Present state of the theory of the MHD equilibrium and stability of stellarator plasmas

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The MHD theory of the equilibrium and stability of the plasma in a stellarator is set forth. Various ways to develop magnetic configurations of the stellarator type are examined. The basic characteristics of the devices presently in operation are reported. The method of averaging over the rapidly varying spatial variable is described. This method is used to study the plasma equilibrium and also the stability with respect to current-driven kink modes and ballooning modes. The limiting plasma pressure, determined jointly by the conditions for plasma equilibrium and stability, is discussed. Methods for raising this limit are described. The present state of the theory of the MHD equilibrium and stability of plasmas in stellarators is summarized. The basic problems for further development of this theory are outlined.

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1. INTRODUCTION

Significant progress has recently been made in research being carried out with the ultimate goal of achieving controlled thermonuclear reactions with magnetic confinement. The basic thrust of this research has been in tokamaks, which have yielded the most impressive experimental results and for which we have reached the best understanding of the physics underlying the events which occur in the plasma. Nevertheless, there is still the possibility that final success in the solution of the controlled-fusion problem may be reached by a different approach. One of these approaches is the stellarator: a closed magnetic confinement system in which a plasma is confined by fields produced by currents flowing in external conductors. The basic advantages of the stellarator are that it does not require excitation of currents in the plasma and that it can operate in a steady state.

The stellarator program has made significant progress in the last few years. In fact, the progress on both the experimental and theoretical fronts has been so significant that stellarator systems are already receiving serious attention as one version of a fusion reactor.

The magnetic confinement system used to confine a hot plasma must obviously keep the plasma at equilibrium (and must do so in such a way that the equilibrium is stable with respect to small perturbations) and must keep the losses of particles and of energy at acceptably low levels. The loss from a confinement system is determined by the rates of diffusion and thermal conductivity; the analysis of these processes is the subject of the theory of transport processes. We will not be discussing that subject in the present paper; we direct the interested reader to, for example, one of the recent reviews.[1] We restrict the present paper to the first two problems: the theory of MHD equilibrium and stability of a plasma. For each of these problems, we attempt to determine the present state of affairs and to identify the most important questions at this point.

We will be discussing only stellarators[1)] with a circular plane magnetic axis. We will not consider systems with three-dimensional axes, such as the ASPERATOR,[2] the DRAKON,[3] and the HELIAX,[4] which are extremely interesting but have not been studied adequately, either theoretically or experimentally. In §2 we discuss methods for producing stellarator magnetic configurations, and we list the basic parameters of such configurations.

One factor which has delayed the development of a theory for plasma confinement in stellarators is the fact that these systems are not axisymmetric, so that the problems are three-dimensional. It is accordingly important to develop methods, both analytic and numerical, capable of dealing with the particular features of plasma confinement in three-dimensional magnetic fields. In §3 we outline the basic approaches to research on various aspects of MHD confinement of plasmas in stellarators.

It had earlier been assumed that the joint satisfaction of

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the conditions for equilibrium and stability of a stellarator plasma would limit the value of $\beta = 8\pi\bar{p}/B^2$—the ratio of the plasma pressure, averaged over the cross section, to the pressure of the magnetic field—to something less than 1% (a magnetic confinement system of the stellarator type would be of practical interest as a possible prototype fusion reactor only if a reasonably dense plasma with $\beta \sim 5\%$ could be stably confined in it). However, a recent theoretical work has refuted that opinion and has demonstrated that a plasma with $\beta \sim 10\%$ can be stably confined in a system with practically feasible parameter values. Questions pertaining to the determination of the limiting plasma pressure are discussed in §4 (equilibrium) and §6 (stability).

There has been essentially no experimental work on the confinement of a high-β plasma in a stellarator. The results set forth in these sections are therefore of interest primarily for predicting future experiments and for choosing the best magnetic configurations for confinement systems of the future (including reactors). On the other hand, many experiments have been carried out on the current-driven kink modes which are responsible for the presence of a longitudinal current in the plasma; this current has recently become the basic approach for producing and heating the plasma. The theortical research on the current-driven kink modes in a stellarator is the subject of §5 of this review. In the same section we will discuss several experimental results obtained from various devices but only to illustrate the basic theoretical positions, since a review of the theoretical work is not a basic purpose of this review. A detailed discussion of the experiments which have been carried out in devices of the stellarator type can be found in, for example, the reviews in Refs. 5 and 6 and a recently published monograph.[7]

In the Conclusion we summarize the present state of the theory of the MHD equilibrium and stability of a plasma in a stellarator, and we outline the basic directions for further development of this theory.

A few words are in order regarding the bibliography at the end of this paper, for which we make no claim of completeness. We have attempted to reflect in this review primarily those studies which have yielded results of fundamental importance or studies which are of interest from the methodological standpoint. The choice of the latter studies is of course colored by the particular tastes of the authors.

2. MAGNETIC CONFIGURATIONS OF STELLARATORS

The stellarator magnetic confinement system was first proposed back in 1951 by Spitzer,[8] who showed that if a toroidal solenoid is twisted into a figure-eight the resulting magnetic configuration will have nested magnetic surfaces with a nonzero rotational transform (which cancels, on the average, the drift of particles due to the nonuniformity of the magnetic field). Koenig[9] proposed yet another way to produce a vacuum rotational transform. In devices of this type, which we will call the "classical stellarator," the longitudinal magnetic field is produced by a solenoid winding, while the rotational transform is produced by a current flowing in n_0 pairs of helical conductors wound on the surface of a torus. The currents in adjacent conductors are equal in magnitude and opposite in sign (Fig. 1).

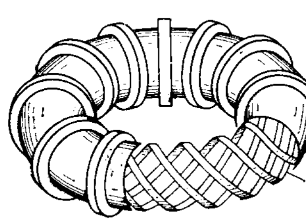

FIG. 1. A stellarator (schematic drawing).

To get a better understanding of the basic properties of the magnetic configuration which is produced in a stellarator, we first consider the graphic case of a straight stellarator, i.e., a system with helical symmetry. In this case the irrotational magnetic fields can be described by a scalar potential $\tilde{\Phi}^M$, which depends in a cylindrical coordinate system (r,φ,z) on only two variables, namely, r and $\omega = \varphi - Ns/n_0$. We also assume that the expression for $\tilde{\Phi}^M$ contains only a single harmonic of ω:

$$\tilde{\Phi}^M = B_0R_0s + B_0\varepsilon(r)\sin n_0\omega, \quad \mathbf{B} = \nabla\tilde{\Phi}^M,$$
$$\varepsilon(r) = \alpha^{-1}\varepsilon_{n0}I_{n0}(n_0\alpha r), \tag{2.1}$$

where $s = z/R_0$, $R_0 = L/2\pi$, L is the length of the system, B_0 is the static longitudinal field, N is the total number of periods of the helical field, ε_{n0} is a constant, I_{n0} is the modified Bessel function of index n_0, and $\alpha = N/R_0n_0$. Strictly speaking, the magnetic field produced by the current flowing in n_0 pairs of helical conductors (n_0 is a "multipolarity") contains an infinite set of harmonics. Far from the current-carrying conductors, however, the approximation of a single harmonic is often competely justified.

Since the magnetic field is helically symmetric in this approximation, the system of equations of a line of force,

$$\frac{dr}{dz} = \frac{B_r}{B_z}, \quad \frac{r d\varphi}{dz} = \frac{B_\varphi}{B_z}, \tag{2.2}$$

has the exact integral

$$A_z + \alpha r A_\varphi = \frac{1}{2}B_0\alpha r^2 - \frac{r\varepsilon_{n0}}{\alpha n_0}I'_{n0}\cos n_0\omega$$
$$= \frac{1}{2}B_0\alpha a_V^2 = \text{const}, \tag{2.3}$$

where A_z and A_φ are components of the vector potential ($\mathbf{B} = \text{curl}\,\mathbf{A}$); the prime means differentiation with respect to r; and a_V is the mean radius of the magnetic surface (Fig. 2 shows schematic intersections with a z = const plane of the magnetic surfaces for stellarators with $n_0 = 1$, 2, and 3). There are two types of magnetic surfaces: those which close around the magnetic axis of the sytem and those which close around the helical conductors. The surface which separates these two families is the "separatrix." The $n_0 = 1$ case is a somewhat special case in that with $n_0 = 1$ the magnetic axis of the system is a three-dimensional curve. At $n_0 > 1$, there is an elliptical singular point at the origin in a z = const cross section, and the z axis is the magnetic axis of the system. Furthermore, there are n_0 hyperbolic points, which form n_0 helical axes: edges of the separatrix which lie opposite the helical conductors through which the current is flowing in the direction opposite the direction of the longitudinal mag-

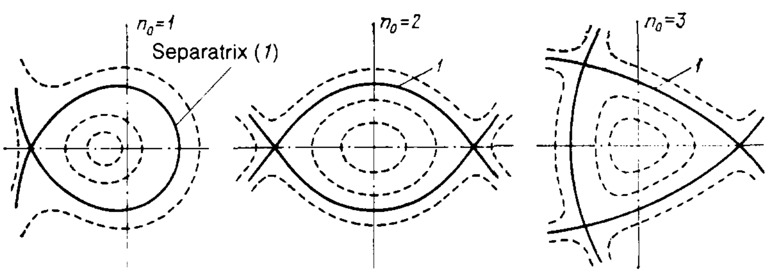

FIG. 2. Intersections of the magnetic surfaces with a $z=\text{const}$ plane for systems with helical symmetry and with $n_0 = 1, 2$, and 3.

netic field. In order to determine unambiguously the behavior of a line of force on a magnetic surface it is necessary to find the second integral of system (2.2). This integral can be expressed in quadrature,[10] but the equations found as a result are extremely complicated. Morozov and Solov'ev[10] have accordingly suggested using an approximate method, an averaging method, to find a second integral. The averaging method of Ref. 11 is actually a convenient change of variables which causes all the quantities in system (2.2) to become the sums of two parts: $r = a_V + \tilde{r}$, $\varphi = \langle\varphi\rangle + \tilde{\varphi}$. Here the tilde marks a quantity which oscillates rapidly along a line of force, while the angle brackets denote an average over the period of the rapidly alternating magnetic field. Here we have $\langle\tilde{r}\rangle, \langle\tilde{\varphi}\rangle = 0$. The oscillating terms are found through a direct integration over the rapidly varying spatial variable, and the problem is thereby reduced to one of finding the average quantities from a system of equations which incorporates terms found by taking an average of the even powers of the oscillating quantitites. In particular, the averaging method makes it a simple matter to find the equation of a line of force within terms of order up to ε^2 inclusively. In this case we find

$$r = a_V + \frac{1}{n_0\alpha^2}\,\varepsilon_{n0} I'_{n_0} \cos n_0\omega,$$

$$\varphi = \tau^* s - \varepsilon_{n0}\left(\frac{1}{\alpha a_V^2}\right)^2 I_{n_0} \sin n_0\omega, \qquad (2.4)$$

where τ^* is the rotational transform (divided by 2π) of the lines of force which lie on the surface with the average radius a_V:

$$\tau^* = \varepsilon_{n_0}^2 R_0 \alpha n_0^4 \left(\frac{1}{2x}\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 I_{n_0}^2, \quad x = n_0\alpha a_V. \qquad (2.5)$$

In the case $x \ll 1$ we find from (2.5)

$$\tau^* \approx \varepsilon_{n_0}^2 \left(\frac{1}{2^{n_0} n_0!}\right)^2 n_0^6 \alpha R_0 x^{2(n_0-2)} \left(n_0 - 1 + \frac{x^2}{2}\ldots\right). \qquad (2.6)$$

It can be seen from (2.6) that only fields with $n_0 = 1$ and 2 produce a nonzero angle of the rotational transform at the magnetic axis, and this angle increases monotonically with distance from this axis. For systems with $n_0 = 1$ and $n_0 = 2$ it is possible to produce a magnetic configuration with a nearly constant angle of the rotational transform, if the condition $n_0\alpha a_{\lim} \ll 1$ holds, where $a_{\lim}$ is the mean radius of the limiting magnetic surface.

The asymptotic averaging method used above is not applicable near the separatrix, there the rotational transform approaches N/n_0, which is the number of pitches of the helical windings along z, and it is not possible to separate the quantities in system (2.2) into slowly varying and rapidly oscillating parts. For this reason, although the averaging method is formally valid, at $a_v \ll a_m$ (a_m is the mean radius of the separatrix) a good agreement is observed at values up to[10] $a_v \lesssim a_m/2$, as is shown by a comparison of the value of τ^* found from (2.5) with the results of numerical calculations based on the exact equations. In cases of practical interest, toroidal effects, various perturbations, etc., will disrupt the helical symmetry, with the further consequence that the separatrix becomes split.[12–14] The inner "branch" of the separatrix bounds the region of surfaces which are closed around the magnetic axis,[2)] while the outer branch bounds the region of magnetic surfaces which are closed around the current-carrying conductors. Between these branches there is a subfamily of magnetic surfaces which are formed at the position of the previous separatrix and which envelop both the helical currents and the magnetic axis and thereby produce a natural diverter layer (Fig. 3).

A magnetic configuration characteristic of a classical stellarator can also be produced in a confinement system which is simpler from the engineering standpoint and which has been called the "torsatron."[15] In a torsatron, the magnetic surfaces are produced by means of n_0 helical conductors in which the currents are flowing in the same direction; these conductors replace both the solenoidal winding producing the longitudinal magnetic field and the $2n_0$ helical conductors which produce the rotational transform and the shear (i.e., the crossing of the magnetic lines of force) in the conventional stellarator. Since the currents are flowing in the same direction in the toroidal geometry, a magnetic field appears perpendicular to the plane of the torus. This field is usually cancelled by a system of annular conductors.

Yet another cancellation method is to wind the current-carrying conductors in a special way on the surface of the torus,[16,17] so that the vertical magnetic field vanishes at the axis of the system without an auxiliary cancellation system. Gourdon *et al.*[16] call this confinement system the "ultimate torsatron" because of the extreme simplicity of the current system.

We have looked at several systems in which the vacuum magnetic surfaces are produced by helical currents. There are, on the other hand, several other ways to produce analogous configurations, in particular, by using a set of discrete magnet coils of special shape. This approach was first suggested by Popov and Popryadukhin,[18] who suggested producing helical magnetic fields by means of a solenoid whose turns are "cranked up" with distance along the geometric axis of this system and have the shape similar to that of the

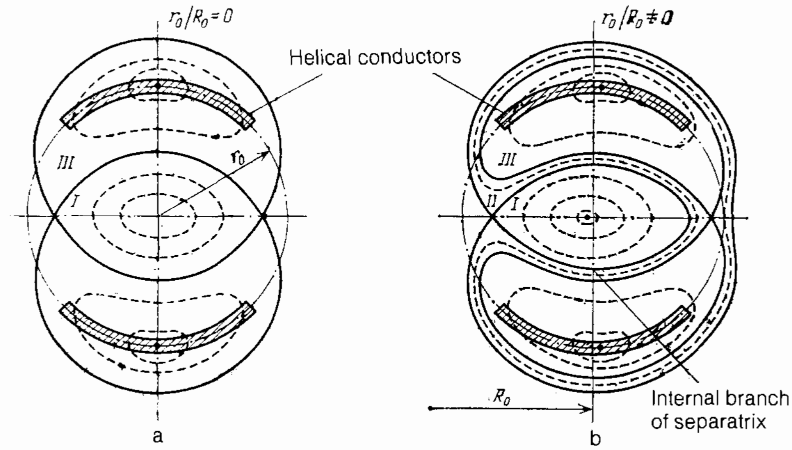

FIG. 3. Splitting of the separatrix in a torsatron in the case of disruption of the helical symmetry by finite toroidal effects.[63] I—Magnetic surfaces which are closed around the magnetic axis of the system; II—transition region; III—magnetic surfaces which are closed around the current-carrying conductors.

magnetic surfaces which are required (coils of elliptical shape are required for producing an $n_0 = 2$ field; triangular coils for an $n_0 = 3$ field; etc.). Since the efficiency at which the rotational transform is produced in a system of this sort has turned out to be too low, it has been suggested[18,19] that τ^* be increased by using an auxiliary solenoidal winding to weaken the longitudinal magnetic field.

More effective from the standpoint of maximizing the rotational transform is the approach of replacing the plane conductors by three-dimensional current rings[20] twisted, for example, into a shape with $r = r_0$, $z = d \cos n_0\varphi$ (Fig. 4), where the constant d is a measure of the level of the modulation. At $n_0 \geqslant 2$ we find a configuration with a nearly plane magnetic axis. If we use a coil with a more complex modulation, e. g., $z = \Sigma_j d_j \cos jn_0\varphi$, and choose the harmonic amplitudes d_j appropriately, we can produce an extremely wide range of magnetic configurations, in particular, some with a rotational transform significantly larger than can be achieved in the case of the single harmonic $j = 1$ (Ref. 21).

All the large experimental devices in operation today in which the magnetic configurations have plane axes use helical conductors, while the systems using module coils are not yet out of the design stage (there is the widespread belief that systems consisting of separate coils will be better for future reactors, so that malfunctioning parts can be replaced without dismantling the entire apparatus).

A circumstance which has contributed to progress in the stellarator program is that all the large experimental devices in operation today use markedly different magnetic configurations. This approach has made it possible to examine a large number of possibilities and to determine the basic physics underlying the behavior of the plasma confined in systems with different properties. For reference we list here the parameters of the largest devices in operation today.

One way to achieve a topologically stable vacuum magnetic configuration[3)] is embodied in the W-VIIA device (in West Germany,[22] with $n_0 = 2$, $\tau^* \leqslant 0.55$, $N = 5$, $R_0 = 200$ cm, $a_p = 9.0$ cm, where a_p is the mean plasma radius). The rotational transform $\tau^* \neq n/m$ remains essentially constant over the cross section of the plasma column (the integers m and n characterize the periodicity of the most probable perturbations along the minor and major circumferences of the torus).[10,23] The same principle is embodied in the JIPP-T2 device (in Japan,[24] with $R_0 = 91$ cm, $a_p = 14$ cm, $n_0 = 2$, $N = 4$, and $\tau^* = 0.14$).

Another principle underlies the development of systems with a large shear ("shear" means a crossing of lines of force in such a manner that the inclination of a line of force changes at a transition from one magnetic surface to another). Although the number of possible resonances is large, the dimensions of the resulting magnetic islands can be kept quite small.

It is widely believed that a zero angle of the rotational transform at the axis would be undesirable, since this circumstance would lead to a splitting of the magnetic axis because of toroidal effects and to the appearance of magnetic islands. Accordingly, the $n_0 = 3$ Uragan-3 torsatron with a diverter [Physicotechnical Institute, Khar'kov, USSR,[25] with $n_0 = 3$, $N = 9$, $\tau^*(0) < 0.25$, $\tau^*(a_p) = 0.6$, $R_0 = 100$ cm, and $a_p = 15$ cm] has provision for producing a nonzero rotational transform at the axis of the system by means of a transverse external field.

In the $n_0 = 2$ L-2 stellarator [Institute of General Physics, Moscow,[26] with $n_0 = 2$, $N = 14$, $R_0 = 100$ cm, $a_p = 11.5$ cm, $\tau^*(0) = 0.2$, and $\tau^*(a_p) = 0.78$] the angle of the rotational transform at the axis is not zero, and there is a moderate rotational transform at the edge: $\tau^* < 1$. The Heliotron-E (Kyoto University, Japan)[27] is the largest device presently in operation. This is a system of the torsatron type with an auxiliary solenoidal winding which makes it possible

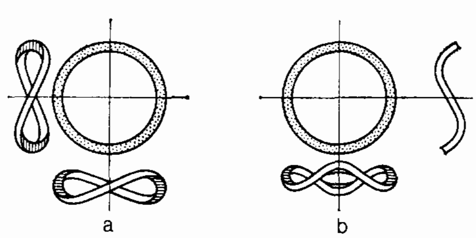

FIG. 4. Module coils which produce (a) $n_0 = 2$ and (b) $n_0 = 3$ helical fields.

to change the angle of the rotational transform by a factor of $(1+h_1)$, where h lies in the interval $-0.3 \leqslant h_1 \leqslant 0.3$. The chamber of the Heliotron-E has a nearly rectangular cross section and rotates with a pitch equal to that of the helical winding. The Heliotron-E has a very large angle of the rotational transform [$\tau^*(0)=0.5$, $\tau^*(a_p)=2.5$ at $h_1=0$, $R_0=220$ cm, $a_p \sim 15\times 30$ cm, $N=19$, and $n_0=2$], maintaining the traditional thrust of the research carried out at Kyoto University, dating back to the Heliotron D and DM devices.[28] (A magnetic confinement system of the heliotron type had been proposed and studied by Uo[29] before torsatrons were developed.[15])

In the following sections of this review we will examine several examples which will make it possible to estimate the parameters of the equilibrium plasma which can be confined without driving MHD instabilities in the systems of the types considered.

3. THEORETICAL METHODS FOR STUDYING PLASMAS IN STELLARATORS

The need to study the equilibrium and stability of a plasma confined in a three-dimensional (three spatial dimensions) magnetic field has led to the development of several specific methods. Historically the first asymptotic method proposed for studying the equilibrium and stability of the plasma in a stellarator was the method of the so-called stellarator approximation, developed by a group of theoreticians at Princeton University.[30–33] The stellarator approximation is based on an expansion in several small parameters: $\varepsilon = |\mathbf{B}^{st}|/B_0$, which is the ratio of the amplitudes of the helical components of the magnetic field to the toroidal component; $\tau n_0/N$, where τ is the total angle of the rotational transform, and n_0/N is the ratio of the multipolarity to the total number of periods of the helical field; $\delta = a_p/R_0$, which is the toroidal ratio; B^J/B_0, is the ratio of the magnetic field produced by the currents flowing through the plasma to the longitudinal magnetic field; and β, the ratio of the plasma pressure to the magnetic pressure. The procedure used here is extremely similar to the averaging method. The set of small parameters of the stellarator approximation was also used in Refs. 34–37, where the longitudinal coordinate was eliminated through a special change of variables. These methods have both been used to study the equilibrium of a plasma and its stability in the linear approximation.

A more general problem was solved in Refs. 38–40, and a system of average equations was derived which reduced the problem to one of studying the axisymmetric case (so that it automatically becomes possible to call upon the entire arsenal of methods which had been developed previously for studying tokamaks). That system of equations can be used to solve a variety of problems, e. g., problems involving equilibrium, stability, and nonlinear processes. Strictly speaking, the advantage of this method over other methods stems from the circumstance that a universal system of average MHD equations is in place, ready to be used to solve specific problems. There is a convenient method for generalizing the results found previously for tokamaks to the case of a stellarator. A similar method was used in Refs. 41 and 42.

As we have already mentioned, the averaging method[10,11] is a special change of variables. We can see the essence of this method as it pertains to the case at hand. The stellarator is a magnetic confinement system in which the plasma is confined by means of a strong, axisymmetric magnetic field and a weak magnetic field which varies rapidly along a line of force (Fig. 5). It is thus natural to assume that all quantities characterizing the plasma can be written as the sum of two parts, e.g., $X = \langle X\rangle + \tilde{X}$, where X is an arbitrary quantity to be determined from the MHD equations, and $\langle\tilde{X}\rangle = 0$. The angle brackets mean an average over the toroidal variable along which the system is periodic.[4)] Making a similar change in the unknown functions, we then find that the number of equations is doubled, but the oscillatory terms can in principle be calculated in any order in ε through a direct integration over the rapidly oscillating spatial variable. Average quantities, on the other hand, are found from a system of equations which incorporates terms which rise upon averaging of the squares or higher even powers of the oscillating quantities. The system of average MHD equations is as follows (for convenience, we are omitting the symbols denoting averages)[38–40]:

$$\rho\,\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c}\,[\mathbf{J}\mathbf{B}] + \frac{1}{c}\,[\mathbf{B}\,[\mathbf{B}\nabla\lambda_p]],$$

$$\mathrm{rot}\,(\mathbf{B}-\mathbf{B}^*)^*) = \frac{4\pi}{c}\,\mathbf{J},\quad \mathrm{div}\,\mathbf{B}=0,\quad \mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\,\frac{\partial\mathbf{B}}{\partial t},$$

$$\frac{d\rho}{dt} = -\rho\,\mathrm{div}\,\mathbf{v},\ \frac{d}{dt}\,(\rho^{-\gamma_0}p)=0,$$

$$\mathbf{E} + \frac{1}{c}\,[\mathbf{v}\mathbf{B}] + \nabla\left[\frac{R_0^2}{B_T}\left\langle \int B_s^{st}\,ds\left(\int \mathbf{B}^{st}\,ds\cdot\nabla\,\frac{\alpha_p}{\sigma_{\parallel}}\right)\right\rangle\right]$$

$$= \frac{\mathbf{J}_{\parallel}}{\sigma_{\parallel}} + \frac{\mathbf{J}_{\perp}}{\sigma_{\perp}} - \frac{\lambda_p[\mathbf{B}\nabla\sigma_{\parallel}]}{\sigma_{\parallel}^2} - \left(\frac{1}{\sigma_{\perp}} - \frac{1}{\sigma_{\parallel}}\right)[\mathbf{B}\nabla\lambda_p],$$

$$\mathbf{B} = \mathbf{B}_T + \mathbf{B}_{ex} + \mathbf{B}^J + \mathbf{B}^*,\quad \mathbf{B}^* = \mathbf{B}_{\perp}^* + \mathbf{e}_s B_s^*,$$

$$\mathbf{B}_{\perp}^* = \mathrm{rot}\,\frac{\Psi^*\mathbf{e}_s}{R},\quad B_s^* = -\langle(\mathbf{B}^{st})^2\rangle\,B_T^{-1},$$

$$\Psi^* = R^2\left\langle B_r^{st}\int B_{\varphi}^{st}\,ds\right\rangle B_T^{-1},\quad \lambda_p = \frac{\Psi^*\alpha_p}{B_0R_0},\quad \alpha_p = \frac{\mathbf{J}\mathbf{B}}{B^2},$$

$$h = \frac{R}{R_0},\quad \langle f\rangle = \oint f\,ds\left(\oint ds\right)^{-1},\quad \mathbf{B}^{st} = \nabla\tilde{\Phi}^{M},$$

$$\tilde{\Phi}^{M} = \sum_i A_i\,(r,\ \varphi)\sin N_i s, \tag{3.1}$$

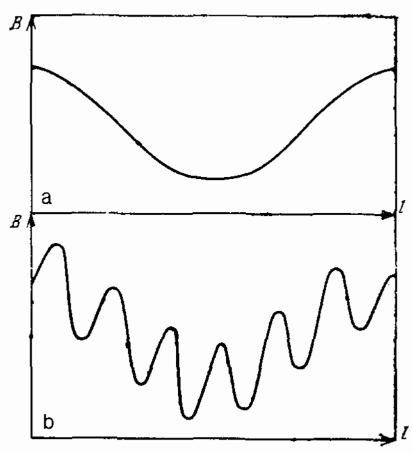

FIG. 5. Variation of the strength of the magnetic field along a line of force in a tokamak (a) and in a stellarator (b).

where N_i $(i = 0, 1, 2, \ldots)$ are integers satisfying $N_{i+1} > N_i$; and we are assuming $N_0 \gg 1$ and $\partial\tilde{\Phi}^M/\partial s \gg \partial\tilde{\Phi}^M/\partial\varphi$. If $\tilde{\Phi}^M$ is helically symmetric, i.e., if

$$\tilde{\Phi}^M = \sum_n \Phi_n^M(r) \sin n\left(\varphi - \frac{N}{n_0}s\right),$$

the quantity B_s^* is expressed in terms of the stellarator angle of the rotational transform,

$$B_s^* = -\int_0^r \frac{NB_0}{n_0 R_0^2 r^3}(r^4 \iota^*)' \, dr,$$

where ρ, p, and $\mathbf{v}$ are the plasma density, pressure, and velocity; $\mathbf{E}$ is the electric field; γ_0 is the adiabatic index; $\mathbf{J}$, $\sigma_\parallel$, and $\sigma_\perp$ are the current density and the longitudinal and transverse conductivities; R is the major radius of the torus; and R_0 is the radius of the geometric axis of the torus. We denote by $\mathbf{B}$ the resultant magnetic field, where B_T is the vacuum, axisymmetric, longitudinal magnetic field ($B_T = B_0/h$, where $B_0 = \text{const}$); $\mathbf{B}_{ex}$ is the sum of the axisymmetric external fields (e.g., the multipole magnetic field used to correct the magnetic axis of the system); $\mathbf{B}^J$ is the magnetic field produced by the currents flowing through the plasma (in particular, by the ohmic-heating current and the diamagnetic currents); and, finally, $\mathbf{B}^*$ is the "effective" magnetic field[5)] characterizing the average effect of the helical components of the magnetic field on the plasma. The quantity Ψ^* in (3.1) is the poloidal flux of the effective magnetic field, divided by 2π (in the absence of the fields $\mathbf{B}_{ex}$ and $\mathbf{B}^J$, the equality $\Psi^* = \text{const}$ determines the equation of the average magnetic surface in vacuum).

In writing the average equations we have, for definiteness, used the quasitoroidal coordinate system r, φ, s (Fig. 6), in which the square of an arc element is given by the following expression:

$$(dl)^2 = (dr)^2 + (r\,d\varphi)^2 + (R\,ds)^2, \tag{3.2}$$

where s is the angular coordinate along the major circumference of the torus, r and φ are polar coordinates in the $s = \text{const}$ plane, $R = hR_0$, $h = 1 + kr\cos\varphi$, and $k = 1/R_0$ is the curvature of the geometric axis of the system. There are two important points to be noted. In deriving (3.1) we used the approximation of small toroidal effects only in deriving the average corrections to Ohm's law, in order to shorten the extremely lengthy procedure of calculating these corrections. The other average equations hold in an arbitrary order in the toroidal effects. Finally, to put the equations in a more compact form we have ignored terms of the order of ε^2 in

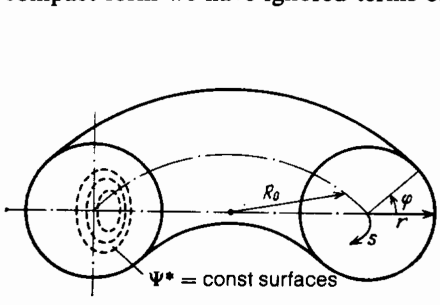

FIG. 6. Quasitorioidal coordinate system.

comparison with unity where this step would lead to a simple renormalization of average quantities. We recall that the averaging method is only a convenient change of variables. In those cases in which it is necessary to find a complete (three-dimensional description of some process, it is sufficient to add to the solution of average equations (3.1) an expression for the rapidly alternating part. An explicit expression for this part is available.[38] However, for a wide range of problems, including, in particular, many involved in research on the equilibrium and stability of plasmas in stellarators, as we will see below, taking into account the rapidly alternating part is superfluous. Consequently, we will not reproduce here the expressions for the rapidly alternating parts; those expressions are lengthy and not graphic. We also note that the system of average equations applies to those cases in which all quantities vary slowly along a line of force of the average field. Since the rapidly alternating field $\mathbf{B}^{st}$ may in general contain several harmonics with different periods along s ($B^{st} \sim \Sigma_j \exp(jN_j s)$, the condition for the applicability of the average equations is written in form

$$R_0 \frac{\partial}{\partial l} \ln X \ll \min\{N_i, |N_j - N_i|\} \quad \text{for} \quad j \neq i,$$

where X is an arbitrary quantity, and $\partial/\partial l$ is the derivative along a line of force of the average magnetic field: $\partial/\partial l = |\mathbf{B}|^{-1}(\mathbf{B}\nabla)$.

The use of the system of average equations thus substantially simplifies the problem, which reduces to a study of the axisymmetric case. For axisymmetric systems—tokamaks—many programs have now been developed for numerically solving problems which arise in the study of equilibrium and stability. Since problems of this type are generally extremely laborious, requiring much computer time, it is customary to replace the complete system of vector MHD equations by a simpler system of scalar equations, which is constructed with the help of a set of small parameters. The basic parameter is $|B_\perp|/B_0$, the ratio of the transverse components of the magnetic field to the longitudinal magnetic field. In this model there are no stable magnetosonic waves with phase velocities $v_A = B_0/(4\pi\rho)^{1/2}$, and the limitations on the time step are accordingly relaxed. The system of simplified nonlinear equations was first used to describe a cylindrical axisymmetric plasma column by Kadomtsev and Pogutse.[43] Several later studies (see, e.g., Refs. 44–46 and the bibliograhies there) have incorporated finite-pressure effects and toroidal effects.

A simplified system of equations analogous to the Kadomtsev-Pogutse equations for a tokamak, which is valid in first order in the toroidal effects and which is convenient in particular for numerical calculations and a nonlinear analysis of stability in a stellarator, was used in Refs. 41 and 42. A system of simplified average equations incorporating finite-pressure and toroidal effects more accurately than in Refs. 41 and 42 was derived in Ref. 40 (see the Appendix). The finite-pressure and toroidal effects are important in, for example, an analysis of the "self-stabilization" of a plasma and the effects of the toroidal nature of the system on equilibrium and stability.

All the approaches outlined above have made use of an expansion in the square amplitude of the helical field. There is yet another method, first proposed by Mercier[47,48] and used in several studies,[49–52] which uses an expansion in powers of a_V, the distance from the magnetic axis. This procedure makes it possible to separate variables in the system of differential equations which arises. In a study of the plasma in a toroidal stellarator there are actually two independent expansion parameters, ka_V and $n_0\alpha a_V$ (the second expansion parameter is unrelated to the curvature, stemming entirely from the period of the helical field). The series which arise in the expansion in the parameter $N_0\alpha a_V$ converge quite well, so that the procedure can be used even in the case $n_0\alpha a_V > 1$ [the formal reason for this circumstance is that one is forced to expand terms $\sim I_{n0}(n_0\alpha a_V)$, where I_{n0} is the modified Bessel function; for example, an error of the order of 1% is achieved when the Bessel function $I_2(x)$ at $x = 2$ is approximated by only the first three terms of the series]. Certain difficulties arise when this method is used because it is necessary to know the exact shape of the magnetic axis; this shape depends on the structure of the vacuum magnetic fields, the plasma pressure, the boundary conditions, and so forth.

So far, we have been discussing the use of several asymptotic methods to study plasmas in stellarators. In recent years, progress in computation technology has made it possible also to carry out some direct three-dimensional calculations. There are several three-dimensional numerical codes[53–55] which can be used to study the equilibrium and stability of a plasma in a stellarator. All are based on a minimization of the potential-energy functional

$$W_p = \int\int\int \left[\frac{B^2}{8\pi} + \frac{p}{\gamma_0 - 1}\right] dV,$$

a first variation of which leads to the equilibrium equations

$$-\nabla p + \frac{1}{c}[\mathbf{J}\times\mathbf{B}] = 0, \quad \frac{4\pi}{c}\mathbf{J} = \operatorname{curl}\mathbf{B}.$$

If the energy functional has a local minimum under corresponding boundary conditions, the given equilibrium is stable.

Despite the increasing power of modern computers, the three-dimensional calculations remain an extremely complicated and subtle problem, frequently lying at the limit of the capabilities of modern computers. It becomes necessary to reach a judicious compromise between the use of direct methods (i.e., three-dimensional numerical methods) and the use of asymptotic methods. One of the main problems which arise when asymptotic methods are used is the questions of the accuracy of these methods. For example, in the derivation of average MHD equations we have retained only the terms which arise when an average is taken of the squares of the oscillating quantities (i.e., the terms which are discarded are of the order of ε^4 and higher). The high accuracy of the asymptotic method is demonstrated by the analysis of Ref. 56, where a numerical study was made of the equilibrium of a plasma in a stellarator by means of both the average MHD equations and the complete (unaveraged) three-dimensional system of equations of single-fluid hydrodynamics. It was demonstrated that the results are almost identical.

4. PLASMA EQUILIBRIUM

The system of average MHD equations in (3.1) makes it a straightforward matter to derive a scalar equation for the equilibrium of a plasma in a stellarator, written as a nonlinear equation for the total poloidal flux Ψ (for convenience, divided by 2π)[57,58]:

$$R^2 \operatorname{div} R^{-2} \operatorname{grad}(\Psi - \Psi^*) \equiv L(\Psi - \Psi^*)$$

$$= -\frac{4\pi}{c}\left[cR^2 p'_\Psi(\Psi) + F'(\Psi)\left(\frac{4\pi}{c}F(\Psi) + RB_s^*\right)\right], \quad (4.1)$$

where the prime means a derivative with respect to the argument.

In the absence of helical fields ($\Psi^* = 0$), Eq. (4.1) becomes the Grad-Shafranov equation[59,60] for a tokamak; in the limit of vanishingly small toroidal effects, it becomes the equation of Greene and Johnson.[61] The expressions for the fields and the currents are

$$\mathbf{B}_\perp = [\nabla\Psi\mathbf{e}_s]R^{-1}, \quad \mathbf{B}_\perp^* = [\nabla\Psi^*\mathbf{e}_s]R^{-1},$$
$$\mathbf{J}_\perp = [\nabla(F(\Psi) - \lambda_p B_0 R_0), \mathbf{e}_s]R^{-1}.$$

Equation (4.1) can be written in a slightly different form, which is more convenient for analyzing the equilibrium in the case of a current-free stellarator. Replacing the function $F(\Psi)$ by a new function, equal to the sth component of the current density, averaged over a magnetic surface,

$$\bar{J}_s(\Psi) = \frac{d}{dS_0(\Psi)}\iint_{\Psi=\text{const}} J_s\, dS_0(\Psi),$$

where $S_0(\Psi)$ is the cross-sectional area of the magnetic surface, we can rewrite the equilibrium equation as

$$L(\Psi - \Psi^*) = -4\pi p'(\Psi)\left[R^2 - \frac{\bar{R}}{\overline{R^{-1}}} + \frac{\bar{R}}{\overline{R^{-1}}}\left(\frac{\bar{B}_s^*}{R_0 B_0 \overline{R^{-1}}} - \frac{\overline{B_s^* R}}{R_0 B_0}\right)\right] - \frac{4\pi}{c}\frac{\bar{J}_s(\Psi)}{\overline{R^{-1}}}, \quad (4.2)$$

where the superior bar means an average over a magnetic surface,

$$\overline{(\cdots)} = \frac{d}{dS_0(\Psi)}\iint_{\Psi=\text{const}} (\cdots)\, dS_0(\Psi)$$

so that all the average quantities are functions of Ψ only. For a current-free stellarator we would have $\bar{J}_s = 0$, and from (4.2) we immediately see one possibility for reducing the effect of pressure on the distortions of the magnetic surfaces. In first order in the toroidal effects we have $R^2 - \bar{R}/\overline{R^{-1}} = R^2 - R_0^2$, and if we formally set $(R^2 - R_0^2)B_0 = R_0^2(B_s^* - \bar{B}_s^*)$, we find that the right side of (4.2) vanishes. In other words, the plasma pressure does not affect the shape of the magnetic surfaces in this approximation. The limiting pressure in a stellarator can therefore be increased by choosing the helical harmonics in a special way. This approach has been proposed for a future modification of the Wandelstein VII stellarator.[62] In our opinion, however, this approach is not adequate, since it results in a degradation of the stability and a significant complication of the topology of the magnetic surfaces.

In studies of the equilibrium and stability of plasmas (particularly by analytic methods) it is frequently convenient to use a coordinate system with straightened lines of force, $x^i = \{a, \vartheta, s\}$, where a is an arbitrary function of the magnetic surface, and ϑ and s are cyclic angular coordinates, which change by 2π when the magnetic axis and the major axis of the torus, respectively, are circumvented. Here we have $B^1 = 0$, and the ratio of the contravariant components of the magnetic field, B^2/B^3, does not depend on ϑ or s. Below, the superscript will refer to the contravariant, and the subscript to the covariant, component of a vector. For definiteness, we are using a as an analog of the minor radius of the torus.

The expression for the square of an arc element in this new curvilinear coordinate system is
$(dl)^2 = g_{11}(da)^2 + 2g_{12}\, da\, d\vartheta + g_{22}(d\vartheta)^2 + g_{33}(ds)^2$.

Straightforward calculations using system (3.1) lead to the following expressions for the magnetic field and the current density[40]:

$$B^2 = \chi'(a)\, g^{-1/2}, \quad B^3 = \Phi'(a)\, g^{-1/2},$$
$$J^i = \frac{1}{g^{1/2}}\Big[-\hat{S}(a, \vartheta),\ I'(a) + \int \hat{S}'\, d\vartheta,\ J'(a) - cp'\,(\widetilde{g^{1/2}})\,(\chi')^{-1}\Big],$$
$$\hat{S} = g_{33} g^{-1/2} \Phi' \frac{\partial \lambda_p}{\partial \vartheta}, \tag{4.3}$$

where $g = \operatorname{Det} g_{ik}$; χ, Φ, I, and J are respectively the azimuthal and toroidal fluxes and currents, reckoned from the magnetic axis, and divided by 2π, for convenience ($I = -F'$, $\chi' = -\Psi'$); and $\chi'/\Phi' \equiv \tau$ is the total angle of the rotational transform.

The magnetic fluxes and the currents are related by

$$cp'\,(g^{1/2} - \widetilde{g^{1/2}}) = I'\Phi' - J'\chi', \tag{4.4}$$

$$-\frac{4\pi}{c} I(a) = + g_{33}\Phi' g^{-1/2} - B_3^* + \frac{4\pi}{c}\int \hat{S}\, d\vartheta, \tag{4.5}$$

$$\frac{4\pi}{c} J'(a) - 4\pi p' \widetilde{g^{1/2}}\,(\chi')^{-1} = (g_{22}\chi' g^{-1/2})' - \frac{\partial}{\partial\vartheta}(\chi' g_{12} g^{-1/2}) + g_{33}^{-1} g^{1/2} L\Psi^*. \tag{4.6}$$

Here and below, the wavy superior bar means the part which depends on ϑ, while the prime means differentiation with respect to a. Equation (4.6) is an analogy of Eq. (4.2) in a coordinate system with straightened lines of force.

There are several different methods for analyzing Eqs. (4.1) and (4.6); correspondingly, the maximum permissible pressure has been determined in several different ways. The first method is to solve (numerically, for the most part) Eq. (4.1) [or (4.2), (4.6)] directly for certain given profiles of the pressure, $p(\Psi)$, and of the current and to find explicit expressions for the magnetic surfaces. In this approach, the limiting pressure $\beta_{\rm equil}$ is often understood as that value of β at which the selected procedure for finding a solution is disrupted. In general, this disruption occurs because of the appearance of multivalued solutions for Ψ, i.e., because of the appearance of an internal separatrix (magnetic islands) or of an external separatrix. The maximum permissible plasma pressure may exceed this value slightly. This assertion requires some clarification. The appearance of an internal separatrix (or of magnetic islands) stems from the presence of dipole diamagnetic currents $J_s \sim (p'/\tau)\cos\varphi$, which eliminate the charge separation caused by the toroidal drift of particles. As a result, the total poloidal field decreases on the inner side of the plasma column and increases on the outer side. At a sufficiently high plasma pressure, $\beta \sim \tau^2\delta$, the magnetic field produced by the diamagnetic currents may be comparable in magnitude to the poloidal component of the average magnetic field. This circumstance will in turn cause the formation of an additional axis and of magnetic islands near a null point of the poloidal field. Significantly, their shape depends strongly on the functions $\tau(a)$ and $p(a)$ (Fig. 7). In stellarators with a large shear ($S = \partial \ln \tau / \partial \ln a|_{a=a_p} \sim 1$), the appearance of an internal substructure of magnetic islands of this type inside the separatrix due to the helical fields may not, in general, lead to a catostrophic degradation of the plasma confinement in such a system.

A second method for analyzing the equilibrium equation is to expand the function $\Psi(r, \varphi)$ in a Fourier series in the poloidal azimuth and to reduce the partial differential equation to a system of ordinary differential equations for the amplitudes of the various harmonics. Calculations show that it is usually sufficient to restrict the analysis to a few (usually, two or three) harmonics. In this formulation, the method is equivalent to describing the magnetic surfaces by means of functions which characterize $\xi(a)$, the displacement of the centers of the magnetic surfaces; $\alpha(a)$, their ellipticity; $\mu(a)$, their triangularity; etc. The parameters characterizing the displacement and distortion of the magnetic surfaces (α, ξ', etc.) and a/R_0—a parameter characterizing the toroidal effects—are generally independent, since the former are determined by the plasma pressure and the strength of the currents flowing through the plasma, in addition to the toroidal effects.[6)] A parameter characterizing the distortion of the magnetic surfaces due to the finite-pressure effects is $\gamma_\vartheta = \beta/\tau^2\delta$; in order of magnitude we have $\xi' \sim \gamma_\vartheta$, $\alpha' \sim \gamma_\vartheta^2$, etc.

We begin with the case of low pressures, $\gamma_\vartheta \ll 1$, which lead to only a displacement of magnitude ξ of the vacuum magnetic surfaces along the major radius (i.e., we are setting $\alpha, \mu \equiv 0$). We assume that ξ is also quite small, and we take it into account only in the linear approximation. (A situation of this sort may hold if, for example, the scalar potential $\tilde{\Phi}^M$ contains a single helical harmonic, and the displacement of the vacuum magnetic surfaces is determined by toroidal effects; another possibility is the case in which, in addition to the fundamental harmonic, with a multipolarity n_0, there are harmonics with $n_0 \pm 1$ which are small in comparison with the fundamental.) In this case it is a straighforward matter to deive from (4.5) and (4.6) a linear equation for the dis-

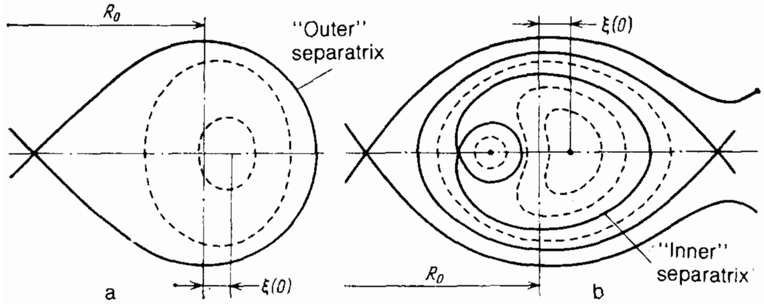

FIG. 7. Qualitative effects of a change in the structure of the magnetic surfaces at $\beta \sim \tau^2\delta$ (a) in a tokamak with $\tau'/\tau < 0$ and (b) in an $n_0 = 2$ stellarator with $a_p\tau'/\tau|_{a=a_p} \sim 1$.

placement of the centers of the magnetic surfaces[38]:

$$(a^3\tau^2\xi')' + [\tau(\tau^{*\prime}a^3)' + 4\pi a p' B_s^{*\prime} R_0^2 B_0^{-3}]\xi = \frac{8\pi R_0 a^2 p'}{B_0^2} + \tau a^2[(\xi^* a^2 \tau^*)' a^{-1}]' - \frac{a^3\tau\tau^J}{R_0}. \quad (4.7)$$

Here $\tau = \tau^* + \tau^J$ is the total angle of the rotational transform, τ^* is the angle of the rotational transform which stems from the helical magnetic field, τ^J is that produced by the currents flowing through the plasma, and ξ^* is the displacement of the centers of the magnetic surfaces in vacuum. For a current-free plasma ($\tau = \tau^*$), it is convenient to seek a solution of Eq. (4.7) in the form $\xi = \xi_p + \xi^*$, i.e., to separate the displacement of the centers of the magnetic surfaces into two parts, one of which, ξ^*, corresponds to the displacement of the centers of the magnetic surfaces in vacuum, while the other part, ξ_p, is related to the plasma pressure in the system and satisfies the following equation, which was first derived by Greene, Johnson, and Weimer[33]:

$$(a^3\tau^{*2}\xi_p')' + [\tau^*(\tau^{*\prime}a^3)' + 4\pi p' B_s^{*\prime} R_0^2 a B_0^{-3}]\xi_p = 8\pi R_0 p' a^2 B_0^{-2}. \quad (4.8)$$

The numerical calculations of Refs. 33 and 34 showed that as the plasma pressure approaches a critical value β_{cr} the equilibrium is disrupted, and we find $\xi_p \to \infty$.

In several particular cases, Eq. (4.8) can be solved analytically. It can be seen from these solutions and also, incidentally, from a general analysis of the linear equation for ξ_p, that in a case with descending pressure profiles there is always a set of values of $\beta(0)$ for which $\xi_p(0)$ becomes infinite [the number of these values is determined by the number of zeros of the solution of the homogeneous version of Eq. (4.8) in the interval $0 < a < a_p$ in the given interval of $\beta(0)$]. The solution of the homogeneous version of the equation, on the other hand, can have zeros only under the condition $a^2p'B_s^{*\prime} > 0$ (and this condition holds automatically in the case of descending pressure profiles, since in a straight stellarator the field averaged over a magnetic surface falls off with distance from the axis of the system; i.e., a straight stellarator has a maximum "average B" or a magnetic hump.

However, it is easy to see from (4.8) that the term $\sim B_s^{*\prime}\xi$ becomes important at $\xi/a_p \sim 1$ (since we have $B_s^{*\prime}/B_0a^{-1} \sim \varepsilon^2 \ll 1$), and this situation goes beyond the applicability of the linear equation.

Let us compare the limiting equilibrium pressure β_{equil}, at which the magnetic axis of the system is displaced a distance equal to the transverse dimension of the plasma column, with that found from Eq. (4.8) without consideration of the effects stemming from the presence in the system of a magnetic hump and a min β_{cr}, at which $\xi_p(0)$ becomes infinite. For shear-free systems we have[39] $\beta_{equil}/\beta_{cr} \approx \delta N\tau^*/n_0 \ll 1$; i.e., the effects of the maximum "average B" in the system are unimportant even in the linear approximation in this case. For a stellarator with a large magnetic shear, β_{cr} can often be smaller than β_{equil}. In this case, we need to analyze the system of nonlinear equations.

Completing the analysis of the linear equation for ξ, we consider the effect of the ohmic-heating current on the equilibrium. It can be seen from (4.7) that with $p = 0$ and $\xi^* = 0$ the displacement increases with increasing τ^J, reaching a limit with $\xi' \sim a/R_0$. Accordingly, in the case of small toroidal effects the ohmic-heating current cannot by itself lead to a significant displacement of the centers of the magnetic surfaces. The first term on the right side of (4.7) becomes larger than the third term even at a rather low pressure, $\beta \sim \tau^2\delta^2$. In this case, an increase in the ohmic-heating current leads primarily to a change in the magnitude and profile of the angle of the rotational transform. The rotational transform of the original vacuum configuration combines with the additional transform produced by the longitudinal current. If the current is directed is such a way that the resultant angle of the rotational transform increases, there will be the further effect of a decrease in the displacement of the centers of the magnetic surfaces.

We showed above that incorporating the term $\sim B_s^{*\prime}$ in the linear equation for the displacement of the centers of the magnetic surfaces is an attempt at an unwarranted accuracy. In order to deal with this term correctly it is also necessay to take into account the other small nonlinear terms of the same order. A system of nonlinear differential equations retaining terms up to γ_ϑ^3 inclusively (in addition to the displacement it was necessary to take into account the small ellipticity $\sim\gamma_\vartheta^2$ which arises from finite-pressure effects) was derived in Refs. 40, 64, and 65. We will not reproduce this extremely lengthy system of equations here; we simply note that incorporating the nonlinear terms has the consequence that the singularity in the limit $\beta \to \beta_{cr}$ which we mentioned disappears, and the displacement ξ_p (Fig. 8) turns out to be an increasing but bounded function of the pressure (a bounded smooth solution is also found for the ellipticity in this case). This circumstance can be given a simple physical explanation. As we have already shown, the solution of the linear equation for ξ_p becomes infinite at

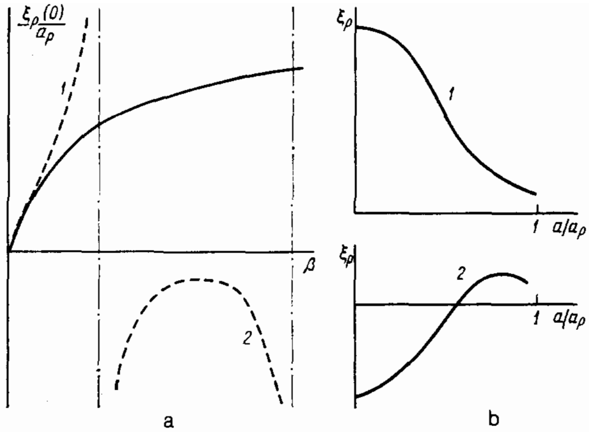

FIG. 8. a: Displacement of the magnetic axis as a function of the parameter β. Solid line—solution of the nonlinear system of equations; dashed line—solution of the linear equation for ξ_p. b: Nature of the solutions of the linear equation for ξ_p corresponding to the branches with the same labels in part a.

$p'B_s^{*\prime} > 0$, i.e., when the magnetic configuration has a maximum "average B." The displacement of the centers of the magnetic surfaces due to the finite β, however (§6), leads to a decrease in the magnetic hump and even to the formation of a magnetic well. It would be surprising if ξ were to become infinite, since the plasma column would go through a position of stable equilibrium (with a minimum "average B").

In general, the limiting equilibrium values β_{equil} are found on the basis of the definition (which is also slightly arbitrary) of the limiting pressure as being that pressure at which, because of distortions of the magnetic configuration, only a reasonably permissible part of the working volume of the plasma is lost, e.g.,[64] $\xi(a_p/2) < a_p/2$.

Although β_{equil} is somewhat ambiguous, all the calculations which have been carried out for it yield a value of the same order of magnitude:

$$\beta_{\text{equil}} = C\tau^2(a_p)\, a_p R_0^{-1}, \qquad (4.9)$$

where the coefficient $C \sim 1$ varies slightly with the plasma parameters and the pressure profile.

If the perpendicular field produced by the diamagnetic currents is cancelled to the maximum extent possible by means of feedback, the limiting equilibrium pressure can be raised to a level three or four times that in[64] (4.9). It must be recalled, however, that a complete cancellation of the fields produced by the diamagnetic currents will degrade the stability (more on this below), and the limiting pressure permitted by the stability conditions may be smaller than β_{equil}.

We also note that, according to the analytic calculations, the angle of the rotational transform τ depends on the displacement ξ with an accuracy to terms of order ξ^2 (in the case $\bar{J}_s = 0$)[65]:

$$\tau(a) = \tau^*\left(1 - \frac{\xi'^2}{2}\right) + \frac{\xi^2}{4a}(3\tau^{*\prime} + a\tau^{*\prime\prime}). \qquad (4.10)$$

It also follows from this that by varying the displacement ξ, e.g., with the help of an external perpendicular field, we can alter the dependence of the angle of the rotational transform on the average radius within certain limits.

The results outlined above were obtained analytically for the most part and are extremely general in applicability. It should be noted that more-detailed numerical calculations, incorporating the particular geometry and parameters of the systems, have been carried out in recent years in order to analyze the behavior of the plasma in various devices, either existing or planned.[56,66,67] Some of these results pertaining to the equilibrium are discussed below.

One of the most curious results of Ref. 56 was the surprisingly good agreement between the results found from the system of average equations in (3.1) and through the solution of the exact three-dimensional equations. Figure 9 shows the displacement of the magnetic axis (divided by the average plasma radius a_p) as a function of the plasma pressure at the magnetic axis, $\beta(0)$ [the parameters adopted for the system here were those corresponding to the ATF stellarator presently under construction in the US: $n_0 = 2$, $N = 12$, $\tau^*(0) = 0.3$, $\tau^*(a_p) = 1$, $R_0 = 210$ cm, and $a_p = 48$ cm]. The calculations were carried out both on the basis of the average equations [under the condition $J_s = 0$ or under the condition that $\tau(a)$ remains constant, i.e., that the flux is conserved] and also with the help of two three-dimensional codes, described in Refs. 54 and 55, which use the approximation of constant flux. We see that all these calculations agree well with each other (the calculations were carried out with a fixed boundary). It also follows from these calculations that we need to take into account the nonlinear effects, which lead to a limitation on the displacement at large values of β, in accordance with the analytic calculations (Fig. 8). Even more surprising was the agreement btween the approximate and the exact theories in the calculated values of the magnetic well, which is a characteristic which is very sensitive to variations of the magnetic field.[56,67]

As mentioned above, the τ profile changes with increasing pressure in a current-free stellarator. The behavior is illlustrated in Fig. 10, which shows profiles of $\tau(a)$ calculated for several values of β for the parameters of the ATF

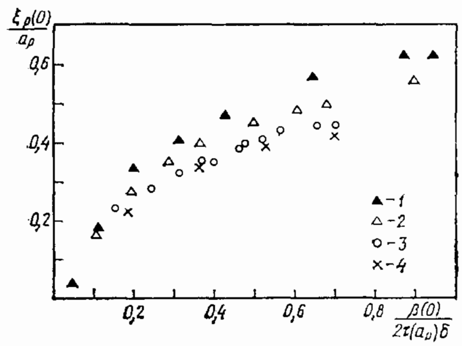

FIG. 9. Displacement of the magnetic axis as a function of the plasma pressure.[56] 1—Calculated from the average equations under the condition $\bar{J}_s = 0$; 2—calculated from the average equations under the condition of flux conservation; 3, 4—calculated from the two three-dimensional codes described in Refs. 54 and 55, respectively.

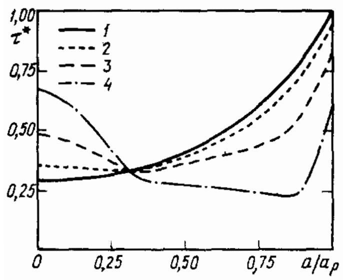

FIG. 10. Angle of the rotational transform as a function of the average radius. 1—$\beta(0) = 2.0\%$; 2—4.3%; 3—10.3%; 4—23.9% (Ref. 67).

stellarator.[67] By choosing appropriate external fields one can change the shape of the magnetic surface in such a way that the $\tau(a)$ profile remains essentially constant with increasing β. It was shown in Ref. 67 that in order to maintain a roughly constant $\tau(a)$ profile (and, in particular, a constant value of τ at the magnetic axis) with increasing β it is necessary to raise the external quadrupole field in order to transform the average magnetic surfaces from a circular shape (at $\beta = 0$) to a bean shape.

In concluding this section of the paper, we wish to stress that the analysis of the equilibrium problem in recent years has shown that in stellarators with reasonable parameter values $[\delta = 1/6 - 1/10, \tau^*(a_p) \lesssim 1]$ it is possible to achieve values $\beta(0) \sim 10$–20%, i.e., values equal to or even larger than those in tokamaks.

One of the problems which has not received adequate study so far is the analysis of the equilibrium in the case in which the plasma boundary coincides with the vacuum separatrix. Since the rotational transform at the separatrix in the case with finite toroidal effects is substantially smaller than the limiting value in the case of helical symmetry (this value is N/n_0), it may be productive to use the averaging method to describe the entire plasma volume bounded by the separatrix (although it is quite possible that small terms of order higher than ε^2 will have to be taken into account in order to obtain results which are correct not only qualitatively but also quantitatively). The ultimate answer will of course have to await a comparison of the results found analytically with those found by the three-dimensional numerical codes.

5. CURRENT-DRIVEN KINK MODES

Two types of instabilities present the greatest danger in a stellarator, as in a tokamak. One type, the interchange of ballooning modes (ideal and dissipative), draw on the thermal energy of the plasma as their source of energy. These are the instabilities which can place an upper limit on the value of β in a stellarator. The following section of this review is devoted to an analysis of the present state of the theory for these modes.

At this point we consider the second type of instability, which is driven in a low-pressure plasma, at $\beta \ll \tau^2\delta$ (this condition is typical of most of the recent experiments), by the longitudinal current which is used to produce and heat the plasma. Several of the manifestations of this instability are reminiscent of the instabilities which occur in tokamaks. Our basic goal will therefore be to call attention to the distinctive features of the current-driven instabilitities in a stellarator. We will be extremely brief in discussing cases which are similar to those in tokamaks; we refer the interested reader to the extremely extensive literature on these questions (see the reviews in Refs. 68–72, for example).

5.1. Linear theory

We restrict the discussion below to modes with small wave numbers, n, $m \ll N$ (n and m are the longitudinal and poloidal wave numbers, respectively), which we can accordingly analyze with the help of the average equations.

In the limit of ideal conductivity, the stability condition can be found by analyzing the equilibrium equation. Here an instability will correspond to the appearance of an additional equilibrium state with a perturbed magnetic field, which satisfies the same boundary conditions.

We first consider the stability of a plasma in a straight stellarator, i.e., we ignore the curvature of the system, but we assume that the longitudinal wave number can take on only integer values (i.e., we are considering a cylinder with identified ends). Writing the perturbation in the form $X \sim X(a)e^{im\vartheta - ins}$ (m and n are integers; $m \neq 0$; and X is some arbitrary perturbed quantity), and using the system of average equations (3.1), we easily find a linear small-oscillation equation:

$$(F_0\xi_1')' + G_0\xi_1 = 0,$$

$$F_0 = a^3\nu^2, \quad G_0 = -(m^2-1)\nu^2 a + (a^3\tau^{*\prime})'\nu - \frac{\nu^2 a^3 n^2}{R_0^2} + 2n^2a^3\nu\left(\nu + \frac{2n}{m}\right)(R_0^2 m^2)^{-1}, \tag{5.1}$$

where we have chosen as the variable the radial displacement of an element of the plasma, ξ_1, and where $\nu = \tau - n/m$. In deriving (5.1) we made several simplifying assumptions: We omitted terms describing finite-pressure effects, and for simplicity we ignored terms $\sim \varepsilon^2$ in comparison with unity and terms $\sim n^2a^2/R_0^2$ in comparison with $m^2(R_0 = L/2\pi$, where L is the length of the plasma column).

In the small-oscillation equation found for a plasma in a stellarator in Ref. 30, there are no terms corresponding to the last two terms, which provide the limiting transition as $\tau^* \to 0$ in the equation for a straight current column.[73] The method for studying Eq. (5.1) is quite simple. As was shown in Ref. 74, a necessary and sufficient condition for stability is that the solution of Eq. (5.1) not vanish in the region with $\nu > 0$.

We first consider a shear-free stellarator (i.e., $\tau^{*\prime} = 0$). If the current density falls off monotonically with increasing radius, and if the current is directed in such a way that an increase in the current is accompanied by an increase in the total angle of the rotational transform, the spectrum of MHD waves in such systems is extremely similar to that of the MHD modes characteristic of a tokamak. The most dangerous perturbation is the $m = 1$ perturbation, for which the negative stabilizing term on the right side of (5.1) vanishes. The satisfaction of the condition $\tau = n$ in the vacuum region outside the plasma unavoidably drives the $m = 1$ kink mode. The stability of $m \geqslant 2$ modes depends strongly on the radial profile of the current density and on the boundary conditions. If the current profile is sufficiently sharp, the plasma column may be stable with respect to these modes under the sole condition $\tau(a_p) < 1$, even in the absence of a conducting wall, which has a strong stabilizing effect on the plasma. If the conducting wall coincides with the plasma boundary, surface modes cannot grow, but the satisfaction of the condition $\tau(a_s) = 1$ inside the plasma column in the case that $a_s < a_p$ unavoidably leads to an instability of the so-called internal kink mode with $m = 1$, $n = 1$. The growth rate of this mode is smaller by a factor of about $(a_s/R_0)^2$ than the growth rate for the kink instability of a plasma with a free boundary. Incorporation of the finite toroidal effects in the limit of vanishing low pressure can lead to stabilization of the internal kink mode, as in tokamaks,[75,76] regardless of the value of τ at the magnetic axis. The curvature of the torus has no substantial effect on the kink modes of a plasma with a free boundary.

We have been discussing the stability of a plasma with respect to current-driven kink modes in ideal MHD. However, if there is a singular point with $\nu(a_s) = 0$ in the plasma, but the condition for the stability of the ideal current-driven kink modes is satisfied, it becomes necessary to carry out a further analysis of the stability, taking the finite conductivity into account. Here there is the possibility of the onset of a slower dissipative instability (the so-called tearing mode), which leads to a splitting of the resonant magnetic surfaces and to the formation of magnetic islands. The condition for stability with respect to tearing modes is written as follows[32]:

$$\Delta' < 0,$$

where

$$\Delta' = \frac{a}{\Psi_1}\frac{\partial \Psi_1}{\partial a}\bigg|_{a_s-\varepsilon_0}^{a_s+\varepsilon_0},$$

and $\varepsilon_0 \to 0$ is the difference between the logarithmic derivatives of the radial component of the perturbed magnetic field $\Psi_1 = a\nu\xi_1$, which satisfies the equation

$$\frac{(a\Psi_1')'}{a} + \left[-\frac{m^2}{a^2} - \frac{(a^3\tau^{J\prime})'}{a^3\nu}\right]\Psi_1 = 0, \tag{5.2}$$

from which terms $\sim a^2n^2/R_0^2$, unimportant in this case, have been omitted. The tearing-mode instability and the instability of a plasma with a free boundary are actually two limiting cases of the same instability.[77] While the magnetic islands arise inside the plasma in the former case, in the latter case the islands form in the vacuum region and cause a deformation of the surface of the plasma column. As in the case of ideal MHD modes, a steepening of the current density profile improves the stability with respect to tearing modes. In principle, as was pointed out in Ref. 78, it is possible to choose an optimum current profile, with flattened regions near the most dangerous resonances with $\tau(a_s) = n/m$, which will be stable with respect to all modes. For example, a current density profile with $q(a_p) = 2.2$ ($q = \tau^{-1}$ is the safety factor) at $\tau^* = 0$, a profile which is stable with respect to all modes even in the absence of a conducting wall, was found in Ref. 79. The presence of the stellarator angle of the rotational transform τ^* reduces the limiting values of q at the edge. At $\tau^* = 0.3$, a stable current profile with $q(a_p) = 1.6$ has been found. A current profile of this type may arise automatically, since the onset of the tearing modes in the nonlinear stage leads to a saturation of the growth of the magnetic islands and to an equalization of the current density along the magnetic surfaces over their cross section.

We turn now to the stability of the current-driven kink modes in a stellarator with a large shear. We approximate τ^* by a very simple function, a quadratic polynomial: $\tau^* = \tau^*(0) + \Delta\tau^* a^2/a_p^2$. We see from (5.1) that with a large shear, i.e., under the condition $\Delta\tau^* \gg n^2a_p^2/m^2R_0^2$, the second term in expression (5.1) for G_0 is larger than the two last terms, which can therefore be ignored.

Figure 11 shows stability diagrams for the current-driven kink modes in a stellarator with a large magnetic shear[81] $S^*(a_p) = a\tau^{*\prime}/\tau^*|_{a=a_p} \sim 1.55$.

As before, the most dangerous perturbation is the $m = 1$ kink mode, which is always unstable in a stellarator

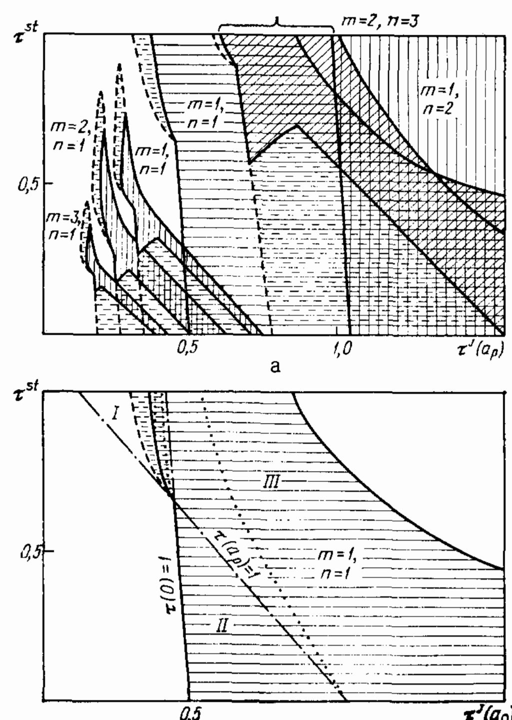

FIG. 11. a: Stability diagram of current-driven kink modes in a stellarator with ohmic heating. Here $\tau^* = \tau^{st}[0.2 + 0.7(a^2/a_p^2)]$, $\tau^J = \tau^J(a_p)[2 - (a^2/a_p^2)]$ (there is a conducting wall at $a_l = 1.5a_p$). The modes with $m = 1$—3, $n = 1$—2, are shown. The regions in which the plasma is unstable according to ideal MHD are hatched with solid straight lines; the regions in which the tearing modes are unstable are shown by hatching with dashed lines. b: The same as in part a, but for the $m = 1$, $n = 1$ mode. In region I, the condition $\nu(a) > 0$ holds at $a > a_s$ (modes of this type are characteristic of only a stellarator with shear). In region II, the condition $\nu(a) > 0$ holds at $a < a_s$ (this case corresponds to an internal kink mode in a tokamak). In region III, there are no singular points inside the plasma column. The dotted line shows the stabiltiy boundaries at $a_l = a_p$.

with shear if there is a region with $\nu(a<a_s)>0$ inside the plasma column; here $\nu(a)$ is a decreasing function of the minor radius (Fig. 11). In contrast with the case discussed earlier, the growth rate for the internal kink mode turns out to be equal in order of magnitude to the growth rate for the model with a free boundary because of the presence of a positive shear ($S^*=a\tau^{*\prime}/\tau^*\geqslant 0$) due to the stellarator magnetic field, τ^*. For the kink modes with $m>1$, the presence of a stellarator angle of rotational transform τ^* which is nonuniform along the radius also results in an increase in the growth rates. In particular, as can be seen from Fig. 11a, an increase in $\tau^{\rm st}$ at a constant $\tau(0)$ has the consequence that the plasma is unstable in ideal MHD, beginning at some $\tau^{\rm st}$. The internal kink modes in a stellarator with shear are thus more unsable than in a tokamak.

Under the condition $\tau^*(a_{\rm p})>\tau^J(a_{\rm p})$, an instability of another type (which we will call an "external mode" can occur. For this instability we have $\nu(a>a_s)>0$ (with $m=1, n=1$, it is region 1 in Fig. 11b). The numerical calculations of Refs. 80 and 81 show that in this case the stability conditions depend strongly on the distribution of the plasma parameters over the cross section of the column and on the boundary conditions. A sufficient condition for the stability of the external modes according to ideal MHD was derived in Ref. 81. It turns out that the stability condition is satisfied best when the resonant surface lies near the boundary of the plasma column. If the resonant magnetic surface lies sufficiently close to the boundary of the plasma column, the $m=1$ tearing mode may also be stable (Fig. 11).

In addition to the situations described above, there could be a situation in which an instability could occur if there were no singular point in the plasma. At small values of τ^* such modes are analogous to kink modes with a free boundary of a staight current column. In a stellarator with a large shear, however, the ideal MHD modes, which have no singular points, may be unstable even if there is a conducting wall at the boundary of the plasma column[81] (Fig. 11b).

It follows from the discussion above that a stellarator angle of the rotational transform which is nonuniform over the cross section of the plasma column plays a double role. On the one hand, the growth rates for the internal modes increase, and several types of unstable waves not seen in a tokamak appear. On the other hand, with increasing τ^* there is a contracting of the current inteval in which the plasma is unstable. When τ^* exceeds a certain value (Fig. 11), a given mode becomes stabilized.

There is yet another purely stellarator effect, which occurs if the ohmic-heating current is directed in such a way that an increase in this current is accompanied by a decrease in the total angle of the rotational transform. If in this case there is inside the plasma column a magnetic surface on which the condition $\tau(a_s)=0$ holds, this is a necessary and sufficient condition for the onset of the $m=1$, $n=0$ instability, as can be shown in a straightforward way with the help of (5.1). This instability was apparently observed in experiments in the Uragan-2 device at Khar'kov.[82]

5.2. Nonlinear development of tearing modes

We have been discussing the stability of tearing modes in the linear approximation. In a plasma with a finite conductivity, the onset of these instabilities leads to a splitting of the magnetic surfaces and to a growth of the magetic islands which form around a new magnetic axes. The presence of a well-developed island structure in a system leads in turn to an increase in the transport coefficients because of the rapid equalization of the plasma parameters over the cross section of an island, which occurs along the magnetic surfaces which form around the new axis.

There are several circumstances which facilitate a qualitative analysis (and, frequently, a quantitative analysis) of the nonlinear dynamics of current-driven modes in a stellarator. First, as in the equilibrium case, the use of the average equations[38] makes it possible to reduce the problem to an axisymmetric problem. Second, since we are dealing with a plasma of vanishingly low pressure, which has no effect on the dynamics of the instability, the only difference between tokamaks and a stellarator with a current in this approximation is that the poloidal flux Ψ is now the sum of Ψ^J, the magnetic flux produced by the longitudinal current, and Ψ^*, the flux of the "effective" magnetic field[7)] (Refs. 40, 83, and 84). For this reason, many of the results derived previously for the case with $\tau^*=0$ also give a qualitatively correct description of the processes which occur in a stellarator.

Rutherford[85] has shown, in a study of an approximate quasilinear system of equations, that when the width of a magnetic island, W, exceeds a scale thickness for the tearing mode, $a_s/T_0^{1/3}(T_0=t_\sigma/t_{\rm A}$ is a measure of the ratio of the skin time $t_\sigma=4\pi a_{\rm p}^2\sigma/c^2$ to the Alfvén time $t_{\rm A}=a_{\rm p}/v_{\rm A}$, and $v_{\rm A}=B_0/[4\pi\rho]^{1/2}$, i.e., under the condition $W>a_s/T_0^{1/3}$, nonlinear terms retard the growth of the islands. A transition occurs from a linear, exponential development of the instability, with growth rate $\gamma\sim t_\sigma^{-3/5}t_{\rm A}^{-2/5}$, to a power-law development, and the thickness of an island increases linearly over time, with the slower, resistive, scale time t_σ.

Next we find an equation which can be used to describe the dynamics of the growth of the islands and to estimate their thickness.[86] As was shown by the calculations in Ref. 86, the limiting dimensions of a magnetic island depend strongly on the particular model chosen for the conductivity. With $\sigma=$ const, either saturation does not occur at all, or the dimensions of an island are comparable to those of the plasma column. In the case of steady-state model with $J/\sigma=$ const, there is a substantial decrease in the limiting thickness of a magnetic island. For conductivity profiles which decay with a scale length $\sim a_{\rm p}$, the steady-state equation for the limiting thickness of a magnetic island is written in the form[87]

$$\Delta'(W_{\max})=0,\ \Delta'(W)=\frac{1}{\Psi_1(a_s)}\left(\frac{\partial\Psi_1}{\partial a}\Big|_{a^+}-\frac{\partial\Psi_1}{\partial a}\Big|_{a^-}\right),$$
$$a^{\pm}=a_s\pm\frac{W}{2}.$$

The quantity $W_{\max}$ falls off rapidly with increasing mode index m. The limiting dimensions of the magnetic islands depend on the shape of the current, falling off for peaked profiles. Finally, we note that finite-Larmor-radius effects have the consequence that a quasisteady perturbation acquires a real frequency,[88] while the width of an island is, as before, determined by the profiles of the current density and the conductivity. Oscillations of this sort have been observed experimentally in tokamaks.[89]

The coments above also apply to stellarators. Analysis of condition (5.3) for the particular case of a stellarator shows[22] that a stellarator angle of rotational transform which is constant over radius improves the stability and reduces the dimensions of the magnetic islands. In experiments on the W-VIIA stellarator it was demonstrated that under the condition $\tau^*(a) = \text{const} > 0.14$ there is a substantial improvement in the stability of the plasma column. The disruptive instability is not seen even at $\tau = 0.8$ at the edge of the plasma column. A similar conclusion, that the stellarator field plays a stabilizing role, was reached from the results of experiments on the JIPP-T2 (Ref. 90), where it was shown that the disruptive instability does not occur if the stellarator fields are sufficiently strong.

The disruptive instability in a tokamak is a complicated nonlinear phenomenon which is not yet completely understood. As a rule, the disruption is related to the presence of large islands with $m = 2$, spanning a significant fraction of the discharge. It is also assumed[91–93] that a disruption is caused by the growth of the $m = 2$, $n = 1$ mode and the nonlinear interaction of this mode with other modes, in particular, the $m = 3$, $n = 2$ mode. When magnetic islands overlap, an ergodic region appears between the surfaces with $\tau = 1/2$ and $2/3$, where the temperature is equalized.

If we assume a similar mechanism for the onset of the disruptive instability, the role played by the stellarator angle of rotational transform τ^* becomes clear. With increasing τ^*, the magnetic surface with $\tau = 1/2$ either moves out of the region inside the conducting wall or, in contrast, goes to the edge of the plasma column, where the current-density gradient is small, and the $m = 2$, $n = 1$ mode turns out to be stabilized.

5.3. Sawtooth oscillations in a stellarator

With increasing longitudinal current in a tokamak, if the angle of rotational transform in the cental region is greater than unity, one observes an interesting physical effect: so-called periodic sawtooth oscillations of the x-ray emission. At the center of the plasma column, the intensity of the x-ray emission initially rises slowly and then drops abruptly (the periodicity of the process is usually a few milliseconds). Outside the surface wth $\tau(a_p) = 1$ the phase of the oscillation changes by 2π; i.e., the intensity of the x-ray emission initially rises rapidly and then decays slowly.[94]

Kandomtsev[95] has carried out a qualitative analysis which explains the sawtooth oscillations. The onset of an instability of the $m = 1$, $n = 1$ mode causes a displacement of the central region and thus the excitation of a helical current near the surface $a = a_s$. As a result of the decay of this current due to the finite conductivity, the lines of force inside a_s begin to reclose with lines of force outside this surface; this reclosure proceeds until the entire region inside this surface is reclosed with the exterior, and the value of τ becomes less than 1 everywhere. Estimates show that an effective mixing of the plasma parameters occurs in the region $a < \sqrt{2}a_s$. Ohmic heating then renews the decrease in the resistance in the central part of the plasma column; the current rises again, and the process is accordingly periodic. Subsequent numerical calculations[96,97] have confirmed the validity of this model. Discrepancies arise only in calculations of the total duration of the process. Since the model is not self-consistent, the value of a_s in a numerical simulation is usually taken from experiment.

Experiments on current heating of plasmas carried out in various stellarators have also revealed sawtooth oscillations.[26,98] As was recently demonstrated by Wakatani *et al.*[99] (who carried out calculations for the parameters of the Heliotron-E—a system with a large magnetic shear), the internal disruption in a stellarator is a process which is similar in many ways to the process which occurs in a tokamak. The only distinction stems from the specific chracteristics of the $m = 1$, $n = 1$ internal mode, whose growth rate in a stellarator with shear is much larger than in a tokamak. As the calculations of Ref. 99 showed, however, nonlinear effects cause a saturation of this mode and thus a reclosure. Unfortunately, there has not yet been a study of how the stellarator angle of rotational transfer affects the quantitative characteristics of the sawtooth oscillations such as their duration and amplitude.

6. STABILITY OF A FINITE-β PLASMA

We have been discussing the stability of current-driven kink modes. Since the stellarator retains its confining properties in the absence of an ohmic current (in fact, these properties generally improve), however, we are presently seeing a trend toward operation without a current. We wish to stress that it is in operation without a current where we can see the advantages of the stellarator as a fundamentally steady-state system. Under these conditions the primary danger is posed by instabilities which stem from the finite pressure and the curvature of the lines of force. These instabilities can impose limits on the plasma pressure in a stellarator. In this section we examine the basic factors which lead to limitations on the pressure and methods for raising the limiting pressure.

As a rule, a distinction is made between modes which are localized near rational magnetic surfaces and at large-scale modes in an analysis of the plasma stability in a toroidal confinement system. The localized modes can be studied analytically, through the derivation of stability conditions expressed in terms of the characteristics of the equilibrium state at the given magnetic surface. The nonlocal modes depend strongly on the distribution of parameters over the entire plasma volume, and they are usually studied by numerical methods.

6.1. Localized modes in an ideally conducting plasma

In this subsection we consider perturbations which are of small scale along the direction perpendicular to the equilibrium magnetic field, i.e., which satisfy the inequality

$$[\mathbf{B}_0\nabla \ln X_1] \gg [\mathbf{B}_0\nabla \ln X_0], \tag{6.1}$$

and which are nearly constant along $\mathbf{B}_0$, i.e., for which we have

$$|[\mathbf{B}_0\nabla \ln X_1]| \gg (\mathbf{B}_0\nabla \ln X_1), \tag{6.2}$$

where the subscripts 1 and 0 specify the perturbed and equilibrium values, respectively.

The condition for the stability of a straight, cylindrical plasma column with respect to flute perturbations (the Suydam criterion) is written as[100]

$$\frac{1}{4}\left(\frac{\tau'}{\tau}\right)^2+\frac{8\pi p'}{aB_0^2}>0. \tag{6.3}$$

The condition for the stability of modes with $n\gg 1$ in a straight stellarator in the case of a single kink harmonic (i.e., if toroidal effects are ignored) is[30]

$$\frac{1}{4}\left(\frac{\tau'}{\tau}\right)^2+\frac{8\pi p'}{\tau^2 aB_0^2}\left[\tau^2+\frac{N}{2n_0a^3}(\tau^* a^4)'\right]>0. \tag{6.4}$$

The appearance of an additional destabilizing term (which, under the condition $N/n_0\gg\tau$, is considerably larger than the first) stems from the circumstance that, as we have already mentioned, a straight stellarator has a maximum "average B." Even in a stellarator with a large magnetic shear, the limiting pressure at which the plasma is still stable, found from (6.4), is extremely low (for the L-2 stellarator of the Institute of General Physics, Moscow, for example, with $\tau^* = 0.2 + 0.5a^2/a_p^2$ and $N/n_0 = 7$, we have $\beta_{\max}\sim 1\%$).

Study of plasmas in closed magnetic confinement systems is complicated by the circumstance that the equilibrium quantities generally depend on ϑ and s [in the case of a straight cylinder which is axisymmetric, perturbations of the type $X\sim X(a)\exp(im\vartheta - ins)$ with different values of m and n are independent]. If the parameter characterizing the coupling of harmonics is small, however, the problem can be solved by a method of successive approximations, limited to a finite number of harmonics. For a tokamak a corresponding parameter is $\gamma_\vartheta = \beta/\tau^2\delta$. Analytic calculations on the stability of a plasma confined in a tokamak with circular magnetic surfaces with respect to perturbations with $m\gg 1$ were carried out for the case $\beta\ll\tau^2\delta$ in Ref. 101, where the fundamental harmonic with $m\gg 1$ was taken into account along with the two adjacent harmonics with $m\pm 1$. The following condition was found

$$\frac{1}{4}\left(\frac{\tau'}{\tau}\right)^2+\frac{8\pi p'}{B_0^2 a\tau^2}(\tau^2-1)>0. \tag{6.5}$$

In the same paper Shafranov and Yurchenko showed that condition (6.5) can also be derived from the so-called Mercier condition[102]: a condition for the stability of a toroidal plasma column of arbitrary cross section with respect to small-scale flute perturbations. It follows from (6.5) that a plasma is stable even if the shear is zero ($\tau' = 0$), provided that the sole condition $\tau < 1$ holds. In this case the ballooning effect which arises because a perturbation on the outer side of the torus is larger than on the inner side, and which makes a contribution to the stability condition which is quadratic in β, is completely cancelled by the deepening of the magnetic well due to the displacement of the magnetic surfaces caused by the fields produced by the diamagnetic currents.

In principle, the condition for the stability of a plasma in a stellarator could be derived by making use of a geometry-independent Mercier condition; this approach would require examining the three-dimensional equilibrium. There is, however, a circumstance which makes it possible to simplify the analysis. It has been shown[103,104] that for a system with helical fields there is, in addition to a coupling of harmonics along the minor azimuthal direction ϑ, a coupling of harmonics along s, with a parameter $\gamma_s = n_0\beta/N\delta$. Under the condition $N\gg n_0\tau$ we have $\gamma_\vartheta\gg\gamma_s$, so that in the case $\beta\lesssim\tau^2\delta$, of practical importance, the ballooning effect at local corrugations of the helical magnetic field can be ignored (because of the repeated alternation of regions with favorable and unfavorable curvature, whose net contribution to the stabiltiy criterion turns out to be negligibly small). Consequently, to derive a condition for the stability of a plasma in a stellarator one can work from the system of average equations and study it by a slightly modified version of the method developed previously for a tokamak. As a result, we find the following stability condition[38] for the case $\beta\ll\tau^2\delta$:

$$R_{\mathrm{M}}=\frac{1}{4}\tau'^2+\frac{8\pi p'}{B_0^2 a}Q_1>0, \tag{6.6}$$

$$Q_1=-\frac{B_s^{*\prime}}{B_0}\frac{R_0^2}{2a}-\xi R_0\frac{(a^3\tau^{*\prime})'}{2a^3\tau}+\frac{R_0}{2a\tau}\left[\frac{(\xi^* a^2\tau^*)'}{a}\right]'+\tau^2-\frac{2\tau-\tau^*}{2\tau}. \tag{6.7}$$

Let us determine the meaning of each of the terms in (6.7). The first destabilizing term occurs because a stellarator has a maximum "average B" if we ignore toroidal effects. The second term is negative if $\tau^{*\prime} > 0$. Since ξ increases with increasing pressure, the second term is also an increasing function of the pressure, causing an improvement in the stability conditions. The mechanism for this sort of self-stabilization can be explained as follows. The displacement of the magnetic surfaces caused by the fields produced by the diamagnetic currents occurs in the direction of increasing major radius. In systems with an angle of rotational transform which increases with increasing radius, a line of force passes more rapidly through the region with unfavorble curvature on the outer side of the torus and is delayed on the inner side. Because of the condition $\xi/a_p < 1$, however, the sum of the first and second terms can be negative only if thre is a sufficiently large magnetic corrugation, i.e., only if $N\tau^*\xi\ll 1$ (a more detailed estimate will be given below).

The third term stems from the structure of the magnetic surfaces in vacuum and may be either positive or negative. In particular, by displacing the magnetic axis of the system (by means of a vertical magnetic field, for example) or by altering the pattern in which the current-carrying helical conductors are wound around the surface of the torus, one can improve the stability conditions. The last two terms are small in comparison with the first under the condition $N\gg n_0\tau$ and have essentially no effect on the stability of the plasma in the system. They are retained solely to allow the correct limit in the Shafranov-Yurchenko criterion[101] as $\tau^*\to 0$. This condition, incidentally, does not impose a restriction on the limiting pressure under the single condition $\tau < 1$. For a certain choice of stellarator parameters, the stability condition in

(6.6) also imposes no restriction on the limiting permissible plasma pressure.

We first consider the case of a plasma without a current ($\tau \equiv \tau^*$). It can be seen from (6.7) that in a system without shear ($\tau^{*\prime} = 0$) there is self-stabilization in this approximation, and the imposition of a uniform vertical external field $B_{0\perp}$ does not give rise to a magnetic well [since in this case we have $\xi^{*\prime} = (B_{0\perp} R_0/B_0 \tau^*)' = 0$)]. A well could be produced only by a term related to a nonuniform displacement $\xi^{*\prime}$ of the vacuum magnetic surfaces. In analyzing the stability of such systems, it is necessary to take the fine stucture of the field into account accurately. As an example illustrating these arguments, we consider a system of helical windings which are wound on the surface of a torus with a minor radius r_0 in accordance with

$$\varphi = \frac{Ns}{n_0} - \hat{C}_1 \frac{r_0}{R_0} \sin \varphi. \tag{6.8}$$

If we then use Eq. (4.8) to eliminate ξ_p from (6.7), condition (6.6) for a plasma with a free boundary can be rewritten as[105]

$$\frac{4R_M}{\tau'^2} = 1 - M_p \beta_0 + N_p \beta_0^2 > 0, \tag{6.9}$$

where the coefficients M_p and N_p depend on the parameters of the magnetic field and the pressure profile, $f(a) = p(a)/p(0)$, $\beta_0 = 8\pi p(0)/B_0^2$:

$$M_p = -\frac{2f'a}{(a\tau^{*\prime})^2} \Big\{ \frac{N}{n_0} (\tau^* a^4)' a^{-3} - 12 (1 + \hat{C}_1) - \frac{R_0^3 B_{0\perp}}{B_0 a^3} \frac{(a^3 \tau^{*\prime})'}{\tau^{*2}} + 2\tau^2 + 3 \Big[(a^3 \tau^{*\prime})' (a^3 \tau^{*2})^{-1} \int_0^a a\tau^* \, da - \frac{a\tau^{*\prime}}{\tau^*} \Big] \Big\},$$

$$N_p = \frac{f'a}{(a\tau^{*\prime})^2} \Big[\frac{(a^3 \tau^{*\prime})'}{a^2 \tau^{*3}} R_0^2 \frac{\tau^*}{a^2} \int_0^a a \, da \int_a^\infty \frac{f'}{\tau^*} da \Big]$$

$$a = a_p. \tag{6.10}$$

For systems with a large shear ($S \sim 1$) the term proportional to $\hat{C}_\perp$ is slightly more complicated. For such systems, however, this term is not very important, and we will not bother to refine it.

The roots of the equation $R_M(\beta_0) = 0$, if they exist, determine two values $\beta_{st}^{(1)}$ and $\beta_{st}^{(2)}$, between which the plasma is unstable. At $\beta_0 < \beta_{st}^{(1)}$ (the first zone) and $\beta_0 > \beta_{st}^{(2)}$ (the second zone) the plasma is stable. If, on the other hand, the parameters of the system are such that the condition $M_p \leqslant 2N_p^{1/2}$ holds, the first and second stability zones merge, and the plasma turns out to be stable at all values of β (Fig. 12).

In a system without shear, the stability condition does not impose a restriction on the limiting permissible pressure if

$$\frac{4N\tau^*}{n_0} + 2\tau^{*2} \leqslant 12 (1 + \hat{C}_1). \tag{6.11}$$

It can be seen from (6.11) that the stability depends extremely strongly on the magnitude and sign of the amplitudes of the satellite harmonics which arise upon a small deviation of the winding pattern from a helix $\varphi = Ns/n_0$.

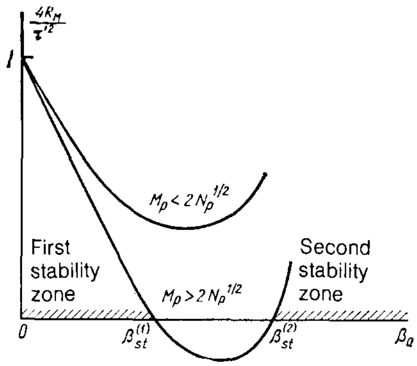

FIG. 12. See the text proper for an explanation.

For systems with a large shear, $\Delta\tau^* \gg \tau^*(0)$ with $p = p(0)(1 - a^2/a_p^2)$ and $\tau^* = \Delta\tau^* a^2/a_p^2$, the condition under which there is no limitation on the pressure becomes[8)]

$$\frac{6N\Delta\tau^*}{n_0} + 2\Delta\tau^{*2} \leqslant 4 \frac{R_0}{a_p} \Big(1 + \frac{2\varepsilon_\perp^{-1} R_0}{a_p \Delta\tau^*}\Big) + 12 (1 + \hat{C}_1), \tag{6.12}$$

where $\varepsilon_\perp = B_{0\perp}/B_0$.

It should be recalled, however, that condition (6.6) was derived in the linear approximation in ξ and is valid, generally speaking, only if $\beta \ll \tau^2 \delta$. For a more rigorous solution of the question of the limiting pressure, we need to take into account the terms of the next higher order in ξ. This analysis (including terms up to ξ'^3 inclusively) was carried out in Ref. 65, where it was shown that under the conditions $\tau^J = 0$, $\tau^* + \tau^*(0) + \Delta\tau^* a^2/a_p^2$, $\Delta\tau^* \gg \tau^*(0)$ the nonlinear terms lead to an additional stabilization, in particular, at the edge of the plasma column, where the most destabilizing term is $\sim B_s^{*\prime}$. Under the condition $\Delta\tau^* = 0$, the nonlinear terms also lead to a stabilization, in agreement with the results found by Mikhaĭlovskiĭ and Shafranov,[107] who observed a self-stabilization in stellarators with a zero shear in the cubic approximation in β. It can therefore be assumed that the nonlinear terms have essentially no effect on conditions (6.11)–(6.12).

We recall that the Mercier condition is not a sufficient condition for the stability of modes with $n \gg 1$ in all cases. The stability condition for a tokamak, (6.5) is known[108,109] (see also the review in Ref. 10) to be only a necessary condition, since the so-called ballooning perturbations are more dangerous. The stability for ballooning perturbations includes a destabilizing term $\sim S\beta^2$.

For a stellarator the condition for the stability of ballooning modes is[58,111]

$$\frac{R_M a^2}{\tau^2} + \frac{S^2}{4} + \frac{S}{2} \Big(\frac{8\pi p' a R_0}{B_0^2 \tau^2} \Big)^2 + \frac{6\pi a p'}{R_0^{-1} B_0^2 \tau^2} \Big(1 - \frac{S}{|S|} \Big) \exp\Big(-\frac{1}{|S|} \Big) > 0. \tag{6.13}$$

The last term was derived under the assumption $|S| \ll 1$ and is generally small in comparison with R_M. For a current-free stellarator with a value of β which is not very large, the quantity S is usually positive, in contrast with the situation in tokamaks. At large values of β, this quantity can be made positive by means of external fields, as was shown in Ref. 67, among other places. Condition (6.13) is less stringent than (6.6), which should accordingly be used to determine the limiting pressure.

We have been discussing for the most part the stability of a plasma at a zero ohmic-heating current. A longitudinal current leads to a degradation of the plasma stability, because with decreasing value of S—a measure of the shear—there is a simultaneous decrease in the depth of the magnetic well.[39] The situation is illustrated in Figs. 13 snd 14, which show how the magnitude of the ohmic-heating current affects the stability of a plasma with a fixed pressure.

6.2. Condition for the stability of dissipative modes

When a plasma is stable in ideal hydrodynamics, the incorporation of a small but nonzero resistance can give rise to new types of unstable oscillations.[112,113] Resistive flute modes are not stablized by shear, but they may be stable if the average-magnetic-well effect outweights the ballooning effect. The condition for the stability of modes of this sort in a stellarator is[39,40] ($\beta \to \tau^2\delta$)

$$\frac{ap'}{B_0^2}\left(Q_1-\frac{R_0\tau'}{\tau}\xi'\right)>0, \qquad (6.14)$$

where Q_1 is given by (6.7). Since the condition $\xi'<0$ generally holds, the second term plays a destabilizing role in a plasma without a current. Analysis[39] of (6.14) shows that the self-stabilization which stems from the shear for the dissipative flute instabilities can occur only in the central part of the plasma column. There is the hope, however, that the growth of resistive modes will not lead to a limitation on the pressure, because of their small-scale nature, although transport processes may be affected.

6.3. Limiting pressure in a stellarator

We have been discussing the stability of local modes. In principle, the sufficient condition for stability with respect to arbitrary perturbations according to ideal MHD may hold in a stellarator without an ohmic-heating current (for a tokamak, a stability condition of this sort would be of no interest, since it could be satisfied only if there were a current-carrying conductor at the axis of the system). The sufficient condition turns out to be more stringent than condition (6.14) for resistive flute modes, and for a stellarator without shear[50] it reduces to the requirement that a vacuum magnetic well be produced throughout the region occupied by the plasma (in this case the limiting pressure turns out to be proportional to the depth of this well). Since the self-stabilitzation associated with shear can produce a magnetic well only in the central part of the plasma column, and the possibility of producing a vacuum magnetic well near the separatrix, at $S^*\sim 1$, appears to us to be extremely problematical, it will generally not be possible to satisfy the sufficient condition for stability in a stellarator with a large shear throughout the region occupied by the plasma. This condition turns out to be extremely severe. Numerical calculations[56,114] demonstrate that large-scale instabilities are stopped in systems with shear at a rather low value of $N\tau^*$ and at $\tau^*(a_p)<1$.

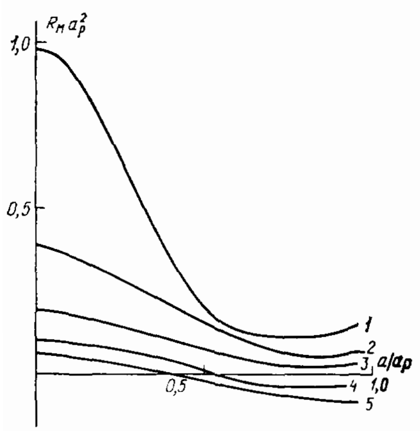

FIG. 13. Effect of the ohmic-heating current on the stability of the plasma at a fixed pressure $\beta=\beta(0)\,[1-(a^2/a_n^2],\beta(0)=0.005$. 1—$\tau^J=0$; 2—0.1; 3—0.2; 4—0.3; 5—0.4. $\tau^*=0.2+0.7\,(a^2/a_p^2)$, $\delta=0.115$.

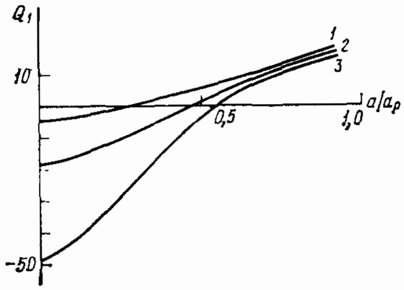

FIG. 14. Effect of the ohmic-heating current on the depth of the magnetic well at a fixed pressure $\beta=\beta(0)\,[1-a^2/_p^2)],\beta(0)=0.005$. 3—$\tau^J=0$; 2—0.1; 1—0.4. $\tau^*=0.2+0.7(a^2/a_p^2)$, $\delta=0.115$.

It may appear that the restrictions imposed on the limiting pressure jointly by the conditions for stability of ideal MHD modes and the conditions for equilibrium are extremely stringent. Actually, since we have $\tau^*\sim N\sim\delta^{-1}$, systems with a small toroidal ratio δ are favorable from the equlibrium standpoint [see (4.9)], while from the standpoint of stability with respect to local modes [see (6.11) and (6.12)], in contrast, we would prefer systems with values of $N\tau^*$ which are not very large, i.e., "steep" tori.

Fortunately, a compromise solution [$\delta^{-1}=6-10, N/n_0=5$–$7$, $\tau^*(a_p)\lesssim 1$] turns out to be completely satisfactory and raises the hope that it will be possible to achieve a stable plasma with $\beta\gtrsim 10\%$ in a stellarator without a current. This value could evidently be increased slightly by means of a transverse external field, which would partially cancel the displacement of the magnetic surfaces due to the finite β, keeping the displacement of the plasma to a level sufficient to produce in the system the magnetic well which satisfies the stability condition.

7. CONCLUSION

The appearance of new methods for studying the plasmas confined in three-dimensional magnetic fields and the refinement of the old methods have resulted in a rapid development of the theory of the MHD equilibrium and stability of plasmas in stellarators. Many of the new results have been obtained through the use of an averaging method. Although the derivation of results from the average equations is based on the method developed earlier for tokamaks, the physical processes which are described by the similar methods frequently turn out to be different. Among the purely "stellarator" effects are the self-stabilization of the plasma due to the shear and the effects which stem from the presence of a magnetic hill. The spectrum of current-drive kink modes in a stellarator has several distinctive features. The theory of current-driven kink modes in a stellarator as it exists today gives a qualitatively correct description of the basic manifestations of the MHD activity of the plasma column. We see no fundamental difficulty along the road to a quantitative description of the processes which occur in the plasma.

We would like to conclude this review with a list of the basic theoretical problems which are still to be solved. One is to study the equilibrium and stability of a plasma in the case in which the plasma boundary is close to or coincides with the vacuum separatrix. This is an extremely important problem, particularly because calculations show that the conditions for the stability of a plasma with a free boundary are quite different from those in the case in which there is a conducting wall.

We have repeatedly cited the results of reseach on the stability of small-scale modes. The situation with regard to research on large-scale modes is not as satisfactory; graphic analytic stability conditions cannot be derived for large-scale modes. Furthermore, the number of adjustable parameters characterizing a stellarator is considerably larger than the corresponding number for a tokamak (this circumstance can, generally speaking, ultimately lead to an optimum choice of plasma parameters). Consequently, an explanation of the entire set of physcial factors determining the stability of large-sclae oscillations will require a really major effort.

Among the foremost theoretical problems are those of studying the nonlinear stage of instabiltities and, in particular, resistive ballooning modes, for which the stability condition may not hold.

We believe that it is extremely important to seek magnetic configurations in which the parameters of the plasma column can be improved (e.g., the limiting value of β can be increased). Here we have looked at systems in which the cross sections of the average magnetic surfaces are approximately circular. It seems quite likely that we should also study magnet systems for which the cross sections of the average magnetic surfaces are not circular (ellipses prolate in the vertical direction, D-shaped surfaces, etc.). Configurations of this sort can be produced by using helical current-carrying conductors on a torid of the appropriate shape and by using a transverse magnetic field which contains harmonics higher than the dipole harmonic.

In summary, it can be asserted that the number of fundamentally unresolved problems remaining in the theory of plasma confinement in stellarators is perhaps no greater than the number of such problems remaining in the theory for the equilibrium and stability of tokamaks. The stellarator work lags behind the tokamak work somewhat only in the specific calculations carried out for various particular cases. There is the hope, however, that even this gap will be bridged in the very near future.

We wish to thank O. P. Pogutse and É. I. Yurchenko for much useful advice and for a rewarding discussion.

APPENDIX. SIMPLIFIED AVERAGE EQUATIONS

The basic small parameter used below is $\Delta_p = |\mathbf{B}_\perp|/B_0$: the ratio of the transverse components of the average magnetic field to the longitudinal magnetic field. In the system of equations below we retain terms $\sim\Delta_p^2$ inclusively and also small terms of second order in the toroidal effects.

We assume that the plasma pressure is quite high, i.e., $\beta\sim\Delta_p$, in which case we have $J_\perp \sim J_\parallel \sim \Delta_p$. We furthermore assume B^*/B_0, $\delta\sim\Delta_p$.

The perturbations of most interest for stability theory are known to be those which vary only slightly along lines of force of the average magnetic field. For such perturbations we have $\partial/\partial z\sim\Delta_p\nabla_\perp$, where $\nabla_\perp = \nabla - \mathrm{e}_s\,\partial/\partial sR$. The maximum growth rate for these instabilities is $\sim\delta/t_A\sim\Delta_p/t_A$, i.e., $t_A\partial/dt\sim\Delta_p$. Finally, we restrict the discussion to the case of a plasma with a high conductivity, i.e., $t_A/t_\sigma \ll \Delta_p$.

The procedure for deriving simplified equations from the system of average MHD equations is esentially the same as the derivation of the corresponding equations for tokamaks.[43,44] We will accordingly omit the straightforward calculations and write the system of simplified equations in the following form[40]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{div}\,\hat{\rho}\nabla_\perp U$$

$$= \mathrm{div}\left\{\frac{1}{4\pi}\left([\boldsymbol{\nabla}_\perp\Psi,\ \mathbf{e}_s R^{-1}] + \boldsymbol{\nabla}_\perp\,\frac{\partial\varkappa}{\partial RB_{\mathrm{T}}s}\right)\frac{1}{RB_{\mathrm{T}}}L(\Psi-\Psi^*)\right.$$

$$\left.+\frac{1}{4\pi}R^{-2}\boldsymbol{\nabla}_\perp\frac{\partial\Psi}{\partial s}+\frac{1}{\hat{B}^2}[\boldsymbol{\nabla}p,\ \mathbf{B}_{\mathrm{T}}]-\frac{1}{2}[\hat{\rho}\boldsymbol{\nabla}v^2,\ \mathbf{B}_{\mathrm{T}}]\right\},$$

$$\rho\,\frac{\mathrm{d}v_s}{\mathrm{d}t} = -\frac{1}{R}\,\frac{\partial}{\partial s}\,p+\frac{1}{4\pi R}(\mathbf{B}_{\mathrm{T}}[\boldsymbol{\nabla}_\perp\hat{K},\ \boldsymbol{\nabla}_\perp\Psi])$$

$$-\frac{1}{4\pi R^3}\left(\boldsymbol{\nabla}_\perp\frac{\partial\Psi^J}{\partial s},\ \boldsymbol{\nabla}_\perp\Psi\right),$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -\frac{\partial\Phi c}{\partial s}+cE_sR+\frac{L(\Psi-\Psi^*)c^2}{4\pi\sigma_\parallel},$$

$$L(U+\Phi c)+\hat{K}LU+\boldsymbol{\nabla}_\perp U\cdot\boldsymbol{\nabla}_\perp\hat{K}=0,$$

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}=0,\quad \hat{\rho}=\frac{\rho}{\hat{B}^2},\quad \hat{B}^2=B_{\mathrm{T}}(B_{\mathrm{T}}+B_s^*),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p\,(\hat{B}^2)^{-\gamma_0}=0,$$

$$\hat{K}=-\frac{\Delta\varkappa}{B_{\mathrm{T}}^2},$$

$$\mathrm{div}\,[-4\pi\boldsymbol{\nabla}_\perp p-(\nabla_\perp\hat{K})\,B_{\mathrm{T}}^2+R^{-2}\Psi^*\boldsymbol{\nabla}_\perp L\Psi^J$$

$$-(R^{-2}\boldsymbol{\nabla}_\perp\Psi^J)\,L\Psi^J]=0. \qquad \mathrm{(A.1)}$$

The magnetic and electric fields, the current density, and the transverse components of the velocity are written in terms of these variables as follows:

$$\mathbf{B}=\mathbf{B}_T+\mathbf{B}_s^*+\hat{K}\mathbf{B}_T+\frac{\nabla_\perp \partial\varkappa}{\partial R B_{Ts}}+\left[\nabla\Psi,\ \frac{\mathbf{e}_s}{R}\right],$$

$$\mathbf{B}_\perp=\operatorname{rot}\mathbf{A},\quad \mathbf{A}=\frac{\Psi\mathbf{e}_s}{R}+[\nabla\varkappa,\ \mathbf{B}_T]B_T^{-2},$$

$$\mathrm{E}=-\frac{1}{c}\frac{\partial A}{\partial t}-\nabla\Phi,$$

$$\frac{4\pi}{c}\mathbf{J}=-\mathbf{e}_s R^{-1}L(\Psi-\Psi^*)+[\nabla_\perp\hat{K},\ B_T]+R^{-2}\nabla_\perp\frac{\partial\Psi}{\partial s},$$

$$\mathbf{v}=\frac{1}{\hat{B}^2}[\nabla_\perp U,\ \mathbf{B}_T],\quad \Psi^J=\Psi-\Psi^*.$$

We thus find a system of eight equations for the eight functions $\hat{\rho}, p, U, \Psi, \Phi, \hat{K}, \varkappa$, and v_s. We need to note here that v_s does not appear in the other equations of this system, and in those cases in which a detailed study of the longitudinal motion is not necessary this equation can be omitted. We also note that in all the equations of the system (A.1) except the last the function $\hat{K}$ appears in such a manner that it need be known only in lowest order, and we can usually set $\hat{K}=-4\pi p/B_0^2$.

The system (A.1) simplifies substantially for the case of a low-pressure plasma. In this case we have $U=-\Phi c$. Finally, if, for simplicity, we set $\rho=$ const and ignore toroidal effects, we find the system of equations

$$\left.\begin{aligned}
&\frac{d}{dt}\Delta U=v_A^2\left[R_0^{-2}\frac{\partial I}{\partial s}+[\nabla I,\ \nabla\Psi]\frac{\mathbf{e}_s}{B_0R_0^2}\right.\\
&\qquad\left.-4\pi[\nabla p,\ \mathbf{e}_sB_0^{-2}]\nabla B_s^*\right],\\
&\frac{d\Psi}{dt}=+\frac{\partial U}{\partial s}+cE_sR+\frac{c^2 I}{4\pi\sigma_\parallel},\\
&\frac{dp}{dt}=0,\quad I=\Delta(\Psi-\Psi^*);
\end{aligned}\right\}\tag{A.2}$$

Here the function I describes the profile of the longitudinal current. In the limit $B^*\to 0$, $\Psi^*\to 0$, the system (A.2) becomes the system of equations first derived by Kadomtsev and Pogutse.[43] The term which explicity contains the pressure drops out of the system (A.2) in the limit $B^*\to 0$. The reason is that a straight stellarator is a magnetic confinement system with a maximum "average B" (a magnetic hump), which has a very strong destabilizing effect even at small values of β. It can be seen from (A.2) that the only distinction between a straight, current-carrying plasma column and a stellarator with a current in the limit of a vanishingly low pressure is that the poloidal flux $\Psi=\Psi^*+\Psi^J$ is now the sum of Ψ^J, the magnetic flux produced by the longitudinal current, and the known function

$$\Psi^*=-B_0\int^a \tau^* a\,da,$$

which is the flux of the "effective" magnetic field.

[1)]We will be using the term "stellarator" to mean a magnetic confinement system in which a system of nested magnetic surfaces in vacuum is produced either by helical conductors carrying a current or by some other external conductors which have an equivalent effect.

[2)]It turns out that the rotational transform at the internal branch of the separatrix is $\tau^*\ll N/n_0$ (so that the averaging method can be used to describe essentially the entire structure of closed magnetic surfaces all the way to the separatrix).

[3)]The basic principles of the confinement of a finite-β plasma will be discussed below.

[4)]In the derivation of the average equations, one makes use of the approximation of a strong magnetic field ($\varepsilon^2\ll 1$), so that the results found by averaging over the period of the rapidly varying field along the axis and along a line of force become equivalent.

[5)]We have combined several terms containing averages of the squares of the components of the rapidly alternating field $\mathbf{B}^{st}$, by introducing the vector $\mathbf{B}^*$, which, like the real field, satisfies the equation div $\mathbf{B}^*=0$.

[6)]Even in the absence of a plasma, the centers of the magnetic surfaces may be displaced from the geometric axis of the torus. The magnitude of this displacement may be determined by toroidal effects, by the pattern in which the current-carrying conductors are wound around the surface of the torus, by the strength of the vertical external magnetic field, etc.

[7)]See also the Appendix.

[8)]Condition (6.12) follows from the conditions $R_M>0$ at the edge of a plasma column with a free boundary, where the condition $R_M>0$ is most difficult to satisfy.

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