Tricentenary of Isaac Newton's "Mathematical Principles of Natural Philosophy"

V. L. Ginzburg

P. N. Lebedev Institute, Academy of Sciences of the USSR, Moscow
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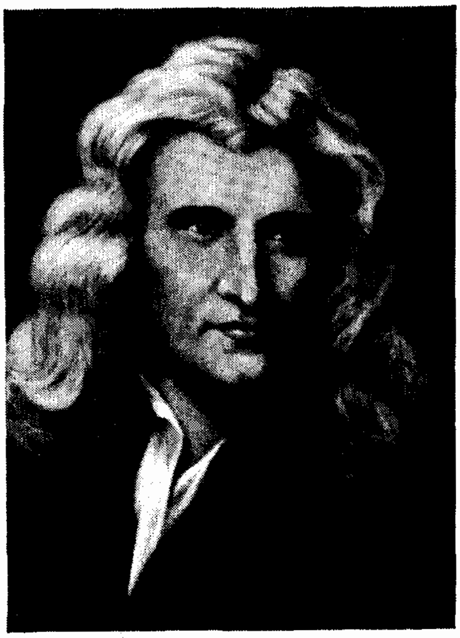
The first edition of Newton's "Philosophiae Naturalis Principia Mathematica" was published in 1687. The present paper is dedicated to the tricentenary of this event, which is important not just in the history of physics, but of science generally. After the Introduction, the paper continues with the following Sections: Before Newton, Principia, Principia and the method of principles, The nature of gravitation, Critique of Newtonian mechanics and its subsequent development, On Newton, Concluding remarks.

"Let Mortals rejoice That there has existed such and so great an Ornament to the Human Race."
(*Epitaph on the monument on Newton's tomb*)

In 1987, it will be 300 years since the publication of "The Mathematical Principles of Natural Philosophy" by Isaac Newton.[1] Lagrange refers to it as the "greatest product of the human mind" and, undoubtedly, on any reasonable scale, it is one of the greatest achievements in the entire history of physics, and of science generally. The "Principia" provided, for the first time, a systematic and sufficiently complete account of classical mechanics, often referred to as Newtonian mechanics. It is precisely with mechanics, i.e., Newtonian mechanics, that the study of physics begins at school. It is also the starting point of more advanced courses of general physics and theoretical physics in colleges and universities (see, for example, Refs. 2 and 3). Apart from mechanics and its applications in astronomy, Newton obtained very important results in optics, and laid the foundations of mathematical analysis (differential and integral calculus). It is therefore not surprising that Newton's name is known to practically everyone. On the other hand, 300 years after the publication of the "Principia" and 260 years after Newton'death,[2] his work in its original form is known to relatively few people.[3] It therefore seems appropriate to take advantage of the "Principia" jubilee to examine if only briefly Newton's researches, especially in the field of mechanics.



ISAAC NEWTON
(1643–1727)

BEFORE NEWTON

The history of physics and astronomy stretches back over the two millenia before Newton. The most prominent and well-known names in this history are Aristotle (384–322 B.C.), Archimedes (about 287–212 B. C.), Hipparchus (second centry B.C.), Ptolemy (about 87–165), Copernicus (1473–1543), Galileo (1564–1642), Kepler (1571–1630), Descartes (1596–1650), and Huygens (1629–1695), who was an older contemporary of Newton. This list could, of course, be extended by adding many other distinguished names (see, for example, Refs. 10 and 11), but this is hardly the place to do so, and we confine ourselves to indicating the overall time scale.

In antiquity, mechanics was typified by propositions such as: the velocity of a body is proportional to the force applied to it, or, the velocity acquired by a freely falling body is proportional to its weight. We now know that both these propositions can be valid only when a body travels through a sufficiently viscous medium or, more precisely, when the product of its mass and acceleration can be neglected in comparison with the force of viscous friction. Ancient mechanics was therefore a generalization of a certain class of experi-

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ment or everyday observation; for example, a leaf from a tree falls more slowly than a stone or a pear. Centuries elapsed before Galileo, who had some precursors in this field,[10] concluded that all bodies fall with the same acceleration independently of their weight, i.e., bodies falling from the same height reach the earth in equal times and with the same velocity (this is, of course, valid only when air resistance can be neglected). Galileo also arrived at the law of inertia, which states that in the absence of external forces a body maintains its velocity or, more precisely, continues to move uniformly and rectilinearly (on the scale of distances that can be achieved on the Earth's surface). Galileo was also able to formulate the principle of relativity or, specifically, the equivalence of certain reference frames, (e.g., those attached to a shore or to a uniformly moving ship) in the description of the motion of moving bodies. We shall not reproduce the various quotations that would illuminate in detail Galileo's arguments and conclusions because this has frequently been done before (see Refs. 10 and 12, the references therein, and Ref. 34. It is probably less well-known that the principle of relativity in mechanics was used in the same form one hundred years before Galileo by Copernicus as proof of the motion of the Earth. In fact, Copernicus states:[T1] "Why should we not admit, with regard to the daily rotation, that the appearance is in the Heavens and the reality in the Earth? .... For when a ship is floating calmly along, the sailors see its motion mirrored in eveything outside, while on the other hand they suppose that they are stationary, together with everything on board" (see Ref. 12 for further details). No less important for further advances was, of course, the transition to the heliocentric frame. But one has to acknowledge that both Copernicus and Galileo retained the dogmatic assumption of uniform circular motion of planets and of their moons. (This was necessary to preserve the epicycles). However, it was only the transformation to the heliocentric system that enabled Kepler to establish, in 1609, his first two laws and, in 1619, his third law of planetary motion. Moreover, it was only on the basis of Kepler's laws that Newton was able (in the "Principia") to discover the law of universal gravitation in a more or less final and general form. Because of the outstanding part played by Kepler's laws, it is opportune to restate them here:

(1) Planets move on ellipses with the Sun at one of the foci.

(2) The radius vector of a planet (or, in other words, the straight line joining the planet to the Sun) sweeps out equal areas in equal times (the law of areas).

(3) The squares of the times T of revolution of the planets around the Sun are in the ratios of the cubes of the semimajor axes a of their elliptic orbits.

In other words, $a^3/T^2 = K$ for all planets in the solar system, where K is the Kepler constant (which is the same for all planets in the solar system, but is different for other analogous systems, e.g., the moons of Jupiter).[4]

Kepler arrived at his laws by abandoning the assumption of uniform circular motion and thus overcame a major barrier that had persisted for almost two millenia. Einstein described Kepler's achievements in words such as "truly, the idea of genius" and "a remarkable outcome," and writes about his "admiration for this remarkable man" (see Ref. 14, pp. 121 and 324). This was not an exaggeration since familiarity with the work and life of Kepler[13] cannot fail to produce feelings of the greatest respect.

Kepler also thought about the reasons for the motion of planets and of the Moon in their orbits and, in this connection, about the nature of gravitation: "I define gravitation (attraction) as a force similar to magnetism, i.e., mutual attraction. The force of attraction increases as the two bodies approach one another. Therefore, bodies resist being separated from one another more strongly when they are still close to one another." And again: "The reasons for the ebb and flow of the ocean can be seen in that the bodies of the Sun and of the Moon attract the waters of the ocean by forces similar to magnetism" (see Ref. 13, p. 105, which gives references to the original papers by Kepler). On the other hand, Kepler considered that, while the force of attaction (universal gravitation) was proportional to mass (we shall use modern language), it was also inversely proportional to the distance between bodies. The correct dependence of the force of gravitation on distance, namely, the law $F \sim 1/r^2$, was found before Newton. Newton himself mentioned in this connection the names of Boulliau (the latinized family name of the Frenchman, Bouilleaud, 1605–1694; the $1/r^2$ law appeared in his book published in 1645), Borelli, and Hooke. The formula $w = v^2/r$ for the centripetal acceleration was also shown before Newton (it was obtained in 1659 by Huygens, but was not published until it appeared in 1673 in his book Horologium Oscillatorium). By combining the force law $F \sim 1/r^2$ and the formula for the centripetal acceleration w, we obtain Kepler's third law for circular orbits, for which $v = (2\pi/T)r$ and $w = (4\pi^2/T^2)r \sim F \sim 1/r^2$ (in this case, it is obvious that $r = a$). This was done by Newton in 1665–1666. These were remarkable years of his life, during which the 22-year-old Newton spent the time of The Plague on a farm in his native Woolsthorpe. Many years later, Newton wrote, ".... for in thos years I was in the prime of my age for invention and minded mathematics and philosophy more than anytime since." [T2] It was at that time that, according to popular anecdote, Newton discovered the law of universal gravitation, suggested to him by an apple falling from a tree. This anecdote is sometimes regarded as a legend, but S. I. Vavilov[6] was inclined to believe it and quotes Stukeley's words, although these refer to Newton in his old age?[5)] "After dinner, the weather being warm, we went into the garden and drank thea, under the shade of some appletrees, only he and myself. Amidst other discourse, he told me, he was just in the same situation, as when formerly, the notion of gravitation came into his mind. It was occasion'd by the fall of an apple, as he sat in a contemplative mood. Why should that apple always descend perpendicularly to the ground, thought he to him self. Why should it not go sideways or upwards, but constantly to the earths centre? Assuredly, the reason is, that the earth draws it. There must be a drawing power in matter: and the sum of the drawing power in the matter of the earth must be in the earths center, not in any side of the earth. Therefore dos this apple fall perpendicularly, or towards the centre. If matter thus draws matter, it must be in proportion of its quantity. Therefore the apple draws the earth, as well as the earth draws the apple. That there is a power, like that we here called gravity, which extends its self thro' the universe." [T3] A similar and better known story was reported by Voltaire, who heard it from Newton's niece.

There have been considerable advances in the study of

Newton during the last decade,[4,5,9] but the author of the present paper has not been able to examine the original material. He must therefore confine himself to remarking that the results obtained by Newton in 1665–1666 were not published at the time, and remained practically unknown for a long time. On the other hand, the problem of gravitation and of the theoretical justification of Kepler's laws became increasingly pressing, and was discussed quite extensively in scientific circles, (for example, at the Royal Society, which was founded in London in 1662, and which published starting in 1665, in English, the first scientific journal in the world the "Philosophical Transactions of the Royal Society"; Newton was elected Fellow of the Society in 1672). Kepler's third law for circular orbits was probably obtained independently by several authors, using the expression $F \sim 1/r^2$ and $w = v^2/r$ (see above). The stumbling block was the derivation of Kepler's first law, i.e., the proof that a force that is inversely proportional to the square of the distance produces (or, more precisely, can produce) motion on an ellipse. In 1684, Halley (1656–1742); of the Halley's comet fame) became aware, after a conversation with Newton, that he, Newton, had long had a proof (derivation) of Kepler's first law. Not unnaturally, Halley considered Newton's results to be exceedingly important for the development of celestial mechanics and prompted him (some say even persuaded him) finally to publish some of his work on mechanics. This was the birth of "Principia." It consists of three books, which were published by the Royal Society in the spring and fall of 1686 and April 1687, respectively. Since the Society had no funds, Halley published the "Principia" (in 1687) at his own expense. 300–400 copies were printed and were quite rapidly distributed (no copies were available on the market by 1691). I. Yu. Kobzarev[7] has estimated that the first edition of the "Principia" was studied during this period by 40 people in continental Europe and by only 10 people in England. There were two further editions of the "Principia" in Newton's lifetime—one in 1713 and another in 1726.

"PRINCIPIA"

To an extent, the last paragraph may be summarized in Newton's own words (in a letter to Hooke[T4]: "If I have seen further (than you and Descartes), it is by standing upon the shoulders of giants." Newton's work in mechanics and the theory of gravitation was based on the efforts of Copernicus, Galileo, and Kepler (and, or course, some others whose names are less well-known). He extended and, in a sense, completed their work. Einstein refers to Newton as the "great systematizer" (Ref. 14, p. 90). Actually, the "Principia" is a fundamental and, indeed, monumental work (the Russian language edition[1] consists of almost 700 pages), covering much that was done by Newton's predecessors, his contemporaries, and Newton himself. However, the essential point here is not the systematization, although this was clearly important, but, first, the general approach, i.e., the use of the method of principles, mentioned below; second, the significant development and generalization of mechanics, including the formulation of the law of "universal gravitation" in its general form; and, third, the solution of many problems, difficult at the time, e.g., problems relating to the motion of the Moon. There was no question of a realistic solution of such problems before Newton. There is no doubt that it is more correct to see Newton's principal achievement as one single entity, and the mention of three main elements of the "Principia" does not ring true (somehow the latter in a way reminds one of the use of the term "deputy" applied not to one but to several persons). However, we shall not linger on mere words, and will concentrate on the many-sidedness and greatness of the "Principia."

We cannot proceed further without considering the content of the "Principia" (as already mentioned, the Russian translation of the "Principia" is not readily available). If we leave on one side Newton's own preface and, in the second edition (1713), the extensive preface by Roger Cotes, the "Principia" starts with "Definitions" and "Axioms, or laws of motion." This is followed by Book I, "The Motion of Bodies," Book II, "The Motion of Bodies (in Resisting Mediums)," and Book III, "the System of the World." A few fragments from the "Principia" [T5] are reproduced below

DEFINITIONS

DEFINITION I

The quantity of matter is the measure of the same, arising from its density and bulk conjointly.

Thus air of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction, and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter every- where under the name of body or mass. And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shown hereafter.

DEFINITION II

The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.

The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple.

....

DEFINITION IV

An impressed force is an action exerted upon a body, in order to change its state either of rest, or of uniform motion in a right line.

This force consists in the action only, and remains no longer in the body when the action is over. For a body maintains every new state it acquires, by its inertia only. But impressed forces are of different origins, as from percusion, from pressure, from centripetal force.

....

SCHOLIUM

Hitherto I have laid down the definitions of such words as are less known, and explained the sense in which I would have them to be understood in the following discourse. I do not define time, space, place, and motion, as being well known to all. Only I must observe, that the common people conceive those quantities under no notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.

I. Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.

II. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position in bodies; and which is commonly taken for immovable space; such is the dimension of a subterraneous, an aerial, or celestial space, determined by its position in respect of the earth.

....

IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of the cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains, is moved. Wherefore, if the earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the earth. But if the earth also moves,the true and absolute motion of the body will arise, partly from the true motion of the earth, in immovable space, partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship, its true motion will arise, partly from the true motion of the earth, in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body in the ship... .

....

And finally, we reproduce the following:

AXIOMS, OR LAWS OF MOTION

LAW I

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

....

LAW II

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, whey they are oblique, so as to produce a new motion compounded from the determination of both.

LAW III

To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

It should be clear from these quotations and, of course, even clearer from the entire text, that the "Principia" is constructed in the spirit of a classical model, namely, Euclid's "Elements of Geometry." This style is generally typical of Newton and was formulated by him well before the "Principia." For example, he used it in "Lectures on Optics," which he gave in 1669–1671, although they were first published only after his death.[16] Of course, the imitation of the form of Euclid's "Elements" is not so much Newton's own style as the style of the time.

The italicized (of course, in accordance with Motte's translation[T5]) parts of laws I, II, and III (the explanations are omitted in the case of laws I and III) constitute the three celebrated "Newton's laws," which are frequently cited even today (see, for example, Ref. 2).[6] This is justifiable, at least in a general course of physics. Newton's laws are given in their original form in Sommerfeld's "Course of Theoretical Physics."[17] Of course, these laws cannot now be used without explanation and comment, which will be provided in the next Section.

Book I of the "Principia" is largely devoted to the solution of problems on the motion of bodies (mass points) under the influence of central forces (and in the absence of resistance). It is essentially an analysis of motion under the influence of a force that is inversely proportional to the square of the distance ($F\sim 1/r^2$). It gives a proof of Kepler's laws and, conversely, shows that motion described by these laws implies $F\sim 1/r^2$ (the first law, i.e., motion on an ellipse with the source of the central force at the focus, suffices for this purpose). We have already mentioned that it was precisely this problem that was presented to Newton by Halley. Newton examines motion not only on ellipses, but also on parabolas and hyperbolas, i.e., any conic section. He also considers the problem of three bodies and of a large number of bodies, and lays down the foundations of perturbation theory. One of the achievements of Book I is the proof (which is not easy and is put forward for the first time) of the following theorem: for the force of attraction $F\sim 1/r^2$, the effect of a sphere of constant mass density, or spherically symmetric density distribution, is the same as if the entire mass were concentrated at the center of the sphere ("a corpuscle placed without the spherical surface is attracted towards the center of the sphere with a force inversely proportional to the square of its distance from that centre"; this is proposition 71, theorem 31 of the "Principia"). Of course, when the theorem of Gauss (1777–1855) is available, the proof is fairly obvious, but we cannot retain *everything* here (more details are given in Ref. 7). We confine ourselves to mentioning that Book III contains the theory of motion of the Moon, taking into account the effect not only of the Earth, but also of the Sun (this theory cannot be considered complete, but is striking in its power, in view of the methods being used). Newton also develops the theory of tides and examines the motion of comets. It was only because of Newton's work that it was possible to show, for the first time in the case of Halley's coment (observed in 1682), that comets travel on elliptic orbits (in the first approximation), and must therefore return.[18,19,39]

Apart from these specific results, which are occasionally referred to as "strong," the concept of the universality of gravity and the formulation of the law of universal gravitation are very important. Newton adds the following remark to his third "rule of reasoning in philosphy" (we shall return to these rules later)[T5]:

Lastly, if it universally appears, by experiments and astronomical observations, that all bodies about the earth gravitate towards the earth, and that in proportion to the quantity of matter which they severally contain; that the moon likewise, according to the quantity of its matter, gravitates towards the earth; that, on the other hand, our sea gravitates towards the moon; and all the planets one towards another; and the comets in like manner towards the sun; we must, in consequence of this rule, universally allow that all bodies whatsoever are endowed with a principle of mutual gravitation. For the argument from the appearances concludes with more force for the universal gravitation of all bodies than for their impenetrability; of which among those in the celestial regions, we have no experiments, nor any manner of observation.

In modern terminology, Newton's law of universal gravitation can be stated as follows: any two bodies (mass points) with masses m_i and m_j attract one another with forces F_{ij} that are proportional to the product of the masses, $m_i m_j$, and inversely proportional to the square of the distance between the bodies, r_{ij}; the force of attraction lies along the line joining the bodies (mass points). Thus, the force acting on mass m_i and due to mass m_j is

$$\mathbf{F}_{ij}(r_{ij}) = \frac{G m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \quad \mathbf{F}_{ji}(r_{ij}) = -\mathbf{F}_{ij}(r_{ij}), \qquad (1)$$

where the gravitational constant is $G = 6.67 \times 10^{-8}$ $\mathrm{dyn \cdot cm^2\, g^{-2}}$ and is independent of the masses of the interacting bodies, i.e., it is a universal constant. The vector $\mathbf{r}_{ij}$ joins the points i and j in the direction from i to j (clearly, $r_{ij} = r_{ji} = |\mathbf{r}_{ij}|$).

The constant G was first determined as a result of terrestrial experiments in 1798 by Cavendish (1731–1810). Kepler's constant K, mentioned above in connection with Kepler's third law, is given by

$$K \equiv \frac{a^3}{T^2} = \frac{GM_\odot}{4\pi^2},$$

where $M_\odot = 1.99 \times 10^{33}$ g is the mass of the Sun. When the mass of the planets cannot be neglected in comparision with the mass of the Sun, the motion of the latter relative to the center of gravity of the planetary system (center of mass) must be taken into account. Kepler's third law is then modified and takes the following form in the two-body problem (see, for example, Ref. 2, Section 59)

$$\frac{a^3}{T^2 (M_\odot + m)} = \frac{G}{4\pi^2},$$

where m is the mass of the planet. The mass of all the planets in the solar system is smaller by the factor 743 than the mass of the Sun, the mass of the Earth is $M_\oplus \equiv M_E = 5.98 \times 10^{27}$ g $= 3 \times 10^{-6} M_\odot$, and the mass of Jupiter is $M_J = 318 M_E \approx 10^{-3} M_\odot$. It follows that, in the first approximation, the Sun may be looked upon as stationary. We recall these well-known facts only to emphasize Newton's understanding of them.

Even a cursory examination of the "Principia" and its role in science cannot, however, be reduced to the mere enumeration of specific theoretical results established by Newton. His approach to physics and the method that he uses (frequently referred to as the method of principles) are very significant. Let us examine this further.

"PRINCIPIA" AND THE METHOD OF PRINCIPLES. THE NATURE OF GRAVITATION

In Book III of the "Principia," Newton formulates what might be described as his methodology, or the method of principles, in the first instance as the "Rules of reasoning in philosophy." [T5]

RULES OF REASONING IN PHILOSOPHY

RULE I

We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.

To this purpose the philosophers say that Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity, and affects not the pomp of superfluous causes.

RULE II

Therefore to the same natural effects we must, as far as possible, assign the same causes.

As to respiration in a man and in a beast; the descent of stones in Europe and in America; the light of our culinary fire and of the sun; the reflection of light in the earth, and in the planets.

RULE III

The qualities of bodies, which admit neither intensification nor remission of degrees, and which are found to belong to all bodies within the reach of our experiments, are to be esteemed the universal qualities of all bodies whatsoever.

Rule III has appended to it an explanation, part of which, relating to universal gravitation, has already been cited. There follows

RULE IV

In experimental philosophy we are to look upon propositions inferred by general induction from phenomena as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions.

This rule we must follow, that the argument of induction may not be evaded by hypotheses.

What is the purpose of the "rules of reasoning" and what is their aim? The answers to these questions can be provided only in the light of the world-outlook and the methodology that was prevalent before Newton and which he tried to fight. He deals, in particular, with Cartesian ideas about Nature and the ways of studying it. These ideas were based not on observations or experiments, but on hypotheses about the nature of phenomena and the reasons for them. Thus, gravitation was said to be due to vortices in some "fine matter" and was wholly treated in the spirit of a contact interaction. However, further progress toward a quantitative theory was completely impossible along this route, especially at the time. To simplify and systematize, it may be said that the program put forward by Descartes and his supporters and followers is, in effect, an attempt to construct a theory of the gravitational field or, specifically, a general theory of relativity, before the creation of mechanics and the theory of universal gravitation by Newton. Newton's great merit was, among other things, his understanding of the true possibilities of the physics of the time; hence his use of the above Rules and, if we are concerned with gravity, the postulation of the law of universal gravitation with its implied action-at-a-distance.

The "General scholium" at the end of the "Principia" is largely concerned with countering the hypohesis of vortices which is "pressed with many difficulties." The scholium recalls, in particular, the observations of planets and comets:[T5] "The motions of the comets are exceedingly regular, are governed by the same laws with the motions of the planets, and can by no means be accounted for by the hypothesis of vortices; for comets are carried with very eccentric motions through all parts of the heavens indifferently, with a freedom that is incompatible with the notion of a vortex." The extensive "preface of the publisher to the second edition" (1713), written by Roger Cotes at Newton's suggestion, is also devoted to countering Newton's opponents and critics. Whilst Newton did not descend to polemical attacks, at least not in the "Principia," Cotes is not constrained in his expressions while defending the Newtonian position and criticizing opponents. For example, he writes[T5]:

"Since, then, all bodies, whether upon earth or in the heavens, are heavy, so far as we can make any experiments or observations concerning them, we must certainly allow that gravity is found in all bodies universally. And in like manner as we ought not to suppose that any bodies can be otherwise than extended, movable, or impenetrable, so we ought not to conceive that any bodies can be otherwise than heavy. The extension, mobility, and impenetrability of bodies become known to us only by experiments; and in the very same manner their gravity becomes known to us. All bodies upon which we can make any observations, are extended, movable, and impenetrable; and thence we conclude all bodies, and those concerning which we have no observations, are extended and movable and impenetrable. So all bodies on which we can make observations, we find to be heavy; and thence we conclude all bodies, and those we have no observations of, to be heavy also. If anyone should say that the bodies of the fixed stars are not heavy because their gravity is not yet observed, they may say for the same reason that they are neither extended nor movable nor impenetrable, because these properties of the fixed stars are not yet observed. In short, either gravity must have a place among the primary qualities of all bodies, or extension, mobility, and impenetrability must not. And if the nature of things is not rightly explained by the gravity of bodies, it will not be rightly explained by their extension, mobility, and impenetrability.

Some I know disapprove this conclusion, and mutter something about occult qualities. They continually are cavilling with us, that gravity is an occult property, and occult causes are to be quite banished from philolosphy. But to this the answer is easy: that those are indeed occult causes whose existence is occult, and imagined but not proved; but not those whose real existence is clearly demonstrated by observations. Therefore gravity can by no means be called an occult cause of the celestial motions, because it is plain from the phenomena that such a power does really exist. Those rather have recourse to occult causes, who set imaginary vortices of a matter entirely fictitious and imperceptible by our senses, to direct those motions.

But shall gravity be therefore called an occult cause, and thrown out of philosophy, because the cause of gravity is occult and not yet discovered? Those who affirm this, should be careful not to fall into an absurdity that may overturn the foundations of all philosophy. For causes usually proceed in a continued chain from those that are more compounded to those that are more simple; when we are arrived at the most simple cause we can go no further. Therefore no mechanical account or explanation of the most simple cause is to be expected or given; for if it could be given, the cause were not the most simple. These most simple causes will you then call occult, and reject them? Then you must reject those that immediately depend upon them, and those which depend upon these last, till philosophy is quite cleared and disencumbered of all causes.

Some there are who say that gravity is preternatural, and call it a perpetual miracle. Therefore they would have it rejected, because preternatural causes have no place in physics. It is hardly worth while to spend time in answering this ridiculous objection which overturns all philosophy. For either they will deny gravity to be in bodies, which cannot be said, or else, they will therefore call it preternatural because it is not produced by the other properties in bodies, and therefore not by mechanical causes. But certainly there are primary properties of bodies, and these, because they are primary, have no dependence on the others. Let them consider whether all these are not in like manner preternatural, and in like manner to be rejected; and then what kind of philosophy we are like to have.

Some there are who dislike this celestial physics because it contradicts the opinions of *Descartes*, and seem hardly to be reconciled with them. Let these enjoy their own opinion, but let them act fairly, and not deny the same liberty to us which they demand for themselves. Since the *Newtonian* Philosophy appears true to us, let us have the liberty to embrace and retain it, and to follow causes proved by phenomena, rather than causes only imagined and not yet proved."

A little later, Cotes uses even sharper language[T5]:

"... for the phenomena can by no means be accounted for by vortices, as our author has abundantly proved from the clearest reasons. So that men must be strangely fond of chimeras, who can spend their time so idly as in patching up a ridiculous figment and setting it off with new comments of their own."

It was only the approach adopted by Newton, i.e., his action-at-a-distance theory of universal gravitation, that led to the successful development of mechanics and astronomy. In his "Autobiographical Notes," written in his 68th year and forming "something in the nature of a self obituary," Einstein made a characteristic remark about this. Discussing Newtonian mechanics and its weak points, Einstein exclaims:

"Newton, forgive me; you found the only way which in your age was just about possible for a man with the highest powers of thought and creativity. The concepts which you created are guiding our thinking in physics even today, although we know that they will have to be replaced by others farther removed from the sphere of immediate experiments, if we aim at a profounder understanding of relationships" (Ref. 17, p. 270).[T6]

In contrast to some of his successors, Newton himself did not consider that his formulation of the law of universal gravitation exhausted the problem and that there was no need for explaining the nature of gravitation. On the contrary, he acknowledged that searches for such an explanation were necessary, but only on a sufficiently firm basis, i.e., above all, observations. This is indicated, in particular, by the following excerpt from the "General scholium" at the end of the "Principia" [T5]:

"Hitherto we have explained the phenomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power. This is certain, that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminuation of its force; that operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes used to do), but according to the quantity of the solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always as the inverse square of the distances. Gravitation towards the sun is made up out of the gravitations towards the several particles of which the body of the sun is composed; and in receding from the sun decreases accurately as the inverse square of the distances as far as the orbit of Saturn, as evidently appears from the quiescence of the aphelion of the planets; nay, and even to the remotest aphelion of the comets, if those aphelions are also quiescent. But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosphy. In this philosophy particular propostions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and of gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea.

And now we might add something concerning a certain most subtle spirit which pervades and lies hid in all gross bodies; by the force and action of which spirit the particles of bodies attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of *this spirit*, mutually propagated along the solid filmaments of the nerves, from the outward organs of sense to the brain, and from the brain into the muscles. But these are things that cannot be explained in a few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic spirit operates."

Newton is possibly even clearer in his opposition to action at a distance, as expressed in a letter to Bentley,[T7] written in 1693:

"Tis unconceivable that inanimate brute matter should (without ye mediation of something else wch is not material) operate upon and affect other matter without mutual contact; as it must if gravitation in the sense of Epicurus be essential and inherent in it. And this is one reason why I desired you would not ascribe innate gravity to me. That gravity should be innate inherent and essential to matter so yt one body may act upon another at a distance through a vacuum without the mediation of anything else by and through wch their action or force may be conveyed from one to another is to me so great an absurdity that I believe no man who has in philosophical matters any competent faculty of thinking can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws, but whether this agent be material or immaterial is a question I have left to ye consideration of my readers."

The phase, "I frame no hypotheses," reproduced above, has very frequently been cited (it is common also to quote the original Latin version *Hypotheses non fingo*), but it is clear from the foregoing that this statement must be understood not as a rejection of the role and necessity of hypotheses, but in the spirit of Newton's methods of fighting unjustified speculations. On the other hand, if "hypothesis" is used in the sense normally employed in modern research, it can be said that Newton "was one of the greatest giants among the relatively small number of inventors of hypotheses. But, actually, he did not frame hypotheses, i.e., unjustifiable speculations and unverified opinions" (Ref. 9, p. 127).

CRITIQUE OF NEWTONIAN MECHANICS AND ITS SUBSEQUENT DEVELOPMENT

Newton constructed the ediface of classical mechanics, which is undoubtedly secure as a whole. There are, however, some weak points in the "Principia," even ignoring the fact that very much more remained to be done in mechanics.

Let us begin with some criticisms. It has been acknowledged, and not without justification, that the very first definition (Definition I), which opens the "Principia" with a definition of mass, is unsatisfactory. A whole literature is devoted to this question.[20,28] For example, Sommerfeld[17] describes this as "only a mock definition, since density itself cannot be defined in any other way than by the amount of matter in unit volume" (Ref. 17, p. 8). Although this criticism is formally well founded, Newton's explanation of Definition I does actually meet some of the objections; it is particularly important that the mass of the body is proportional to its weight. In general, as far as we know, the definition of mass adopted by Newton has led to no errors or even any particular obscurities. Actually, according to Definition II, the momentum of a body (its "quantity of motion") is $\mathbf{p} = m\mathbf{v}$. Further, the second and third laws of Newton lead to the conclusion that, when two bodies isolated from their environment interact, the resultant momentum $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ is conserved, which means that the mass ratio

$m_1/m_2 = -\Delta v_2/\Delta v_1$ can be measured by measuring the changes in the velocities of the bodies that result from their interaction, e.g., a collision. It is precisely this definition of mass, taken in conjunction with some particular chosen mass as the standard, that has been discussed in particular detail by E. Mach (1838–1916), who used accelerations a_1 and a_2 of the masses instead of the velocity changes $\Delta\mathbf{v}_1$ and $\Delta\mathbf{v}_2$ (see Ref. 20 for further details). Since everything here is based on Newton's laws, it is not surprising that his definition of mass, which has been justly criticized, has not led to any objectionable consequences.

We have mentioned the definition of measurement of the "quantity of mass," but not the definition of mass itself. Mass is a measure of the inertia of a body, and appears in the expression for the momentum $\mathbf{p} = m\mathbf{v}$ and in the second law of Newton (clearly, this involves the inertial mass; we shall not be concerned here with the question of "heavy mass"). We have already indicated how the mass m can be measured (see Refs. 2, 20, and 38 for further details). However, and it would be incorrect to say, as is often done in the literature, that the mass (or, more precisely, the mass ratio) is none other than the ratio of the corresponding accelerations, or the ratio of the velocity changes.

Newton's Definition III, in which he actually names the product of the mass and acceleration, $m\mathbf{a}$, the "force of inertia" [T8] is unfortunate. The ill-starred concept of the force of inertia has given rise to many disputes, the echoes of which can be heard even today (see, for example, Refs. 21–23 and 35). However, we cannot pause to consider this problem in greater detail, and merely note that, in our view, which is also the most widely held, the concept and phrase "force of inertia" are appropriate and useful only in a noninertial frame of reference. When the acceleration of such a frame relative to an inertial frame is $\mathbf{a}_0$, Newton's third law in the noninertial frame assumes the same form as in the inertial frame (see below for further details), but with an extra term on the right-hand side. This term represents the "force of inertia" acting on the mass and given by

$$\mathbf{F}_i = -m(\mathbf{a}_{\text{tr}} + \mathbf{a}_{\text{cor}}),$$

where $\mathbf{a}_{\text{tr}} = \dot{\mathbf{v}}_0 + [\dot{\boldsymbol{\omega}}\times\mathbf{r}] + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is the translational acceleration and $\mathbf{a}_{\text{cor}} = 2\boldsymbol{\omega}\times\mathbf{v}_{\text{rel}}$ is the Coriolis acceleration[2] (there is little point in examining here these well-known expressions in detail). When we are concerned with the foundations of mechanics and with its principles, it is not only sufficient, but probably most rational, to confine our attention to the consideration of inertial frames of reference. Transformations to other reference frames are very important in practice, but constitute mere kinematic conversions.

We have now arrived at the fundamental question, namely, the choice of the reference frame in which Newton's laws are valid in the form in which we have formulated them here. Newton clearly understood, of course, that the choice of the reference frame and the specification of the method of measurement (determinations) of time were essential in the study of the motion of bodies. For example, if we take the first law literally, and consider that a given body travels inertially, i.e., uniformly and rectilinearly, in a given reference frame, its motion relative to other frames will, in general, be nonuniform and nonrectilinear (with the exception, of course, of reference frames traveling with constant velocity $\mathbf{V} = \text{const}$ relative to the original inertial frame). This is why Newton actually introduced the concepts of absolute space and absolute time, and his laws of motion are valid in terms of these concepts. At the same time, Newton understood that it was impossible to specify a way of determining absolute position and absolute time. This is quite clear from the text of the "Principia" and, for example, from the following fragment concerned with absolute space[T5]:

"But because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places, we estimate all motions, considering bodies as transferred from some of those places into others. And so, instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body really at rest, to which the places and motions of others may be referred."

To us, the metaphysical character of absolute space and absolute time is obvious. However, 300 years ago, the prevailing atmosphere was different and other concepts were predominant. Newton must have seen, and probably wanted to see, beyond the "relative," "apparent," and "commonplace" something absolute existing independently of the possibilities of measurement and observation. This approach turned out to be possible and fruitful because Newton actually chose frames of reference that were very close to inertial frames. Just such was the frame initially chosen in astronomy with the center coincident with the center of the Sun (or the center of mass of the solar system), and with axes pointing to the "fixed stars." For many experiments on the Earth's surface, this surface and its immediate surroundings are very close to an inertial frame. Newton's absolute space was "materialized" in theories based on the idea of a stationary aether. It was only the general theory of relativity, created at the beginning of this century, that finally banished from physics both absolute space and absolute time, understood as absolutely invariable and somehow external to matter and all fields. However, even well before this, in the eighteenth and nineteenth centuries, the Newtonian idea of absolute space and time had already been subjected to stringent criticism and, significantly, the idea of inertial frames of reference, in which Newton's laws were valid, was introduced. It will be useful to consider, if only briefly, the modern form of presenting the foundations of Newton's mechanics (details can be found in books and articles; see Refs. 12, 24–27, 35–37, and the references cited therein).

We know from experiment that there are frames of reference (and clocks based on the observation of periodic processes such as the rotation of the Earth or the oscillation of a pendulum), in which a body (mass) that is sufficiently distant from other masses continues to travel rectilinearly and uniformly. This proposition may be regarded as the first law of dynamics. It is possible that Newton had something similar in mind. There is, however, another explanation of why Newton formulated the first law separately. He may have been trying to emphasize that, in the absence of forces, a body can either remain at rest, or can travel with constant velocity (it appears that this was not well understood before

Newton). In one way or another, the separate formulation of the first law (the law of inertia) has provided grounds for some misunderstanding and criticism. Thus, if we consider that the forces are known, the first law is a direct consequence of the second because, when there are no forces, the momentum is $m\mathbf{v} = \text{const}$, i.e., the velocity is $\mathbf{v} = \text{const}$, at least if $m = \text{const}$. On the other hand, how can we tell that a force is not acting on a body? If we use an inertial frame of reference, the absence of forces is immediately known because the body moves inertially, i.e, $\mathbf{v} = \text{const}$. However, to avoid a circular argument, we cannot reverse the formulation of the problem and consider without further elaboration that the frames of reference in which masses move with constant velocity are inertial frames. As usual, therefore, we introduce inertial frames by exploiting the following properties, based on observation: forces decrease with distance. The following procedure is more logical. To be specific and brief, let us take the example of the solar system. In the "astronomical frame of reference" (with the origin at the center of mass of the solar system and axes pointing at the fixed stars), the forces between all the masses (planets or moons) satisfy Newton's third law, i.e., in terms of the notation introduced above, we have for any two masses m_i and m_j

$$\mathbf{F}_{ij}(r_{ij}) = -\mathbf{F}_{ji}(r_{ij}). \tag{2}$$

In this case, we are concerned with the forces of gravity (1). However, it is well-known that (2) is valid in a much wider domain (for electrostatic forces and in macroscopic mechanics, in which we consider springs, collisions, and so on).

Clearly, when (1) and (2) are valid, a body that is sufficiently distant from all other bodies will travel inertially. Consequently, to a sufficient precision, the astronomical frame of reference is an inertial frame. The second law of Newton is valid in this frame in the form

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_j \mathbf{F}_{ij}(r_{ij}), \tag{3}$$

where $\mathbf{r}_i$ is the radius vector of the ith body ($r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$). Equation (3) is invariant under the Galilean transformation

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{V}t, \quad t' = t, \quad \mathbf{V} = \text{const}. \tag{4}$$

It is thus clear that there is an infinite number of inertial frames moving with constant velocity $\mathbf{V}$ relative to the original frame—in our case, the astronomical frame.

On the other hand, we do not,of course, have to use the astronomical reference frame as our initial inertial frame. This is especially natural only in celestial mechanics. Any frame that falls freely without rotation is known to be an inertial frame in a sufficiently uniform gravitational field (examples include a freely falling elevator, or a spacecraft with engines turned off). It is, of course, assumed in this that nongravitational forces are absent, or better still, play no part (further details, including estimates of the "degree of inertiality" of particular reference frames, can be found in Refs. 35 and 36). An essentially similar definition of an inertial frame is one in which space can be regarded as homogeneous and isotropic and time as homogeneous (i.e., we are referring to the Minkowski space-time; see Ref. 3, Section 3, and Ref. 37). The question of inertial frames of reference in classical mechanics is, in general, well understood at present.

The equations of motion (3) are valid when the forces $\mathbf{F}_{ij}$ are known or given (and, of course, only in such cases), and can be used, at least in principle, to find the trajectories $\mathbf{r}_i(t)$ of all the "mass points." We need not consider here the relatively trivial necessity of specifying the initial conditions, or the possibility of using reference frames obtained by a suitable transformation of coordinates. There is also an obvious, at least in principle, generalization to more complicated mechanical systems, such as solids, different mechanisms, and so on.

The foundations of classical mechanics are thus perfectly definite and clear. Further refinements, mostly from the logical point of view, can be introduced into the "Principia," but do not affect in any way its fundamental importance. On the whole, Newton's approach to studies in the field of mechanics was correct, although he was not, of course, able to foresee all the possible refinements and generalizations that came later.

The following dilettante opinion is widely held (and the author must admit to having adhered to it in the past). Since Newton "invented" analysis (differentiation and integration) or, at any rate, was in full command of its foundations and was one of the principal pioneers in this area, he made use of it in the "Principia." Of course, this conclusion stems from the fact that, today, we cannot even imagine solving problems in mechanics without integrating the differential equations of motion given by (3). In actual fact, however, Newton did not explicitly use analysis in the "Principia"! The entire "Principia" is based, from the mathematical point of view, on geometrical methods and construction. On the other hand, the fact that Newton was familiar with the elements of analysis when he wrote the "Principia" appears to have had a profound influence on his work, and some of the results could only have been obtained by exploiting this facility. In the "Scholium" in Book II of the "Principia," Newton mentions that he has a number of mathematical methods that had previously been unknown.[T5] It is quite possible that Newton did not use analysis explicitly in order to make the presentation easier to assimilate by readers, to whom analysis was unknown (Newton had not published his mathematical work prior to issuing the "Principia"; the first publication, put out by Leibnitz (1646–1716), on the subject of analysis appeared in 1684). There is also a somewhat different point of view (see, in particular, the very informative Ref. 7): Newton was unable to make extensive use of analytic methods because they had not been developed. It was not until 1736, i.e., 50 years after the publication of the "Principia," that Euler (1707–1783) wrote his "Mechanics, or the Science of Motion, Analytically Presented," which contained methods similar to those employed today. Subsequent advances in this direction are associated with the names of Clairaut (1713–1765), d'Alembert (1717–1783), Lagrange (1736–1813), and Laplace (1749–1827). They all studied celestial mechanics, they repeated Newton's derivations by other methods, and they produced improved calculations of the motion of the Moon, the planets, and their moons. Doubts arose about the validity of the $F \sim 1/r^2$ law for the force of universal gravitation whenever there were difficulties in such calculations, or there were simply discrepancies when calculations were compared with observations. This was not unreasonable because this law was not "derived" or based on some model. However, the discrepancies disappeared as the calculations became more accurate. It seemed that Laplace, in his five-volume "Celestial Mechanics,"[8] completed the Newtonian theory of motion in the solar system. In some sense, this was actually so, although advances in celestial mechanics have continued to this day. They have received a particular stimulus by the launching of artificial earth satellites and different space probes (including, of course, the probes sent out in 1986 to examine Halley's comet). However, it was not until 1859 that the first steps were taken outside the range of validity of Newtonian mechanics, although this historical event was not recognized at the time. We have in mind here the anomaly in the motion of Mercury, discovered by Leverrier (1811–1877). In particular: even when all the known perturbations due to the other planets were taken into account, Leverrier's calculations showed that the perihelion of Mecury was advancing for some unknown reasons by an additional 38 seconds of arc (38″) per century. According to the more accurate 1882 data, the advance of the perihelion of Mercury was 43″ per century. When modern values of the velocity of light and of the astronomical unit were employed, the advance of the perihelion of Mercury, calculated from Einstein's formula mentioned below, turned out to be 42.98″ per century.[41] This agrees with observations to within about $\pm 0.1''$ per century. Attempts were then made to explain this anomaly in terms of different factors such as perturbations due to an unknown planet or an asteroid ring, the oblateness of the Sun, deviations from the $F \sim 1/r^2$ law of gravitation, and so on. However, the question remained open until 1915, when Einstein showed (Ref. 29, p. 439) that the general theory of relativity, which he had just completed, led to a relativistic advance of the perihelion of planets without additional assumptions. Einstein wrote: "The calculation gives a rotation of 43″ per century for the planet Mercury, whereas astronomers indicate $45'' \pm 5$ as the unexplainable difference between observations and Newton's theory. This means that there is now complete agreement with observations" (Ref. 29, pp. 446–447). Had Einstein used the more accurate observational data already available at the time (rotation by 43″), the agreement with the theory would have been better still. However, even without this, it was a major triumph for the general theory of relativity. The relatively dry phrasing of the above quotation from Einstein, common in scientific literature, does not, of course, reflect the feelings of its author. According to Pais, who wrote the best of the biographies of Einstein known to me,[30] the explanation of the advance of the perihelion of Mercury "was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life." (Einstein himself wrote in this connection: "For a few days, I was beside myself with joyous excitement"; Ref. 30, p. 253.

If we ignore the slight cloud on the distant horizon presented by the anomaly in the motion of Mercury, the nineteenth century was a period of triumph for Newtonian mechanics. Attempts were even made to use it as a basis for an explanation of electromagnetic phenomena. We now know that absolutism is unjustified, or simply incorrect, in classical or Newtonian mechanics, as it is anywhere else.

The accuracy of the results of classical mechanics is restricted by the dependence of mass m on velocity v (special theory of relativity, 1905) and by the requirement that the Newtonian gravitational potential φ must be small in comparison with the square of the velocity of light, c^2 (general theory of relativity, 1915). When applied to the solar system (details would hardly be appropriate here), these two conditions can be reduced to the single condition.

$$\frac{v^2}{c^2} \sim \frac{|\varphi|}{c^2} \ll 1 \tag{5}$$

(it is well-known that $v^2 = |\varphi| = GM/r$ in the case of circular motion in the gravitational field). On the solar surface, $|\varphi_\odot|/c^2 = 2.12 \times 10^{-6}$ $(M_\odot = 1.99 \times 10^{33}$ g, $r_\odot = 6.96 \times 10^{10}$ cm); on the Earth's orbit, $v^2/c^2 \simeq |\varphi|/c^2 \simeq 10^{-8}$ $(v \simeq 3 \times 10^6$ cm/s); and for close satellites of the Earth or on its surface, $v^2/c^2 \simeq |\varphi|/c^2 = 7 \times 10^{-10}$ $(v \simeq 8 \times 10^5$ cm/s). It follows that relativistic effects are very small within the solar system, i.e., conditions (5) are readily satisfied (they have been confirmed by observations to within the limits of experimental precision[31,32]). The gravitational interaction between particles such as electrons, protons, and so on, is negligible, but the velocity v of such particles can approach values close to c. It is then necessary to use the special theory of relativity, in which the mass is given by

$$m = \frac{m_0}{[1-(v^2/c^2)]^{1/2}}.$$

In the Newtonian formulation, the second law (see above) is concerned not with the product of mass and acceleration but, in modern language, with the equation

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}. \tag{6}$$

We may therefore say that Newton's formulation allows for the possibility that mass may be a function of velocity. Sommerfeld wrote, in this connection, that "Newton's formulation prophetically turns out to be the correct one" Ref. 17, p. 5). Unfortunately, the present author is unaware of any remarks about the possible velocity dependence of mass in Newton's publications or letters. In the light of Newton's definiton of mass, it seems unlikely that he would have considered a velocity-dependent mass. There is therefore little point in bringing "prophecy" into it.

However, in other ways, Newton can be regarded as a true prophet. This is indicated, for example, by the two remarks quoted, among others, by S. I. Vavilov as epigraphs to the chapters of his book[33,T5]:

"Are not gross bodies and light convertible into one another...? The changing of bodies into light, and light into bodies, is very conformable to the course of Nature, which seems delighted with transmutations."... .

"Do not bodies act upon light at a distance, and by their action bend its rays; and is not this action (caeteris paribus) strongest at the least distance?"

Of course, to evaluate these remarks correctly, we must recall that Newton tended toward the corpuscular theory of light. From this point of view, light corpuscles are created (emitted) and absorbed, and have mass, both inertial and gravitational. It is therefore not accidental that, as far back as the end of the eighteenth century, the corpuscular theory of light was used as a basis for introducing what we would now call "black holes," and the deflection of light rays by the field of the Sun was calculated at the beginning of the nineteenth century (the calculated deflection was, however, smaller by a factor of two as compared with the prediction of the general theory of relativity; see Ref. 31).

The precision of classical mechanics is also restricted by effects treated in quantum theory, the foundations of which were laid in 1900 by Planck (1858–1947). The criterion that will suffice for our purposes here, which ensures that quantum effects can be neglected in mechanics, is that the wavelength λ of "matter waves" (the de Broglie wavelength) must be small in comparison with typical linear dimensions and distances L:

$$\lambda = \frac{2\pi\hbar}{mv} \ll L, \qquad (7)$$

where $h = 2\pi\hbar = 6.63\times10^{-27}$ erg·s is Planck's constant, m is the mass of the body or particle under consideration, and v is its velocity. For the orbiting Earth, $L \sim 10^{13}$ cm (semimajor axis of the Earth's orbit—astronomical unit of length is 1 a.u. $= 1.496\times10^{13}$ cm). Moreover $m \sim M_E \sim 5\times10^{27}$ g and $v \sim 3\times10^{6}$ cm·s^{-1}, so that $\lambda \sim 10^{-60}$ cm. Clearly, the wavelength λ is so small in this case that there is very little sense in using it at all. When $m \sim 1$ g and $v \sim 1$ cm·s^{-1}, we find that $\lambda \sim 6\times10^{-27}$ cm. However, for electrons ($m \simeq 10^{-27}$ g), it turns out that $\lambda \sim 10/v$ cm, and quantum-mechanical effects may be significant under macroscopic conditions (not to mention atomic scales $L \sim 10^{-8}$ cm).

It is logically conceivable that classical mechanics is restricted not only, so to speak, on the relativistic and quantum sides [conditions (5) and (7)], but also in certain other ways. For example, it has been suggested more than once that classical mechanics may not be valid for galactic clusters, or galaxies themselves, because of the enormous masses and distances involved in such cases (for further details, see, for example, Ref. 32, p. 149). There is no real evidence for this, but, if such effects are present (this seems to us highly unlikely), they would not be significant in our present context. It is also quite clear that, even in the absence of these unknown limitations, Newtonian mechanics is not absolute, and is only valid to the extent that relativistic and quantum-mechanical effects can be neglected. In a certain range, however, Newtonian mechanics is exact and complete, or it is convenient to say that it is accurate and complete within the range of its validity (see Ref. 32, p. 299). We may thus conclude that the role and and value of Newtonian mechanics are not transient and will remain forever. Our debt to Newton and to his "Principia" will also be remembered in perpetuity.

ON NEWTON. CONCLUDING REMARKS

Great men and, especially, the greatest of them (they undoubtedly include Newton), are the subject of constant attention and interest. There are many reasons for this. First, there is the natural desire to look into the "creative laboratory" and try to understand the psychology of exceptional people. There is also the further significant fact that the life of celebrities is often, and for various reasons, much richer than that of ordinary folk. Considerable amounts of information about them frequently survive in the form of manuscripts and other material "records," as well as recollections of contemporaries (these are often contradictory and do not inspire particular confidence). Finally, we have to admit that there is a specific interest in the life of celebrities. The origin of this interest was clearly reflected by Goethe who noted that "mediocrity has no greater consolation than the idea that even men of genius are not immortal." [T9] Goethe also emphasized the trivial truism that "even a great man is still only a man." [T9] And all humans have weaknesses. Moreover, the correlation between "evil and genius," or between great talent and moral countenance, even if it exists, is not at all common.[9)]

One way or another, Newton is the favorite subject of biographic study, especially in recent years.[5,9,15] The superficially uneventful life led by Newton, the unmarried Puritan who never ventured outside England, was internally enormously rich. Luckily, it was also a very long life (Newton died in his 85th year). To begin with, we have before us a young man endowed with exceptional talent, who had a difficult childhood, and then a man capable of the titanic effort necessary to write the "Principia" (see below). Finally, and to some extend unexpectedly, we see the Warden, and later Master, of the Royal Mint in London.[10)]

At different times in his life, Newton's behavior and, to some extent, his character too, seem to have undergone considerable changes. In many ways, Newton's disputes about priority (especially with Hooke and Leibnitz) produce a disagreeable impression. It is distressing to read[6] about his relationship with the well-known astronomer John Flamsteed (1646–1719), the first Astronomer Royal, who built the Greenwich Observatory. Newton was rude and unjust to Flamsteed, but Flamsteed himself writes about Newton in very uncomplimentary terms (S.I. Vavilov refers to Flamsteed's description of Newton as a caricature). This illustrates the burning passions that were involved in the disputes between Newton and Flamsteed. In 1712, Falmsteed received from Halley 300 copies of the stellar catalog compiled by Flamsteed but published at Newton's insistence against Flamsteed's wishes. Flamsteed burnt all 300 copies.

True, to present Newton as not only a genius but also a model of all desirable human qualities would be to distort historical truth. This in no way contradicts the epitaph at the head of this article: Newton was indeed "an ornament to the human race," but his personal characteristics were often contradictory. It seems that he was "suspicious, irritable, and had an inflated picture of his own excellence" (Ref. 9, p. 97). However, apart from his remarkable talent (and, of course, it is the talent that is the most important), there were also some quite different traits. For example, rather than concentrate on unsavory priority and other disputes, the present author was much more impressed with the testimony of Newton's emanuensis (and relative) Humphrey Newton who performed his duties between 1685 and 1689. Here are some excerpts from his letters (Ref. 6, pp. 120–121).[T10]

"In such time he wrote his *Principia Mathematica*, which stupendous work, by his order, I copied out before it went to the press. .....

His carriage then was very meek, sedate, and humble, never seemingly angry, of profound thought, his countenance mild, pleasant, and comely. ....

He always kept close to his studies, very rarely went a visiting, and has as few visitors.... I never knew him to take any recreation or pastime either in riding out to take the air, walking, bowling, or any other exercise whatever, thinking all hours lost that was [sic] not spent in his studies, to which he kept so close that he seldom left his chamber except at term time, when he read in the schools as being Lucasianus Professor, where so few went to hear him, and fewer that understood him, that ofttimes he did in a manner, for want of hearers, read to the walls. Foreigners he received with a great deal of freedom, candour, and respect. When invited to a treat, which was very seldom, he used to return it very handsomely, and with much satisfaction to himself. So intent, so serious upon his studies, that he ate very sparingly, nay, ofttimes he has forgot to eat at all, so that going into his chamber, I have found his mess untouched of which, when I have reminded him, he would reply—'Have I?' and then making to the table, would eat a bit or two standing, for I cannot say I ever saw him sit at table by himself. At some seldom entertainments, the Masters of Colleges were chiefly his guests. He very rarely went to bed till *two* or *three* of the clock, sometimes not until *five* or *six*, lying about *four* or *five* hours, especially at spring and fall of the leaf, at which times he used to employ about six weeks in his elaboratory, the fire scarcely going out either night or day; he sitting up one night and I another, till he had finished his chemical experiments, in the performances of which he was the most accurate, strict exact...

Sir Isaac at that time had no pupils nor any chamber-fellow, for that, I would presume to think, would not have been agreeable to his studies. He was only once disordered with pains at the stomach, which confined him for some days to his bed, which he bore with a great deal of patience and magnanimity, seemingly indifferent either to live or die. ....

Sir Isaac's elaboratory.....was well furnished with chemical materials as bodies, receivers, heads, crucibles, etc., which was [sic] made very little use of, the crucibles excepted, in which he fused his metals; he would sometimes, tho' very seldom, look into an old mouldy book which lay in his elaboratory, I think it was titled *Agricola de Metallis*, the transmuting of metals being his chief design, for which purpose antimony was a great ingredient. Near his elaboratory was his garden which was kept in order by a gardener. I scarcely ever saw him do anything as pruning, etc., at it himself. When he has sometimes taken a turn or two, has made a sudden stand, turn'd himself about, run up the stairs like another Archimedes, with an ϵὔρηκα fall to write on his desk standing without giving himself the leisure to draw a chair to sit down on..... When he read in the schools he usually staid about half an hour; when he had no auditors, he commonly returned in a 4th part of that time or less...... In his chamber he walked so very much that you might have thought him to be educated at Athens among the Aristotelian sect..... He never slept in the daytime that I ever perceived; I believe he grudged the short time he spent in eating and sleeping.' ....

His thoughts were his books; tho' he had a large study seldom consulted with them." What intensity of effort, devotion to science, and great enthusiasm!

The "high" that culminated in the "Principia" was followed by a difficult period. Newton's mother, to whom, by all accounts, he was greatly attached, died in 1689. Then there was the fire in his room, which apparently destroyed many valuable manuscripts. Finally, between 1690 and 1693, he suffered from a psychological disorder, apparently a persecution mania. Newton himself, his relatives and pupils, and his eighteenth-century biographers, carefully concealed the fact of his illness, but there is no doubt about it. It is possible that the illness was the result of the enormous stress involved in creating the "Principia." There has also been the suggestion (how true, we do not know) that Newton poisoned himself during his chemical and alchemical experiments and this was the reason for his mental disorder.

By 1694, Newton has resumed his previous level of activity, but he had clearly undergone a change. As already noted, Newton left for London in 1696 and he was then only 53 years old (another way of putting this is: he was already 53 years old; the average expectation of life at birth was then much shorter than it is today). For one reason or another, Newton's scientific activity was sharply reduced after the "Principia," although he continued to be preoccupied with science as well as with the direction of the Royal Mint and with religious and other activities. What was the reason for this departure from science? The question is difficult to answer. The phenomenon is not uncommon after the attainment of a certain age and the completion of plans laid down in earlier life. However, it is interesting that there was no loss of brilliance. This is clear from the following example. In 1696, the famous mathematician, Johann Bernoulli (1667–1748) proposed the brachistochrone problem, i.e., the determination of the curve which a mass, moving under the influence of the force of gravity, will follow to reach a given point in the shortest time. Bernoulli and Leibnitz considered that the mathematical formalism that they had just developed was powerful enough to solve the problem and that Newton, preoccupied with the minting of coins, would not be able to solve it.[19] But his is not what happened. On his return home from the Mint, the tired Newton took up the challenge and found a solution the same evening. The solution was sent for publication, anonymously, the following morning. However, when Bernoulli saw the solution, he exclaimed, "I recognize the claw of the lion." [T11]—there was no doubt as to who was the author (this account of the history of the brachistochrone problem is based on Ref. 19; actually, this episode is much richer in content and the account presented in Ref. 19 does not seem to be completely accurate; this is unimportant here, since our only aim is to emphasize Newton's exceptional powers). Newton retained the clarity of his though right to the end. Toward the end of his life, he became kinder to people and very sensitive to cruelty and injustice.[9] Not long before his death, Newton put the following celebrated words to an unknown companion: "I don't know what I may seem to the world, but, as to myself, I seem to have been only a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." [T12]

S. I. Vavilov ended his valuable biography of Newton[6] with these words. We shall conclude in a different way, returning to Einstein's evaluation of Newton and his work. This is a natural choice because Einstein has frequently been compared with Newton, and is often called the second Newton and the third Archimedes (Newton being the second Archimedes).[11] Or course, such comparisons are invidious, but there is no doubt that there has been no greater physicist than Einstein since the time of Newton. This is what he wrote in one of his papers in 1927 (Ref. 14, p. 82)[T9]:

"In a few days, it will be two hundred years since Newton's death. We must recall the image of this shining genius; he showed the West a way of thinking, of experimental investigation, and of practical construction, the like of which has not been seen before or since. Newton not only created brilliant methods; he reached perfection in his mastery of the empirical material of his time, and was exceptionally creative in finding mathematical and physical demonstrations. In all this he commands our deepest respect. But the figure of Newton signifies more than the sum total of his own achievements, since fate has placed him at the turning point of the intellectual evolution of mankind. To show this graphically, we recall that a completed system of physical causality, capable of reflecting the deeper propeties of the external world, did not exist before Newton."

It seems to us that this evaluation of Newton's contribution is in harmony with the famous couplet by the English poet Alexander Pope (1688–1744):[12)]

Nature and nature's laws lay hid in night.
God said: "Let Newton be!" And all was light.

In conclusion, the author wishes to take this opportunity to thank B. A. Volkov, I. Yu. Kobzarev, I. S. Shapiro, and I. A. Yakovlev for reading the manuscript and commenting upon it.

Translator's Notes.

[T1] Nicolaus Copernicus, *On the Revolutions*, The Macmillan Press, London, 1978, p. 16.

[T2] The Portsmouth Collection, Sec. I, Div. X, No. 41.

[T3] William Stukeley, *Memoirs of Sir Isaac Newton's Life*, Taylor and Francis, 1936, p. 19.

[T4] From a letter to R. Hooke, 5 February 1675/6 (see T7).

[T5] "Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World," translated by A. Motte and revised by F. Cajori, Cambridge Unviersity Press, 1934.

[T6] Taken from P. A. Schilpp (editor), *Albert Einstein: Philosopher-Scientist*, Tudor, New York, 1949, p. 684.

[T7] Taken from "The correspondence of Isaac Newton," Vol. 3, edited by H. W. Turnbull, Cambridge University Press, 1961, p. 253; see also Ref. 40.

[T8] Motte's translation of Definition III is: "The *vis insita*, or innate force of matter, is a power of resisting, by which every body, as much in it lies, continues in its present state, whether it be of rest, or of moving uniformly forwards in a right line." The explanation that follows this definition includes the sentence: ".... This *vis insita* may, by a most significant name, be called inertia (*vis inertiae*)...."

[T9] Untraced quotation.

[T10] L. T. More, *Isaac Newton, A Biography*, Scribners, 1934, pp. 246–251.

[T11] "Tanquam ex ungue leonem."

[T12] J. Spence, *Anecdotes, Observations and Characters of Books and Men*, edited by S. S. Singer, London, 1820, p. 54

[1)] Isaac Newton, *Philosophiae Naturalis Principia Mathematica*, 1687. This Latin title can be translated as "Mathematical Principles of Natural Philosophy." The phrase "natural philosophy" or "the philosophy of nature," is roughly equivalent to what we now call "science." The Russian text of the present paper follows the Krylov translation of the "Principia." [1] The English translation follows Motte's translation.[T5]

[2)] Isaac Newton was born on 4 January 1643 and died 31 March 1727 according to the new (i.e., the Gregorian) calendar. According to the old (i.e., the Julian) calendar, used in England up to 1752, Newton was born on 27 December 1642 and died 20 March 1727.

[3)] Apart from the fact that the Russian translation of the "Principia" [1] is not readily accessible, this great book is very difficult to read in any case. Newton's correspondence and various other materials relating to his work were published only quite recently[4]; this also applies to a detailed biography.[5] In the Russian language, there is a biography,[6] a brochure,[7] a collection,[8] and a bibliographic collection.[9]

[4)] Instead of the semimajor axis, Kepler used the mean distance between the Sun and the planet, which amounts to the same thing. Kepler's second law was established by him at the end of 1601 and the beginning of 1602, while the first law was established in 1605. The above date (1609) refers to the time of publication by Kepler of his "New Astronomy," which contained his first two laws.[13] Or course, we shall not always be able in the present article to give similar details, which are unimportant for a general understanding of the essence of our topic.

[5)] The conversation with William Stukeley (physician and Fellow of the Royal Society) took place on 15 April 1726,[15] when Newton was already 83.

[6)] The original (Latin) formulation of these laws graces one of the walls of the Large Physics Lecture theatre of the Physics Faculty of Moscow State University, as it probably does in other lecture rooms throughout the world.

[7)] For example, the amount of mass is defined in Ref. 2. I shall frequently refer to this course because it is an "approved" textbook, currently used in physics departments of our colleges and universities. It is a pleasure to acknowledge that this textbook is in our opinion of the required high standard (although, it is true, that we are concerned here only with the foundations of mechanics).

[8)] The edition was completed in 1825. We note, in passing, that the phrase "celestial mechanics" was introduced by Laplace himself[28] in 1798.

[9)] It is generally believed that there is no such correlation. It seems to us that a correlation does actually exist, and is positive (see Ref. 32, p. 286.

[10)] Newton became Warden of the Royal Mint in 1696, and this is the reason why he left Cambridge for London. Newton's duties were in no way a sinecure. He devoted considerable effort over a number of years to his work at the royal Mint and increased its output by a factor of 8 without, it is said, introducing a single new machine. While remaining Master of the Royal Mint since 1699, Newton was also President of the Royal Society between 1703 and his death in 1727.

[11)] As far back as 1910, Planck called Einstein the "Copernicus of the Twentieth Century," which is a little surprising because the general theory of relativity had not yet been created. Clearly Planck had in mind the special theory of relativity, the role played by which to some extent is actually analogous to the achievements of Copernicus.

[12)] The Russian translation of these lines by S. Ya. Marshak, runs as follows:

"Byl étot mir glubokoĭ t'moĭ okutan.
Da budet svet! I vot yavilsya N'yuton."

This is truer to the original than the version quoted in Ref. 6.

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[1] Isaac Newton, *Mathematical Principles of Natural Philosophy* [Russian translation from Latin with notes and commentaries by A.N. Krylov]. the most recent publication of the translation is given in A. N. Krylov, Collected Papers, Vol. 7, Academy of Sciences of the USSR, M.,-L., 1936.

[2] D. V. Sivukhin, A General Course of Physics. Mechanics (in Russian), M.,-L., 1979.

[3] L. D. Landau and E. M. Lifshitz, Theoretical Physics, Vol. 1. Mechanics (in Russian), Nauka, 1973 [English translation by Pergamon; various eds.].

[4] *The Correspondence of Sir Isaac Newton*, Vols. I–VII, Cambridge University Press, 1959–1977.

[5] R. Westfall, *Never at Rest: A Biography of Sir Isaac Newton*, Cambridge University Press, 1982.

[6] S. I. Vavilov, Isaak N'yuton (Isaac Newton), Academy of Sciences of the USSR, M.,-L., 1945. This book is included in: S. I. Vavilov, Collected Works (Moscow, Academy of Sciences of the USSR, 1956, Vol. 3) and was reissued by the USSR Academy of Sciences in 1961.

[7] I. Yu. Kobzarev, Newton and His Times [in Russian], Znanie, M., 1978.

[8] Isaac Newton, A Collection of Papers on the Tricentenary of His Birth (in Russian), USSR Academy of Sciences, M.,-L., 1943.

[9] Contemporary Historical-Scientific Research (Newton): Referetivnyi Sb. INION AN SSSR, Moscow, 1984 (in Russian); see also Naukovedenie: Referativnyĭ Zh. No. 5, 67, 73 (1980).

[10] M. Gliozzi, Storia della Fisica [Russ. transl., Mir, M., 1970].

[11] A. Pannekuk, History of Astronomy (in Russian), Nauka, M., 1966.

[12] V. L. Ginzburg, "The heliocentric system and the general theory of relativity," in V. L. Ginzburg, On the theory of Relativity (in Russian), Nauka, M., 1979.

[13] Yu. A. Belyĭ, Johann Kepler (in Russian), Nauka, M., 1971.

[14] A. Einstein, Collected Works (in Russian), Nauka, M., 1967, Vol. 4.

[15] V. P. Kartsev, Isaac Newton (in Russian), Molodaya Gvardiya, M., 1987.

[16] I. Newton, Lectures on Optics [translated into Russian from Latin by S. I. Vavilov, Academy of Sciences of the USSR, M.,-L., 1946].

[17] A. Sommerfeld, "Lectures on theoretical physics," Vol. 1, Mechanics, Academic Press, 1964 [Russ. transl., IL, Moscow, 1947].

[18] L. S. Marochnik, Rendezvous with a Comet (in Russian), Nauka, M., 1985.

[19] O. Gingerich, "Newton, Halley and the comet," Sky and Telescope, **71**, 230 (1986).

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[22] A. Yu. Ishlinskiĭ, Mechanics of Relative Motion and the Force of Inertia (in Russian), Nauka, M., 1981; Mechanics (Ideas, Problems, Applications) (in Russian), Nauka, M., 1985.

[23]L. M. Sedov, Essays on the Foundations of Mechanics and Physics (in Russian), Znanie, M., 1983.
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[26]A. Voss, Die Prinzipien der rationellen Mechanik/Enzyklopädie der Mathematische, Wissenschaften, Leipzig, 1901–1908, Vol. 4, Part 1.
[27]M. Laue, Articles and Speeches [Russ. transl., Nauka, M., 1969, pp. 153, 266, 282].
[28]B. A. Vorontsov-Vel'yaminov, Laplace (in Russian), Nauka, M., 1985.
[29]A. Einstein, Collected Scientific Works [Russ. transl., Nauka, 1965, Vol. 1].
[30]A. Pais, "Subtle is the Lord: The Science and the Life of Albert Einstein," Oxford University Press, 1982.
[31]V. L. Ginzburg, "Experimental verification of the general theory of relativity," Usp. Fiz. Nauk **128**, 435 (1979) [Sov. Phys. Usp. **22**, 514 (1974)]. See also Collection of Papers mentioned in Ref. 12.
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[33]S. I. Vavilov, "Experimental foundations of the theory of relativity," Collected Works (in Russian), USSR Academy of Sciences, M., 1956, Vol. 4.
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[37]R. W. Brehme, "On force and the inertial frame," Am. J. Phys. **53**, 952 (1985).
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[39]Historical-Astronomical Studies (in Russian), Nauka, M., 1986, p. 41.
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Translated by S. Chomet