Hydrodynamic instability

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The current state of the classical theory of hydrodynamic instability is examined by viewing the mathematical theory (as well as experimental data) concerning the randomization of motions of liquids and gases as a problem in bifurcation theory of families of dynamic systems. Along with a discussion of the theory of linear operators encountered in hydrodynamics (a theory which is still not entirely complete), the author also gives illustrations of powerful nonlinear methods used in the analysis of hydrodynamic instability, such as Landau's amplitude equations and V. I. Arnold's variational method. The multiplicity of possible scenarios for randomization of fluid motions is noted, of which the most thoroughly investigated is M. Feigenbaum's universal sequence of period-doubling bifurcations. Recent experimental data concerning the bifurcations of G. Taylor flow between rotating cylinders and E. Lorentz flow in the case of convection in a planar fluid layer are analyzed.

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INTRODUCTION

In the seven years following the publication in Uspekhi of our review[1] on the onset of turbulence, a number of theoretical and experimental results pertaining to this problem have been obtained which warrant the attention of physicists.

The theory of bifurcations formed the foundation for theoretical study of the randomization of fluid motions. In this area, the discovery by M. Feigenbaum[2] of a universal sequence of period-doubling bifurcations has had the greatest impact; in the case of one-dimensional quadratic mappings it turned out to be possible to sort out rather fully also their super-critical stochastic behavior with "lacunae" of periodic regimes. It has become clear that there is a multiplicity of possible scenarios for randomization of motion. It is pleasant to note the "rehabilitation" of the scenario devised by L. D. Landau[3] and E. Hopf[4]: If one considers bifurca-

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tions not of the quasiperiodic trajectories but of the tori which contain them, the "reproach" that this scenario lacks structural stability is eliminated.

However, the theoretical work is perceived to be primarily general-mathematical in character, as well as lacking in relevance to real-world hydrodynamics. Therefore we deemed it expedient to combine a presentation of this theoretical work with a review of the contemporary state of the classical theory of hydrodynamic instability, proceeding chronologically through Rayleigh, O. Reynolds, W. Heisenberg and G. Taylor, and containing along with a discussion of the theory of linear operators encountered in hydrodynamics (which is still not entirely complete) a description of such powerful nonlinear methods as L. D. Landau's[3] amplitude equations and V. I. Arnol'd's[5] variational method.

It is also useful to draw the attention of a wide circle of physicists to the specifics of hydrodynamic instability in geophysical flows, for which stratification of the density in the presence of the gravitational field and of the rotation of the planet are important—the Taylor instability, the Richardson number and the so-called baroclinic instability, i.e., the growth of perturbations due to the available potential energy of the primary flow, with it being specific to the situation that the sufficient condition for the stability of a zonal flow (as established by the nonlinear method of V. I. Arnol'd) is provided by transition to a rotating frame of reference which moves faster than the flow.

At the same time, we will not consider here the well-known nonlinear instability processes—self-modulation, self-focusing, decaying and explosive instability, etc. (see, e.g., the book of Ref. 8), since, in contrast to bifurcation theory, these processes are of little help so far in helping to decipher the mechanisms by which motion is randomized; in addition, these phenomena are by no means specifically characteristic only of hydrodynamics.

We also will not consider the so-called "coherent structures," long-known in studies of the mechanics of turbulence under the name of "macrostructural elements." In recent years, they have been subjected to active experimental study (see, e.g., the review by B. Cantwell[93] and the books of Refs. 94, 95); however, there is no theory for these structures as yet, except for general considerations on self-organization (as opposed to randomization; see, e.g., Chapter 24 of the book of Ref. 8) and the numerical calculation techniques developed by O. M. Belotserkovskiĭ.[96]

The experimental possibilities for investigating the processes of randomization of fluid flows are enriched by new methods of imaging flows, by the use of laser Doppler anemometers in conjunction with computers, and by using exotic liquids (classic liquid helium and others). However, it appears that the new experimental results are for the time being fewer in number than the theoretical results; among these should be mentioned, first of all, confirmation of the theory of period-doubling bifurcations, and the observation (in circular Couette flow) of more complex behavior than is included in the existing theoretical scenarios.

Much of the material in Secs. 1–2, 4 and 6 is contained in the monograph of Ref. 6 and in the small book of L. A. Dikiĭ[7]; that in Secs. 3 and 5–6 in the recent books of M. I. Rabinovich and D. I. Trubetskov[8] (devoted to nonlinear oscillations) and A. Lichtenberg and M. Lieberman[9] (devoted primarily to Hamiltonian systems).

1. INSTABILITY OF IDEAL-FLUID FLOWS

For adiabatic small oscillations of a fluid relative to a quiescent state (or a fluid in steady-state motion described by one of the solutions of the linearized hydrodynamic equations), the time independence of the total energy implies that the frequency of these oscillations σ is real, so that a small initial perturbation neither grows nor decays with time. This is correct under the condition that the density stratification is stable, i.e.,

$$N^2 = -\frac{g}{\rho}\left(\frac{\partial \rho}{\partial z} + \frac{g\rho}{c_0^2}\right) > 0, \tag{1.1}$$

where z is the vertical coordinate, increasing upward; ρ is the density of the liquid; g is the acceleration of gravity; c_0 is the velocity of sound; N is the so-called Väisälä-Brunt frequency.

1.1. Static (Taylor) instability

Let us investigate the opposite case of an unstable density stratification, i.e., $N^2 = -M^2 < 0$. We here limit ourselves to the Boussinesq approximation, replacing the continuity equation by the acoustic-wave filtering condition that the velocity field be solenoidal: $\nabla \cdot \mathbf{u} = 0$, so that the linearized hydrodynamic equations take the form

$$\operatorname{div} \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla p''}{\rho_*} + g\eta' \nabla r, \quad \frac{\partial \eta'}{\partial t} = \frac{M^2}{g} u_r, \tag{1.2}$$

where ρ_* is the so-called potential density, determined by the relation $\partial \ln \rho_* / \partial z = M^2/g$; $p'' = (\rho_*/\rho_0)p'$ is the normalized deviation p' of the pressure from the hydrostatic distribution $p_0(z)$; $\eta' = (p' - c_0^2 \rho')/\rho_0 c_0^2$ is the dimensionless entropy perturbation; $\rho' = \rho - \rho_0$ is the density perturbation. The local energy equation takes the form

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\rho_* u^2 - \rho_* \frac{g^2 \eta'^2}{M^2}\right) + \operatorname{div} p'' \mathbf{u} = 0. \tag{1.3}$$

As a specific example, for a layer $0 \leqslant z \leqslant h$ between horizontal solid plates we find

$$\frac{\partial}{\partial t}\left(\int \frac{\mathbf{u}^2}{2} \rho_* \, dV - \int \frac{g^2 \eta'^2}{2M^2} \rho_* \, dV\right) = 0, \tag{1.4}$$

i.e., the *difference* of the total kinetic and thermobaric energies of the perturbations does not change with time; as for the behavior of each of these two terms in the energy taken separately as given in (1.4), no conclusions of any kind can be drawn (they both grow exponentially with time). We will investigate only two-dimensional perturbatiions of motion in the (x,z) plane; using the fact that the velocity is divergenceless, we introduce a flow function ψ, setting $u = -\partial\psi/\partial z$, $w = \partial\psi/\partial x$. After calculating the curl of the velocity (assuming the potential density ρ_* is quasiconstant) and eliminating η' with the help of the third equation (1.2), we obtain for ψ the equation

$$\frac{\partial^2 \Delta\psi}{\partial t^2} - M^2 \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{1.5}$$

with the boundary condition $w=0$, i.e., $\psi=0$ for $z=0, h$. Since M^2 does not depend on x and t, an elementary solution to this equation can be sought in the form $\psi=\psi(z)\exp\times[i(kx-\sigma t)]$. We obtain from (1.5) an equation for the complex amplitude $\psi(z)$

$$\sigma^2\left(\frac{\partial^2\psi}{\partial z^2}-k^2\psi\right)-k^2M^2\psi=0. \qquad (1.5')$$

For quasi-constant M, the solution which reduces to zero for $z=0$ has the form $\psi\sim\sin mz$, where $m=-ik[1+(M^2/\sigma^2)]^{1/2}$. In order for ψ to vanish at $z=h$ also, we must set $mh=\pi n$, $n=0, \pm 1, \pm 2,...$ from which we have

$$\sigma=\pm iM\left(1+\frac{\pi^2n^2}{k^2h^2}\right)^{-1/2}, \qquad (1.6)$$

so that for each fixed k there is a denumerable set of elementary solutions, and for all of them the quantity σ turns out to be pure imaginary. Thus, in this case there always exist elementary solutions which grow with time. This phenomenon is called static (Taylor) instability. Its mechanism involves an acceleration due to the Archimedes force as liquid particles shift from their equilibrium positions along the vertical direction. Thus, according to the third equation (1.2) the interaction of the vertical velocity u_r with the unstable gradient of the potential density M^2/g leads to growth of the perturbation η' of the entropy, and according to the second equation (1.2) its contribution $g\eta'$ to the Archimedes force leads to growth in u_r.

1.2. The Rayleigh equation

The growth of small perturbations can be due not only to an unstable density distribution in the equilibrium state, but also to certain kinds of velocity distributions in this state. For example, let us address the question of stability of plane-parallel steady-state flow of an incompressible ideal fluid, directed along the x axis and having a velocity $\mathbf{u}_0=[U(z),0,0]$. The linearized hydrodynamic equations take the form

$$\operatorname{div}\mathbf{u}=0, \quad \frac{\partial\mathbf{u}}{\partial t}+U\frac{\partial\mathbf{u}}{\partial x}+w\frac{\partial\mathbf{u}_0}{\partial z}=-\frac{\nabla p'}{\rho_0}. \qquad (1.7)$$

The local energy equation now has the form

$$\frac{1}{2}\frac{\partial}{\partial t}\rho_0|\mathbf{u}|^2+\frac{1}{2}\frac{\partial}{\partial x}\rho_0U|\mathbf{u}|^2+\rho_0uw\frac{\partial U}{\partial z}+\operatorname{div}p'\mathbf{u}=0, \qquad (1.8)$$

from which, e.g., for a layer $0\leqslant z\leqslant h$ between horizontal solid walls we obtain

$$\frac{\partial}{\partial t}\int\frac{|\mathbf{u}|^2}{2}\rho_0\,dV=-\int\rho_0uw\frac{\partial U}{\partial z}\,dV, \qquad (1.9)$$

so that the kinetic energy of the perturbation can change with time due to the work done by the stresses $-\rho_0uw$ on the gradient of the equilibrium velocity $\partial U/\partial z$. As above, we investigate only two-dimensional perturbations $u=-\partial\psi/\partial z, w=\partial\psi/\partial x$. Then after calculating the curl of Eq. (1.7) we obtain an equation for ψ analogous to (1.5):

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta\psi-\frac{\partial^2U}{\partial z^2}\frac{\partial\psi}{\partial x}=0. \qquad (1.10)$$

However, the question of how solutions ψ to this equation behave turns out to be much more complicated than in the case of (1.5). Let us again seek an elementary wave-like solution in the form $\psi=\psi(z)\exp[i(kx-\sigma t)]$. Then we obtain the so-called *Rayleigh equation* for the complex amplitude $\psi(z)$:

$$(U-c)\left(\frac{\partial^2\psi}{\partial z^2}-k^2\psi\right)-\frac{\partial^2U}{\partial z^2}\psi=0, \qquad (1.11)$$

where $c=\sigma/k$ is the phase velocity. This equation, to begin with, has a singular point z_0; for "neutral" perturbations (real c) the coefficient $U-c\approx U'(z_0)(z-z_0)$ of the leading derivative $\partial^2\psi/\partial z^2$ can reduce to zero at this point, so that the derivative $\partial\psi/\partial z\sim(\psi U''/U')_0\ln(z-z_0)$ is multivalued there; the problem then arises of choosing the correct branch of the solution. Secondly, as a rule, the set of discrete eigenvalues of c turns out to have only a finite number of elements, so that we can no longer determine the stability of a given solution to Eq. (1.10) by representing it as a superposition of the corresponding elementary wave solutions. True, there is still a continuous spectrum of eigenvalues c in this case; it is easy to convince oneself of this by writing Eq. (1.1) in the form

$$(U-U''\mathscr{L}^{-1})\psi_1=c\psi_1, \quad \mathscr{L}=\frac{\partial^2}{\partial z^2}-k^2, \quad \psi_1=\mathscr{L}\psi \qquad (1.11')$$

(here and henceforth, the dashes denote derivatives with respect to z), from which it is clear that c is an eigenvalue of the sum of a multiplication operator $U(z)$ (having a real continuous spectrum which fills the interval $U_{\min}\leqslant c\leqslant U_{\max}$) and the completely continuous operator $U''\mathscr{L}^{-1}$ (the addition of which does not change the continuous spectrum). However, the operator $U''\mathscr{L}^{-1}$ is not self-adjoint, and the theorems on expanding an arbitrary function using its discrete and continuous spectrum do not hold here. Thirdly, because of the lack of self-adjointness of this operator even the real eigenvalues c, say, of the discrete spectrum, if they are multiples can lead to instability in the form of "secular" perturbations which increase linearly with time.

1.3. Lyapunov stability

The example of the Rayleigh equation shows that it is expedient to start from a more general criterion for hydrodynamic instability than the presence of eigenvalues of the linearized equations which have negative imaginary parts. So as to give a more general definition, let us introduce the concept of the *phase space* of the fluid, whose "points" M are complete sets of independent (i.e., not coupled by synchronous relations) thermodynamic fields which characterize the instantaneous states of the moving fluid. In the case of an incompressible fluid, this is the velocity field $\mathbf{u}(\mathbf{x})$ in the region of space occupied by the fluid which satisfy the required boundary conditions. In the general case we add to these the fields of the density $\rho(\mathbf{x})$, entropy $\eta(\mathbf{x})$ and impurity concentration $s(\mathbf{x})$. The evolution of the fluid flow in time is imaged in the phase space by some curve $M=M(t)$, i.e., the phase trajectory of the flow; for steady-state flow it consists of a single point, while for a periodic flow it forms a closed curve (a cycle). The ensemble $M(t)=F^tM(0)$ of phase trajectories traced out through every point of the phase space $M = M(0)$ and extended along the entire time axis determines a group of transformations F^t of the phase space into itself, called the *phase flow*. It describes the evolution of all the fluid flows in the given geometry for all possible initial conditions.

Let us introduce into the phase space the norm $\|M\|$ for its elements. Then the general definition of the stability of a phase trajectory $M = M_0(t)$, according to A. M. Lyapunov, is the following: for any arbitrarily small positive number ε there exists a positive number $\delta = \delta(\varepsilon)$ such that for any trajectory $M = M(t)$ with initial value $M(0)$ which satisfies the condition $\|M(0) - M_0(0)\| < \delta$, for all times $t > 0$ the inequality $\|M(t) - M_0(t)\| < \varepsilon$ holds. It is not hard to convince oneself that from the presence of even one unstable infinitesimal wave perturbation $M'(t) = M(t) - M_0(t)$ (with an eigenfrequency σ with negative imaginary part $\gamma = \operatorname{Im} \sigma < 0$) there follows the instability of the trajectory $M_0(t)$ according to Lyapunov. For, even if the perturbation $M'(t)$ is small, it actually grows according to the law $e^{|\gamma| t}$ of the linear theory; then, as a rule, this growth slows down and reaches some finite limit. Decreasing the amplitude of the initial perturbation only prolongs this process, but does not change its ultimate limit. This nonuniformity with respect to time of the convergence to zero is what implies the absence of Lyapunov stability. Since in reality small perturbations are always present, the linear instability of a flow $M_0(t)$ implies that the flow is in practice unrealizable (such as, e.g., the state of rest in the presence of an unstable stratification; essentially it is this fact that underlies the basic significance of the concept of hydrodynamic instability.

In many cases one can prove the opposite assertion—that linear stability guarantees Lyapunov stability. Thus, for Eq. (1.10) the following theorem holds (L. A. Dikiĭ[7]): A two-dimensional plane-parallel flow with a monotonic velocity profile $U(z)$, $0 \leqslant z \leqslant h$, in which neither $U(0)$ nor $U(h)$ are eigenvalues, can be unstable only in the presence of a discrete spectrum of complex or multiple real eigenvalues.

The idea of the proof involves solving the Cauchy problem for Eq. (1.10) (assuming ψ depends on x as e^{ikx}) for arbitrary initial values $\psi(z,0)$, by finding the Laplace transform in time and using the Green's function $G^*(z,\zeta,s)$ of the transformed equation. So as to elucidate whether or not the function $\psi(z,t)$ is bounded as $t \to \infty$, we must investigate the behavior of the inverse Laplace transform of $G^*(s)$. It can be proved that the function $G^*(s)$ can be analytically continued from the upper half s-plane to the lower half-plane everywhere except the points $s = U(0)$, $U(h)$, $U(z)$ and $U(\zeta)$, where it has a pole whenever s equals an eigenvalue of the homogeneous equation (of the same multiplicity as the eigenvalue); if $s = U(0)$ and $s = U(h)$ are not eigenvalues, then there is a finite number of eigenvalues in the upper half-plane in any given neighborhood of the interval $(U(0), U(h))$. To calculate $G(t)$ from $G^*(s)$ we must lower the integration contour in the s plane by a fixed distance below the real axis, allow it to rise from there along the edges of the vertical branch cuts to the points $s = U(0)$, $U(z)$, $U(\zeta)$, $U(h)$ and go around these points along circles with radii of the order of c/t. Then as $t \to \infty$ the integrals along the horizontal parts of the contour decay exponentially, while along the circles and vertical segments they remain bounded; we then need only add the sum of the residues at the poles between the old and new contours of integration—at those points s which are eigenvalues. For a simple pole the residue is proportional to $G^*(s)e^{-ikst}$, for a k-multiple pole it is proportional to $\partial^{k-1}[G^*(z)e^{-ikst}]/\partial s^{k-1}$. These residues pick out the wave solutions, which are the only reason for an unbounded growth of $\psi(z,t)$; and this proves the theorem of L. A. Dikiĭ.

1.4. The Rayleigh and Fjortoft theorems

We will note without proof that for a flow with "type A" velocity profile—monotonic $U'(z) > 0$ and with one point of inflection $z = \bar{z}$ (so that $\bar{U}'' = 0$) in which $U'(z)$ is a maximum (so that $\bar{U}''' < 0$)—to be stable, it is necessary and sufficient that no complex eigenvalues c be present at $k = 0$, or that the condition of N. Rosenbluth and A. Simon[10] hold:

$$[U'(U-\bar{U})]^{-1}\Big|_0^h + \int_0^h U''(U')^{-2}(U-\bar{U})^{-1}\,dz > 0. \quad (1.12)$$

In particular, it is sufficient that there exist a constant K_0 such that $(U - K_0)U'' \geqslant 0$ (R. Fjortoft[11]), or simply that no point of inflection be present (Rayleigh, 1880). Conversely, a flow with a "type A" velocity profile for $0 \leqslant z \leqslant h/2$ which is symmetric about the point $z = R/2$ is always unstable (V. Tollmien[12]). Tangential discontinuity in the velocity $U(z) = U_0$ for $z > 0$ and $(-U_0)$ for $z < 0$ are also unstable (the so-called *Helmholtz instability*).

The Rayleigh and Fjortoft theorems can be proved for arbitrary initial data without considering elementary wave solutions. Thus, from Eq. (1.10) the following integral relation can be derived for a solution ψ which depends on x as e^{ikx}:

$$\frac{\partial}{\partial t}\int (|\psi'|^2 + k^2|\psi|^2)\,dz - 2\operatorname{Re} ik \int U(\psi'' - k^2\psi)\psi^*\,dz = 0. \quad (1.13)$$

Let there exist a constant K_0 such that $(U - K_0)(U'')^{-1}$ is everywhere continuous. Multiplying (1.10) by $(\psi^{*\prime\prime} - k^2\psi^*)(U - K_0)(U'')^{-1}$, integrating over z, taking the real part and adding it to (1.13), we obtain

$$\frac{\partial}{\partial t}\int [|\psi'|^2 + k^2|\psi|^2 + (U - K_0)|\psi'' - k^2\psi|^2(U'')^{-1}]\,dz = 0, \quad (1.14)$$

so that when Fjortoft's condition holds, the smallness of this integral implies the smallness of the mean square values of ψ, ψ' and ψ'' at any moment of time, i.e., stability of the flow.

1.5. Arnol'd's variational method

The meaning of the quadratic integral invariant in (1.14) can be understood once we consider, following V. I. Arnol'd,[5] the Lyapunov stability of nonlinear *two-dimensional* flow of an incompressible ideal fluid, for which the function $\psi(x,z,t)$ satisfies the equation for the curl:

$$\frac{\partial \Delta\psi}{\partial t} + \frac{\partial(\psi, \Delta\psi)}{\partial(x, z)} = 0 \quad (1.15)$$

(Linearizing this equation relative to $\psi_1 = \psi + \int^z U(z)dz$ leads back to Eq. (1.10)) with the boundary conditions that the boundary is impenetrable $\psi(\Gamma) = \text{const}$, and that the circulation along the boundary $\oint \frac{\partial \psi}{\partial n} d\Gamma = \text{const}$ is conserved. Here, not only the kinetic energy (per unit mass) $\mathscr{E} = \frac{1}{2}\int |\nabla\psi|^2 dxdz$ is time-independent, but also any functional $\mathscr{F} = \int \Phi(\Delta\psi)dxdz$ of the vorticity. In order to investigate the stability of a steady-state flow with a flow function ψ_0 [which we will assume is a monotonic function of the vorticity $\psi_0 = \Psi(\Delta\psi_0)$], we will choose the conserved functional $F(\psi) = \mathscr{E} + \mathscr{F}$, so that for $\psi = \psi_0$ F has an extremum. It is easy to convince oneself that in order to reduce the first variation δF to zero one must set $\Phi' = \Psi$. For plane-parallel flow with velocity $u_0 = U(z)$ we will have $\Psi'(\Delta\psi_0) = U(z)/U''(z)$, and in a frame of reference moving relative to the original one with constant velocity K_0 we obtain

$$F(\psi) - F(\psi_0) = \frac{1}{2}\int [(\nabla\delta\psi)^2 + (U-K_0)(U'')^{-1}(\Delta\delta\psi)^2]\, dx\, dz \ldots . \quad (1.16)$$

If the Fjortoft condition holds, this quadratic form in $\delta\psi$ is positive and can be adopted as the square of the norm $\|\delta\psi\|^2_{(1)}$. For the linearized equations, according to (1.14) it will be an exact invariant. For the nonlinear equations the left side of (1.16) also acts as the square of the norm $\|\delta\psi\|^2_{(1)}$. However, these norms are equivalent in the sense that there exist positive constants $C_1 \leqslant C_2$ which ensure the inequality $C_1\|\delta\psi\|_{(1)} < \|\delta\psi\|_{(2)} < C_2\|\delta\psi\|_{(1)}$. If the initial norm $\|\delta\psi\|^2_{(1)}$ is small, then the invariant norm $\|\delta\psi\|^2_{(2)}$ is also small; then $\|\delta\psi\|^2_{(1)}$ stays small for all times, so that in this case the nonlinear Lyapunov stability follows from the linear stability.

1.6. The Richardson number

In natural flows one often encounters cases in which the stabilizing action of a stable stratification $N_0^2(z) > 0$ competes with the destabilizing influence of an unstable velocity profile $U(z)$. In these cases, in place of Eqs. (1.5) or (1.10) we obtain the following combination of them:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta\psi - U''\frac{\partial\psi}{\partial x}\right] + N_0^2\frac{\partial^2\psi}{\partial x^2} = 0. \quad (1.17)$$

For elementary wave solutions of the form $\psi(z)\exp \times [ik(x-ct)]$ we obtain

$$(U-c)[(U-c)(\psi'' - k^2\psi) - U''\psi] + N_0^2\psi = 0. \quad (1.18)$$

Let us denote $U-c$ by W and introduce a new unknown function $\Psi = \psi W^{-1/2}$. Then Eq. (1.18) is brought to the form

$$(W\Psi')' - \left[\frac{1}{2}U'' + k^2W + U'^2W^{-1}\left(\frac{1}{4} - \mathrm{Ri}\right)\right]\Psi = 0, \quad (1.19)$$

where

$$\mathrm{Ri} = N_0^2(U')^{-2} \quad (1.20)$$

is the so-called *Richardson number*. We multiply Eq. (1.19) by the complex conjugate quantity Ψ^* and integrate the result over the layer $0<z<h$ for boundary conditions $\Psi = 0$ (i.e., $w = 0$) at both ends of this segment; we then write the imaginary part of the equation so obtained:

$$(\mathrm{Im}\, c)\int_0^h \left[|\Psi'|^2 + k^2|\Psi|^2 - \left(\frac{1}{4} - \mathrm{Ri}\right)|U'\Psi W^{-1}|^2\right] dz = 0. \quad (1.21)$$

If $\mathrm{Ri}(z) \geqslant 1/4$ everywhere, then this equation can hold only if $\mathrm{Im}\, c = 0$, i.e., if there are no unstable wave solutions. Thus, the condition $\mathrm{Ri} \geqslant 1/4$ is sufficient for stability of the stratified flow (J. Miles[13] and L. Howard[14]).

Up to now we have considered only two-dimensional wave perturbations u, $w(x,z)$ of plane-parallel flows $\mathbf{u}_0 = \{U(z),0,0\}$, because their stability is sufficient to ensure the stability of three-dimensional wave perturbations also (only the projection $\mathbf{u}_0\mathbf{k}k^{-1}$ of the primary flow affects the waves $\exp[i(k_1x + k_2y - \sigma t)]$. With the help of the invariant (1.14) we can also demonstrate this for perturbations which have arbitrary time dependence: if the two-dimensional perturbation of velocity and vorticity is bounded, then the three-dimensional perturbation of the velocity multiplied by U'' and the z-component of vorticity are bounded (while the other components of the latter can grow linearly with time).

1.7. Axially-symmetric flows

Let us also discuss briefly the stability of stationary axially-symmetric flows parallel to the x-axis having the velocity profile $U(r)$, $r = (y^2 + z^2)^{1/2}$. Using cylindrical coordinates (x,r,φ), we consider small perturbations of the form $\mathbf{u}(r)\exp[ik(x-ct) + in\varphi]$. Then for the complex amplitude $f = u_r(r)$ from (1.7) we obtain the following analog to the Rayleigh equation:

$$(U-c)\left(\frac{\partial}{\partial r}\frac{r}{n^2+k^2r^2}\frac{\partial rf}{\partial r} - f\right) - r\frac{\partial Q}{\partial r}f = 0,$$
$$Q = \frac{r}{n^2+k^2r^2}\frac{\partial U}{\partial r} \quad (1.22)$$

with the boundary conditions $f \to 0$ as $r \to \infty$ (for an axially-symmetric jet in an unbounded space) or $f(R) = 0$ (for a circular pipe of radius R). Multiplying (1.22) by $rf^*(U-c)^{-1}$, integrating over r and taking the imaginary part, we obtain

$$(\mathrm{Im}\, c)\int r^2|f|^2|U-c|^{-2}\frac{\partial Q}{\partial r}dr = 0, \quad (1.23)$$

from which it is clear that in order to have an instability (i.e., for the condition $\mathrm{Im}\, c \neq 0$ to hold) it is necessary for the derivative $\partial Q/\partial r$ to change sign at some point (Rayleigh, 1892). For example, in Poiseuille flow in a circular pipe, $U(r) = U_{\max}(1 - r^2/R^2)$, and growing non-axially-symmetric wave perturbations ($n \neq 0$) are impossible (for $n = 0$, $\partial Q/\partial r \equiv 0$ here and the Rayleigh equation has no eigenvalues in general). If $\partial Q/\partial r$ changes sign at a point r_c and $U(r_c) = U_c$, then in order to have instability it is necessary for the Fjortoft condition $(U - U_c)\partial Q/\partial r < 0$ to hold somewhere in the flow. Here, the stability problem, apparently, reduces (as above) completely to an investigation of the discrete spectrum of the Rayleigh equation. We note that G. Batchelor and A. Gill,[15] who studied axially-symmetric jets, and H. Sato and O. Okada,[16] who studied axially-symmetric wakes, found only the $n = 1$ possibility for instability.

2. BAROCLINIC INSTABILITY

In the previous section we saw that adiabatic perturbations of the velocity field of an incompressible fluid can grow only because of the kinetic energy of the primary flow; see the energy Eq. (1.9). Such instabilities are termed *barotropic*, since they are characteristic of barotropic fluids (in which ρ is a function only of p); this is because in such fluids the barotropic potential energy which arises because of the two-dimensional compressibility is very small. In baroclinic fluids, however (i.e., fluids in which ρ depends not only on p but on T and s) it is possible also to have so-called *baroclinic instability*—the growth of a perturbation due to available potential energy of the primary flow state. This instability plays a major role, in particular, in the formation of synoptic processes in the terrestrial atmosphere and in the World Ocean.

2.1. The potential vorticity equation

As an example of this, let us consider *quasihydrostatic* flows in the atmosphere, using the hydrodynamic equations to describe these flows in isobaric coordinates (in which the x axis points east, the y axis points north and the pressure p is used as a vertical coordinate). We then will be assuming that these processes are also *quasigeostrophic* in the sense that pressure differentials approximately counterbalance the Coriolis force. Then (see, e.g., the book of Ref. 17) the following equation can be derived from the hydrodynamic equations for the so-called potential vorticity $\tilde{\Omega} = \mathscr{L}\psi + f$:

$$\frac{\partial \mathscr{L}\psi}{\partial t} + \frac{\partial(\psi, \mathscr{L}\psi + f)}{\partial(x, y)} = 0, \qquad (2.1)$$
$$\mathscr{L}\psi = \Delta\psi + \frac{f_0^2}{c_0^2}\frac{\partial}{\partial p}\frac{p^2}{\alpha^2}\frac{\partial\psi}{\partial p}, \quad \psi = g f_0^{-1} z,$$

where f is the Coriolis parameter (twice the vertical projection of the angular velocity vector of the rotation of the Earth); f_0 is its quasiconstant local value; z is the height of the isobaric surfaces; $\alpha^2 = H^2 N^2 c_0^{-2}$ is the dimensionless Väisälä-Brunt frequency (H is the so-called thickness of the homogeneous atmosphere). The difference between this equation and (1.15) lies in the replacement of the two-dimensional Laplace operator Δ by a three-dimensional operator $\mathscr{L}$ (which for stable stratification is elliptic), and in a correction term $(\partial f/\partial y)\partial\psi/\partial x$ caused by the rotation and sphericity of the Earth. The boundary condition at the Earth's surface (reduction of the vertical velocity to zero) reduces to the form[17]:

$$\frac{d}{dt}\left(p\frac{\partial z}{\partial p} + \alpha^2 z\right) = 0 \quad \text{for} \quad p = p_0. \qquad (2.2)$$

We will show that the nonlinear equations (2.1)–(2.2) have a quadratically-integrable invariant—the total energy

$$\mathscr{E} = \frac{1}{2}\int\left[(\nabla_h z)^2 + \frac{f_0^2}{\alpha^2 c_0^2}p^2\left(\frac{\partial z}{\partial p}\right)^2\right]dx\,dy\,dp + \frac{1}{2}\int\frac{f_0^2 p_0}{c_0^2}z^2\,dx\,dy. \qquad (2.3)$$

Indeed, if we differentiate this expression with respect to time, and integrate quantities like $(\partial z/\partial x)\partial(\partial z/\partial t)/\partial x$ by parts and use the Gauss-Ostrogradskiĭ formula, we obtain

$$\frac{\partial\mathscr{E}}{\partial t} = -\frac{f_0}{g}\int z\frac{\partial\tilde{\Omega}}{\partial t}\,dx\,dy\,dp + \int z\mathbf{n}\nabla\frac{\partial z}{\partial t}\,dp\,d\Sigma + \int\frac{f_0^2}{\alpha^2 c_0^2}\left(\frac{\partial}{\partial t}p\frac{\partial z}{\partial p} + \alpha^2 z\right)pz\,dS, \qquad (2.4)$$

where Σ is the lateral (curved) surface of the cylinder, $\mathbf{n}$ is the normal unit vector to Σ, and S is the sea-level pressure $p = p_0$. As a consequence of (2.1), the first term of this equation is proportional to the integral of $\partial(z^2,\tilde{\Omega})/\partial(x,y)$ which after integration by parts leads to an integral of $(z^2\tilde{\Omega}_x)_y - (z^2\tilde{\Omega}_y)_x$, and then to the integral $\int dp \oint z^2(\tilde{\Omega}_x\,dx + \tilde{\Omega}_y\,dy)$; therefore, $\int dp \oint z^2 d\tilde{\Omega} = 0$, because on the lateral surface if $p = \text{const}$ we must have $z = \text{const}$ (the impenetrability condition). The second term in (2.4) reduces to zero because the circulation is constant around the boundary $\oint \mathbf{n}\nabla z\,dl = \text{const}$. By using (2.2) in place of (2.1) we can prove that the third term is zero as is the first term. So, $\partial\mathscr{E}/\partial t = 0$.

We note that $z - F(p)$ is again a solution of Eq. (2.1) with the boundary condition (2.2), with F being an arbitrary function of p. Therefore, in (2.3) we can subtract from $\partial z/\partial p$ an arbitrary function of p, and from $\hat{z}$ an arbitrary constant c. From this follows the invariance with time of the average $\partial\bar{z}/\partial p$ of $\partial z/\partial p$ over x and y for each p, and also the average value $\bar{z}$ for $p = p_0$. If we set $F(p) = \bar{z}(p)$ and $C = \bar{z}(p_0)$, we obtain the minimum value of the energy (2.3):

$$\mathscr{E} = \frac{1}{2}\int\left[(\nabla_h z)^2 + \frac{f_0^2 p^2}{\alpha^2 c_0^2}\left(\frac{\partial z}{\partial p} - \frac{\partial\bar{z}}{\partial p}\right)^2\right]dx\,dy\,dp + \frac{1}{2}\int\frac{f_0^2 p_0}{c_0^2}(z - \bar{z})^2\,dx\,dy. \qquad (2.3')$$

Here the first term corresponds to kinetic energy, while the second corresponds to the baroclinic and the third to the barotropic available potential energies. The ratios of these quantities are $1:(L/L_R)^2:(L/L_0)^2$, where L is the horizontal scale of the perturbation, $L_R = HN_0 f_0^{-1}$ is the so-called Rossby deformation radius and $L_0 = cf_0^{-1}$ is the scale of barotropic perturbations introduced by A. M. Obukhov.[18]

2.2. Barotropic stability

We begin investigating the question of stability of quasi-geostrophic flows with the simpler case of barotropic flows. A method of deriving these flows was formulated in Ref. 19: it consists of taking the limit $\alpha \to 0$, in which limit the derivatives $\partial z/\partial s$ no longer depend on p. Then as a result of this, Eq. (2.1) takes the form

$$\frac{\partial \mathscr{L}_1\psi}{\partial t} + \frac{\partial(\psi, \mathscr{L}_1\psi + f)}{\partial(x, y)} = 0, \qquad (2.5)$$
$$\mathscr{L}_1\psi = \Delta\psi - \frac{f_0^2}{c_0^2}\psi, \quad \psi = g f_0^{-1} z.$$

The potential vorticity (averaged over the thickness of the atmosphere) equals $\tilde{\Omega} = \mathscr{L}_1\psi + f$, while the total energy (2.3′) loses its second term (the baroclinic available potential energy). We will analyze the stability of stationary zonal barotropic flows with potential vorticity $\tilde{\Omega}_0$ by the method of V. I. Arnol'd, where $\tilde{\Omega}_0$ depends monotonically on latitude so that the current function can be cast in the form $\psi_0 = \Psi(\tilde{\Omega}_0)$. Then in analogy with (1.14), the first variation of the functional $F(\psi) = \mathscr{E} + \int\Phi(\tilde{\Omega})dx\,dy$ reduces to zero for $\Phi' = \Psi$, while the second turns out to be equal to

$$\delta^2F = \int \left[(\nabla\delta\psi)^2 + \frac{f_0^2}{c_0^2}(\delta\psi)^2 + \Psi'(\tilde{\Omega}_0)(\delta\tilde{\Omega})^2\right] dx\,dy, \quad (2.6)$$

from which it is clear that for a flow to be stable (i.e., reducing F to a minimum for $\psi = \psi_0$, or fulfilling the condition $\delta^2F > 0$) it is sufficient that the inequality $\Psi'(\tilde{\Omega}_0) > 0$ to be fulfilled. Usually the opposite inequality $\Psi'(\tilde{\Omega}_0) < 0$ holds in general atmospheric circulation, because zonal flow is directed from west to east, i.e., ψ_0 increases from north to south, while the behavior of $\tilde{\Omega}_0$ is determined essentially by the term f which grows from south to north: ψ_0 turns out to be a decreasing function of $\tilde{\Omega}_0$. But this behavior can be reversed if we transfer to a frame of reference rotating relative to the origin with constant angular velocity which *exceeds that of the zonal flow*. In this case, the following analog of the Rayleigh theorem is true: for a zonal barotropic flow to be stable, it is sufficient that the potential vorticity be a monotonic function of latitude (S. L. Kuo[20]).

2.3. Baroclinic stability

Sufficient conditions for stability of zonal baroclinic flows can be determined in a fully analogous fashion. In this case, according to (2.2) there is also a boundary invariant

$$\left(\frac{\partial\psi}{\partial p} + \frac{\alpha^2}{p}\psi\right)_{p=p_0}$$

and the stability of a zonal flow with the current function Ψ_0 is ensured by a minimum for $\psi = \psi_0$ of the functional

$$F(\psi) = \mathscr{E} + \int \Phi(p, \tilde{\Omega})\,dx\,dy\,dp + \int_{p=p_0} \Gamma\left(\frac{\partial\psi}{\partial p} + \frac{\alpha^2}{p}\psi\right) dx\,dy, \quad (2.7)$$

where $\Phi(p,\tilde{\Omega})$ is an arbitrary function of two variables, while Γ is an arbitrary function of one variable. In order to reduce the first variation δF to zero for $\psi = \psi_0$ we assume that for every p $\tilde{\Omega}_0$ is a monotonic function of latitude, so that we can set $\psi_0 = \Psi(p,\tilde{\Omega}_0)$, and that

$$\left(\frac{\partial\psi_0}{\partial p} + \frac{\alpha^2}{p}\psi_0\right)_{p=p_0}$$

also is a monotonic function of latitude, so that for $p = p_0$ we can set

$$\psi_0 = X\left(\frac{\partial\psi_0}{\partial p} + \frac{\alpha^2}{p}\psi_0\right).$$

Then it is not difficult to convince oneself that in order to satisfy the requirement that $\delta F = 0$ we must set $\partial\Phi/\partial\tilde{\Omega} = \Psi$, and

$$\Gamma' = -\left(\frac{f_0^2}{c_0^2}\frac{p^2}{\alpha^2}\right)_{p=p_0} X$$

and the second variation δ^2F takes the form

$$\delta^2F = \int\left[(\nabla_h\delta\psi)^2 + \frac{f_0^2}{c_0^2}\frac{p^2}{\alpha^2}\left(\frac{\partial\delta\psi}{\partial p}\right)^2 + \frac{\partial\psi_0}{\partial\tilde{\Omega}_0}(\delta\tilde{\Omega})^2\right] dx\,dy\,dp + \left(\frac{f_0^2}{c_0^2}\frac{p^2}{\alpha^2}\right)_{p=p_0} \int_{p=p_0}\left[\frac{\alpha^2}{p}(\delta\psi)^2 - X'\left(\frac{\partial\delta\psi}{\partial p} + \frac{\alpha^2}{p}\delta\psi\right)^2\right] dx\,dy. \quad (2.8)$$

For stability of a zonal flow, i.e., $\delta^2F > 0$, one might think that it is enough to fulfill both conditions: $\partial\psi_0/\partial\tilde{\Omega}_0 > 0$ for every p, and $X' < 0$ for $p = p_0$. As in (2.6), the first of these conditions is ensured by going to a rotating frame of reference which "leads" the zonal flow. However, the second condition reduces, roughly speaking, to an unnatural requirement on the growth of the surface density from the poles to the equator. Let us replace it by the condition

$$\left(\frac{\partial\psi}{\partial p} + \frac{\alpha^2}{p}\psi\right)_{p=p_0} = \text{const}$$

(roughly speaking, this is the requirement that the surface density be constant); then the variations of this expression will equal zero, and for stability of zonal baroclinic flow it will be sufficient that for each p the potential vorticity decreases from north to south (J. Charney and M. Stern[21]).

2.4. Linear baroclinic instability

However, these sufficient conditions for baroclinic stability are apparently fulfilled rather rarely in the terrestrial atmosphere and the World ocean. In order to study the unstable perturbations, we linearize Eqs. (2.1) and (2.2) relative to a stationary plane-parallel flow along the x axis with velocity $-\partial\psi_0/\partial y = U(y,p)$:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\Delta\psi + \frac{f_0^2}{c_0^2}\frac{\partial}{\partial p}\frac{p^2}{\alpha^2}\frac{\partial\psi}{\partial p}\right) + \left(-\frac{\partial^2U}{\partial y^2} + \frac{\partial f}{\partial y} - \frac{f_0^2}{c_0^2}\frac{\partial}{\partial p}\frac{p^2}{\alpha^2}\frac{\partial U}{\partial p}\right)\frac{\partial\psi}{\partial x} = 0,$$

$$\left(\frac{\partial}{\partial t} + \hat{U}\frac{\partial}{\partial x}\right)\left(\frac{\partial\psi}{\partial p} + \frac{\alpha^2}{p}\psi\right) - \left(\frac{\partial U}{\partial p} + \frac{\alpha^2}{p}U\right)\frac{\partial\psi}{\partial x} \quad \text{for} \quad p = p_0. \quad (2.9)$$

The energy equation for the perturbations, which is analogous to (1.9), now takes the form

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\int\overline{\left[(\nabla_h\psi)^2 + \frac{f_0^2p^2}{\alpha^2c_0^2}\left(\frac{\partial\psi}{\partial p}\right)^2\right]}dy\,dp + \int_{p=p_0}\frac{f_0^2p_0}{c_0^2}\overline{z^2}\,dy\right\} = \int\left(\overline{\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}}\frac{\partial U}{\partial y} + \frac{f_0^2p^2}{\alpha^2c_0^2}\overline{\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial p}}\frac{\partial U}{\partial p}\right)dy\,dp, \quad (2.10)$$

where the bar over the terms denotes the average value with respect to x. The second term on the right side is specific for baroclinic instability; it is proportional to the quantity $-\overline{v'\rho'}\partial\rho_0/\partial y$ and agrees with it in sign. Thus, e.g., for $\partial\rho_0/\partial y > 0$ it is necessary that $\overline{v'\rho'} < 0$ to have an instability, so that as perturbations grow the heavier particles ($\rho' > 0$), in sinking, must shift to the south ($v' < 0$) while the lighter particles ($\rho' < 0$) conversely must rise and shift to the north

($v'>0$). Concentrating in the following on this effect, we will assume that U depends only on p, and seek wave solutions ψ of Eq. (2.9) in the form $\psi(p)\exp[ik_1(k-ct)+ik_2y]$. Then, setting $k_1^2+k_2^2=k^2$, we obtain for the complex amplitude $\psi(p)$:

$$(U-c)\left(\frac{f_0^2}{c_0^2}\frac{\partial}{\partial p}\frac{p^2}{\alpha^2}\frac{\partial\psi}{\partial p}-k^2\psi\right)+B\psi=0, \qquad (2.11)$$
$$B=\frac{\partial f}{\partial y}-\frac{f_0^2}{c_0^2}\frac{\partial}{\partial p}\frac{p^2}{\alpha^2}\frac{\partial U}{\partial p},$$

and for $p=p_0$,

$$(U-c)\left(\frac{\partial\psi}{\partial p}+\frac{\alpha^2}{p}\psi\right)-F\psi=0,\quad F=\frac{\partial U}{\partial p}+\frac{\alpha^2}{p}U. \qquad (2.12)$$

Equation (2.11) differs from the Rayleigh equation essentially only in the term $\partial f/\partial y$. However, the boundary condition (2.12) is now more complex: It contains the eigenvalue c. Replacing it by the condition $\partial\psi/\partial p+(\alpha^2\psi/p)=0$, we can prove stability for $B>0$ (i.e., the Rayleigh theorem or a special case of the Charney-Stern theorem), or: if B changes sign once, then the flow is stable for $(U-K)B<0$ where K is the value of U at the point where B changes sign (an analog of the Fjortoft theorem). The condition (2.12), however, must give rise to instability, but only of a type in which there cannot be more than one growing wave solution for each K. We can prove this by a finite-difference approximation of equations (2.11), (2.12), in which the segment $0\leqslant p\leqslant p_0$ is broken up by the points $p_1,\dots,p_{N-1}$ into N equal parts of length $\delta=p_0/N$, and the equations are written in the equivalent form

$$(U_n-c)\left(r_{n-1/2}^2\frac{\psi_{n-1}-\psi_n}{\delta^2}-r_{n+1/2}^2\frac{\psi_n-\psi_{n+1}}{\delta^2}-k^2\psi_n\right) \qquad (2.11')$$
$$+B_n\psi_n=0,$$
$$(U_0-c)\left(\frac{\psi_0-\psi_1}{\delta}+s_0^2\psi\right)-F_0\psi_0=0, \qquad (2.12')$$

where $r_{n-1/2}^2$ is some average value of $f_0^2p^2/\alpha^2c^2$ between the points p_n and p_{n-1}, while $s_0^2=(\alpha_2/p)_{p=p_0}$. Then the following theorem holds (L A. Dikiĭ[22]): *If all $B_n>0$ or all $B_n<0$ (the Rayleigh condition), or if the sequence B_n changes sign once, and if there exists a constant K_0 for which $(U_n-K_0)B_n<0$ (the Fjortoft condition), then Eq.* (2.11') *with the boundary conditions*

$$\psi_N=\psi_{N-1}, \qquad (2.13)$$
$$\psi_1-\psi_0=K(c)\psi_0,\quad K(c)=a+\frac{K}{b-c},\quad a\geqslant 0, \qquad (2.14)$$

has no more than one pair of non-real complex-conjugate eigenvalues c.

The idea of the proof involves the fact that the eigenvalues are obtained graphically as intersections of the function $M(c)=(\psi_1-\psi_0)/\psi_0$ and the hyperbola $K(c)$, while $M(c)$ is a rational fraction whose denominator is a polynomial of degree $N-1$ without any roots that are not real.

3. BIFURCATIONS

The topological features of the phase flow $M(t)=F^tM(0)$, which describes the evolution of all fluid flows in a given geometry for all possible initial data, usually depend on some parameter μ which characterizes the degree to which the phase flow is out of equilibrium: in the case of a viscous fluid, this is the *Reynolds number* $\mathrm{Re}=\nu^{-1}LU$ (where L and U are typical length and velocity scales while ν is the kinematic coefficient of viscosity), i.e., a typical magnitude of the ratio of the inertial forces to the viscous forces, or something analogous to this ratio.

3.1. Topological features

We include among the distinctive features of the phase flow the non-wandering phase points (a point is non-wandering if any neighborhood of it is intersected by some phase trajectory at least twice; in particular, *fixed points* which correspond to steady-state solutions of the equations of hydrodynamics, are such points, along with *periodic points* which lie on closed trajectories which correspond to solutions periodic in time); *limit points of trajectories*:

$$M_\omega=\lim_{t_K\to\infty}F^{t_K}M$$

(if such limits exist) and the *limit sets* Ω_M made up of these points (if $M\in\Omega_M$, the point M is said to be *stable in the sense of Poisson*); *invariant sets* (filled by complete trajectories; this also includes their boundaries, so that we may consider some of them *closed* sets. Limit sets are closed invariant sets; a nonempty closed invariant set which has no subset which has this property is said to be *minimal*); *recurrent* points M, for which for any $\varepsilon>0$ there exists a $T>0$ such that an ε-neighborhood of the segment of the trajectory $\{F^tM\}$, $t\in[\tau,\tau+T]$ for every τ contains the entire trajectory (by the Birkhoff theorem: For a point M to be recurrent it is necessary and sufficient that the closed trajectories F^tM form a minimal set); *attractors*, i.e., minimal sets Λ of non-wandering points having neighborhoods for which all trajectories which originate in the neighborhoods asymptotically approach Λ (attractors which differ from finite sums of smooth manifolds are said to be *strange*).

The special features of phase flows enumerated above can be extremely varied. For example, in the simplest two-dimensional linear dynamic system $\dot{\mathbf{u}}=A\mathbf{u}$, depending on the eigenvalues λ_1,λ_2 of the matrix A the fixed point $\mathbf{u}=0$ can be a *vertex* (λ_1,λ_2 real and the same sign), a *saddle point* (λ_1 and λ_2 real and differing in sign), a *focus* (λ_1 and λ_2 complex conjugates) or a *center* (λ_1 and λ_2 imaginary and conjugate) (see Fig. 1). In particular, near a saddle point the trajectories are hyperbolae, both of whose asymptotes pass through the saddle point, with one of them serving as an axis of compression (the so-called *unstable manifold* of the saddle point, consisting of the point $\mathbf{u}=0$ and two leaving tra-

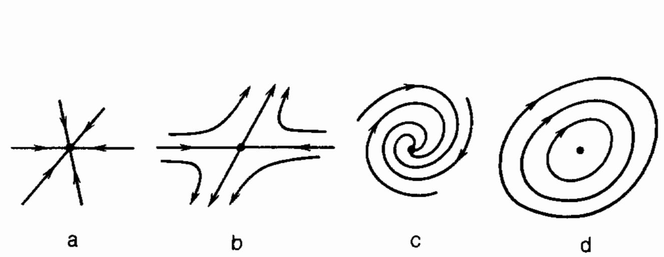

FIG. 1. Fixed points $u=0$ of the two-dimensional linear equation $\dot{\mathbf{u}}=A\mathbf{u}$: a—vertex, b—saddle point, c—focus, d—center.

jectories, i.e., the *unstable separatrixes*) and the other an axis of stretching (the stable manifold consisting of $\mathbf{u}=0$ and two entering trajectories, i.e., the *stable separatrixes*).

As the defining parameter μ is varied (e.g., growth of Re) the phase flow deforms, and for some critical values $\mu_{1cr}, \mu_{2cr},...$ some of its features appear or disappear, or undergo qualitative changes. Such changes in the topological features of the phase flow are called its *bifurcations* (or "catastrophes").

3.2. Interchange of stability

Let us exhibit the simplest bifurcations by considering the evolution of small perturbations $\mathbf{u}(\mathbf{x},t)$ of a steady-state flow $\mathbf{u}_0(\mathbf{x})$ of a viscous fluid, which are solutions of the linearized hydrodynamic equations:

$$\mathbf{u}(\mathbf{x},t)=A(t)\,\mathbf{f}_0(\mathbf{x}),\ A(t)=e^{-t\sigma},\ \sigma=\pm\omega+i\gamma. \quad (3.1)$$

For small Re, the steady-state flow $\mathbf{u}_0(\mathbf{x})$ usually constitutes a stable focus in the phase space. This means that all eigenvalues of the linearized equations have negative imaginary parts $\gamma<0$, so that any small perturbation (3.1) decays with time. As Re increases, the imaginary parts γ of some eigenvalues grow, and we can find some critical value Re_{1cr} for which some one of the eigenvalues of the linearized equations $\sigma(\mathrm{Re})$ first crosses the real axis of the complex σ-plane, i.e., $\gamma(\mathrm{Re}_{1cr})=0$. The corresponding perturbation (3.1) will be *neutral*, i.e., neither decaying nor growing with time.

It may turn out that in this case we will have $\omega(\mathrm{Re}_{1cr})=0$ at the same time, i.e., $\sigma(\mathrm{Re}_{1cr})=0$ as a whole; this means that $A(t)=1$ and $\mathbf{u}(\mathbf{x},t)=\mathbf{f}_0(\mathbf{x})$, i.e., the perturbed velocity field $\mathbf{u}_0(\mathbf{x})+\mathbf{u}(\mathbf{x},t)=\mathbf{u}_0(\mathbf{x})+\mathbf{f}_0(\mathbf{x})$ describes the field of a new steady-state flow; it is then said that for $\mathrm{Re}=\mathrm{Re}_{1cr}$ a bifurcation occurs, i.e., an *interchange-of-stability*. This bifurcation is observed, e.g., in the development of *thermal convection* in a fluid layer heated from below (where the state of rest $\mathbf{u}_0(\mathbf{x})=0$ is first transformed into *steady-state* convection in the form of "rollers" or Bénard cells), and also in a *Taylor flow*, i.e., a circular Couette flow between two coaxial rotating cylinders (where stationary laminar flow is transformed into steady-state toroidal rolling Taylor vortices). We will discuss these flows in detail in what follows.

3.3. Normal Andronov-Hopf bifurcation

If $\omega(\mathrm{Re}_{1cr})=\omega_1\neq 0$, then the perturbation (3.1) for $\mathrm{Re}=\mathrm{Re}_{1cr}$ is a neutral wave. For $\mathrm{Re}>\mathrm{Re}_{1cr}$ there will exist eigenvalues σ with positive imaginary parts $\gamma>0$, i.e., perturbations which grow exponentially with time, so that the flow $\mathbf{u}_0(\mathbf{x})$ will be unstable relative to small perturbations. E. Hopf[23] proved that in phase spaces of dynamic systems of a quite general form there exists a one-parameter family of closed trajectories for values of Re in a certain neighborhood of Re_{1cr}; the application of this theorem to the hydrodynamic equations was proved by N. N. Brushlinskaya.[24] The appearance of closed trajectories for $\mathrm{Re}>\mathrm{Re}_{1cr}$ (these are limit cycles, which correspond to flows which are periodic in time), i.e., the conversion of a stable focus into a limit cycle (Fig. 2a) is called a *normal Hopf bifurcation* (this bifurcation was discovered 13 years earlier by A. A. Andronov).

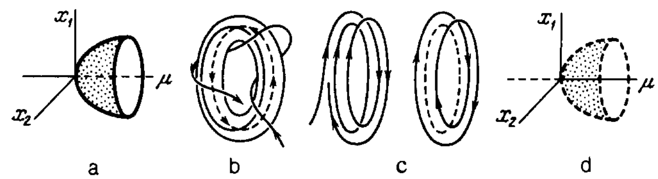

FIG. 2. a—Normal Andronov-Hopf bifurcation. b—Creation of a two-dimensional invariant torus. c—Period-doubling bifurcations. d—Inverse Andronov-Hopf bifurcation.

3.4. The Landau equation

L. D. Landau[3] described the transition of an unstable small perturbation (3.1) to a periodic flow. While the perturbation is small, its amplitude $A(t)$ satisfies the equation $\partial|A|^2/\partial t=2\gamma|A|^2$, but for finite $|A|$ the right side must contain additional terms from its expansion in powers of A and A^*. The high-frequency oscillations given in (3.1) with frequencies $|\omega_1|\gg\gamma$ should be excluded by averaging over time (i.e., over a period τ from the interval $[2\pi|\omega_1|^{-1},\gamma^{-1}]$); then the cubic terms drop out, while of the fourth power terms only those which are proportional to $|A|^4$ are saved. To this accuracy we obtain the following Landau equation and its solution:

$$\frac{\partial|A|^2}{\partial t}=2\gamma|A|^2-\delta|A|^4,\qquad \delta>0, \quad (3.2)$$

$$|A(t)|^2=A_0^2A_\infty^2\,[A_0^2+(A_\infty^2-A_0^2)\,e^{-2\gamma t}],\quad A_\infty=\left(\frac{2\gamma}{\delta}\right)^{1/2} \quad (3.3)$$

For a small initial value A_0 the amplitude $|A(t)|$ initially grows exponentially according to the linear theory as $A_0e^{\gamma t}$; subsequently, this growth slows down, and for $t\to\infty$ it reduces to a limiting value A_∞ which does not depend on A_0, which for small $\mathrm{Re}-\mathrm{Re}_{1cr}$ is proportional to $(\mathrm{Re}-\mathrm{Re}_{1cr})^{1/2}$ (since $\gamma\sim\mathrm{Re}-\mathrm{Re}_{1cr}$ and $\delta\neq 0$). Thus, for small $\mathrm{Re}-\mathrm{Re}_{1cr}>0$ the perturbation reduces as time passes to a periodic oscillation $\mathbf{u}_1(\mathbf{x},t)$ with finite amplitude A_∞ and arbitrary phase (determined by the random initial phase of the perturbation and thus constituting a degree of freedom of the limiting flow).

3.5. Bifurcation of periodic flows

If we write the hydrodynamic equations linearized around a periodic solution $M(t)$ with period T_1 in the symbolic form $\partial M'/\partial t=f'M'$, where f' is a bounded linear operator which is continuous and periodic with period T_1 as a function of t, then for every perturbation $M'(t)$ of the periodic solution we will have $M'(t+T_1)=U(T_1)M'(t)$, where $U(T_1)$ is a bounded linear operator, referred to as the *monodromy operator*. Its eigenvalues ρ_n (Re) are the so-called *multipliers*; one of them is trivially equal to unity and need not be considered henceforth. If all $|\rho_n|<1$, then every perturbation for each circuit of the closed trajectory decreases, so that the periodic flow is stable; if, however,

$|\rho_n|>1$ for at least one n, then it is unstable. Thus, as Re increases the bifurcations of periodic motions occur by passage of the multipliers ρ_n (Re) in the complex plane through the unit circle.

This means that as Re grows we may reach a new critical value Re_{2cr} for which a pair of multipliers assume the values $\exp(\pm i\alpha)$ (where $\alpha\neq 0,\pi,\pi/2,2\pi/3$ so that resonances are excluded). Then a second kind of normal bifurcation occurs: a periodic flow $\mathbf{u}_0(\mathbf{x})+\mathbf{u}_1(\mathbf{x},t)$ becomes unstable relative to some perturbation of the form $e^{-i\sigma t}\,\mathbf{f}_1(\mathbf{x},t)$, where $\mathbf{f}_1$ is a function periodic in time with period $2\pi/\omega_1$, while the eigenvalue σ has a real part $\pm\omega_2$. For fairly large $\mathrm{Re}-\mathrm{Re}_{2cr}$ this perturbation will grow with time to a finite limit—a quasiperiodic motion with two periods $2\pi/\omega_1$ and $2\pi/\omega_2$, and two degrees of freedom (the oscillation phases). Thus, from the closed trajectory we form trajectories on a two-dimensional torus (Fig. 2b). If then a subsequent normal bifurcation follows, the trajectories lie on a three-dimensional torus, etc.

If for $\mathrm{Re}=\mathrm{Re}_{2cr}$ one of the multipliers passes through the unit circle at the point $\rho=-1$, then $U(T_1)M'(t)=-M'(t)$, i.e., a small perturbation simply changes sign after one circuit around the trajectory $\mathbf{u}_0(\mathbf{x})+\mathbf{u}_1(\mathbf{x},t)$. Then after a second such circuit we obtain $M'(t+2T_1)=-U(T_1)M'(t)=M'(t)$, i.e., the perturbed trajectory is closed. Thus, in this case for $\mathrm{Re}=\mathrm{Re}_{2cr}$ there occurs *bifurcation by period doubling*—from a periodic motion of period T_1 there arises a stable periodic motion with twice the period $2T_1$, while the original motion becomes unstable (Fig. 2c). In this same way, we can have a subsequent bifurcation with period doubling, etc.

3.6. The inverse Andronov-Hopf bifurcation

If the one-parameter family of closed trajectories predicted by the Andronov-Hopf bifurcation theorem appears already for $\mathrm{Re}<\mathrm{Re}_{1cr}$, then the coefficient δ in the Landau expansion (3.2) must be negative while the coefficient $\gamma\sim\mathrm{Re}-\mathrm{Re}_{1cr}$ will be negative for $\mathrm{Re}<\mathrm{Re}_{1cr}$ and positive for $\mathrm{Re}>\mathrm{Re}_{1cr}$. This means that for $\mathrm{Re}<\mathrm{Re}_{1cr}$ Eq. (3.2) takes the form

$$\frac{\partial\,|A|^2}{\partial t}=-2|\gamma|\,|A|^2+|\delta|\,|A|^4, \tag{3.3}$$

from which it is clear that the closed trajectory is unstable: trajectories lying inside it spiral in toward a fixed point [putting this another way, perturbations with small amplitude $|A|<A_1=(2|\gamma|/|\delta|)^{1/2}$ decay with time], while trajectories outside it spiral outward and move to another region of the phase space (that is, perturbations with finite amplitudes $|A|>A_1$ grow with time so that for $\mathrm{Re}_{1cr}>\mathrm{Re}>\mathrm{Re}_{1cr}-\alpha^2|A|^2$ the motion turns out to be unstable relative to such perturbations). For increasing $\mathrm{Re}<\mathrm{Re}_{1cr}$ the closed trajectory shrinks, and when Re passes through the value Re_{1cr} it disappears—this phenomenon is called *inverse bifurcation* (Fig. 2d). For $\mathrm{Re}>\mathrm{Re}_{1cr}$ Eq. (3.2) with the coefficients $\gamma>0$ and $\delta<0$ has the solution

$$|A(t)|^2=A_0^2A_1^2\,[(A_0^2+A_1^2)\,e^{-2\gamma t}-A_0^2]^{-1},\quad A_1=\left(\frac{2\gamma}{|\delta|}\right)^{1/2}, \tag{3.4}$$

which becomes infinite at a finite time $t_1=(1/2\gamma)\ln[1+(A_1^2/A_0^2)]$. However, it is clear that even before this the Eq. (3.2) ceases to be useful and must be supplemented by the next terms in the Landau expansion. Available examples show that after an inverse bifurcation, for $\mathrm{Re}>\mathrm{Re}_{1cr}$ the motion apparently will rapidly acquire a nonperiodic character.

4. INSTABILITY OF FLOWS OF VISCOUS LIQUIDS

We first address the question of the stablility of steady-state plane-parallel flows of an incompressible viscous fluid with a velocity $\mathbf{u}_0=\{U(z),0,0\}$, so that a perturbation of the velocity field $\mathbf{u}(\mathbf{x},t)$ will satisfy Eq. (1.7) with the addition to the left side of the equation of motion of the term $\nu\Delta\mathbf{u}$ which describes the acceleration of the viscous forces. Let us refer to this as Eq. (1.7').

4.1. The Orr-Sommerfeld equation

Then for two-dimensional elementary wave solutions of these equations, in place of the Rayleigh equation (1.1) we obtain the following so-called *Orr-Sommerfeld equation*:

$$(U-c)\left(\frac{\partial^2\psi}{\partial z^2}-k^2\psi\right)-\frac{\partial^2U}{\partial z^2}\psi$$
$$=-\frac{i\nu}{k}\left(\frac{\partial^4\psi}{\partial z^4}-2k^2\frac{\partial^2\psi}{\partial z^2}+k^4\psi\right), \tag{4.1}$$

in dimensionless variables this equation will have the same form and only ν will be replaced by $(\mathrm{Re})^{-1}$. We note that if we eliminate the unknowns u, v, and p' from the four equations (1.7'), which describe a *three-dimensional* wave perturbation depending on x, y, and t through $\exp(ik_1[x-ct]+ik_2y)$, we can obtain an equation like (4.1) for w in which k^2 is simply replaced by $k_1^2+k_2^2$ and ν by νkk_1^{-1} (that is, Re by $k_2k^{-1}\mathrm{Re}$). This means that if for a given value Re of the Reynolds number there is an unstable three-dimensional wave perturbation with eigenvalue c, then there exists a two-dimensional wave perturbation with the same c which is unstable at a smaller Reynolds number $k_1k^{-1}\mathrm{Re}$ (H. Squire[25]). Let us note in passing that in eliminating unknowns we lose solutions of Eq. (1.7') for which $w=0$, that is to say solutions for which $k_1u+k_2v\equiv 0$. In this case, u and v satisfy the equation

$$\frac{\partial^2u}{\partial z^2}-k^2u-\frac{ik_1}{\nu}(U-c)\,u=0 \tag{4.2}$$

with boundary condition $u=0$ for $z=0,h$. For $k_2\neq 0$ this equation has an additional spectrum of eigenvalues c which is absent from that of the two dimensional perturbations; however, it corresponds only to stable perturbations, i.e., $\mathrm{Im}\,c<0$ (V. A. Romanov[26]). Actually, if we multiply (4.2) by u^*, integrate over z and take the real part of the resulting equation, we obtain

$$-\int_0^h\left(\left|\frac{\partial u}{\partial z}\right|^2+k^2|u|^2\right)dz-(\mathrm{Im}\,c)\frac{k_1}{\nu}\int_0^h|u|^2\,dz=0, \tag{4.3}$$

from which it is also clear that $\mathrm{Im}\,c<0$. Thus, as Re increases the two-dimensional wave perturbations lose their stability before the three-dimensional ones, and in analyzing stability with respect to small wave perturbations it is sufficient to study the eigenvalue spectrum of the Orr-Sommerfeld Eq. (4.1) with the boundary conditions $\psi = \partial\psi/\partial z = 0$ at $z = 0,h$. Furthermore, in contrast to the Rayleigh Eq. (1.11), Eq. (4.1) has no singularities and apparently any two-dimensional solution to Eq. (1.7′) is a superposition of elementary wave solutions (although a complete proof of this assertion has not been published).

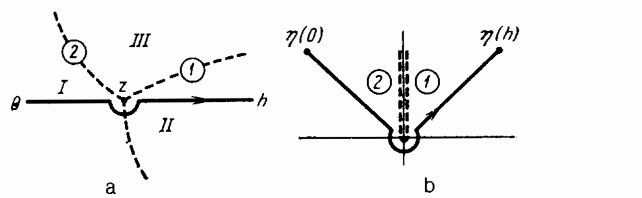

FIG. 3. a—Stokes lines for real c. b—Their image under the mapping $\eta(z)$.

4.2. The vanishing-viscosity principle

It turns out that the following *vanishing-viscosity principle* holds: Wave solutions to Eq. (1.7) which are either neutral or growing with time are the limits of corresponding solutions to Eqs. (1.7′) as $\nu \to 0$, while solutions to (1.7′) which decay reduce to the corresponding solutions to (1.7) only outside of some range of values of z (the "inner boundary layer"), within which their behavior is determined by the viscosity, no matter how small it is.

So as to formulate this principle in terms of the eigenvalues c of the Orr-Sommerfeld equation, we assume that the function $U(z)$ is determined not only within the segment $0 \leqslant z \leqslant h$ but is also analytically continued to some neighborhood of this segment in the complex z plane. We introduce the function $r(z) = [i(U-c)]^{1/2}$. Let us refer to arcs in the z-plane which connect the points $z = 0$ and $z = h$ as allowable if along them $U - c \neq 0$ and $\mathrm{Re} \int r dz$ varies monotonically. We will say that c is an eigenvalue along the arc if for this c there exists a solution to the Rayleigh equation which is analytic along this arc and reduces to zero at its ends.

Then the vanishing-viscosity principle follows from the following theorem: if as $\nu \to 0$ an eigenvalue of the Orr-Sommerfeld equation tends to such a limit c for which we can construct an allowable arc in the z-plane, then c is an eigenvalue of the Rayleigh equation along this arc; conversely, if c is an eigenvalue of the Rayleigh equation along some allowable arc, then this c is the limit as $\nu \to 0$ of eigenvalues of the Orr-Sommerfeld equation (C. Lin,[27] W. Wasow[28]).

So as to determine what sort of arc is allowable, let us consider first of all a real c, and let z_c be a root of the equation $U(z) = c$. Then in a neighborhood of z_c we have $U - c \approx U_c'(z - z_c)$. Let $U(z)$ be a monotonically-increasing function so that $U_c' > 0$. We set

$$\eta(z) = \int_{z_c}^{z} r\, dz \approx e^{\pi i/4} (U_c')^{1/2} \times \frac{2}{3} |z - z_c|^{3/2} \exp\left[\frac{3i}{2} \arg(z - z_c)\right]. \quad (4.4)$$

The three curves $\mathrm{Re}\,\eta = 0$ (the so-called *Stokes lines*) emerge from the critical point $z = z_c$ at equal angles and divide its vicinity into sectors I, II in which the interval of $0 \leqslant z \leqslant h$ lies, and III which does not contain it (Fig. 3a). Any two sectors are mapped by the function $\eta(z)$ into a plane cut along a ray whose "cut edges" are the images of the Stokes lines. In Fig. 3b we show the image of sectors I and II on the plane cut along the positive part of the imaginary axis. In this plane, the points $\eta(0)$ and $\eta(h)$ can be connected by a curve on which $\mathrm{Re}\,\eta$ varies monotonically. Its image in the z plane circles the critical point from below; therefore this is an allowable arc.

For $U'(z) < 0$, on the contrary, we must circle the critical point from above. For a profile $U(z)$ symmetric with respect to the point $z = h/2$, there are two critical points, and one must be circled from below (when $U_c' > 0$) and the other from above (when $U_c' < 0$). If $\mathrm{Im}\, c > 0$ (instability) and at the critical point $U' > 0$ on the real axis, then z_c and all of sector III lie above, while for $U' < 0$ they lie below the real axis, which thus is also an allowable arc. If, however, $\mathrm{Im}\, c < 0$ (stability), then the interval $0 \leqslant z \leqslant h$ intersects the three sectors, and $\mathrm{Re}\,\eta$ varies nonmonotonically along it, and the direction to circle the critical point on the allowable arc (along with all of sector III) is determined by the sign of U' near it, the same as above. The eigenfunctions of the Orr-Sommerfeld equation in sectors I and II reduce to the eigenfunctions of the Rayleigh equation; in sector III, however, they apparently oscillate rapidly for small ν; in the neighborhood of the intersection of the Stokes lines with the real axis inner boundary layers are formed.

4.3. Planar Couette and Poiseuille flows

Passing to the investigation of specific plane-parallel flows, we begin with plane-parallel Couette flow with a linear velocity profile $U(z) = Az$. According to all calculations performed in the past, this flow is linearly stable, that is no normal bifurcations occur from it, and $\mathrm{Re}_{1cr} = \infty$ (although a full proof of this has not been published). At the same time, it is well known from experiments that Couette flow is apparently unstable relative to finite perturbations which lie inside a certain "neutral surface" in the three-dimensional space of the parameters (k, Re, A). This region was approximately analyzed by S. Kuwabara[29] and T. Ellingsen, B. Gjevik and E. Palm.[30] According to the first of these authors, the instability occurs only for $\mathrm{Re} = (1/2)\nu^{-1}Ah^2 > \mathrm{Re}_{cr,\,min} \approx 45\,000$ in rather small regions of the (k,A) plane.

Planar Poiseuille flow, with a parabolic velocity profile $U(z) = 4U_{max}(z/h)(1 - z/h)$ in an ideal fluid, is linearly stable; it was observed by W. Heisenberg[31] that this flow possesses a linear *instablity* in a viscous fluid for large Re (normal bifurcation), which at first sight appears paradoxical (since it would seem that viscosity can only be a stabilizing factor). However, subsequently the existence of this instability was verified by C. Lin[32] and other authors, who investigated the "neutral curve" in the (k,Re) plane within which $\gamma > 0$ [where k is measured in units of $(h/2)^{-1}$,

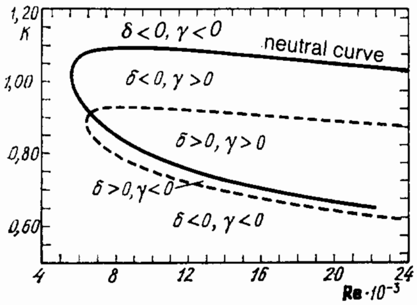

FIG. 4. Instability regions for planar Poiseuille flow from S. Pekeris and B. Shkoller (1967).[34]

$\mathrm{Re} = (1/2)\nu^{-1}hU_{\max}$ and γ is given in (3.1)]; see the continuous curve shown in Fig. 4. In this case a value of $\mathrm{Re}_{1\mathrm{cr}} \approx 5800$ was obtained, and $k_{\mathrm{cr}} \approx 1$, while both branches of the neutral curve approach the abscissa $k = 0$ as $\mathrm{Re} \to \infty$, the upper curve as $\mathrm{Re}^{-1/11}$, the lower as $\mathrm{Re}^{-1/7}$. So, perturbations with fixed and not-too-large k first turn out to be stable, and then as Re increases they pass into the region of instability; however, for very large Re (including $\mathrm{Re} = \infty$, i.e., the ideal fluid) they again become stable. We should also mention Poiseuille-Couette flow

$$U(z) = U\left(\frac{h}{2}\right)\left[(4-\alpha)\frac{z}{h} - (4-2\alpha)\left(\frac{z}{h}\right)^2\right]. \quad (4.5)$$

For $\alpha = 0$ we obtain Poiseuille flow; as α increases, $\mathrm{Re}_{1\mathrm{cr}}$ increases rapidly and becomes infinite at α as small as ≈ 0.55—far from the Couette flow regime.

However, planar Poiseuille flow is nevertheless not the best example of a normal bifurcation: the experimental data do not confirm the existence of the "neutral curve" mentioned above, but rather show that the loss of stability occurs at a value $\mathrm{Re} \sim 1000$–2500, i.e., much lower than the $\mathrm{Re}_{1\mathrm{cr}}$ of linear theory. This causes us to suspect the presence of an inverse bifurcation, and thus instability relative to perturbations of finite amplitude. And, in fact, W. Reynolds and M. Porter,[33] along with S. Pekeris and B. Shkoller,[34] have investigated the coefficient $\delta = \delta_1 + \delta_2 + \delta_3$ in the Landau expansion (δ_1 describes the influx of energy from a more unstable perturbation which deforms the laminar flow, δ_2 the generation by this perturbation of higher harmonics and δ_3 the distortion of its form) and found $\delta < 0$ (because of δ_3) everywhere in the (k,Re) plane except in the region bounded in Fig. 4 by the dotted curve. Thus, the curves $\gamma(k,\mathrm{Re}) = 0$ and $\delta(k,\mathrm{Re}) = 0$ divide the plane into four regions with various combinations of signs of γ and δ. Particularly interesting is the region $\gamma > 0$, $\delta > 0$, in which finite-amplitude periodic motions can exist. According to D. Meksyn,[35] $\mathrm{Re}_{\mathrm{cr,min}} \sim 1000$ is obtained for certain three-dimensional flows.

4.4. Instability of the boundary layer

In contrast, a normal bifurcation has been quite reliably observed in the flow within the boundary layer next to a flat surface, which is treated in calculations as approximately plane-parallel. The "neutral curve" for this bifurcation was first calculated by W. Tollmien[36] in 1929, for a profile $U(z)$ made up of linear and parabolic segments, and in 1930 for the Blasius profile (for $\mathrm{Re}_{1\mathrm{cr}} = 420$, where $\mathrm{Re} = \nu^{-1}\delta^* U$; δ^* is the so-called displacement thickness and U is the (uniform) flow velocity of the approaching fluid). This same calculation was performed again by H. Schlichting, and later by other authors; its validity was confirmed experimentally by G. Schubauer and H. Skramstad,[37] who used perturbations $\psi(z)\exp[i(kx - \omega t)]$, produced by a vibrator with a fixed (real) frequency ω, which for $\mathrm{Im}\, k < 0$ grow downstream (however, since $\omega = ck$ the "neutral curves" $\mathrm{Im}\, k = 0$ and $\mathrm{Im}\, \omega = 0$ coincide). The "neutral curve" here is qualitatively the same as in Fig. 4, but later calculations showed that for velocity profiles $U(z)$ with inflection points the flow is more unstable: for $\mathrm{Re} \to \infty$ the upper branch of the "neutral curve" tends toward an asymptote $k = k_\infty > 0$, so that a perturbation with $k < k_\infty$ which is unstable for some $\mathrm{Re}_{\mathrm{cr}}$ remains unstable for all $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ up to the ideal fluid (see Fig. 8b below).

The experiments of P. Klebanoff, K. Tidstrom and L. Sargent[38] along with other authors showed that later evolution of the unstable Tollmien-Schlichting wave in the boundary layer near a flat surface led for some $\mathrm{Re}_{2\mathrm{cr}}$ to a second normal bifurcation—the superimposing of a three-dimensional flow onto the initially two-dimensional flow, which is periodic in the transverse y direction and has a group velocity along x which is close to the phase velocity of the first wave. This secondary wave grows downstream with extraordinary speed, creating vortices with longitudinal axes which give rise to an abrupt transverse redistribution in the intensities of pulsations in the longitudinal velocity u (Fig. 5); the wave then becomes nonlinear. This last effect leads to focusing of the secondary wave packet at the crest of the first wave, and also to the disappearance of those parts of the secondary wave with positive anomalies in the longitudinal velocity $U > 0$, leaving only the parts with negative pulses $U < 0$ (this effect was explained by M. Landahl[39]). First a single pulse forms within one cycle of oscillation of the vibra-

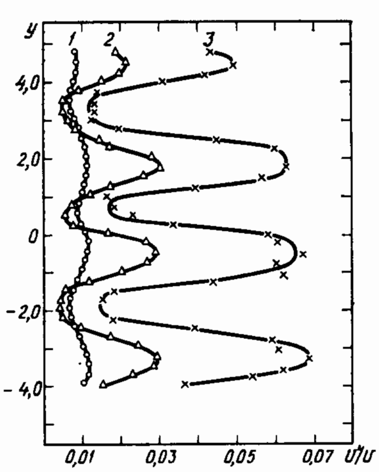

FIG. 5. The growth downstream of pulsations in the longitudinal velocity in a secondary wave within the boundary layer above a planar surface. 1—for $x = 7.6$ cm; 2—for $x = 15.2$ cm; 3—for $x = 19$ cm.

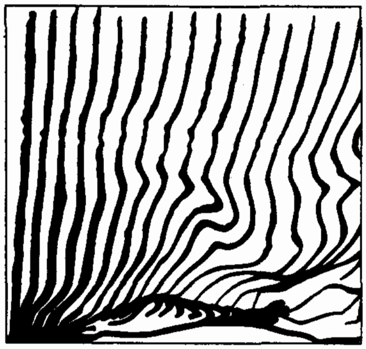

FIG. 6. Bubble "isochrones" (i.e., lines of constant time: side view) in a boundary layer near a planar surface, generated by a vertical filament (the left edge of the photograph; the flow is from left to right).

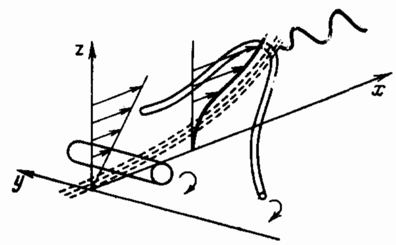

FIG. 7. Formation of a horseshoe-shaped vortex.

tor, then two downstream etc.; apparently, after the formation of four pulses in a cycle the flow becomes chaotic.

Making the flow visible by using chains of micron-sized bubbles periodically generated by electric current in a platinum filament and microinjection of a dye (S. Klein *et al.*[40–45]), and also with the help of suspended particles (E. Corino and R. Brodky[46]), indicated the presence of two interacting small-scale flows in the viscous sublayer next to the wall: vortices with transverse axes and the same sign as U', and streams of decelerated fluid in the troughs of the secondary transverse waves.

These streams are formed at heights $z^+ = \nu^{-1}u_* z = 2.5 - 10$ (u_* is the "velocity of friction" at the wall), and have a width $\delta y^+ = 10$–30 and are separated by spacings of $\Delta y^+ \sim 100$. They slowly float to the surface under the action of the longitudinal vortices, but, because of the negative pressure gradients under the transverse vortices above them drifting by, they break free of the wall and penetrate upward into the more rapidly-moving fluid, creating in the instantaneous velocity profile deformations with points of inflection (Fig. 6). After this, oscillations arise in the stream and shortly thereafter its end "explodes," creating chaotic motion (mainly at a height $z^+ = 10$–30 and at a distance $\delta x^+ \sim 1000$–1500 down from the separation point). It is calculated that these "explosions" are responsible for almost all the generated turbulent energy.

When separation of the stream occurs, the central part of the transverse vortex connected with it floats up and departs downstream, creating a "horseshoe" (Fig. 7; the longitudinal cross section of the "leg" of this "horseshoe" is visible in the photograph in Fig. 6). The upper part of the "horseshoe," which overtakes from above, the next downstream decelerating stream causes a negative pressure gradient over it; this leads to its separation and generates a new "horseshoe." The superposition of two "horseshoes" causes them to combine; more often, however, they cross and generate an "explosion": this leads to local "randomization" and transfer of energy to the small-scale region of the spectrum.

4.5. Flows in an unbounded region

Of the various plane-parallel flows in an unbounded region we first discuss the planar jet, e.g., with a velocity profile $U(z) = U_0[\cosh(z/h)]^{-2}$, which is always unstable relative to antisymmetric perturbations, with $\mathrm{Re}_{\mathrm{cr,min}} = (\nu^{-1}hU_0)_{\mathrm{cr}} \approx 3.7$ and $hk_{\mathrm{cr}} \approx 0.25$; as Re grows, the upper branch of the "neutral curve" increases monotonically (Fig. 8c). The laminar "mixing zones" which smooth out a tangential discontinuity in the velocity are unstable for any Re: the lower branch of their "neutral curves" coincides with the entire positive axis $k = 0$, while the upper branch rises monotonically with increasing Re (Fig. 8d).

4.6. Poiseuille flow in a pipe

For axially-symmetric flows, we will limit ourselves here to considerations of Poiseuille flow in a circular pipe. In this case, the question of eigenvalues of the Orr-Sommerfeld type of equation, which generalizes the Rayleigh Eq. (1.22) to the case of a viscous fluid, turns out to be very complex mathematically, and has been analyzed only in a few special cases (axially-symmetric perturbations with $n = 0$ and a few others).

Unstable perturbations of this flow have not been observed, so that Poiseuille flow in a pipe, like planar Couette flow, is apparently linearly stable, i.e., $\mathrm{Re}_{1\mathrm{cr}} = \infty$.

At the same time, experiments starting with those of Reynolds himself (in 1883) showed beyond any doubt that this flow will always remain laminar, i.e., regardless of the size of the perturbation at the entrance of the pipe, only if $\mathrm{Re} = 2\nu^{-1}RU_{\max} < \mathrm{Re}_{\mathrm{cr,min}} \sim 2000$; above a certain $\mathrm{Re}_{\mathrm{cr}}$ it loses its stability—evidently relative to finite perturbations, since a decrease in the initial perturbation can "drag" the laminar regime out to very large Re (according to W. Pfanniger,[47] even out to $\mathrm{Re} = 100\,000$). The loss of stability oc-

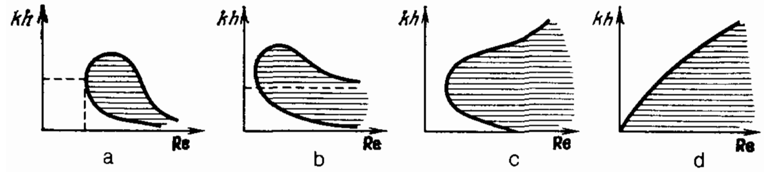

FIG. 8. Regions of linear instability in the (kh, Re) plane: a—for plane Poiseuille flow; b—for a boundary layer near a plane surface; c—for a planar jet; d—for a planar mixing zone.

curs in the form of the appearance of intermittent "turbulent bottlenecks" which are short with respect to x but which occupy the entire cross section of the pipe, and which moving downstream lengthen and fuse with one another.

The region of instability in the space of the parameters (k, Re, A) for this flow was studied by A. Davey and H. Nguyen.[48] We note that in the presence along the axis of the pipe of a rod on the surface of which the velocity has to become zero the flow behaves in a manner similar to that of planar Poiseuille flow as shown in Fig. 4. It acquires both a region of linear instability and a region of inverse bifurcation.

4.7. Wakes behind bodies

To conclude Section 4, we enumerate the bifurcations of a wake behind a cylindrical body of circular cross-section perpendicular to the incident current, as a generic example of the behavior of wakes behind bodies with a viscous fluid streaming past them. For $\mathrm{Re} \sim 10$ an *exchange of stability* occurs: in place of a monotonic smooth stream behind the cylinder, a pair of steady-state vortices forms. For $\mathrm{Re} > 40$, these vortices alternately detach from the cylinder, and are replaced by new vortices; the detached ones flow downstream, forming a *Karman vortex street.* For $\mathrm{Re} > 100$ the vortices are rapidly replaced by turbulent regions which alternately detach from the boundary layers. For $\mathrm{Re} > 10^5$ the boundary layers are turbulent even prior to detachment, and the detachment point moves downstream. The turbulent wake then constricts and the drag decreases (the *drag crisis*). For $\mathrm{Re} \sim 10^6$ the turbulent wake widens and the drag increases again.

Finally, for $\mathrm{Re} \sim 10^7$ the wake begins to oscillate as a whole. If the fluid has a free surface, all these phenomena can change their form, and so-called "ship waves" can be superimposed on them in addition. In a stratified fluid they will all be accompanied by the generation of various kinds of internal waves.

5. RANDOMIZATION

In a viscous fluid, a finite number of lower (large-scale) modes of the motion determines all the remaining modes, since the higher (small-scale) modes are strongly damped due to viscosity and only replicate with decreased amplitude the fundamental modes of oscillation (in particular, they have the same kind of spectrum—discrete or continuous).

5.1. Finite dimensionality

In connection with these issues, E. Hopf[4] has advanced the hypothesis that every set of phase trajectories of the Navier-Stokes equation is attracted for $t \to \infty$ to a *finite-dimensional* set. For two-dimensional flows of a viscous fluid this hypothesis can be proved (C. Foias and G. Prandtl,[49] O. A. Ladyzhenskaya[50]). Let us recall the natural estimate of the number of degrees of freedom in fully-developed (i.e., large Re) local three-dimensional turbulence $N \sim (\mathrm{Re}/\mathrm{Re}_{cr})^{9/4}$ (L. D. Landau and E. M. Lifshitz[51]). An analogous estimate for two-dimensional turbulence with spectral transfer of "entrophy" (i.e., the square of the vorticity) to the small-scale region $N \sim \mathrm{Re}/\mathrm{Re}_{cr}$ is less accurate, since this spectral transfer is not entirely localized. A more general estimate of the dimension of the attractors of the two-dimensional Navier-Stokes equations will be given in Sec. 5.7 below. Thus, the hydrodynamic equations of a viscous fluid can be written in the form

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}, \mathrm{Re}), \quad \mathbf{u} = (u^1(t), \ldots, u^N(t)), \tag{5.1}$$

where $u^k(t)$, e.g., are the coefficients in the Galerkin approximation.

5.2. Dissipation

Viscosity gives rise not only to finite-dimensionality of the phase space but also to *dissipation* of the phase flow, i.e., an average compression of the phase volume "downstream." Let $\delta V(\mathbf{u}_0, 0) = \prod_{k=1}^{N} \delta u_0^k$ be a small initial element of the phase volume around the point $\mathbf{u}_0$ and $\delta V(\mathbf{u}_0, t)$ be its value after displacement down the phase flow over a time t. From small t we will have $\delta V(\mathbf{u}_0, t) = \prod_{k=1}^{N} \delta u^k$ where $\delta u^k \approx (\partial u^k/\partial u_0^k)\delta u_0^k$. Since $\partial \delta u^k/\partial t \approx (\partial \dot{u}^k/\partial u_0^k)\delta u_0^k$, the relative rate of change of δV with time equals

$$\Lambda(\mathbf{u}_0) \equiv (\delta V)^{-1} \frac{\partial(\delta V)}{\partial t} = \sum_{k=1}^{N} (\delta u^k)^{-1} \frac{\partial(\delta u^k)}{\partial t} \approx \sum_{k=1}^{N} \frac{\partial \dot{u}^k}{\partial u_0^k} = \operatorname{div} \mathbf{F}(\mathbf{u}_0). \tag{5.2}$$

At different phase points $\mathbf{u}_0$ this quantity can be either positive (expansion) or negative (compression). The phase flow is called dissipative if for every $\mathbf{u}_0$

$$\Lambda_0(\mathbf{u}_0) = \lim_{t \to \infty} t^{-1} \ln \left| \frac{\delta V(\mathbf{u}_0, t)}{\delta V(\mathbf{u}_0, 0)} \right| < 0 \tag{5.3}$$

(or, for a more general definition, if every sphere of sufficiently large radius with its center at the origin in phase space is an absorbing region). Because of dissipation, attractors have zero phase volume (and dimensionality smaller than N).

5.3. Definition of randomness

Of special interest to us here are the strange attractors, on which phase trajectories display the following properties of randomness:

(1) An extremely sensitive dependence on initial conditions, due to *exponential divergence* of trajectories which are initially close together (and leading to their unpredictability or nonreproductibility for initial conditions which are given with arbitrarily high (but finite) precision). (2) The everywhere-denseness at the attractor of almost all trajectories, i.e., their arbitrarily close approach to any of the attractor's points (which implies that they return infinitely often to the attractor), and the property that any initial nonequilibrium probability distribution (measure) over the phase space (or, more precisely, over the region of attraction of the strange attractor) reduces to some limiting equilibrium distribution at the attractor (an invariant measure). (3) The *mixing*

property: For any (measurable) subsets A and B of the attractor, the probability after emerging from A of arrival at B is proportional after a long time to the measure of B:

$$\lim_{t\to\infty} P\{(F^t A)\cap B\} = P(A)\,P(B), \tag{5.4}$$

where the symbol $\cap$ denotes set intersection. A consequence of the mixing property is the fact that the time-averaged value $\langle\Phi[u(t)]\rangle$ of any function $\Phi(u)$ defined on the strange attractor is independent of the initial conditions $\mathbf{u}_0$ (for almost all $\mathbf{u}_0$), and that this average value coincides with the average $\overline{\Phi}(u)$ over the invariant measure (*ergodicity*):

$$\langle\Phi\rangle \equiv \lim_{T\to\infty} T^{-1}\int_0^T \Phi[\mathbf{u}(t)]\,dt = \int \Phi(\mathbf{u})\,P(d\mathbf{u}) \equiv \overline{\Phi}. \tag{5.5}$$

One mark of the mixing property is a rather *rapid decay of the correlation functions* as $\tau\to\infty$:

$$B^{jl}(\tau) = \langle[u^j(t) - \langle u^j\rangle][u^l(t+\tau) - \langle u^l\rangle]\rangle \tag{5.6}$$

which is to say *continuity* of their Fourier transforms with respect to τ, i.e., their *spectral functions*.

It appears expedient to have the term *turbulence* refer to the random evolution [in the sense of (1)–(3) above] of the flow of a (viscous) fluid which possesses *vorticity*. Stochastic *potential* flows of a fluid are by preference referred to as *random wave fields*, while for nonhydrodynamic systems one should preferably restrict oneself, where necessary, to the adjective *stochastic*.

5.4. Hyperbolicity

Let us now focus on the first property of randomness. Exponential divergence of nearby trajectories as the phase volume is compressed is possible if expansion occurs along some directions u^k in the phase space $\{\mathbf{u}\}$ and compression occurs along others, i.e., the nonwandering phase points must be similar to two-dimensional saddle points. Such points are called *hyperbolic*. A fixed point $\mathbf{u}_0$ is hyperbolic if the Jacobian $A(\mathbf{u}_0) = \{\partial F^k/\partial u^l\}$ at this point has K eigenvalues with positive and N–K eigenvalues with negative real parts, where $0<K<N$. The sets $W^s_{\mathbf{u}_0}$ and $W^n_{\mathbf{u}_0}$ of phase points $\mathbf{u}_0$, through which trajectories pass tending to $\mathbf{u}_0$ as $t\to+\infty$ and $t\to-\infty$, are referred to as the stable and unstable manifolds of the point $\mathbf{u}_0$, respectively; in a small neighborhood of the point $\mathbf{u}_0$ they are subspaces, spanned by the N–K and K eigenvectors of the matrix $A(\mathbf{u}_0)$ which correspond to its eigenvalues with negative and positive real parts, respectively.

Points of intersection of the stable and unstable manifolds $W^s_{\mathbf{u}}$ and $W^n_{\mathbf{u}}$ which differ from the point $\mathbf{u}$ itself are called *homoclinic* (while intersections of the stable manifold $W^s_{\mathbf{u}}$ for one fixed point $\mathbf{u}$ with the unstable manifold $W^n_{\mathbf{v}}$ for another fixed point $\mathbf{v}$ are called *heteroclinic points*). In their neighborhood, the structure of the phase flow can be particularly complex (see below).

A periodic trajectory is a hyperbolic set if some multipliers (see Sec. 3) are within while others are outside the unit circle (we again do not consider the trivial multiplier one). The sets W^s and W^n of trajectories attracted to a limit cycle as $t\to+\infty$ and $t\to-\infty$ are referred to as the stable and unstable manifolds of the limit cycle. If the sum of their dimensions equals N–1, then their intersection is said to be *transversal*.

As an illustration of homoclinic structures, let us consider the intersection of a stable and an unstable manifold of a periodic trajectory in three-dimensional phase space. Here it is convenient to use the so-called *Poincare mapping*, which in its general form for an N-dimensional phase space involves listing the sequence of points $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ ... of intersections of trajectories (in the same direction, without tangency) with some *secant-like* N-1 dimensional surface Σ in the phase space, which determines a mapping $\mathbf{v}_{n+1} = \Pi(\mathbf{v}_n, \mathrm{Re})$ of the surface Σ into itself. Because a solution $\mathbf{u}(\mathbf{u}_0,t)$ of Eq. (5.1) exists for all t, this mapping is *invertible*.

In the case $N=3$, the secantlike surface Σ can be taken to be some plane. A periodic trajectory on Σ corresponds to a fixed point of the mapping Π, while the stable and unstable manifolds correspond to the stable and unstable separatrixes (Fig. 9, in which for clarity we have connected the sequence of intersection points of Σ with the phase trajectory with smooth curves). If there is a common point on the separatrixes, then all its images as $n\to\infty$ and preimages as $n - \infty$ are also intersection points of the separatrixes, so that they form a denumerable set. They belong to the so-called *homoclinic* trajectory, which is doubly asymptotic in the sense that for $t\to\pm\infty$ it either unwinds from the original periodic trajectory or winds upon it. On an approach to a fixed point where the motion becomes exponentially slow, the homoclinic points "condense," while the amplitudes of oscillation of the separatrixes increases. In the neighborhood of a homoclinic trajectory, any small phase volume as $t\to\pm\infty$ undergoes a complex deformation and "blurring," so that almost all trajectories diverge exponentially. This local instability of the trajectories enclosed within a bounded volume of phase space is what leads to a particularly complex phase flow which here includes a denumerable set of periodic trajectories and a nondenumerable set of trajectories doubly asymptotic to them.

An invariant set is hyperbolic in general if at each of its points $\mathbf{u}$ the space $T_{\mathbf{u}}$ tangent to the phase space (i.e., the linear space of tangent vectors at the point $\mathbf{u}$) is a direct sum of the one-dimensional subspace $E^1_{\mathbf{u}}$ spanned by the phase velocity vector and the stable and unstable subspaces $E^s_{\mathbf{u}} = \{\xi\}$ and $E^n_{\mathbf{u}} = \{\eta\}$ such that the action on their elements of the differential of the phase flow DF^t (which is a

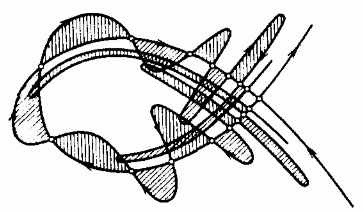

FIG. 9. Homoclinic trajectory on a secant-like surface. The shaded parts of the plane are mapped into one another.

linear mapping from $T_{\mathbf{u}}$ to $T_{\mathbf{F}^t\mathbf{u}}$) obeys the inequalities.

$$\begin{aligned} &\|\xi\|^{-1}\|\mathbf{D}\mathbf{F}^t\xi\| \leqslant ae^{-ct}, \quad \|\eta\|^{-1}\|\mathbf{D}\mathbf{F}^t\eta\| \geqslant be^{ct} \quad \text{for} \quad t\geqslant 0,\\ &\|\xi\|^{-1}\|\mathbf{D}\mathbf{F}^t\xi\| \geqslant be^{-ct}, \quad \|\eta\|^{-1}\|\mathbf{D}\mathbf{F}^t\eta\| \leqslant ae^{ct} \quad \text{for} \quad t\leqslant 0, \end{aligned} \tag{5.7}$$

where a, b, c are certain positive constants which do not depend on $\mathbf{u}$. The phase points $\mathbf{v}$ for which $\mathbf{F}^t\mathbf{v}$ unrestrictedly approaches $\mathbf{F}^t\mathbf{u}$ as $t\to\infty$ (or as $t\to-\infty$) form the stable manifold $W^s_{\mathbf{u}}$ (or the unstable manifold $W^n_{\mathbf{u}}$) of the point $\mathbf{u}$; this manifold is tangent at this point to $E^s_{\mathbf{u}}$ (or $E^n_{\mathbf{u}}$). The union of these manifolds for all $\mathbf{u}$ for a given trajectory makes up the corresponding manifolds W^s and W^n for that trajectory. Their intersections, just as above, form homoclinic structures.

5.5. Structural stability

Thus, exponential divergence of neighboring trajectories in dissipative systems is connected with the presence of hyperbolic sets in the phase spaces of these systems. Are these sets characteristic of many dynamic systems, or are they, on the contrary, exceptions? In the latter case, a small perturbation of such a system (say, by the "noise" always present in nature) would deprive it of this property. In connection with this, it is useful to employ the concept introduced by A. A. Andronov and L. S. Pontryagin[56] of a *structurally stable* (or "coarse-grained") system, which for (5.1) is formulated as follows: For any $\varepsilon>0$ there is a $\delta>0$ such that for any perturbed system $\dot{\mathbf{u}}=\mathbf{F}_1(\mathbf{u})$ which is "distant" from the original system (in terms of a certain metric $\|\mathbf{F}_1-\mathbf{F}\|$) by no more than δ there exists a mutually single-valued and mutually continuous transformation of the phase space into itself, which shifts its points by no more than ε and maps the trajectories of the unperturbed system into the trajectories of the perturbed system. Structurally stable systems form an open set in the space of all possible dynamic systems.

In one- and two-dimensional phase spaces the so-called *M. Morse-S. Smale systems* possess structural stability. In these systems, the sets of nonwandering points consist only of a finite number of fixed points and closed trajectories, which are all hyperbolic, and the stable and unstable manifolds corresponding to any such points are transversal (i.e., either they do not intersect, or the sum of spaces tangent to them at each point $\mathbf{u}$ of their intersection forms the complete tangent space $T_{\mathbf{u}}$).

In phase spaces with a large number of dimensions, the hypothesis of S. Smale[53] says that in order to have structural stability, it is necessary and sufficient that for each transformation F^t of the phase space arising from the phase flow the set Ω of nonwandering points should be hyperbolic, while the set of periodic points should be everywhere dense in Ω (this is the so-called "axiom A"); furthermore, each stable and each unstable manifold of points from Ω should be transversal. The sufficiency of this condition has been proved in a quite general form; the necessity, however, has at this time been proved only within the context of a more restricted definition of structural stability.

Thus, for $N>3$ it is in some sense typical for phase flows to have an infinite set Ω of hyperbolic nonwandering points, with a set of periodic trajectories everywhere dense in Ω (D. V. Ansonov[54] has even found flows in which the entire phase space forms a hyperbolic set).

5.6. Cantor sets

The set Ω can have an extremely complicated geometric structure. For example, Smale[55] has proved that in a wide class of typical dynamic systems, each homoclinic point belongs to some invariant subset K of the set Ω which is a "Cantor discontinuum," i.e., a closed nowhere-dense set with no isolated points.

The standard example of a Cantor set is constructed in the following way: (1) remove the *middle third* of the segment [0,1], i.e., the open (without endpoints) interval (1/3,2/3); this gives us a set K_1 made up of the two segments [0,1/3] and [2/3,1]; (2) remove from each of the two segments of the set K_1 its middle third; this gives us the set K_2 consisting of four segments; ... (n) remove from each of the 2^{n-1} segments of K_{n-1} its middle third; we obtain the set K_n consisting of 2^n intervals; etc. The intersection of all the sets K_n then forms the Cantor set K. If each number on the interval [0,1] is written in base 3, i.e., as $\sum_{n=1}^{\infty} a_n 3^{-n}$ where all the a_n take one of the values 0.1, and 2, then K consists of points for which all the a_n equal to either 0 or 2. This means that a one-to-one correspondence can be set up between K and the set of all binary sequences, i.e., the entire interval [0,1]. That is, the set K is nondenumerable (it has the power of the continuum).

As an example of the formation of a Cantor set in phase space, let us consider a dynamic system which in a fixed time T gives rise to a mapping $\Pi(\mathbf{u})$ which takes the interior U of a two-dimensional torus into itself, such that $\Pi(\mathbf{u})$ is the interior of a torus contained in U with one loop, as shown in Fig. 10. The circle S, being the cross-section of the body U, thus is mapped into two small circles $\Pi(S)$ within S. A second iteration $\Pi^2(S)$ gives two even smaller circles inside the circles of $\Pi(S)$, etc. The intersection of all the iterates $\Pi^n(S)$ gives a Cantor set of points in S, so that the intersection of all the iterates $\Pi^n(U)$ is a Cantor set of curves—the so-called *one-dimensional solenoid* of R. Williams.

5.7. Fractality

The Cantor set is *fractal*, i.e., its *Hausdorff dimension* $\dim_{\mathrm{H}} K$ exceeds the usual *topological* dimension (which in its case equals zero). The quantity $\dim_{\mathrm{H}} K$ is defined using the

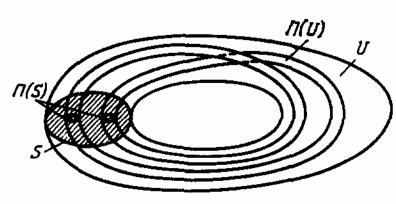

FIG. 10. A mapping $\Pi(U)$ of the interior U of a two-dimensional torus into itself which gives rise to a strange attractor.

Hausdorff α-measure of the set A:

$$\mathrm{mes}_{\mathrm{H},\alpha}(A)=\lim_{\varepsilon\to 0}\inf_{\Gamma(A)}\sum[D(U)]^{\alpha}, \tag{5.8}$$

where the lower bound is taken over finite or denumerable covers Γ of the set A by spheres U with diameters $D(U)\leqslant\varepsilon$. The dimension $\dim_{\mathrm{H}}K$ is defined as that number α_0 such that the measure (5.8) for $\alpha>\alpha_0$ equals zero and for $\alpha<\alpha_0$ is infinite. If in (5.8) we use only covers made up of spheres of diameter ε and we take the upper or lower bound (i.e., the largest or smallest limit for a subsequence of values $\varepsilon\to 0$), then we obtain the upper *capacitive* α-measure $\overline{\mathrm{mes}}_{c,\alpha}(A)$ or the lower *capacitive* α-measure $\underline{\mathrm{mes}}_{c,\alpha}(A)$; then $\overline{\dim}_c A$ or $\underline{\dim}_c A$ is the lower bound of values of α for which the upper or lower measure equals zero (or the upper bound of values α for which the measure equals infinity). Then $\dim_{\mathrm{H}}A\leqslant\underline{\dim}_c A\leqslant\overline{\dim}_c A$. If the latter two values coincide here, then they define the *capacity* of the set A:

$$\dim_c A=\lim_{\varepsilon\to 0}\frac{\ln N_\varepsilon(A)}{\ln(1/\varepsilon)}, \tag{5.9}$$

where $N_\varepsilon(A)$ is the smallest number of spheres of diameter ε which cover the set A. The sets K_N used above to construct K consist of $N_\varepsilon=2^n$ intervals of length $\varepsilon=3^{-n}$, so that $\dim_c K=\lim(\ln 2^n)/(\ln 3^n)=\ln 2/\ln 3\approx 0.631$.

Yu. S. Il'yashenko,[97–99] O. A. Ladyzhenskaya,[100] and A. V. Babin and M. I. Vishik,[101–103] using rigorous methods, have obtained estimates of the Hausdorff dimension of the attractors for two-dimensional Navier-Stokes equations with periodic boundary conditions (in Refs. 97, 98 for the Galerkin approximations to these equations), expressed in the form $C_1\mathrm{Re}\leqslant\dim\Lambda\leqslant C(\mathrm{Re})$.[4]

5.8. Lyapunov exponents

The exponential divergence of neighboring trajectories averaged over time can be qualitatively characterized by the so-called *Lyapunov exponents*, for which at points $\mathbf{u}(t)$ of a trajectory with initial conditions $\mathbf{u}(0)=\mathbf{u}_0$ we introduce the tangent vector $\mathbf{w}=\mathbf{w}(\mathbf{w}_0,t)$ with initial conditions $\mathbf{w}(\mathbf{u}_0,0)=\mathbf{w}_0$ so that $\|\mathbf{w}(\mathbf{u}_0,t)\|$ characterizes the projection onto the direction $\mathbf{w}$ of the distance at time t between trajectories with neighboring initial points $\mathbf{u}_0$ and $\mathbf{u}_0+\mathbf{w}_0$. The vector $\mathbf{w}$ satisfies Eq. (5.1) linearized relative to $\mathbf{u}(t)$, which is of the form $\dot{\mathbf{w}}=A[\mathbf{u}(t)]\mathbf{w}$ where $A=\{\partial F^k/\partial u^l\}$ is the Jacobian at the point $\mathbf{u}(t)$. This equation has a complete system of fundamental solutions $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_N$, and for each of these we determine its own Lyapunov exponent:

$$\sigma_i(\mathbf{u}_0)=\lim_{t\to\infty}t^{-1}\ln\frac{\|\mathbf{w}_i(\mathbf{u}_0,t)\|}{\|\mathbf{w}_i(\mathbf{u}_0,0)\|}, \tag{5.10}$$
$$\|\mathbf{w}_i(0)\|\to 0.$$

We list the solutions in order of decreasing σ: $\sigma_1\geqslant\sigma_2\geqslant\ldots\geqslant\sigma_N$. One of the first researchers to use the theory of Lyapunov exponents to analyze random motions was V. I. Oseledets.[104] The divergence of the phase flow (5.3) equals $\Lambda_0=\sum_{i=1}^{N}\sigma_1$, so that for dissipative flows not only $\sigma_N<0$ but in addition the modulus of the sum of negative σ_i is larger than the sum of positive σ_i. If a trajectory $\mathbf{u}(t)$ is attracted to a fixed point (or to a periodic or quasiperiodic trajectory), then all σ_i (or all except $\sigma_1=0$, where the vector $\mathbf{w}_1$ is directed along the limiting trajectory) are negative.

If a trajectory is attracted to a strange attractor Λ, then necessarily $\sigma_1>0$. If on the attractor there exists an invariant measure and ergodicity holds, then the time-averaged values in (5.10) do not depend on $\mathbf{u}_0$ and can be replaced by an average over the invariant measure. Let k be an integer such that $\sum_{i=1}^{k}\sigma_i\geqslant 0$ and $\sum_{i=1}^{k+1}\sigma_i<0$. Then the quantity

$$\dim_L\Lambda=k+\sum_{i=1}^{k}\frac{\sigma_i}{|\sigma_{k+1}|}. \tag{5.11}$$

is referred to as the *Lyapunov dimension* of the attractor Λ. T. Li and J. Yorke[56] have advanced the hypothesis that this quantity coincides with the Hausdorff dimension of the set Λ (defined as the lower bound of the Hausdorff dimensions of sets of unit invariant measure, which is a bound for the Lebesgue measure in phase space). Numerical calculations for several two-dimensional mappings and a single three-dimensional flow have shown that the quantities (5.11) and (5.9) practically coincide.

6. SCENARIOS FOR RANDOMIZATION

Until now, the specific sequence of bifurcations which leads as Re increases to the conversion of steady-state (laminar) flow to random (turbulent) flow has not been identified, with the required degree of rigor, for even a single viscous-fluid flow geometry. The material presented in Sec. 4 shows that for different flow geometries such bifurcation sequences can apparently be extremely varied. The most probable hypotheses concerning the sequence of bifurcations which lead to randomness we will call *scenarios*; let us investigate a few of these.

6.1. Landau-Hopf scenario

This scenario involves a sequence of normal bifurcations which gives rise to a limiting quasiperiodic flow $\mathbf{u}[\mathbf{x},\varphi_1(t),\ldots,\varphi_N(t)]$. This flow has a period of 2π in each of the arguments $\varphi_k(t)=\omega_k t+\alpha_k$ with frequencies $\omega_1,\ldots,\omega_N$ which are in general incommensurate, and occupies a region in phase space corresponding to all possible sets of initial conditions (phases) $\alpha_1,\ldots,\alpha_N$. The flow is ergodic in the sense that a trajectory belonging to such a flow will in the course of time come arbitrarily close to any point of this region (because at times $t_k=2\pi k/\omega_1$, $k=0,1,2,\ldots$, where $\varphi_1(t)=\alpha_1$, the phase of any other oscillation $\varphi_2(t_k)=2\pi k\omega_2/\omega_1$ after reduction to the interval $[0,2\pi]$ can have a value arbitrarily close to any previously given point).

The temporal velocity correlation functions will in general not reduce to zero at infinity; however, they at first decrease rapidly (as $N^{-1/2}$), and the time T until the next maximum (the *Poincare recurrence time*) is extremely long: $T\sim e^{\alpha N}$ where $\alpha\sim 1$ (V. I. Arnol'd[57]). Much worse is the fact that the sequence of normal bifurcations and the resulting quasiperiodic motion do not possess structural stability (J. Eckmann[58]), and are atypical even in the sense that phase flows which are subject to this scenario do not form a *Baire set* in the space of all phase flows (i.e., a denumerable intersection of open everywhere-dense sets). In connection with this, G. Sell[59] suggests that we make this scenario more precise as a sequence of bifurcations of K-dimensional tori $T^k \to T^{k+1}$, without requiring that the flow on these tori be quasiperiodic. Such bifurcations are possible, and in the presence of strange attractors they also possess structural stability in a certain sense.

6.2. Ruelle-Takens scenario

This scenario involves the appearance of a strange attractor after three normal bifurcations. It possesses structural stability: for every sufficiently small neighborhood of a phase flow with an n-dimensional invariant torus, for $n \geqslant 3$ there exists an open set of phase flows with a strange attractor satisfying axiom A (while the "smallness" of the neighborhood is understood in the sense of the C^{n-1} norm:

$$\|\mathbf{F}\| = \max_{\substack{(\mathbf{u}) \\ 0 \leqslant k \leqslant n-1}} |D^{(k)}\mathbf{F}|,$$

where $D^{(k)}\,\mathbf{F}(\mathbf{u})$ denotes any derivative of order k, and for $n \geqslant 4$ in the sense of the C^∞ norm). In 1971 this theorem was proved for $n \geqslant 4$ (and attracted universal attention to the study of randomizing processes in dissipative systems), while for the case $n = 3$ it was generalized in a paper by the same authors (Ruelle and Takens) together with S. Newhouse.[61]

The proof of the Ruelle-Takens theorem is based on the possibility of approximating a flow with torus T^k by a flow with a closed trajectory wound around the torus (where all ω_i/ω_k for $i = 1, ..., k-1$ are small rational fractions), into which is imbedded a Cantor attractor, e.g., of the Williams solenoid type. In other words, in this mechanism the three-frequency motion is destroyed by means of nonlinear synchronization (generation of resonances) of its higher harmonics. This mechanism is apparently too "soft" (in the words of B. V. Chirikov in the book of Ref. 9, too "gentle") for turbulence to arise. In the experiments which have been performed, the observed randomization is more similar to disruption of a two-frequency motion T^2—i.e., probably through synchronization of *beating* between the two periods, and then either period-doubling bifurcations of the resulting cycle or coalescence and disappearance of stable-saddle-point cycles (and formation of an attractor from the homoclinic structure of the saddle-point cycle or from folds in an originally nonsmooth torus).

6.3. Feigenbaum scenario

This scenario involves the appearance of a strange attractor as a result of an infinite sequence of period-doubling bifurcations. Let us investigate these bifurcations first on the example of a one-dimensional non-invertible (single-valued and continuous) mapping $x_{n+1} = \Pi(x_n, \mu)$ of the segment $0 \leqslant x \leqslant 1$ into itself, where the functions Π has one quadratic extremum on this interval, which we will assume is a maximum. Iterations of such a mapping are conveniently portrayed on a so-called Königs-Lameria diagram (Fig. 11), on

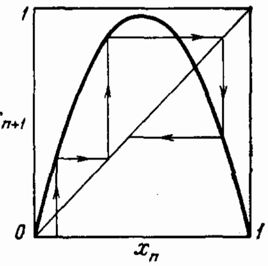

FIG. 11. The Königs-Lameria diagram.

which the value x_n is plotted on the abscissa and the value x_{n+1} is plotted on the ordinate; the graph of $\Pi(x)$ is drawn along with the bisectrix of the coordinate angle, and between these two curves the *Lameria ladder* is constructed, so that the image x_{n+1} becomes the pre-image of x_{n+2}, $n = 0,1,2,...$.

The point $x_* = \Pi(x_*)$ where the graph of $\Pi(x)$ intersects the bisectrix is a fixed point of the mapping. x_* is *stable* if there is a sufficiently small neighborhood of x_*, all points of which converge to x_* when iterated. A sufficient condition for this is $|\Pi'(x_*)| < 1$ (in this case, when the graph of $\Pi(x)$ intersects the bisectrix at an angle either smaller than 45°—for which the iterations converge monotonically to x_*, or larger than 135°—for which the iterations converge in an oscillatory way). The set of points $x_{i+1} = \Pi(x_i) \neq x_1$, $i = 1,2, ..., n-1$ where $x_{n+1} = \Pi(x_n) = x_1$ form an n-fold cycle (its Lameria ladder is closed). Each of its points is a fixed point of the n-fold-iterated map Π_n; this cycle is stable if $|\Pi_n'(x)| < 1$ for one of these points. Retaining only the quadratic part of $\Pi(x)$, the mapping can without loss of generality lead to the *logistic equation*

$$x_{n+1} = \mu x_n\,(1 - x_n). \tag{6.1}$$

We are interested only in the range $1 < \mu \leqslant 4$ (since for $\mu \leqslant 1$ all trajectories are attracted to the point $x = 0$, while for $\mu > 4$ values $\Pi(x) > 1$ appear). For these values of μ the mapping has two fixed points—an unstable one $x = 0$ and the point $x_0 = 1 - \mu^{-1}$ whose stability is determined by the eigenvalue of the linearized version of Eq. (6.1). It is not difficult to convince oneself that $\lambda = 2 - \mu$, so that the region of stability $|\lambda| < 1$ of the point x is $1 < \mu < 3$. For $\mu \geqslant \mu_1 = 3$ this point is unstable and in addition a two-fold cycle appears—the two roots x_1^0, x_1^1 of the equation $x = \Pi_2(x)$ which are different from x_0. A plot of this function shows two maxima, while the minimum between them is entirely similar to the inverted graph of $\Pi(x)$: for $\mu_1 < \mu < \mu_2 \approx 3.45$ the points x_1^0 and x_1^1 are stable, while for $\mu \geqslant \mu_2$ they become unstable and in addition in their neighborhood two twofold cycles (x_2^0, x_2^1) and (x_2^2, x_2^3) of the mapping Π_2 appear, generating a fourfold cycle for the mapping Π which for $\mu_2 < \mu < \mu_3$ is stable, etc.

Thus, for the values μ_n, $n = 1,2,3, ...$, period-doubling bifurcations occur: a 2^{n-1}-fold cycle loses stability while a stable 2^n-fold cycle appears. M. Feigenbaum observed that the sequence μ_n converges (to a limit $\mu_\infty \approx 3.57$) asymptotically as a geometric progression with a rather large geomet-

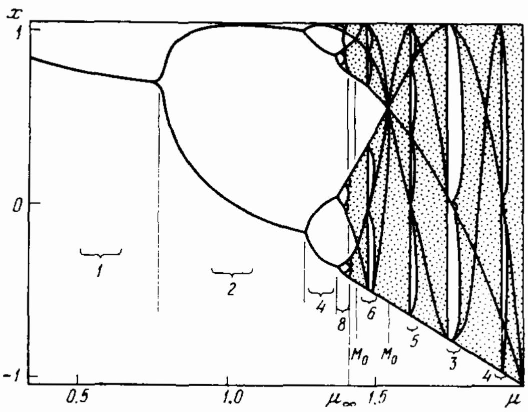

FIG. 12. Bifurcation diagram for the mapping $x_{n+1} = 1-\mu x_n^2$ of the segment $-1 \leqslant x \leqslant 1$ into itself for $0.35 \leqslant \mu \leqslant 2$.

ric ratio:

$$(\mu_n - \mu_{n-1})(\mu_{n+1} - \mu_n)^{-1} \sim \delta = 4.6692\ldots . \quad (6.2)$$

Furthermore, using functional equations he proved that this law and the value of δ are *universal*: they are correct not only for the mapping (6.1) but also for any mapping $\Pi(x)$ of the general form described above. Thus, knowing μ_0 and μ_1, we can predict

$$\mu_\infty \approx \mu_0 + \frac{\delta}{\delta - 1}(\mu_1 - \mu_0).$$

M. Feigenbaum proved this universality also for the distribution on the segment $0 \leqslant x \leqslant 1$ of the periodic points of the sequence of $2^0, 2^1, 2^2, 2^3, \ldots$-fold cycles x_0, (x_1^0, x_1^1), $(x_2^0, x_2^1, x_2^2, x_2^3), \ldots$. These points are stable on the intervals $3 \leqslant \mu < \mu_1, \mu_1 \leqslant \mu < \mu_2, \mu_2 \leqslant \mu < \mu_3, \ldots$; on a plot $x(\mu)$ is represented by segments of curves which at the points $\mu_1, \mu_2, \mu_3, \ldots$ divide (asymmetrically) in two, in sum forming a "tree" [(Fig. 12), constructed numerically for a quadratic mapping equivalent to (6.1)]. The universality of this "tree" consists of the fact that each sequential doubling is similar to the preceding one with similarity coefficients for the different branches equal to $(-\alpha^{-1})$ or α^{-2}, where $\alpha = 2.5029\ldots$ (the similarity is also asymptotic; however, it is already established practically after the first few iterations:

$$\frac{x_n^k - x_n^{k+2^{n-1}}}{x_{n+1}^k - x_{n+1}^{k+2^n}} = \begin{cases} -\alpha, & 0 \leqslant k < 2^{n-1}, \\ \alpha^2, & 2^{n-1} \leqslant k < 2^n. \end{cases} \quad (6.3)$$

For large n, replacing (by interpolation) the sequence $x_n^0, x_n^1, \ldots$ by a function with continuous argument $x_n(t)$ and representing it in the form of a Fourier series:

$$x_n(t) = \sum_k X_n^k e^{2\pi i k t/2^n}, \quad (6.4)$$

we can establish universality of the similarity law for the amplitude $|X_n^k|$. Thus, we obtain

$$X_{n+1}^k = \frac{1}{2^{n+1}} \int_0^{2^{n+1}} x_{n+1}(t)\, e^{-2\pi i k t/2^{n+1}}\, dt$$
$$= \frac{1}{2 \cdot 2^n} \int_0^{2^n} [x_{n+1}(t) + (-1)^k x_{n+1}(t+2^n)]\, e^{-\pi i k t/2^n} dt. \quad (6.5)$$

For even $k = 2l$, we obtain from (6.3) that $x_{n+1}(t) \approx x_{n+1}(t+2^n) \approx x_n(t)$ and then (6.5) takes the form $X_{n+1}^{2l} \approx X_n^l$, i.e., at this frequency the Fourier amplitude does not change for all succeeding bifurcations. For odd $k = 2l+1$, we derive from (6.3)–(6.5) the recursion relation

$$X_{n-1}^{2l+1} = -\frac{1}{2\alpha}[1-(-1)^l i]\left[1+(-1)^l \frac{i}{\alpha}\right] S,$$
$$S = \frac{1}{\pi i} \sum_m \left[(2m+1) - \frac{1}{2}(2l+1)\right]^{-1} X_n^{2m+1}. \quad (6.6)$$

Assuming that the modulus of the amplitude X_n^{2m+1} is a smooth function of m and that its phase is random, when we replace $2l+1$ by a continuous argument ξ in (6.6) we obtain the similarity law

$$X_{n+1}(\xi) = \frac{1}{\gamma}\left|X_n\left(\frac{\xi}{2}\right)\right|,$$
$$\gamma = 2\alpha^2(1+\alpha^2)^{-1/2} \approx 4.65. \quad (6.7)$$

For $\mu > \mu_\infty$, at certain μ there appear trajectories (in pairs—one stable, one unstable) for periodic motion (whose periods are, in order of occurrence, 1,6,5,3, ... ; see Fig. 12), each of which then undergoes a series of period-doubling bifurcations with its own limit point, In addition, there now appear on the segment $0 \leqslant x \leqslant 1$ bands of random motion, while for the values $\mu_\infty < \ldots < \mu_n^* < \mu_{n-1}^* < \ldots < \mu_1^*$ these bands undergo inverse period-doubling bifurcations for which the number of bands decreases by a factor of 2 while the bands themselves broaden (and fuse), according to a similarity law with the same constants δ and α as were given above. Thus, after the $(n+1)$st bifurcation, the mean-square width of a band equals $W_{n+1} = [(\alpha^{-2}/2) + (\alpha^{-4}/2)]^{1/2} W_n$, from which $W_n = W_0 \beta^{-n}$ where $\beta = \gamma/\sqrt{2} \approx 3.29$. Thus, the dispersion of the random part of the motion is proportional to $W^2 \sim \beta^{-2n} \sim (\mu_n^* - \mu_\infty)^m$ where $m = 2 \ln \beta / \ln \delta \approx 1.544\ldots$ [here we have used the similarity law (6.2)]. Thus, randomness does not appear discontinuously but rather grows gradually as μ increases (and not monotonically, but rather interrupted by regions in which self-organizing periodic motion occurs).

The one-dimensional mapping has a single Lyapunov exponent (5.10):

$$\sigma(x_0, \mu) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \ln \left| \frac{\partial \Pi(x_n, \mu)}{\partial x_n} \right|. \quad (6.8)$$

It is independent of x_0 almost everywhere; for $\mu < \mu_\infty$ it is negative (since a limit cycle is present), while for $\mu - \mu_\infty > 0$ it is predominantly positive and is initially proportional to $(\mu - \mu_\infty)^k$, where $k = \ln 2/\ln \delta \approx 0.4498\ldots$ (B. Huberman and J. Rudnick[62]), and then becomes a complicated function of μ.

The invariant measure (probability density) $p(x)$ for a mapping $\Pi(x)$ with one extremum, as a consequence of conservation of the "number of trajectories," satisfies the relation $p(x)dx = p(x_1)dx_1 + p(x_2)dx_2$, where x_1 and x_2 are two preimages of the point x. From this we obtain the functional equation

$$p(x) = p(x_1)\left|\frac{\partial\Pi(x_1)}{\partial x_1}\right|^{-1} + p(x_2)\left|\frac{\partial\Pi(x_2)}{\partial x_2}\right|^{-1}, \quad (6.9)$$

which can be solved numerically (by the iterative method). For a triangular mapping $\Pi(x)$ with $\Pi(1/2) = 1$, the obvious solution is $p(x) = 1$. Mapping (6.1) for $\mu = 4$ is converted into a triangular mapping $\Pi(\bar{x})$ if we replace the variable $\bar{x} = 2\pi^{-1}\arcsin x^{1/2}$, so that for (6.1) we have

$$p(x) = \frac{\partial\bar{x}}{\partial x} = \pi^{-1}[x(1-x)]^{-1/2} \quad (6.10)$$

(whose Landau exponent equals $\sqrt{2}$). Random motion with such an invariant measure is found to have the mixing property. For $\mu_\infty \leqslant \mu < 4$ the distributions $p(x)$ turn out to be complicated (in particular, they have gaps). For $\mu = \mu_\infty$ the attractor is ergodic, but it does not possess the mixing property (J. Eckmann[58]), which seems to be a deficiency of this scenario.

Let us now investigate the two-dimensional mapping $x_{n+1} = \Pi(x_n, \mu)$ where $x = (x,y)$ and the Jacobian $\partial(x_{n+1}, y_{n+1})/\partial(x_n, y_n)$ is different from zero (an invertible mapping) and whose modulus is less than unity (i.e., it is dissipative). As an example, we can consider the mapping

$$x_{n+1} = \Pi(x_n, \mu) + y_n, \quad y_{n+1} = \beta x_n, \quad |\beta| < 1, \quad (6.11)$$

where $\Pi(x)$ is a non-invertible one-dimensional map. The Jacobian of (6.11) equals $-\beta$, and, since the Jacobian of the kth iteration $(-\beta)^k \to 0 (k \to \infty)$, it is clear that the limiting attractor lies in the neighborhood of the curve $x = \Pi(y/\beta)$. Bifurcations of the two-dimensional mappings are subject to the same universal Feigenbaum laws, so that it is sufficient to limit ourselves to the quadratic mappings which, it is not difficult to show, for a constant Jacobian always can be brought to the form (6.11) with a function $\Pi(x)$ of the form (6.1).

Such a mapping has two fixed points—the unstable one $x = y = 0$ and the stable one $x = 1 - (1-\beta)\mu^{-1}, y = \beta x$. For a certain $\mu = \mu_1(\beta)$ the second of these points loses its stability and in addition a twofold cycle appears, i.e., a fixed point of the second iteration $x_{n+2} = \Pi_2(x_n)$. In the neighborhood of its fixed points this second iteration, by renormalizing the values of x, μ and β, leads to the same functional form as the original mapping (while the renormalized $\beta_2 = \beta^2$). Therefore there occurs a sequence of period-doubling bifurcations with the Feigenbaum asymptotic law of similarity involving the same constants δ and α, along with the same point of accumulation μ_∞ (for $\beta_\infty = 0$) and subsequent behavior analogous to what was described above for $\mu > \mu_\infty$. For the following mapping, which is equivalent to Eqs. (6.11) and (6.1):

$$x_{n+1} = 1 - \mu x_n^2 + y_n, \quad y_{n+1} = \beta x_n \quad (6.12)$$

M. Henon[63] has computed 5×10^6 iterations (for $\mu = 1.4$ and $\beta = 0.3$), which make up a set of curves on the (x,y) plane which has all the appearances of a Cantor structure, although the randomness of this attractor has not been rigorously proved yet (this has been proved by M. Misiurewicz[64] when x_n^2 is replaced in (6.12) by $|x_n|$). Here, the first Lyapunov exponent has been determined numerically, and the

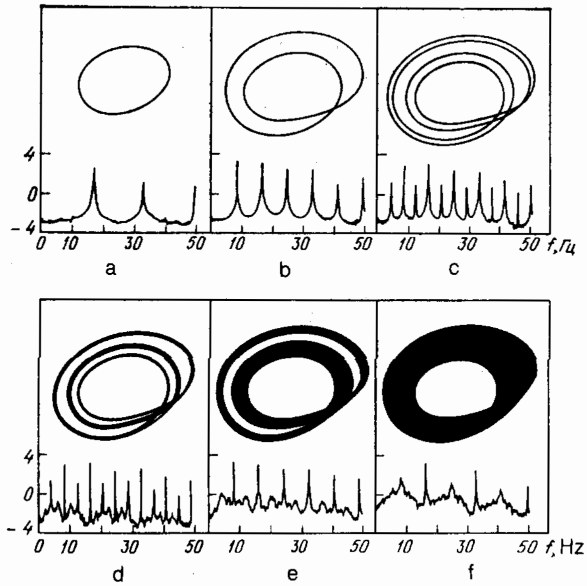

FIG. 13. The phase flow of O. Rössler projected into the (x,y) plane, and the spectral density of $z(t)$ for (a) $\mu = 2.6$, (b) 3.6, (c) 4.1, (d) 4.23, (e) 4.30 and (f) 4.6.

second is equal to $\sigma_2 = \ln\beta - \sigma_1$, while the Lyapunov dimension of the attractor (5.11) is found to be equal to $1 + \sigma_1/|\sigma_2| \approx 1.26$.

The functional equations of M. Feigenbaum can be generalized also to the case of the $(N-1)$-dimensional mappings of the Poincaré sequence $\mathbf{x}_{n+1} = \Pi(\mathbf{x}_n, \mu)$ for N-dimensional dissipative phase flows: if for some μ_1 in such flows a period-doubling bifurcation occurs, then with subsequent growth of μ_1 there will occur an infinite sequence of such bifurcations which satisfy similarity laws with the universal constants δ and α, and with a certain limit point μ_∞ at which random motion arises (at first ergodic, but not having the mixing property). As an example, we present the three-dimensional phase flow of O. Rössler[65]:

$$\dot{x} = -y - z; \quad \dot{y} = x + \frac{y}{5}, \quad \dot{z} = \frac{1}{5} - \mu z + xz. \quad (6.13)$$

In Fig. 13 we show the results of analogue modeling of this system projected on the (x,y) plane, along with the spectral densities for $z(t)$ for three subcritical and three supercritical values of μ (here $\mu_\infty \approx 4.20$); the direct and inverse period-doubling bifurcations are clearly visible in the spectra. In the section of this projection along the line $y = 0$ we obtain an almost one-dimensional, approximately quadratic mapping $x_{n+1} = \Pi(x_n)$; in reality, this "curve" has some thickness,

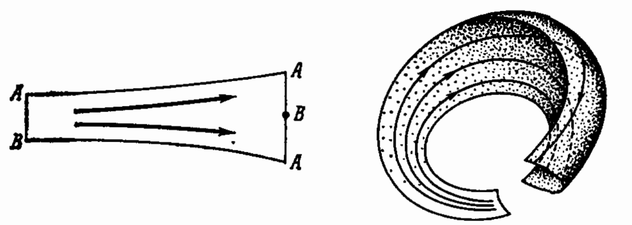

FIG. 14. Topology of the Rössler attractor.

because the attractor consists of an infinite set of sheets with Cantor structure; topologically, it is obtained from a sheet whose width is doubled as it spreads across the trajectories and which is folded double in the longitudinal direction, after which its right edge is attached to its left edge (Fig. 14).

6.4. Pomeau-Manneville scenario

This scenario involves the appearance of motions which are alternately periodic and random in time after an inverse tangential bifurcation (fusion and disappearance of stable and unstable fixed points of the Poincaré mapping, i.e., stable and unstable periodic phase trajectories); the time intervals in which stochastic motion occurs are random in length, while the periodic-motion intervals have lengths proportional to $|\mu-\mu_{cr}|^{-1/2}$. An example of this is the bifurcation where the stable and unstable trajectories with the period 3 fuse for the mapping (6.1) at the value $\mu_c \approx 3.83$, which acts as an accumulation point for the sequence of *inverse* bifurcations for the doubling of this period.

M. I. Rabinovich and D. I. Trubetskov[8] have pointed out that it is possible to explain the alternation by a special form of the Poincaré mapping $x_{n+1}=\Pi(x_n,\mu)$. Let the graph of $\Pi(x)$ consist of three curved segments: 1—a steeply-rising segment, 2—a part with a minimum, and 3—a sharp decrease. For $\mu<\mu_c$ let part 2 intersect the bisectrix at two points; they correspond to stable and unstable periodic trajectories of the phase flow being mapped. As μ increases, let this portion rise upward so that at $\mu=\mu_c$ its two intersection points with the bisectrix fuse. For a very small increase $\mu>\mu_c$ there appears between the bisectrix and Sec. 2 such a narrow gap that a Lameria ladder drawn within it has many steps: in the course of many iterations (i.e., a long time), the flow trajectories stay close to the original periodic one. When it leaves Sec. 2, large steps appear in the ladder (sharp fluctuations in the trajectory); and on returns to Sec. 2 long intervals over which the motion is close to periodic again occur.

6.5. Circular Couette flow

The most detailed illustrations of the scenarios listed above come from specific measurements on the loss of stability of circular Couette flow (in the gap between two coaxial rotating cylinders) and of a layer of fluid heated from below (the development of thermal convection).

Circular Couette flow in cylindrical coordinates r, φ, z with the z axis directed along the cylinders' axis has the velocity components

$$\left.\begin{aligned} u_{0r}=u_{0z}=0,\quad u_{0\varphi}=U(r)=Ar+Br^{-1},\\ A=(\Omega_2R_2^2-\Omega_1R_1^2)(R_2^2-R_1^2)^{-1},\\ B=-R_1^2R_2^2(\Omega_2-\Omega_1)(R_2^2-R_1^2)^{-1},\end{aligned}\right\}\qquad(6.14)$$

where $R_1<R_2$ are the radii, Ω_1 and Ω_2 are the angular velocities of rotation of the inner and outer cylinders. In an *ideal fluid*, because of the law of conservation of angular momentum of a fluid particle $mrU=$ const in its displacement from r_0, to $r>r_0$ its velocity will become $r_0U(r_0)/r$, and if the centrifugal acceleration acting on it $r_0^2U^2(r_0)/r^3$ will turn out to be larger than its equilibrium value $U^2(r)/r$ under the

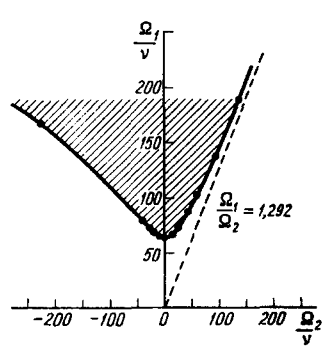

FIG. 15. Region of instability for circular Couette flow in the (Ω_1,Ω_2) plane for $R_2/R_1=1.13$.

condition $\partial(rU)^2/\partial r<0$, the flow will be unstable (Rayleigh, 1916). This criterion reduces to the form $(\Omega_2^2R_2^2-\Omega_1^2R_1^2)U<0$, so that in the case that the cylinders rotate in opposite directions the flow is always unstable (since somewhere in the gap U changes sign), while if they rotate in the same direction the instability criterion takes the form $\Omega_2/\Omega_1<(R_1/R_2)^2$ (Fig. 15).

In a viscous fluid, we obtain the following equation for the complex amplitudes of small wavelike velocity perturbations of the form $\mathbf{u}(r)\exp[i(kz+n\varphi-\omega t)]$ with arbitrary k and integer n

$$\left[\frac{\partial}{\partial r}\mathcal{L}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)-k^2\left(\mathcal{L}-\frac{1}{r^2}\right)\right]u_r$$
$$=\left[\frac{2k^2}{\nu r}\left(U-\frac{in\nu}{r}\right)-in\frac{\partial}{\partial r}\mathcal{L}\frac{1}{r}\right]u_\varphi,$$
$$\left[\frac{n^2}{r}\mathcal{L}\frac{1}{r}+k^2\left(\mathcal{L}-\frac{1}{r^2}\right)\right]u_\varphi$$
$$=\left\{\frac{2k^2}{\nu r}\left[\frac{r}{2}\left(\frac{\partial U}{\partial r}+\frac{U}{r}\right)-\frac{in\nu}{r}\right]+\frac{in}{r}\mathcal{L}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\right\}u_r,$$
$$\mathcal{L}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}-k^2+\frac{i\omega}{\nu}-\frac{inU}{\nu r}\qquad(6.15)$$

with the boundary conditions $u_r=\partial u_r/\partial r=u_\varphi=0$ for $r=R_1, R_2$. These equations have no singularities, so that obviously for fixed parameters (including k and n) there always exists a discrete spectrum of eigenvalues ω_j and the corresponding set of eigenfunctions is complete. For a long time it was assumed that as Re increased the first modes to lose their stability must be the axially-symmetric perturbations (with $n=0$). It has been established (although not yet proved with sufficient completeness) that the first bifurcation is a *change of stability*—a transition from the flow (6.14) to a new *steady-state* flow of the form $u(r)\exp(ik_{cr}z)$ of toroidal rolling *Taylor vortices* [first investigated by G. Taylor[67] for small d/R_0, where $d=R_2-R_1$ and $R_0=1/2(R_1+R_2)$]. There then occur regions of instability related to these vortices of the type shown in Fig. 15 [which show that the viscosity here plays a stabilizing role; however, for $\Omega_2/\Omega_1<(R_1/R_2)^2$ and sufficiently large Re instability develops all the same]. For a long time it was assumed that these regions agreed well with the experimental data (see the points on the border of the cross-hatched region in Fig. 15).

However, it later turned out that for *negative* and not-too-small values of Ω_2/Ω_1 perturbations with $n\neq0$ lose their stability somewhat earlier than the axially-symmetric ones. Thus, according to the calculations of E. Krüger, A. Gross and R. DiPrima[68] for $d/R = 1/20$, on the rays $\Omega_2/\Omega_1 = -0.7; -0.8; -0.9; -1.0$ and 1.25, as Re grows the first perturbations to lose stability are those with $n = 0,1,3,4$, and 5. This is confirmed by the experimental data, in particular that of H. Snyder.[69]

Both theoretical calculations (J. Stuart,[70] A. Davey,[71] A. Davey, R. DiPrima and J. Stuart[72]) and experimental data (R. Donnelly and K. Schwarz,[73] H. Snyder and R. Lambert[74]) show that for $\mathrm{Re} = \nu^{-1}\Omega_1 R_1 d > \mathrm{Re}_{1cr}$ the growth in intensity of the axially-symmetric Taylor vortices is excellently described by the Landau Eq. (3.2).

Experiments specifically designed to verify the scenarios performed by J. Gollub and H. Swinney,[75] R. Fenstermacher, H. Swinney and J. Gollub,[76] V. S. Lvov, and A. A. Predtechensky,[77] and A. Brandstater *et al.*,[105] showed that at some Re_{2cr} the Taylor vortices become unstable, and azimuthal "buckling" waves appear on them. As Re increases, at subsequent (apparently normal Hopf) bifurcations there appear an additional one to three independent frequencies (and in addition, the discrete peaks in the fluctuation spectrum of the velocity $u_r(t)$ at the middle of the gap broaden, while the continuous background spectrum grows, which is not predicted by the Landau-Hopf or Ruelle-Takens scenarios). After the next bifurcation for $\mathrm{Re} = \mathrm{Re}_c$ there remains only the randomized motion with a continuous spectrum.

The quantitative characteristics of the bifurcations observed in these experiments vary, in particular, as a function of the size of the cylinders and of the initial data. Thus, for the first of these the following values were chosen $R_1 = 22.54$ mm, $R_2 = 25.40$ mm, $d = R_2 - R_1 = 2.86$ mm, $d/R_1 = 0.14$ and the height of the cylinders $h = 20d$, $\Omega_2 = 0$, $\mathrm{Re} = 2501$. Taylor vortices appeared for $R^* = \mathrm{Re}/\mathrm{Re}_c = 0.051$ (their number being $kh/2\pi = 17$); for $R^* = 0.064$ buckling waves appeared on them [four over the circumference, with dimensionless frequency $f_1^* = 2\pi f_1/\Omega_1 = 1.30$, while in the spectrum of $u_r(t)$ six of the wave's harmonics were visible]; for $R^* = 0.54$ another (low) frequency f_2 appeared; as R^* increased it decreased to zero for $R^* = 0.78$ where a third frequency $f_3^* \approx (2/3) f_1^*$ appeared; for $R^* \sim 1$ randomization set in, reversibly and without hysteresis.

The experiments of Lvov and Predtechensky used $R_1 = 17.5$ mm, $R_2 = 27.5$ mm, $d = 10$ mm, $d/R_1 = 0.57$, $h = 30d$, $\Omega_2 = 0$. Taylor vortices appeared for $\mathrm{Re} \approx 74$ (22–36 in number, for slow acceleration 28–30). For 30 vortices buckling waves appeared in the interval $\mathrm{Re} = 995$–1015 (six over the circumference, $f_1 = 1.93\Omega_1$, two harmonics); then for $\mathrm{Re} = 1040$ the second harmonic $f_2 = 0.55\Omega_1$ and also combination harmonics appeared in the spectrum of $u_\varphi(t)$ and alternated with the first harmonic; then the third frequency $f_3 = 0.95\Omega_1$ appeared. For $\mathrm{Re} = 1901$ there were no longer any distinct maxima in the spectrum (but separate sharp peaks appeared and disappeared as Re increased further). The behavior of the flow with 28 vortices was entirely different (in particular, for $\mathrm{Re} = 1100$–1200 it made a transition to a state with 29 vortices). In sum, the evolution of Taylor vortices although being similar in general outline to the Ruelle-Takens scenario nevertheless in detail turned out to be significantly more complicated.

6.6. Thermal convection

Let us now investigate thermal convection in a horizontal layer $0 \leqslant z \leqslant h$, described by the Boussinesq equation

$$\operatorname{div}\mathbf{u} = 0 \quad \frac{d\mathbf{u}}{dt} = -\frac{\nabla p'}{\rho_0} - \mathbf{g}\alpha T' + \nu\Delta\mathbf{u}, \quad \frac{dT'}{dt} = \chi\Delta T' \quad (6.16)$$

(where α is the coefficient of thermal expansion) with a fixed boundary value for the temperature $T'_{z=0} = T_0$ and $T'_{z=h} = T_1$. A steady-state solution of these equations takes the form $\mathbf{u} = 0$, $T' \equiv T'_s = T_0 - (T_0 - T_1)z/h$. Introducing the Rayleigh number $\mathrm{Ra} = g\alpha(T_0 - T_1)h^3/\nu\chi$ and the Prandtl number $\sigma = \nu/\chi$, measuring length in units of h, time in units of h^2/χ and setting $T' - T'_s = (T_0 - T_1)\vartheta/\mathrm{Ra}$, we bring (6.16) to dimensionless form

$$\operatorname{div}\mathbf{u} = 0 \quad \frac{d\mathbf{u}}{dt} = -\nabla\Pi + \sigma\vartheta\nabla r + \sigma\Delta\mathbf{u}, \quad \frac{d\vartheta}{dt} = w\,\mathrm{Ra} + \Delta\vartheta, \quad (6.17)$$

where Π is the dimensionless deviation of the pressure from its steady-state hydrostatic distribution. Linearizing these equations (which reduces to replacing d/dt by $\partial/\partial t$ and seeking the unknown functions in the form $f(z) \times \varphi(x,y)\exp(-i\sigma\omega t)$, where $\Delta\varphi + k^2\varphi = 0$, we obtain for the complex amplitude $\vartheta(z)$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\left(\frac{\partial^2}{\partial z^2} - k^2 + i\omega\right)\left(\frac{\partial^2}{\partial z^2} - k^2 + i\sigma\omega\right)\vartheta + k^2\,\mathrm{Ra}\,\vartheta = 0 \quad (6.18)$$

with boundary conditions $\vartheta = \vartheta'' = \vartheta''' - (k^2 - i\sigma\omega)\vartheta' = 0$ on one face and $\vartheta = \vartheta'' = \vartheta^{\mathrm{IV}} = 0$ on the other. The first bifurcation here turns out to be a *change of stability* (A. Pellew and R. Southwell[78]). Rayleigh established this for a layer with two free surfaces, where he obtained $\mathrm{Ra}_{1cr} = 27\pi^4/4 \approx 657.5$ and $k_{cr} = \pi/\sqrt{2} \approx 2.2$; later the values $\mathrm{Ra}_{1cr} \approx 1708$ and $k_{cr} \approx 3.12$ were obtained for a layer with rigid surfaces, while for a rigid lower surface and free upper surface $\mathrm{Ra}_{1cr} \approx 1100$, signaling the appearance of a *steady-state motion* which is periodic in x and y. Its form $\varphi(x,y)$ (convective "rollers," square or hexagonal cells, etc.) is not determined in the linear theory.

A. Schlüter, D. Lorz and F. Busse[79] established that for very small $\mathrm{Ra} - \mathrm{Ra}_{1cr} > 0$ of all the steady-state convective motions only the "rollers" are linearly stable (in a narrow band in the k, Ra plane). E. Palm[80] and subsequent authors obtained more general results by establishing that the formation of hexagonal convective *Bénard cells* (1900) is determined by the temperature dependence of the material properties of the liquid, above all the viscosity: $\nu' = |\partial\nu/\partial T|$. For a perturbation of the form

$$w = \left[A_1(t)\cos ky + A_2(t)\cos\frac{kx\sqrt{3}}{2}\cos\frac{ky}{2}\right]\sin lz \quad (6.19)$$

Palm derived a system of equations of Landau type

$$\dot{A}_1=\gamma A_1-\frac{\nu'}{4}A_2^2-\delta_1 A_1^3-\left(2\delta_2-\frac{\delta_1}{2}\right)A_1A_2^2,$$
$$\dot{A}_2=\gamma A_2-\nu' A_1A_2-\delta_2 A_2^3-(4\delta_2-\delta_1)A_1^2A_2, \tag{6.20}$$

having a stationary solution $A_2 = \pm 2A_1$ corresponding to the hexagonal cells. It was shown that for $\mathrm{Ra}_{1cr} < \mathrm{Ra}_1 < \mathrm{Ra} < \mathrm{Ra}_2$ only the hexagonal cells were stable, while for $\mathrm{Ra}_2 < \mathrm{Ra} < \mathrm{Ra}_3$ both the hexagonal cells and the "rollers" were stable; for $\mathrm{Ra} > \mathrm{Ra}_3$ only the rollers were stable. As $\nu' \to 0$, Ra_1, Ra_2 and Ra_3 reduce to Ra_{1cr}, so that we recover the results of Schlüter, Lortz, and Busse.

Let us investigate convective "rollers" extended along the y axis, so that for such a flow Eq. (6.17) turn out to be two-dimensional; after introducing the flow function ψ in the (x,z) plane we are led to the form

$$\frac{\partial\Delta\psi}{\partial t}+\frac{\partial(\psi,\Delta\psi)}{\partial(x,z)}=\sigma\frac{\partial\theta}{\partial x}+\sigma\Delta^2\varphi,$$
$$\frac{\partial\vartheta}{\partial t}+\frac{\partial(\psi,\vartheta)}{\partial(x,z)}=\mathrm{Ra}\frac{\partial\psi}{\partial x}+\Delta\theta. \tag{6.21}$$

As $\mathrm{Ra} > \mathrm{Ra}_{1cr}$ increases, the "roller" motion ceases to be a steady-state one and grows with time. Initially it can be described by the three modes

$$\psi=\pi^{-1}\left(\frac{R_0}{k}\right)^{1/3}X\cdot\sqrt{2}\sin kx\cdot\sin\pi z,$$
$$\vartheta=R_0(Y\cdot\sqrt{2}\cos kx\cdot\sin\pi z-Z\sin 2\pi z), \tag{6.22}$$

where $R_0 = k^{-2}(\pi^2+k^2)^3$. F. Busse[81] has proved that for infinitesimal perturbation amplitudes, the other modes are of a higher order of smallness. Taking this into account, substituting (6.22) into (6.21) and neglecting interactions with all other modes, we obtain for X, Y, Z the system of equations

$$\dot{X}=-\sigma X-\sigma Y,\quad \dot{Y}=-Y+rX-XZ,$$
$$Z=-bZ+XY, \tag{6.23}$$

where the dot implies differentiation with respect to $(k^2R_0)^{1/3}t$; $b=4\pi^2(k^2R_0)^{-1/3}$ is a geometric factor (for the Rayleigh value $k=\pi/\sqrt{2}$ it equals 8/3); $r=R_0^{-1}\,\mathrm{Ra}$ is the relative Rayleigh number. This system, first obtained by E. Lorenz,[82] is famous as the first example of a system with a strange attractor (an infinite-sheeted attractor of Cantor structure was suspected earlier by the same Lorenz, but he did not obtain a rigorous mathematical proof of this).

Let us emphasize that the Lorenz equations (to which we will return later) describe real convection only for small r. G. Willis and J. Deardorff[83] have established that for $\mathrm{Ra}_{2cr} \approx 3\mathrm{Ra}_{1cr}$ transverse waves appear on the convective "rollers," as happens on the toroidal Taylor rollers mentioned above. Calculations based on the linear theory were carried out by F. Busse,[81] for the nonlinear theory such calculations were performed by J. McLaughlin and P. Martin,[84] who first calculated an eight-mode motion with rollers (6.22) and one harmonic along the y axis, and constructed a Landau expansion (3.2) for them (for which $\delta > 0$ was obtained, corresponding to a normal bifurcation). Secondly, they numerically calculated a 39-mode motion with rollers from (6.22) and four harmonics along the y axis, and for $k_1/\pi = 0.072$, $k_2/\pi = 0.1$ and $\sigma = 1$ they obtained: $r_{2cr} \approx 1.25$; for $r = 1.4$, a periodic motion, for $r_{2cr} = 1.45$ a slightly nonperiodic motion, for $r = 1.5$ and 1.55 again a periodic motion, for $r = 1.6$ an abruptly nonperiodic regime (while when the fourth harmonic is excluded, $r = 1.6$, 2 and even 20 are still within the periodic regime).

Let us now investigate the Lorenz equations (6.23). The divergence of the phase flow (5.2) is negative for these equations: $\Lambda = -(\sigma+b+1)$, so that all trajectories migrate towards a certain set of zero volume. The quantity $W = [X^2+Y^2+(Z-r-\sigma)^2]^{1/2}$ satisfies the condition $\dot{W} \leqslant -C_1W + C_2$ with positive C_1 and C_2, so that all trajectories enter the sphere $W \leqslant 2C_2/C_1$. The system does not change under the substitution $(X,Y,Z) \to (-X,-Y,-Z)$. For $r < 1$, the unique fixed point is the stable vertex 0 at the coordinate origin. For $r \geqslant 1$ (the onset of convection) it loses its stability (becoming a saddle point with a two-dimensional stable manifold and two unstable one-dimensional ones—the separatrixes Γ^+ and Γ^-), and two new fixed points appear C^+, $C^- = (\pm[b(r-1)]^{1/2}, \pm[b(r-1)]^{1/2}, r-1)$, towards which the separatrixes move. For $\sigma < b+1$ they are stable, while for $\sigma > b+1$ (following Lorenz, we will henceforth investigate the case $\sigma = 10$, $b = 8/3$) they are stable for $1 < r < r_3 = \sigma(\sigma+b+3)(\sigma-b-1)^{-1} \approx 24.74$, while for $r > r_3$ they lose their stability. According to the linear theory, for $r = r_3$ it is here possible to have neutral equilibrium with frequency $\omega = [2b\sigma(\sigma+1)(\sigma-b-1)^{-1}]^{1/2}$, let us say $\delta X = A\cos\omega t$. For r somewhat smaller than r_3 a small nonlinear correction of order $|A|^2$ is added to δX, for which J. McLaughlin and P. Martin[84] set up a Landau equation (3.2) accurate to order σ^{-1}, obtaining $\gamma = (b/2\sigma^{1/2})[(r-1)^{1/2}-(r_3-1)^{1/2}]$ and $\delta = -37/72\ \sigma$, so that the bifurcation at $r = r_3$ is inverse.

Below we report the results of V. C. Afraĭmovich, V. V. Bykov and L. P. Shil'nikov[85] (Fig. 16). For $r = r_1 \approx 13.92$ they found a bifurcation for which the separatrixes return to the saddle point. For $r > r_1$, out of the loops of the separatrixes there appear saddle-point periodic motions L^+, L^- around the foci C^+, C^- (at the same time, there appears an invariant set of curves Ω_1 which is *not* an attractor and which has Cantor structure, including a denumerable set of saddle-point periodic motions); the separatrixes Γ^+, Γ^- intersect and move towards the foci C^-, C^+. For $r = r_2 \approx 24.06$ the separatrixes Γ^+, Γ^-, in place of the foci, are curled around

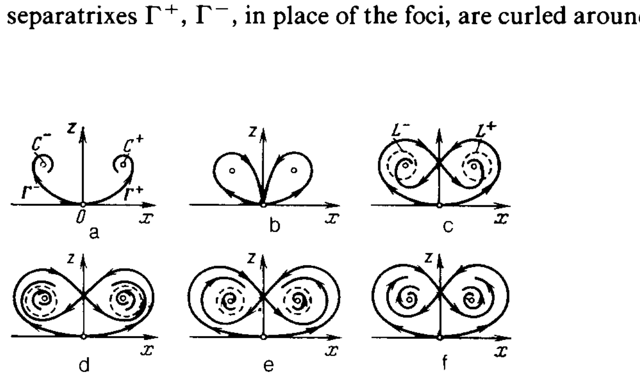

FIG. 16. Bifurcations in the Lorentz system: (a) for $1<r<r_1$; (b) for $r=r_1$; (c) for $r_1<r<r_2$; (d) for $r=r_2$; (e) for $r_2<r_3$; (f) for $r=r_3$.

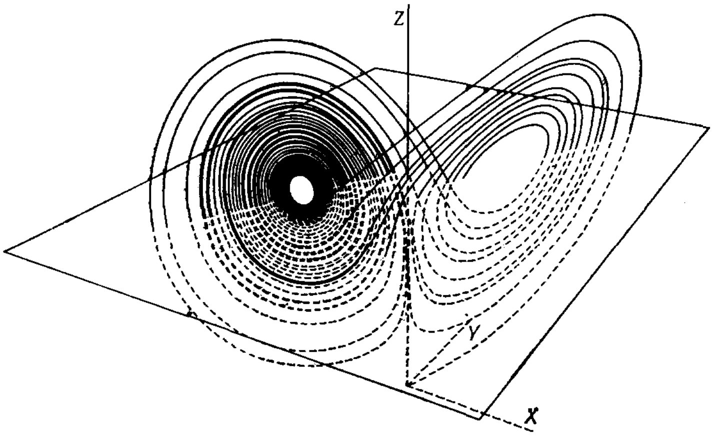

FIG. 17. An example of a trajectory on the Lorentz attractor for $r = 28$. The X, Y plane corresponds to $Z = 27$.

the cycles L^-, L^+, and instead of Ω, there appears an infinite Lorenz attractor Ω_2 whose region of attraction is limited by the stable manifolds of the cycles L^-, L^+ (so that excitation of randomness is "hard"). For $r > r_2$ it is stable, including 0, Γ^+, Γ^- and therefore there is no structural stability; on this attractor the periodic motions are everywhere dense (capable of undergoing a sequence of period-doubling bifurcations and of disappearing as r grows only by way of adhesion to the loops of the separatrixes). For $r = r_3$, the cycles L^+, L^- contract to the points C^+, C^- and the latter lose their stability. For $r_3 < r < r_4 \approx 220$ the Lorenz attractor is the unique stable limit set (we note that as r decreases from r_4 to r_2, the phase point $M(t)$ stays within the attractor, while for $r < r_2$ it loses stability and $M(t)$ moves toward C^+ or C^-).

An example of a trajectory in the attractor (for $r = 28$, intersecting the plane $Z = 27$) is shown in Fig. 17. It starts at the coordinate origin, circles C^+, and then unwinds and is drawn to C^-, leaves C^- and spirals toward C^+, etc., while the period of rotation around C^+ or C^- equals 0.62 and the radii of the spirals change by 6% per rotation. As pointed out by Lorenz himself, for this example the Poincaré mapping $Z_{n+1} = \Pi(Z_n)$ of successive maxima has the triangular form, while $|\Pi'(Z)| > 1$ everywhere; it is ergodic and has the mixing property (L. A. Bunimovich and Ya. G. Sinai[86]). Topologically, the Lorenz attractor consists of two sheets, which widen out across the trajectories, while the right-hand edge of each of them adheres to the left edges of both sheets (Fig. 18). H. Mori and H. Fujisaka[87] have calculated the Lyapunov exponent σ_1 for the Lorenz attractor as a function of r (the second one σ_2 is zero, the third $\sigma_3 = \Lambda - \sigma_1$ is negative and the Lyapunov dimension of the attractor equals $2 + \sigma_1|\sigma_3|^{-1}$). For $r < 1$ it is negative and for $r = 1$ (the appearance of the convective "rollers") it reduces to zero; for $1 < r < r_3$ it is again negative and for $r = r_3$ it reaches zero. However, for $r = r_2$ greater than zero there also appears a new branch of σ_1 which grows for $r > r_2$ and for very large r is multiply interrupted by "lacunae" of zero values, corresponding to periodic motion (see the example in Fig. 19 for $b = 4$, $\sigma = 16$; here, for $r = 40$ the Lyapunov dimension is 2.06).

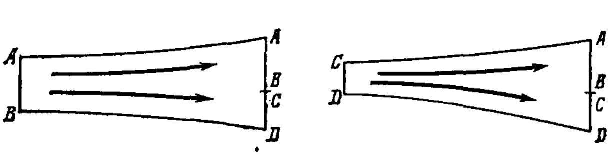

FIG. 18. Topology of the Lorentz attractor.

In recent years experimental conditions for studying convection have sharply improved as a consequence of the use of normal (non-superfluid) liquid helium, whose heat capacity is large compared to the walls and whose temperature sensitivity is 10^{-7} (the groups of G. Ahlers[88] and A. Libchaber[89]), and the use of laser Doppler anemometers with sensitivities of 10^{-4} cm/sec (the groups of J. Gollub[90] and P. Berge[91,106]) and computers. This made it possible to observe (in the spectra) quasiperiodic motion with 2–3 incommensurate frequencies, a period-doubling bifurcation sequence with the similarity law (6.7) for the amplitudes (an example from the paper by J. Gollub, S. Benson and J. Steinman, is shown in Fig. 20), the alternation property, absolute instability of thin films, edge effects ("rollers" perpendicular to walls), etc.

CONCLUSION

We very much wanted to introduce into the definition of turbulence in Sec. 5.3 the requirement that it be a *multi-*

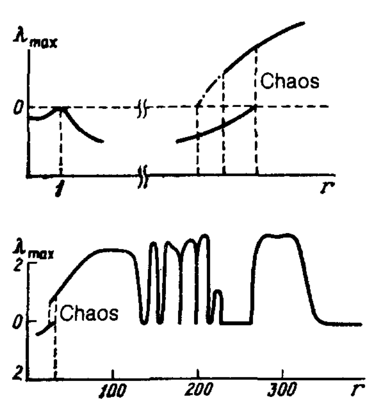

FIG. 19. Graph of the Lyapunov exponent $\sigma_1(r)$ for $b = 4$, $\sigma = 16$.

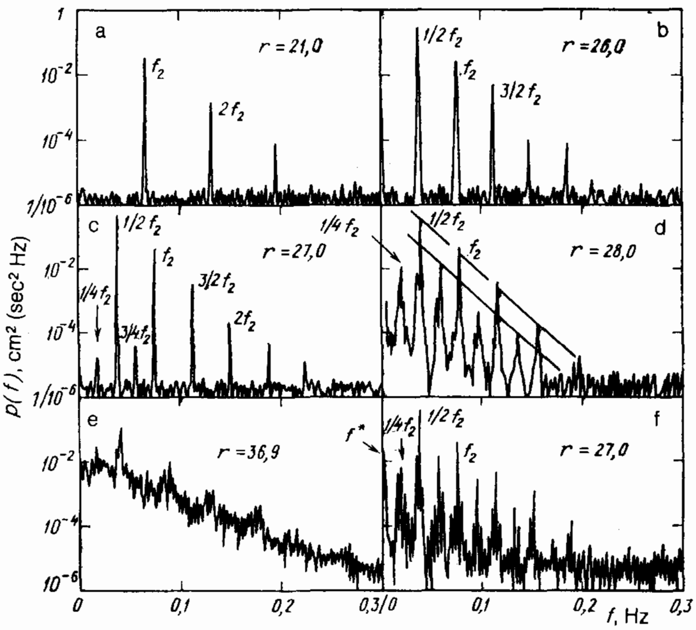

FIG. 20. Period-doubling bifurcations in convective "rollers."

mode phenomenon, i.e., that it should be chaotic in its spatial structure at any fixed instant of time, since, e.g., the stochastic motion in geometrically regular structures of convective "rollers" described by the equations of E. Lorenz is too far removed from our intuitive concepts of turbulent flows in liquids and gases one of the most important features of which is the possibility of cascade processes whereby conserved quantities—energy, enstrophy, potential enstrophy, conserved impurities, wave action, etc.—are transferred down the spectrum of scales. The model which corresponds best to these ideas is A. M. Obukhov's chain model[92] of linked triplets (in which stochastic behavior in the sense of Sec. 5.3 is possible for each of the triplets). However, we have refrained from imposing such a supplementary requirement, preferring to retain it for defining *developed* turbulence and agreeing to relate the theory of strange attractors to processes whereby turbulence *arises*.

What would we like to expect in the near future? Firstly, the resolution of unsolved problems in the theory of linear instability (a few of these were pointed out in Sec. 4); thus, e.g., it is desirable to give a qualitative explanation for the fundamental differences in the behavior of Poiseuille flow in channels and pipes. Secondly, a mathematical elaboration of new scenarios for randomization (which are doubtless many and diverse), including the interpretation of mechanisms for randomization after inverse Hopf bifurcations, and also a mathematical description of processes of explosive randomization of small-scale flow patterns in a viscous boundary layer at a solid wall. However, it is our feeling that the mathematical part of the theory is already on a hopeful track. Thirdly, and perhaps this is something whose lack is most keenly felt, laboratory investigations of the onset of turbulence using new experimental techniques (e.g., laser-Doppler flow measurements in glass pipes).

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