High-frequency asymptotic behavior of radiation spectra of moving charges in classical electrodynamics

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When a charge is moving in free space or in an inhomogeneous and nonstationary medium, it generates electromagnetic radiation. The spectrum of this radiation depends on the expression specifying the motion of the charge, and also on the laws according to which properties of the medium are changing in time and space. The asymptotic behavior of the radiation spectrum, i.e., the high-frequency behavior of spectral intensity, is studied. It is shown that if a charge moves along a smooth trajectory, or if the variation of the medium properties is described by a smooth function, the radiation spectrum at high frequencies decreases exponentially. Therefore, the radiation spectrum of a charge, moving along a smooth trajectory in a medium with a smooth inhomogeneity and (or) nonstationarity, drops abruptly to zero, starting from a certain value of the frequency. By a smooth trajectory we mean a trajectory of a charge moving according to the law $\mathbf{r} = \mathbf{r}(t)$, where the vector-function $\mathbf{r}(t)$ is continuous together with all its derivatives. Similarly, a medium with smooth inhomogeneities (or smooth nonstationarity) is described by functions, which are continuous together with all their derivatives of arbitrary order. A method is described that allows one to determine the upper limit of the radiation spectrum, i.e., the value of the frequency beginning with which an exponential decay of the spectrum takes place.

TABLE OF CONTENTS

INTRODUCTION

It is well known that a moving particle is a source of electromagnetic radiation. An exception from this rule is the case of a particle moving uniformly in vacuum, when it does not radiate. For accelerated motion of a charge in vacuum radiation is always present. A charge moving uniformly in a homogeneous refracting medium emits radiation only in the case when the velocity of the charge exceeds the phase velocity of electromagnetic waves in that medium. This is the well-known Vavilov-Cherenkov radiation.[1] When a charge moves in an inhomogeneous and (or) nonstationary medium, radiation arises both in the case of accelerated motion and in the case of uniform motion of a charge with any velocity. In the particular case when a uniformly moving charge crosses a plane boundary between two media, "transition" radiation occurs.[2]

The radiation from a charge in vacuum is sufficiently well-studied for a number of cases, for which the law of the particle motion is known. Some of these cases are of practical interest. For example, the problem of radiation from a charge moving along the circumference of a circle has a direct relation to the radiation from accelerated particles moving along an orbit in a synchrotron chamber. This radiation has received the name of synchrotron radiation. It is used widely in physics, biology and engineering.[3–5] Synchrotron radiation must be taken into account in studying a number of astrophysical phenomena.[6] The interest in undulator radiation has increased over the last several years. This radiation arises for a special class of trajectories, namely, for such

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equations of motion $\mathbf{r}=\mathbf{r}(t)$ which satisfy the relation $\mathbf{r}(t+T)=\mathbf{r}(t)+\mathbf{L}$, where $\mathbf{L}$ is the distance over which the charge moves during the time T. The undulator radiation of relativistic particles has a number of unique properties, and is increasingly often used in the research on and processing of various materials, biological objects, and for other purposes.[9,10]

Here we restrict our consideration by giving only a list of the types of motion for which the radiation of a charge has been studied in sufficient detail. By the present time, radiation corresponding to other types of motion has also been studied, but here we shall not discuss the corresponding papers and refer the reader to the literature.[2–4,6,7]

Radiation produced during the motion of a charge has different characteristics—spectrum, angular distribution, intensity, polarization—for different types of motion of a particle. However, for a rather large class of trajectories it is possible to make some general statements about the behavior of intensity and radiation spectrum at high frequencies. These statements made in Refs. 8, 14, 17 can be reduced to the statement that for smooth trajectories the intensity of radiation at sufficiently high frequencies decreases with an increase of the frequency ω faster than any finite power of ω. Such a conclusion follows from the properties of the Fourier transformation. In fact, let us assume that the expression describing the motion of a charge is given in the form $\mathbf{r}=\mathbf{r}(t)$, where $\mathbf{r}(t)$ is the radius–vector determining the position of the particle at the moment t. The current produced by the motion of the charge is proportional to its velocity $\mathbf{v}(t)=\mathrm{d}\mathbf{r}(t)/\mathrm{d}t$. Let $\mathbf{r}(t)$ be a smooth function, i.e., it and all its derivatives are continuous. It is easy to demonstrate (see below) that the amplitude of the wave radiated by a moving charge is proportional to the Fourier component of the current produced by the motion of the particle, i.e., for the case of a smooth trajectory it is expressed in terms of the Fourier component of a smooth function (by a smooth function we shall understand in the future a function continuous together with all its derivtives). In that case a theorem holds, according to which the Fourier component of a smooth function at sufficiently high values of the frequency decreases with an increase of frequency faster than any integer power of ω.[13] This means that the intensity of radiation falls off rapidly with an increase of frequency, beginning with a certain value of the frequency.

For the types of radiation given above (i.e., for synchrotron, magnetic brehmstrahlung and undulator radiation) there is, in fact, a rapid intensity decrease (exponential) of the radiation at high frequencies and, therefore, these results are in agreement with the above-mentioned theorem dealing with the asymptotic properties of the Fourier transform. However, the question arises whether the intensity of radiation falls off exponentially for all smooth types of motion. In this note we shall demonstrate that, for smooth trajectories, the intensity of high-frequency radiation in almost all cases (exceptions will be indicated) falls off exponentially.

It will be shown also that, if a charge moves in a homogeneous and non-stationary medium with slowly varying parameters, the radiation spectrum arising during this kind of motion also falls off exponentially at high frequencies.

1. RADIATION OF ELECTROMAGNETIC WAVES BY A CHARGE MOVING ACCORDING TO A SPECIFIED LAW

Let us consider the case when the source of the field is a point charge moving according to a specified law. We denote the magnitude of the charge by q.

Assume that at the moment t the charge is at the point

$$\mathbf{r}=\mathbf{r}\,(t), \tag{1}$$

where $\mathbf{r}(t)$ is a given function of time.

The expression (1) specifying the motion determines not only the position, but also the velocity $\mathbf{v}(t)$ of the charge at any moment of time t

$$\mathbf{v}\,(t)=\frac{\mathrm{d}\mathbf{r}\,(t)}{\mathrm{d}t}. \tag{2}$$

The motion of a charge described by the law (1) and (2) corresponds to the charge density

$$\rho=q\delta\,(\mathbf{r}-\mathbf{r}\,(t)) \tag{3}$$

and current density

$$\mathbf{j}=q\mathbf{v}\,(t)\,\delta\,(\mathbf{r}-\mathbf{r}\,(t)), \tag{4}$$

where $\delta(\alpha)$ is a delta function.

Determination of the electromagnetic field produced by a point charge moving according to the expressions (1) and (2) is, thus, reduced to the solution of the system of Maxwell equations for the electromagnetic field, with the charge density and current density having the form (3) and (4).

Since we are interested in the spectral decomposition of the field, we shall describe the radiation field by the Fourier component of the vector potential $\mathbf{A}_\omega(\mathbf{r})$:

$$\mathbf{A}_\omega(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{A}(\mathbf{r},t)e^{i\omega t}\,\mathrm{d}t. \tag{5}$$

For the case when the motion of the charge is determined by expressions (1) and (2), the expression for the vector potential $\mathbf{A}_\omega(\mathbf{r})$ is given in the book of Landau and Lifshitz[3]:

$$\mathbf{A}_{\omega\mathbf{j}}(\mathbf{r})=\frac{qe^{i\mathbf{kr}}}{cr}\int_{-\infty}^{\infty}\mathbf{v}\,(t)\exp\,[i\,(\omega t-\mathbf{kr}\,(t))]\,\mathrm{d}t. \tag{6}$$

This expression is valid for large distances $\mathbf{r}$ from the area where the charge is moving.

Let us consider the structure of the expression (6) for the vector potential. The factor

$$\frac{e^{i\mathbf{kr}}}{r}$$

describes a spherical wave diverging from the area in which the charge is moving. The quantity k entering the exponent is the wave vector corresponding to this spherical wave:

$$k=\frac{\omega}{c}. \tag{7}$$

The amplitude of the spherical wave (6) is proportional to the quantity

$$\mathbf{I}=\int_{-\infty}^{\infty}\mathbf{v}'(t)\exp\,(i\omega t-i\mathbf{kr}\,(t))\,\mathrm{d}t. \tag{8}$$

Here $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are, respectively, the position and velocity of a charge at the moment t, the vector $\mathbf{k}$ is equal in absolute value to ω/c and is directed from the area where the charge is moving towards the observation point:

$$k=\frac{\omega}{c}\frac{\mathbf{r}}{r}. \tag{9}$$

The amplitude $\mathbf{I}$, which is determined by formula (8), is equal, up to a constant factor, to the Fourier component of the density of the current produced by the motion of a charge described by expressions (1), (2). To prove that, let us expand the current density $\mathbf{j}(\mathbf{r},t)$ which is determined by formula (4), into the Fourier integral in terms of all the variables:

$$\mathbf{j}_{\mathbf{k},\omega}=\frac{1}{(2\pi)^4}\int \mathbf{j}(\mathbf{r},\ t)\exp(i\omega t-i\mathbf{kr})\,\mathrm{d}t\,\mathrm{d}\mathbf{r}. \tag{10}$$

If we substitute expression (4) for $\mathbf{j}(\mathbf{r},t)$ into this formula, and integrate over entire space, we shall obtain

$$\mathbf{j}_{\mathbf{k},\omega}=\frac{1}{(2\pi)^4}\mathbf{I}, \tag{11}$$

where $\mathbf{I}$ is the amplitude (8) of the radiated spherical wave (6).

Knowing the quantity $\mathbf{I}$, it is possible to determine the intensity of radiation at the frequency ω into an element of the solid angle $\mathrm{d}\Omega$[12]

$$\mathrm{d}\varepsilon_{\mathbf{n},\,\omega}=\frac{q^2}{4\pi^2c}\,|[\mathbf{kI}]|^2\,\mathrm{d}\omega\,\mathrm{d}\Omega, \tag{12}$$

where $\mathbf{n}$ is a unit vector in the direction of radiation.

From expression (12) one can see that the intensity of radiation is determined by the value of $\mathbf{I}$ (8). For this reason, asymptotic behavior of the spectral intensity $\mathrm{d}\varepsilon_{\mathbf{n}},\omega$ is determined by the behavior of the amplitude $\mathbf{I}$ at high frequencies.

2. HIGH-FREQUENCY ASYMPTOTIC BEHAVIOR OF THE AMPLITUDE I

We shall consider now the asymptotic behavior of the amplitude $\mathbf{I}$ at high frequencies, assuming that the motion of the charge (1) is determined by a smooth function, i.e., that the function $\mathbf{r}(t)$ and all its derivatives are continuous.

Expression (8) for $\mathbf{I}$ can be considered as the frequency Fourier component of the function $\mathbf{v}(t)\exp[-i\mathbf{kr}(t)]$. If $\mathbf{r}(t)$ is a smooth function, then $\mathbf{v}(t)\exp[-i\mathbf{kr}(t)]$ is also a smooth function. Therefore, based on general properties of the Fourier transformation, it is possible to make a statement that with an increase of the frequency ω the quantity $\mathbf{I}$ tends to zero faster than any integer power of ω. The analysis of expression (8) for $\mathbf{I}$ allows one to obtain more specific conclusions about type of dependence of $\mathbf{I}$ on the frequency ω for high frequencies.

Let us estimate the amplitude $\mathbf{I}$ for large values of frequency ($\omega\to\infty$). Later we describe in more detail the conditions under which the frequency can be assumed to be sufficiently large.

The amplitude $\mathbf{I}$, as can be seen from expression (8), is determined by an integral with the integrand containing the function

$$\exp\,(i\omega t\ -i\mathbf{kr}\ (t)). \tag{13}$$

For high values of the frequency this function oscillates rapidly in the range of variation of the variable t where the phase $\Phi(t)$ of expression (13)

$$\Phi\ (t)\ =\ \omega t\ -\mathbf{kr}\ (t) \tag{14}$$

varies with time. For this reason, the integral over the corresponding range is small. If the integration path contains a point, where the phase (14) does not change, the integral over the segment adjacent to that point is different from zero and determines the value of $\mathbf{I}$ (8). The point t_0, in the vicinity of which the phase (14) does not change, can be found from the equation

$$\frac{\mathrm{d}\Phi\,(t)}{\mathrm{d}t}=\omega-\mathbf{kv}\,(t_0)=0. \tag{15}$$

This equation does not have real roots. In fact, since the value of the wave vector $\mathbf{k}$ is determined by expression (7), we can rewrite equation (15) in the form

$$\omega\left(1-\frac{v\,(t_0)}{c}\cos\theta\right)=0, \tag{16}$$

where θ is the angle between the wave vector and the charge velocity $\mathbf{v}$. Since the charge velocity $\mathbf{v}$ cannot exceed the speed of light in vacuum c, and the magnitude of $\cos\theta$ is not larger than one, the second term in (16) is always smaller than the first, and for this reason equation (16) does not have any real roots t_0. However, if $\mathbf{v}(t)$ is a smooth function, equation (16) can have a complex root. Let us write this complex root in the following way:

$$t_0\ =\ t_1\ +\ it_2, \tag{17}$$

where t_1 is the real part of the root, and t_2 is the imaginary part.

We note here that for the case of a charge moving uniformly in the vacuum, equation (16) does not have any roots. If a particle moves uniformly in a homogeneous refracting medium, equation (16) coincides with the condition for the existence of Vavilov-Cherenkov radiation.

We expand the phase (14) in expression (8) for $\mathbf{I}$ in a power series in the vicinity of the point t_0, keeping only the first three terms

$$\Phi\,(t)=\omega t-\mathbf{kr}\,(t)$$
$$=\Phi\ (t_0)+\frac{\partial\Phi\ (t_0)}{\partial t}(t-t_0)+\frac{\partial^2\Phi\ (t)}{2\partial t^2}(t-t_0)^2+\ \ldots \tag{18}$$

Since equation (15) is satisfied at the point t_0, the second term in expansion (18) must be zero. Further, substituting expression (18) in (8) and taking $\mathbf{v}(t)$, evaluated at the point t_0, out of the integral, we obtain

$$\mathbf{I}=\mathbf{v}\,(t_0)\exp\,[i\,(\omega t_0-\mathbf{kr}\,(t_0))]$$
$$\times\int_{-\infty}^{\infty}\exp\left[-\frac{i}{2}\,\mathbf{a}\,(t_0)\,\mathbf{k}\,(t-\ t_0)^2\right]\mathrm{d}t, \tag{19}$$

where $\mathbf{a}(t_0)$ is the particle acceleration at $t=t_0$. The integral in expression (19) can be easily calculated, and we obtain

$$\mathbf{I}=\left(\frac{2\pi}{|\mathbf{a}\,(t_0)\,\mathbf{k}|}\right)^{1/2}\mathbf{v}\,(t_0)\exp\,(i\omega t_0-i\mathbf{kr}\,(t_0))\exp\,(i\beta_m) \tag{20}$$

where β_m is the angle determining the direction of the integration path in the method of steepest descent.[15] We shall not give values of β_m, since the absolute value of the phase factor $\exp(i\beta_m)$ which enters (20) is equal to unity, and in future we will be interested in the radiation spectrum determined by expression (12) and containing only the absolute value **I**.

The described procedure coincides with the calculation of **I** for large values of ω by the method of steepest descent.[15]

Expression (20) determines the value of the amplitude **I** of the radiated wave (6) at high frequencies.

Since the quantity t_0 in expression (20) is complex, $t_0 = t_1 + it_2$ [see expression (17)], $\mathbf{r}(t_0)$, too, is a complex function. Assume that

$$\mathbf{r}(t_0) = \mathbf{r}_1 + i\mathbf{r}_2, \tag{21}$$

here $\mathbf{r}_1$ and $\mathbf{r}_2$ are real vectors. Taking into account (17) and (21), we find that **I**, determined by expression (20), decays exponentially as the frequency increases, with the damping factor being of the form:

$$\exp[-(\omega t_2 - \mathbf{k}\mathbf{r}_2)]. \tag{22}$$

In future we shall assume that t_2, i.e., the imaginary part of t_0 in expression (17), is small. Then we can assume that

$$\mathbf{r}(t_0) = \mathbf{r}(t_1) + it_2\frac{\partial\mathbf{r}(t_1)}{\partial t_1} = \mathbf{r}(t_1) + it_2\mathbf{v}(t_1). \tag{23}$$

From here we obtain that the imaginary part of the vector $\mathbf{r}(t_0)$ is equal to

$$\mathbf{r}_2 = t_2\mathbf{v}(t_1). \tag{24}$$

Then the damping factor (22) takes the form

$$\exp[-t_2(\omega - \mathbf{k}\mathbf{v}(t_1))]. \tag{25}$$

This factor can be presented in the form

$$e^{-t_2/J_\varphi}, \tag{26}$$

where

$$T_\varphi = (\omega - \mathbf{k}\mathbf{v}(t_1))^{-1} \tag{27}$$

is the quantity that determines the order of magnitude of the time interval, during which a charge moving with the velocity $\mathbf{v}(t_1)$ radiates waves with almost equal phases. The quantity T_φ is called the radiation formation time.

Let us now find the physical meaning of the quantity t_2 in the exponent in expression (26).

According to the definition (17), the quantity t_2 is the imaginary part of the quantity t_0, where t_0 is a point of stationary phase, satisfying equation (16). We recall that here we assume that the quantity t_2 is small. In this case, as it can be seen from (19), the quantity t_2 is equal, in order of magnitude, to the time interval, during which a charge radiates waves reaching the observation point. The quantity t_1, i.e., the real part of t_0, determines the order of magnitude of the instant of time, near which radiation takes place.

We shall illustrate the above description by the following examples.

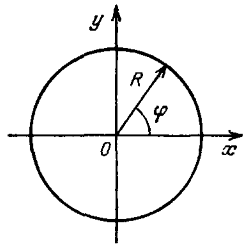

FIG. 1.

2.1. Synchrotron radiation

Assume that a charge is moving along the circumference of a circle of radius R with angular velocity ω_0. We choose the coordinate axes as shown in Fig. 1. The position of the particle on the circumference is determined by the angle $\varphi = \omega_0 t$. We assume that the particle is moving along the circumference counter-clockwise. The position of a particle at the moment t is determined by expressions

$$x = R\sin\omega_0 t,\quad y = R\cos\omega_0 t. \tag{28}$$

From here it is easy to determine the velocity components of the particle

$$v_x = v_0\cos\omega_0 t,\quad v_y = -v_0\sin\omega_0 t, \tag{29}$$

where

$$v_0 = R\omega_0. \tag{30}$$

Radiation by the particle is determined by the integral (8), where, in the case under consideration, one must substitute the values of the coordinates $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ from (28) and (29).

Let us consider the amplitude (8) for the case of high frequencies. Assume that the wave vector is directed parallel to the x axis. In that case equation (16) for the saddle point has the form

$$1 - \frac{v_x(t_0)}{c} = 0, \tag{31}$$

or

$$1 - \frac{v_0\cos\omega_0 t_0}{c} = 0. \tag{32}$$

We expand $\cos\omega_0 t_0$ in powers of t, keeping only the terms of not higher than second order:

$$\cos\omega_0 t_0 = 1 - \frac{1}{2}(\omega_0 t_0)^2 + \ldots.$$

Substituting this expansion into (32), we obtain

$$2\left(1 - \frac{v_0}{c}\right) = -(\omega_0 t_0)^2. \tag{33}$$

We assume that the velocity of the charge is close to the velocity of light. Then

$$2\left(1 - \frac{v_0}{c}\right) = \frac{1}{\gamma^2}, \tag{34}$$

where

$$\gamma=\left(1-\frac{v_0^2}{c^2}\right)^{1/2} \qquad (35)$$

is the so-called Lorentz factor. In future we shall assume that the Lorentz factor is large compared with 1 ($\gamma \gg 1$).

Taking into account (34), we obtain from expression (35)

$$t_0=\frac{i}{\omega_0\gamma}. \qquad (36)$$

The value of the saddle point t_0 is, therefore, purely imaginary in our example. If we assume, as we have done above, that $t_0 = t_1 + it_2$, we obtain $t_1 = 0$, $t_2 = 1/\omega_0\gamma$.

This result admits a simple physical interpretation. As is known, a charge moving with relativistic velocity radiates waves in the direction of motion in the narrow interval of angles $\Delta\theta \sim 1/\gamma$. If we consider the radiation with the wave vector directed in the positive direction of the x axis, such radiation is collected from the portion of the circumference having an angular extent $1/\gamma$ and located near the point $\varphi = 0$. The time that it takes for the charge to traverse this portion of the circumference is approximately equal to $1/\omega_0\gamma$, i.e., to the imaginary part of the expression (36) for t_0. The real part of t_0, equal in this case to zero, gives the position of that portion of the trajectory, from which the radiation is collected ($t = 0$, $\varphi = \omega t = 0$).

We estimate now the behavior of the amplitude of the radiated wave at high frequencies. For this we use expression (22). Substituting the values t_1 and t_2 from (36), we obtain the relationship

$$e^{-\omega/3\omega_0\gamma^3}, \qquad (37)$$

where we assume that $\omega \gg \omega_0\gamma^3$. The amplitude of the radiated wave decreases, therefore, as the frequency ω increases, in accordance with (37). Formula (37) shows also the frequencies for which the asymptotic behavior (37) is valid. The frequency ω must be sufficiently large, so that the exponent would be large compared with one.

Since the radiated field decreases as the frequency increases in accordance with (37), the energy of radiation also decreases as the frequency increases proportionally to the square of the factor (37), i.e., in accordance with

$$e^{-2\omega/3\omega_0\gamma^3}. \qquad (38)$$

Rigorous theory[3] gives for the case of high frequencies the same exponentially decaying frequency dependence.

Until now we had assumed that the wave vector of a radiated wave lies in the plane of the circular orbit of an electron. Assume now that the wave vector **k** forms an angle θ with the plane of the orbit, and that the orbit lies in the x,y plane, and the wave vector **k** is parallel to the x,z plane (Fig. 2). The motion of the charge is described as before by expression (28). We denote by θ the angle between the wave vector **k** and the plane of the orbit (or, what is the same, the angle between **k** and the x axis). In that case equation (16), which determines the point of the stationary phase, has the form

$$1-\frac{v_0}{c}\cos\omega_0 t_0\cdot\cos\theta=0. \qquad (39)$$

The value $t = t_0$, for which equation (39) is satisfied, is, as in

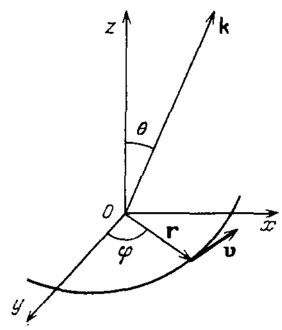

FIG. 2.

the previous case ($k_z = 0$) purely imaginary:

$$t_0=\frac{i}{\omega_0}\left[\frac{2(1-\beta\cos\theta)}{\beta\cos\theta}\right]^{1/2}. \qquad (40)$$

At $\theta = 0$ this value becomes equal to expression (36), if we take into the account the assumption that the charge velocity is close to the speed of light, $\beta \approx 1$. If we determine now the position of the charge $\mathbf{r}(t_0)$ at the instant t_0, determined by expression (40), and substitute values of t_0 and $\mathbf{r}(t_0)$ into (20), we obtain that in this case the dependence of the amplitude of the radiated wave on the frequency for large values of the frequency has the form

$$\exp\left\{-\frac{\omega}{3\omega_0}\left[\frac{8(1-\beta\cos\theta)^3}{(\beta\cos\theta)^3}\right]^{1/2}\right\}. \qquad (40')$$

Intensity of radiation is proportional to the square of this quantity.

2.2. Radiation from a charge with smoothly varying velocity

Assume that a point charge is moving along an axis, with the velocity of motion changing according to the following law

$$\mathbf{v}(t)=\frac{\mathbf{v}_1+\mathbf{v}_2}{2}+\frac{\mathbf{v}_2-\mathbf{v}_1}{2}\,\mathrm{th}\,\frac{t}{T}. \qquad (41)$$

For negative values of time, large in absolute value, the equation describing the motion (41) gives

$$\mathbf{v}(t)|_{t\to-\infty}=\mathbf{v}_1.$$

For large positive values of time we obtain from (41)

$$\mathbf{v}(t)|_{t\to\infty}=\mathbf{v}_2.$$

Therefore, expression (41) describes the motion, in which the initial velocity is equal to $\mathbf{v}_1$, and the final velocity is equal to $\mathbf{v}_2$. The transition from $\mathbf{v}_1$ to $\mathbf{v}_2$ occurs smoothly during a time interval, equal in order of magnitude to T. Knowing the velocity of the charge (41), it is easy to find its position as a function of time

$$\mathbf{L}(t)=\frac{\mathbf{v}_1+\mathbf{v}_2}{2}\,t+\frac{\mathbf{v}_2-\mathbf{v}_1}{2}\,T\ln\mathrm{ch}\,\frac{t}{T}. \qquad (42)$$

The motion of the charge according to (41), (42) is accompanied by radiation. Let us estimate the behavior of the radiated field at high frequencies. Equation (16) for the saddle point has the form

$$\frac{c}{\cos\theta}=\left(\frac{v_1+v_2}{2}+\frac{v_2-v_1}{2}\,\mathrm{th}\,\frac{t_0}{T}\right). \qquad (43)$$

Solving this equation, we obtain for t_0 the value

$$t_0=t_1+it_2=\frac{T}{2}\ln\frac{1-(v_1/c)\cos\theta}{1-(v_2/c)\cos\theta}+i\pi\frac{T}{2}. \qquad (44)$$

Thus,

$$t_1=\frac{T}{2}\ln\frac{1-(v_1/c)\cos\theta}{1-(v_2/c)\cos\theta},\quad t_2=\frac{\pi T}{2}. \qquad (45)$$

The imaginary part of the saddle ponit t_0 is equal in order of magnitude to the time T, during which the acceleration of the charge takes place. Since the variation of the speed is accompanied by radiation, the quantity t_2 gives the effective time, during which radiation takes place.

Using the asymptotic expression (20) for the amplitudes of radiated waves at high frequencies, and also expressions (41), (42), (44), and (45), we obtain that the dependence on the frequency is determined by the relation:

$$l\approx\begin{cases}\exp\left[-\pi\omega T\left(1-\frac{v_2}{c}\cos\theta\right)\right] & \text{for}\quad v_2>v_1,\\ \exp\left[-\pi\omega T\left(1-\frac{v_1}{c}\cos\theta\right)\right] & \text{for}\quad v_2<v_1.\end{cases} \qquad (46)$$

In expressions for the intensity of radiation, the exponents in (46) are doubled. Obviously the decay law (46) is valid also when the following inequalities hold

$$\omega\gg\begin{cases}\left(\pi T\right)^{-1}\left(1-\frac{v_2}{c}\cos\theta\right)^{-1} & \text{for}\quad v_2>v_1,\\ \left(\pi T\right)^{-1}\left(1-\frac{v_1}{c}\cos\theta\right)^{-1} & \text{for}\quad v_1>v_2.\end{cases} \qquad (47)$$

These asymptotic properties of radiation can also be obtained from rigorous theory.[12]

2.3. Radiation from a charge moving uniformly along a given bounded segment taking into account smooth acceleration at the beginning of the path and smooth deceleration at the end of the path

Consider now the case when a charge is initially at rest, and then starts accelerating smoothly, and within the time interval T reaches the velocity $\mathbf{v}$. During the time interval 2 T_0 the charge moves with the constant velocity $\mathbf{v}$, then decelerates, and during the time interval T the charge velocity again decreases to zero. We shall consider the following law of velocity variation:

$$\mathbf{v}(t)=\frac{\mathbf{v}}{2}\left(\operatorname{th}\frac{t+T_0}{T}-\operatorname{th}\frac{t-T_0}{T}\right). \qquad (48)$$

Here T is the time during which acceleration takes place and time of the subsequent deceleration, and 2 T_0 is the time during which the charge moves with the constant velocity $\mathbf{v}$. It is assumed that $T_0\gg T$.

If the time T of acceleration or deceleration tends to zero, the equation describing the motion of the charge changes, and in the limit when $T=0$ the velocity of the charge changes by a jump. At the beginning of the motion $t=-T_0$, the velocity of the charge changes abruptly from 0 to $\mathbf{v}$, and at the end of the motion ($t=T_0$) changes just as abruptly, from $\mathbf{v}$ to zero after the charge has travelled the distance $2vT_0$ with constant velocity. Radiation of the charge during the abrupt change in velocity was studied for the first time by I. E. Tamm.[11] It was shown that in this case, the radiation spectrum at high frequencies oscillates with the frequency, but does not tend to zero. It is obvious that if the velocity varies smoothly the radiation spectrum at high frequencies, as has been already shown, must decay according to the exponential law.

Let us find the high-frequency asymptotic behavior of the spectrum arising in the case of motion according to expression (48). The saddle point t_0 is determined by equation (16), which in the case under consideration takes the form

$$\frac{c}{\cos\theta}=\frac{\mathbf{v}}{2}\left(\operatorname{th}\frac{t_0+T_0}{T}-\operatorname{th}\frac{t_0-T_0}{T}\right) \qquad (49)$$

Solving this equation, we obtain

$$t_0=t_1+it_2=\frac{T}{2}\{i\pi+\ln[b\mp(b^2-1)]^{1/2}\}, \qquad (50)$$

where

$$b=\operatorname{ch}\frac{2T_0}{T}-\frac{v}{c}\cos\theta\operatorname{sh}\frac{2T_0}{T} \qquad (51)$$

and it is assumed that $T_0\gg T$.

If the velocity of the particle varies according to expression (48), the dependence of the position on time is determined by the expression

$$\mathbf{L}(t)=\frac{\mathbf{v}T}{2}\ln\frac{\operatorname{ch}[(t+T_0)/T]}{\operatorname{ch}[(t-T_0)/T]}. \qquad (52)$$

Taking into account expressions (20), (48), and (51), we obtain the asymptotic dependence of the radiation amplitude $\mathbf{l}$ on the frequency ω

$$\mathbf{l}\sim\exp\left[-\pi\omega T\left(1-\frac{v}{c}\cos\theta\right)^{-1}\right]=\exp\left(-\frac{\pi T}{T_\varphi}\right), \qquad (53)$$

where T is the time during which the velocity of the particle changes slowly (the acceleration time at the beginning or the deceleration time at the end), and T_φ is the time of the formation of radiation at the frequency ω [see expression (27)].

If the acceleration (or deceleration) time T is equal to zero, exponential decay (53) does not occur. Therefore, the assumption of instantaneous acceleration or deceleration leads to the loss of short-wave asymptotic behavior.

Expression (53) is valid if

$$\omega\gg\frac{1}{\pi T[1-(v/c)\cos\theta]}, \qquad (54)$$

or

$$T\gg T_\varphi$$

Radiation taking place when the charge moves according to (48), can be calculated in closed form. Substituting the equation of motion into expression (8) for the amplitude of the radiation field and taking into account that the position of the charge depends on the time according to (52), we obtain the following expression for the amplitude of the radiation field[24]

$$\mathbf{l}=\frac{\mathbf{v}T}{4}\left(1-e^{-4T_0/T}\right)\frac{\pi\omega T}{\operatorname{sh}(\pi\omega T/2)}F$$

$$\left(1-\frac{i\omega T}{2}\frac{v}{c}\cos\theta,\ 1+i\omega\frac{T}{2},\ 2;\ 1-e^{-4T_0/T}\right), \qquad (55)$$

where $F(\alpha,\beta,\gamma;z)$ is the hypergeometric function.[23] One might try to determine the asymptotic behavior of the radiation spectrum directly with the aid of the exact formula (55), but we found it simpler to use the method of steepest descent right away.

2.4. Harmonic oscillator of a finite amplitude

Consider a point charge oscillating along the z-axis according to

$$z = A \sin \omega_0 t. \tag{56}$$

Obviously, the velocity of the motion depends on the time in the following way:

$$v = \frac{dz}{dt} = A\omega_0 \cos \omega_0 t. \tag{57}$$

Let us determine the asymptotic behavior of the radiation spectrum in the case of motion described by equations (56), (57). Equation (16) for the saddle point in that case has the form

$$A\omega_0 \cos \omega_0 t_0 = \frac{c}{\cos\theta}. \tag{58}$$

This equation coincides, in essence, with the already considered equation (32), which determines the stationary phase point for synchrotron radiation. Performing similar calculations, we obtain for the asymptotic behavior of the amplitude of the radiation from a harmonic oscillator, the expression

$$I \sim e^{-\omega/3\omega_0\gamma^3}; \tag{59}$$

the same formula also follows from the rigorous theory.[4]

3. HIGH-FREQUENCY ASYMPTOTIC BEHAVIOR OF THE RADIATION SPECTRUM OF CHARGES MOVING UNIFORMLY IN NONSTATIONARY AND INHOMOGENEOUS MEDIA

We shall now study the high-frequency asymptotic behavior of the radiation spectrum of a charge moving uniformly in a medium with a dielectric permittivity ε which varies smoothly in time or in space. We consider first the case of a nonstationary medium with the refractive index $n = \varepsilon^{1/2}$ varying smoothly with time. We assume that the change of the dielectric permittivity of the medium is occurring sufficiently slow, so that the characteristic time of the change of n is much greater than the period of the radiated wave and the medium relaxation time. This will allow us to use the approximation of geometrical optics, and also a quasistationary value of the dielectric permittivity.

Assume now that the charge q is moving with the constant velocity v along the z axis in a medium the refractive index of which depends on time according to $n = n(t)$. Using Maxwell equations it is easy to obtain the equation for the electric displacement $\mathbf{D}$:

$$\Delta\mathbf{D} - \frac{n^2(t)}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2} = 4\pi\left(\operatorname{grad}\rho + \frac{n^2(t)}{c^2}\frac{\partial\mathbf{j}}{\partial t}\right), \tag{60}$$

where the charge density ρ and current density j are described by the following expressions:

$$\begin{aligned} \rho &= q\delta(\mathbf{r} - \mathbf{v}t), \\ \mathbf{j} &= q\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t). \end{aligned} \tag{61}$$

We expand the electric displacement $\mathbf{D}$ into the space Fourier integral of the form

$$\mathbf{D}(\mathbf{r}, t) = \int d^3\mathbf{k}\mathbf{D}_\mathbf{k}(t)\, e^{i\mathbf{k}\mathbf{r}}; \tag{62}$$

a similar expansion is used also for the right side of equation (59). Then, taking (61) into account, we obtain for the spatial Fourier component of the displacement $\mathbf{D}_\mathbf{k}$ the following equation:

$$\mathbf{D}_\mathbf{k}'' + \tilde{n}^2(t)\,\omega_0^2\mathbf{D}_\mathbf{k} = i\frac{q}{2\pi^2}\left(\mathbf{v}(\mathbf{k}\mathbf{v}) - \tilde{n}^2(t)\,c^2\mathbf{k}\right)e^{-i(\mathbf{k}\mathbf{v})t}, \tag{63}$$

where $\omega = \mathbf{k}c$, $\tilde{n} = 1/n$, and the prime means differentiation with respect to time. As has been shown in Ref. 16, radiation in a nonstationary medium is determined only by the transverse component of the displacement vector

$$\mathbf{D}_\mathbf{k}^{\mathrm{tr}} = \mathbf{D}_\mathbf{k} - \mathbf{k}\,(\mathbf{k}\mathbf{D}_\mathbf{k})k^{-2}. \tag{64}$$

For this reason we shall obtain and solve the equation for $\mathbf{D}_\mathbf{k}^{\mathrm{tr}}$. It has the form

$$(\mathbf{D}_\mathbf{k}^{\mathrm{tr}})'' + \tilde{n}^2(t)\,\omega_0^2\mathbf{D}_\mathbf{k}^{\mathrm{tr}} = Be^{-i(\mathbf{k}\mathbf{v})t}, \tag{65}$$

where

$$B = i\frac{q}{2\pi^2}(\mathbf{k}\mathbf{v})\left(v - \frac{\mathbf{k}(\mathbf{k}\mathbf{v})}{k^2}\right).$$

Introducing the new function $y(t)$ such that $\mathbf{D}_\mathbf{k}^{\mathrm{tr}} = By$, we obtain for y the following equation:

$$y'' + \tilde{n}^2(t)\,\omega_0^2 y = e^{-i(\mathbf{k}\mathbf{v})t}. \tag{66}$$

Since we are interested in the high-frequency asymptotic behavior of the radiation spectrum, we shall use the method of geometrical optics in the following discussion. Solutions of equation (66) without the right side obtained in this approximation have the form[17]

$$y_{1,2}(t) = [n(t)]^{1/4}\exp\left(\pm i\omega_0\int^t \tilde{n}(t')\,dt'\right). \tag{67}$$

Solutions (67) represent two waves propagating in opposite directions (along and against the direction of the vector $\mathbf{k}$). Further, using (67) and the fact that as $t \to -\infty$ radiation is absent, we obtain a solution of the inhomogeneous equation (66):

$$\begin{aligned} y(t) = \frac{1}{2i\omega}\Big[&-y_1(t)\int_{-\infty}^{\infty} y_2(\tau)\,e^{-i(\mathbf{k}\mathbf{v})\tau}d\tau \\ &+ y_2(t)\int_{-\infty}^{\infty} y_1(\tau)\,e^{-i(\mathbf{k}\mathbf{v})\tau}d\tau \\ &+ \int_t^{\infty}(y_2(\tau)\,y_1(t) - y_2(t)\,y_1(\tau))e^{-i(\mathbf{k}\mathbf{v})\tau}d\tau\Big]. \end{aligned} \tag{68}$$

The last term in (68) is proportional to the Fourier component of the field of a uniformly moving charge; this term is not related to radiation. The waves radiated by a uniformly moving charge in a nonstationary medium are determined by the first two terms in (68); we denote these terms by $Y_1(\mathbf{k})$ and $Y_2(\mathbf{k})$, respectively. In order to find the angular and spectral distribution of the radiation energy it is necessary to take into account that in the direction $\mathbf{k}$ (at an angle θ

to the z axis) not only the wave $Y_2(\mathbf{k})$ is radiated but also the wave $Y_2(-\mathbf{k})$. For this reason, from (67) and (68) it follows that the high-frequency asymptotic behavior of the radiation spectrum is determined by the square of the absolute value of the asymptotic expression (for $\mathbf{k}\to\infty$) of the following integral:

$$\int_{-\infty}^{\infty} [n(\tau)]^{1/2} \exp\left[i\left(kc\int^{\tau} \tilde{n}(t')\,dt' - kv\cos\theta\,\tau\right)\right] d\tau. \quad (69)$$

Comparison of formulas (69) and (8) shows that the integrands in them are, essentially, analogous. Therefore, the general conclusions drawn by us regarding the asymptotic behavior of integral (8) remain valid also for integral (69). Namely, the point τ_0, in which the phrase of the exponent in (69) does not change, is determined by equation

$$kc\tilde{n}(\tau_0) - kv\cos\theta = 0 \quad (70)$$

or

$$1 - \frac{v}{c} n(\tau_0)\cos\theta = 0 \quad (70')$$

If the condition for the existence of Vavilov-Cherenkov radiation is not fulfilled (and for sufficiently high frequencies it is not fulfilled, since in this case $n<1$), the solution of equation (70) necessarily has a nonzero imaginary part:

$$\tau_0 = \tau_1 + i\tau_2, \quad (71)$$

and this in the end determines the exponential decay of radiation energy for large values of $\mathbf{k}$, and, as in the case of (22), the damping factor can be presented in the form

$$\exp\left(-kc\left[F_2(\tau_0) - \frac{v}{c}\cos\theta\cdot\tau_2\right]\right), \quad (72)$$

where $F_2(\tau_0)$ is the imaginary part of the function

$$F = \int^{\tau_0} \tilde{n}(t')\,dt.$$

Using (72), it is easy to obtain the asymptotic behavior of radiation spectra for a number of specific models, describing the time variation of the index of refraction of a nonstationary medium (for example, for the case of a smooth transition of the index of refraction from a constant initial to a constant final value, or for the case when it changes periodically with time). However, as can be seen from (69) and (72), such calculations are analogous to calculations carried out above, and we do not give them here. We remark only that the rigorous solution of the problem of radiation by a uniformly moving charge in the case of a smooth variation of dielectric permittivity of a medium from one constant value to another is obtained in Ref. 18.

Consider now the asymptotic behavior of the transition radiation spectrum of a uniformly moving charge in an inhomogeneous medium. For simplicity we consider the case when the dielectric permittivity of the medium depends only on one coordinate z, and the charge also is moving along the z axis with the velocity v. Since we are interested in high-frequency radiation, we shall again use the approximation of geometrical optics. The quasiclassical solution of the formulated problem is given in Ref. 19. Namely, the spectral intensity of the x rays radiated forward is proportional to the square of the absolute value of the following integral:

$$\int_{-\infty}^{\infty} \exp\left[i\left(\frac{\omega}{v}u - \int^{u}\lambda(z')\,dz'\right)\right] du, \quad (73)$$

where ω is the frequency of the radiated wave, $\lambda(z) = [(\omega^2/c^2)\varepsilon(z) - \varkappa^2]^{1/2}$, $\varkappa$ is the transverse (relative to the z axis) component of the wave vector. Obviously, expression (73) is analogous to expressions (8) and (69). For this reason we can assert also for the case of transition radiation (if λ is a smooth function of z) that the high-frequency asymptotic behavior of the spectrum will be exponential.

In conclusion we note that in all the problems solved to date concerning transition radiation in media with dielectric permittivity smoothly varying in space (for example, radiation from a smeared-out boundary,[20] in a plate with smeared-out boundaries,[21] resonance radiation[22]) specifically exponential asymptotic behavior is obtained.

CONCLUSIONS

In this note we have considered a general property of the radiation spectra which arise during the motion of a charged particle. We can formulate this property in the following way: if the expression describing the motion of the particle is determined by a smooth function, then the spectrum of radiation falls off at high frequencies exponentially. We have derived the asymptotic formula for the indicated case (i.e., on the assumption that the specific expression describing the motion of the charged particle is given by an analytic function) and studied the examples of some specific equations of motions that are of physical interest. Some of the equation of motion that have been considered are, possibly, only models helping to clarify the essential characteristics of the radiation spectrum. This can be said, for example, concerning equations of motion (41) and (48). However, the asymptotic behavior of the radiation spectrum discussed above is general, and does not depend on a specific model. Spectral measurements in the cases of synchrotron and undulator radiations are in agreement with this conclusion.[25,26]

In our article we have limited our treatment to studying radiation from a single particle moving along a given trajectory. If the source of radiation is a system of moving particles, the intensity of radiation is more complicated (some of these examples are considered in the article of N. P. Klepikov[27]). However in the case, when the laws of motion of all the particles are expressed by smooth functions, the radiation spectrum of the system will decay exponentially, beginning with some frequency characteristic for the given system.

The radiation spectrum of a charge moving uniformly in an inhomogeneous nonstationary medium has a similar asymptotic behavior, if the temporal and spatial variation of the properties of the medium is described by a smooth function.

Therefore, exponential decay of the radiation spectrum at high frequencies is a general property of radiation for a rather large class of processes, for which a slow variation of parameters either of the medium or of the moving source is typical.

In conclusion we note that both quantum and classical theories lead, generally speaking, to various predictions, concerning the high-frequency behavior of the radiation spectrum. According to classical theory, spectral intensity of radiation is different from zero (although it is small) for arbitrarily high frequencies. According to the quantum theory of radiation, however, the radiation spectrum disappears abruptly at the frequency $\omega_b = E/\hbar$, where E is the energy of the radiating particle, $\hbar$ is the Planck constant. Taking this circumstance into account our results are applicable in the following case. Assume that the spectral intensity of radiation, calculated by classical theory, decreases exponentially, beginning with the frequency, which we denote by Ω. Then, if $\Omega \ll E/\hbar$, classical theory gives an exponential decrease of spectral intensity in the frequency range from Ω to approximately $E/\hbar$, and this result does not contradict quantum theory.

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