Topology, manifolds, and homotopy: Basic concepts and applications to n-field models

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Usp. Fiz. Nauk **149**, 259–273 (June 1986)

The basic concepts of topology, the theory of manifolds, and the theory of homotopy, which are employed in the study of extended objects both in field theory and in the physics of the condensed state, are presented. The physical systems (magnetic materials and nematics) are described first. Their topological properties illustrate the general topological concepts which are introduced later. Thus two goals are pursued. On the one hand, a detailed analysis of examples completes the brief exposition of the general mathematical questions, while on the other the physical systems automatically become objects of topological study. In particular, the topological defects in the field of the order parameter of ordered systems are classified with the help of homotopic groups.

Physical examples. Topology and mappings. Manifolds. Topology of **n***-field models. Basic concepts of homotopy theory. Classification of singularities in the order-parameter field. Homotopic invariants.*

1. INTRODUCTION

In recent years, the ideas and methods of modern algebra and topology have been increasingly permeating physics. One of the main sources of this process of "mathematization" of physics is the problem of the construction of a theory of strong interactions. Attempts to solve this problem have led to the idea that the strongly interacting particles—the hadrons—consist of quarks and that the hadrons must be regarded as particles of finite size, unlike the traditional description of elementary particles as point particles. As a result, there has arisen an area of physics which can be called the physics of extended objects. The first examples of topological extended objects were kinks, solitons, and vortices, which were studied in 1973 by Nilsen and Olesen in an abelian Higgs model by analogy with vortices in a superconductor.

The next step was taken by 't Hooft and Polyakov in 1974. They found in the Giorgi-Glashow model a solution to the classical field equations; this solution is now called the 't Hooft-Polyakov monopole. An even more important result was obtained in 1975. In a purely gauge nonabelian field theory, similar to the standard electrodynamics without charged particles, Belavin, Polyakov, Shvarts, and Tyupkin discovered a quasiparticle which has finite dimensions in both space and time. This quasiparticle was given the name "instanton." Among all the numerous consequences of the discovery of the 't Hooft-Polyakov monopole and the instanton, here we call attention to only one: the concept of the topological quantum number or the topological invariant, which is one of the most important concepts of topology, has entered into physics.

At the same time (1975) Wu and Yang used the theory of fiber bundles to study the Dirac monopole. They showed that the condition for charge quantization is a purely topological property of the theory. In addition, Wu and Yang demonstrated the force and elegance of fiber theory. With its help a number of interesting results have now been obtained in the theory of gauge fields.

In this paper, some basic concepts of topology, the theory of manifolds, and homotopy theory, which are now used in the study of extended objects both in field theory and in the physics of the condensed state, are described.

The paper consists of the following sections. The physical systems (magnetic materials and nematic liquid crystals), whose topological properties serve as an illustration of the general topological concepts introduced later, are described first. Then two goals are pursued. On the one hand, the brief exposition of the general mathematical questions is supplemented by a detailed analysis of examples, while on the other the physical systems automatically become objects of topological study. The overall goal is to give a general idea of the mathematics which plays a significant role in the development of new physical ideas. The relationship between topology and physics has already been briefly mentioned above and is considered in somewhat greater detail in the conclusion. A brief commentary on the list of references, which contain the details as well as additional references, is also given in the conclusion.

2. PHYSICAL EXAMPLES

Before considering the mathematical questions we shall examine some physical examples, to which we shall refer in what follows. These examples are magnetic materials and nematic liquid crystals. Depending on the specific model the indicated systems can be described by a one, two- or three-dimensional vector **n**, defined in a plane or in a three-dimensional space. In the case of a magnetic material **n** is an ordinary vector, while in the case of a nematic **n** is the director, i.e., the directions **n** and $-$**n** are regarded as physically equivalent (in a magnetic material **n** gives the direction of magnetization, while in a nematic **n** indicates the predominant orientation of the long axes of prolate molecules, which are the constituent elements of the nematic).

We shall study the following three **n**-field models (of magnetic materials and nematics):

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1. Two-dimensional: $\mathbf{n}(\boldsymbol{\rho}) = \{n_1(\boldsymbol{\rho}), n_2(\boldsymbol{\rho})\}$, $\boldsymbol{\rho} = \{x, y\}$.

2. Three-dimensional: $\mathbf{n}(\mathbf{r}) = \{n_1(\mathbf{r}), n_2(\mathbf{r}), n_3(\mathbf{r})\}$, $\mathbf{r} = \{x, y, z\}$.

3. Planar: $\mathbf{n}(\boldsymbol{\rho}) = \{n_1(\boldsymbol{\rho}), n_2(\boldsymbol{\rho}), n_3(\boldsymbol{\rho})\}$, $\boldsymbol{\rho} = \{x, y\}$.

In these models there can exist configurations of the field $\mathbf{n}$ such that the field $\mathbf{n}$ cannot be transformed by a continuous deformation into a uniform distribution (i.e., $\mathbf{n} = \mathbf{n}_0 = \text{const}$) over the entire space. Depending on the dimensionality of the model such configurations of the $\mathbf{n}$-field are associated with the presence in the distribution of the $\mathbf{n}$-field of singular (at which the field $\mathbf{n}$ is not determined) or nonsingular points or lines, called hedgehogs, disclinations, vortices, etc.

In nematics, under a microscope, disclinations look like dark filaments, floating in the sample. Some are mobile, while others appear to be fixed. The term "nematic," which comes from the Greek word "nema" meaning thread, arose precisely from observations of such a pattern. The term was introduced by Friedel.

Figure 1 shows typical configurations of the director field near a disclination line (the line is perpendicular to the plane of the figure).

All types of disclinations shown in Fig. 1 exist in a two-dimensional nematic, and the disclinations of the type shown in Figs. 1 d–f are unstable in three-dimensional nematics: they "flow into the third dimension." This process is shown in Fig. 2 for the disclination illustrated in Fig. 1e.

A two-dimensional magnetic material does not have the singularities illustrated in Figs. 1 a–c, since this would cause the $\mathbf{n}$-field to be discontinuous along a semi-infinite line, as can be seen in Fig. 3. In a three-dimensional magnetic material the remaining disclinations (Figs. 1 d–f) are unstable, just as in a nematic.

If one full turn is made following a closed circuit around a disclination line, the vector $\mathbf{n}$ turns through an angle πN, where N is an integer; in addition, for the cases shown in Figs. 1 a–c N is an odd number, while for the cases shown in Figs. 1 d–f, N is an even number. The integer N is called the Frank index. In a three-dimensional nematic only the disclinations with an odd Frank index are stable, while in a three-dimensional magnetic material there are no stable linear singularities at all.

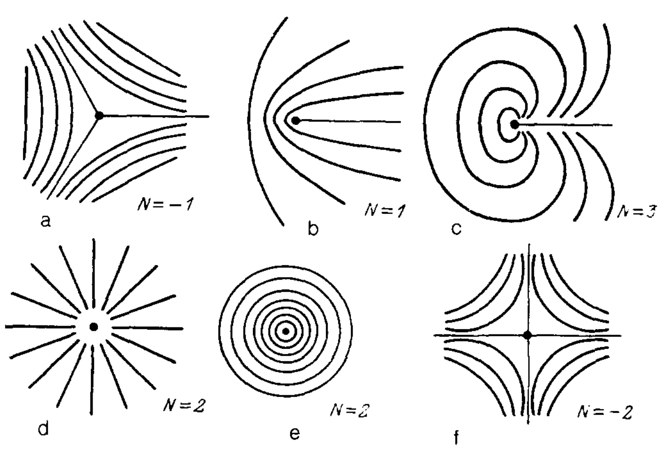

FIG. 1.

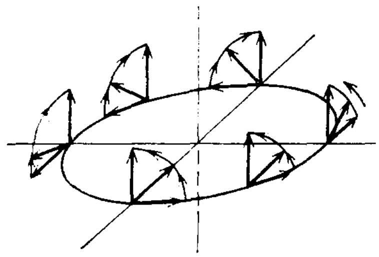

FIG. 2.

An example of a point singularity (hedgehog) in a three-dimensional nematic is shown in Fig. 4a. In a magnetic material the radial distribution of $\mathbf{n}$ can be of two types (Figs. 4b and c).

Another important case is the nonsingular point vortex in a planar nematic or magnetic material (Fig. 5a), as well as the linear vortex or linear soliton in the three-dimensional case (Fig. 5b). We recall that in magnetic materials substituting $-\mathbf{n}$ for $\mathbf{n}$ gives different types of vortices. The point vortex in the planar model can be regarded as the cross section of a linear soliton in the three-dimensional model. Such vortices were first studied in the physics of liquid crystals, but the special interest in such field configurations arose after the appearance of the work of Belavin and Polyakov in 1975, concerning the study of a planar Heisenberg magnetic material, which is also a representative of a wide class of field-theory models with an interaction of the geometric type. Belavin and Polyakov found exact N-vortex solutions in the model which they studied. These solutions exhibit many properties of instantons, found in gauge field theories.

The meaning of the term "instanton" (from the English word instant) can be understood from Fig. 5a by replacing one of the coordinates, for example y, by the time t: the Belavin-Polyakov vortex has finite dimensions in both space and time.

Some topological properties of the models described above will be discussed below. We shall now proceed to the mathematical introduction.

3. TOPOLOGY AND MAPPINGS

From the formal standpoint no preliminary information is needed to begin the study of topology. In practice, however, any course in topology presumes a knowledge of

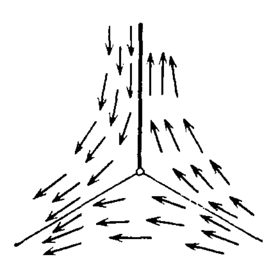

FIG. 3.

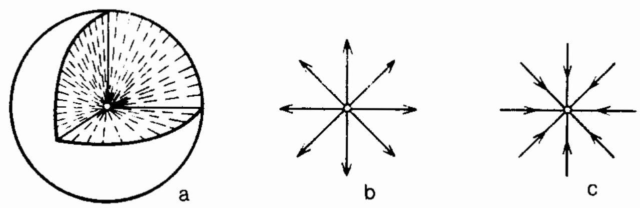

FIG. 4.

Euclidean space and continuous functions or mappings. For the example of the n-dimensional real Euclidean space, denoted by the symbol R^n ($R = R^1$ is the real line, R^2 is the plane, and R^3 is the standard three-dimensional space), it is easy to obtain an idea of the basic concepts of topology—such as neighborhood, connectedness, compactness, and topological equivalence. This is the introductory path to topology proposed by Steenrod and Chinn in their book "First Concepts of Topology."

The Euclidean space R^n is a topological space. The concept of a topological space, however, is much more abstract and general than the familiar concept of Euclidean space. The definition of a topological space includes axioms which are based on the concept of a neighborhood of a point or an open set. To understand the definition and basic properties of topological spaces, one must regard a space as a set of points, each of which has an open neighborhood, while continuous mappings of spaces must be regarded as mappings under which points which are close to one another in one space are mapped into points which are close to one another in another space. The convenient examples of Euclidean spaces also help in mastering this viewpoint.

One of the central concepts in modern geometry is the concept of a manifold. A manifold is a topological space in which every point has a neighborhood which is in one-to-one correspondence with an open region of the Euclidean space. The simplest example of a manifold is the Euclidean space R^n. A simple, but nontrivial, example of a manifold is the n-dimensional sphere, denoted by the symbol S^n, in the space R^{n+1} (S^1 is a circle; S^2 is the ordinary sphere in R^3). Unlike the Euclidean space, a single coordinate system cannot be introduced for a sphere (see below). It should also be noted that the definition of a manifold requires the concept of a differentiable function. The book by Milnor and Wallace "Differential Topology," which, like the book by Steenrod and Chinn mentioned above, is part of the popular series on modern mathematics published by Mir, is a fully accessible source of information on topological spaces, manifolds, homotopy, etc. The reader who is not familiar with topology can, if necessary, consult these books (see also the list of references at the end of the paper).

In set theory the starting concepts are *sets* (or collections or families) and *elements* (or points) of the sets. The notation $x \in A$ indicates that x is an element of the set A. The notation $\{x \in A: P(x)\}$ specifies the subset of all elements $x \in A$ for which the assertion $P(x)$ holds. For example, $A = \{x: x \in A\}$. The set which contains no elements, or the empty set, is denoted by the symbol $\varnothing$ (for example, $\varnothing = \{x: x \neq x\}$). We also define the *union* $A \cup B$ and the *intersection* $A \cap B$ of two sets A and B: $A \cup B = \{x: x \in A \text{ or(and) } x \in B\}$ and $A \cap B = \{x: x \in A \text{ and } x \in B\}$. It is also said that the sets A and B do not intersect, if $A \cap B = \varnothing$. The set $A \times B = \{(x,y): x \in A, y \in B\}$, where the pair (x,y) is an element of the set $A \times B$, is called the *direct* (*Cartesian*) *product* of the two sets A and B.

The symbol $f: A \to B (b = f(a)$ for $a \in A$ and $b \in B)$ denotes a *mapping* (function) of the set A into the set B.

For two mappings $f: A \to B$ and $g: B \to C$ we define the composition $g \circ f$ (or gf), which is the mapping of A into C: $(g \circ f)(a) = g(f(a))$ for $a \in A$.

A *topology* on a set X is defined as the family of subsets τ from X which satisfy the following conditions:

1. $\varnothing$ and X belong to τ.
2. If U and U' belong to τ, then $U \cap U' \in \tau$.
3. The union of any family from τ belongs to τ. The elements of the system τ are called *open* sets in the topology τ. The set X together with the topology τ on it is called a *topological space*.

For what follows it is sufficient to introduce the intuitively understandable concept of the usual topology on the set of real numbers R. The usual topology is defined as the family of all sets each of which has the property that if a point is contained in a set, then some neighborhood around the point is also contained in the set. An example of such a family is any open subset A of the set of real numbers $R (A \subset R)$, i.e., the subset A in which for every point x there exist numbers a and b such that $a < x < b$ and the set $\{y: a < y < b\}$ (interval) is a subset of the set A.

Consider the set of real numbers R with the usual topology and construct the direct product of n copies of the space R. This yields an n-dimensional Euclidean space R^n, the topology on which is defined as the product of the topologies of the spaces R. Examples of open sets in the space R^n are the n-dimensional cubes, regarded as a direct product of open intervals in R.

A *neighborhood* of a point $x \in X$ is defined as any subset of X which contains an open set containing x.

A topological space is said to be *Hausdorff* if any two points in the space have nonintersecting neighborhoods.

Topological spaces are studied with the help of *continuous* mappings, amongst which the *one-to-one continuous mappings* are especially important. Such mappings are called *homeomorphisms*. If $f: X \to Y$ is a homeomorphism and $f(X) = Y$, then the spaces X and Y are said to be *homeomor-*

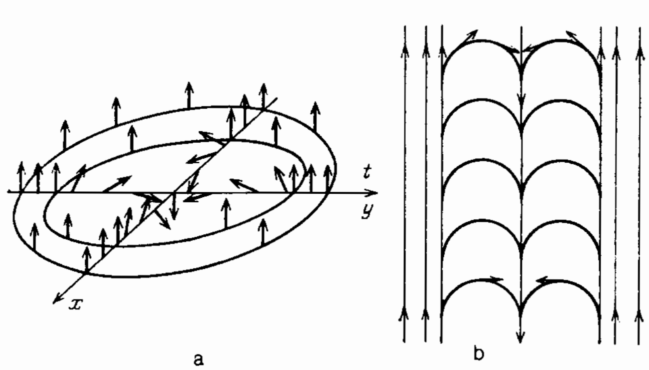

FIG. 5.

phic or *topologically equivalent*. Thus topology is not concerned with every topological space separately, but rather with classes of spaces consisting of homeomorphic spaces. Here the properties of all spaces homeomorphic to a given space are of special interest. Such properties are called *topological invariants*.

A topological space X is said to be *connected* if the set X is not the union of two nonintersecting sets. Connectedness is a topological invariant.

We shall give several more definitions, in order to indicate the class of topological spaces which are encountered in physical applications.

A subset A of a topological space X is said to be *closed* if the *complement* $X \setminus A$ is open.

The *closure* $\overline{A}$ of a set A of a topological space X is the smallest closed set containing A. For example, the segment $[0, 1]$ is the closure of the open interval $(0,1)$.

A *covering* of a set X is a system of subsets in X whose union coincides with all of X. A space is said to be *compact* if every open covering of the space contains a finite subcovering.

Heine-Borel-Lebesgue theorem. A subset of the real R^n and complex C^n n-dimensional Euclidean spaces is compact if and only if it is closed and bounded. For example, the closed interval $[0, 1] \subset R$ is compact.

A topological space is said to be *locally compact* if every point in it has a neighborhood whose closure is compact.

The spaces R^n and C^n are locally compact (but they are not compact). The sphere S^n is compact.

It is often necessary to embed a given space into a compact space. A well-known example is Riemann's sphere, which is constructed from the Euclidean plane by adding to it a point at infinity.

Single-point compactification of as topological space X is the set X^* obtained by adding to X one point denoted by the symbol ∞: $X^* = X \cup \{\infty\}$.

The space X^* is compact. The space X^* is Hausdorff if and only if X is Hausdorff and locally compact.

For what follows we can confine our attention to *locally compact Hausdorff spaces*.

In physical applications, as a rule, one is concerned with metric spaces, which are Hausdorff. However, the metric can be defined by different methods, and often a metric does not have to be introduced at all. The topology of a Hausdorff space, i.e., the separability of any two points in the space, is no less obvious than the distance between two points.

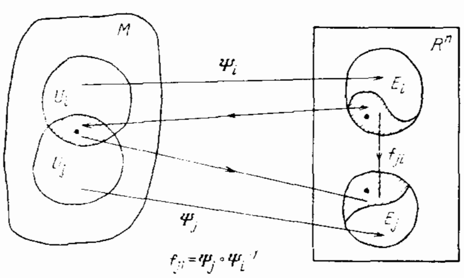

FIG. 6.

Nevertheless topological spaces and continuous mappings are too general for physical applications. In physics, a coordinate description is used (the coordinate description does not require the introduction of a metric), while mappings, as a rule, must be differentiable a definite number of times (even if this is not mentioned). Hence, manifolds, i.e., spaces in which every point can be described by a set of numbers or coordinates, are an appropriate class of topological spaces for physical applications.

4. MANIFOLDS

We shall give a definiton of a manifold. A Hausdorff space X is said to be an n-dimensional manifold if every point of X has a neighborhood which is homeomorphic to some open set of an n-dimensional Euclidean space F^n ($F^n = R^n$ or C^n).

The simplest examples of manifolds are the Euclidean spaces R^n and C^n.

In order to gain a better understanding of the properties of manifolds, the definition given above must be made more precise.

The *atlas of an n-dimensional manifold M* is a family of open sets $\{U_j\}$, covering M, and homeomorphisms Ψ_j: $U_j \to E_j$, where E_j is a region in F^n, such that the mapping

$$f_{ji} = \Psi_j \circ \Psi_i^{-1}\colon \Psi_i(U_i \cap U_j) \to \Psi_j(U_i \cap U_j)$$

is a homeomorphism (Fig. 6).

The pair (U_j, Ψ_j) is called a *chart*.

The mapping f of an open set from F^n into F^m ($f(x) = \{f_1(x^1, ..., x^n), ..., f_m(x^1, ..., x^n)\}$) is said to be a mapping of class C^r, $r = 1, 2, ..., \infty$, if f is continuously differentiable r times (if all functions f_i are differentiable with respect to each coordinate x^j).

A differentiable manifold of class C^r is a manifold for which all homeomorphisms f_{ji} and $(f_{ji})^{-1} = f_{ij}$ are mappings of class C^r. If $r = \infty$, then the manifold is called a *differentiable* or *smooth* manifold, or simply a manifold.

Let a chart (U_i, φ_i) of an n-dimensional manifold M in the vicinity of a point $p \in U_i \subset M$ be given. The Cartesian coordinates of the point $\varphi(p) x^j = x_j(\varphi_i(p))$, $x = \{x^1, ..., x^n\} \in F^n$ are called the *local coordinate system* in U_i, and the set U_i is called a *coordinate neighborhood*.

In the case of a Euclidean space the local coordinates are the standard Cartesian coordinates.

Let us examine a simple but nontrivial example of a manifold—the sphere S^n in R^{n+1}, defined by the equation

$$x_1^2 + \ldots + x_{n+1}^2 = a^2. \qquad (1)$$

The sphere is an example of a manifold on which it is impossible to define coordinates globally, i.e., on the entire manifold. However, if any point on the sphere is discarded, then it is possible to establish a one-to-one correspondence between the remaining points on the sphere S^n and the Euclidean space R^n, i.e., it is possible to introduce local coordinates. Local coordinates can be introduced with the help of the so-called *stereographic projection* (Fig. 7). Let the discarded point be the north pole of the sphere S^n with the coordinates $x_N = \{0, ..., 1\}$. Every point of the set $U_- = S^n \setminus \{x_N\}$ can

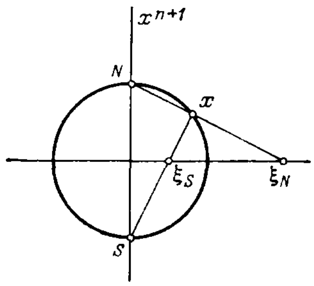

FIG. 7.

be put into correspondence with the point $\xi = \{\xi^1, ..., \xi^n\} \in R^n$ of the equatorial plane $x^{n+1} = 0$ with the help of a ray emanating from the north pole and passing through the given point of the sphere x and the point ξ corresponding to it in the equatorial plane. The relationship between the local coordinates $\xi_N = \{\xi^1_N, ..., \xi^n_N\}$ in U_- and the Cartesian coordinates $x = \{x^1, ..., x^{n+1}\}$ of the sphere S^n in R^{n+1} is given by the formula

$$\xi^i_N = \frac{ax^i}{a - x^{n+1}}, \quad i = 1, \ldots, n. \tag{2}$$

The analogous formula for the stereographic projection from the south pole is given by the formula

$$\xi^i_S = \frac{ax^i}{a + x^{n+1}}, \quad i = 1, \ldots, n, \tag{3}$$

where $\xi_S = \{\xi^1_S, ..., \xi^n_S\}$— are local coordinates in the region $U_+ = S^n \setminus \{x_S\}$, where $x_S = \{0, ..., -1\}$.

The regions U_- and U_+ are coordinate neighborhoods with the local coordinates ξ_N and ξ_S, respectively.

The atlas $\{U_-, U_+\}$ covers the entire sphere S^n. The charts of this atlas are the pairs (U_-, Ψ_N) and (U_+, Ψ_S), where the mappings Ψ_N and Ψ_S are defined by the corresponding formulas (2) and (3). The homeomorphisms $\Psi_N \circ \Psi_S^{-1}$ and $\Psi_S \circ \Psi_N^{-1}$, defined by the formulas

$$\xi^i_N = \frac{a^2}{|\xi_S|^2} \xi^i_S, \tag{4}$$

$$\xi^i_S = \frac{a^2}{|\xi_N|^2} \xi^i_N, \tag{5}$$

enable the transformation in the region $U_- \cap U_+$ from the local coordinates ξ_S to the coordinates ξ_N and vice versa.

So, the sphere S^n with a discarded point is homeomorphic to the space R^n. Let us supplement R^n with a point at infinity. Then the stereographic projection establishes a one-to-one correspondence between the points S^n and $R^n \cup \{\infty\}$. We shall denote the topological equivalents S^n and $R^n \cup \{\infty\}$ by $S^n = R^n \cup \{\infty\}$.

Let an m-dimensional manifold M and an n-dimensional manifold N of class C^r be given.

The mapping $f: M \to N$ is called a *differentiable mapping of class* C^k $(k \leqslant r)$, if for every chart (U_j, φ_j) in M and every chart (V_i, Ψ_i) in N the mapping from $\varphi_j(U_j)$ into $\Psi_i(V_i)$ is a differentiable mapping (from R^m into R^n) of class C^k. We shall call a mapping of class C^∞ a *differentiable mapping*.

The homeomorphism $f: M \to N$ is called a *diffeomorphism* if f and f^{-1} are differentiable; the manifolds M and N are said to be *diffeomorphic* (or *smoothly equivalent*).

Examples of differentiable or smooth manifolds are spheres. For convenience, a smooth manifold can be thought of as the smooth surface of some object. It seems intuitively clear that if two smooth manifolds are homeomorphic, then they are also diffeomorphic or smoothly equivalent. This is indeed so in the case of the ordinary sphere S^2 and, for example, in the case of an ellipsoidal surface. However, Milnor found examples of smooth manifolds which are homeomorphic but not diffeomorphic to the seven-dimensional sphere S^7. Roughly speaking, this has the following significance. Milnor's seven-dimensional manifolds are just as smooth as the sphere S^7 and they "look" like S^7—they are topologically equivalent to it, but there do not exist smooth mappings which map the indicated manifold into the sphere S^7. Milnor's discovery was an important event in mathematics, and is not irrelevant to physics. Milnor's manifolds are fiber bundles *with the group SU(2) and the base space* S^4. *Instantons* in field theory were discovered precisely for SU(2) gauge fields, defined on the space-time manifold in the form of a four-dimensional sphere S^4 (the compactified space R^4).

Here it is useful to say a few words about fiber bundles or fiber spaces. A fiber bundle is a generalization of the direct product of spaces. For example, the rectangle is the direct product of two segments and the cylinder is the direct product of the segment I and the circle S^1. Spaces such as the rectangle and the cylinder are called trivial fiber bundles. In the case of the cylinder the circle S is called the base space of the fiber and the segment I is called the fiber. The simplest example of a nontrivial fiber bundle is the Möbius band. Unlike the cylinder or the rectangle the Möbius band is not a direct product, but it is a direct product locally, i.e., the Möbius band consists of rectangles sewn together by the topology of the fiber bundle. For the Möbius band, like in the case of the cylinder, the base space of the fibers is the circle S^1 and the fibers are the segments I. The nontrivial topology of the Möbius band arises as a result of the effect in the fiber of the symmetry transformation of a segment relative to its center. The point is that when the Möbius band is constructed by gluing together strips of paper, the ends of the strips are joined after they are twisted relative to one another by 180°. This is precisely what distinguishes the Möbius band from the cylinder. It is said that a group, called the structural group of the bundle, operates in the band.

In general, we can make the following statements about fiber bundles. A fiber bundle is a generalization of the direct product of spaces, which in this terminology is called a trivial bundle. The cross section in a nontrivial bundle is a natural generalization of the concept of a function (of a vector field) to the case when the function cannot be defined in the entire space, called the base space of the bundle. This situation can arise when the base space of the bundle (for example, a sphere) does not permit the introduction of a single coordinate system without singular points. In this case, the sections are defined on overlapping neighborhoods, covering the base space (for a sphere, these are the southern and northern hemispheres), and in the regions of overlapping (on the equator, in the case of the sphere) the sections are sewn together by the topology of the fiber bundle. Once the principal bundle is constructed, for example, much can be learned about the general properties of the physical theory even before the Lagrangian is constructed. For this it is sufficient to study only the base space of the bundle (the space-time manifold) and the structural group of the bundle (the symmetry group of the physical theory).

5. TOPOLOGY OF n-FIELD MODELS

We shall now return to the **n**-field models discussed previously. The collection of unit vectors $\mathbf{n}\in R^{n+1}$ is an n-dimensional sphere S^n. If the directions **n** and $-\mathbf{n}$ are equivalent, then the sphere is transformed into another manifold, called a real projective space, which is denoted by the symbol RP^n. RP^n is the sphere S^n whose antipodal points are equivalent points. Another representation of RP^n is the collection of all straight lines passing through the origin in the space R^{n+1}.

Thus for the models under study we have the following mappings:

1. a) Two-dimensional magnetic material $\mathbf{n}$: $R^2\to S^1$.
   b) Two-dimensional nematic $\mathbf{n}$: $R^2\to RP^1$.
2. a) Three-dimensional magnetic material $\mathbf{n}$: $R^3\to S^2$.
   b) Three-dimensional nematic $\mathbf{n}$: $R^3\to RP^2$.
3. a) Planar magnetic material $\mathbf{n}$: $R^2\to S^2$.
   b) Planar nematic $\mathbf{n}$: $R^2\to RP^2$.

It should be noted that the circle S^1 and the projective straight line RP^1 are topologically equivalent. This is evident from the following construction. RP^1 can be constructed by joining the ends of a semicircle. The result is, once again, a circle. However, taking into account the symmetry transformation, as was evident in the example of disclinations, makes the magnetic material and nematic different.

In a greater number of dimensions the sphere S^n and RP^n are topologically not equivalent. In particular, linear singularities do not occur in a magnetic material at all.

The planar model of a nematic or a magnetic material is an example of a nonsingular configuration of the field **n**. A nonsingular vortex is obtained by giving at infinity a uniform distribution of the field $\mathbf{n}$: $\mathbf{n}\to\mathbf{n}_0=\text{const}$ as $|\mathbf{r}|\to\infty$. Physically, this denotes the fact that the distribution of the field **n** approaches the ground-state or vacuum state at infinity. The indicated boundary conditions mathematically determine the compactification of the plane R^2, i.e., the attachment of a point at infinity to it. Since $R^2\cup\{\infty\}=S^2$, the nonsingular vortices in planar models give an example of mappings of the sphere into a sphere or into RP^2

$$\mathbf{n}:\ S^2\to S^2 \quad \text{or} \quad \mathbf{n}:\ S^2\to RP^2.$$

Fixing analogous boundary conditions in the three-dimensional case $\mathbf{n}\to\mathbf{n}_0$ as $|\mathbf{r}|\to\infty$ transforms R^3 into the sphere S^3 and leads to the mappings

$$\mathbf{n}:\ S^3\to S^2 \quad \text{and} \quad \mathbf{n}:\ S^3\to RP^2$$

for a magnetic material and a nematic, respectively. The simplest nontrivial example of a field configuration for the mapping of S^3 into S^2 or RP^2 is the nonsingular ring-shaped vortex.

A more precise study of the mappings examined above requires the use of homotopy theory.

6. BASIC CONCEPTS OF HOMOTOPY THEORY

A *homotopy* or *deformation* is a set of mappings f_t $(0<t<1)$ of class C^k of the space X into the space Y. Two mappings f: $X\to Y$ and g: $X\to Y$ are said to be *homotopic* if there exists a homotopy f_t: $X\to Y$ such that $f_0=f$ and $f_1=g$. The set of homotopic mappings forms a homotopic class of mappings of X into Y, and the set of all mappings of X into Y is thus divided into *homotopic classes of mappings* of X into Y.

The mapping f of a space X into some point in the space x_0 $(f: X\to x_0)$ is called a *constant mapping*. The mapping of a space X into itself is called the *identity mapping* and is denoted by the symbol $1_x: X\to X$.

Example. The mappings $f(x)=x$ and $g(x)=0$ from R^n into R^n are homotopic. The homotopy is given by the formula $f_t(x)=(1-t)x$, $0\leqslant t\leqslant 1$, whence it follows that $f_0=f$ and $f_1=g$ ($f_0=1^n_R$ is the identity mapping, f_1: $R^n\to 0$ is the constant mapping of R^n into the origin of the coordinates).

Let two spaces X and Y and two mappings f: $X\to Y$ and g: $Y\to X$ be given. The spaces X and Y are said to be *homotopically equivalent* if there exist mappings f and g such that the compositions $f\circ g$ and $g\circ f$ are homotopic identity mappings 1_Y and 1_X, respectively $y\in Y$, $(g\circ f)(x)=g(f(x))$, $x\in X$). We shall denote the homotopic equivalence of X and Y by $X\sim Y$.

The space X is said to be *contractable* if the identity mapping of X is homotopic to the constant mapping. As demonstrated above, the space R^n is contractable or homotopically equivalent to one point $x_0\in R^n$: $R^n\sim x_0$.

Any two mappings of an arbitrary space into a contractable space are homotopic and therefore any mapping is homotopic to a constant mapping. This quite obvious assertion is important for the constructions given below.

A space is said to be *simply connected* if any closed contour in this space can be contracted to a point.

Examples. The circle S^1 is not simply connected. The sphere S^n for $n\geqslant 2$ is simply connected. The space RP^2 or the sphere S^2 whose antipodal points are the same point are not simply connected, since any contour connecting the antipodal points on such a sphere is closed, but it cannot be contracted to a point. This, in particular, is what distinguishes a magnetic material from a nematic.

An important generalization of the concept of simple-connectedness is *n-connectedness*, i.e., contractability to a point of n-dimensional closed surfaces that can be represented by n-dimensional spheres S^n. A contractable space (such as R^m) is n-connected for all n.

Thus it is clear that the mappings of a sphere S^m into some space X for all $m\geqslant 0$ (the zero-dimensional sphere $S^0=\{-1, 1\}$, i.e., the boundary of the interval $[-1, 1]$) characterize definite topological and homotopic properties of the space X.

Let us examine the set of homotopic classes of mappings $[S^m, X]$ of the sphere S^m into the topological space X. If

$X = R^l$, then the set of mappings $[S^m, R^l]$ is homotopic to constant mappings or, in other words, a sphere of any dimension in a Euclidean space can be contracted into a point.

As a nontrivial example we shall examine the set of homotopic classes of mappings $[S^1, S^1]$ of a circle into a circle. If the circle is represented by a curve $|z| = 1$ in the complex plane, then an example of the mapping of the circle into itself is the mapping $z \to z^n$. Then every integer n corresponds to a homotopic class of mappings from the set $[S^1, S^1]$ or, in other words, any mapping $S^1 \to S^1$ is homotopic to the mapping $z \to z^n$ for some n. The number n conveniently characterizes the homotopic class of mappings of a circle into a circle for which the number n is the number of loops in the winding of the first circle on the second circle (thread $\to$ spool), and in addition all possible windings with loops of different length and form (no knots) belong to the same class.

Thus there exists a one-to-one correspondence between homotopic classes of mappings $S^1 \to S^1$ and the group of integers Z.

We recall the definition of a group. A set G is a *group* if it contains the *multiplication operation* $g \cdot h \in G$ for all g, $h \in G$ and a) the multiplication operation is *associative* $(f \cdot (g \cdot h) = (f \cdot g) \cdot h$, where $f, g, h \in G)$; b) there exists an *identity element* e, for which $e \cdot g = g \cdot e$ for any $g \in G$; and, c) for all $g \in G$ there exists an *inverse element* g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

A group is called *abelian* or *commutative* if $g \cdot h = h \cdot g$. In this case $g \cdot h$ is often written as $g + h$, the identity element is denoted by the symbol 0, and the element inverse to g is denoted by the symbol $-g$.

Examples. The set of all integers Z is an abelian group with respect to addition. The set $R \setminus \{0\}$ is an abelian group with respect to multiplication. The set $Z_2 = \{-1, 1\}$ is an abelian group with respect to multiplication.

For the example of the mapping $S^1 \to S^1$ it was shown that the elements of the set $[S^1, S^1]$ are in a one-to-one correspondence with the elements of the group of integers Z. It turns out that the set of homotopic classes of mappings $[S^n, X]$ of a sphere of arbitrary dimension n into the space X also corresponds to some group whose structure depends on the dimension of the sphere S^n and the topology X. For this reason, the special notation $\pi_n(X)$, is introduced for the set $[S^n, X]$, and the sets $\pi_n(X)$ are called *n-dimensional homotopic groups*. For $n \geqslant 2$ the groups $\pi_n(X)$ are abelian. In the example studied above $X = S^1$ and $\pi_1(S^1) = Z$. The latter indicates that the group $\pi_1(S^1)$ is isomorphic to the group of integers. The group $\pi_n(X)$ for $n = 1$ has a special name: $\pi_1(X)$ is called the fundamental group of the space X.

The isomorphism $\pi_n(X) = 1$ or $\pi_n(X) = 0$ (in the case of an abelian group) indicates that the group $\pi_n(X)$ is trivial, i.e., it is isomorphic to the group consisting only of the identity element. For any contractable space X $\pi_n(X) = 1$ for all n. As in the case of the mapping $S^1 \to S^1$, for the mapping $S^n \to S^n$ we have the isomorphism $\pi_n(S^n) = Z$. Since any contour on the sphere S^2 can be contracted to a point, $\pi_1(S^2) = 1$ and in general $\pi_i(S^n) = 1$ for $i < n$.

For homeomorphic spaces, homotopic groups are isomorphic. For example, $\pi_1(RP^1) = \pi_1(S^1) = Z$.

Homotopic groups are isomorphic also for homotopically equivalent spaces. For example,

$$\pi_1(R^2 \setminus x_0) = \pi_1(R^3 \setminus R) = \pi_1(S^1) = Z,$$
$$\pi_2(R^3 \setminus x_0) = \pi_2(S^2) = Z.$$

The group $\pi_1(RP^2) = Z_2$. The fact that the fundamental group of the projective space RP^2 consists of two elements indicates the following: on a sphere whose antipodal points are equivalent there exist only two classes of closed paths. All closed contours which can be contracted to a point on the ordinary sphere correspond to the identity element in the group $\pi_1(RP^2)$. The nontrivial element corresponds to contours connecting the antipodal points of a sphere. All such contours are closed, and they cannot be contracted to a point. They are homotopic to one another and they are homotopic, for example, to some great semicircle. We also point out that $\pi_i(RP^2) = \pi_i(S^2)$ for $i > 1$.

In spite of the clarity of the definition of homotopic groups, their calculation is a very complicated problem, even in the simplest cases. The homotopic groups of a large number of spaces have now been calculated.

7. CLASSIFICATION OF SINGULARITIES IN THE ORDER-PARAMETER FIELD. HOMOTOPIC INVARIANTS

Ordered systems, such as magnetic materials, liquid crystals, superfluid liquids, etc., are described in terms of an order parameter. In the case of magnetic materials the order parameter is a unit vector $\mathbf{n}$ and in the case of nematics the order parameter is the director. The range of the order parameter, called the space of the order parameter, in these cases is S^2 and RP^2, respectively. The definition domain is the coordinate space R^3. The existence of point or linear singularities in the order-parameter field alters the topology of the coordinate space. In the case of a point singularity the order parameter is not defined at some point, for example, at the origin. From the topological viewpoint this indicates that instead of R^3 we have $R^3 \setminus \{0\} \sim S^2$. In the case of a linear singularity the line R is removed from R^3: R: $R^3 \setminus R \sim S^1$. Thus the existence of linear and point singularities gives rise to mappings of the circle S^1 and the sphere S^2 into the space of the order parameter M. Therefore the groups $\pi_1(M)$ and $\pi_2(M)$ characterize the topological singularities in the order-parameter field. If, for example, $\pi_1(M) = 1$, then any distribution of the order-parameter field with a linear singularity can be transformed by means of a continuous deformation or homotopy into a uniform distribution. Such singularities are said to be topologically unstable.

In the case of magnetic materials the order-parameter space is $M = S^2$ and $\pi_1(S^2) = 1$ and, therefore, there are no stable linear singularities in the field of the vector $\mathbf{n}$ (in a three-dimensional magnetic material). In a two-dimensional magnetic material $M = S^1$ and instead of linear singularities we have point singularities, which are also characterized by the group π_1. Now, however, the group $\pi_1(S^1) = Z$, and every integer N, described by one or another homotopic class of mappings S^1 into S^1, corresponds to a point singularity for which the integer N is equal to the number of revolutions of the vector $\mathbf{n}$ by 2π for one full turn following a closed circuit around the singularity (in terms of the director, this is a disclination with an even Frank index).

In describing a point singularity in a three-dimensional magnetic material we arrive at a mapping of S^2 into S^2 and the group $\pi_2(S^2) = Z$. In general, the point singularity in an $(n+1)$-dimensional vector field of the unit vector $\mathbf{n}$, defined on the space $R^{n+1}\setminus x_0 \sim S^n$, is characterized by the mapping of S^n into S^n (since the set of all $(n+1)$-dimensional unit vectors is the sphere S^n) and by the group $\pi_n(S^n) = Z$. The integer $N \in Z$, with which each homotopic class of mappings $S^n \to S^n$ is associated, is called the *degree of the mapping* of the vector field and is denoted by the symbol $\deg \mathbf{n}(\mathbf{r})$. The degree of the mapping indicates how many times the vector $\mathbf{n}$ runs over the sphere S^n in moving over the region $R^{n+1}\setminus x_0 \sim S^n$. If $\deg \mathbf{n} = 0$, then the vector field has no singularities and any nonuniform distribution of the vector field can be transformed by a continuous deformation into the uniform distribution ($\mathbf{n}(\mathbf{r}) = \mathbf{n}_0 = \text{const}$ in the entire space R^{n+1}). In the case when $\deg \mathbf{n} = 1$, we have, for example, a radial distribution of the vector field (hedgehog) with a singularity at the origin of coordinates. Such a distribution cannot be transformed into a uniform distribution by any continuous deformation, and in general with $\deg \mathbf{n} \neq 1$. The degree of the mapping is a *homotopic* (and topological) *invariant*, i.e., it remains constant under homotopic mappings $S^n \to S^n$. We note that the definition of the degree of a mapping can be generalized to the case of differentiable mappings of manifolds with the same dimensions.

In a two-dimensional nematic the order-parameter space is $M = RP^1 = S^1$ and $\pi_1(RP^1) = Z$. The difference from a magnetic material lies in the fact that the integer N, characterizing some homotopic class of mappings of S^1 into RP^1, equals the number of revolutions of the vector $\mathbf{n}$ by an angle π for one full turn following a closed circuit around a singular point, i.e., a two-dimensional nematic contains disclinations with both even and odd Frank indices.

For a three-dimensional nematic $M = RP^2$ and $\pi_1(RP^2) = Z_2$, and hence we have one type of stable linear singularity: disclinations with an odd Frank index. They can be transformed into one another with the help of a homotopy, and topologically they are indistinguishable. Physically, disclinations differ with respect to their energy (the energy depends on the Frank index). Point singularities in a three-dimensional nematic are the same as those in a magnetic material (up to the equivalence of the directions $\mathbf{n}$ and $-\mathbf{n}$).

In a planar magnetic material with the boundary conditions $\mathbf{n} \to \mathbf{n}_0$ as $|\mathbf{r}| \to \infty$ we have the mapping of $R^2 \cup \{\infty\} = S^2$ into $M = S^2$, characterized by the group $\pi_2(S^2) = Z$. Thus nonsingular vortices in a planar magnetic material, like point singularities in a three-dimensional magnetic material, are classified by integers or by the degree of the mapping of S^2 into S^2.

In the three-dimensional case with uniform boundary conditions at infinity $\mathbf{n} \to \mathbf{n}_0$ as $|\mathbf{r}| \to \infty$ we arrive at the mapping $R^3 \cup \{\infty\} = S^3$ into S^3, called the Hopf mapping, and the group $\pi_3(S^2)$. The group $\pi_3(S^2) = Z$, but this is more difficult to prove than in the cases studied above. The integer characterizing the mapping of S^3 into S^2 is called the *Hopf invariant*. The simplest nontrivial configuration of the $\mathbf{n}$ field is a nonsingular ring-shaped vortex for which the Hopf invariant equals unity. Everything said above remains equally valid for nematics, for which one need only take into account the equivalence of the directions $\mathbf{n}$ and $-\mathbf{n}$.

8. CONCLUSIONS

In the preceding sections we described the basic concepts of modern algebra and topology, which, in recent years, have been introduced into physics. We demonstrated the application of topological methods in the solution of physical problems for the example of the classification of singularities in the order-parameter field in magnetic materials and nematics. We must now emphasize once again what we stated at the outset. The introduction of the language and methods of topology into physics is linked with the development of new ideas in physics. The most graphic (and very important) consequence of the permeation of topology into physics was probably the fact that physics has been enriched not only with new methods but also new physical quantities, examples of which are the topological invariants. In field theory topological invariants are called topological quantum numbers or topological charges. They obey definite conservation laws, and this makes them similar to quantities such as the electric charge, spin, isospin, etc.

In ordered systems (magnetic materials, liquid crystals, and superfluid liquids) the topological charge is an observable quantity (the physical meaning of the topological charge corresponds to the graphic geometric interpretation). In elementary particle physics the situation is much more complicated. Extended objects (vortices, monopoles, instantons, etc.), which are carriers of topological quantum numbers or topological charge, appear quite naturally in classical field theories, but the assignment of quantum-mechanical properties to these objects encounters great difficulties. At the present time there does not exist an unequivocal physical interpretation of topological extended objects. Some of them (for example, monopoles) are viewed as some exotic elementary particles, and others serve as models for hadrons. The instanton, for example, is an exact solution of the Yang-Mills equations and from the mathematical viewpoint has attractive properties. On the other hand, the instanton describes the process of tunneling between quantum mechanical vacuums of the Yang-Mills theory, which leads to the breaking of some symmetries (for example, parity). One of the consequences of instanton effects is the prediction of a new heavy boson, called the axion.

The list of topological results in field theory is long and continues to grow. Some, comparatively recent, results have already had a significant effect on physical ideology, but many topological results have yet to be interpreted in the quantum theory and their connection to observed quantities has yet to be determined.

It is also important to note that everything discussed above is only a part of the process of geometrization of physics, which has been going on in recent years, and in addition it is a part which interacts actively with the other parts of this process.

This process was started in the general theory of relativity, which gave a geometric interpretation of the gravitational interaction. The modern viewpoint that the carriers of interactions are geometric objects, the so-called gauge fields, was formulated by Yang and Mills in 1954. Gauge fields, or Yang-Mills fields, are analogous to the vector potential in electrodynamics and hence it may be conjectured that the Yang-Mills fields, like photons, should be carriers of some interaction. This idea has been intensively developed, and in 1967 Weinberg and Salam independently constructed a theory of weak and electromagnetic interactions, in which the Yang-Mills "photons," called $W^{\pm}$ and Z^0 bosons, were assigned the role of carriers of weak interactions (such as the β decay of the neutron). The first experimental confirmation, though indirect, of the Weinberg-Salam theory was obtained in 1973. It remained to discover the gauge bosons directly. This was done in 1983.

Extended objects are objects in gauge theories, such as the Weinberg-Salam theory or the theory of strong interactions, called chromodynamics, i.e., theories which to some extent have been experimentally confirmed. The discovery of nontrivial topological consequences in such theories indicates both the rich content of gauge theories and, possibly, the future development of the theory of elementary particles.

In conclusion, we shall indicate some publications which could be useful in studying the questions touched upon in this paper.

One of the best textbooks on general topology is the book by Kelley,[1] though Refs. 2 and 3 could be more suitable for an initial introduction to topology. A fully accessible exposition of the methods of differential geometry used in physics is given in a recently published, but already popular book, by Schutz.[4] An excellent exposition of homotopy theory can be found in the book by Dubrovin, Novikov, and Fomenko,[5] where many mathematical questions closely related to physics are also examined. Some information on manifolds and homotopic groups can be obtained from the review by Ol'shanetskiĭ,[6] which also contains additional references. Volovik and Mineev[7] have described in detail the topological approach to the study of spatially nonuniform states such as vortices and disclinations in ordered systems (magnetic materials, nematics, superfluid He^3, etc.). An introduction to gauge theories can be found in Refs. 8 and 9. Monopoles and instantons are described in a paper by Prasad, a translation of which into Russian is given in Ref. 10. Vortices or Belavin-Polyakov instantons are studied in detail in a review by Perelomov.[11]

The basic concepts of algebra and topology are given in Ref. 12: 1. Sets and mappings. 2. Topological spaces and continuous mappings. 3. Manifolds. 4. Topological groups and Lie groups. 5. Tangent spaces of differentiable manifolds. 6. Homotopy theory (homotopic groups, degree of mappings, Hopf invariant).

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Translated by M. E. Alferieff