta_0$ $\times \left[1 - \frac{\tilde{\alpha}\lambda}{2J^{(0)}S} \sin^2 \theta_0 \right],$ $H_z < H_{z,c};$ $\theta \equiv 0, H_z > H_{z,c}$	$H_{z,c} \equiv H_{z,c}^{(0)}$	$\epsilon_k = Ck,$ $C^2 = 2\tilde{\alpha}JS^2 \sin^2 \theta_0$ $\times \left[1 + \frac{\tilde{\alpha}}{J^{(0)}S} \left\{ \lambda - \frac{1}{2[1 - (2S)^{-1}]} \right\} \sin^2 \theta_0 \right],$ $H_z < H_{z,c}$	$\frac{2\mu S}{v_0} \cos \theta$	$\frac{2\mu S}{v_0} \sin \theta$
2. Easy-axis ferromagnetic material in a transverse field	$\sin \theta = \sin \theta_0$ $\times \left[1 + \frac{\tilde{\alpha}\lambda}{2J^{(0)}S} \sin^2 \theta_0 \right],$ $H_x < H_{x,c};$ $\theta \equiv \frac{\pi}{2}, H_x > H_{x,c}$	$H_{x,c} \equiv H_{x,c}^{(0)}$	$\epsilon_0 = 2\tilde{\alpha}S$ $\times \left\{ \begin{array}{l} \left(1 - \frac{\lambda\tilde{\alpha} \sin^2 \theta_0}{J^{(0)}S} \right) \cos \theta_0, H_x < H_{x,c}, \\ \frac{\mu H_x}{\tilde{\alpha}S} \left[\frac{\mu (H_x - H_{x,c})}{\tilde{\alpha}S} \right]^{1/2}, H_x > H_{x,c}, \end{array} \right.$ $U^2 = 2\tilde{\alpha}JS^2$ $\times \left\{ \begin{array}{l} 2 - \sin^2 \theta - \frac{2\tilde{\alpha} \sin^2 \theta}{J^{(0)}S} (\lambda \cos^2 \theta \\ + \frac{\sin^2 \theta}{4[1 - (1/2S)]}), H_x < H_{x,c}, \\ 2\mu (H_x - H_{x,c})(\tilde{\alpha}S)^{-1} + 1 - \tilde{\alpha}(2J^{(0)}S)^{-1}, \\ H_x > H_{x,c} \end{array} \right.$	$\frac{2\mu S}{v_0}$	$\frac{2\mu S}{v_0} \cos \theta$
3. Isotropic antiferromagnetic material near the spin-flip transition	$\cos \theta = \cos \theta_0$ $\times \left[1 - \frac{\lambda}{2S} \sin^2 \theta_0 \right],$ $H_z < H_{z,c};$ $\theta \equiv 0, H_z > H_{z,c}$	$H_{z,c} \equiv H_{z,c}^{(0)}$	$\epsilon_h = Ck,$ $C^2 = 2J J^{(0)} S^2 \sin^2 \theta_0 \left(1 + \frac{\lambda}{S} \sin^2 \theta_0 \right),$ $H_z < H_{z,c}$	$\frac{2\mu S}{v_0} \cos \theta$	$\frac{2\mu S}{v_0} \sin \theta$

energy of the elementary excitations and of the magnetization on the site spin. Furthermore, the angle θ , between the quantization axis and the selected axis of the crystal, decreases with increasing field in an easy-plane ferromagnetic or antiferromagnetic material, and it vanishes at the point of the phase transition (at $H = H_c$). Correspondingly, the ground state at $H > H_c$ coincides with the classical ground state, so there is no quantum shift of the transition point, and the longitudinal magnetization at $T = 0$ reaches its nominal value at the transition point. In an easy-axis ferromagnetic substance, in contrast, the angle θ is $\pi/2$ in a transverse field at $H = H_c$; i.e., both of the states (at $H < H_c$ and $H > H_c$) contain a Bose condensate. Quantum fluctuations, which exist in both phases, shift the transition point down the field scale⁶⁰:

$$\frac{\Delta H_c}{H_c^{(0)}} = -\frac{\tilde{\alpha}}{2J^{(0)}S} \left\{ W + \frac{1}{2S[1-(2S)^{-1}]} \right\} - \frac{1}{3\sqrt{2}\pi S} \left(\frac{\tilde{\alpha}}{J} \right)^{3/2}, \quad (5.12)$$

where $H_c^{(0)} = \tilde{\alpha}S$. The reason for the sign of ΔH_c is that the phase at $H > H_c$, in which the magnetization is directed along the field, is "disordered" in the sense that it belongs to the paramagnetic region on the phase diagram (Fig. 1). Fluctuations, on the other hand—regardless of whether they are quantum-mechanical or classical—should broaden the region in which the disordered phase exists. A further manifestation of quantum fluctuations in strong fields is the circumstance that the longitudinal magnetization does not reach its nominal value at any finite field, simply approaching it in a square-root fashion.

A nonzero temperature $T \neq 0$ leads to a renormalization of the magnetic characteristics, regardless of how the ground state is constructed. The diagram equations for the temperature renormalizations are the same as for the quantum renormalizations [see (5.9)–(5.11)], but it now becomes necessary to consider the discrete nature of the Matsubara frequencies $2\pi nT$ (the values of n are integers). After a summation over the frequency, the result is the appearance of additional terms containing a Bose function n_k . The results calculated in Refs. 22, 35, 61, and 62 are given in Table VI. We wish to call attention to the existence of two temperature intervals far from the Curie point ($T \ll JS$): $T \ll \tilde{\alpha}S \sin^2\theta$ and $T \gg \tilde{\alpha}S \sin^2\theta$. At comparatively high temperatures, the renormalization is dominated by magnons with fairly large characteristic wave vectors, for which the spectrum essentially "does not feel" the anisotropy ($\varepsilon_k \approx JS(\mathbf{k}\mathbf{a})^2$). In this case the temperature renormalizations of all quantities are determined by the number of thermal magnons and are proportional to $T^{3/2}$. At the lowest temperatures, the major role is played by extremely long-wave magnons, whose energy depends strongly on the anisotropy (in an easy-plane ferromagnet, for example, these would be the magnons with $\varepsilon_k \propto k$). The situation is more complicated here because (first) the difference between ε_k and A_k is manifested and (second) cubic anharmonicities contribute substantially to the renormalization.¹⁵ In this interval the temperature laws depend on both the model and the selected characteristic. In an easy-axis ferromagnetic material, for example, all the temperature renormalizations (except the renormalization of the point of the orientational transition) contain an exponential

factor $\exp(-\varepsilon_0/T)$ (since these renormalizations are small, we do not give the exact expressions in Table VI). In systems with a Goldstone spectrum (in easy-plane ferromagnetic and antiferromagnetic materials), in contrast, the temperature renormalizations are always of a power-law nature. Furthermore, the spin-wave velocity, the longitudinal magnetization, and the transverse magnetization are renormalized in different ways in this case [$\Delta C(T) \propto T^4 \ln T$; $\Delta M_z(T) \propto T^4$; and $\delta M_1(T) \propto T^2$, respectively]. It is also useful to note the analogy with a superfluid Bose liquid, in which one observes similar functional dependences for, respectively, the sound velocity, the density of the normal component, and the density of particles in the condensate.⁸ Admittedly, this analogy would hold, strictly speaking, only in the case $H \neq 0$ because there are always ternary anharmonicities in a superfluid Bose liquid [these anharmonicities give rise to a logarithmic factor in $\Delta C(T)$]. In a zero field, the renormalization of the spin-wave velocity is $\Delta C(T) \sim T^4$.

The difference between the temperature dependence of the longitudinal magnetization and that of the transverse magnetization reflects the fact that the temperature corrections to M_z are finite in a space of any dimensionality, since at $T \neq 0$ the quantity M_z is essentially a paramagnetic characteristic, and its value is determined by the external field, while the temperature corrections to M_1 —i.e., to the order parameter—diverge in a space of dimensionality $D \leq 2$, in accordance with Mermin-Wagner theorem (§6). Correspondingly, the correction to M_1 is determined by the number of particles above the condensate ($\sum_k A_k \varepsilon_k^{-1} n_k$), while the correction to M_z (like the spin-wave velocity, which also remains finite, regardless of the dimensionality) is determined by the energy of the quasiparticles ($\sum_k \varepsilon_k n_k$).

It can be seen from Table VI that the value of the angle θ also depends on the temperature; i.e., the quantization axis shifts with a variation in T . A situation of this sort is characteristic of any oscillator with anharmonicities of odd powers. In essence, there was essentially also a shift of the quantization axis at $T = 0$ because of the zero-point vibrations. The two effects are identical in nature; the only distinction is that the amplitude of the quantum fluctuations increases with increasing reciprocal spin $1/S$, while the amplitude of the classical fluctuations increases with increasing T .

The table which we are discussing does not give equations for the damping of spin waves. The reason is that most of the equations are very lengthy, and there are a multitude of different limiting cases. We refer the interested reader to the original papers^{61,63–65} and content ourselves with some general assertions here.

First, the damping is nonzero even at $T = 0$ in the case $H \neq 0$, because of quantum effects.

Second, the damping is always (under the conditions $\alpha/J \ll 1$ and $T/JS \ll 1$) weaker than the corrections to the energy.

Third, at $H \neq 0$ the damping is caused primarily by three-particle anharmonicities. At $H = 0$ there are no such anharmonicities (the corresponding terms in the Bose Hamiltonian contain a factor of $\sin 2\theta$), and the damping is determined by four-particle anharmonicities.

The picture which we have drawn here is correct when dipole interactions are ignored. Incorporating them alters the situation for $H = 0$: A damping at $T = 0$ appears, and at a nonzero temperature the damping due to three-particle

TABLE VI. Temperature renormalizations of the magnetic characteristics of several systems containing zero-point vibrations. The asymptotic expressions for the functions $\tilde{\varphi}_{\theta, \epsilon, M}$ differ from the asymptotic expressions for the functions $\varphi_{\theta, \epsilon, M}$, which are shown in this table, solely in the substitution $\tilde{\alpha} \rightarrow J(0)$.

Model	$H_C(T)/H_C(0)$	$\cos \theta(T)/\cos \theta(0)$	$\epsilon_h(T)/\epsilon_h(0)$	$M_{\parallel}(T)/M_{\parallel}(0)$	$M_{\perp}(T)/M_{\perp}(0)$
1. Easy-plane ferromagnetic material in a longitudinal field	$1 - \frac{2}{S} \tau^{3/2} \zeta \left(\frac{3}{2} \right)$	$1 + \frac{2}{S} \varphi_{\theta}(T),$ $\varphi_{\theta}(T) = \begin{cases} \tau^2 \left(\frac{2J}{\tilde{\alpha} \sin^2 \theta_0} \right)^{1/2} \zeta(2), \\ T \ll \tilde{\alpha} S \sin^2 \theta_0. \\ \tau^{3/2} \zeta \left(\frac{3}{2} \right), \\ T \gg \tilde{\alpha} S \sin \theta_0 \end{cases}$	$\frac{C(T)}{C(0)} = \left(1 - \frac{\epsilon_h = Ck,}{S} \frac{1 + \cos^2 \theta_0}{\sin^2 \theta_0} \varphi_{\epsilon}(T) \right)^{1/2}$ $\varphi_{\epsilon}(T) = \begin{cases} 24\pi^2 \tau^4 \left(\frac{2J}{\tilde{\alpha} \sin^2 \theta_0} \right)^{5/2} \zeta(4) \\ \times \left(1 - \frac{18 \cos^2 \theta_0}{1 + \cos^2 \theta_0} \ln \frac{\tilde{\alpha} S \sin^2 \theta_0}{T} \right), \\ \tilde{\alpha} S \sin^2 \theta_0 \gg T; \\ \tau^{3/2} \zeta \left(\frac{3}{2} \right), \tilde{\alpha} S \sin^2 \theta_0 \ll T \end{cases}$	$1 + \frac{1}{S} \varphi_M(T),$ $\varphi_M(T) = \begin{cases} 24\pi^2 \tau^4 \\ \times \left(\frac{2J}{\tilde{\alpha} \sin^2 \theta_0} \right)^{5/2} \zeta(4), \\ \tilde{\alpha} S \sin^2 \theta_0 \gg T, \\ \tau^{3/2} \zeta \left(\frac{3}{2} \right), \\ \tilde{\alpha} S \sin^2 \theta_0 \ll T \end{cases}$	$1 - \frac{1}{S}$ $\times (2\varphi_{\theta}(T) - \varphi_M(T))$
2. Easy-axis ferromagnetic material in a transverse field	$1 - \begin{cases} \frac{\pi^{3/2} \sqrt{2}}{S} \tau^{3/2} \left(\frac{T}{\tilde{\alpha} S} \right)^{1/2}, \\ T \ll \tilde{\alpha} S, \\ \frac{2}{S} \tau^{3/2} \zeta \left(\frac{3}{2} \right), T \gg \tilde{\alpha} S \end{cases}$		$1 - O(e^{-\epsilon_0/T})$		
3. Isotropic antiferromagnetic material near the spin-flip transition	$1 - \frac{2}{S} \left(1 + \frac{W-1}{S} \right)^{-1} \times \tau^{3/2} \zeta \left(\frac{3}{2} \right)$	$1 + \frac{2}{S} \tilde{\varphi}_{\theta}(T)$	$\frac{C(T)}{C(0)} = \left(1 - \frac{\epsilon_h = Ck,}{S} \frac{1 + \cos^2 \theta_0}{\sin^2 \theta_0} \tilde{\varphi}_{\epsilon}(T) \right)^{1/2}$	$1 + \frac{1}{S} \tilde{\varphi}_M(T)$	$1 - \frac{1}{S}$ $\times (2\tilde{\varphi}_{\theta}(T) - \tilde{\varphi}_M(T))$

anharmonicities of dipole origin generally becomes predominant.

The magnon lifetime (and the method for determining it) depend on the relation between the magnon energy $\varepsilon_{\mathbf{k}}$ and the characteristic lifetime τ_{char} of the thermal quasiparticles with which the magnon interacts. In the case $\varepsilon_{\mathbf{k}} \tau_{\text{char}} \gg 1$ (the ballistic regime), the damping is due entirely to the first nonvanishing order of perturbation theory (usually the second), while at $\varepsilon_{\mathbf{k}} \tau_{\text{char}} \ll 1$ (the hydrodynamic regime) it is frequently important to go to a higher-order perturbation theory.¹⁶⁾ In particular, the change in the expression for the damping at the transition to the hydrodynamic region¹⁷⁾ for an easy-plane ferromagnetic material at $H = 0$ is extremely important⁶⁴⁾:

$$\gamma(\mathbf{k}) \propto \varepsilon_{\mathbf{k}} T^4 \quad \text{for } \varepsilon_{\mathbf{k}} \tau_{\text{char}} \gg 1, \quad (5.13)$$

$$\gamma(\mathbf{k}) \propto \varepsilon_{\mathbf{k}}^2 T^{-1} \quad \text{for } \varepsilon_{\mathbf{k}} \tau_{\text{char}} \ll 1, \quad (5.14)$$

and we have $\tau_{\text{char}} \propto T^{-5}$. We complete this section of the paper by presenting several results on systems in which the zero-point vibrations are not parametrically small. We begin with the case of an isotropic antiferromagnetic substance. Its classical spectrum

$$\varepsilon_{\mathbf{k}} = J(0) S (1 - v_{\mathbf{k}}^2)^{1/2} \quad (5.15)$$

contains two Goldstone modes [$\mathbf{k} = 0$ and $\mathbf{k} = (\pm \pi/a, \pm \pi/a, \pm \pi/a)$], which reflect the invariance of the Hamiltonian under spin rotations and the interchange of the sublattices.^{37,38,67} The symmetry of the spectrum under the substitution $k_i \rightarrow (\pi/a_i) - k_i$ allows us to shrink the cell in the \mathbf{k} space of the antiferromagnetic substance and to treat the spectrum as consisting of two coincident branches (the more customary interpretation). In applying a magnetic field we disrupt the invariance of the sublattices: The substitution of one sublattice for the other requires the simultaneous replacement of the spin directions, which in turn changes the Zeeman energy. This change in turn has a further consequence: The symmetry of the spectrum with respect to the point $k_i = \pi/2a_i$ ($i = x, y, z$) is disrupted. If we go to a shrunken cell in \mathbf{k} space, we find that a gap appears on one of the branches.

As we have already noted, the contribution of zero-point vibrations in an antiferromagnetic material is numerically small. For example, the first (in terms of $1/S$) correction to the spin-wave velocity is⁶⁸⁾

$$\frac{\Delta C}{C} = \frac{1}{2NS} \sum_{\mathbf{k}} \left(1 - \frac{\varepsilon_{\mathbf{k}}}{J(0)S} \right); \quad (5.16)$$

for a simple cubic lattice, this correction is $0.097/2S$.

The temperature correction to the velocity in the lowest order in $1/S$ is determined by the quasiparticle energy ($\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}}$) and is proportional to T^4 , as in an easy-plane ferromagnet.^{69,70} A specific feature of an antiferromagnetic material—the presence of *two* Goldstone modes in the spectrum—arises in calculations in the next order in $1/S$: Incorporating the interaction between low-energy modes with $\mathbf{k} \approx 0$ and $k_i \approx \pi/a$ gives rise to a factor which is logarithmic in the temperature,⁷¹⁾

$$\frac{\Delta C}{C} = -\frac{\pi^2}{2460 \sqrt{3} S} \left(\frac{T}{JS} \right)^4 \left(1 + \frac{40 \sqrt{3}}{3\pi^2 S} \left| \ln \frac{T}{JS} \right| \right). \quad (5.17)$$

We recall that the corresponding renormalization in a ferro-

magnetic substance gives rise to a constant factor $Q(S)$ (Table III).

The damping of antiferromagnetic magnons has been calculated in many papers, primarily for isotropic⁶³⁾ and easy-plane⁷²⁾ antiferromagnetic materials. Among the fundamental points we will mention only that calculations in second-order perturbation theory here lead to the correct result for the damping of a magnon at small wave vectors $ka \ll (T/J(0)S)^3 \ll 1$:

$$\gamma(\mathbf{k}) \propto \varepsilon_{\mathbf{k}}^2 T^3 \left| \ln T \cdot (J(0)S)^{-1} \right|. \quad (5.18)$$

Going over to hydrodynamics thus does not require taking into account the finite lifetime of the virtual magnons.⁶³⁾ We would like to call attention again to an important difference between the behavior of an isotropic antiferromagnetic material and that of an easy-plane ferromagnetic substance. Although the spectrum for each is linear as $k \rightarrow 0$, there are two Goldstone modes in the case of the antiferromagnetic substance, so the antiferromagnetic material has properties similar to those of an isotropic ferromagnetic material.

In completing this discussion of the antiferromagnetic material we would like to see how Adler's principle is manifested in this case.¹⁸⁾ We switch to bosons, and we diagonalize the quadratic form by means of a uv transformation, so that we will have a seed Goldstone spectrum. Any of the four amplitudes will then contain, along with a term quadratic in the energy in the numerator, the root of the product of energies in the denominator, so that as certain wave vectors tend toward zero the amplitudes will diverge, and it might appear that Adler's principle was being violated. This is of course not the case. The explanation is that Adler's principle is formulated for phonon-like Goldstone bosons, for which the seed Green's function is proportional to $(\omega^2 + k^2)^{-1}$. The Green's function of the magnons, in contrast, contains the first power of the frequency, $G(k, \omega) \propto (|k| - i\omega)^{-1}$. The physical reason is that the spins in a spin wave rotate in a common direction. Correspondingly, in calculations by perturbation theory with a magnon Green's function the denominators have a "deficiency" of virtual-quasiparticle energies in comparison with that which would be found in calculations with a phonon-like Green's function. This deficiency is exactly offset by the energies which appear in the denominators of the amplitudes of antiferromagnetic substances. Accordingly, we should speak of Adler's principle only in application to the numerators of the corresponding amplitudes, and in total accordance with the requirements of this principle the expansion of each of the numerators in momenta (or energies) begins with the quadratic terms.

We turn now to an easy-plane ferromagnetic material with a large anisotropy constant ($\alpha \gtrsim J$). At a large value of α , zero-point vibrations can completely change the structure of the classical ground state. To see this, we will analyze the case $\alpha \gg J$ with $H = 0$. In the zeroth order approximation in the exchange interaction, the problem then becomes a single-particle problem. We assume that the atomic spin is an integer spin. The ground state is then a singlet ($S_1^z = 0$), separated from the first, $2N$ -fold-degenerate, excited state (for one of the atoms, $S_{10}^z = +1$ or -1) by an energy gap $\sim \alpha$. Turning on a weak exchange interaction could not liquidate this gap, of course. The existence of this gap is unambiguous evidence of a complete smearing of the long-range order in

the ground state, since the spectrum in the ordered phase is definitely a Goldstone spectrum (see the discussion at the beginning of this section). With $\alpha \sim J$, an order-disorder phase transition clearly must occur.^{56,73}

For systems with half-integer spins the situation is different. In such systems, the ground state is 2^N -fold degenerate ($S_i^z = +1/2$ or $-1/2$) in the single-particle approximation. The incorporation of even a weak exchange interaction leads to an ordering in the x, y plane. In this case, therefore, there is no phase transition involving a change in the anisotropy constant: The system remains in the ordered phase at all times.

Applying a longitudinal field $H_z = H$ gives the phase diagram of an easy-plane ferromagnetic substance an extremely unusual shape,^{22,35,74} even at $T = 0$. As we move up to stronger fields we do not monotonically approach the melting point of the Bose condensate, as in Fig. 1; we instead cross S regions with spins which are ordered in the x, y plane. These regions are separated by regions in which there is no long-range order (Fig. 6).

This alteration of ordered and disordered regions results from the circumstance that turning on a magnetic field in the single-particle problem causes a splitting of all the levels (except the $S_i^z = 0$ level) by an amount proportional to the field and to the z projection of the spin. As the field is strengthened, levels with different values of S_i^z turn out to be the lowest-energy levels in succession. Near each of the crossing points of the levels $S_i^z = m$ and $S_i^z = m + 1$ (there are a total of $[s]$ such points) the ground state is $2N$ -fold degenerate, either exactly or approximately (depending on whether we are dealing with strictly the crossing point itself or a small neighborhood of it). Since the distance between the two lowest levels is smaller than the exchange integral which smears the upper level into a band, there will be a long-range magnetic order in the system. If the exchange is slight, all levels other than the two lowest ones are unimportant, so the system in its ordered phase is essentially equivalent to an XY magnetic substance with $S = 1/2$ in a longitudinal field. There is a long-range order in the x, y plane as long as the field is below the critical value.

Interestingly, the field dependences $M_z(H)$ and $M_1(H)$ at $\alpha \gg J$ are reminiscent of the H dependence of the Hall and dissipative conductivities under the conditions of the quantum Hall effect⁷⁴ (Fig. 7). This is not simply a for-

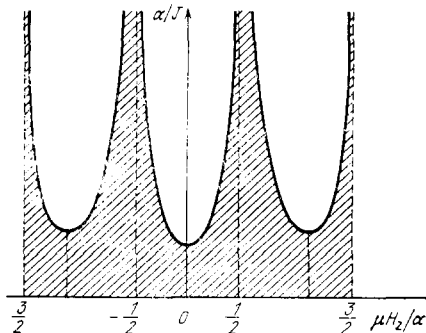


FIG. 6. Phase diagram of an easy-plane ferromagnetic material with a single-ion anisotropy and with $S = 2$ in the plane of the variables α/J and $\mu H_z / \alpha$. The hatched regions are the regions in which a long-range order exists in the x, y plane.

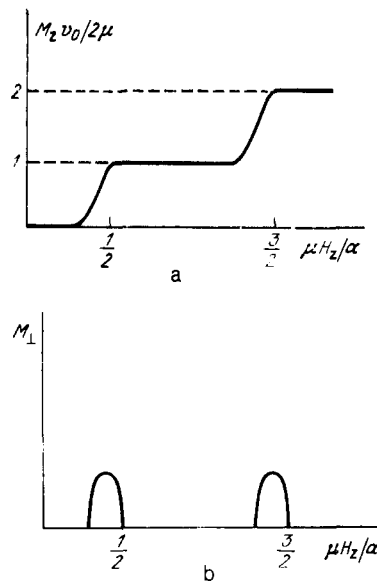


FIG. 7. Field dependence of (a) the longitudinal magnetization M_z and (b) the transverse magnetization M_1 of an easy-plane ferromagnetic material with a large single-ion anisotropy constant. The plateau on the plot of $M_z(H)$ and the simultaneous vanishing of $M_1(H)$ arises because of the finite size of the correlation radius in the paramagnetic ("localized") phase.

tuitous agreement, since the reason for the appearance of the strict plateau on the plot of $M_z(H)$ [and for the simultaneous vanishing of $M_1(H)$] is the same as that in the Hall effect: If the spins in the x, y plane have a finite correlation radius (i.e., under "localization" conditions, with $M_1 = 0$) the interaction between the single-particle levels which results from the transverse components of the exchange interaction does not give rise to a macroscopic effect, i.e., does not impart a field dependence to M_z . At a nonzero temperature, the entire picture will of course be smoothed over slightly.

Experimentally, the transition from a singlet state to magnetic order in a magnetic field has been observed in several magnetic substances with $S = 1$ (Ref. 75). Furthermore, a field dependence of M_z similar in shape to that predicted theoretically has been seen experimentally⁷⁶ for $\text{FeSiF}_6 \cdot 6\text{H}_2\text{O}$ ($S = 2$).

6. DISTINCTIVE FEATURES OF REDUCED-DIMENSIONALITY SYSTEMS

We have been discussing systems which are anisotropic in spin space but isotropic in coordinate space. Actually, the situation is different in many cases: The values of the exchange integrals along different directions are quite different. Quite frequently, the hierarchy of exchange interactions is so clearly expressed that one or two exchange integrals can simply be ignored. We are then led to the model of a reduced-dimensionality (two-dimensional or one-dimensional) magnetic substance as a zeroth order approximation.

In the strict sense of the word, of course, there are no reduced-dimensionality entities. For example, no matter how anomalously small certain components of the exchange interaction may be there is always a dipole-dipole interaction between magnetic moments, which decays slowly over distance.¹⁹⁾ Nevertheless, there are essentially always conditions under which the reduced-dimensionality properties of

TABLE VII. Some substances which fall in the categories of 3D, 2D, and 1D magnetic substances, are studied in this review. The data regarding the assignment of a specific substance to a particular category are taken from the proceedings of recent international conferences on magnetism and from some reviews.^{118,119}

Type of substance	Dimensionality	Substance
Ferromagnetic materials: exchange	3	EuO, EuS, $\text{Cu}(\text{NH}_4)_2\text{Br}_4 \cdot 2\text{H}_2\text{O}$
	2	K_2CuF_4 , $\text{Cu}(\text{CH}_3\text{NH}_3)_2\text{Cl}_4$
	1	$(\text{C}_6\text{H}_{11}\text{NH}_3)\text{CuCl}_3$, $(\text{C}_6\text{H}_{11}\text{NH}_3)\text{CuBr}_3$
Easy-plane with single-ion anisotropy	3	$\text{Dy}(\text{Ni})_5$, $\text{FeSiF}_6 \cdot 6\text{H}_2\text{O}$
	2	Rb_2CuCl_2
Easy-plane with unlike-ion anisotropy (of the XY model type)	1	CsNiF_3 , RbFeCl_3
	3	$\text{Fe}[\text{Se}_2\text{CN}(\text{C}_2\text{H}_5)_2]_2\text{Cl}$
Easy-axis with a single-ion anisotropy	2	CoCl_2
	1	$(\text{CH}_3)_4\text{NNiBr}_3$
Easy-axis with unlike-ion anisotropy (of the Ising-model type)	3	$\text{NiZnF}_8 \cdot 6\text{H}_2\text{O}$, FeP
	2	AgCrSe_2 , NaCrSe_2
Easy-axis with unlike-ion anisotropy (of the Ising-model type)	3	$\text{Tb}(\text{OH})_3$, $\text{Fe}[\text{S}_2\text{CH}(\text{C}_2\text{H}_5)_2]_2\text{Cl}$
	2	FeCl_2
Antiferromagnetic materials: exchange	1	$\text{CoCl}_2 \cdot 2\text{NC}_5\text{H}_5$, $\text{CoCl}_2 \cdot 2\text{H}_2\text{O}$
	3	RbMnF_3 , KMnF_3
Easy-plane with single-ion anisotropy	2	BaMnF_4 , K_2NiF_4
	1	$\text{CsMnCl}_3 \cdot 2\text{H}_2\text{O}$, RbNiCl_2 , CsNiCl_3
	3	CsMnF_3 , MnCO_3 , FeBO_3
Easy-plane with unlike-ion anisotropy (of the XY-model type)	2	$\text{Ba}(\text{Ni})_2(\text{PO}_4)_2$
	1	$(\text{CH}_3)_2\text{NH}_2\text{MnCl}_3$, $(\text{CH}_3)_4\text{NMnCl}_3$
	3	$\text{Co}(\text{C}_5\text{H}_5\text{NO})_6(\text{ClO}_4)_2$, $\text{Co}(\text{C}_5\text{H}_5\text{NO})_6(\text{BF}_4)_2$
Easy-axis with a single-ion anisotropy	2	$\text{CoBr}_2 \cdot 6\text{H}_2\text{O}$, $\text{CoCl}_2 \cdot 6\text{H}_2\text{O}$
	1	Cs_2CoCl_4 , PrCl_3
Easy-axis with unlike-ion anisotropy (of the Ising-model type)	3	MnF_2
	2	Rb_2NiF_4 , Tl_2NiF_4
	1	$\text{CsMnCl}_3 \cdot 2\text{H}_2\text{O}$
Complex structures: metamagnetic materials	3	DyPO_4 , FeF_2
	2	CoCs_2Br_3 , Rb_2CoF_4
helimagnetic materials	1	CsCoCl_3 , $\text{RbFeCl}_5 \cdot 2\text{H}_2\text{O}$
	3	$[(\text{CH}_3)_3\text{NH}]\text{CoCl}_3 \cdot 2\text{H}_2\text{O}$, FeCl_2 , FeBr_2
	3	MnO_2 , MnAu_2 , FeCl_3

magnetic substances are seen in their full glory. This is clearly the case, for example, when the temperature is high in comparison with the energy of the interaction which disrupts the reduced dimensionality, but low in comparison with the large exchange integral. There are also other examples in which a specific reduced-dimensionality behavior is not masked by the three-dimensional nature of the sample. Several magnetic substances which can be categorized as reduced-dimensional are listed in Table VII.

The aspect of reduced-dimensionality systems which attracts most of our attention is the significant increase in the role played by fluctuational effects, both temperature and quantum effects. We will discuss them separately, beginning with the case $T \neq 0$, in which classical fluctuations play a governing role. For two-dimensional systems, a manifestation of these fluctuations is the well-known Mermin-Wagner theorem⁷⁷ regarding the absence of long-range order from easy-plane and isotropic two-dimensional Heisenberg ferromagnetic substances at $T \neq 0$. This theorem shows that, in contrast with three-dimensional magnetic substances, the nature of the low-temperature behavior of 2D systems depends strongly on the dimensionality of the order parameter n . The assertion that there is no order does not apply to easy-axis ferromagnetic materials with a single-component order

parameter²⁰⁾ ($n = 1$). It applies only to systems in which the order parameter is a vector for which there is no special direction in the plane ($n = 2$) or in space ($n = 3$). An interesting aspect of 2D magnetic substances is that the absence of a long-range order does not mean that their magnetic properties at low temperatures are similar. In easy-plane magnetic substances ($n = 2$) there is a fixed temperature T_B at which the correlation radius becomes infinite, if we approach from the side of higher temperatures. Although there is no ordering at any temperature according to the Mermin-Wagner theorem, at $T < T_B$ we are dealing with a new phase: The system in a sense freezes at the point of the phase transition, where everything is prepared for ordering. The correlation radius and the susceptibility are infinite, and the spin correlation function decays by a power law.⁷⁹⁻⁸¹ These properties persist at all temperatures below T_B , except that the exponent of the power-law decay of the correlation function decreases monotonically with decreasing temperature.⁸² At $n > 2$, on the contrary, no phase transition occurs down to $T = 0$; i.e., at all nonzero temperatures the magnetic substance is in a paramagnetic phase with a finite spin correlation radius.^{83,84} The reason for the amplification of fluctuations effects in an isotropic magnetic substance ($n = 3$) in comparison with those in an easy-plane magnetic substance

TABLE VIII. Temperature renormalizations of the spin-wave energy for 2D magnetic substances with various numbers of order-parameter components (see Table IV regarding θ_0).

Model	Temperature renormalization of the magnon energy $\varepsilon_{\mathbf{k}}(T)/\varepsilon_{\mathbf{k}}(0)$
1. 2D easy-axis ferromagnetic material	$\varepsilon_{\mathbf{k}} = (\varepsilon_0^2 + U^2 k^2)^{1/2}$
2. 2D easy-plane ferromagnetic material	$\frac{U(T)}{U(0)}, \frac{\varepsilon_0(T)}{\varepsilon_0(0)} \sim 1 - O(e^{-\varepsilon_0/T})$ $\varepsilon_{\mathbf{k}} = Ck, \quad \frac{C(T)}{C(0)} = \left(1 - \frac{2}{S} (1 + 2 \operatorname{ctg}^2 \theta_0) \varphi_{\varepsilon}^{(2)}(T)\right)^{1/2}$ $\varphi_{\varepsilon}^{(2)}(T) = \begin{cases} \tau \left(\frac{T}{\tilde{\alpha} S \sin^2 \theta_0}\right)^2 \frac{1 + 18 \operatorname{ctg}^2 \theta_0}{1 + 2 \operatorname{ctg}^2 \theta_0} \zeta(3), & T \ll \tilde{\alpha} S \sin^2 \theta_0, \\ \tau \ln \frac{T}{\tilde{\alpha} S \sin^2 \theta_0}, & T \gg \tilde{\alpha} S \sin^2 \theta_0 \end{cases}$
3. 2D isotropic ferromagnetic material	$\varepsilon_{\mathbf{k}} = Jk^2,$ $\frac{J(T)}{J(0)} = 1 - \frac{(2T/\pi J(0) S^2)^2 \ln ka }{1 - (2T/\pi J(0) S^2)^2 \ln ka }$
4. 2D isotropic antiferromagnetic material	$\varepsilon_{\mathbf{k}} = Ck;$ $\frac{C(T)}{C(0)} = 1 - \frac{1}{2} \left(\frac{2T}{\pi J(0) S^2}\right)^2 \ln ka \times \left(1 + O\left(\frac{2T}{\pi J(0) S^2}\right) \ln ka \right)$

($n = 2$) is the larger number of degrees of freedom of the order parameter (and thus the existence of additional low-energy excitations). The situation can be seen most simply by comparing easy-plane magnetic substances and antiferromagnetic substances. In each case there are low-energy excitations with $k\alpha \ll 0$, but in the antiferromagnetic substance there are in addition some low-lying excitations with wave vectors close to π/a , and all the specific features of fluctuation effects in an antiferromagnetic substance result from the interaction of low-energy magnons with $\mathbf{k} \approx 0$ and $k_i \approx \pi/a$.

The difference in the behavior of easy-plane and isotropic magnetic substances at low temperatures is manifested in a difference in the structure of the long-wave part of the spectrum. If the correlation radius is infinite (an easy-plane magnetic substance), spin waves exist at all k . If, on the other hand, the correlation radius ζ is finite (isotropic magnetic substances), there are no spin waves as weakly damped collective excitations at $k\zeta < 1$ (over distances greater than the correlation length). The spectrum in this part of \mathbf{k} space is of a diffusive nature, as in a paramagnet. The nature of the long-wavelength excitations governs the form of the basic temperature corrections to the seed ($T = 0$) spin-wave spectra. Expressions for the corrections to the spectra of various 2D magnetic substances^{35,71,85,86} are given in Table VIII. The corrections in the cases of easy-plane ferromagnetic materials are finite and small at low temperatures; these results are a natural reflection of the infinite nature of the correlation radius. The finite velocity of the spin waves actually serves as an "order parameter" in the phase in which the spin correlations fall off by a power law.

In isotropic magnetic substances, in contrast, the corrections to the spectrum grow logarithmically with decreasing wave vector k and become comparable to the seed values of the energy at

$$k \sim \zeta^{-1} = a^{-1} \exp(-2\pi J S^2 T^{-1}), \quad (6.1)$$

where a is the interatomic distance. The quantity ζ can naturally be treated as a correlation radius of the system. Although a perturbation theory would not be valid at $k\zeta < 1$, this assertion is justified by the circumstance that, according to the exact solution,²¹⁾ the fluctuations grow even outside the range of applicability of perturbation theory.

According to the calculations of Refs. 85 and 86, the damping of spin waves is always slight when perturbation theory is applicable. In isotropic magnetic substances at the applicability limit of perturbation theory (at $k\zeta \sim 1$) the damping is comparable to the real part of the energy.

Let us return to easy-plane ferromagnetic substances. The decrease in the velocity (or rigidity) of the spin waves with increasing temperature (Table VIII) suggests that the transition to the paramagnetic phase occurs in the standard way: as a second-order phase transition, when the rigidity vanishes. However, this is not the case! The transition from a power-law decay to an exponential decay of the correlation functions occurs as an infinite-order phase transition (a Kosterlitz-Thouless phase transition or "Berezinskii-Kosterlitz-Thouless phase transition"). The disruption of the rigidity, i.e., the formation of a finite correlation length, results from the creation of some unusual elementary excitations: vortices. At low temperatures, the vortices collect in pairs (a vortex plus an antivortex). As the temperature is raised, the average distance between the centers of the vortices in a pair increases, and the transition temperature is determined quite accurately from the condition under which the existence of a single vortex is preferred from the thermodynamic standpoint. The energy loss $\Delta E = \pi J \ln(L/a)$ which results from the formation of a single vortex (L is the dimension of the sample) is canceled exactly at the transition point by the entropy term, $2T_B \ln(L/a)$ (the center of the vortex can be placed anywhere in the sample of area L^2). Hence

$$T_B = \frac{\pi}{2} J. \quad (6.2)$$

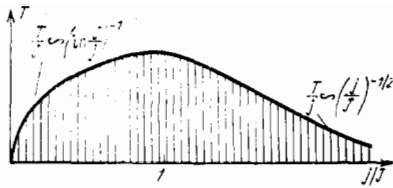


FIG. 8. Phase diagram of a 2D isotropic ferromagnetic material in terms of the variables T and j/J . Here j is the exchange integral between layers; the hatched region is that in which an ordered phase exists.

More-accurate calculations show that the estimate of T_B given by (6.2) differs only slightly from the actual value.^{81,82}

To wrap up this discussion of the role played by classical fluctuations in 2D systems we would like to point out that turning on a weak exchange interaction along the third axis quickly gives rise to a long-range order in an easy-plane ferromagnetic material at all temperatures below T_B , while in isotropic magnetic substances it gives rise to a phase transition at $T_c \propto J/\ln J/j$, where j is the exchange integral between layers (Fig. 8).

In 1D systems the classical fluctuations completely rule out a phase transition at $T \neq 0$, regardless of the number of components in the order parameter. The reason is that in a fairly long chain a breakup into domains is always favored from the thermodynamic standpoint: The energy loss from the formation of a single domain wall, $\Delta E \sim 2J$, is offset by the entropy increase ($TS \sim T \ln N$). At low temperatures, however, the dimensions of the domains are large, and they grow with decreasing temperature, exponentially in an Ising magnetic substance and by a power law in isotropic and planar magnetic substances. If the wave vectors are not too small, one can thus define the spin waves as elementary excitations above an ordered state (within a domain). In easy-plane and easy-axis magnetic substances, the temperature corrections to the velocity of these magnons are nonzero ($\Delta C/C \propto T^2$ in an easy-plane magnetic substance), while in an isotropic magnetic substance the correction to the spectrum is $\Delta \epsilon_k / \epsilon_k \propto (T/J)^2 / ka$. The range of applicability of the spin-wave description thus becomes narrower [ka must be greater than $(T/J)^2$].

In real systems a weak interaction between chains would naturally lead to a phase transition. It occurs at $T_c^{(1)} \propto (Jj)^{1/2}$, where j is the exchange integral between chains.

We now consider the role played by quantum fluctuations in reduced-dimensionality systems. In 2D systems these fluctuations do not play a critical role: The quantum effects are manifested more strongly than in 3D systems, but "there are not enough of them" to completely wash out the order in the ground state. We are of course talking about analogs of those 3D systems in which quantum fluctuations are weak. In 1D systems, quantum fluctuations are more important. Their manifestation at $T = 0$ is reminiscent of that of classical fluctuations in 2D systems at $T \neq 0$: A long-range order exists in the ground state for systems with $n = 1$ but not for systems with $n > 1$ (Refs. 33, 37, 87 and 88). At $n = 2$ the correlation radius in the ground state is infinite, and the spin correlation functions fall off with increasing distance by a power law, with an exponent which depends on

the reciprocal spin $1/S$. The reciprocal spin plays the same role for quantum fluctuations as the temperature plays for classical fluctuations. In isotropic magnetic substances ($n = 3$) the situation is more complicated than in the corresponding 2D systems. In ferromagnetic materials there are no quantum fluctuations at all, while in antiferromagnetic substances the structure of the ground state for an arbitrary spin has not yet been finally resolved. According to a perturbation theory⁷¹ in $1/S$, the quantum corrections to the Green's function have a logarithmic dependence on the wave vector—of the same nature as the temperature corrections to the spectrum of a 2D ferromagnetic material (Table VIII)—

$$G(k, \omega) \propto \frac{1}{\omega^2 + \epsilon_k^2} \left[1 + \frac{(\pi^2 S^2)^{-1} |\ln ka|}{1 - (\pi S)^{-1} |\ln ka|} \right]. \quad (6.3)$$

If we assume that the fluctuation corrections continue to grow even outside the range of applicability of perturbation theory, we can conclude that the spin correlation functions in the ground state of 1D antiferromagnetic substances decay exponentially, i.e., that the ground state of a one-dimensional antiferromagnetic material is paramagnetic. This system is a spin liquid. On the other hand, it has been established quite accurately that the correlation radius in the ground state is infinite for all antiferromagnetic materials with half-integer site spin.^{87,89,90} It has been suggested^{91,92} that the analogy with 2D magnetic substances beyond the range of applicability of perturbation theory (perturbation theory was the basis for the conclusion that a finite correlation radius forms as a result of fluctuations) holds only for integer values of the spin S (this suggestion was first made by Haldane⁹¹). Numerical simulations^{89,93-95} confirm this point of view.²²⁾

The introduction of a weak exchange interaction between the spin chains for easy-plane magnetic substances and for isotropic antiferromagnetic materials with half-integer spins quickly gives rise to a long-range order in the ground state (for magnetic substance with $n = 2$, we again see a manifestation of an analogy with classical 2D systems). In an isotropic antiferromagnetic material with integer spins, the paramagnetic ground state is stable with respect to the introduction of a sufficiently weak exchange interaction between chains. A finite value of the interchain exchange integral j is required if the correlation radius is to "grow all the way to infinity":

$$\frac{j_{cr}}{J} \propto e^{-\pi S}. \quad (6.4)$$

Admittedly, we cannot rule out the further possibility that when an interchain exchange is introduced the long-range order will appear immediately (at an arbitrarily small j), but not as the result of a phase transition. That possibility seems unlikely to us. Figure 9 shows the phase diagram proposed for an anisotropic Heisenberg antiferromagnetic material.

In concluding this section of the review we note that at sufficiently low temperatures, despite the absence of long-range order in several cases, the spectrum of spin waves with energies $\epsilon \sim T$ is always slightly renormalized.²³⁾ For this reason, the equations given in Table VII are completely suitable for calculating the macroscopic characteristics of reduced-dimensionality magnetic substances. The existence of a region of very small wave vectors, in which there are no weakly damped collective excitations, is manifested in the

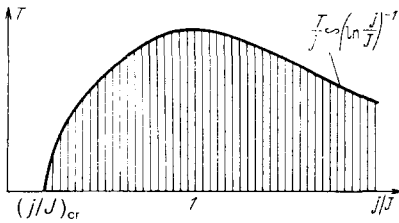


FIG. 9. Phase diagram of an antiferromagnetic material with an integer site spin in terms of the variables J and j/J . Here j is the exchange integral between chains. The hatched region is the region in which an ordered phase exists. The decrease in the transition temperature at $j \gg J$ is a consequence of the quasi-two-dimensional nature of the antiferromagnet in this limit [$(j/J)_{cr} \propto e^{\pi s}$].

form of small corrections to the equations derived in the spin-wave approximation.

7. QUANTUM PHASE TRANSITIONS

As we stated in the Introduction, the existence of phase transitions in the HT plane (Fig. 1) shows that a transition from a paramagnetic phase to a ferromagnetic phase occurs not only as the temperature is varied but also as the magnetic field is varied. When such a transition occurs at a low temperature (and a transition may occur even at $T = 0$), it is called an *orientational transition*. We need to recall that at any nonzero temperature an orientational transition is not a specific transition; i.e., such a transition is not fundamentally different from a transition at the Curie point. From the theory of phase transitions we know that the critical behavior at $T \sim T_c(H)$, where $T_c(H)$ is the transition temperature at $H \neq 0$, is determined by fluctuational modes with small wave vectors²⁴ $ak \sim |(T_c(H) - T)/JS|^{1/2**}$. In the immediate vicinity of the transition point, the distribution function of these modes is always classical,

$$n_{\mathbf{k}} \approx \frac{T}{\epsilon_{\mathbf{k}}} \quad \text{for} \quad \left| \frac{T_c(H) - T}{T_c(H)} \right| \ll 1, \quad (7.1)$$

and it is specifically for this reason that the nature of the critical behavior remains the same over the entire transition line. The critical indices are the same as at $H = 0$. They depend on the dimensionality of the space, D , and the number of components of the order parameter n . The values of these indices are given in any sufficiently detailed monograph on critical phenomena (see Refs. 96 and 97 for example). We would simply like to recall that the critical dimensionality D_c , below which the Landau theory does not apply, is 4.

At a low temperature, the classical fluctuations are important only in a very narrow interval near the critical point (the width of this interval tends toward zero along with the temperature). However, as can be verified easily, and as examples show, there are always zero-point vibrations which create their own fluctuation region on at least one side of the point of an orientational transition in a spin system. Far from the transition point, a slight deviation from ideality in a Bose gas of magnons is sufficient to keep the renormalizations for the zero-point vibrations small. As the transition point is approached, the quantities to which renormalizations are calculated (e.g., the energy of a spin wave and the angle between the directions of the magnetic moment and the external field) decrease, so the quantum renormaliza-

tions are important near the critical point, and we need to go beyond perturbation theory.

At sufficiently low temperatures the fluctuation region which results from the zero-point vibrations is wider than the classical fluctuation region. In a 2D easy-axis ferromagnetic material in a transverse magnetic field, for example, the classical fluctuation region corresponds to $|\Delta H_x/H_x| < T/J(0)S$, while the quantum fluctuation region corresponds to $|\Delta H_x/H_x| < (\tilde{\alpha}/J(0)S)^3$. These regions are comparable at $T \sim T_{cr} = \tilde{\alpha}(\tilde{\alpha}/J(0)S)^2$. At $T \ll T_{qu}$, a magnetic substance behaves essentially everywhere in the fluctuation region as it does at zero temperature. In other words, the critical behavior is shaped completely by quantum fluctuations.

Again in the case of quantum fluctuations there is a concept of a critical dimensionality D_c^{qu} . The value of D_c^{qu} is always smaller than D_c^{cl} since the growth of the interaction between quantum fluctuation modes is weakened. The reason for this effect is that large-scale fluctuations play a role near the critical point, so the spins of blocks consisting of a large number of cells serve as "elementary" spins.⁹⁸ The spin of each such block is very large, and it is for this reason that the quantum effects are weakened.²⁵ However, the growth of the interaction between fluctuations at a sufficiently small dimensionality of the space is not completely liquidated.

Quantum orientational transitions are far more sensitive than classical transitions to the number of components of the order parameter n (Ref. 99). In particular, the critical dimensionality varies with n (Refs. 100 and 101):

$$D_c^{cr} = 3 \quad \text{for} \quad n = 1, \quad D_c^{cr} = 2 \quad \text{for} \quad n = 2. \quad (7.2)$$

A further weakening of the interaction between fluctuations in modes with a two-component order parameter occurs because for these systems the approach to the transition point is simultaneously an approach to a state without zero-point vibrations, which occurs above the transition point. In systems with a single-component order parameter, quantum fluctuations exist on both sides of the critical point.

The difference in the structure of the "paramagnetic" phase (the presence or absence of zero-point vibrations) leads not only to a difference in the critical dimensionalities but also to a difference in the values of the critical indices. For $n = 1$ there is a simple rule: The indices of the transition at $T = 0$ are the same as those of the transition at $T \neq 0$ in the same system, but in a space of dimensionality one unit greater.^{99,100,102} For $n = 2$ there is no corresponding rule: The critical behavior has no classical equivalent. In this case the only space with a dimensionality less than the critical dimensionality is one-dimensional space. The indices for it can be determined exactly¹⁰³:

$$\gamma = 1, \quad \nu = \frac{1}{2}, \quad \eta = 0, \quad \beta = \frac{1}{4}, \quad \delta = 5, \quad \alpha = \frac{1}{2}. \quad (7.3)$$

Furthermore, a specific feature of the phase transition for an easy-plane magnetic substance—the approach to a complete ferromagnetic ordering—makes it possible to determine the indices exactly as functions of an arbitrary dimensionality D (Refs. 104–107).

A nontrivial critical behavior is of course observed in systems with a dimensionality equal to the critical dimensionality. For orientational transitions this circumstance is seen in (for example) the logarithmic divergence of the longitudinal susceptibility (which is analogous to the heat ca-

capacity in the case of a transition along the temperature scale). The power of this logarithm is¹⁰⁸⁻¹¹⁰

$$\begin{aligned} n = 1, \quad D = 3: \quad \chi_{\parallel} &\propto \ln^{1/5} \frac{H_c}{\Delta H}, \\ n = 2, \quad D = 2: \quad \chi_{\parallel} &\propto \ln \frac{H_c}{\Delta H}. \end{aligned} \quad (7.4)$$

In completing this section of the review, we note that the equivalence of D -dimensional quantum and $(D + 1)$ -dimensional classical critical indices also prevails in the case of easy-plane ferromagnetic materials, but in transitions in the anisotropy constant⁵⁶ rather than in orientational transitions (cf. §5 and Fig. 6). Admittedly, this is true only for certain discrete values of the magnetic field H :

$$\begin{aligned} \mu H_z^m = \frac{\alpha}{2m} \{3m^2 - S^2 - S + [S(S+1) - m(m+1)]^{1/2} \\ \times [S(S+1) - m(m-1)]^{1/2}\}, \end{aligned} \quad (7.5)$$

where $m = -S + 1, \dots, S - 1$ ($H = 0$ also appears in this sequence for integer values of S). The equivalence of the quantum and classical indices can be seen by writing a Ginzburg-Landau functional for quantum phase transitions (as can be done, for example, by a coherent-state technique). The quantum nature of the situation is reflected by the appearance in this functional of an additional variable—the Matsubara frequency—in addition to the wave vector. If the field H_z satisfies condition (7.5), the frequency ω appears in the form ω^2 in the quadratic part of the functional. In other words, the wave vector \mathbf{k} in a sense acquires an additional component, with the result that the D -dimensional quantum and $(D + 1)$ -dimensional classical critical behaviors become equivalent. If, on the other hand, the field H_z differs from that given by (7.4), then the quadratic part of the functional will contain, in addition to ω^2 , a term which is linear in ω and which turns out to be governing at low frequencies. This term forms the specific quantum critical behavior: the same behavior as that near orientational transitions in easy-plane magnetic substances.

8. DIPOLE INTERACTION AS A SOURCE OF ZERO-POINT VIBRATIONS

In speaking of relativistic interactions up to this point we have restricted the discussion to the short-range interaction forces between atoms. In other words, we have considered the single-ion and unlike-ion anisotropies. We know that between magnetic moments there is always a dipole-dipole interaction, which falls off slowly with distance ($\propto r^{-3}$). Strictly speaking, this dipole interaction can never be omitted. However, if the dimensionless anisotropy constant is large in comparison with 4π (as was tacitly assumed above), the role of the dipole interaction is unimportant in many cases. On the other hand, incorporating the dipole interaction leads to several *qualitative* changes. Let us examine these changes in more detail.

We would like to begin by emphasizing that when the dipole interaction is taken into account there will be zero-point vibrations in any spin structure. This assertion means that in reality there is always the complexity of determining the ground and low-lying excited states and their energies.²⁵ In the ferromagnetic materials with a large exchange interaction which we have studied, the amplitude of the zero-point vibrations due to the dipole interaction is small, and we

can use the approximation of low particle density, which was demonstrated in the preceding sections. A correct calculation will of course require an accurate inclusion of the strong exchange renormalizations. That situation does not, however, change the results in the lowest order in μ^2/Ja^3 compared to the semiclassical equations.⁶⁰

To see the qualitative changes caused by the incorporation of a long-range effect, we write the known expressions for the energy of a spin wave in a uniaxial ferromagnetic material at $H = 0$ (Ref. 25):

$$\begin{aligned} \varepsilon_{\mathbf{k}}^2 = [2\tilde{\alpha}S + J(0)S(1 - \nu_{\mathbf{k}})] \\ \times [2\tilde{\alpha}S + J(0)S(1 - \nu_{\mathbf{k}}) + 16\pi\mu^2Sa^{-3}\sin^2\theta_{\mathbf{k}}]. \end{aligned} \quad (8.1)$$

Analogously, in a uniaxial antiferromagnetic material we would have¹¹¹

$$\begin{aligned} \varepsilon_{\mathbf{k}}^{(1)} = 4\tilde{\alpha}S(\tilde{\alpha}S + J(0)S) + (J(0)S)^2(1 - \nu_{\mathbf{k}}^2) \\ + 16\pi\mu^2S(3a^3)^{-1}[2\tilde{\alpha}S + J(0)S(1 - \nu_{\mathbf{k}})], \\ \varepsilon_{\mathbf{k}}^{(2)} = 4\tilde{\alpha}S(\tilde{\alpha}S + J(0)S) + (J(0)S)^2(1 - \nu_{\mathbf{k}}^2) \\ + 16\pi\mu^2S(3a^3)^{-1}(1 - 3\sin^2\theta_{\mathbf{k}}) \\ \times [2\tilde{\alpha}S + J(0)S(1 - \nu_{\mathbf{k}})]. \end{aligned} \quad (8.2)$$

Here $\theta_{\mathbf{k}}$ is the angle between the wave vector \mathbf{k} and the quantization axis Z .

The long-range effect leads to a nonanalytic dependence on the components of the wave vector: $\sin^2\theta_{\mathbf{k}} = \mathbf{k}_z^2/\mathbf{k}^2$. As a result, even at $\mathbf{k} = 0$ there exists a band of allowed energy values.²⁶⁾ As can be seen by comparing (8.1) and (8.2), the role played by the dipole interaction in an antiferromagnetic substance is less important than that in a ferromagnetic one. The reason is that the gap in an antiferromagnetic substance is larger than that in a ferromagnetic one, because of the exchange intensification, while the dipole energy does not experience this intensification. The zero-point vibrations caused by the dipole interaction lead to the result that even in the case of an isotropic ferromagnetic material the magnetization at $T = 0$ does not reach saturation at any finite field²⁵:

$$M_H = 2\mu Sa^{-3} [1 - (2S)^{-1}Q(H)], \quad (8.3)$$

where, in the specified limiting cases, we have

$$\begin{aligned} Q(H) = \sqrt{\pi} \left(\frac{\mu^2}{Ja^3} \right)^{3/2}, \quad H \ll \frac{2\mu S}{a^3}, \\ = \frac{32\pi}{15\sqrt{2}} \left(\frac{\mu^2}{Ja^3} \right)^2 \left(\frac{JS}{\mu H} \right)^{1/2}, \quad H \gg \frac{2\mu S}{a^3}. \end{aligned} \quad (8.4)$$

In some of the anisotropic magnetic substances discussed above, there were no zero-point vibrations in strong fields. Incorporating a dipole interaction of course causes such vibrations. As a particular result, the strength of the critical field at $T = 0$ is lowered, and there is a deviation of the magnetization in the ground state from its nominal value.

The dependence of the gap in the spin-wave spectrum on the direction of the vector \mathbf{k} leads to a change in the temperature properties of magnetic substances at extremely low temperatures,¹¹² $T \ll 4\pi\mu^2S/a^3$. In an isotropic ferromagnetic material at $H = 0$, for example, there should be a violation of the Bloch law $\Delta M \propto T^{3/2}$ as the temperature is reduced:

$$\Delta M(T) = -\frac{\pi}{96} \frac{2\mu S}{a^3} \left(\frac{Ta^3}{4\pi\mu^2 S} \right)^{1/2} \left(\frac{T}{JS} \right)^{3/2}, \quad T \ll \frac{4\pi\mu^2 S}{a^3}. \quad (8.5)$$

The dipole-dipole interaction has an even stronger effect on the kinetic properties of magnetic substances.

First, the magnon lifetime in an isotropic ferromagnet is finite even at $T = 0$ because of the dipole interaction.^{113,114}

Second, this interaction always—regardless of the spin structure—generates three-magnon anharmonicities. The result is to reduce substantially the lifetime of a spin wave in several cases (in which the anharmonicities due to the anisotropy energy begin at the four-magnon level).^{41,44,113–115}

Third, the existence of a band of allowed energies as $\mathbf{k} \rightarrow 0$ makes possible an elastic scattering of a magnon with $\mathbf{k} = 0$ and its conversion into a magnon with $\mathbf{k} \neq 0$. This effect leads to a broadening of the ferromagnetic-resonance line²⁷ (Ref. 2).

The dipole-dipole interaction also affects the behavior of magnetic substances near phase-transition points. The effect is not simply one of a renormalization of the transition point; there are also changes in the critical indices. Specifically, the indices of a D -dimensional magnetic substance with a dipole-dipole interaction are the same as those of a $(D + 1)$ -dimensional magnetic substance when long-range effects are ignored.¹¹⁷ The reason for this correspondence is that the low-energy fluctuations which are important in a second-order phase transition are characterized by a dispersion law with

$$\varepsilon_{\mathbf{k}} \propto \left[(\mathbf{ak})^2 + \theta_{\mathbf{k}}^2 + \left| \frac{H - H_c}{H_c} \right| \right], \quad (8.6)$$

from which we see that $\theta_{\mathbf{k}}$ is playing the role of a $(D + 1)$ th component of the vector \mathbf{k} .

As is clear from the discussion above, the dipole-dipole interaction is particularly important for states with extremely small wave vectors. The nonanalytic dependence of the energy of a spin wave on the components of the wave vector in the limit $\mathbf{k} \rightarrow 0$ focuses our attention on (first) the question of the role played by the finite size of the sample and (second) the applicability limits of the magnetostatic approximation.

These questions go beyond the scope of the present review.

9. CONCLUSION

In this review we have studied the structure of the ground and low-lying excited states of a system of spins pinned at the sites of a crystal lattice. The most important interaction between spins is the exchange interaction. In addition, there are relativistic interactions (spin-spin and spin-orbit) which disrupt the isotropy in spin space. Specifically, we studied the properties of systems which are describable by various versions of Hamiltonian (2.3). What we have been attempting to say in this review can basically be summarized in two assertions.

- In most cases, a classical ground state (complete ordering at $T = 0$) is partially disrupted (or even completely disrupted, in the case of reduced dimensionalities) by quantum fluctuations: zero-point vibrations of spins. The amplitude of the zero-point vibrations (even in $3D$ systems) is, strictly speaking, small, provided that there is a parameter in terms of

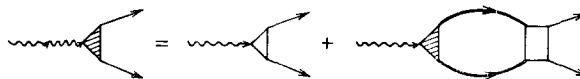


FIG. 10. One of the magnon-phonon interaction processes for which the exchange renormalization of the vertex is important. The vectors, which are graphical representations of the Green's functions of magnons, are pointed in the same direction. A wavy line represents the Green's function of a phonon.

which the ground state is close to a ferromagnetic state.

- Since the exchange interaction between magnons is not small, when we take it into account we cannot restrict the analysis to low-order perturbation theories (if we are talking about arbitrary values of the spin S).

A method for taking the exchange normalizations into account exactly (summing ladder sequences of diagrams) is demonstrated in the Appendix for the particular example of an isotropic ferromagnetic material. The results of corresponding calculations for easy-axis and easy-plane ferromagnetic materials are given in the corresponding tables. We wish to emphasize that the assertion that the exchange renormalizations must be taken into account fully applies to all situations in which two or more magnons are created as the result of some process (e.g., the decay of a phonon into two magnons) (Fig. 10). The result of the renormalization is not universal: Each vertex renormalizes in its own way.

In concluding this review of the role played by the interaction between magnons in magnetic insulators we would like to point out that, although the topic which we have selected is a particular case of a many-body system, it serves as an example which demonstrates many common types of behavior which are being studied in both quantum statistical physics and field theory. In particular, the simple model of a Heisenberg Hamiltonian can be used to check the predictions of the modern theory of critical phenomena. On the other hand, the variety of ways for working with spin operators and the graphic value of the results means that (first) we can construct approximate solutions in the form of expansions in small parameters which actually exist, and (second) we can frequently derive exact analytic solutions,^{87,88,103} which can serve as "reference points" for general theories. Furthermore, magnetic systems are convenient subjects for numerical calculations.^{89,92-95} Finally, as we have already mentioned, nearly each one of the variety of particular cases which we have discussed here has a representative in the vast assortment of available magnetic crystals.

APPENDIX. TOTAL SCATTERING AMPLITUDE IN AN ISOTROPIC HEISENBERG FERROMAGNETIC SUBSTANCE

We restrict the discussion to the case²⁸ $T = 0$, and we write out explicitly the integral equations for the total amplitude which are shown in diagram form in (4.3). We first separate from their kernels and free terms the particular terms which do not depend on the spin factor $q = 4S\{1 - [1 - (1/2S)^{-1}]\}$ (Table I), and we make a convenient change in notation ($r_{\mathbf{pk}}^{\mathbf{pk}}$ is the total vertex):

$$\tilde{r}_{\mathbf{pk}}^{\mathbf{pk}} = -\frac{J^{(1)}}{4} r_{\mathbf{pk}}^{\mathbf{pk}}, \quad \tilde{r}_{\mathbf{p}, \mathbf{k}}^{\mathbf{m}, \mathbf{p}+\mathbf{k}-\mathbf{m}} \equiv \tilde{r}_{\mathbf{p}, \mathbf{k}}^{\mathbf{m}} \equiv \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}}. \quad (A1)$$

The last equation is a consequence of the Hermitian nature of the Bose Hamiltonian. Here are the explicit equations:

$$\begin{aligned} \tilde{r}_{\mathbf{p}, \mathbf{k}}^{\mathbf{p}, \mathbf{k}} &= \nu_{\mathbf{p}-\mathbf{k}} + 1 - \nu_{\mathbf{p}} - \nu_{\mathbf{k}} + \frac{1}{2NS} \\ &\times \sum_{\mathbf{m}} (\nu_{\mathbf{m}-\mathbf{p}} + \nu_{\mathbf{m}-\mathbf{k}} - \nu_{\mathbf{m}} - \nu_{\mathbf{m}-\mathbf{p}-\mathbf{k}}) I_{\mathbf{m}}^{-1} \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}}, \\ &- \frac{2-q}{4NS} \sum_{\mathbf{m}} \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} + (1-q)(\nu_{\mathbf{p}} + \nu_{\mathbf{k}}) \\ &\times \left(1 + \frac{1}{2NS} \sum_{\mathbf{m}} \frac{\tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}}}{I_{\mathbf{m}}} \right), \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} &= \nu_{\mathbf{m}-\mathbf{p}} + \nu_{\mathbf{m}-\mathbf{k}} - \nu_{\mathbf{m}} - \nu_{\mathbf{m}-\mathbf{p}-\mathbf{k}} \\ &+ \frac{7}{2NS} \sum_{\mathbf{s}} (\nu_{\mathbf{s}-\mathbf{m}} + \nu_{\mathbf{s}+\mathbf{m}-\mathbf{p}-\mathbf{k}} - \nu_{\mathbf{s}} - \nu_{\mathbf{s}-\mathbf{p}-\mathbf{k}}) \\ &\times I_{\mathbf{s}}^{-1} \tilde{r}_{\mathbf{s}}^{\mathbf{p}, \mathbf{k}} - I_{\mathbf{m}} \left(\frac{2-q}{2} - \frac{q}{4NS} \sum_{\mathbf{s}} \frac{\tilde{r}_{\mathbf{s}}^{\mathbf{p}, \mathbf{k}}}{I_{\mathbf{s}}} \right), \end{aligned} \quad (\text{A3})$$

where

$$I_{\mathbf{q}} = \nu_{\mathbf{p}} + \nu_{\mathbf{k}} - \nu_{\mathbf{q}} - \nu_{\mathbf{p}+\mathbf{k}-\mathbf{q}}, \quad \mathbf{q} = \mathbf{m}, \mathbf{s}. \quad (\text{A4})$$

We can show that the sum of the last two terms in (A.2) is zero. Summing the two sides of Eq. (A.3) over the wave vector \mathbf{m} , and using the obvious condition $\sum_{\mathbf{q}} \nu_{\mathbf{q}} = 0$, we find

$$\begin{aligned} \left(1 - \frac{q}{4S} \right) \frac{1}{N} \sum_{\mathbf{m}} \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} \\ = -\frac{q}{2} (\nu_{\mathbf{p}} + \nu_{\mathbf{k}}) \left(1 + \frac{1}{2NS} \sum_{\mathbf{m}} \frac{\tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}}}{I_{\mathbf{m}}} \right). \end{aligned} \quad (\text{A5})$$

That the last two terms in (A2) cancel each other out can now be seen from the obvious relation (see the definition of q)

$$2-q = q \left(1 - \frac{q}{4S} \right).$$

As a result, Eq. (A2) simplifies considerably:

$$\begin{aligned} \tilde{r}_{\mathbf{p}, \mathbf{k}}^{\mathbf{p}, \mathbf{k}} &= \nu_{\mathbf{p}-\mathbf{k}} + 1 - \nu_{\mathbf{p}} - \nu_{\mathbf{k}} + \frac{1}{2NS} \\ &\times \sum_{\mathbf{m}} (\nu_{\mathbf{m}-\mathbf{p}} + \nu_{\mathbf{m}-\mathbf{k}} - \nu_{\mathbf{m}} - \nu_{\mathbf{m}-\mathbf{p}-\mathbf{k}}) I_{\mathbf{m}}^{-1} \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}}. \end{aligned} \quad (\text{A6})$$

It can be seen from this last equation that the total amplitude $\tilde{r}_{\mathbf{p}, \mathbf{k}}^{\mathbf{p}, \mathbf{k}}$ does in fact satisfy Adler's principle, i.e., it vanishes if even one of the wave vectors \mathbf{p}, \mathbf{k} vanishes.

We further note that Eq. (A6) is invariant under the substitution

$$\tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} \rightarrow \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} = \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} + \lambda I_{\mathbf{m}}, \quad (\text{A7})$$

where λ is some arbitrary factor.

It is natural to choose λ in such a way that the integral equation for the quantity $\tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}}$ is as simple as possible. We see from (A4) that it is convenient to take

$$\lambda = \frac{2-q}{2} - \frac{q}{4NS} \sum_{\mathbf{s}} \frac{\tilde{r}_{\mathbf{s}}^{\mathbf{p}, \mathbf{k}}}{I_{\mathbf{s}}}. \quad (\text{A8})$$

With this choice of λ , and taking into account the condition $\sum_{\mathbf{s}} \nu_{\mathbf{s}+\mathbf{q}} = 0$, we find, in place of (A3),

$$\begin{aligned} \tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} &= \nu_{\mathbf{m}-\mathbf{p}} + \nu_{\mathbf{m}-\mathbf{k}} - \nu_{\mathbf{m}} - \nu_{\mathbf{m}-\mathbf{p}-\mathbf{k}} \\ &+ \frac{1}{2NS} \sum_{\mathbf{s}} \tilde{r}_{\mathbf{s}}^{\mathbf{p}, \mathbf{k}} (\nu_{\mathbf{s}-\mathbf{m}} + \nu_{\mathbf{s}+\mathbf{m}-\mathbf{p}-\mathbf{k}} - \nu_{\mathbf{s}} - \nu_{\mathbf{s}-\mathbf{p}-\mathbf{k}}) I_{\mathbf{s}}^{-1}. \end{aligned} \quad (\text{A9})$$

We are interested in the total amplitude for two-particle scattering at small momenta of the interacting magnons. Consequently, it is sufficient to retain only the terms $\propto p^2 k^2$ in the second (integral) term in (A6). Since we have

$$\nu_{\mathbf{m}-\mathbf{p}} + \nu_{\mathbf{m}-\mathbf{k}} - \nu_{\mathbf{m}-\mathbf{p}-\mathbf{k}} - \nu_{\mathbf{m}} \propto \mathbf{p}\mathbf{k}, \quad (\text{A10})$$

we can assume the following, at the accuracy of this treatment, in Eqs. (A6) and (A9):

$$I_{\mathbf{m}, \mathbf{s}} \approx 2(1 - \nu_{\mathbf{m}, \mathbf{s}}), \quad \nu_{\mathbf{s}-\mathbf{p}-\mathbf{k}} \approx \nu_{\mathbf{s}}, \quad \nu_{\mathbf{s}+\mathbf{m}-\mathbf{p}-\mathbf{k}} \approx \nu_{\mathbf{s}+\mathbf{m}}. \quad (\text{A11})$$

We seek a solution of integral equation (A9) in the form

$$\tilde{r}_{\mathbf{m}}^{\mathbf{p}, \mathbf{k}} = \sum_{i=1}^3 p_i k_i C_{\mathbf{m}}^i. \quad (\text{A12})$$

Substituting it into (A6) we find the result

$$\begin{aligned} \tilde{r}_{\mathbf{p}, \mathbf{k}}^{\mathbf{p}, \mathbf{k}} &= \frac{1}{3} \sum_{i=1}^3 \left[p_i k_i + \frac{1}{4} p_i^2 k_i^2 \left[1 + \frac{1}{NS} \sum_{\mathbf{m}} C_{\mathbf{m}}^x \right. \right. \\ &\quad \left. \left. \times \cos m_x (1 - \nu_{\mathbf{m}})^{-1} \right] \right]. \end{aligned} \quad (\text{A13})$$

The term in square brackets is the Dyson factor $Q(S)$. To determine $C_{\mathbf{m}}^x$, we return to Eq. (A9). Using (A12), we put it in the form

$$C_{\mathbf{m}}^x = \frac{1}{3} \cos m_x + \frac{1}{4NS} \sum_{\mathbf{s}} C_{\mathbf{s}}^x (\nu_{\mathbf{s}-\mathbf{m}} + \nu_{\mathbf{s}+\mathbf{m}} - 2\nu_{\mathbf{s}}) (1 - \nu_{\mathbf{s}})^{-1}. \quad (\text{A14})$$

The kernel of integral equation (A14) can be factorized; i.e., the substitution

$$C_{\mathbf{m}}^x = \lambda_1 + \lambda_2 \cos m_x + \lambda_3 (\cos m_y + \cos m_z) \quad (\text{A15})$$

reduces integral equation (A14) to a system of three algebraic equations with three unknowns. Substituting the solution of this system into (A13), we find the known expression for the total amplitude for two-particle scattering:

$$\tilde{r}_{\mathbf{p}, \mathbf{k}}^{\mathbf{p}, \mathbf{k}} = \frac{1}{3} \sum_i \left(p_i k_i + \frac{1}{4} p_i^2 k_i^2 Q(s) \right), \quad (\text{A16})$$

where the values of $Q(S)$ are given in the text proper [see (4.5)–(4.7)].

¹⁾The comparative simplicity of the spectrum of the various branches of elementary excitations clarifies the interactions between them: The interaction must be taken into account near intersections of branches (near resonances).

²⁾The justification here is that we are interested in excitations with energies $\varepsilon \ll \varepsilon_0$ (as discussed above).

³⁾The formalism presented below, which is suitable for calculations for an arbitrary spin, can also be applied to more complex structures, e.g., helimagnetic and metamagnetic materials.

⁴⁾Admittedly, in certain substances (CeBi and CeSb) the Jahn-Teller effect leads to a marked change in the exchange integral along one direction while causing essentially no changes in the lattice constants (as was pointed out by V. L. Pokrovskii).

⁵⁾In discussing antiferromagnetic materials here and below, we will omit the absolute-value sign from the exchange integral J .

⁶⁾In the case of a slight deviation of a Bose gas from ideality, we could use

another transformation—the Dyson-Maleev transformation^{27,28}—equally successfully. Just which transformation is chosen is largely a matter of taste and custom for the particular researcher working with Bose analogs of spin Hamiltonians.

⁷¹Admittedly, calculations show that in an isotropic antiferromagnetic material the density $\kappa = \langle a_i^\dagger a_i \rangle$, calculated in the harmonic approximation is numerically very small: $\kappa = 0.078$ (Refs. 37 and 38).

⁶⁵We have in mind macroscopic zero-point vibrations, which differ from the zero-point vibrations of an individual particle. The latter always exist, since we have $\max S^z = S$ and $\hat{S}^2 = S(S+1)$.

⁹¹According to Adler's principle,⁴⁰ the amplitude for the scattering of Goldstone bosons should vanish as the momenta of the corresponding quasiparticles tend toward zero.

¹⁰¹For clarity, we should point out that at small wave vectors \mathbf{k} the complete interaction amplitude of the magnons, although small, of the order of the small quantity \mathbf{k} , is by no means the same as the seed value, since virtual magnons with arbitrary wave vectors contribute to the renormalization of the amplitude.

¹¹The replacement $\alpha \rightarrow \alpha[1 - (2S)^{-1}]$ is a natural one since for $S = 1/2$ the term $\alpha \sum_i (S_i^z)^2$ reduces to a constant.

¹²The non-Hermitian Bose Hamiltonian used by Dyson can be obtained from a Heisenberg Hamiltonian through the transformation proposed by Maleev.²⁸

¹³Let us refine a point: In its pure form, the "disappearance of a Bose condensate" occurs in the course of an order-disorder phase transition in the anisotropy (more on this below). In the case of orientational transitions, the disappearance of the condensate is accompanied by the vanishing of the total density of "particles."

¹⁴As can be seen particularly clearly in 2D space: The total number of particles above the condensate, $\sum_{\mathbf{k}} n_{\mathbf{k}}$, diverges, in contrast with the total number of quasiparticles, $\sum_{\mathbf{k}} n_{\mathbf{k}}$.

¹⁵Unless otherwise specified, we are assuming that the magnetic field is not zero.

¹⁶In this interval it is simpler and more common to work directly from macroscopic hydrodynamic equations.⁶⁶

¹⁷This change in the damping at the transition to the hydrodynamic region is also found in an isotropic ferromagnetic material. However, we will not discuss that question in §4, since the transition in an isotropic ferromagnetic material is manifested only in a change in the degree of the logarithmic factors.⁶⁴

¹⁸The discussion below applies equally well to all magnetic substances with a Goldstone spectrum in which there are zero-point vibrations.

¹⁹The only exceptional cases are some specific one-dimensional and two-dimensional formations such as long organic molecules, isolated dislocation lines, or monatomic films on a nonmagnetic substrate.

²⁰Ordering begins at $T = T_c \neq 0$ in a 2D easy-axis ferromagnetic material. The transition indices have been calculated by Onsager.⁷⁸

²¹An exact solution was derived⁸⁴ for a nonlinear $O(3)-\sigma$ field-theory model in $(1+1)$ -space (one temporal and one spatial coordinate). The Euclidean version of this model is equivalent to 2D isotropic magnetic substances.

²²Evidence in favor of the suggestion that there is a difference between the structures of the ground state of antiferromagnetic materials with integer and half-integer spins (even if $S \gg 1$) comes from an analysis in terms of the σ -model with an additional term which is a topological invariant.⁹²

²³We are not considering here the possibility, which we just mentioned, of a paramagnetic ground state.

²⁴There might be transitions in which a soft mode corresponds to a finite wave vector \mathbf{k}_0 . For such transitions, the estimate given in the text proper pertains to the deviation from \mathbf{k}_0 .

²⁵We recall a trivial truth: There are no quantum effects at all at $S = \infty$.

²⁶At a fixed $\mathbf{k} \neq 0$ in a noncubic magnetic substance there always exists a band of allowed energy values, but in the absence of a dipole interaction the width of this band would vanish in the limit $\mathbf{k} \rightarrow 0$.

²⁷For magnons with $\theta_{\mathbf{k}} = 0$, a process of this sort would not be possible. The lifetime of these quasiparticles is determined exclusively by intrinsic dissipation mechanisms, so this lifetime is far longer than at $\theta_{\mathbf{k}} \neq 0$ (Ref. 116).

²⁸Knowledge of the scattering amplitude at $T = 0$ makes it possible to calculate the renormalizations of the inverse-effective-mass tensor and of the free energy (see the text proper).

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