

A topological approach to the determination of macroscopic field vectors

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Passage from the microscopic to the macroscopic description of the electromagnetic field is achieved with the use of Maxwell equations in integral form. Certain integrals in these equations depend on the topological relations between the integration domains and the molecule volumes. This dependence gives rise to two methods of field-vector averaging, and leads to the doubling of the number of field vectors for the macrodescription. The physical meanings of the vectors **D** and **H** and of the equations into which they enter are elucidated more clearly and more fully in such an exposition.

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1. ON THE VECTORS E AND D

1.1. It is a tradition, going back to H. A. Lorentz, to derive the Maxwell equations for a molecular medium in the macroscopic approximation by averaging the corresponding differential equations for the microscopic quantities through integration over the volume.¹⁻⁵

But we can use for this purpose the equations for the microscopic quantities in integral form; this procedure possesses important methodological advantages.

Giving rise to the macroscopic vectors **E** and **D** is the Maxwell equation

$$\oint_{\Sigma} \mathbf{E}_m \, d\mathbf{\Sigma} = 4\pi \int_v \rho_f \, dv + 4\pi \int_v \rho_{pol} \, dv. \quad (1)$$

Here and below we use the system of Gaussian units. The subscript *m* denotes the microscopic value of the vector; ρ_{pol} is the polarization charge density of the molecules; and ρ_f is the density of the free, or better still of the nonpolarization, charges.

The total polarization charge in the volume of any molecule is equal to zero. Therefore, only those molecules which are intersected by the surface Σ bounding the volume *v* make a contribution to the last integral in (1), which reduces this integral to an integral over the surface Σ . By shifting sections of Σ by microscopic distances of the order of the intermolecular separation, we can include in the resulting volume substantial excess positive or negative charges: therefore, the ρ_{pol} integrals taken over a set of microscopically differing volume form a set of random quantities with a relatively large spread in the values.

But there are among the indicated set of volumes two subsets for which the integrals in question assume determinate values.

First, these are the volumes whose bounding surfaces are fairly smooth, and are randomly disposed among the molecules. Such surfaces are denoted below by the letter *S*. In the general case the sections ΔS cut across the volumes of the molecules themselves. The number of such intersections for a small section ΔS is a random quantity, but the total number of intersections for many sections loses the random character. This is a consequence of the statistical properties of a molecular medium and the law of large numbers.

Secondly, the integrals over those volumes whose bounding surfaces nowhere intersect the molecules themselves assume determinate values. Such surfaces are denoted below by the letter σ (in Fig. 1, by *C*). For the volumes bounded by surfaces of the type σ , the ρ_{pol} integrals are always exactly equal to zero.

In (1) the integral over \mathbf{E}_m is the only other quantity sensitive to microscopic changes in the surface: the polarization charges contribute to the \mathbf{E}_m vector flux from a closed surface of the type *S*, but not to the flux from a closed surface of the σ type.

1.2. As is well known, the mutual relationships between two surfaces remain topologically equivalent if the surfaces do not intersect, or, conversely, their intersections are preserved in the course of the variation of their shape and disposition.⁶ Therefore, we can assert that the *S* and σ surfaces differ in the types of topological relations they have with the molecular volumes. These relations are important for the

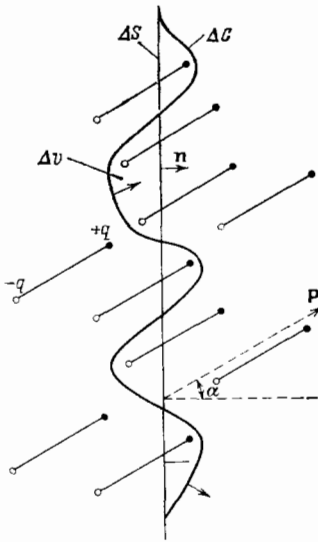


FIG. 1. For the definition of the vectors \mathbf{E} and \mathbf{D} .

ρ_{pol} and E_m integrals, and give rise to two methods of \mathbf{E}_m -vector averaging in the transition to the macroscopic description.

The \mathbf{E}_m -vector fluxes through surfaces of the S type give rise to the vector $\mathbf{E}(x, y, z, t)$. This is a vector whose components along the normals \mathbf{n} to arbitrarily oriented small plane areas ΔS (Fig. 1) with centers at the point (x, y, z) satisfy the equation

$$E_n \Delta S = \int_{\Delta S} \mathbf{E}_m \cdot d\mathbf{S} \quad (2)$$

for the moment of time t .

According to the established terminology, ΔS is a physically infinitely small area.

The \mathbf{E}_m -vector fluxes through surfaces of the σ type give rise to the vector \mathbf{D} . But the σ surfaces are not explicitly considered in the macroscopic description. Therefore, in determining \mathbf{D} , we map the \mathbf{E}_m -vector fluxes through the σ -type surfaces onto the S surfaces. This is achieved through the following definition.

The vector $\mathbf{D}(x, y, z, t)$ is a vector whose components along the normals \mathbf{n} to the above-indicated small areas ΔS satisfy the equation

$$D_n \Delta S = \int_{\Delta \sigma} \mathbf{E}_m \cdot d\boldsymbol{\sigma}, \quad (3)$$

where $\Delta \sigma$ is a surface that does not intersect the molecules, and rests on the contour of the area ΔS . In the general case it is assumed that the surface $\Delta \sigma$ differs microscopically from ΔS , and passes on either side of the latter with the same probability (see Fig. 1).

1.3. Let us derive from (1) the relation between the macroscopic quantities. Let us assume that the surface Σ coincides with a surface of the σ type. Then the last term in (1) vanishes.

Using again the definition (3), we obtain

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = 4\pi \int_V \rho \, dv,$$

where ρ is the volume averaged free-charge density.

1.4. Let us find the relation between the vectors \mathbf{D} and \mathbf{E} . From (2) and (3) we obtain

$$(D_n - E_n) \Delta S = \int_{\Delta \sigma} \mathbf{E}_m \cdot d\boldsymbol{\sigma} - \int_{\Delta S} \mathbf{E}_m \cdot d\mathbf{S}. \quad (4)$$

Let us denote the volume enclosed between $\Delta \sigma$ and ΔS by Δv . Using (1), we can reduce the right-hand side of (4) to integrals taken over the volume Δv :

$$(D_n - E_n) \Delta S = 4\pi \int_{\Delta v} \rho_f \, dv + 4\pi \int_{\Delta v} \rho_{\text{pol}} \, dv. \quad (5)$$

It is clear that we must assign different signs to the portions of the volume Δv that are located on different sides of the area ΔS (see Fig. 1). Therefore, the ρ_f integral vanishes, but the uncompensated polarization charges of either sign that are obtained from the molecules intersected by the area ΔS (see Fig. 1) make a positive contribution to the ρ_{pol} integral (see Fig. 1).

It can be shown that the last integral in (5) remains unchanged when ρ_{pol} undergoes any changes that preserve the magnitude of the molecule's first-order electric moment. This assertion is equivalent to the neglect of the higher-order electric moments of the molecules in the traditional exposition.^{2,4} Therefore, let us, without loss of generality of the final result, assume that the polarization charges are point charges $\pm q$ located at a distance a from each other, and producing identically oriented dipoles (Fig. 1). The number of dipoles intersected by the area ΔS is equal to $\nu a \Delta S \cos \alpha$, where ν is the number of molecules per unit volume and α is the angle between the direction of the dipoles and the normal \mathbf{n} to the area ΔS . Consequently, the charge that falls within the volume Δv is equal to $\nu q a \Delta S \cos \alpha$. Let us note that $\nu q a \cos \alpha = P_n$ is the component of the polarization vector \mathbf{P} in the direction of the normal. From these relations we obtain a relation of the nature of a theorem:

$$\mathbf{D} - \mathbf{E} = 4\pi \mathbf{P}. \quad (6)$$

1.5. It is clear that the averaging (2) is the same as volume averaging. Indeed, the averaging (2) over sufficiently close and parallel areas ΔS leads to fairly close E_n values; therefore, the averaging over one area with center at the point (x, y, z) will be equivalent to averaging over a family of parallel areas located at equal and arbitrarily close distances in a small volume Δv with center at the point (x, y, z) , and this amounts to the traditional volume averaging.

The averaging (3) does not possess such a property.

2. ON THE VECTORS \mathbf{B} AND \mathbf{H}

2.1. Giving rise to the vectors \mathbf{B} and \mathbf{H} for the macroscopic description of the magnetic field is the Maxwell equation:

$$\oint_A \mathbf{B}_m \cdot d\mathbf{A} = \frac{1}{c} \frac{\partial}{\partial t} \int_{\Sigma} \mathbf{E}_m \cdot d\boldsymbol{\Sigma} + \frac{4\pi}{c} \int_{\Sigma} \mathbf{j}_{\text{cond}} \cdot d\boldsymbol{\Sigma} + \frac{4\pi}{c} \int_{\Sigma} \mathbf{j}_{\text{mol}} \cdot d\boldsymbol{\Sigma}; \quad (7)$$

here \mathbf{j}_{mol} is the density of the molecular currents closed within the boundaries of each molecule and \mathbf{j}_{cond} is the conduction current density.

In (7) the topological effects are given by the last integral. This integral receives contributions from only those molecular currents whose lines of flow surround the contour

Λ (the contour l in Fig. 2). This reduces it to a contour integral, and microscopic deformations of this contour can be substantially reflected in its value. The integrals over \mathbf{j}_{mol} assume determinate values only on contours whose topological relations with the volumes of the molecules are the same as for the surfaces of the S and σ types.

Firstly, these are contours formed by sufficiently smooth curves randomly disposed among the molecules. Such curves are denoted below by the letter L . In the general case the sections ΔL intersect the molecules. The total number of such intersections for a number of sections loses its random character. This results in the integral assuming a determinate value.

Secondly, these are contours formed by curves that go around the molecules. Such curves are denoted below by the letter λ . For such contours the integrals under consideration are always equal to zero.

As a consequence, the integral over the vector \mathbf{B}_m also depends on the topological relations, which gives rise to two methods of averaging of the latter.

The circulation around curves of the L type lead to the vector $\mathbf{B}(x, y, z, t)$. This is a vector whose projections onto arbitrarily oriented segments ΔL (Fig. 2) with center at the point (x, y, z) satisfy the equation

$$B_n \Delta L = \int \mathbf{B}_m dL, \quad (8)$$

where \mathbf{n} is a vector that indicates the direction of the segment ΔL . The circulation of the vector \mathbf{B}_m around curves of the λ type gives rise to the vector \mathbf{H} . The determination of it is somewhat complicated, since the curves of the σ type, like the surfaces of the σ type, are not explicitly considered in the macroscopic description. The vector $\mathbf{H}(x, y, z, t)$ is a vector whose components along the above-indicated segments satisfy the equation

$$H_n \Delta L = \int_{\Delta \lambda} \mathbf{B}_m d\lambda, \quad (9)$$

where $\Delta \lambda$ is a segment of the curve going around the molecules, and drawn from the beginning to the end of the segment ΔL in such a way that its separation from the latter is minimal (Fig. 2).

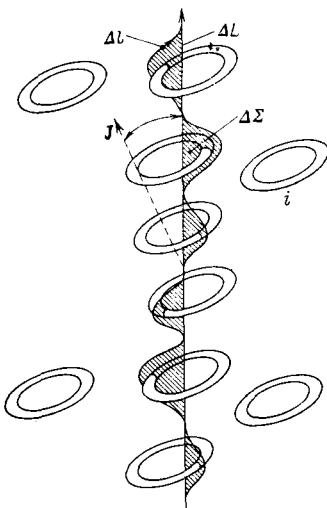


FIG. 2. For the definition of the vectors \mathbf{B} and \mathbf{H} .

The averaging (8) is equivalent to the traditional volume averaging: the averaging (9) does not possess such a property.

2.2. Let us derive from (7) the equation for the macroscopic quantities. Let the contour Λ be formed by a curve of the λ type, and let the surface Σ coincide with a surface of the σ type. In that case the last integral in (7) vanishes. Using again the definitions (9) and (3), we obtain

$$\oint_L \mathbf{H} dL = \frac{1}{c} \frac{\partial}{\partial t} \int_S \mathbf{D} dS + \frac{4\pi}{c} \int_S \mathbf{j} dS, \quad (10)$$

where \mathbf{j} is the mean conduction current density.

2.3. Let us find the relation between the vectors \mathbf{B} and \mathbf{H} . From the definitions (8) and (9) it follows that

$$(\mathbf{B}_n - \mathbf{H}_n) \Delta L = \int_{\Delta L} \mathbf{B}_m dL - \int_{\Delta \lambda} \mathbf{B}_m d\lambda. \quad (11)$$

Segments of the straight line ΔL and the curve $\Delta \lambda$ form a closed contour bounding some surface $\Delta \Sigma$. Therefore, using (7), we can represent the right-hand side of (11) by integrals taken over the surface $\Delta \Sigma$:

$$(\mathbf{B}_n - \mathbf{H}_n) \Delta L = \frac{1}{c} \frac{\partial}{\partial t} \int_{\Delta \Sigma} \mathbf{E}_m d\Sigma + \frac{4\pi}{c} \int_{\Delta \Sigma} \mathbf{j}_{\text{cond}} d\Sigma + \frac{4\pi}{c} \int_{\Delta \Sigma} \mathbf{j}_{\text{mol}} d\Sigma. \quad (12)$$

The surface $\Delta \Sigma$ consists of elementary segments connected by the molecules that are threaded by the segment ΔL . Each of these segments is a surface drawn across that portion of the segment ΔL which is located within some molecule and that portion of the curve $\Delta \lambda$ which goes around the molecule on the outside (see Fig. 2). Owing to the random orientation of the elementary segments, the \mathbf{E}_m - and \mathbf{j}_{cond} -vector fluxes through the entire surface $\Delta \Sigma$ will be equal to zero.

Contributions to the last integral in (12) are made by only those molecules which are threaded by the contour. This obtains only along the segment ΔL . Let us assume, without loss of generality in the final result, that all the molecules are of the same kind and are identically oriented, and that the molecular currents are linear. The last assumption is equivalent to the routine neglect of the higher-order magnetic moments of the molecules. In this case, for the mean number of molecules threaded by the segment ΔL , we can write $\nu b \Delta L \cos \beta$, where b is the area enclosed by the contour of the molecular current i and β is the angle between the direction of the segment ΔL and the normal to the area b (Fig. 2). Multiplying this number by $4\pi i/c$, we find the last term in (12): $4\pi i \nu b \cos \beta / c$. Let us note that $i \nu b \cos \beta / c = J_n$ is the component of the magnetization vector \mathbf{J} in the direction of the segment ΔL . Finally, we obtain the theorem

$$\mathbf{B} - \mathbf{H} = 4\pi \mathbf{J}. \quad (13)$$

The last two Maxwell equations for the microscopic quantities do not depend on the topological characteristics. In this case any averaging is equivalent to averaging over portions of the planes ΔS and segments of the straight lines ΔL , and this leads only to the macroscopic vectors \mathbf{E} and \mathbf{B} .

3. CONCLUSIONS

3.1. The traditional approach is the formal mathematical approach that leads to the macroscopic dependences by the shortest means. This approach is justified and necessary in an in-depth study of the subject.

But for an initial acquaintance with the laws of electromagnetism, such an approach is not effective, since it does not elucidate the obvious qualitative aspects of the important concepts and relations. This shortcoming of the traditional method has been pointed out by Ya. I. Frenkel² and I. E. Tamm.³

A shortcoming of the above-exposed method is the increased extensiveness, since we must consider different averaging procedures corresponding to different integrals and different topological conditions. But this extensiveness is compensated by a deeper understanding. Let us, for the purpose of illustration, emphasize certain facts.

3.2. The transition to the macroscopic description is necessitated not by the difficulty of the microscopic description of large bodies, a difficulty which is usually cited in the traditional exposition,^{2,5} but by the statistical laws governing the integrals of the microscopic quantities in the Maxwell equations in the case when the domains of integration are of sufficiently large dimensions.

3.3. The macroscopic vectors \mathbf{E} , \mathbf{D} , \mathbf{B} , and \mathbf{H} emerge as equally necessary statistical characteristics of the field in the molecular medium, and in the process the simple force meaning of the vectors \mathbf{E}_m and \mathbf{B}_m is lost.

In the conventional exposition the statistical nature of the vectors \mathbf{D} and \mathbf{H} is not revealed, and they are introduced as auxiliary quantities devoid of a clear meaning.

3.4. In the traditional exposition the vector \mathbf{D} is represented by the sum

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad (14)$$

so that we can then write concisely

$$\operatorname{div} \mathbf{D} = 4\pi\rho. \quad (15)$$

We then lose sight of the instructive features of the similarity of, and difference between, Eqs. (14) and (15), or better still, (6) and (15).

The similarity lies in the fact that (6) and (15) can be regarded as particular cases of (1). As to the difference between them, it is due to their different forms and the topological character of the volumes v . In (15) it is, roughly speaking, a unit volume bounded by a surface of the σ type. It encloses a charge equal to ρ . In (6) it is a volume bounded by

a unit area ΔS and an adjoining surface $\Delta\sigma$. The total charge enclosed by them amounts to the polarization charges cut off by the area ΔS (Fig. 2). It is numerically equal to the component of the vector \mathbf{P} along the normal to ΔS . It is the dependence of this charge on the orientation of the area that gives rise to the vectorial character of Eq. (6).

3.5. Analogous similarity and difference features obtain for the equations

$$\mathbf{B} - \mathbf{H} = 4\pi\mathbf{J}, \quad (13')$$

$$\operatorname{rot} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{j}. \quad (16)$$

In order to show this, let us scalar-multiply (13) and (16) by the unit vector \mathbf{n} , but we shall assume that \mathbf{n} is the directed unit segment $\Delta\mathbf{L}$ in the case of (13) and the oriented unit area ΔS in the case of (16). We can consider the equalities obtained to be Eq. (11) for the case in which the integration domains Σ and Λ have different shapes.

For (13) the curve Λ is formed by the unit segment $\Delta\mathbf{L}$ and the adjoining segment $\Delta\lambda$; they circumscribe a surface $\Delta\Sigma$ (Fig. 2). For (16) the surface Σ is the surface $\Delta\sigma$ adjoining the unit area ΔS , and bounded by a λ -type curve that adjoins the edge of ΔS .

In the above-presented arguments the currents generated by the motion of the polarization charges are taken into account implicitly, since the net current through a σ surface is equal to zero. But if for the derivation of (10) and (16) we use as Σ in (7) a surface of the S type, then these currents must be taken into account explicitly.

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