Khokhlov's method in the theory of nonlinear waves

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R. V. Khokhlov's (1926–1977) papers on nonlinear optics and nonlinear acoustics are reviewed.

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1. NONLINEAR WAVES AND NONLINEAR OSCILLATIONS

The nonlinear wave is perhaps one of the most popular entities in modern physics.

Two striking examples of nonlinear waves—the Riemann shock wave and the Scott Russell soliton—arose in physics back in the last century.

Over the long history of research on nonlinear wave processes, however, the comparatively short time interval from the late 1950s to the early 1970s was a time of particularly marked progress. It would be no exaggeration to say that it was during this interval that the basic physical ideas were completely formulated, and a successful mathematical apparatus of the theory of nonlinear waves was developed.

The primary reason for this leap forward was undoubtedly experimental progress: the development of powerful sources of radiation and the development of effective nonlinear materials. This was the case in rf physics, optics, and acoustics.

In the mid-1950s, rf physicists acquired the capability to observe nonlinear electromagnetic waves in the laboratory. Electromagnetic shock waves were produced in artificial lines loaded with nonlinear ferrite cores. Parametric amplifiers and traveling-wave converters were developed from lines with semiconductor diodes and microwave systems with long electron beams.

Frequency doubling in a crystal was first observed in 1961. This experiment was the founding of laser nonlinear optics. Progress in the development of high-power lasers and of nonlinear optical materials caused nonlinear optical phenomena also to acquire much practical importance in only three years. Tunable parametric light sources were developed, and optical frequency multipliers with an efficiency reaching considerably in excess of 10% were also developed.

High-power ultrasonic generators were constructed in the early 1950s. These generators substantially increased the opportunities in nonlinear acoustics. They made it possible to undertake a systematic study of nonlinear acoustic phenomena in liquids and solids.

During the same period, nonlinear waves in plasmas attracted increased interest. The challenge presented by experiments posed some serious questions for the theory.

Here is how the situation which had developed was described by R. V. Khokhlov in a paper[1] published in a 1965 issue of Uspekhi and dedicated to the memory of L. I. Mandel'shtam: "The mastery of the millimeter- and submillimeter ranges of electromagnetic waves and recent progress in the development of lasers have had the consequence that nonlinear wave processes have begun to play in fact a governing role .... It has become necessary to derive a nonlinear wave theory which would, after the example of the nonlinear theory of oscillations, generalize the multitude of individual phenomena in the behavior of various devices . ... ."

These words summarize the rf-physics "oscillation" approach to the theory of nonlinear wave processes, which has proved extremely successful and heuristic.

During the development of this theory, especially in the first stage, when the basic physical concepts were forming, the theoretical approaches—the principles of "thinking nonlinearly"—which had developed from the nonlinear theory of oscillations (and which were vividly described in a paper by A. A. Andronov dedicated to the memory of L. I. Mandel'shtam[2]) played a decisive role. In a paper[1] which we have already mentioned, Khokhlov wrote the following, in noting the contribution which had been made to research on nonlinear wave processes by the group at the Lebedev Physics Institute led by N. G. Basov and A. M. Prokhorov, by the rf physicists at Gor'kiĭ, led by A. V. Gaponov, and by the rf

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physicists at Moscow State University, "It is fair to say that this problem [the derivation of a theory of nonlinear waves—S. A.] has now been solved to a large extent. Some graphic qualitative concepts and representations have been found, some guiding "wave" concepts have been developed, and a systematic study has been made of a [long] list of nonlinear wave phenomena. ..."

Khokhlov himself made a hugh contribution to this field of research. The theoretical methods which he developed, the physical ideas which he introduced, and the specific results which he along with his colleagues and students obtained in nonlinear optics, nonlinear acoustics, x-ray optics, and γ-ray optics laid the foundation for scientific directions which have been developing rapidly up to the present day and which have had a strong influence on the nature and style of physical and applied research in these fields. Rem Viktorovich Khokhlov's life ended early, and I never had an occasion to hear him make any sort of appraisal of his own work or, especially, say anything with regard to a classification or hierarchy of his papers.

Time puts everything in its place, and it is time for a detailed analysis of Khokhlov's scientific work. However, the present paper, which is being published in the sixtieth year after the birth of Khokhlov, has a narrower scope. We are attempting here to trace the fate of the ideas which he expressed in the early 1960s in his first studies of the theory of nonlinear waves.

1986 is twenty five years after the publication of two fundamental studies by Khokhlov, "On the theory of shock radio waves in nonlinear lines" [3] and "On the propagation of waves in nonlinear dispersive lines." [4] Despite the "narrow" rf-physics title (and here we are seeing Khokhlov's scientific style—we find in his work almost nothing in the way of a purely methodological study; the methods which he developed in these papers were used immediately to solve specific problems), it is fair to classify the first of these papers as one of the fundamental studies in the field of nonlinear acoustics. In evaluating this paper in Ref. 4 from today's viewpoint, on the other hand, we should emphasize two points in particular. That paper included a formulation of the method of slowly varying amplitudes for analyzing a particular case of the interaction of plane waves in highly dispersive media. In addition, there is a detailed analysis of the conditions for a nearly 100% pumping of energy of the fundamental wave into the second harmonic: a prediction which has been confirmed completely in recent experiments with high-power laser systems developed for controlled fusion research.

The papers in Refs. 3 and 4 undoubtedly contain the most systematic and clear extension to wave problems of the approaches developed in the nonlinear theory of oscillations. This is, however, by no means the total importance of those papers. Khokhlov brought to the nonlinear theory of waves not only a brilliant mastery of the apparatus of the theory of nonlinear oscillations but also an original method which he had developed for analyzing and radically simplifying the so-called truncated equations describing the behavior of nonlinear systems. This method, which is now called the "Khokhlov method," has proved particularly effective in analyzing transients and the stability of complex nonlinear systems.

FIG. 1. R. V. Khokhlov at a seminar in the department of wave processes at Moscow State University (1976).

In the theory of nonlinear waves, problems of this type are of primary interest. In the overwhelming majority of cases we are interested in the dynamics of nonlinear interactions or self-effects which develop as waves propagate: a spatial "transient process," an analog of the temporal transient process in an oscillatory system.[1)] Khokhlov's method has accordingly found a variety of applications in the theory of nonlinear waves. The possibilities of this method were first demonstrated particularly strikingly by Rem Viktorovich Khokhlov himself, who derived some physically transparent results in several difficult problems in the theory of nonlinear systems.

2. METHOD OF STEP-BY-STEP SIMPLIFICATION OF TRUNCATED EQUATIONS: KHOKHLOV'S METHOD

Khokhlov obtained these results perhaps as early as 1951–1953. He began as a "pure" theoretician. The mathematical side of things was prominent in his studies, particularly in his early studies.[2)] His primary interests at the time, however, were problems in the nonlinear theory of oscillations.

In the early 1950s, the state of affairs in the nonlinear theory of oscillations was undoubtedly completely satisfactory. Approximate methods had been developed for analyzing nonlinear phenomena, and these methods had been given a firm footing. Results obtained in this work have become parts of the classical monographs.[7,8] However, the complete clarity regarding the general approach to nonlinear problems was still far from guaranteeing the effective derivation of specific results. This comment applies in particular to the nonautonomous operation of nonlinear oscillatory systems and research on transients under very nonlinear conditions.

FIG. 2. International Conference on Quantum Electronics (1968). R. V. Khokhlov is at left front; to his right is N. Bloembergen (Harvard University).

Khokhlov took up these problems. The first in the series became the classical problem of the synchronization of a self-excited oscillator by an external force. The reference here is to a system which is described by a nonlinear oscillator equation with a right side, which is an external, and generally a modulated force:

$$\ddot{x} - 2\delta(x)\dot{x} + \omega_0^2 x = \omega_0^2 E \cos \omega t. \tag{1}$$

Using the conventional averaging method, introducing the slowly varying amplitude and phase $A(t)$ and $\varphi(t)$ (the "Van der Pol variables")

$$x(t) = A(t) \cos [\omega t - \varphi(t)],$$

we transform from dynamic equation (1) to the truncated equations for the slowly varying amplitude and phase:

$$\dot{A} = \delta(A) A + \frac{E\omega_0}{2} \cos \varphi, \tag{2a}$$

$$\dot{\varphi} = \omega - \omega_0 - \frac{E\omega_0}{2A} \sin \varphi. \tag{2b}$$

Equations (2), which approximate (1) within terms $(\delta/\omega_0)^2$, are simpler than (1), but they also cannot be solved in quadrature even for $E = \text{const}$. The case of a small amplitude of the external force is particularly difficult. Consequently, back in the early 1950s, the study of (2) was essentially limited to a qualitative analysis in the phase plane; this work was carried out by Andronov and Vitt (see Ref. 7). This situation of course did nothing to satisfy the needs of experiments. In the 1950s, the possibility of using synchronized oscillators as active filters came under widespread discussion, so that specific data were required on the synchronization of an oscillator by modulated signals, e.g., with $E = E(t)$.

The most obvious way out of the difficulty is numerical integration. However, in a paper[9] published in Doklady Akademii Nauk in 1954, Khokhlov showed that in cases of practical interest there is no need to resort to numerical integration: Equations (2) are amenable to yet another, "secondary," simplification, and then they can be solved in quadrature. The idea underlying these simplifications is estimating the comparative order of magnitude of the small terms on the right sides of (2a) and (2b). In the case of synchronization by a weak signal, $E/A_0 < \delta/\omega_0$ (A_0 is the amplitude of the steady-state self-excited oscillations), and at deviations from the resonant frequency not too much larger than the locking region, we have $|\omega - \omega_0| \sim E\omega_0/2A$.

In this case, therefore, the right sides of (2a) and (2b) are small quantities of different orders of magnitude. The right side in (2b) is far smaller than that in (2a), so that the phase changes more slowly than the amplitude. Over a time of the order of $\tau \approx (|d\delta/dA|A_0)^{-1}$, the amplitude nearly reaches the steady-state value A_0, and the equation for the slowly varying coordinate—the phase—becomes

$$\dot{\varphi} = \omega - \omega_0 - \frac{E\omega_0}{2A_0} \sin \varphi; \tag{3}$$

the theory for the synchronization by a weak signal thus reduces to the anlaysis of a single first-order "phase" equation.

Equation (3) can be integrated easily. We then find the corrections to the "rapidly varying" amplitude.

The secondary simplification of the truncated equations based on the separation of the "slow" and "fast" variables in Van der Pol space has proved to a very successful approach. Khokhlov used this method to solve several difficult problems involving the mutual synchronization of oscillators,[3)] the fractional synchronization of oscillators, the pulling of self-excited oscillations, and of the stability of self-excited oscillations, including the stability of the self-excited oscillations in a molecular generator. A very attractive aspect of this secondary simplification is undoubtedly the "physical content" and graphic nature of the computation procedure itself. The results are also simple to interpret. For example, the only transparent interpretation of the mechanism for the locking of a vacuum-tube oscillator by an external force was found with the help of phase equation (3). It should be noted that Khokhlov considered these points very important.

It would be completely justified to say that in his series of studies on nonlinear oscillations Khokhlov demonstrated a higher level of "nonlinear physical thinking," at the Mandel'shtam level.

At the same time, in these papers one can clearly see that the effort is being directed toward obtaining a physical result and analyzing a specific phenomenon.

Rem Viktorovich Khokhlov published almost nothing in the way of methodological papers, and he was essentially uninterested in formal generalization of the methods which he proposed. His general attitude toward the secondary simplification and its possibilities can be found perhaps only in his doctoral dissertation,[10] which he defended in 1962. But at the time, Refs. 3 and 4 had already been published, and the prospects for the method in the theory of nonlinear waves had already become clear.

Following Ref. 10, we can characterize the overall situation in this way.

A rather general property of the system of first-order truncated equations, which describe the motion of a nonlinear system in the space of the Van der Pol variables (x_i, y_i),

$$x_i = F_i\,(x_1, \ldots, x_n, y_1, \ldots y_n, \mu t), \quad (4a)$$

$$y_i = \Phi_i\,(x_1, \ldots, x_n, y_1, \ldots, y_n, \mu t) \quad (4b)$$

[system (2), discussed above, is a particular case of (4)], is the circumstance that for some of these equations the right sides of the truncated equations are small. For example, we have $F_i \sim \mu$, where $\mu \ll 1$ is a characteristic small parameter. This small parameter makes it possible to select from a complex motion of a nonlinear system some rapid changes in certain variables and some slow changes in others. It is thus possible to *lower* the order of the system (and we again run into the example of the synchronization problem discussed above).

A general analysis of (4), a justification of the separation of fast and slow variables, and methods for calculating the corrections to the fast motions are given in the dissertation.[10] The relationship between the new method and earlier studies, particularly studies in the theory of relaxation oscillations, is also examined there.[4)]

The principles for the separation of motions and the step-by-step simplification of the truncated equations can be illustrated most simply by the example in which the right sides of (4) are independent of t, and there are only the two variables (x,y).

In this case we find from (4) an equation for the paths in the phase plane:

$$\frac{dy}{dx} = \frac{1}{\mu}\,\frac{\Phi(x, y)}{F(x, y)}\,. \quad (5)$$

According to (5), except for a small neighborhood of $\Phi(x,y) = 0$ in the x,y plane the integral curves are nearly vertical ($\mu \ll 1$). In certain steps the motion thus occurs in such a way that the coordinate y rapidly reaches the values corresponding to an essentially constant variable x, while in other steps the coordinate x varies slowly, and y adjusts in a quasistatic way to these changes. In each step, of course, the truncated equations resulting from the second simplification are operative and can be used to calculate the characteristics of the motion in detail.

We wish to emphasize once more that the step-by-step simplification of the truncated equations is *a more radical procedure* than a transformation from the original dynamic equations to the truncated ones: In the step-by-step simplification, the order of the system of differential equations is lowered.

The technique of a step-by-step analysis of the truncated equations was used very effectively by Khokhlov and, later, by many of his successors in a long list of nonlinear wave problems. As we have already mentioned, it was a study of a transient nonlinear process which was the most prominent feature of most of these studies.

Studies in the theory of nonlinear waves and various studies in nonlinear wave phenomena in the rf range, optics, and acoustics undoubtedly occupy the leading position in the extensive scientific heritage of Khokhlov.

A characteristic feature of these studies, particularly those in nonlinear optics, is an intimate relationship with experiment. Here we are thinking primarily of the experiments which were carried out at the University with the personal involvment of Khokhlov himself. This circumstance had a very strong influence on the choice of, and the evaluation of the significance of, problems and methods for solving them.

In this short paper, which is devoted primarily to the scientific fate of his theoretical studies, we will restrict the discussion to nonlinear optics and nonlinear acoustics.

In an effort to maintain a historical perspective, I would like to recall some events from one of the most productive stages in the development of the physics of nonlinear waves. Although these events are not in the remote past, they are nevertheless already becoming a part of history.

3. APPROXIMATE EQUATIONS OF NONLINEAR OPTICS

The present stage of the development of nonlinear optics began with the invention of lasers. The twenty-fifth anniversary of the laser in 1985 was simultaneously an anniversary of nonlinear optics. The optical nonlinearity of an active medium decisively determines many unique properties of laser light. It was just a year after the operation of the first laser that a nonlinear optical response of an unexcited passive medium was detected: The extremely wide opportunities for studying and using nonlinear phenomena in essentially all branches of optics were discovered. In the years since this event, the sphere of nonlinear optics has expanded to a huge extent, as has the spectrum of problems which have been studied with the help of nonlinear optics. Many of these problems (in particular, those such as the physics of nonlinear agencies acting on matter and laser-induced phase transitions) long ago went beyond the scope of physical and applied optics in their conventional interpretation. One of the most important trends has been a close interaction between, and a mutual borrowing of methods from, quantum electronics and nonlinear optics.

The twenty-fifth anniversary of the laser was the subject of a special issue of *Uspekhi*. The history of the development of the basic ideas of quantum electronics and the physics of lasers was outlined in papers by Basov[12] and Prokhorov.[11]

Research on laser nonlinear optics[5)] was begun in 1962 in the physics department at Moscow State University with a study of the parametric interactions of light waves.[13] Although originally the primary motivation for this work was to develop a continuously tunable optical generator (such generators, spanning the near-IR range were put into operation in 1965 at Moscow State University), a theoretical apparatus of nonlinear wave optics was also being developed in parallel. The foundations for this theoretical work were essentially laid down by Khokhlov in Ref. 4, on the generation of the second harmonic in a long nonlinear line. That study was carried out a year before the experiment of Franken *et al.*[16] on the doubling of the frequency of the beam from a ruby laser in a quartz crystal.

At this point, however, it became necessary specifically to take into account some factors of primary interest for optics: the anisotropy of the response [as was first demonstrated by Giordmaine[17] and by Terhune *et al.* (see Ref. 18), it is the anisotropy which makes it possible to synchronize the phases of waves with very different frequencies in crystals], the nonlocal nature of the responses (in space and time), the spatial boundedness of the beams, etc. In other words, it was necessary to develop regular methods for studying nonlinear wave processes described by equations for a field,

$$[\nabla[\nabla \mathbf{E}]] + \frac{1}{c^2}\frac{\partial^2 \mathbf{P}^{(L)}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}^{(NL)}}{\partial t^2} = 0, \qquad (6)$$

and by a system of constitutive equations describing a linear response,

$$\mathbf{P}^{(L)}(r, t) = \int_0^\infty \hat{\chi}^{(1)}(t')\,\mathbf{E}(r, t-t')\,dt', \qquad (7)$$

and a nonlinear response, $\mathbf{P}^{(NL)} = \mathbf{P}^{(NL)}(r,t)$, of the medium. In nonlinear optics, in the overwhelming majority of cases, still today, the nonlinear response (which was first analyzed in 1962 in the fundamental papers by Bloembergen and coworkers[19] (see also Ref. 20) can be written as an expansion in the small parameter[6)] $\mu = E/E_a \ll 1$ (E_a is the characteristic "atomic" field):

$$\mathbf{P}^{(NL)}(r, t) = \int_0^\infty dt' \int_0^\infty \hat{\chi}^{(2)}(t', t'')\,\mathbf{E}(r, t-t') \times \mathbf{E}(r, t-t'-t'')\,dt'' + \int_0^\infty dt' \int_0^\infty dt'' \int_0^\infty \hat{\chi}^{(3)}(t', t'', t'') \times \mathbf{E}(r, t-t')\,\mathbf{E}(r, t-t'-t'')\,\mathbf{E}(r, t-t'-t''-t''')\,dt''' + \ldots, \qquad (8)$$

so that we have $P^{(NL)} \ll P^{(L)}$.

Spatial-dispersion effects have been ignored in (7) and (8); the reader is referred to Refs. 50, 57, and 58 regarding effects of a nonlocal nature of the nonlinear response and a spatial dispersion.

The nonlinearity of the response leads to an energy exchange between waves with different frequencies and wave vectors (wave interactions) and nonlinear changes in the frequencies and angular spectra of quasimonochromatic, nearly plane waves (self-effects). In the course of the interactions and the self-effects, there is generally a nonlinear change in the polarization of the waves: Nonlinear polarization effects arise.

A natural way to simplify the system (6) + (8) is to use the method of slowly varying amplitudes. By virtue of the relation $\mathbf{P}^{(NL)} \ll \mathbf{P}^{(L)}$, the changes in the complex amplitudes which are caused by the nonlinearity can be regarded as slow both in space (at the scale of the wavelength) and in time (at the scale of the oscillation period).

Analysis of the methodological problems which arise in the first stage (a correct account of the dispersion of the medium, of the vector nature of the fields, of the anisotropy of the medium, etc.) was offered in the book in Ref. 14, which was submitted for publication in late 1963. Approximate equations of nonlinear optics were formulated in that book.

3.1. Plane waves: space-time analogies. For plane waves which are propagating through an anisotropic, slightly nonlinear medium,

$$\mathbf{E}(r, t) = \sum_n \mathbf{E}_n = \sum_n \mathbf{e}_n A_n(\mu t, \mu \mathbf{r}) \exp[i(\omega_n t - \mathbf{k}_n \mathbf{r})] + \text{c.c.} \qquad (9)$$

in the first approximation of dispersion theory, the system of integrodifferential equations (6)–(8) reduces to a system of truncated partial differential equations (cf. Ref. 14):

$$\mathbf{s}_n[\mathbf{e}_n[\mathbf{k}_n\mathbf{e}_n]]\frac{\partial A_n}{\partial t} + [\mathbf{e}_n[\mathbf{k}_n\mathbf{e}]]\nabla A_n + (\mathbf{e}_n\hat{\alpha}\mathbf{e}_n)A_n + \mathscr{P}^{(NL)}(\omega_n)e^{i\Delta_n \mathbf{r}} = 0 \qquad (10)$$

$\mathscr{P}^{(NL)}$ is a spectral component of the nonlinear polarization at the frequency ω_n, α_n is a tensor describing the loss in the medium, $\mathbf{s}_n$ is the ray vector, Δ_n is the difference between the wave vectors, and $\mathbf{k}_n$ is a wave vector).

Equations (10) are written for the case in which the nonlinearity has essentially no effect on the polarization state of the waves; opposite situations are of course also possible.[7)]

The application to (10) of the qualitative methods of the theory of nonlinear oscillations and the method of a secondary simplification results in a fairly clear picture of the generation of optical harmonics, the modulation of light by traveling waves, and the parametric amplification and generation of light. A corresponding series of studies was carried out in the nonlinear optics laboratory at Moscow State University in 1961–1963, and the results are summarized in Ref. 14.

As an example we examined the results of an analysis of the standard problem of the generation of a second harmonic of a plane sinusoidal wave.[4,14]

We are dealing here with an interaction in a medium with a quadratic nonlinearity of two waves, whose frequencies differ by a factor of two, so that the total field is

$$\mathbf{E} = \mathbf{e}_1 A_1 \exp[i(\omega t - k_1 z)] + \mathbf{e}_2 A_2 \exp[i(2\omega t - k_2 z)].$$

The truncated equations for the real amplitudes A_1 and A_2 and the phase $\Phi = \varphi_1 - \varphi_2 - \Delta z$ (here φ_1 and φ_2 are the phases of the interacting waves, and $\Delta = 2k_1 - k_2$ is the phase deviation) cannot be solved analytically in their general form.

Figure 3 shows integrated curves of the problem, in the plane $X = A_2 \sin\Phi, Y = A_2 \cos\Phi$ for the case $\Delta = 0$ (exact phase synchronization) and for various ratios of the phase deviation Δ to the nonlinearity parameter

$$\sigma = k_1\chi^{(2)}[A_1^2(0) + A_2^2(0)]^{1/2} = k_1\chi^{(2)}A_0.$$

Analysis in the phase plane of Fig. 3 along with a secondary simplification of the truncated equations made it possible in certain steps[8)] to calculate the spatial scales and to evaluate the efficiency of an optical frequency doubler [and of a degenerate parametric amplifier, if $A_2(0) \gg A_1(0)$] under various conditions.[14]

Two of the results remain extremely pertinent even today. It follows from Fig. 3 that in the case $\Delta = 0$ there can be a complete pumping of energy from the fundamental wave into the second harmonic (the phase path leaving the origin in Fig. 3a).

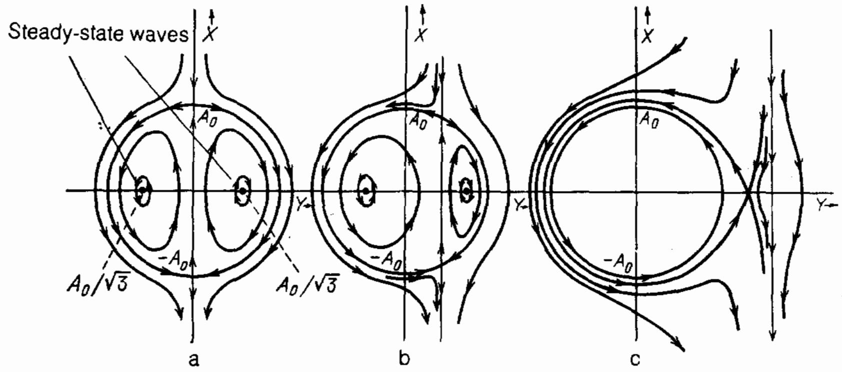

FIG. 3. Integrated curves of the truncated equations describing the generation of the second harmonic of a sinusoidal plane wave in a medium with a quadratic nonlinearity, plotted in the coordinates $X = A_2 \sin \Phi$, $Y = A_2 \cos \Phi$ (Refs. 4 and 14). a: $\Delta = 0$. b: $\Delta/\sigma < 1$. c: $\Delta/\sigma \gg 1$. Motion away from the origin of coordinates describes a doubling of the frequency; the pointers show stable points of the center type which correspond to the excitation of steady-state waves.

The experimental implementation occurred just recently. It was necessary to resolve several subtle questions associated with the stability of the phase relations during intense energy transfer.[59,60]

In addition to revealing the conditions for an intense energy exchange between the waves with frequencies ω and 2ω (depending on the boundary conditions, these conditions correspond to a frequency doubling with a high efficiency or an intense parametric amplification), analysis of the phase plane in Fig. 3 predicts the existence of steady-state nonlinear waves. These waves correspond to stable points of the nature of centers which are shown by the pointers.

The physical meaning is that under certain conditions the waves at the second harmonic and at the fundamental frequency propagate with constant amplitudes. Superficially, matters appear as they would if the medium were linear, but actually what is involved is a nonlinear interference effect.

We should point out that recent experiments, even in more complex situations, provide an increasing number of examples in which it is possible to observe steady-state waves which are the result of an interference between different nonlinear effects.[61,62]

Also related to the truncated equations in (10) are some extremely graphic physical pictures, which are especially pertinent to rf physics. Indeed, for unmodulated plane waves, system (10) is a system of nonlinear ordinary differential equations—nonlinear oscillations in a system with many degrees of freedom are described in exactly the same way. Consequently, nonlinear oscillatory and wave problems which are analogs arise. Corresponding to the spatial coordinate in the wave problem is the time in the oscillation problem; corresponding to the wave number k is the frequency ω; corresponding to boundary conditions are initial conditions; etc.

This space-time analogy was undoubtedly of heuristic value, especially in the first stage[9)] (Refs. 14 and 23).

In general, the oscillation approach and the ideas from the physics of nonlinear oscillations initially had a significant effect on the direction of the research and led to a clear differentiation between the studies carried out in nonlinear optics by the rf physicists and the optical physicists.

The rf physicists emphasized the study and practical use of the nonlinear interactions of light waves. The optical physicists, in contrast, were initially more interested in the self-effects and the nonlinear corrections to absorption and refraction. Today, of course, the scientific past of the people working in nonlinear optics can no longer be determined in such a simple way.

3.2. Diffraction of light in a nonlinear medium. Equations of the type in (10) can be used to derive a geometrical-optics theory of nonlinear optical phenomena, but the fundamental question of diffraction in a nonlinear medium remains beyond their scope.

The problem of correctly incorporating diffraction in the equations of nonlinear optics became an urgent one as early as 1963. The goal was to describe the nonlinear interactions which occur in strongly focused laser beams. A focusing which leads to an intensification of the optical field naturally increases a local nonlinear effect. On the other hand, there is a decrease in the size of the region with the strong field, and the length of the region of nonlinear interaction is reduced. Where is the optimum? This was the formulation of the diffraction problem of the "optimum focusing" in generators of optical harmonics and sum and difference frequencies in parametric light sources.

Nonlinear diffraction problems at a more fundamental level also arose in 1964: Active research on the self-focusing of light began.

Khokhlov was one of the first to point out that the "local" or Young treatment of diffraction is best suited to nonlinear problems, since its mathematical formulation calls in no way on the superposition principle, while the parabolic-equation method of Leontovich[24] is closely related to the method of slowly varying amplitudes.

In late 1963, Khokhlov suggested that Sukhorukov use the parabolic-equation method to examine the diffraction of light in a linear anisotropic medium.[25] A nonlinear parabolic equation describing the propagation of a wave beam in a nonlinear medium was derived in Ref. 26.

What was involved here was an extension of the method of slowly varying amplitudes to problems involving the nonlinear propagation of wave beams. In a slightly nonlinear medium, the changes in the complex amplitude along a beam (the beam vector **s**) in a light beam of finite cross section can be assumed to occur more slowly than changes across the beam, where there is transition into a shadow region.

For bounded beams, expression (9) should therefore be replaced by[25,26]

$$\mathbf{E}=\sum_n \mathbf{e}_n A_n\,(\mu \mathbf{r} s_n,\ \mu^{1/2}[\mathbf{r} s_n])\exp[i(\omega_n t-\mathbf{k}_n \mathbf{r})]+\text{c.c.} \tag{11}$$

(for simplicity, we are assuming sinusoidal waves).

Substituting (11) into (6), we find the following system of *second-order* truncated equations in place of (10), in the first approximation in μ:

$$[\mathbf{e}_n[\mathbf{k}_n\mathbf{e}_n]]\,\nabla A_n+\frac{i}{2}\Delta_\perp A_n+(\mathbf{e}_n\hat{\alpha}\mathbf{e}_n)A_n+\mathscr{P}^{(\mathrm{NL})}(\omega)\,e^{i\Delta_n \mathbf{r}}=0, \tag{12}$$

where $\Delta_\perp$ is the Laplacian corresponding to differentiation along the direction perpendicular to the beam.[10)]

The effectiveness of the approach based on (12) was demonstrated by a group at Moscow State University in the problem of optical harmonic generation in sharply focused laser beams.[26,27] In 1964–1965, Kovrigin *et al.* (see Ref. 21) developed optical harmonic generators with output power levels which were record highs at the time. The theory derived in Refs. 26 and 27 has made it possible to draw a complete picture of the nonlinear interaction and one which is in excellent agreement with experiment.

A completely new class of diffraction problems—new for optics and also for the physics of wave processes in general— are those that arose after the prediction of self-focusing of light.[29–31]

Over a period of more than five years starting in 1964, self-focusing remained perhaps the most intriguing of the nonlinear effects. The complex, time-varying picture of the competition between nonlinear refraction and diffraction which was observed experimentally made it difficult to interpret the experimental data even qualitatively.

During these years, research on self-focusing occupied a prominent position in the work in the nonlinear optics laboratory at Moscow State University. One of the first experiments to observe self-focusing directly was carried out at the University.[32] Theoretical work resulted in the development of steady-state and time-varying pictures of the phenomenon.[33,34] This work was summarized in a review[35] in 1967. The mathematical apparatus used in Refs. 33 and 34 was naturally based on the nonlinear parabolic equation.

The method of step-by-step simplification and the choice of suitable models (one model which proved extremely useful is the model of a Gaussian beam with a slowly varying amplitude and a slowly varying width) led to a systematic picture of steady-state aberration-free self-focusing, made it possible to incorporate nonlinear aberrations, and made it possible to evaluate the role played by higher-order nonlinearities in the formation of the focal region.

In 1966 Khokhlov introduced the important idea of an instability of a light beam which is undergoing self-focusing in a medium with a fast-response nonlinearity. This idea went a long way in determining the future development of the theory.

A dry recitation of the sequence of publications can of course do nothing to convey the lively and often extremely emotional atmosphere which prevailed at the time. We will accordingly conclude this section of the paper by recalling one episode in the development of the theory of self-focusing. This particular espisode took place in 1965–1966.

Several phenomena, which were at first glance unexpected, were reported in experimental papers published in 1965 by Pilipetskiĭ and Rustomov,[32] Townes *et al.*,[36] and Bloembergen and Lallemand.[37] One of these phenomena was a low "self-focusing efficiency": Only a very small part of the beam was "captured" into the self-focusing "filaments," i.e., narrow channels with a high light intensity. The origin and behavior of the filaments themselves were also unclear in many ways. At first it seemed natural to regard these filaments as nonlinear wave guides of a sort.

Zel'dovich and Raĭzer[38] suggested that these anomalies might be due to the finite relaxation time of a nonlinear response; such a relaxation would of course mean that the nonlinear waveguide would "sprout" at a finite rate, and there would be an incomplete capture of the laser beam.

Zel'dovich and Raĭzer[38] informed our own group about their work before publishing it. A calculation was carried out[34] on the self-focusing of a short laser pulse in a relaxing medium in early 1966. This was a rather difficult problem.

For a beam of the type

$$\mathbf{E}=\mathbf{e}A(t,z,r)\exp\{i[\omega t-kz-ks(t,r,z)]\}$$

the parabolic equation is equivalent to two equations: one for the real amplitude A and one for the correction to the eikonal s:

$$\frac{2}{u}\frac{\partial s}{\partial t}+2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{n_2}{n_0}p+\frac{1}{k^2A}\left(\frac{\partial^2 A}{\partial r^2}+\frac{1}{r}\frac{\partial A}{\partial r}\right), \tag{13}$$

$$\frac{1}{u}\frac{\partial A}{\partial t}+\frac{\partial A}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A}{\partial r}+\frac{A}{2}\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)=0; \tag{14}$$

here $u=\omega/k$ is the group velocity in a linear medium, and $n_2 p=n-n_0$ is the nonlinear increment in the refractive index.

In a relaxing medium, the behavior of the quantity p is described by the equation

$$\tau\frac{\partial p}{\partial t}+p=A^2. \tag{15}$$

A solution of equations (13)–(15) was found for the self-similar problem in Ref. 34 (here complete use was made of the arsenal of step-by-step analysis). The solution drew a picture of the self-focusing of a short laser pulse in a relaxing medium. Figure 4, taken from Ref. 34, shows the spatial structure of the field of a short laser pulse in a nonlinear medium.

The nonlinear refraction which is operative over a time τ compresses the tail of the pulse; at the front of the pulse, the diffraction-level divergence is retained. The front propagates as in a linear medium: A sort of nonlinear precursor arises which is essentially analogous in many ways to a Sommerfeld precursor in linear optics.

The picture in Fig. 4 did not contradict experiment. Nevertheless, it was later found that it did not bear directly

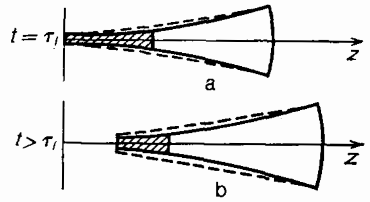

FIG. 4. Dynamics of the onset of self-focusing of a short laser pulse of length τ_l in a medium with a slow nonlinear response. Two phases are shown in the evolution of a light pulse which is incident on the nonlinear medium at $t=0$.[34] a: $t=\tau_l$. b: $t>\tau_l$. The hatched part of the beam is compressed as the result of self-focusing. The unhatched region is a nonlinear precursor. The dashed lines show the shape of the beam in a linear medium.

on the self-focusing of nanosecond laser pulses which was being studied at the time. Lugovoĭ and Prokhorov[39] showed that the anomalies in question and the self-focusing filaments themselves could be explained in a vastly simpler way: in terms of a quasistatic motion of focal spots. The debate between the positions of waveguide filaments and moving foci was resolved in favor of the moving foci in the case of nanosecond pulses.

The picture discussed in Refs. 34 and 38 and shown in Fig. 4 here was taken up again in the early 1970s. It was found (see the reviews of Refs. 40, 41) that this picture gives a satisfactory description of the self-focusing of picosecond laser pulses.

Here the theory found a complete quantitative confirmation. The ideas introduced in Ref. 34 regarding the self-effect of light in a medium with a slowly relaxing nonlinear response were used in the late 1960s to develop a theory of thermal self-focusing and self-defocusing of light in absorbing media.[42–44] In the years which followed, the work on thermal defocusing developed rapidly at Moscow State University. It became clear[43–45] that what was involved consisted essentially of basic effects in nonlinear atmospheric optics.

Distinctive finite-response-time self-effects of wave packets must be dealt with in laser amplifiers. Khokhlov was one of the first to call attention to the possibility of making use of these self-effects to produce high-power short light pulses. He and Il'inova[119] analyzed the nonlinear formation of stationary pulses in an amplifying medium.

The apparatus of nonlinear wave optics which was constructed in 1961–1966 by the efforts of many groups, in the Soviet Union and elsewhere, not only led to a satisfactory description of a broad range of new physical phenomena but also became the foundation of effective methods for designing nonlinear-optics devices (e.g., Refs. 46–49), for designing systems of adaptive nonlinear optics,[52] and for analyzing arrangements for static and dynamic holography.[53] The long list of wave interactions and self-effects of light waves is the foundation of nonlinear laser spectroscopy (e.g., Refs. 54, 55, and 122).

4. METHOD OF A SLOWLY VARYING PROFILE: KHOKHLOV'S METHOD IN NONLINEAR ACOUSTICS

The present stage of research on nonlinear acoustics began in the mid-1950s. By this time, high-power ultrasonic generators had been constructed in several laboratories in the USSR and the USA. This apparatus made it possible to observe strong nonlinear effects in liquids and solids.

The first Soviet experiments which were carried out in this direction were carried out in Burov's laboratory, where a unique wide-aperture ultrasonic radiator with an intensity $\sim 6\cdot 10^2$ W/cm^2 in a pulse was developed.[63]

A strong nonlinear absorption of this radiation in viscous liquids was detected in Refs. 63 and 64. That absorption was caused by the generation of higher harmonics.

In the years which followed, Krasil'nikov, Zarembo *et al.* also made some direct observations of distortion of the shape of waves and carried out experiments on the generation of intense shock waves in liquids and solids. The first stage of this research is summarized in Ref. 65.

The theoretical interpretation of nonlinear-acoustic experiments at the time, however, usually appealed to the Riemann solution for a plane wave in a nondissipative medium, expansions of this solution, and simple spectral representations.[65] This state of affairs naturally did not meet the experimental needs.

In nonlinear acoustics, as in nonlinear optics, a clear need arose for the development of theoretical methods capable of fully incorporating dissipation (viscosity and thermal conductivity), the finite relaxation time of a response, diffraction phenomena, a temporal deviation from a monochromatic situation, and incomplete temporal and spatial coherence of intense radiation.

These problems were solved by the method of a slowly varying profile, which Khokhlov developed in 1960–1961. What is actually involved here is an effective universal approach in the theory of nonlinear waves which are propagating through media with a slight dispersion. The general principles of this method were formulated by Khokhlov in Ref. 3, which we have already cited.

We should note that in Ref. 3 the method was presented in the example of the problem of the generation of shock radio waves in nonlinear lines. The work was stimulated by research by some rf physicists at Gor'kiĭ, who were the first to observe shock radio waves in the laboratory and who developed a theory of electromagnetic shock waves.[66,67] As early as 1961, however, Khokhlov, along with Soluyan,[68] used the method of a slowly varying profile to study the formation of acoustic shock waves in a dissipative medium.

The idea behind the method is graphic and rich in physical content (we will explain it here in acoustic terms).

Let us consider a wave which is traveling in the direction $x>0$ in a nonlinear, slightly dispersive medium. In this case, in contrast with nonlinear optics, a broad spectrum of harmonics arises because the dispersion is weak. For this reason, a representation of the solution as the sum of a limited number of quasimonochromatic waves [see (9)] and the method of slowly varying amplitudes in its conventional form are not effective.

Khokhlov proposed a new approach to the construction of a solution: representing it as a wave with a profile which varies slowly over the distance traversed (in general, this profile can be arbitrary). In a slightly nonlinear medium, a wave has to travel a distance much greater than the wavelength for the nonlinearity and dissipation to cause any significant distortion of its profile.

When the nonlinearity and attenuation are taken into account, the "slowness" is equivalent to small values of two properties: the Mach number $\mathbf{Ma} = u_0/c_0$ (u_0 is the vibrational velocity amplitude) and the dissipation parameter $d = \alpha\lambda$ (α is the attenuation coefficient).

A plane wave with a slowly varying (evolving) profile is described by

$$u = u\left(t - \frac{x}{c_0},\ \mu x\right); \tag{16}$$

here $\mu \sim \mathbf{Ma} \ll 1$, $\mu \sim d \ll 1$ is a small parameter which is used to incorporate the slowness in the dependence of the wave shape on the coordinate x. If we seek a solution of the system of hydrodynamic equations of a viscous, heat-conducting liquid in the form in (16), and if we retain terms $\sim\mu$, we find the approximate equation

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} = \Gamma \frac{\partial^2 V}{\partial \theta^2}. \tag{17}$$

Here $z = x/x_{\text{dis}}$ is the distance, expressed in the units of the discontinuity formation length, $V = u/u_0$, $\theta = \omega\tau$, $\tau = t - (x/c_0)$, and $\Gamma = \alpha x_{\text{dis}}$ is the reciprocal of the acoustic Reynolds number.

The evolution of an initially sinusoidal wave in a slightly nonlinear dissipative medium was first analyzed with the help of Eq. (17) in Ref. 68 (the analysis of an analogous problem in Ref. 3 was carried out for a medium with a "low-frequency absorption"). The basic phases of this evolution over distance traversed by the wave are illustrated by Fig. 5. Initially, the nonlinearity steepens the front and leads to the formation of a nearly discontinuous wave; then dissipation becomes dominant, the fronts become rounded, and the wave becomes essentially sinusoidal again.

The method of step-by-step analysis made it possible to calculate[3,68] the scale lengths of each of these stages, to find the width of the shock fronts and to draw a complete quantitative picture of the phenomenon.

In the years which followed, Khokhlov, his graduate students, and his colleagues from the Acoustic Institute made successful use of the method of a slowly varying profile to solve a long list of problems in nonlinear acoustics.

Equation (17) was generalized to the cases of cylindrical and spherical waves:

$$\frac{\partial V}{\partial z} + \frac{V}{nz} - V\frac{\partial V}{\partial \theta} + \Gamma\frac{\partial^2 V}{\partial \theta^2} = 0 \tag{18}$$

(the value $n = 1$ corresponds to a converging spherical wave, while $n = 2$ corresponds to a converging cylindrical wave). Equation (18) was used to analyze the nonlinear distortion of converging and outgoing waves. An equation describing nonlinear waves in a relaxing medium was also formulated:

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} = D\frac{\partial}{\partial \theta}\int_{-\infty}^{\theta} \exp\left(-\frac{\theta - \theta'}{T}\right)\frac{\partial V}{\partial \theta'}\, d\theta'; \tag{19}$$

here $T = \omega\tau_{\text{rel}}$ is the dimensionless relaxation time, and $D = x_{\text{dis}}/x_\tau$ is the ratio of the nonlinearity length to the typical dispersion length.

The generalization of these results to bounded wave beams was an important step.

Khokhlov was the first to formulate a nonlinear equation which also describes the diffraction in a slightly dispersive medium.

Under the assumption that the changes in the profile along and across the beam axis are slow [cf. (11); we recall that at that point we were talking about changes in the amplitude!],

$$u = u\left(t - \frac{x}{c_0},\ \mu x;\ \mu^{1/2}\mathbf{r}\right) \tag{20}$$

[as in the derivation of (12), the field varies more rapidly across the beam then along it], we find the following equation from the equations of the mechanics of ideal continuous media:

$$\frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta}\right) = \frac{N}{4}\Delta_\perp V. \tag{21}$$

Here $N = x_{\text{dis}}/x_{\text{diff}}$ is the ratio of the scale length of the nonlinearity to the diffraction scale length. At $N > 1$, the diffraction has a strong effect on the formation of the shock wave.

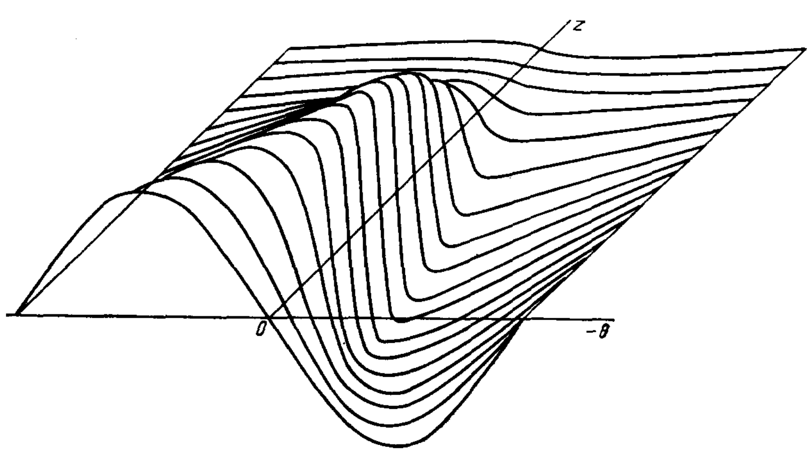

FIG. 5. Evolution of the profile of an orginally sinusoidal wave in a slightly nonlinear dissipative medium.[68,71] The nonlinearity is seen initially to lead to the formation of a nearly discontinuous wave. As the wave propagates further, the shock front becomes "smeared." Finally, at a sufficiently large distance, the shape of the wave becomes essentially sinusoidal again.

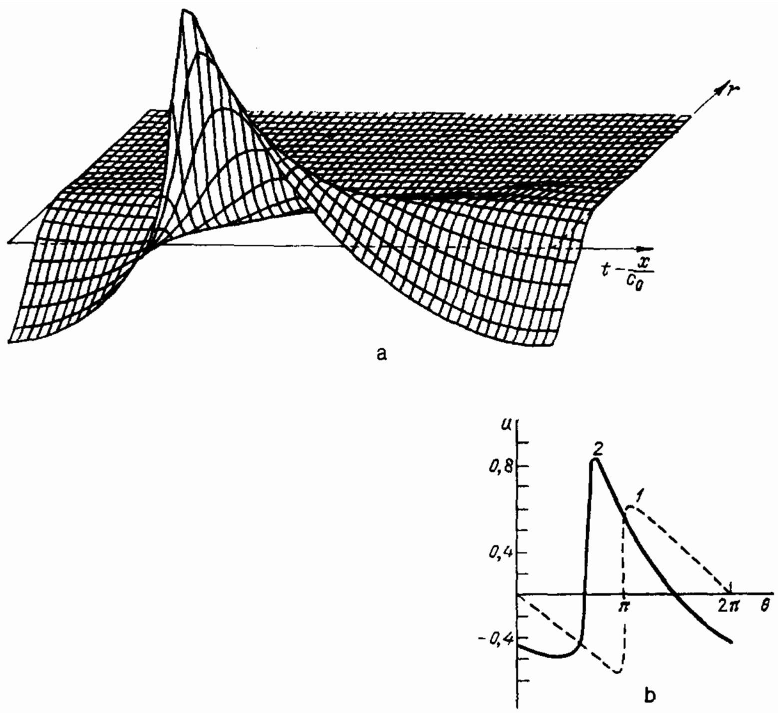

FIG. 6. Generation of shock waves in a bounded sound beam.[72] a: Form of the excitation as a function of two variables t and r (an axisymmetric sound beam). b: Profiles of the nonlinear wave (one period). 1—Plane wave; 2—bounded beam.

In this case, in contrast with the case of a plane wave, the regions of compression and rarefaction are distorted in an asymmetric way, the amplitude of the compression phase is higher than that of the rarefaction phase, while the duration of the latter is shorter (Fig. 6). The reason for these results lies in the phase shifts between the harmonics caused by diffraction and the difference in the scale lengths of the diffraction divergence for the different harmonics.

Modern theoretical nonlinear acoustics is thus constructed on the basis of Eqs. (17)–(21). The evolutionary nature of these equations means that we can also construct combinations of them which incorporate different combinations of diffraction, dissipative, and relaxation processes; external sources ("driven waves"); etc. The many applications of (17)–(21) were demonstrated by Khokhlov and his coworkers in a long list of illustrative problems. The results of this work and a classification of the hierarchy of nonlinear acoustic equations are given in the review in Ref. 69 and in the monograph by Rudenko and Soluyan.[70]

The results of a numerical analysis of the approximate equations of nonlinear optics carried out by a group at Moscow State University have been summarized in monographs of Refs. 71, 72. In concluding this section of the paper, we would like to stress two circumstances.

In contrast with nonlinear optics, nonlinear acoustics has a long history. In continuum mechanics and in hydrodynamics, the propagation of finite-amplitude perturbations had been discussed repeatedly long before experiments were begun in the 1950s on the nonlinear propagation of ultrasound in condensed media. General ideas regarding the nonlinear distortion of a finite-amplitude wave were formulated a long time ago by Riemann. Several evolutionary equations which are widely used in modern nonlinear acoustics and in fact in the physics of nonlinear waves in general were derived in connection with research on hydrodynamic problems, waves on water, etc. Important progress in the theory of nonlinear waves was reached in connection with research on wave processes in plasmas.[73,74]

Equation (17) was originally derived by Burgers in 1940, and it was derived for waves in plasmas[73] in the early 1960s. Equation (19) with $T \ll 1$ reduces to the Korteweg-de Vries equation.

The idea of using these equations to analyze the behavior of finite-amplitude acoustic waves is encountered in papers published in the mid-1950s. However, Khokhlov was the first systematically to derive these equations for acoustic waves from the system of hydrodynamic equations with viscosity and heat conduction, to derive detailed solutions, and to carry out a detailed analysis with reference to problems in nonlinear acoustics.

The nonlinear equation with diffraction in (21) is called the "Khokhlov-Zabolotskaya equation." It and its solutions were first given in a paper published in 1969. Equation (19) again has essentially no predecessors.

Rudenko *et al.* conclude their review in Ref. 69 as follows: "The approximation of a slowly varying profile is a powerful tool for analyzing a variety of problems which arise in nonlinear acoustics. Many problems have already been solved, but many more are still waiting their turn." These words, written in 1974, could be repeated today, a decade later, without reservation.

Nonlinear optics and nonlinear acoustics are generally regarded as two "poles" in the physics of nonlinear waves. In the former case, the picture of the nonlinear interaction is formed by the dispersion properties of the medium. Only a few waves at greatly different frequencies effectively interact. In the latter case, the nonlinear interaction spans many octaves of the acoustic spectrum.

Recent years have seen increasing indications that these poles are moving closer together. The advent of ultra high-power femtosecond pulses and the development of new nonlinear materials suggest that it may be possible to generate optical shock waves.[28,77] However, in acoustics particularly strong tendencies toward a convergence with nonlinear optics are found. Media with an artificial dispersion, special configurations of interacting beams in solids, etc., can be used to develop parametric amplifiers and generators of sound, acoustic harmonic generators, and other analogs of nonlinear-optics devices. We should point out that Khokhlov was very interested in work in this direction.[75]

Important results have recently been obtained in this field by Bunkin and his coworkers; their new results are reviewed in this issue of Uspekhi.[76]

5. RANDOM NONLINEAR WAVES

The nonlinear interactions of random waves play a fundamental role in solid state physics and hydrodynamics. Ideas regarding the interaction of phonons have been developed into a theory of the thermal conductivity of crystals. The interactions with thermal phonons result in an attenuation of sound in solids. The research on these processes dates back to the classical studies by Peierls and Landau (see Ref. 80, for example, for a recent review). In terms of waves we are evidently dealing here with nonlinear interaction, which are caused by the anharmonicity of the crystal lattice, between various spectral components of acoustic noise and a regular wave.

The pumping of energy along the spectrum which is characteristic of the onset of turbulence is related to a cascade of three-frequency interactions of randomly modulated waves.[73,74,81,82]

In these cases we are of course dealing with incoherent (essentially uncorrelated phases) interactions of random waves. They are described by the truncated equations derived for intermediate intensities (kinetic equations) from the equations for the amplitudes presented in the preceding sections of this paper through an averaging over phase (e.g., Refs. 73 and 95). In contrast, in the rf, optical, and acoustic research on nonlinear wave processes, dating back to the mid-1950s, the emphasis was originally placed on the interactions and self-effects of regular waves and on coherent nonlinear effects.

As in the physics of linear waves, however, the statistical phenomena in the sources of intense radiation (the "field statistics"), the fluctuations in the parameters of the medium, and the unavoidable presence of intrinsic fluctuation fields (which we will lump together as the "statistics of the medium" for brevity) have forced stochastic generalizations.

Statistical problems had a significant place in the scientific program of the nonlinear optics laboratory at Moscow State University from the very outset.

This direction of the research was affected by (in addition to the objective factors listed above[11)]) the traditions of the Soviet school of the theory of nonlinear oscillations and the advanced level of Soviet work on statistical rf physics.[83] Khokhlov and the present author were strongly influenced by our participation over many years in a seminar on statistical rf physics led by S. M. Rytov.

The first paper of the "statistical cycle,"[85] which was reported at Rytov's seminar and published in 1961, dealt with an area where rf physics bordered optics. Problems involving the passage of noise through systems with variable parameters attracted considerable interest in those years.

In Ref. 86 it was suggested that an rf parametric oscillator might be used to arrange a "phase quantization" by virtue of the phase selectivity of a parametric system and the pronounced suppression of fluctuations in the phase of the signal being amplified. It was also shown that a nonlinearity makes it possible simultaneously to achieve a substantial lowering of the level of amplitude fluctuations. The new field states formed by the parametric generator thus turned out to be extremely promising in the technology of phase measurements and in systems for detecting weak signals in the presence of noise.[86]

Essentially a wave analogy of a parametrically excited resonator (Section 3 of this review and Ref. 23) was analyzed in Ref. 85. Specifically, the subject of that paper was a parametric traveling-wave amplifier with sinusoidal pumping. It was natural to expect that such a system would have important advantages in terms of the rate of "phase quantization." The idea was essentially quite simple: What was actually involved was the problem for which integral curves are plotted in Fig. 3. If we have $A_2(0) \gg A_1(0)$ at $z = 0$ (the wave at the frequency 2ω is the "pump," while that at ω is the "signal"), and if we have selected the phase difference correctly, the signal will be amplified. Such an amplifier, like a parametric generator, has a sharp phase selectivity. The result of this selectivity is a pronounced modification of the statistical characteristics of the phase of the signal. The one-dimensional distribution of the phase of the signal, which was uniform at the entrance, is converted into a bimodal distribution (Fig. 7). At a large gain, two stable phase states arise; in modern terms, we would say that there is a resonator-free phase bistability.

In Ref. 85, considerable attention was paid to a method for solving statistical nonlinear wave problems,[12)] to analyzing the role played by fluctuations in the boundary conditions and a distributed random force, and to fluctuational transitions from one stable to another. Extensive use was made here of the method of secondary truncation.

Interest in unusual states of the field at the output from a degenerate parametric amplifier (Fig. 7) and their use in interferometry and in information systems has recently increased, now in connection with quantum optics.

In terms of a standard description of noise as a randomly modulated oscillation (Refs. 83 and 95, for example),

$$x(t) = a(t)\cos\omega t - b(t)\sin\omega t \tag{22}$$

the onset of the phase distribution shown in Fig. 7 would mean

$$\langle a^2\rangle \neq \langle b^2\rangle, \tag{23}$$

i.e., that the fluctuations in the quadrature components are not identical. It is not difficult to see that at a large gain (Fig.

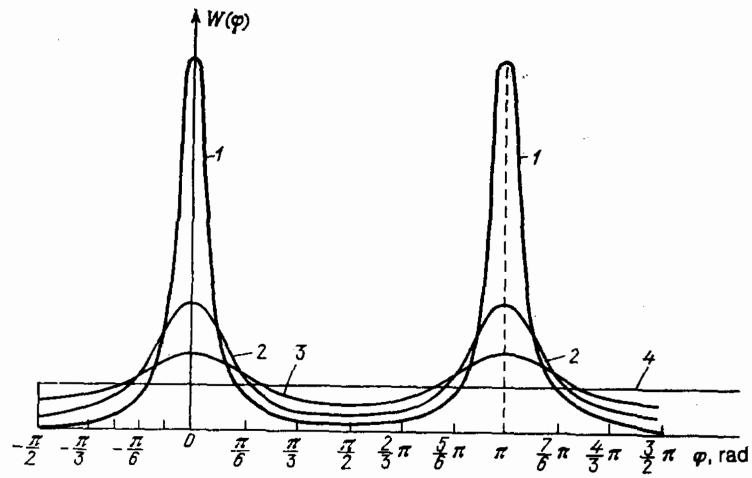

FIG. 7. One-dimensional phase distribution $W(\varphi)$ in a parametric optical traveling-wave amplifier for various amplification parameters. The phase distribution at the entrance is uniform (4). The gain decreases from 1 to 3.

7) one of the components will be essentially completely suppressed. Specifically these properties are exhibited by the "squeezed states" which are presently the subject of active research in quantum optics[88–91]: states for which fluctuations of one of the quadrature components are smaller than the quantum fluctuations in a coherent state. We wish to emphasize that we are talking here about quantized fields, so that the states found in Ref. 85 should be regarded as their classical counterpart.[13)]

It should also be kept in mind that in essentially all the experiments which are presently being carried out to search for squeezed states three- and four-photon parametric interactions are being used in optics. The problems posed about twenty-five years ago involving the conversion of random optical fields by parametric systems with a regular pump have become the subject of detailed theoretical and experimental research.

For the physics of nonlinear waves itself, on the other hand, the nonlinear propagation of random waves—a "noisy pump"—is of particular interest. How does a random modulation of a wave (in space and time) affect the course of a nonlinear process? What is the difference between the interactions and self-effects of random waves from the interactions and self-effects of regular waves? Answers to these questions have emerged from statistical nonlinear optics and statistical nonlinear acoustics.

In nonlinear optics, a fundamental consequence of the "field statistics," has turned out to be the decay of the phase correlations between the interacting waves as they propagate which is caused by the dispersion of the group velocity. In Ref. 92, the decorrelation process was studied in detail on the example of the generation by nonmonochromatic radiation of the second harmonic. The coherent process near the source gives way to an incoherent interaction of essentially uncorrelated waves at $z \gg l_{\rm coh} = \tau_c/\nu$; here $\tau_c \approx 1/\Delta\omega$ is the correlation time of the fundamental wave, and $\nu = v_1^{!} - v_2^{!}$ is the difference between group velocities. The situation is illustrated in Fig. 8a, which shows the average intensity of the second harmonic, $\langle I_2 \rangle$, as a function of the distance traversed in the nonlinear medium for a regular fundamental wave and for a noisy fundamental wave. These curves were plotted for the case of a large phase difference $\Delta = 2k_1 - k_2$ (Sec. 3 and Fig. 3). Here the coherent process is relatively inefficient, and fast spatial beats of the fundamental wave and the harmonic occur. In the incoherent process, on the other hand, there is a monotonic growth of the harmonic. At $\Delta = 0$, the situation is the opposite: In the coherent process we have $\langle I_2 \rangle \sim z^2$, and in the incoherent process we have $\langle I_2 \rangle \sim z$.

Some qualitatively new effects arise in stimulated scattering in the field of a noisy pump. These effects have attracted considerable practical interest for stimulated Raman scattering. A systematic dynamic theory of stimulated Raman scattering was derived by Khokhlov and Platonenko already in the early 1960s (Ref. 93). An intense light wave (the pump), propagating through a Raman-active medium

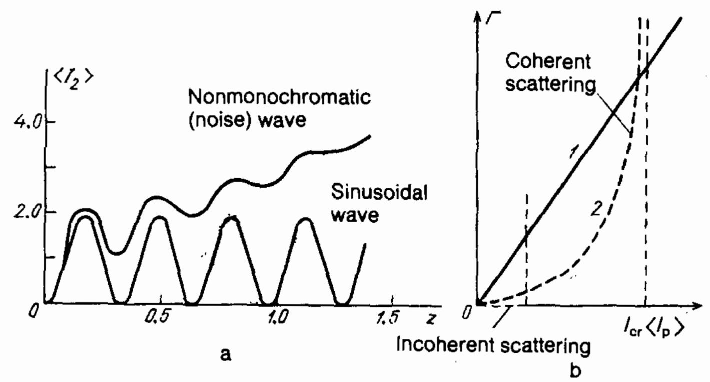

FIG. 8. Nonlinear processes in a noise pump field. a: Average intensity of the second harmonic excited by a noise wave, $\langle I_2 \rangle$, as a function of the distance (z) traversed in the nonlinear dispersive medium for the case of a pronounced phase difference ($\Delta/\sigma > 1$) (Fig. 3). Also shown here is a plot for a sinusoidal wave (from Ref. 92; see also Ref. 95). b: Growth rate of the stimulated Raman scattering as a function of the pump intensity in a dispersive medium. 1—Monochromatic pump; 2—wide-band noise pump (from Ref. 94; see also Ref. 95).

(which might be a gas, a condensed medium, or a plasma), derives an instability induced by a Raman resonance and an exponential increase in the field at the Stokes frequency.

For a sinusoidal pump (of intensity I_p) the intensity of the Stokes wave is

$$I_S = I_0 e^{\Gamma z}; \quad (24)$$

in this case the growth rate is $\Gamma \sim I_p$.

A theory of stimulated Raman scattering in the field of a noisy pump was derived in Ref. 94. It should be noted that what is involved here is perhaps one of the most difficult problems in statistical nonlinear optics (there is a review in Ref. 95). Figure 8b shows one of the principal results of the theory: plots of the growth rate as a function of the average intensity of the noisy pump (shown for comparison is the behavior for a monochromatic pump). We see that there are two characteristic regions. At $\langle I_p \rangle \ll I_{cr}$, the noise pump is far less efficient than a sinusoidal pump of the same intensity: Decorrelation is dominant, and there is an incoherent situation in which the instability caused by the stimulated Raman scattered is significantly suppressed. These comments may be taken as a direct recommendation of a method for suppressing stimulated scattering. Specifically these ideas underlie some schemes for laser fusion in which wide-band laser systems are used to heat targets.[102]

At $\langle I_p \rangle \sim I_{cr}$, on the other hand, the nonlinearity suppresses the incoherence effects (this effect is a particular feature of parametric processes; there is no such suppression in harmonic generation): A noisy pump of sufficiently high power operates as if it were monochromatic. The latter circumstance is extremely important to nonlinear optics in the UV range. The output from high-power UV lasers (particularly excimer lasers) has a low temporal and spatial coherence. A stimulated-Raman converter designed in an appropriate way would make it possible to substantially raise the output brightness.

A long list of statistical problems involves the self-effects of noisy waves. One of the first of these problems was that of the instability of a plane wave in a medium with a cubic nonlinearity, posed by Khokhlov (see Ref. 103); a theory was derived by Bespalov and Talanov.[104] Interest in the self-focusing and self-defocusing of random waves (essentially an emphasis or smoothing of surges in the noise) has recently increased greatly in connection with the developing of high-power laser systems for laser fusion, technological advances, and problems of nonlinear atmospheric optics. Considerable progress has been achieved since the developing of methods for numerically solving problems in statistical nonlinear optics. In the nonlinear optics laboratory at Moscow State University, a series of studies in this field has been carried out by Kandidov and his coworkers (Ref. 96).

In speaking of the self-effects of randomly modulated wave packets, we should single out for special mention some studies of the formation of optical ("Schrödinger") solitons from a noise. These questions are also discussed in a review in this issue of Uspekhi.[28]

Work on statistical nonlinear acoustics was begun in the mid-1970s.[97] The problems which arise here are quite different from the problems of statistical nonlinear optics, both in formulation and in methods of solution.

The essential absence of dispersion brings to the forefront problems involving the nonlinear evolution of wide continuous spectra, which are similar in their formulation to problems in the theory of weak turbulence.[70,97,98] Another interesting circle of questions involves the interaction of a regular signal and noise in a weakly dispersive medium.[99] Such an interaction could be utilized to suppress low-frequency acoustic noise: There is a nonlinear "up-conversion" (in nonlinear-optics terminology) along the frequency scale. The model of an interaction of a signal and noise also makes it possible to analyze wave aspects (the three-dimensional nature of the fields, the role played by the phase relations, diffraction, etc.) in the attenuation of sound in solids,[99] which were analyzed by means of kinetic equations some time ago by Landau and Rumer (see Ref. 100).

We should point out that very recent studies aimed at finding a wave picture of the space-time evolution of nonequilibrium phonon fields has received new motivation. High-power short laser flashes excited highly nonequilibrium phonons (a nonequilibrium acoustic noise) and coherent strain waves in absorbing crystals. The interaction of regular and random acoustic fields is of considerable interest in connection with the problem of generating extremely short hypersonic pulses and acoustic solitons, for solid state spectroscopy, etc.[101] The list of problems in statistical nonlinear acoustics is rapidly getting longer (see also the reviews by Rudenko[98] and Bunkin *et al.*[76] in this issue of Uspekhi.

6. THEORY OF THE γ-RAY LASER; COHERENT x-RAY OPTICS

Khokhlov was actively involved in these problems starting in the early 1970s.

In the work at Moscow State University, the emphasis was on a search for methods of achieving a population inversion of nuclear levels. Khokhlov evaluated the possibilities of a laser approach in Ref. 105.

The problem has proved exceedingly difficult and has not yet been solved.

In addition, work has been undertaken on resonators and on the development of feedback arrangements for γ-ray lasers. These studies have undoubtedly had a significant effect on x-ray and γ-ray optics, introducing new ideas and theoretical approaches in these fields.

One possible version of a laser, which attracted the interest of Khokhlov, made use of long-lived nuclear isomers.[105] The excited nuclei had to be situated in the lattice of a high-quality crystal in order to achieve the necessary gain.

The wave problem of a laser oscillator thus reduces to an analysis of coherent scattering or, in the terminology of x-ray optics, dynamic diffraction in an amplifying medium.

Here it is of course necessary to allow for the finite cross section of the wave beam, the pulsed nature of the radiation, and the finite temporal and spatial scales of the coherence—factors which are ignored in the conventional theory of dynamic diffraction, which is based on the plane-wave approximation.

A group at Moscow State University made effective use in research on dynamic diffraction of approaches and methods developed in nonlinear optics. The coupled-wave equations describing two-wave and multiwave x-ray diffraction have much in common with the equations of nonlinear optics.

The parabolic equation was used in Ref. 106 to construct a theory of the dynamic diffraction of a bounded beam and to evaluate aspects of a γ-ray laser using a "needle-shaped" crystal. A theory of the diffraction of an x-ray beam with finite temporal and spatial coherence radii was derived in Ref. 107, and a study was made of the transition from a coherent "dynamic" scattering to an incoherent "kinematic" scattering (see also Ref. 95).

Some new methods for measuring the coherence parameters of x radiation and γ radiation were also proposed in Ref. 107.

Subsequent work in the nonlinear optics laboratory at Moscow State University emphasized the x-ray optics of surface waves, x-ray surface diagnostics,[108,109] and pulsed methods for x-ray spectroscopy and structural analysis—the attainment of a high time resolution.

Some totally new possibilities arise when they are used in combination with laser methods for acting on matter and laser diagnostic methods, particularly in the picosecond and femtosecond time ranges (see the review in Ref. 110).

7. AUTOWAVES; PRONOUNCED NONLINEARITIES

In the 1970s, Khokhlov became primarily concerned with research on laser methods for controlling physicochemical processes in bulk and at surfaces (see the review by Gordiets and Panchenko[124]), on lasers of new types, γ-ray optics, and the interaction of extremely intense optical fields with matter.

At his initiative, work was begun at the University on research on laser photobiology and the use of lasers to sound the ocean. At about the same time, new directions were arising in the physics of nonlinear waves, stimulated by the discovery of periodic structures which arise spontaneously in nonequilibrium media and the stochastic behavior of dynamic systems—directions associated with manifestations of a *pronounced nonlinearity* (the development of ideas in these fields was recently reviewed by A. V. Gaponov and M. I. Rabinovich[112]). These studies of course aroused much interest in the nonlinear optics laboratory at Moscow State University.

I recall a lively discussion of the famous Belousov reaction[111] at a seminar in the department of wave processes and Khokhlov's comments on the theory of "dissipative structures." While serving as an opponent in A. M. Zhabotinskiĭ's defense of his doctoral dissertation in 1971, Khokhlov first suggested the term "autowaves" for this new class of nonlinear wave processes.

This term has now become generally accepted. It successfully reflects the physics of the matter, establishing a relationship with another example of the appearance of "order from chaos": self-excited oscillations.

In nonlinear optics, presently dominated by problems in the physics of nonlinear effects on matter (*nonlinear excitation*), the *nonlinear relaxation* of highly excited states, *nonlinear spectroscopy*, and the *nonlinear diagnostics* of atoms and molecules of plasmas and condensed media,[41,51,54,55,110,114,122,123] particular interest has been attracted to manifestations of pronounced nonlinearities in passive[14)] media.

Would it be possible to arrange conditions such that the local nonlinear response of a medium can no longer be regarded as small? Would it be possible to use this response for a laser initiation of phase transitions and the initiation of structural transitions which are unrelated to changes in temperature?

These questions arose rather a long time ago in nonlinear optics (see Ref. 77 for a review of the work through 1978). A pronounced nonlinearity is usually "bought" at the cost of a pronounced increase in the relaxation time of the response, as in liquid crystals.[50] Today we are seeing more and more indications that "giant" optical nonlinearities which are fast are also possible. Of primary interest in this regard is the resonant nonlinear response of free and bound excitons, which is cubic in the field.[121]

The latter circumstance has stimulated theoretical research on self-effects in a medium with a strong local response. A convenient model system is a nonlinear medium whose local response is described by the Duffing equation. Now, however, the response of a nonlinear oscillator is not calculated by perturbation theory, as was ordinarily done in nonlinear optics; instead, a nonlinear shift of the resonant frequency is considered.[115,121] A natural result of this analysis is the appearance of a wave bistability. Bistability and hysteresis effects in a local response, well known already in the classical theory of nonlinear oscillations, "adjust themselves" to accommodate a propagating wave (see also Ref. 22, which we have already cited). At high intensities, the emergence of the stochastic nature of the oscillations of nonlinear oscillator[116] will also lead to the wave process becoming stochastic.[115]

We note, however, that so far there has been no reliable observation in optics of a bistability and a stochastic nature of traveling waves due to a fast nonlinearity.

The overwhelming majority of the experiments which have been carried out involve studies of the optical bistability[113] and dynamic chaos[120] in systems with feedback: optical resonators filled with a *slightly nonlinear* medium. The situation is the same with regard to the observation of autowave processes in optics. So far, the only successful actual experiments have been carried out in systems with an external feedback.

8. CONCLUSION

In discussion Khokhlov's papers and the studies of his students and colleagues, we cannot overlook the remarkable scientific atmosphere back then—an atmosphere for which Khokhlov himself was responsible in a very substantial way.

The work carried out in the nonlinear optics laboratory at Moscow State University which we have been discussing here developed within the framework of a close interaction and exchange of ideas with many research groups.

I warmly recall how, back in 1962, Khokhlov and I were invited, at the initiative of A. M. Bonch-Bruevich, to the State Optics Institute to discuss some work on parametric light generators. This was a time when the difficulties confronting the development of these generators were far more obvious than the actual paths to success.

Friendly help from the State Optics Institute served in many ways to promote the organization of laser experiments at Moscow State University. International relations also expanded. In 1965, data (which set records at the time) on the development of high-power optical frequency multipliers and the achievement of parametric light amplification in anisotropic crystals were reported at the International Conference on Physical Problems of Quantum Electronics in Puerto Rico.[21]

The number of people working in the nonlinear optics laboratory grew rapidly, and in 1965 a department of wave processes was organized at the University. This department was chaired by Khokhlov. Recalling the time, L. V. Keldysh writes,[126] "Rem Viktorovich Khokhlov was undoubtedly one of the primary attractions; at conferences, in his own department, at seminars, in his own study—everywhere, he was surrounded by people ... . Although he had not yet received all the titles and ranks which he was to receive in the years to come, he was already the acknowledged leader in this embryonic field of physics. And his leadership was not exclusively scientific but also spiritual."

Two episodes go a long way toward conveying Khokhlov's style of operation at the time. One episode is related to nonlinear optics.

In early 1962, a theoretical paper[13] which we have already mentioned was completed. This paper raised the possibility of developing tunable optical generators and a variety of applications. It also became clear, however, that success in these fields would be impossible unless we carried out our own experiments.

Khokhlov plunged into the experiments although, as a theoretician, he of course ran into some difficulties at a psychological level. The group at Moscow State University, almost completely lacking in any optical experience, published the results of the first experiments on laser nonlinear optics as early as 1963.

The other episode occurred in 1967, at the beginning of research on selective laser effects on matter.

Khokhlov played a very great role in the development of this direction of laser physics. He deserves the credit for the first suggestions regarding the laser control of chemical reactions and the selective optical excitation of nonequilibrium molecular ensembles. We must emphasize that he began this study at a time when wave nonlinear optics was blooming.

It must be acknowledged that at the time this "chemical" direction did not inspire much enthusiasm in many of his colleagues, who tended instead to perceive it as a digression from the primary direction. The situation today is well known, and we have discussed it in this review. Khokhlov undoubtedly had a fine sense of the logic of scientific development.

We should add that it has not been in all cases in the physics of selective effects on matter that events took the path that Khokhlov was discussing back in 1967.

Other investigations made much progress by pursuing ideas introduced by other people. They found uniformly benevolent discussions and support in the seminar led by Khokhlov in the department of wave processes. Khokhlov was never susceptible to wounds to his pride; he was sincerely happy to hear about new ideas and new results. His scientific ethics were irreproachable. His genuine involvement and his readiness to listen and learn endeared him to many people. From the very outset, a genuinely scientific and benevolent atmosphere permeated the activities of the organizational committees of the hugh All-Union Conferences on Nonlinear Optics (since 1976, they have been the All-Union Conferences on Coherent and Nonlinear Optics) which Khokhlov headed for almost a decade, the Vavilov Conferences on Nonlinear Optics, and the Council on Coherent and Nonlinear Optics of the Academy of Sciences of the USSR, which Khokhlov founded.

The first All-Union Conference on Nonlinear Optics was convened at the initiative of B.I. Stepanov near Minsk in 1965. About 80 people participated. The Third Conference, held in 1967 in Erevan and directed by Khokhlov, has undoubtedly become one of the most exemplary in the history of international conferences on laser physics and nonlinear optics (see Ref. 127 for a brief review of this conference).

In a foreword to a collection of works of the Enrico Fermi International School on Nonlinear Spectroscopy,[125] the School's director, N. Bloembergen, writes, "We are dedicating the Russian edition of Nonlinear Spectroscopy to the memory of Rem Viktorovich Khokhlov, one of the pioneers in nonlinear optics ... . The rapid development of nonlinear optics and the striking progress achieved in this field in the Soviet Union are largely associated with his name. His contribution to nonlinear optics has been recognized by a Lenin Prize.[15)] ... His memory remains with many scientists throughout the world. His personal and work relationships promoted the development of a deep international understanding."

In 1978, the Ninth Conference was held in Leningrad; that Conference was dedicated to the memory of Khokhlov.

The Twelfth Anniversary Conference on Coherent and Nonlinear Optics convened in 1985 at Moscow State University.[128] More than a thousand Soviet and foreign participants attended the opening session in the University's auditorium.

We can say quite confidently that today nonlinear optics and nonlinear spectroscopy, like quantum electronics and laser physics in general, are in a growth stage, and this is a growth with a very large derivative. New ideas continue to appear at an intense pace, new methods are being developed, new faces are appearing on the scene, and new groups are forming.

Rem Viktorovich Khokhlov's works and the physical ideas which he conceived are destined for a long life.

[1)]We should emphasize that we are talking about an analogy in the fullest sense of the word. The establishment of graphic and heuristic space-time analogies in the theory of nonlinear systems is also the result of the successful extension of asymptotic methods from the theory of oscillations to wave systems.

[2)]His first studies were in the theory of time-varying phenomena in acoustic waveguides[5] (the paper was presented in *Doklady Akademii Nauk* by L. D. Landau) and a derivation of asymptotic expressions for the associated Laguerre functions[6] (that paper was presented in *Doklady Akademii Nauk* by V. A. Fok).

[3)]In one of these problems, which was a problem of practical importance at the time—the synchronization of reflex klystrons—Khokhlov showed how a mutual synchronization makes it possible to extend the frequency range of electronic tuning.

[4)]In this connection, Khokhlov, in his dissertation,[10] discusses in detail some studies by N. A. Zheleztsov *et al.* on relaxation oscillators.

[5)]In speaking here of the history of research on laser nonlinear optics at Moscow State University we must not fail to mention that as early as the 1920s or 1930s direct experiments were conceived and carried out at the University to search for a contribution of a *nonlinear response* to the absorption and refraction of light. Around 1930, S. I. Vavilov, his coworkers, and his students carried out some clear and purposeful experiments aimed at detecting nonlinearities in the absorption and refraction of light in fluorescent crystals and glasses. This reseach by Vavilov is summarized in Ref. 15; see Ref. 56 for more details.

[6)]See Sec. 7 of this paper regarding possibilities for a manifestation of a strong local nonlinear response in optics.

[7)]One clear example is a nonlinear optical activity, specifically, a gyrotropy induced by an intense light field. In this case the truncated equations are written for vector amplitudes[14,57]—in a slightly nonlinear medium, the polarization state changes slowly. Today, nonlinear polarization effects are attracting increasing interest. Nonlinear optical activity, in particular, has become an effective method in the nonlinear spectroscopy of semiconductors.[58]

[8)]The ideas of a secondary truncation have been used extremely effectively in a method developed by A. S. Chirkin for solving the truncated equations of nonlinear optics. That method is based on the "given-intensity approximation." [79]

[9)]For example, in the paper "On the flip-flop properties of nonlinear waveguide systems," [22] published in 1962, there was a mention of the possibility of developing nonlinear wave systems in which the amplitude or phase of the propagating wave takes on two or several stable states: wave analogs of flip-flops. Reference 22 called attention to the possibilities of a very fast switching of "wave flip-flops." In modern terms, what was involved there was a resonator-free amplitude or phase optical bistability or multistability. Unfortunately, it was extremely difficult back in the 1960s to implement such ideas experimentally. Just recently, the situation has changed radically: Optical flip-flops with a speed $\sim 10^{-12}$ s and a switching energy $\sim 10^{-12}$ J have been constructed.[78,113]

[10)]A similar procedure is widely used also to derive approximate equations describing the nonlinear propagation of short wave packets subject to a dispersive spreading (a temporal analog of diffraction). In this case, however, truncated equations not only of second order but also of higher orders appear. A review published in *Uspekhi*[28] covers the nonlinear optics of short laser pulses and space-time analogies in the nonlinear theory of modulated waves.

[11)]In nonlinear optics, it became necessary to deal with their manifestations unexpectedly quickly: already in the interpretation of the first experiments on optical harmonic generation it was found that in certain respects a multimode laser should be regarded as a generator of intense optical noise.[84]

[12)]Khokhlov remained interested in these problems also in later years. In 1977 he published some studies of the use of a probability functional to describe random waves.[87]

[13)]Work on a search for squeezed states of optical fields (the first successful experiments have already been carried out in this direction[91]) is of practical interest. There is the hope that it will be possible to lower the level (set by quantum noise) of the minimum detectable optical signal in measurement and information systems. Physical research on new quantum states of field and their classical analogs has also received further stimulus. Although from the fundamental standpoint the question of whether quantum states of the field always have classical analogs remains an open question (furthermore, one frequently hears of quantum fields which have no classical analogs), in those cases in which this analogy can be examined it has proved extremely useful. In particular, the statistical properties of classical fields for which (23) holds have been studied in great detail[95]—in far more detail than for quantum fields. We note in particular that in classical mechanics relation (23) implies that a process is time-varying.

[14)]In the torrent of papers on dynamic chaos, strange attractors, and self-excited oscillatory systems we are seeing progressively more frequently citations of Khokhlov's early papers on the theory of stability in complex self-excited oscillatory systems, carried out back in 1955–1958. For example, in a recent review by Abraham *et al.*[118] on instabilities and a stochastic situation in lasers, one of the first references is to a paper[117] by Khokhlov in which unstable regimes of a molecular generator were found.

[15)]In 1985, Khokhlov was posthumously awarded a State Prize of the USSR for his work in nonlinear acoustics.

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