

Spin waves and quantum collective phenomena in Boltzmann gases

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Usp. Fiz. Nauk **148**, 433–471 (March 1986)

The concept of the quantum gas is introduced and illustrated by numerous examples. The fundamentals of the theory of collective phenomena in quantum Maxwellian gases are surveyed in a simple and readily assimilated form, and possible experimental studies are outlined. Particular attention is devoted to weakly-damped spin waves. The spectrum of these waves is calculated and the magnetic susceptibility generalized. The results obtained are compared with experimental data on spin waves in gaseous $H\uparrow$, $^3He\uparrow$, and $^3He\uparrow$ - 4He quantum solutions. It is shown that, at low temperatures, spin-polarized Boltzmann gases exhibit long-range spin correlations which fall off as r^{-1} at large distances. The equations of spin dynamics are solved for arbitrary temperatures and degrees of polarization, both in the weakly damped and diffusion regimes. The thermodynamics of spin-polarized gases and some of the features of transport phenomena are examined. Paramagnetic resonance and other collective effects in binary quantum gases are discussed. Magnetic and structural thermodynamic phase transitions in binary Maxwellian gases are predicted. Collective phenomena in semimagnetic semiconductors and analogous effects in the spectroscopy of Rydberg atoms and levitating electrons are discussed.

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1. INTRODUCTION: WHAT IS A QUANTUM GAS?

It has recently become clear that essentially quantum-mechanical effects may become manifest in low-density systems at temperatures much higher than the quantum degeneracy temperature, i.e., in objects which traditionally have been described by classical statistical physics. It is very important to note that many of these effects are not described by small quantum corrections to classical formulas, but provide the main contribution to the quantitative description of the physical phenomenon which, in this sense, has no classical analog. There is a relatively larger number of such phenomena, ranging from specific oscillations of magnetization to structural and magnetic phase transitions in equilibrium thermodynamics. These effects are highly nontrivial in a gas whose particles obey the Boltzmann-Maxwell statistics, and are attracting increasing attention among both theoreticians and experimentalists, especially since some rather surprising

phenomena have recently been confirmed experimentally. Boltzmann gases exhibiting macroscopic quantum phenomena will be designated as "quantum gases," and the significance of this phrase will become clearer as we continue with our review.

As usual, a "gas" will be understood to be a system in which the contribution of free particles to the total energy is much greater than the contribution due to the interaction between them. This situation occurs when either the particle interaction is weak or the gas density is low, although the interaction at short distances may be very strong. The former case is encountered infrequently in nature and we shall not consider it here (although, under certain particular conditions, the Coulomb interaction between particles may be regarded as weak, and is ignored in perturbation theory). We shall, in fact, confine our attention to systems of particles with a short-range interaction potential and low density N ,

such that $Nr_0^3 \ll 1$, where r_0 is the interaction range. The availability of this small parameter, natural for a gas, means that all the macroscopic properties of the system can be described in terms of the corresponding virial expansions. We shall also consider that the gas temperature T is high enough, i.e., $T \gg \varepsilon_d$, where ε_d is the degeneracy temperature, so that all the particles obey Boltzmann statistics. We shall thus be concerned with quantum effects in the classical temperature range.

When applied to a Boltzmann gas, the phrase "quantum gas" means, broadly, that the interaction between its particles must be described quantum-mechanically and not classically. This imposes an upper bound on the temperature, namely, $T \lesssim T^*$ where T^* can be found for a power-type potential by comparing quasiclassical and quantum virial corrections in gas thermodynamics and is given by¹

$$\hbar (mT^*)^{-1/2} = \xi(T^*), \quad U(\xi) = T^*, \quad (1.1)$$

where m is the particle mass and $U(r)$ is the interaction potential between two particles, so that, when $r = \xi$, the interaction energy of the two particles is comparable with the temperature, i.e., it is of the order of the average kinetic energy of a free particle. Since, in a gas, we always have $\xi(T^*) \ll N^{-1/3}$, it is certain that $T^* \gg \varepsilon_d$, which is in accord with the above assumption. We shall use the phrase "quantum gas" in a narrower sense, namely, we shall use it to describe a gas in which the mean de Broglie wavelength Λ of the particles in the system is much greater than the linear size r_0 of an individual atom. Actually, unless the gas has already been condensed by reducing the temperature, we shall enter the very interesting region

$$\varepsilon_d \ll T \ll \frac{\hbar^2}{mr_0^2}, \quad (1.2)$$

which is available to us because of the availability of the small gas parameter $Nr_0^3 \ll 1$. Condition (1.2) is equivalent to the following hierarchy of typical lengths in the system:

$$N^{-1/3} \gg \Lambda \gg r_0, \quad \Lambda \equiv \frac{\hbar}{mv_T} = \hbar (mT)^{-1/2}, \quad (1.3)$$

which means, in fact, that we are dealing with a nondegenerate quantum gas and not a quantum liquid. Nevertheless, although the gas molecules do obey Boltzmann statistics, the particle delocalization scale in the region defined by (1.2) is found to be greater than the size of the molecules ($\Lambda \gg r_0$), so that one would expect the system to exhibit qualitatively new (as compared with a classical gas) and fundamentally quantum properties.

Since the great majority of gases condense well before (1.2) begins to be satisfied, the leading candidates for media displaying quantum-mechanical effects are the gaseous isotopes of hydrogen and helium (H_2 , D_2 , HD , 3He , and 4He), the traditional objects used in low-temperature physics, which have appreciable saturated vapor pressures even in the temperature region defined by (1.2). Although there may be a great variety of macroscopic quantum-mechanical phenomena in tenuous systems at low enough temperatures (1.2) (some will be examined in Section 5), the most interesting effects are those relating to the nonequilibrium dynamics of magnetization in spin-polarized gases. In many

ways, this is explained by recent experimental advances in the study of the spin dynamics of polarized gas systems. Spin polarization, i.e., the establishment of unequal populations of different spin states, can have a very substantial effect on the macroscopic properties of quantum gases. Many interesting effects displayed by such gases are due precisely to the presence of spin polarization in the system. The most widely studied systems are, at present, spin-polarized atomic hydrogen and helium-3, which are commonly denoted by $H\uparrow$ and $^3He\uparrow$. Gaseous $H\uparrow$ can be polarized and stabilized against possible recombination into the molecular state H_2 by applying a strong enough external magnetic field. Kagan *et al.* (see, for example, Refs. 2 and 3) have examined in detail the possible recombination channels and stability conditions for the spin-polarized atomic state. Because of the hyperfine interaction, there are four different modifications of spin-polarized hydrogen of which the longest lived are the states $|a\rangle = |\uparrow\downarrow\rangle - \eta|\downarrow\uparrow\rangle$ and $|b\rangle = |\uparrow\uparrow\rangle$, where \uparrow and \downarrow represent the components of electron and nuclear spins and η is a small mixing parameter. Both states are, at present, the subject of extensive experimental investigation.^{4,5} The Amsterdam⁶ and Massachusetts⁷ groups have produced very high densities in gaseous $H\uparrow$ ($4.5 \times 10^{18} \text{ cm}^{-3}$), and the hope is that it may be possible to detect phenomena due to the quantum-mechanical degeneracy of the gas, e.g., Bose condensation and superfluidity, in $H\uparrow$. Moreover, gaseous $H\uparrow$ is the perfect material for the detection of macroscopic quantum effects even in the classical temperature range, well away from degeneracy, because theory suggests that it does not condense even at absolute zero, so that we may be confident that the region defined by (1.2) exists, and there is no prior condensation of the gas.

Another very convenient candidate for the spin-polarized quantum gas is $^3He\uparrow$. Its nuclear magnetic moment is very small, so that gaseous 3He cannot be appreciably polarized by applying an external magnetic field. Different dynamic methods of producing nuclear spin polarization, e.g., optical pumping, injection of a polarized beam, and so on,⁸ have therefore become very topical. Recent experiments by the Paris group^{9,10} have resulted in the very high degree of polarization $\alpha = 70\%$ at room temperature and $\alpha = 25\text{--}50\%$ at 4.2 K. The production of high levels of polarization in a gas is of fundamental importance because many collective effects can be observed only for high enough values of α . The relativistic dipole-dipole interaction in a gas is very small, and the depolarization of 3He atoms occurs mainly on the walls of the chamber. Hence, by applying suitable coatings to the chamber wall (solid H_2 , D_2 , Ne, superfluid 4He , and so on), it is possible to increase the magnetic moment relaxation time in $^3He\uparrow$ very substantially. In the experiments of Barbe *et al.*,¹¹ the longitudinal relaxation time τ_s , i.e., the time for which the magnetic polarization persists after the optical pump has been turned off, is more than two days at 4.2 K. Since the time necessary to establish equilibrium with respect to the spin τ_s is immeasurably greater than the gas Maxwellization time, the spin-polarized Boltzmann 3He exists for a long time after the pump has been turned off. The total magnetic moment persists for $t \lesssim \tau_s$, and the usual

thermodynamic relations can be used as for a true equilibrium system.

The solution of ^3He in superfluid ^4He is an example of a condensed medium that can be successfully described by the quantum gas theory. Because of the particular form of the diagram of state of solutions, the ^3He impurity atoms do not "condense" into a dense liquid even at $T = 0$. It follows that we can always enter the temperature range (1.2) in which the dissolved ^3He atoms may be looked upon as a quantum gas of impurity quasiparticles.^{12,13} The difference from gaseous ^3He , for which the saturated vapor pressure at low enough temperatures begins to decrease exponentially, is that the gas of impurity excitations can be cooled down to an arbitrarily low temperature at constant concentration because ^3He has a finite solubility in He II at $T = 0$, which means that realistically attainable magnetic fields can be used to polarize the nuclear spins of ^3He in solution. In a mixture of quantum gases, and, in particular, in a binary gas, we encounter not only a trivial increase in the number of components in the system, but also qualitatively new properties and effects that are not present in single-component systems. The new effects are particularly well-defined whenever the mass of the particles of one of the components of the quantum gas mixture is much greater than the mass of the particles belonging to the other component. Accessible examples include cold, weakly-ionized plasma containing not only electrons and ions, but also an appreciable number of neutrals, and a gas of highly-excited Rydberg atoms in an atmosphere of a foreign gas.¹⁴ In the former case, the light component consists of the free plasma electrons which are scattered by the short-range potential representing their interaction with the neutrals. In (1.2), which defines the extent to which the Maxwell gas is a quantum gas, the mass m must be interpreted as the reduced mass, i.e., practically the mass m_e of the electron, whereas r_0 is of the order of the scattering amplitude of a slow electron on a neutral, i.e., it is of the order of the atomic scale. In the second case, quantum effects in the system give rise to the refraction of the bound optical electron in the Rydberg atom by the foreign gas particles.

Nondegenerate semiconductors containing point defects¹⁴ are convenient objects for the experimental investigation of collective quantum phenomena in binary Boltzmann systems. When the concentrations of the conduction electrons and defects acting as the components of the system are not too high, all the qualitative characteristics due to the scattering of electrons by defects can be obtained in the form of the corresponding virial expansions. Moreover, since the electron mass is small, the temperature range (1.2) in which quantum effects are expected is substantially greater for the systems enumerated above. For example, for typical values of the effective mass of the electron in a semiconductor ($m^* \approx 0.1m_e$), the upper limit in (1.2) is of the order of 10^6 K, which covers practically the entire range of existence of a nondegenerate crystalline semiconductor.

2. SPIN WAVES IN POLARIZED PARAMAGNETIC GASES

One of the most surprising quantum effects in spin-polarized Boltzmann gases is the presence of weakly-damped

spin waves. The possibility of these magnetization waves, their spectrum, and the conditions for their existence in $\text{H}\uparrow$, $^3\text{He}\uparrow$, and Maxwellian plasma¹¹ were first noted in our previous paper.¹⁵ A more detailed derivation and a solution of the equations of spin dynamics for two-component spin-polarized Boltzmann systems were reported in Refs. 1 and 14. In 1982, Lhuillier and Laloë¹⁶ formulated the macroscopic equations for magnetization in the form of the Leggett equations¹⁷ for a Fermi fluid, which enabled them to describe spin oscillations under the conditions of strong damping, and to follow the transition to pure diffusion at high temperatures. The phenomenological equations of motion of a magnetic moment in spin-polarized Fermi systems are also given by Meyerovich.¹⁸ Quite recently, Lee, Freed, *et al.*, working at Cornell University,¹⁹ used nuclear magnetic resonance to confirm experimentally for the first time the existence of weakly-damped spin waves in gaseous $\text{H}\uparrow$ for $T < 0.8$ K, $N \sim 10^{16} \text{ cm}^{-3}$, and $H = 7.7$ Tl, where H is the external magnetic field. Levy and Ruckenstein²⁰ have developed the quasiparticle approach to the description of gaseous $\text{H}\uparrow$ and have given a qualitative interpretation of some of the experimental results reported in Ref. 19. At the same time, the Paris group²¹ detected experimentally highly-damped collective spin modes in gaseous $^3\text{He}\uparrow$ for $T = 2-6$ K, $N = 10^{16}-10^{18} \text{ cm}^{-3}$, and $\alpha \approx 0.3$. A systematic examination of collective effects, and a derivation and analysis of the equations of spin dynamics and long-range magnetic correlations in spin-polarized quantum gases and quantum liquids, is given in Ref. 22.

We note that a still earlier paper²³ exploited the fact that, in a low-density Fermi liquid, quasiparticles are weakly damped even well away from the Fermi surface, and an attempt was made to apply the transport equation of Fermi liquid theory to a nondegenerate system of impurity excitations in a $^3\text{He}-^4\text{He}$ solution. It became clear later that the small damping of the quasiparticles was not a sufficient basis for the conclusion that the equations of the theory of Fermi liquids were valid in the classical temperature range. Nevertheless, the reverse conclusion did turn out to be valid, i.e., in the region in which collective effects provided the main contribution, and spin waves were weakly damped, long-lived quasiparticles could also be defined. For this reason, although the formal application of the Fermi-liquid transport equation to the description of the nondegenerate $^3\text{He}-^4\text{He}$ solution in a magnetic field would not appear to be completely valid (especially since collective spin effects can also occur in a Maxwellian gas of Bose particles) and would not provide a complete elucidation of the conditions for the existence of spin waves, it could lead to the correct expression for the spin-wave spectrum. Unfortunately, an error crept into the calculations reported in Ref. 23 (it was subsequently corrected in Refs. 15 and 24).

The frequency of spin waves in a quantum gas can be either high or low and, as we shall see later, in contrast to the zero-sound in a Fermi liquid, the damping of magnetization waves is independent of the parameter $\omega\tau$, where ω is the oscillation frequency and $\tau \sim (Nr_0^2\nu_T)^{-1}$ is the gas-kinetic relaxation time. The quantum nature of these waves becomes qualitatively particularly clear in the high-frequency

case $\omega\tau \gg 1$. Actually, in a classical gas with a short-range interaction between the particles, the description of scattering processes reduces to the fact that all changes in particle states occur only at the instant of collision while, in the interval between collisions, i.e., within the mean free path, the gas molecules move freely without any external disturbance. It follows that, in the classical picture, there can be no high-frequency processes with periods shorter than the time τ between collisions. The physics of such phenomena in quantum gases becomes clear if we consider as an example the well-known quantum-mechanical problem of refraction of a monochromatic beam of slow particles by a tenuous system of scattering centers.²⁵ The solution of this problem shows that the difference between the real part of the refractive index n and unity is proportional to the real part of the scattering amplitude $f(0, E)$ and the density N of scattering centers [$\text{Re}n - 1 \sim N \text{Re}f(0, E)$], whereas the imaginary part of the refractive index, describing the dissipation of the beam over one mean free path, satisfies the relation $\text{Im}n \sim N \text{Im}f(0, E)$. According to the optical theorem, in the case of scattering, $\text{Im}f(0, E) = q\sigma/4\pi\hbar$, where q is the momentum of a beam particle and σ is the total scattering cross section. Hence, for slow, long-wave particles for which $qr_0/\hbar \simeq r_0/\Lambda \gg 1$, we always have $|\text{Im}f(0, E)| \ll |\text{Re}f(0, E)|$, i.e., $\text{Re}n - 1 \gg \text{Im}n$, so that, in the leading approximation, the beam experiences quantum refraction over distances shorter than the mean free path, but does not dissipate. The correction $\text{Re}n - 1$ to the real part of the refractive index is an indication of the presence of a certain addition to the real energy of the beam particle, which is linear in the forward scattering amplitude and is a functional of the distribution of the scattering centers. This means that the beam particle is always influenced by the entire ensemble of scattering centers. In this sense, the addition to the energy of the beam particle may be looked upon as being due to a particular quantum-mechanical self-consistent field of the Fermi-liquid type. The presence of this field, which is due to the quantum-mechanical refraction of gas particles by one another assures the possibility of collective quantum phenomena even in the classical temperature range (1.2).

2.1. Collective effects and the transport equation

From the quantitative point of view, the description of collective modes can be reduced to the determination of virial corrections to the kinematic part of the Boltzmann transport equation (but not the collision integral!). The fact that such corrections are necessarily present will become clear even from the following simple considerations. Suppose we have carried out the virial expansion for pressure and, by differentiating this expression, have determined the virial corrections to the velocity of sound in the gas, which are due to the interaction between the particles. We now wish to determine these corrections from the transport equation. This means that we must pass to the hydrodynamic equations. By integrating the transport equation with respect to the momenta, we obtain the continuity equation. The current equation (the equation of Euler) is found, as usual, by multiplying the transport equation by the momentum and

integrating over phase space. The above integrals of the right-hand side of the Boltzmann equation, i.e., of the collision integral in its usual form, must be equal to zero because the number of particles and the total momentum are both conserved. Hence, to determine the required addition to the velocity of sound, we must take into account the virial corrections to the transport equation, which do not vanish in the course of the above integrations and contribute to the macroscopic relationships expressing the conservation of both momentum and particle number. As a convention, we shall apply these corrections to the kinematic (left-hand) part of the Boltzmann equation. Since we are concerned, in the present case, with the hydrodynamic region, the required corrections will be determined by the thermodynamic virial expansions. Actually, the kinematic virial corrections appear in the linearized transport equation because this equation must be linearized in the small deviations from the true distribution function of the nonideal gas and not from the Maxwell distribution function. Thus, the collision integral vanishes precisely when the true distribution function is substituted into it. The distribution function of a nonideal gas is a functional of the Maxwell distribution function^{1,22,26} (virial expansion of the distribution function), and this leads to the renormalization of the Boltzmann equation and to the appearance in it of kinematic virial terms of the Fermi-liquid type.

The well-known Bogolyubov method can be used to derive fairly simply from first principles the kinetic equation for a nonideal gas, which contains the kinematic virial corrections. Using the Fermi renormalization procedure, it can be written in the following traditional form in the classical long-wave limit^{14,15,22}:

$$\frac{\partial n_s}{\partial t} + \frac{i}{\hbar} [\tilde{\epsilon}_s, n]_s + \frac{1}{2} \left(\frac{\partial \tilde{\epsilon}_s}{\partial \mathbf{p}} \nabla n_s + \nabla n_s \frac{\partial \tilde{\epsilon}_s}{\partial \mathbf{p}} \right) - \frac{1}{2} \left(\nabla \tilde{\epsilon}_s \frac{\partial n_s}{\partial \mathbf{p}} + \frac{\partial n_s}{\partial \mathbf{p}} \nabla \tilde{\epsilon}_s \right) = \text{St } n_s, \quad (2.1)$$

where the distribution function $n_s(\mathbf{p})$ and the renormalized energy $\tilde{\epsilon}_s(\mathbf{p})$ of gas particles are spin operators, $[\tilde{\epsilon}_s, n]_s$ is the commutator in spin space, $\tilde{\epsilon}_s$ is a functional of n_s , i.e.,

$$\tilde{\epsilon}_s = \frac{p^2}{2m} - 2\beta\mathbf{S} \cdot \mathbf{H} - \frac{4\pi\hbar^2}{m} S_{p_s'} \sum_{p'} f_{ss'}(0, E) n_{p'}(\mathbf{p}'), \quad (2.2)$$

$\text{St } n_s$ represents the collision integral, and \mathbf{H} is the external magnetic field. Since Eqs. (2.1) and (2.2) are of the Fermi-liquid form, one is tempted to find collective modes analogous to those present in the degenerate Fermi liquid. The fundamental question is then whether one can retain in the transport equation for the high-temperature Maxwell gas the self-consistent quantum corrections responsible for the collective phenomena. It is readily seen that, in the quantum gas, i.e., when (1.2) is satisfied, the retention of these virial corrections is valid.

In point of fact, we have already noted that it follows from the optical theorem that, in the leading approximation in $r_0/\Lambda \ll 1$, the energy (2.2) of a gas particle turns out to be real. In this approximation, the forward-scattering ampli-

tude f_{ss} , $(0, E)$ is an energy-independent constant, namely, the s -scattering length (with the sign reversed), whose numerical value is of the order of the atomic size r_0 . Since we are retaining the specific gradient terms $\nabla \bar{\epsilon}_s \partial n_s / \partial \mathbf{p}$ on the left-hand side of the transport equation (2.1), we must, in principle, retain the nonlocal corrections to the collision integral on the right-hand side of the Boltzmann equation, i.e., we must take into account the expansion of the collision integral in terms of the gradients of the distribution function. In the gas-kinetic approximation, this can be written in the form

$$\text{St } n_s \propto \delta n_s N r_0^2 v_{\text{T}} (1 + k\Lambda), \quad (2.3)$$

where k^{-1} is the characteristic scale of a spatial inhomogeneity. In the temperature range defined by (1.2), nonlocal corrections to $\text{St } n_s$ in (2.3) turn out to be of the same order as the terms that were discarded in the transport equation and were due to the imaginary additions to $\bar{\epsilon}_s$, i.e., negligible in comparison with the real kinematic corrections to $\bar{\epsilon}_s$ in (2.2), provided $r_0/\Lambda \ll 1$. Hence, inclusion of the quantum-mechanical self-consistent field in the Boltzmann equation with a local collision integral is wholly justified for $T \ll \hbar^2 / m r_0^2$. Direct calculation will readily show that $\bar{\epsilon}_s = E / \delta n_s$ in the above approximation, where E is the total energy. This provides a still better illustration of the analogy between the transport equation containing the virial corrections for the quantum Maxwellian gas and the equations of the Landau theory of the Fermi fluid.

Of course, both kinematic corrections and the collision integral are determined by the particle interaction and, in this sense, the subdivision into them is arbitrary. However, the physical significance of these two parts of the transport equation is different. $\text{St } n_s$ describes the change in the state of the particles during a scattering event, whereas the self-consistent term represents the effect of the interaction on the motion of particles between collisions. In the quantum temperature range, the formal difference for $r_0 \ll \Lambda$ between the virial kinematic corrections and the collision integral is particularly appreciable and ensures that the self-consistent corrections are linear in the scattering amplitude while $\text{St } n_s$ is quadratic in f . The analogy between the transport equation of the Boltzmann gas (2.1) for $|a|/\Lambda \ll 1$ and the equations of the Landau theory of the Fermi fluid can be seen in the relation $\bar{\epsilon}_s = \delta E / \delta n_s$ and in the fact that the correction to the particle energy is expressed in terms of the zero-angle scattering amplitude. However, it also has a more fundamental physical significance. The basic results of the theory of the Fermi fluid are strictly valid only for highly degenerate Fermi systems, when all the quasiparticles are near the Fermi surface. As the temperature of the system increases, so that $T \gtrsim \epsilon_d$, damping due to the finite lifetime of the quasiparticles in the liquid becomes important. However, Galitskii²⁷ has shown that, for $N^{1/3} r_0 \ll 1$, the damping of the excitations in tenuous Fermi systems is proportional to the square of the gas parameter ($N^{2/3} r_0^2$) and is small even well away from the Fermi surface, right up to high momenta $p \lesssim \hbar / r_0$. Hence, even in a Boltzmann gas, the damping of Fermi excitations is small right up to the boundary of the quantum region $T \lesssim \hbar^2 / m r_0^2$, and we have well-defined long-

lived quasiparticles with the energy spectrum (2.2). In the leading approximation in $|a|/\Lambda \ll 1$, the quasiparticle distribution function is identical with the distribution function of the true gas particles because the renormalizing constant of the Green function of the low-density Fermi liquid differs from unity by an amount of the order of $N^{2/3} r_0^2$. As $T \rightarrow 0$, all the results obtained for the Fermi gas go over to the corresponding results²⁷ for the low-density Fermi liquid. When the transport equation is justified for the Fermi liquid, there is then no need to compare the self-consistent gradient term on the left-hand side of the equation with the gradient corrections in the collision integral because the latter always contains the small factor $(T/\epsilon_d)^2$ and vanishes as $T \rightarrow 0$.

However, the cooperative properties of the quantum Boltzmann gas must not be totally identified with the high temperature "echo" of the properties of the Fermi liquid regarded as a set of fermions. Actually, in the region defined by (1.2), where quantum degeneracy effects in particle statistics can be neglected, Eq. (2.1), taken with (2.2), is valid for particles of arbitrary spin, independently of whether they are fermions or bosons.

At higher temperatures $T \gtrsim \hbar^2 / m r_0^2$, both the real and imaginary self-consistent corrections to $\bar{\epsilon}_s$ are found to be of the same order. The pseudopotential method used to determine the virial corrections to the kinematic part of the transport equation then becomes ineffective. Moreover, when $r_0 \gtrsim \Lambda$, nonlocal gradient corrections to $\text{St } n_s$, which, in this case, are proportional to $N r_0^2 v_{\text{T}} k r_0 \delta n_s$, become greater than or comparable with the virial kinematic corrections, so that the inclusion of the self-consistent quantum-mechanical field in the transport equation with the local collision integral at high temperatures takes us outside the range of precision. To avoid misunderstanding, we emphasize once again that, when the hydrodynamic parameters of a nonideal gas are determined from the transport equation, e.g., the velocity of sound (but not the collective Fermi liquid modes, such as zero sound!), the kinematic virial corrections can be taken into account for arbitrary temperature because the exact collision integral vanishes after integration during the derivation of the macroscopic hydrodynamic equations.

Analysis of the transport equation (2.1)–(2.2) shows that, unfortunately, even in the quantum temperature range defined by (1.2), the propagation of high-frequency waves such as zero sound, or longitudinal spin waves with linear dispersion relation, in a single-component Maxwell gas is impossible because of strong collisionless Landau damping. However, in some cases, the propagation of weakly-damped collective modes with a linear dispersion relation is still possible in two-component quantum gases.¹⁴

2.2. Weakly damped spin waves. The wave spectrum

Nevertheless, weakly-damped collective spin waves are always present in a spin-polarized quantum gas and are transverse relative to the spin polarization of the magnetization wave. For a number of reasons, these waves are hydrodynamic in character even for $\omega\tau \gg 1$, and constitute a weakly inhomogeneous precession of the macroscopic magnetic moment vector. The transport equation (2.1)–(2.2) can be

made much simpler in the case of transverse spin waves. Actually, since we shall confine our attention to weakly homogeneous solutions of the transport equation as $k \rightarrow 0$, we need only retain the largest gradient term in (2.1). It is readily verified that, because $Nr_0^3 \ll 1$, the term $\nabla n_s \cdot \partial \tilde{\epsilon}_s / \partial \mathbf{p}$ on the left-hand side of the Boltzmann equation will be much greater (at any temperature) than the nonlocal corrections in the collision integral and the virial kinematic corrections $\nabla \tilde{\epsilon}_s \cdot \partial n_s / \partial \mathbf{p}$. Hence, the main contribution to the transport equation is due to the spin commutator $[\tilde{\epsilon}_s, n_s]$, and to the term $\nabla n_s \cdot \partial \tilde{\epsilon}_s / \partial \mathbf{p}$ in which we can now neglect the particle interaction, so that it takes the usual form corresponding to the free motion of gas molecules between successive collisions:

$$\frac{\partial n_s}{\partial t} + \mathbf{v} \nabla n_s + \frac{i}{\hbar} [\tilde{\epsilon}_s, n_s] = \text{St } n_s, \quad \mathbf{v} \equiv \frac{\mathbf{p}}{m}. \quad (2.4)$$

The quantity $\text{St } n_s$ in (2.4) is now the usual local collision integral without gradient corrections. The collision integral which governs the collisional relaxation damping of spin waves can be neglected in the transport equation (2.4) provided

$$\hbar^{-1} |[\tilde{\epsilon}_s, n_s]| \gg |\text{St } n_s|.$$

Using (2.2) for the energy, and simple gas-kinetic estimates in the τ -approximation for $\text{St } n_s$, we obtain the following criterion:

$$1 \gg |\alpha| \gg \frac{r_0}{\Lambda}, \quad (2.5)$$

where α is the degree of polarization characterizing the unequal populations of spin states, i.e., the equilibrium (or quasi-equilibrium) magnetic moment of the gas. Thus, for a system of fermions with spin $1/2$, the degree of polarization is conveniently defined by $\alpha = (N_+ - N_-)/N$, where N_+ and N_- are the numbers of particles per unit volume with spins, respectively, along and against the spin polarization vector. The criterion (2.5) is fundamental because it shows that weakly-damped magnetization waves can propagate in spin-polarized gases only in the quantal temperature range (1.2) and for a high enough degree of polarization of the gas. Under these conditions, the real virial correction in the spin commutator on the left-hand side of the Boltzmann equation is greater than the imaginary collisional term on the right-hand side of (2.4). When $T \gtrsim \hbar^2/mr_0^2$, spin waves are always strongly damped because the criterion (2.5) is not satisfied. We emphasize that, even when (1.2) is satisfied, the inclusion in the scattering amplitude $f_{ss}(0, E)$ of not only the leading term (the s -scattering length) but also the higher-order terms in $r_0/\Lambda \ll 1$, i.e., inclusion of higher-moment scattering while neglecting damping, takes us outside the range of precision because (2.5) is then violated for the corresponding terms. In a previous paper,¹⁵ the criterion given by (2.5) was derived from the condition for the existence of a region in which both Landau damping and collisional absorption of spin waves were small.

We shall now derive explicitly the spectrum of transverse spin waves in a gas. To be specific, we shall perform the calculation for a gas consisting of spin $1/2$ particles. The equilibrium polarization density matrix for the spin-polarized

Maxwell gas can then always be written in the form of a linear function of Pauli matrices:

$$n_s^{(0)}(\mathbf{p}) = (1 + \alpha \sigma \vec{\mathcal{M}}) n_0(\mathbf{p}), \quad \sigma = 2S, \quad (2.6)$$

where $\vec{\mathcal{M}}$ is a unit vector in the direction of the spin polarization and $n_0(\mathbf{p})$ is the Maxwellian distribution function

$$n_0(\mathbf{p}) = \frac{N}{2} \left(\frac{2\pi\hbar^2}{mT} \right)^{3/2} e^{-p^2/2mT}. \quad (2.7)$$

In principle, it is a relatively simple matter to introduce into (2.6) the virial corrections to the equilibrium density matrix.¹ However, these corrections can be neglected in the determination of the spin-wave spectrum in the leading approximation. Small deviations of the density matrix n_s from its equilibrium value (2.6) will be sought in the form

$$\delta n_s = n_s - n_s^{(0)} = \lambda(\mathbf{p}) \sigma, \quad \lambda(\mathbf{p}) \propto \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t). \quad (2.8)$$

In accordance with the foregoing, we can write (2.2) for the renormalized energy $\tilde{\epsilon}_s$ in the quantal temperature range (1.2) in the form

$$\tilde{\epsilon}_s = \frac{p^2}{2m} + g_0 N - \left[g_0 N \alpha \vec{\mathcal{M}} + 2g_0 \sum_{\mathbf{p}} \lambda(\mathbf{p}) + \beta H \right] \sigma, \\ g_0 \equiv \frac{2\pi a \hbar^2}{m}, \quad (2.9)$$

where we have introduced the s -scattering length a for the collisions between the gas atoms. The next step is to substitute for n_s and $\tilde{\epsilon}_s$ from (2.6)–(2.9) into the transport equation (2.4), and linearize it in the small deviations $\lambda(\mathbf{p})$. By taking the z -axis to be in the direction of the vector $\vec{\mathcal{M}}$, and transforming to the circular variables $\lambda_{\pm} = \lambda_x \pm i\lambda_y$, we obtain the equation describing the dynamics of transverse magnetization in a spin-polarized gas:

$$(\omega + \Omega_H - \Omega_{\text{int}} - \mathbf{k} \cdot \mathbf{v}) \lambda_{-}(\mathbf{p}) - \frac{4\alpha}{\hbar} g_0 n_0(\mathbf{p}) \sum_{\mathbf{p}'} \lambda_{-}(\mathbf{p}') = \text{St } \lambda_{-}, \\ \Omega_{\text{int}} = -\frac{2g_0 N \alpha}{\hbar} = -\frac{4\pi a \hbar N \alpha}{m}, \quad \Omega_H = \frac{2\beta H}{\hbar}. \quad (2.10)$$

The equation for $\lambda_{+}(p)$ is obtained from (2.10) by introducing the replacements $\omega \rightarrow -\omega$, $\mathbf{k} \rightarrow -\mathbf{k}$. When (2.5) is satisfied, and collisional damping of spin waves can be neglected, the expression given by (2.10) leads directly to the dispersion relation

$$1 - g_0 \frac{4\alpha}{\hbar} \sum_{\mathbf{p}} \frac{n_0(\mathbf{p})}{\omega + \Omega_H - \Omega_{\text{int}} - \mathbf{k} \cdot \mathbf{v}} \equiv D(\omega, \mathbf{k}) = 0. \quad (2.11)$$

Integration in (2.11) must be performed in accordance with the usual rule for going around poles. In the long-wavelength region, $kv_T \ll |\Omega_{\text{int}}|$, $|\omega| \ll |\Omega_{\text{int}}|$, and after some simple algebra we find from (2.11) that, in the linear approximation in $r_0/\Lambda \ll 1$, the dispersion relation for the spin modes is^{1,22}

$$\omega' = \text{Re } \omega = \Omega_H + \frac{(kv_T)^2}{\Omega_{\text{int}}}. \quad (2.12)$$

The collisionless spin-wave absorption coefficient due to Landau damping then turns out to be exponentially small:

$$\omega'' \equiv \text{Im } \omega = \text{Im } D(\omega', \mathbf{k}) \left[-\frac{\partial}{\partial \omega'} \text{Re } D(\omega', \mathbf{k}) \right]^{-1},$$

$$\omega'' = -\left(\frac{\pi}{2}\right)^{1/2} \frac{\Omega_{\text{int}}^2}{k v_T} \exp\left(-\frac{\Omega_{\text{int}}^2}{2k^2 v_T^2} - 1\right), \quad |\omega''| \ll |\omega'|. \quad (2.13)$$

The condition for the validity of (2.12) and (2.13) is $k v_T \ll |\Omega_{\text{int}}|$ and determines the range of wave vectors in which, when (2.5) is satisfied, weakly-damped spin waves can propagate in the gas. Formally, small Landau damping is assured by the k -independent term Ω_{int} in the denominator of the singular part of $D(\omega, \mathbf{k})$. The quantity Ω_{int} is a measure of the molecular quantal field and of collective magnetic phenomena in the spin-polarized gas. Moreover, it follows from the obvious condition

$$N^{-1/3} |\Omega_{\text{int}}| v_T^{-1} \sim \frac{e_d}{T} \frac{|a|}{\Lambda} \ll 1 \quad (2.14)$$

that all the values of the wave vector $k \ll |\Omega_{\text{int}}|/v_T$, for which (2.12)–(2.13) are valid, satisfy the necessary conditions of macroscopic and quasiclassical behavior, i.e.,

$$1 \gg k N^{-1/3} \gg k \Lambda, \quad (2.15)$$

which enable us to look upon spin waves in a quantum gas as oscillations of the macroscopic magnetic moment in a continuous medium. In the short-wave region $k v_T \gtrsim |\Omega_{\text{int}}|$, where the term Ω_{int} in the dispersion relation (2.11) can be neglected, collisionless absorption prevents the propagation of undamped spin waves.

In the real world, the principal mechanism producing spin-wave damping is usually collisional relaxation. Since the numerical value of magnetization is conserved in the exchange approximation, the integration of (2.10) over momentum space for uniform distributions with $\mathbf{k} = 0$ ensures that the collision integral $\text{St } \lambda_-$ becomes equal to zero, i.e., the uniform precession of the magnetic moment in the external field is undamped. Hence, the collisional relaxation time for transverse spin waves and small \mathbf{k} is definitely a function of \mathbf{k} such that $\tau_{\text{coll}}(\mathbf{k} \rightarrow 0) \rightarrow \infty$. For weak collisional absorption (2.5), the transport equation (2.10) will readily yield the imaginary addition to the spin-wave spectrum which is due to the damping of the waves by collisions between gas particles²².

$$\omega = \Omega_H + \frac{(k v_T)^2}{\Omega_{\text{int}}} \left(1 - \frac{i}{\Omega_{\text{int}} \tau}\right), \quad \tau_{\text{coll}} = \frac{1}{\omega''} = -\left(\frac{\Omega_{\text{int}}}{k v_T}\right)^2 \tau, \quad (2.16)$$

where τ is a relaxation time which has the gas-kinetic order of magnitude $\tau^{-1} \sim N a^2 v_T$, so that, when (2.5) is satisfied, we have $|\Omega_{\text{int}}| \tau \gg 1$. To describe transverse spin waves in a Maxwell gas, we have discarded in the transport equation (2.10) all the kinematic gradient corrections due to interactions, although they could have been taken into account in the temperature range (1.2). The solution of the complete transport equation (2.1), including these virial corrections,²² confirms the validity of (2.10).

2.3. Weakly-damped diffusion modes. Arbitrary temperature

Although, by virtue of (2.5), weakly-damped spin waves exist in a gas only in the quantal temperature range

(1.2), there is also definite interest in the equations of spin dynamics at higher temperatures $T \gtrsim \hbar^2/mr_0^2$ at which there is a transition from the spin-wave regime to the pure diffusion-type spreading of magnetization, especially since it is precisely this experimental situation that is usually encountered in gaseous $^3\text{He}\uparrow$. We have already noted that Eqs. (2.1) and (2.2) cannot be used directly at high temperatures. Nevertheless, we can neglect all the interaction-related gradient corrections and retain in the dynamic equation for the transverse magnetization the terms describing free motion of particles, and the terms representing interactions, but only in the local spin commutator and the local collision integral, as was done in deriving (2.10). The possibility of this simplification is entirely due to the low gas density $N r_0^3 \ll 1$ and is in no way related to the temperature. However, even in this case, the evaluation in a general form of the kinematic virial corrections and of the collision integral in terms of the exact scattering amplitude is a very complicated problem. Nevertheless, at least in the linearized transport equation, the self-consistent kinematic correction can be obtained in explicit form. To do this, the virial correction is written as a functional that is linear in δn_s , and the transport equation is then used to deduce the linear equations of hydrodynamics. It is assumed in this procedure that the collision integral satisfies all the macroscopic conservation laws, i.e., it vanishes after the corresponding integrations. On the other hand, the macroscopic hydrodynamic equations can be formulated by starting with only the thermodynamic functions for which the corresponding virial expressions are well known. By comparing the results obtained by these two methods, it is possible to establish the particular form of the linear functional for the kinematic virial correction to the Boltzmann equation. Of course, such corrections can be used in conjunction with the transport equation to find the virial expansion for the hydrodynamic sound velocity at arbitrary temperature. It turns out that the kinematic virial terms in the transport equation arise simply from the renormalization of δn_s due to the virial functional corrections to the equilibrium distribution function. For the spin degrees of freedom, the macroscopic Landau-Lifshitz equations play the part of the equations of hydrodynamics. The correspondence between the transport equation with the virial corrections in the spin commutator at arbitrary temperature and the Landau-Lifshitz equations was demonstrated in Ref. 1.

In view of the foregoing, and using the results reported in Refs. 1 and 22, we can readily verify that the equations of spin dynamics for the off-diagonal components of the polarization density matrix, i.e., for the transverse magnetization, can be written at arbitrary temperature as before in the form given by (2.4), where

$$\tilde{\varepsilon}_s = \frac{p^2}{2m} - \beta \boldsymbol{\sigma} \cdot \mathbf{H} + \frac{\delta E_{\text{int}}}{\delta n_s}, \quad (2.17)$$

and the interaction energy E_{int} is given by the virial expansion²²

$$E_{\text{int}} = F_{\text{int}} = \sum_{\mathbf{p}, \mathbf{p}'} \Phi_{ss'}(\mathbf{p}, \mathbf{p}') n_s(\mathbf{p}) n_{s'}(\mathbf{p}'),$$

$$\Phi_{ss'}(\mathbf{p}, \mathbf{p}') = \psi(\mathbf{p}, \mathbf{p}') + \zeta(\mathbf{p}, \mathbf{p}') \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}',$$

$$\psi(\mathbf{p}, \mathbf{p}') = \frac{1}{4} [3A_+(\mathbf{q}) + A_-(\mathbf{q})], \quad (2.18)$$

$$\zeta(\mathbf{p}, \mathbf{p}') = (1/4) [A_+(\mathbf{q}) - A_-(\mathbf{q})],$$

$$2\mathbf{q} = \mathbf{p} - \mathbf{p}',$$

$$A_{\pm}(\mathbf{q}) = -\frac{4\pi\hbar^2}{m} \left\{ \text{Re } f_{\pm}(0, q) \right. \\ \left. + \frac{mT}{\hbar} \left[\text{Re } f_{\pm}(\theta, q) \frac{\partial}{\partial q} \text{Im } f_{\pm}(\theta, q) \right. \right. \\ \left. \left. - \text{Im } f_{\pm}(\theta, q) \frac{\partial}{\partial q} \text{Re } f_{\pm}(\theta, q) \right] \right\};$$

where $f_+(\theta, q)$ and $f_-(\theta, q)$ are the scattering amplitudes for the two particles in the triplet and singlet states (spin 1/2). If necessary, the quantities $A_{\pm}(\mathbf{q})$ can be readily expressed in terms of the scattering phase shifts:

$$f_{\pm}(\theta, q) = \frac{i\hbar}{2iq} \sum_l (2l+1) (e^{2i\delta_l} - 1) P_l(\cos\theta), \quad (2.19)$$

where the sum in (2.19) is evaluated over odd values of l for $f_+(\theta, q)$ and over even values of l for $f_-(\theta, q)$. The collision integral $\text{St } n_s$ satisfies the macroscopic conservation law for the absolute magnitude of magnetization although its specific structure is unknown in the general case. By transforming (2.4) with $\bar{\epsilon}_s$ taken from (2.17) and (2.18), and the small perturbations of the density matrix from (2.8), we obtain the following dynamic equation for the circular component of magnetization:

$$(\omega + \Omega_H - \mathbf{k}\mathbf{v}) \lambda_{\pm}(p) + \frac{8\alpha}{\hbar} \\ \times \sum_{\mathbf{p}'} \zeta(\mathbf{p}, \mathbf{p}') [n_0(\mathbf{p}) \lambda_{\pm}(\mathbf{p}') - n_0(\mathbf{p}') \lambda_{\pm}(\mathbf{p})] = \text{St } \lambda_{\pm}. \quad (2.20)$$

It is readily seen that, when $T \ll \hbar^2/mr_0^2$, so that the interactions are largely reduced to s -scattering, the expression for $\bar{\epsilon}_s$ given by (2.17)–(2.18) and Eq. (2.10) become identical with (2.9) and (2.10), respectively.

We shall now find the formal solution of (2.20), neglecting the collisional right-hand side. This solution, which we shall note by $\tilde{\omega}(\mathbf{k})$, will be sought in the form of an expansion in powers of a small wave vector, $\tilde{\omega}(\mathbf{k}) = \omega_0 + \omega_1 + \omega_2 + \dots$, $\lambda_{\pm} = \lambda_{\pm}^{(0)} + \lambda_{\pm}^{(1)} + \lambda_{\pm}^{(2)} + \dots$, i.e., we shall neglect the exponentially small Landau damping. Equation (2.10) then yields immediately

$$\left. \begin{aligned} \omega_0 &= \Omega_H, & \omega_1 &= 0, & \omega_2 &= -\frac{2}{N} \sum_{\mathbf{p}} \mathbf{k} \cdot \mathbf{v} g(\mathbf{p}) n_0(\mathbf{p}), \\ \lambda_{\pm}^{(0)} &= b n_0(\mathbf{p}) \alpha, & b &= \text{const}, & \lambda_{\pm}^{(1)} &= \lambda_{\pm}^{(0)} g(\mathbf{p}), \\ \frac{8\alpha}{\hbar} \sum_{\mathbf{p}'} \zeta(\mathbf{p}, \mathbf{p}') n_0(\mathbf{p}') [g(\mathbf{p}') - g(\mathbf{p})] - \mathbf{k} \cdot \mathbf{v} &= 0. \end{aligned} \right\} \quad (2.21)$$

It is readily verified that, even when the exact collisional equation (2.10), including the right-hand side, is solved, the imaginary terms describing the damping of spin waves appear only in the second order in \mathbf{k} . The general solution of the phenomenological equations of spin dynamics, obtained in Ref. 16 in the first approximation of the Chapman-Enskog method, can be written in the form

$$\omega = \Omega_H + \frac{D_0 k^2}{1 + (\mu\beta N \alpha)^2} (\mu\beta N \alpha - i), \quad (2.22)$$

where D_0 is the classical spin diffusion coefficient for longitudinal magnetization and μ is a parameter that is independent of α and is due to the self-consistent molecular field. Gas-kinetic estimates such as (2.5), and the form of the general solution of (2.22), leads to the conclusion that the formal collisionless solution $\tilde{\omega}(\mathbf{k})$ corresponds in (2.21) to the limit as $\alpha \rightarrow \infty$, i.e.,

$$\tilde{\omega}(\mathbf{k}) = \frac{D_0 k^2}{\mu\beta N \alpha} + \Omega_H. \quad (2.23)$$

Since, as can be seen from (2.21), $g(\mathbf{p}) \sim 1/\alpha$ and $\omega_2 \sim 1/\alpha$, the parameter μ is independent of α , as should be the case, and this enables us to determine it from (2.21)–(2.23) by formally going to the limit as $\alpha \rightarrow \infty$. By using (2.23) to express μ in terms of $\tilde{\omega}(\mathbf{k})$, we can deduce from (2.2) the dispersion relation for spin waves at arbitrary temperature and degree of polarization:

$$\omega = \Omega_H + \frac{D_0 k^2}{1 + (D_0 N \alpha \gamma)^2} (-D_0 N \alpha \gamma - i), \\ \frac{1}{\gamma} = \frac{\hbar}{12} \int v^2 A(p) n_0(p) \frac{d^3 p}{(2\pi\hbar)^3}, \\ \frac{1}{3} \int \zeta_1(p, p') n_0(p') \frac{v'}{v} \frac{d^3 p'}{(2\pi\hbar)^3} A(p') \\ - A(p) \int \zeta_0(p, p') n_0(p') \frac{d^3 p'}{(2\pi\hbar)^3} = 1, \\ \zeta(\mathbf{p}, \mathbf{p}') = \sum_l \zeta_l(p, p') P_l(\cos\theta), \quad (2.24)$$

where $\zeta_0(p, p')$ and $\zeta_1(p, p')$ are the first harmonics in the expansion of the function $\zeta(\mathbf{p}, \mathbf{p}')$ in terms of the Legendre polynomials $P_l(\cos\theta)$. In the quantal temperature range (1.2), but for small polarizations $\alpha \lesssim |a|/\Lambda$, the spin waves are again found to be highly damped. The dispersion relation for the magnetic moment in this case is deduced from (2.22), (2.23), and (2.16), and is given by

$$\omega = \Omega_H + \frac{D_0 k^2}{1 + (\Omega_{\text{int}} \tau)^2} (\Omega_{\text{int}} \tau - i), \quad \mu\beta N \alpha = \Omega_{\text{int}} \tau = \frac{D_0 \Omega_{\text{int}}}{v_T^2}, \quad (2.25)$$

where $\tau = D_0/v_T^2$ is the diffusion relaxation time introduced in (2.16).

2.4. Magnetic polarization and the collision integral

It is important to note that, when spin-wave damping is taken into account, the expressions given by (2.22)–(2.25) are explicitly valid only provided α is not too high. Actually, even at low temperatures, the collisional term $\text{St } n_s$ is always found to contain two different components.²⁸ The first has the traditional form of a collision integral and contains the δ -functions of the total momentum and energy of the two interacting particles, and can be interpreted as the difference between the “influx” and “outflux” of particles in the given quantal state:

$$I_1(\lambda_{\perp}) = (2\pi\hbar)^3 \left(\frac{2a}{m} \right)^2 \int d\mathbf{p}_1 \frac{d\mathbf{p}'_1 d\mathbf{p}'_2}{(2\pi\hbar)^6} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2) \\ \times \delta \left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{p'^2_1}{2m} - \frac{p'^2_2}{2m} \right) \\ \times [\lambda_{\perp}(\mathbf{p}'_1) n_0(\mathbf{p}'_2) - \lambda_{\perp}(\mathbf{p}_1) n_0(\mathbf{p}_2) \\ + \lambda_{\perp}(\mathbf{p}_2) n_0(\mathbf{p}_1) - \lambda_{\perp}(\mathbf{p}'_2) n_0(\mathbf{p}'_1)] \sim N a^2 v_T \lambda_{\perp}. \quad (2.26)$$

In deriving (2.26), we have allowed for the fact that the equilibrium polarization density matrix is diagonal and that perturbations of the off-diagonal elements of the density matrix, i.e., of the transverse magnetization $\lambda_{\perp} = (\lambda_x, \lambda_y)$, are not coupled to density perturbations, as is the case for the fluctuations in λ_z . The second term $I_2(\lambda_{\perp})$ does not contain the δ -function of the total energy and is unrelated to relaxation processes. It must be looked upon as the next order correction in $|a|/\Lambda \ll 1$ to the spin commutator on the left-hand side of the transport equation:

$$I_2(\lambda_{\perp}) = (2\pi\hbar)^3 \left(\frac{2a}{m}\right)^2 \int d\mathbf{p}_2 \frac{d\mathbf{p}'_1 d\mathbf{p}'_2}{(2\pi\hbar)^6} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2) \\ \times \frac{2m}{p_1^2 + p_2^2 - p_1'^2 - p_2'^2} 2\alpha \overleftrightarrow{\mathcal{M}} \{ \lambda_{\perp}(\mathbf{p}_1) n_0(\mathbf{p}_2) - \lambda_{\perp}(\mathbf{p}_2) n_0(\mathbf{p}_1) \} \\ \sim N a^2 v_T \alpha \lambda_{\perp}. \quad (2.27)$$

It is clear from (2.27) that the term $I_2(\lambda_{\perp})$ is small in comparison with the spin commutator $\hbar^{-1}[\varepsilon, n]_s$ to the extent that $|a|/\Lambda \ll 1$ for any α , so that $I_2(\lambda_{\perp})$ can always be neglected in the quantal temperature range when the real part of the frequency (2.16) of weakly-damped spin waves is evaluated. On the other hand, it follows from (2.26) and (2.27) that $I_1(\lambda_{\perp}) \gg I_2(\lambda_{\perp})$ provided $\alpha \ll 1$, so that the evaluation of spin-wave damping in terms of the coefficient D_0 with the aid of the usual collision integral $I_1(\lambda_{\perp})$, as was done in deriving (2.22)–(2.25), is admissible only for $\alpha \ll 1$. Hence, the formulas given by (2.25) and $\tau = D_0/v_T^2$ are valid only for low values of the polarization. When $1 \gtrsim \alpha \gg |a|/\Lambda$, the expression for the spin-wave spectrum (2.16) has the same form as before except that, when the time τ is evaluated, the collision integral must include not only the term $I_1(\lambda_{\perp})$, but also $I_2(\lambda_{\perp})$. The inclusion of $I_2(\lambda_{\perp})$ in $\text{St}\lambda_{\perp}$ essentially signifies that we are evaluating the relaxation time τ as a function of α . At high temperatures, the specific expression for the collision integral in terms of the exact scattering amplitude $f(\theta, E)$ (as in the case of the real kinematic corrections in the spin commutator on the left-hand side of the Boltzmann equation) is still unknown although the gas-kinetic estimates such as (2.26)–(2.27) remain valid, as before. Hence, even at arbitrary temperatures, the formulation of the equations of spin dynamics in terms of the usual diffusion coefficient D_0 given by (2.22), where μ is independent of α , is valid only for low polarizations $\alpha \ll 1$.²⁾ Since, under these conditions, the equations of motion of magnetization (2.20) are valid for any temperature (and not merely in the quantal region), we have the basic possibility of observing spin waves, even in the highly-damped diffusion regime, in any paramagnetic gas such as O_2 , NO , NO_2 , ClO_2 , alkali-metal vapor, and so on, in an external magnetic field. The practical realization of this type of experiment is, however, very difficult because the exchange interaction, which is a quantitative measure of the effect, falls off rapidly with increasing temperature.

The derivation of the dynamic equations (2.20) and (2.24) for arbitrary temperature is a good illustration of the successful combination of the transport equation method (the quasiparticle approach) and the phenomenological formulation of spin dynamics. It is important to emphasize,

however, that the macroscopic equations of motion for the magnetic moment can only be formulated on the assumption that a self-consistent exchange field is present in the spin-polarized Maxwell gas, and the introduction of this field must be justified within the framework of the transport equation method. Moreover, the range of validity of the results, $k v_T \ll |\Omega_{\text{int}}|$, cannot be found from macroscopic equations because they do not include the collisionless Landau damping, which defines this range. After all the necessary conditions have been elucidated on the basis of the transport equation, the phenomenological scheme turns out to be very convenient when spin dynamics is investigated in specific experimental situations.

2.5. Long-range correlations in a gas. Phenomenological equations

We now turn to spin waves in the undamped regime. The criterion (2.5) is equivalent to $|\Omega_{\text{int}}|\tau \gg 1$. The spin-wave eigenfrequency ω can then either be small ($\tilde{\omega} \ll \tau^{-1} \ll |\Omega_{\text{int}}|$) or large ($\tau^{-1} \ll \tilde{\omega} \ll |\Omega_{\text{int}}|$, where $\tilde{\omega} = \omega - \Omega_H$). In a classical gas, and when the long-range van der Waals forces are neglected, all the spatial (single-time) correlation functions decrease exponentially over distances of the order of the atomic size r_0 . This is why any macroscopic superstructures or collective modes cannot be present in such systems. On the other hand, in spin-polarized quantum gases, the very existence of undamped, high-frequency ($\omega\tau \gg 1$), magnetic-moment waves signifies the presence of long-range spatial correlation between the spins of the gas molecules. The self-consistent effective field produced by all the gas particles, and significant only at low enough temperatures (1.2) at which nondissipative quantum-mechanical refraction begins to play the dominant part, is thus seen to give rise to nonlocal terms in the total free energy and, even in the exchange approximation, to the possibility of long-range correlations and, occasionally, long-range order in a Boltzmann gas. Moreover, like the spectrum of fluctuations in magnetization, the coefficients of the gradient terms in the free energy are expressed in terms of a local quantity, namely, the zero-angle scattering amplitude. By specifying the macroscopic energy of an inhomogeneity, we can formulate the phenomenological equations, i.e., the Landau-Lifshitz equations, for the spin waves in a gas. For greater clarity, we begin with the $H = 0$ case, i.e., the situation where the spin polarization of the gas is produced by a dynamic pump or in a time much shorter than τ_s after the external field is turned off (if the system had been polarized simply by an external magnetic field).

We shall seek the linear response of the system to this perturbation by introducing the effective field $\mathcal{H}(\mathbf{r}, t) \sim \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$ and using the transport equation (2.4). The method employed to obtain (2.10) and (2.11) can then again be used to calculate the circular components $\delta M_{\pm} = \delta M_x \pm i\delta M_y$ of the magnetization induced by the external magnetic field ($\delta M_{\pm} = \chi_{\pm} \mathcal{H}_{\pm}$):

$$\chi_{-}(\omega, \mathbf{k}) = \chi_{+}^{*}(-\omega, -\mathbf{k}) = \beta^2 \frac{1 - D(\omega, \mathbf{k})}{g_0 D(\omega, \mathbf{k})}. \quad (2.28)$$

The pole of (2.28) determines the spectrum of magnetiza-

tion waves, as should be the case. The elements of the general susceptibility matrix $\chi_{ik}(\omega, \mathbf{k})$, defined by the usual relation

$$\delta M_i(\omega, \mathbf{k}) = \chi_{ik}(\omega, \mathbf{k}) \mathcal{E}_k(\omega, \mathbf{k}) \quad (i, k = x, y), \quad (2.29)$$

can be expressed in terms of $\chi_{\pm}(\omega, \mathbf{k})$ as follows:

$$\chi_{xx} = \chi_{yy} = \frac{1}{2}(\chi_+ + \chi_-), \quad \chi_{yx} = -\chi_{xy} = -\frac{i}{2}(\chi_+ - \chi_-). \quad (2.30)$$

The dynamic magnetic formfactor of the system

$$S_{ih}(\omega, \mathbf{k}) = \int d^3r \int_{-\infty}^{\infty} dt \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})] S_{ih}(t, r),$$

$$S_{ih}(t, r) = \langle \delta M_i(t_1, \mathbf{r}_1) \delta M_h(t_2, \mathbf{r}_2) \rangle,$$

$$t \equiv t_1 - t_2, \quad r = |\mathbf{r}_1 - \mathbf{r}_2| \quad (2.31)$$

can be expressed with the aid of the fluctuation-dissipation theorem in terms of the χ_{ik} :

$$S_{ih}(\omega, \mathbf{k}) = \frac{2\hbar}{1 - e^{-\hbar\omega/T}} \text{Im} \chi_{ih}(\omega, \mathbf{k}). \quad (2.32)$$

In the long-wavelength region $kv_T \ll |\Omega_{\text{int}}|$, we can use (2.28)–(2.32) together with $D(\omega, \mathbf{k})$, given by (2.11) with $\Omega_H = 0$, to show that²²

$$S_{xx}(\omega, \mathbf{k}) = S_{yy}(\omega, \mathbf{k})$$

$$= 2\pi \frac{\beta^2 N \alpha}{1 - e^{-\hbar\omega/T}} \left[\delta \left(\omega - \frac{k^2 v_T^2}{|\Omega_{\text{int}}|} \right) - \delta \left(\omega + \frac{k^2 v_T^2}{|\Omega_{\text{int}}|} \right) \right]. \quad (2.33)$$

The δ -functions in (2.33) represent the contribution of magnons to the dynamic formfactor of the quantum Boltzmann gas. Spin waves are then a collective Bose branch of long-wavelength elementary excitations in the system. The gas particles themselves (single-particle excitations) can then be either fermions, as in the case of $^3\text{He}\uparrow$, or bosons, as for $\text{H}\uparrow$. It then follows from the obvious inequality

$$N|a|^3 \left(\frac{\hbar^2}{ma^2} \right) \ll \varepsilon_d \ll T \ll \frac{\hbar^2}{ma^2} \quad (2.34)$$

that, in a quantum gas, we always have $\hbar|\Omega_{\text{int}}| \ll T$. Since for undamped spin waves $|\omega| \ll |\Omega_{\text{int}}|$, the contribution of magnons to the thermodynamics is always cut off at low frequencies $\omega \ll T/\hbar$, i.e., it is negligible.

The static structure factor

$$S_{ih}(k) = \int e^{i\mathbf{k}\cdot\mathbf{r}} S_{ih}(0, r) d^3r = \int_{-\infty}^{\infty} (2\pi)^{-1} S_{ih}(\omega, \mathbf{k}) d\omega \quad (2.35)$$

determines the purely spatial single-time correlation between transverse spin fluctuations in the quantum gas. Trivial integration of (2.35) and (2.33), subject to $\hbar|\Omega_{\text{int}}| \ll T$, yields

$$S_{xx}(k) = S_{yy}(k) = \beta^2 N \alpha \text{cth} \frac{\hbar k^2 v_T^2}{2|\Omega_{\text{int}}|T} = 2\pi|a| \left(\frac{2\beta N \alpha}{k} \right)^2. \quad (2.36)$$

It can be shown by direct calculation, using (2.28)–(2.32) and (2.35), that $S_{xy}(k) = S_{yx}(k) = 0$, as expected. The Fourier transform of (2.36) yields the correlation function $S_{ik}(r)$ in the coordinate representation²²:

$$S_{ih}(r) = 2|a| \frac{(\beta N \alpha)^2}{r} \delta_{ih}, \quad (2.37)$$

which falls off with distance in the same way as in a cubic ferromagnet with localized spins. However, in an ordinary ferromagnet, the exchange interaction produces spontaneous symmetry breaking and a transition to a magnetically ordered state in which the spin correlation function is proportional to r^{-1} at large distances. The change in magnetic symmetry under the influence of an optical pump is then energetically inconvenient and must be forced by external agencies. Nevertheless, in a spin-polarized quantum gas, magnetic exchange correlations follow the same power law r^{-1} at macroscopic distances. This slow reduction in correlations in a gas with a short-range potential is the macroscopic manifestation of the specific self-consistent field due to the quantum-mechanical refraction of particles. The final expressions for the correlation function (2.36)–(2.37) and for the magnon energy $\hbar\omega'$ (2.12) do not include the Planck constant. This is a reflection of the fact that, despite the fundamentally quantal origin of spin waves in a gas, these waves behave in the statistical sense as classical fluctuations, and may be looked upon as a spatially-inhomogeneous precession of the macroscopic magnetic moment.

The correlation between the spins of different particles over macroscopic distances gives rise to a specific macroscopic inhomogeneity energy, which is a measure of the increase in the free energy of the gas when inhomogeneities in the spatial distribution of the magnetic moment appear in it. The corresponding change in free energy due to fluctuations in transverse magnetization in an isotropic gas can be written in the form

$$\Delta F = \frac{1}{2} \int \varphi(r) \delta \mathbf{M}(\mathbf{r}_1) \delta \mathbf{M}(\mathbf{r}_2) d^3r_1 d^3r_2, \quad r = |\mathbf{r}_1 - \mathbf{r}_2|. \quad (2.38)$$

It is readily shown in the theory of classical fluctuations³⁹ that the Fourier component of the function $\varphi(r)$ and the statistical formfactor are related by

$$S_{ih}(k) = \frac{\delta_{ih} T}{\varphi(k)}. \quad (2.39)$$

The main contribution to the static structure factor for $kv_T \ll |\Omega_{\text{int}}|$ is given by (2.36), so that it follows directly that the function $\varphi(k)$ in (2.39) is given by

$$\varphi(k) = \frac{T}{2\pi|a|} \left(\frac{k}{2\beta N \alpha} \right)^2. \quad (2.40)$$

Usually, the free energy associated with slow changes in the direction of the vector \mathbf{M} along the system is written in the form of an expansion in powers of magnetization gradients. In a homogeneous isotropic gas, the first term of this expansion has the traditional form

$$\Delta F = \frac{1}{2} A \delta_{ih} \int \frac{\partial \mathbf{M}}{\partial r_i} \frac{\partial \mathbf{M}}{\partial r_h} d^3r. \quad (2.41)$$

Taking the Fourier transform of (2.41), and comparing (2.41) with (2.38), we find that

$$Ak^2 = \varphi(k), \quad A = \frac{T}{2\pi |a|} \frac{1}{(2\beta N\alpha)^2} \quad (2.42)$$

By introducing the energy of the magnetic inhomogeneity and by specifying the coefficient A in (2.41)–(2.42), we are able to provide a complete description of long-wavelength oscillations in magnetization in a purely phenomenological manner, using the linearized Landau-Lifshitz equations:

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{2\beta}{\hbar} \left[\frac{\delta}{\delta \mathbf{M}} \Delta F, \mathbf{M} \right]. \quad (2.43)$$

Actually, it is readily verified that, by substituting for ΔF from (2.41) into the linearized equation (2.43), with the coefficient A given by (2.42), we obtain the spin-wave spectrum $\omega'(\mathbf{k})$ in (2.12). This means that, despite the collisionless and possibly high-frequency ($\omega\tau \gg 1$) nature of the spin oscillations, an essentially hydrodynamic situation prevails in the long-wavelength region $kv_T \ll |\Omega_{\text{int}}|$, especially since the distribution function for these oscillations also has a hydrodynamic form, as is readily seen from (2.4) and (2.10). There is no difficulty in including the dissipative terms in (2.43), as well. These terms lead to collisional absorption in the spin-wave spectrum (2.16) and (2.25). We emphasize, however, that this method will actually produce only the linearized and not the exact macroscopic equations of spin dynamics. Since $\tilde{\epsilon}_s$ and E are related by the operation of functional differentiation, i.e., $\tilde{\epsilon}_s = \delta E / \delta n_s$, the determination of the spin-wave spectrum from the total free energy and the macroscopic equations requires the inclusion in F of the contribution due to spin fluctuations $A(\nabla \mathbf{M})^2$, but when this is introduced into the transport equation with $\tilde{\epsilon}_s$, the range of validity of the theory is exceeded.

Since the expression for $S_{ik}(k)$ given by (2.36) is strictly valid only for sufficiently small values of the wave vector, such that $kv_1 \ll |\Omega_{\text{int}}|$, Eq. (2.37) is suitable for the description of spin correlations only for large enough distances $r \gg r_{\text{int}} \equiv v_T / |\Omega_{\text{int}}|$. The characteristic parameter with the dimensions of length, r_{int} , which automatically appears in the long-wavelength expansion when collective magnetic phenomena in a gas are examined, can be interpreted as a correlation scale in the self-consistent quantal field. This scale is a measure of the difference between correlation properties of the quantum gas and point correlations (or, more precisely, correlations over atomic distances) in a classical gas. This difference is due to precisely the collective effect of quantum-mechanical refraction. Thus, the function $\varphi(k)$ is expanded into a series in even powers of kr_{int} that corresponds to an exponential reduction in the function $\varphi(r)$ at large distances ($r \gg r_{\text{int}}$), which determines the energy of the magnetic inhomogeneity. [In principle, the function $\varphi(r)$ may also acquire small power-type van der Waals "tails" at very large distances, but these will not be considered here.] Because the inequalities (2.14) and (2.15) are naturally valid for a gas, the sphere of radius r_{int} (the correlation zone) will then always contain a macroscopically large number of particles, $Nr_{\text{int}}^3 \gg 1$. This provides a qualitative explanation of long-range spin correlations in the system. There is a similar criterion for plasmas in which the Debye sphere acts as

the correlation zone and contains a large number of charged particles. This ensures that low-frequency ($\omega\tau \gg 1$) waves can propagate in plasma.

We note that the range of validity of all the formulas based on the exchange approximation and the long-wavelength expansion for which $kr_{\text{int}} \ll 1$, is bounded at the lower end by values of the wave vectors that are not too low because relativistic effects provide a contribution even when the magnetic dipole-dipole interaction is neglected. Actually, oscillations in the magnetization \mathbf{M} are accompanied by a magnetic field \mathbf{H} which can be described by the Maxwell equations. Since $v_T \ll c$, we have $\omega \ll ck$ throughout the wave vector range $kr_{\text{int}} \ll 1$, so that the magnetic field \mathbf{H} may be looked upon as quasistationary. The corresponding magnetostatic equations

$$\text{rot } \mathbf{H} = 0, \quad \text{div} (\mathbf{H} + 4\pi\mathbf{M}) = 0 \quad (2.44)$$

ensures that the spin wave has a specific relativistic anisotropy energy, i.e., the magnetostatic energy. The possibility of neglecting magnetostatic energy thus means that the relativistic gap in the spin-wave spectrum corresponds to $Ak^2 \gg 1$. If we substitute for A from (2.42), we obtain the following criterion:

$$1 \gg (kr_{\text{int}})^2 \gg \frac{r_e}{|a|} \frac{m}{m_e}, \quad (2.45)$$

where $r_e = e^2/m_e c^2 = 2.8 \times 10^{-13}$ cm is the classical radius of the electron. The inequalities given by (2.45) are written for gas particles with electron spin. On the other hand, for particles with nuclear spins, the electron mass m_e on the right-hand side of (2.45) and in r_e must be replaced with the proton mass m_p , as in the case of $^3\text{He}\uparrow$.

The introduction of a constant external magnetic field \mathbf{H} gives rise to the gap Ω_H in the spin-wave spectrum (2.12). Hence, to include the external field \mathbf{H} in all the formulas given in this section, we must replace ω with Ω_H . It is readily seen that this produces a screening of spin fluctuations at very large distances:

$$S_{ik}(r) = \delta_{ik} \frac{2|a|}{r} (\beta N\alpha)^2 \begin{cases} e^{-r/r_H}, & a < 0, \\ \cos \frac{r}{r_H}, & a > 0, \end{cases} \quad \beta H \ll T, \quad (2.46)$$

where the magnetic correlation length is on a macroscopic scale, and is given by

$$r_H^2 = \frac{1}{4\pi\alpha} \frac{T}{2\beta H} \frac{1}{N^{2/3} |a|} N^{-2/3} \gg N^{-2/3}. \quad (2.47)$$

2.6. Magnetic resonance. Theory and experiment

Spin waves are most readily detected and investigated in magnetic resonance experiments in which a measurement is made of the absorption of the energy of the alternating magnetic field. The magnetic resonance spectrum, and the absorption line shape corresponding to the excitation of standing spin waves, depend on the geometry of the resonator and the boundary conditions prevailing in the particular experimental situation. Nevertheless, certain qualitative conclusions can be deduced in the most general case whatever the specific boundary conditions may be. The energy

dissipated in the system when an alternating magnetic field propagates through it is proportional to the imaginary part of the generalized magnetic susceptibility. Inclusion of the weak collisional absorption of spin waves broadens the δ -function in the expression of $\text{Im}\chi_-(\omega, \mathbf{k})$ into the Lorentz curve. Using the well-known result

$$\lim_{\gamma \rightarrow 0} \frac{\gamma}{x^2 + \gamma^2} = \pi \delta(x) \quad (2.48)$$

and the dispersion relation (2.16), we find from (2.28) that, near the pole,

$$\text{Im}\chi_-(\omega, \mathbf{k}) = \frac{\beta^2}{|g|} \text{Im} \left(\frac{\partial D}{\partial \omega} \right)^{-1} \frac{1 - D(\omega, \mathbf{k})}{\omega + \Omega_H + (k^2 v_T^2 / \Omega_{\text{Int}}) - i\tau_{\text{coll}}^{-1}}, \quad (2.49)$$

or, finally,

$$\text{Im}\chi_+(\omega, \mathbf{k}) = \frac{\beta^2 N \alpha}{\hbar} \frac{\tau_{\text{coll}}^{-1}}{[\omega - \Omega_H - (k^2 v_T^2 / \Omega_{\text{Int}})]^2 + \tau_{\text{coll}}^{-2}}. \quad (2.50)$$

The Lorentzian (2.50) describes the line shape in the case of resonance absorption. The maximum of the Lorentzian at $\omega = \omega'(k)$ is shown by (2.12) to correspond to the maximum line intensity I :

$$I = \frac{\beta^2 N \alpha \tau_{\text{coll}}}{\hbar} \sim \frac{\beta^2 \hbar N^2 \alpha^2}{k^2 T^2 m^{1/2}}, \quad (2.51)$$

and the frequency difference between the maximum and the point at which the intensity has fallen by a factor of 2 gives the absorption linewidth:

$$\Delta\omega = \tau_{\text{coll}}^{-1} \sim \frac{m^{1/2} k^2 T^{3/2}}{\hbar^2 N \alpha^2}. \quad (2.52)$$

Thus, the absorption linewidth increases and the intensity I decreases with increasing k , i.e., increasing line number in the magnetic resonance spectrum. Moreover, it follows from (2.51)–(2.52) that, as the temperature T is reduced, or the polarization α is increased, the lines become narrower and stronger, and the observed number of lines increases. This is precisely what is observed in NMR experiments¹⁹ with polarized atomic hydrogen. These experiments have revealed the presence of fine resonance lines corresponding to the excitation of spin waves for the nuclear spin degrees of freedom, due to $|a\rangle \rightarrow |b\rangle$ transitions. The density and temperature of gaseous $\text{H}\uparrow$ in the experiments reported in Ref. 19 was varied in the ranges $N = 3 \times 10^{15} - 5 \times 10^{16} \text{ cm}^{-3}$ and $T = 0.16 \text{ K} - 0.7 \text{ K}$. A typical NMR spectrum of gaseous $\text{H}\uparrow$ is shown in Fig. 1. It is interesting to note that gas-kinetic estimates (2.51)–(2.52) indicate that neither the intensity nor the linewidth corresponding to weakly-damped spin waves depends on the parameters characterizing the interaction between the gas particles, whereas the positions of the lines in the spectrum, i.e., the resonance frequencies, are determined precisely by these parameters. The exact formulas for the line shape in the quantum temperature range (1.2) can be expressed in terms of D_0 with the aid of (2.25):

$$I = D_0 \frac{N \alpha}{\hbar} \left(\frac{\beta \Omega_{\text{Int}}}{k v_T} \right)^2, \quad \Delta\omega = \left(\frac{k v_T}{\Omega_{\text{Int}}} \right)^2 \frac{1}{D_0}. \quad (2.53)$$

Levy and Ruckenstein²⁰ used a numerical solution of the macroscopic equations of spin dynamics with specific boundary conditions to achieve good fit to the experimental

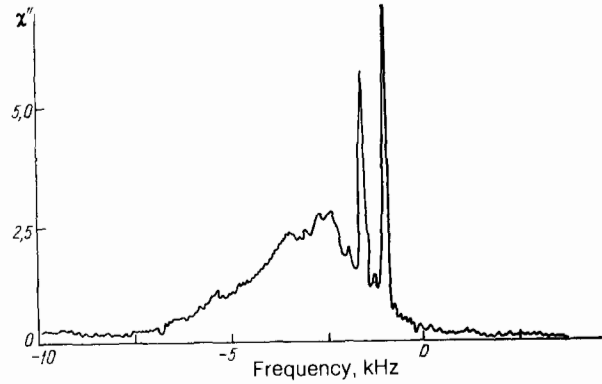


FIG. 1. Typical NMR spectrum of gaseous $\text{H}\uparrow$ based on the experimental data reported in Ref. 19 for $T = 245 \text{ mK}$, $N = 3.2 \times 10^{16} \text{ cm}^{-3}$, $\alpha \approx -1$.

results and a quantitative interpretation of the experimental data from Ref. 19.

It is readily seen from the form of the spin-wave dispersion relation (1.12) that, when $\alpha > 0$, the first spin-wave line (it is also the strongest and narrowest) lies on the “blue” wing (high-frequency end) of the NMR spectrum. Conversely, when $\alpha < 0$, the narrowest and strongest line appears on the “red” wing (low-frequency end) of the NMR spectrum. Actually, when $\alpha < 0$, the frequencies of all the spin waves with different k are greater than Ω_H , and the first standing wave with the minimum value of k (the strongest line) corresponds to the lowest frequency. The reverse situation prevails for $\alpha > 0$ (see the schematic given in Fig. 2). Hence, the experimental NMR spectrum¹⁹ shown in Fig. 1 corresponds to $\alpha > 0$, i.e., to a repulsion between the gas particles, and this is indeed the case for the interaction between two hydrogen atoms with parallel spins. The profound analogy between the phenomenological equations of motion of magnetization and the Schrödinger equation, noted in Refs. 19 and 20, leads to the same interpretation of the experimental results. It is also clear from the form of the dispersion relation that, if the direction of the spin polarization vector is reversed relative to the external magnetic field ($\vec{M} \rightarrow -\vec{M}$), i.e., if we introduce the formal replacement $\alpha \rightarrow -\alpha$, the principal (strongest) line will shift from the “blue” to the “red” wing, and the entire NMR spectrum will be mirror-reflecting. This experiment on the change in the sign of α was performed in Ref. 19 and produced the expected results (Fig. 3).

Searches for spin waves in gaseous $^3\text{H}\uparrow$ by the Paris group²¹ were complicated by the fact that, under their conditions ($T = 2-6 \text{ K}$, $\alpha = 30-50\%$), the criterion (2.5) was not well satisfied and the spin waves were known to be highly damped.³⁾ Nevertheless, they succeeded in detecting one spin wave corresponding to the first excited mode⁴⁾ in the diffusion regime, and extracted from the experimental data the magnitude of the parameter μ which plays an exceedingly important part in the formulation of the macroscopic dynamic equations. Figure 4 shows the experimental values of μ and its calculated temperature dependence.³⁰ The broken line shows the function $\mu(T)$ calculated in the s -wave approximation from (2.25), extrapolated to this temperature range.

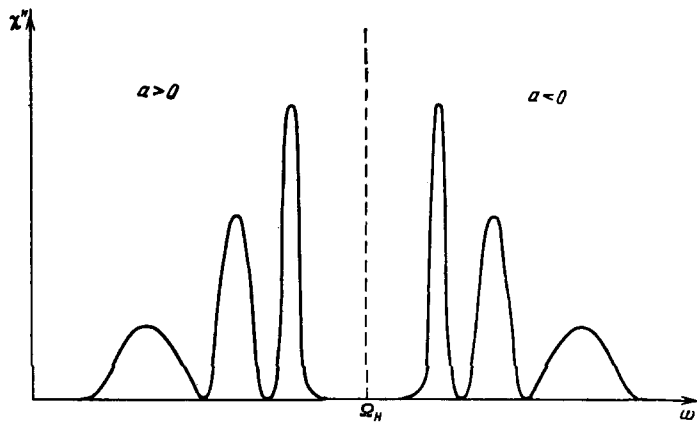


FIG. 2. Schematic showing the disposition of the main (strongest) line in the NMR absorption spectrum as a function of the sign of the scattering length. When this scheme is used to interpret the NMR spectra of gaseous H \uparrow , it must be remembered that the degree of polarization is given by $\alpha = (N_a - N_b)/(N_a + N_b)$, where N_a and N_b are the particle densities in states $|a\rangle$ and $|b\rangle$ so that, for $\alpha < 0$, the direction of the equilibrium nuclear magnetization is parallel to \mathbf{H} and we can use the spin-wave spectrum given by (2.12).

Gully and Mullin³¹ used spin echo experiments to show the presence of collective quantum-mechanical effects in a quantum Boltzmann gas of ^3He atoms dissolved in superfluid ^4He . Since, as noted in the Introduction, an external magnetic field can be used to produce appreciable polarization of the solution, condition (2.5) for weak damping of spin waves, i.e.,

$$1 \gg \text{th} \frac{\beta H}{T} \gg \frac{|a|}{\Lambda}, \quad T \gg e_d, \quad (2.54)$$

presupposes the use of strong enough but practically attainable fields for which $(\beta H / \epsilon_d) \gg N^{1/3} |a|$. The temperature range in which weakly-damped magnetization waves can propagate in nondegenerate solutions is defined by

$$e_d \left[\frac{\beta H}{e_d N^{1/3} |a|} \right]^{2/3} \gg T \gg e_d. \quad (2.55)$$

In the experiment reported in Ref. 31, the polarization of the solution was $\alpha \approx 30\%$, and this was achieved with an external magnetic field $H = 89$ kOe. The experimental data³¹ were used to determine the temperature dependence of the parameter $\Omega_{\text{int}} \tau = \mu \beta N \alpha$, which is the main quantitative measure of the self-consistent exchange field. Figure 5 shows part of the data at the highest temperatures for which quantum degeneracy of impurity ^3He quasiparticles can be completely ignored. The solid curve in Fig. 5 was calculated from (2.25) with $a = -1.5 \text{ \AA}$ and illustrates the satisfactory

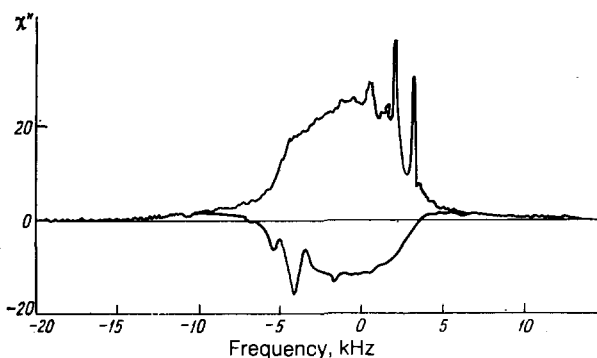


FIG. 3. Change in the NMR spectrum of gaseous H \uparrow when the spin-polarization vector is reversed in direction¹⁹ for $N = 1.5 \times 10^{16} \text{ cm}^{-3}$, $T = 246$ mK.

agreement between experimental data and the theory as applied to weak ^3He - ^4He solutions. New effects due to spin-wave processes, for example, concentrational suppression of spin waves,²² may also be expected in the region of the Fermi degeneracy in ^3He -He II solutions. However, phenomena occurring in the quantum degeneracy region are outside the scope of the present review.

Spin-polarized quantum gases placed in an external magnetic field may exhibit long-lived dissipative helicoidal superstructure.²² This occurs when, for a given a , the spin-polarization vector (produced, for example, by an optical pump) is oriented relative to \mathbf{H} ($\mathcal{M} \uparrow \uparrow \mathbf{H}$ or $\mathcal{M} \uparrow \downarrow \mathbf{H}$, depending on the sign of a) so that the two terms in the spin-wave spectrum (2.12) have different signs. When the magnetic field strength \mathbf{H} is not too high, i.e.,

$$\Omega_H \ll |\Omega_{\text{int}}|, \quad \Omega_H |\Omega_{\text{int}}| v_T^{-2} \equiv k_H^2 \ll |\Omega_{\text{int}}|^2 v_T^{-2}, \quad (2.56)$$

the spin-wave frequency is then zero at $k = k_H$ in the wave-vector range in which undamped oscillations in magnetization are still present. However, this does not signify a thermodynamic instability because we are dealing with a quasiequilibrium polarized state. This means that, over the long relativistic longitudinal relaxation time that is necessary to establish true thermodynamic equilibrium, the dynamically induced polarization will vanish altogether, and the system will assume a new state with $\mathcal{M} \uparrow \uparrow \mathbf{H}$ and polarization α determined by the external field. Nevertheless, a dissipative helicoidal superstructure with spatial period of $2\pi/k_H$ and lifetime given by

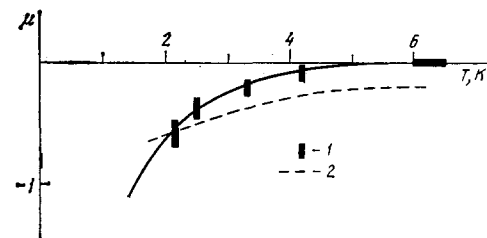


FIG. 4. Exchange parameter μ of gaseous $^3\text{He} \uparrow$ as a function of temperature. Solid curve corresponds to the theoretical calculation presented in Ref. 30; 1—experimental data,²¹ 2—calculations for s -scattering in the τ -approximation, normalized to the value of μ at $T = 2.15$ K.

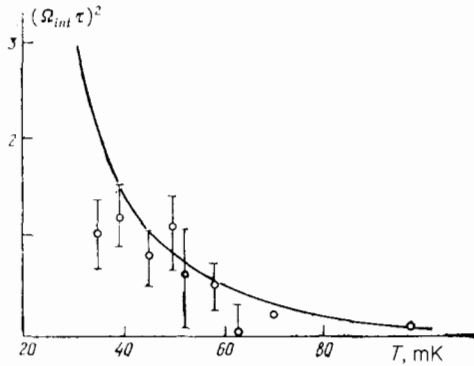


FIG. 5. Temperature dependence of the self-consistent exchange field in weak nondegenerate ${}^3\text{He}\uparrow$ - ${}^4\text{He}$ solution based on the experimental data from Ref. 32 for $H = 8.9$ T. The solid curve represents s -wave calculations.

$$t = \tau \frac{|\Omega_{\text{int}}|}{\Omega_H} \gg \tau \quad (2.57)$$

is possible.

Experiments on nuclear spin waves in gaseous $\text{H}\uparrow$ and ${}^3\text{He}\uparrow$ constitute more than simply a verification and identification of collective modes in tenuous Boltzmann systems. They actually confirm the existence of a new class of objects in the physics of condensed states, namely, the quantum gas in which macroscopic quantum phenomena occur even in the classical temperature range. The properties of such systems are only just beginning to be studied. However, all the experiments have been concerned with nuclear magnetism alone. On the other hand, there is undoubted interest in the possible experimental discovery of collective phenomena in nondegenerate systems of particles with electron spin.

3. MAGNETOMECHANICAL EFFECT. TRANSPORT PHENOMENA

Although the effect of spin polarization on the thermodynamic functions of a gas does not result in qualitatively new phenomena similar to spin waves, it does suggest the possibility of certain interesting effects that are within the reach of experimental possibility. To be specific, we shall consider gaseous ${}^3\text{He}\uparrow$, polarized by an optical pump in the absence of an external magnetic field ($\mathbf{H} = 0$). Owing to the availability of the natural small parameter $Nr_0^3 \ll 1$, all the thermodynamic functions of the gas can be written in the form of the corresponding virial expansions that are equivalent to the functional expansion into a series in powers of the ideal-gas distribution function. For binary collisions, the contribution of particle interactions to the free energy of a gas can be written in the form of (2.18). The possibility of a rigorous determination of the thermodynamic functions for arbitrary values of α is based on the following fact. In the nonrelativistic approximation, the particle interaction potential is independent of the particle spins. Moreover, in the second virial approximation that we have used, the interaction function $\Phi_{ss'}(\mathbf{p}, \mathbf{p}')$ is not a functional of the density matrix $n_s(\mathbf{p})$, as is the case in, say, a dense Fermi liquid, but is determined explicitly by the two-particle scattering amplitude in vacuum. Hence, the function $\Phi_{ss'}(\mathbf{p}, \mathbf{p}')$ does not de-

pend on the polarization α , and is equal to its value in the unpolarized gas. The spin dependence of the interaction function is entirely determined by exchange effects. The dependence of F_{int} on α is governed by the polarization dependence of the populations n in the ideal Maxwell gas, which appear in the virial expansion (2.18). Substituting for n_s from (2.6) in (2.18), we obtain

$$F_{\text{int}} = 4! \sum_{\mathbf{p}, \mathbf{p}'} [\psi(\mathbf{p}, \mathbf{p}') + \alpha^2 \zeta(\mathbf{p}, \mathbf{p}')] n_0(\mathbf{p}) n_0(\mathbf{p}'). \quad (3.1)$$

Thus, the polarization dependence of the interaction corrections to the thermodynamics of a gas of spin $\frac{1}{2}$ particles reduces to a quadratic function for arbitrary (not necessarily small!) values of α .

The total free energy of the spin-polarized gas is determined by the sum of F_{int} , given by (3.1), and the term representing the contribution of the ideal Boltzmann gas, which includes the quantum-mechanical corrections that are quadratic in N_+ and N_- . Integration in (3.1) in the light of what we have said about the total free energy yields

$$\begin{aligned} F(\alpha) &= F(0) + \frac{NT}{2} \left[\frac{2}{(18\pi)^{1/2}} \left(\frac{e_d}{T} \right)^{3/2} \alpha^2 \right. \\ &\quad \left. + \ln(1 - \alpha^2) + \alpha \ln \frac{1 + \alpha}{1 - \alpha} \right] + Y(T) N^2 \alpha^2, \\ F(0) &= -NT \ln \left[\frac{2e}{N} \left(\frac{mT}{2\pi\hbar^2} \right)^{3/2} \right] \\ &\quad + \frac{1NT}{(18\pi)^{1/2}} \left(\frac{e_d}{T} \right)^{3/2} + X(T) N^2, \\ e_d &= \frac{(3\pi^2 N)^{2/3} \hbar^2}{2m}, \\ X(T) &= (\pi m T)^{-3/2} \int \psi(\mathbf{q}) e^{-q^2/mT} d^3q, \\ Y(T) &= (\pi m T)^{-3/2} \int \zeta(\mathbf{q}) e^{-q^2/mT} d^3q. \end{aligned} \quad (3.2)$$

The dependence of α on any thermodynamic parameter of the gas can be calculated readily from (3.2). Thus, the pressure is given by

$$\begin{aligned} P(\alpha) &= NT \left[1 + \frac{1}{(18\pi)^{1/2}} \left(\frac{e_d}{T} \right)^{3/2} (1 + \alpha^2) \right. \\ &\quad \left. + \frac{X(T)}{T} N + \alpha^2 \frac{Y(T)}{T} N \right]. \end{aligned} \quad (3.3)$$

Differentiating this with respect to density, we obtain the hydrodynamic sound velocity in the gas ($\tau_s^{-1} \ll \omega \ll \tau^{-1}$):

$$s^2 = \frac{1}{m} \left(\frac{\partial P}{\partial N} \right)_{T, \alpha} = \frac{T}{m} \left(1 - 2 \frac{P - NT}{NT} \right). \quad (3.4)$$

The deviation of the paramagnetic susceptibility of the system from the Curie-Weiss law for an ideal gas can also be expressed in terms of the function $P(\alpha)$:

$$\frac{P(\alpha) - P(0)}{NT\alpha^2} = \frac{\chi_{\text{id}} - \chi}{2\chi_{\text{id}}}, \quad \chi_{\text{id}} = \frac{\beta^2 N}{T}. \quad (3.5)$$

In the quantal temperature range (1.2), in which the particle interaction in the gas reduces mostly to s -scattering, we find from (3.3) that

$$\begin{aligned} P(\alpha) &= NT \left[1 + \frac{1}{(18\pi)^{1/2}} \left(\frac{e_d}{T} \right)^{3/2} (1 + \alpha^2) + \pi (N\alpha^2) \right. \\ &\quad \left. \times \left(\frac{\Delta}{T} + 1 \right) (1 - \alpha^2) \right], \\ \Delta &= \frac{\hbar^2}{m\alpha^2}. \end{aligned} \quad (3.6)$$

It is clear from (3.3) and (3.6) that the pressure correction due to the magnetic polarization of the gas is of the order of the virial addition in the unpolarized gas, multiplied by α^2 . This means that, for large enough values of α , the magneto-mechanical effect is definitely within the range of modern experimental techniques. In the limiting case where $T \ll \hbar^2 / m r^2$ and $\alpha \rightarrow 1$, we find that

$$\frac{P(1) - NT}{P(0) - NT} = 2. \quad (3.7)$$

Magnetokinetic effects due to the enormous increase in the mean free path and in the transport coefficients when the gas is highly polarized are among the most interesting phenomena in spin-polarized Fermi gases. It was predicted in Ref. 32 for the degenerate quantum gas consisting of ^3He atoms dissolved in superfluid ^3He , and was described in detail in the reviews in Refs. 24 and 33. The physics of this phenomenon can be summarized as follows. In the case of s-scattering, which predominates in the quantal temperature range (1.2), the only contribution to the scattering cross section is that due to collisions between particles with oppositely directed spins. When the gas is completely polarized, the s-scattering cross section is therefore zero, and the main contribution to gas kinetics is due to p-scattering. The corresponding p-scattering cross section is smaller than the s-scattering cross section by the factor $(\Lambda/a)^4 \gg 1$. Consequently, when the gas is completely polarized, the mean free path l and the transport coefficients are greater by the same factor. It is quite obvious that this effect is in no way related to the form of the particle distribution function (which can be either the Fermi or the Maxwell function, i.e., the gas can be degenerate or nondegenerate). It can be fully explained in terms of the quantum-mechanical scattering of slow Fermi particles for $|a|/\Lambda \ll 1$. A detailed discussion of transport phenomena in a nondegenerate spin-polarized gas in the hydrodynamic approximation ($l \ll d$, where d is the typical geometric size of the gas-containing vessel) is given in Refs. 16 and 34.

We shall now examine some of the features of transport phenomena under Knudsen conditions, which may turn out to be relevant in the study of the transport properties of ^3He - ^4He solutions and gaseous $^3\text{He}\uparrow$ at the lowest possible temperatures and densities. Actually, for an unpolarized 0.01% ^3He - ^4He II solution, the mean free path is $l \sim 10^{-3}$ cm. Hence, when the solution is completely polarized at $T \sim 10$ mK $\gtrsim \varepsilon_d$ (whatever the method used to polarize it), the value of l rises to something of the order of 10 cm, and this is greater than any reasonable linear size of ordinary experimental cells. In this situation, the transport coefficients are determined not by the particle interaction but by the geometric shape and size of the vessel containing the gas. In the plane-parallel geometry, and if we adopt the very crude assumption that there is complete accommodation, the viscosity η and thermal conductivity κ are given by the well-known expressions³⁵

$$\eta = Nd \left(\frac{mT}{2\pi} \right)^{1/2}, \quad \kappa = 2Nd \left(\frac{T}{2\pi m} \right)^{1/2}. \quad (3.8)$$

Using the first approximation of the Chapman-Enskog method for $1 - \alpha \ll 1$ in the hydrodynamic region¹⁶

$$\eta(\alpha) = \frac{4\eta(0)}{5(1-\alpha)}, \quad \kappa(\alpha) = \frac{50\kappa(0)}{86(1-\alpha)}, \quad (3.9)$$

and the well-known expressions for $\kappa(0)$ and $\eta(0)$ given in Ref. 35, we can readily determine the polarization α_c corresponding to the transition from the hydrodynamic to the Knudsen regime when the characteristic geometric size d is such that

$$\frac{1}{Na^2} \ll d \ll \frac{1}{Na^2} \left(\frac{\Lambda}{a} \right)^4. \quad (3.10)$$

The value of α_c is given by

$$1 - \alpha_c = \frac{\sqrt{2}}{2^{10}} \frac{4425}{43} \frac{1}{Nda^2} \approx \frac{0.14}{Nda^2}, \quad (3.11)$$

in the case of the thermal conductivity, and by

$$1 - \alpha_c = \frac{\sqrt{2}}{16} \frac{1}{Nda^2} \approx \frac{0.09}{Nda^2} \quad (3.12)$$

in the case of viscosity. It is clear that the transition to the Knudsen regime in the spin-polarized gas becomes even more important in the case of Fermi degeneracy because, even when $\alpha = 0$, the mean free path then contains the additional large factor $(\varepsilon_d/T)^2 \gg 1$. For degenerate ^3He - ^4He solutions, the corresponding calculations were reported in Ref. 36. The temperature dependence of viscosity and thermal conductivity under the conditions defined by (3.10) is shown schematically in Figs. 6 and 7 for $\alpha < \alpha_c$ and $\alpha > \alpha_c$. Of course, the Knudsen regime can, in principle, be implemented even in an unpolarized gas ($\alpha = 0$) if the capillary is narrow enough, or the particle density high enough, so that $d \ll l$ from the very beginning. However, the greatest interest attaches to the transition from the hydrodynamic to the Knudsen regime under the influence of spin polarization.

We must also note the interesting possibility of identifying the Knudsen regime in the ^3He - ^4He solution by performing thermodynamic instead of transport measurements.³⁶ When $d \ll l$, the ^3He quasiparticles become "stuck" in the capillary, and this gives rise to a thermomechanical effect, i.e., to a pressure gradient at the end of the capillary when a temperature gradient is applied. A qualitative measure of the effect can readily be established with the aid of (3.6):

$$\frac{\Delta P}{\Delta T} = S = N \ln \left[\frac{e}{N} \left(\frac{m^* T}{2\pi \hbar^2} \right)^{3/2} \right] + \frac{3}{2} N, \quad \alpha \rightarrow 1, \quad (3.13)$$

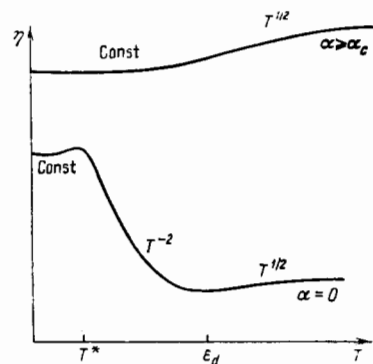


FIG. 6. Schematic temperature dependence of the viscosity of a Fermi gas under the conditions of restricted geometry for different values of polarization. $\eta(\alpha_c)/\eta(0) \sim Na^2 d \gg 1$ when $T \gg \varepsilon_d$, and $\eta(\alpha_c)/\eta(0) \approx 2^{4/3}$ when $T \ll T^*$, where $T^* = (Na^2 d)^{-1/2} \varepsilon_d$.

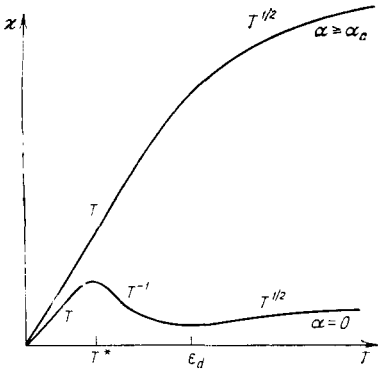


FIG. 7. Schematic temperature dependence of the thermal conductivity of a Fermi gas under the conditions of restricted geometry for different values of α . $\kappa(\alpha_c)/\kappa(0) \sim Na^2 d \gg 1$ when $T \gg \epsilon_d$, and $\kappa(\alpha_c)/\kappa(0) \approx 2^{2/3}$ when $T \ll T^*$.

where it is assumed that the temperature is low enough for the phonon and roton contributions to the entropy to be negligible.

4. PARAMAGNETIC RESONANCE IN BINARY GASES

It is well-known that the paramagnetic resonance frequency of a Fermi liquid placed in an external magnetic field does not depend on parameters characterizing the interaction between the particles of the liquid and, as in the case of the ideal gas, is exclusively determined by the gyromagnetic ratio.³⁷ The same conclusion is valid for an interacting Boltzmann gas. However, the situation is radically altered in the case of two-component (multicomponent) systems consisting of different particles or of molecules of the same material but in different internal states. The resonance frequencies are then functions both of the interaction between the particles belonging to the different components and of the state of the system, i.e., they depend on the temperature and the partial pressures of the two components. If, on the other hand, the gyromagnetic ratios of the molecules of the two components are equal, the exchange interaction between the subsystems leads to the appearance of a new resonance frequency which corresponds to the additional degree of freedom in the two-component magnet, all the particles of which have the same magnetic moments.

If the magnetic field is not too strong, so that $\hbar\gamma_{1,2}H \ll T$, where $\gamma_{1,2}$ are the gyromagnetic ratios of particles belonging to the first and second components, respectively, the free-energy density F in the exchange approximation for the uniformly magnetized two-component system is given by the phenomenological expansion

$$F = \frac{M_1^2}{2\chi_{11}} + \frac{M_2^2}{2\chi_{22}} + \frac{M_1 M_2}{\chi_{12}} - (M_1 + M_2)H - \frac{H^2}{8\pi}. \quad (4.1)$$

We can then use the Landau-Lifshitz equations and take into account the small gas parameter $N_k r_0^3 \ll 1$, $k = 1, 2$ to deduce from (4.1) the frequencies of spatially uniform oscillations in magnetization, i.e., the spectrum of the magnetic resonance frequencies¹

$$\omega_1 = \gamma_1 H \left(1 - \frac{\chi_{22}^{(0)}}{\chi_{12}} \right), \quad \omega_2 = \gamma_2 H \left(1 - \frac{\chi_{11}^{(0)}}{\chi_{12}} \right), \quad (4.2)$$

when $\gamma_1 \neq \gamma_2$, or

$$\omega_1 = \gamma H, \quad \omega_2 = \gamma H \left(1 - \frac{\chi_{11}^{(0)} + \chi_{22}^{(0)}}{\chi_{12}} \right), \quad (4.3)$$

when $\gamma_1 = \gamma_2 \equiv \gamma$, where $\chi_{11}^{(0)}$ and $\chi_{22}^{(0)}$ are the static paramagnetic susceptibilities of the two components in the ideal gas approximation, i.e.,

$$\chi_{kk}^{(0)} = \frac{\gamma_k^2}{T} \frac{S_k(S_k+1)}{3} N_k, \quad k = 1, 2, \quad (4.4)$$

where N_k and S_k are, respectively, the number of molecules per unit volume and the spin of the molecule of the k th component. For the resonance solution given by (4.3) with $\gamma_1 = \gamma_2$, the oscillations of the first type (frequency ω_1) constitute the usual precession of the total magnetic moment of the system $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ around the applied magnetic field with the Larmor frequency γH . In oscillations of the second type, on the other hand, the total magnetization \mathbf{M} is conserved and does not oscillate although the magnetic moments of each of the components, \mathbf{M}_1 and \mathbf{M}_2 , do execute periodic motions with frequency ω_2 given by (4.3). The relative motion of the vectors \mathbf{M}_1 and \mathbf{M}_2 occurs so as to maintain $\mathbf{M} = \text{const}$. The thermodynamic virial expansions can then be used to obtain the exact expressions relating the magnetic virial cross coefficient (and, hence, the resonance frequencies) and the two-particle scattering amplitude of the binary interaction.¹ When the spin of the particles belonging to one of the components is $S_2 = 1/2$, these expressions are

$$\chi_{12}^{-1} = \frac{1}{\gamma_1 \gamma_2} \frac{1}{(2\pi m_{12} T)^{3/2}} \int \xi(\mathbf{q}) e^{-q^2/2m_{12}T} d^3q \equiv \frac{Y_{12}(T)}{\gamma_1 \gamma_2},$$

$$\xi(\mathbf{q}) = \frac{2}{2S_1+1} (A_+ - A_-), \quad m_{12} = \frac{m_1 m_2}{m_1 + m_2}, \quad (4.5)$$

where the functions $A_{\pm}(\mathbf{q})$ are given by (2.18) with m replaced by $2m_{12}$ and $f_{\pm}(\theta, q)$ is the scattering amplitude when the resultant spin of the pair of colliding particles is $S_1 \pm 1/2$. Thus, when $S_1 \neq 1/2$, we finally have

$$\omega_1 = \gamma_1 H \left[1 - \frac{\gamma_2}{4\gamma_1} \frac{Y_{12}(T)}{T} N_2 \right],$$

$$\omega_2 = \gamma_2 H \left[1 - \frac{S_1(S_1+1)}{3} \frac{\gamma_1}{\gamma_2} \frac{Y_{12}(T)}{T} N_1 \right], \quad (4.6)$$

and when $S_1 = 1/2$, i.e., $\gamma_1 = \gamma_2 \equiv \gamma$, we have

$$\omega_1 = \gamma H, \quad \omega_2 = \gamma H \left[1 - \frac{Y_{12}(T)}{4T} N \right], \quad N = N_1 + N_2. \quad (4.7)$$

At low temperatures $T \ll \hbar^2/mr_0^2$, the function $Y_{12}(T)$ is a slowly-varying function of temperature:

$$Y_{12}(T) = \frac{2}{2S_1+1} \frac{2\pi\hbar^2}{m_{12}} (a_+ - a_-)$$

$$\times \left[1 + \frac{2m_{12}T}{\hbar^2} (a_+^2 + a_-^2 + a_+ a_-) \right], \quad (4.8)$$

and the shift of the resonance frequencies in the leading approximation is inversely proportional to T :

$$\frac{\chi_{kk}^{(0)}}{\chi_{12}} = \frac{2\pi}{3} S_k(S_k+1) \frac{\gamma_k^2}{\gamma_1 \gamma_2} \left(\frac{\Lambda_{12}}{a_2} \right)^2 N_k a_2^3,$$

$$\Lambda_{12} = \frac{\hbar}{m_{12} v_T} = \frac{\hbar}{(m_{12} T)^{1/2}}, \quad a_2 = \frac{2(a_+ - a_-)}{2S_1+1}. \quad (4.9)$$

When the transport virial corrections in the spin commuta-

tor are taken into account, the solution of the system of transport equations (2.20) with $\mathbf{k} = 0$ for each of the components of the binary mixture again leads to the homogeneous paramagnetic resonance frequencies¹ given by (4.6) and (4.7).

We note that, when $\gamma_1 = \gamma_2 \equiv \gamma$, the formulas given by (4.7) and (4.9) do not contain any parameters that distinguish one type of particle from another, such that the effect would vanish if these parameters were allowed to tend to zero. Nevertheless, the above expressions are invalid when the two components are identical (one-component gas), and the effect disappears. The point is that there is no group of parameters that would allow us to perform a continuous physical transition from different to identical particles. Particles are always identical or always different (however small the change in the parameters), and the transition can only occur discontinuously. In this sense, the situation is similar to that encountered in the well-known Gibbs paradox involving the entropy of a mixture of two identical gases.

There is a relatively wide range of gases whose molecules are paramagnetic even in the ground state. Among inorganic substances, there are the well-known O_2 (ground-state term $^3\Sigma$), NO (ground-state term $^2\Pi$), and the triatomic gases NO_2 and ClO_2 with total spin $S = 1/2$. Recent experimental work has shown that spin-polarized atomic hydrogen $H\uparrow$ can be described as a long-lived, structurally stable, paramagnetic gas. There is also a whole series of gaseous nuclear paramagnets, for example, 3He , HD , $o\text{-}H_2$ (ortho-hydrogen), and $p\text{-}D_2$ (paradeuterium). The above formulas are valid, at least in principle, for all these gases. However, since the quantum-mechanical corrections to the free energy of the gas decrease with increasing temperature, all the above effects are particularly appreciable in gases that condense at the lowest possible temperature and, at the same time, have an appreciable saturated vapor pressure. Moreover, the rotational quantum Δ_{rot} of the molecules of some of these gases is found to be considerably smaller than the boiling temperature (even at the very lowest pressures), so that the rotational degrees of freedom of the molecules will be definitely excited, which gives rise to strong coupling between the electron magnetic moment and the angular momentum of the molecule. This, in turn, gives rise to a considerable complication of the paramagnetic resonance picture and of the identification of the phenomena discussed above. For example, $\Delta_{rot}(O_2) = 2.1\text{ K}$ and $\Delta_{rot}(NO) = 2.4\text{ K}$.

Thus, for a clean observation of the predicted effects, it is convenient to use different paramagnetic isotopes of hydrogen whose molecules are characterized by a large rotational quantum, much greater than the boiling temperature: $\Delta_{rot}(H_2) = 85.4\text{ K}$, $\Delta_{rot}(D_2) = 43\text{ K}$, and $\Delta_{rot}(HD) = 64\text{ K}$. However, since the hydrogen isotopes enumerated above and 3He are nuclear paramagnets, and the range of nuclear forces is much smaller than the dimensions of the molecule, there is very little overlap between the nuclear wave functions during collisions between different molecules, and the corresponding exchange interaction is utterly negligible. A totally different situation occurs when the colliding molecules include identical atoms. The exchange of complete

atoms rather than nuclei is then possible in a reaction zone of the order of the molecular dimensions. The order of magnitude of the exchange scattering amplitude is then determined by the size of the molecules, i.e., its scale is atomic rather than nuclear. This situation can occur, for example, in two-component mixtures of $o\text{-}H_2$ and HD (identical H atoms) or $p\text{-}D_2$ and HD (identical D atoms). The saturated vapor pressure of these materials at $T \sim 10\text{ K}$ corresponds to the concentration $N_k \sim 10^{18}\text{--}10^{19}\text{ cm}^{-3}$, so that estimates based on (4.9) for $|a_2| \sim 1\text{ \AA}$ lead to the following paramagnetic resonance line shift: $|\Delta\omega_k|/\gamma_k H \sim 10^{-5}\text{--}10^{-4}$.

Another interesting system that may exhibit analogous effects is spin-polarized atomic hydrogen ($H\uparrow$) because, after stabilization in the magnetic field, the hyperfine interaction ensures that the gas is a mixture of atoms in different spin states, namely, $|a\rangle = |\uparrow\downarrow\rangle - \eta|\downarrow\uparrow\rangle$, $|b\rangle = |\uparrow\uparrow\rangle$, i.e., $H\uparrow$ may be looked upon as a peculiar binary gas. Two magnetic dipole transitions are then allowed, namely, $|a\rangle \rightarrow |d\rangle$ and $|b\rangle \rightarrow |c\rangle$, where $|d\rangle = |\downarrow\downarrow\rangle$, $|c\rangle = |\downarrow\uparrow\rangle + \eta|\uparrow\downarrow\rangle$. The resonance line corresponding to the $|b\rangle \rightarrow |c\rangle$ transition has been observed experimentally.³⁸ The same atoms in ground and excited states can play the part of different particles of the binary gas, and this actually occurs in gas-discharge devices.

Quantitatively, the resonance-frequency shift is particularly large when one of the components of the binary system is a simple gas of electrons whose mass m_e is smaller by three orders of magnitude than that of the atom. Here, we nearly always encounter the ultraquantal condition $T \ll \hbar^2/mr_0^2$, and quantitative descriptions can be based on (4.8) and (4.9) in the limit of low-energy scattering. Such systems will be examined in the next section.

5. BINARY QUANTUM GASES WITH AN ELECTRON COMPONENT

As already noted, when one of the components of a binary gas is a light electron gas, the system may be expected to exhibit both an increase in the quantitative measure of collective effects and a considerable expansion of the corresponding temperature range, as well as the appearance of qualitatively new phenomena. A typical example of this kind of multicomponent quantum gas is cold, weakly-ionized plasma containing electrons, ions, and neutrals.

5.1. Ferromagnetism and charge-density waves in a weakly-ionized Maxwell gas

It is commonly considered that long-range magnetic order in a system of particles with a short interaction range requires either a high particle density, so that the separation between the particles is comparable with the exchange interaction range, or a low enough temperature, much lower than the Fermi degeneracy temperature, so that the system includes long-range interactions such as those of Ruderman, Kittel, Kasuya, and Yosida. Our assertion is that the quantal virial corrections describing the interaction between the particles in a tenuous multicomponent gas containing a light component can lead to a spontaneous phase transition and to the appearance of a magnetic or structural order even at high

temperatures, well away from the quantum degeneracy temperature.¹⁴

It is readily seen that, owing to the availability of the small gas parameter $Nr_0^3 \ll 1$, the paramagnetic state of a one-component gas is stable at any temperature, both for $T \gtrsim \varepsilon_d$ as well as for $T \lesssim \varepsilon_d$. The situation is totally different in the weakly-ionized gas containing neutrals, electrons, and ions because of the availability of a further small parameter, namely, $m_e/m_n \ll 1$, where m_e is the mass of the electron and m_n the mass of the neutral. The interplay between these two small parameters can lead to an unstable paramagnetic state of the weakly-ionized gas in the classical temperature range. To be specific, we consider the case where the neutral has electron spin 1/2 and the plasma ions have no magnetic moment. Since typical values of the ionization potential I are lower than or comparable with the quantum-mechanical energy $\hbar^2/m_e r_0^2$, the condition for weak ionization of the gas, $T < I$, implies that the condition $T \ll \hbar^2/m_e r_0^2$ has been satisfied. At such temperatures the virial additions to the free energy of the weakly-ionized polarized gas, which are due to the interaction between electrons and neutrals, can be satisfactorily calculated in the s -wave approximation¹⁴:

$$F_{\text{on}}^{(\text{int})} = \frac{\pi \hbar^2 N_e N_n}{2m_e} (a_1 + a_2 \alpha_e \alpha_n) \vec{\mathfrak{M}}_e \vec{\mathfrak{M}}_n, \quad (5.1)$$

where N_e, N_n are the electron and neutral densities, α_e, α_n are the polarizations of the two components, $\vec{\mathfrak{M}}_e, \vec{\mathfrak{M}}_n$ are unit vectors in the direction of the magnetization for electrons and neutrals ($a_1 = 3a_+ + a_-, a_2 = a_+ - a_-$), and a_+ and a_- are the triplet and singlet s -scattering lengths, respectively. The total free energy of the ionized gas with polarized components can then be written in the form

$$F(\alpha_e, \alpha_n) = F(0, 0) + \frac{T}{2} [N_e Z(\alpha_e) + N_n Z(\alpha_n)] + \frac{\pi \hbar^2 a_2 N_e N_n}{2m_e} \alpha_e \alpha_n \vec{\mathfrak{M}}_e \vec{\mathfrak{M}}_n, \quad (5.2)$$

$$Z(\alpha) = \ln(1 - \alpha^2) + \alpha \ln \frac{1 + \alpha}{1 - \alpha},$$

where the terms $Z(\alpha_e)$ and $Z(\alpha_n)$ describe the contributions of the ideal gas of electrons and neutrals, respectively. Virial corrections due to the scattering of neutrals by neutrals and the weak electron-electron exchange corrections will be neglected because the retention of these discarded terms would produce additional terms proportional to $N_n r_0^3 \ll 1$ and $e^2 N_e^{1/3} / T \ll 1$ in the final expressions. Moreover, virial corrections due to the mutual scattering of neutrals are smaller by the factor $m_n/m_e \gg 1$ than the terms in (5.1). The subsequent procedure consists of minimizing the expression for $F(\alpha_e, \alpha_n)$ in (5.2). Minimization of free energy with respect to the angle θ between the magnetizations $\vec{\mathfrak{M}}_e$ and $\vec{\mathfrak{M}}_n$ shows that, when $a_2 < 0$, we have the ferromagnetic structure $\vec{\mathfrak{M}}_e \cdot \vec{\mathfrak{M}}_n = 1$, i.e., $\theta = 0$, whereas, for $a_2 > 0$, we have ferrimagnetic-type order with $\vec{\mathfrak{M}}_e \cdot \vec{\mathfrak{M}}_n = -1$, i.e., $\theta = \pi$. The conditions for minima, $\partial F / \partial \alpha_e = 0$ and $\partial F / \partial \alpha_n = 0$, are equivalent to the set of equations

$$\begin{cases} \operatorname{arctg} \alpha_n - \frac{\alpha_e \gamma_{2e} E_2}{T} = 0, & E_2 = -\frac{\pi \hbar^2}{2m_e a_2^2}, \\ \operatorname{arctg} \alpha_e - \frac{\alpha_n \gamma_{2n} E_2}{T} = 0, & \gamma_{2e} = N_e |a_2|^3, \quad \gamma_{2n} = N_n |a_2|^3. \end{cases} \quad (5.3)$$

The consistency condition for the equations in (5.3) defines the magnetic phase transition temperature¹⁴

$$T_{c2} = \frac{\pi \hbar^2}{2m_e a_2^2} (N_e |a_2|^3)^{1/2} (N_n |a_2|^3)^{1/2}. \quad (5.4)$$

The range of validity of Boltzmann statistics, used to calculate T_{c2} from (5.4), i.e., the conditions $T_{c2} \gg \varepsilon_d^{(e)}, \varepsilon_d^{(n)}$, corresponds to the following inequalities for the electron concentration:

$$1 \gg N_n |a_2|^3 \gg \left(\frac{N_e}{N_n}\right)^{1/2} \gg \frac{m_e}{m_n} (N_n^{1/3} |a_2|)^{-1}. \quad (5.5)$$

It is clear from (5.5) that it is precisely the presence of the small parameter $m_e/m_n \ll 1$ that ensures the possibility, in principle, of the magnetic phase transition in the classical temperature range, provided the neutral-particle concentration is high enough:

$$1 \gg (N_n^{1/3} |a_2|)^4 \gg \frac{m_e}{m_n}. \quad (5.6)$$

Using (5.3) with (5.5), we obtain the equilibrium polarizations of the two components. Near the transition point, these polarizations turn out to be

$$\alpha_e = (6\tau)^{1/2}, \quad \alpha_n = \left(\frac{N_e}{N_n}\right)^{1/2} \alpha_e \ll \alpha_e, \quad \tau = \frac{T_{c2} - T}{T_{c2}} \ll 1. \quad (5.7)$$

The expression given by (5.4) is essentially the equation defining the phase transition point because the electron concentration N_e and the neutral concentration N_n in the plasma, which appear in (5.4), are themselves functions of temperature. If we use the Saha formula to calculate N_e and N_n , it is difficult to satisfy (5.4)–(5.6) at the same time. Nevertheless, under suitably chosen conditions, a solution of (5.4) subject to (5.5) and (5.6) turns out to be possible. All the same, a more convenient system for the observation of magnetic phase transitions in the classical temperature range is a nondegenerate semimagnetic semiconductor, which will be considered in the next section. We also note that the uniformly magnetized state of a quantum Maxwell gas may turn out to be unstable against a transition to a spatially inhomogeneous phase. However, this question would take us outside the framework of the present review.

A further interesting quantum phenomenon in a weakly-ionized gas is due to the structural phase transition and the appearance in the system of a spatially-periodic superstructure. This phenomenon involves not only the interaction between electrons and neutrals, but also the Coulomb interaction between charged particles, as well as the presence of the ion component in the gas. The physics of this effect is as follows. If the particles did not have an electric charge, the conditions for the thermodynamic stability of the system against stratification, or a spontaneous change in the density of the components at $T = \text{const}$, could be reduced to the inequality

$$\det \left\| \frac{\partial^2 F}{\partial N_k \partial N_l} \right\| = \det \| A_{kl} \| > 0 \quad (k, l = e, n, i). \quad (5.8)$$

We shall use the index i to label quantities referring to the ion component. To keep the final expressions as simple as possible, we shall, as before, neglect the virial corrections due to

the scattering of neutrals and ions, which we are entitled to do because of the large mass of the particles. From (5.8), (5.1), and (3.2), it then readily follows that, for $T < T_{c1}$, where

$$T_{c1} = \frac{\pi \hbar^2}{2m_e a_1^2} (N_e |a_1|^3)^{1/2} (N_n |a_1|^3)^{1/2}, \quad (5.9)$$

the system becomes thermodynamically unstable, and the electron component and neutral particles cannot coexist in the same mixture. In the real situation, electrons and ions are charged, so that the spontaneous separation of the electron and neutral components for $T < T_{c1}$, which is accompanied by a reduction in that part of the free energy that does not depend on the particle charge and the Coulomb interaction between the particles, gives rise to a departure from the overall electrical neutrality of the plasma in regions of macroscopic size. This type of redistribution of electrons affects the macroscopic homogeneity of the plasma. It is accompanied by the appearance of a macroscopic electric field and produces an increase in the Coulomb correlation energy of electrons and ions. These two competing mechanisms are responsible for the phase transition involving the appearance of a spatially periodic structure in the distribution of the electric field and of the densities of all the gas components, i.e., charge-density waves.⁵⁾

Thus, when charged components are present, the conditions given by (5.8) must also take into account the above macroscopic electric field. The criterion for thermodynamic equilibrium then reduces to the condition for constant chemical potentials of the gas components in the electric field, which must be augmented by the self-consistent Poisson equation for this field:¹⁴

$$\begin{aligned} A_{ee} \delta N_e + A_{e1} \delta N_1 + A_{en} \delta N_n &= e \varphi, \\ A_{ne} \delta N_e + A_{n1} \delta N_1 + A_{nn} \delta N_n &= 0, \\ A_{ie} \delta N_e + A_{i1} \delta N_1 + A_{in} \delta N_n &= -ze \varphi, \\ \Delta \varphi + 4\pi e (z \delta N_1 - \delta N_e) &= 0, \end{aligned} \quad (5.10)$$

where φ is the electric potential and ze and $(-e)$ are the ion and electron charges, respectively, so that $N_e = zN_1$. Eliminating δN_k , $k = i, n, e$ from (5.10), and calculating A_{kl} from (5.1) and (3.2), we obtain

$$\Delta \varphi + [d_1^{-2} + d_e^{-2} (1 - G)^{-1}] \varphi = 0, \quad G = N_e N_n \left(\frac{\pi \hbar^2 a_1}{2m_e T} \right)^2, \quad (5.11)$$

where the Debye lengths of ions and electrons are respectively given by

$$d_e = \left(\frac{T}{4\pi N_e e^2} \right)^{1/2}, \quad d_1 = \left(\frac{T}{4\pi N_1 z^2 e^2} \right)^{1/2}. \quad (5.12)$$

Equation (5.11) determines the screening of the electrostatic field in a weakly-ionized gas. The corresponding screening length D is given by

$$D^{-2} = d_1^{-2} + d_1^{-2} (1 - G)^{-1}. \quad (5.13)$$

Within the temperature interval

$$T_{c1}^* < T < T_{c1}, \quad T_{c1}^* = T_{c1} (1 + z^{-1})^{-1} \quad (5.14)$$

we have $D^{-2} < 0$ and Eq. (5.11) has an oscillatory solution corresponding to charge-density waves of wavelength $\lambda_s = 2\pi |D|^{-1}$ whose value is determined by (5.13). This means that, in the range $T_{c1}^* < T < T_{c1}$, a new spatially periodic phase appears in the weakly-ionized gas and, at the points $T = T_{c1}^*$ and $T = T_{c1}$, a phase transition actually takes place between the state in which correlations between the particles decrease exponentially with distance and the state in which the single-time correlation function oscillates and falls off with distance in accordance with a power-type law, which is a macroscopic manifestation of the appearance of long-range order in the system. As the temperature is varied from T_{c1}^* to T_{c1} , the period λ_s of the superstructure undergoes a formal variation from ∞ to 0. Of course, a charge-density wave is physically meaningful only when λ_s has the macroscopic scale defined by $\lambda_s N_k^{1/3} \gg 1$, $k = e, n, i$. The formulas given by (5.11)–(5.14) and (5.9) are readily generalized to the case of nonisothermal plasma, and Eq. (5.11) can also be deduced within the framework of the traditional approach to the Debye length.¹⁴

When $T \ll \hbar^2 / m_e r_0^2$, the interaction between electrons and neutrals ensures that there are quantum kinematic corrections which must be taken into account in the transport equation together with the self-consistent Coulomb field. The corresponding set of equations¹⁴ enables us to determine the corrections to the spectra of all the high-frequency oscillations in the weakly-ionized plasma. Here, we reproduce only the expression for the longitudinal permittivity:

$$\begin{aligned} \varepsilon(\omega, \mathbf{k}) &= (1 - GF_e F_n)^{-1} \{1 + (kd_e)^{-2} F_e + (kd_1)^{-2} F_1 \\ &\quad - GF_e F_n [1 + (kd_1)^{-2} F_1]\}, \\ F_m &= 1 + \frac{\chi_m}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{e^{-z^2} dz}{z - x_m - i0}, \\ x_m &= \frac{\omega}{\sqrt{2} kv_{Tm}} \quad (m = e, n, i). \end{aligned} \quad (5.15)$$

In the static limit $\omega \ll kv_{Te}, kv_{Ti}, kv_{Tn}$, we find from (5.15) that

$$\begin{aligned} \varepsilon(0, \mathbf{k}) &= (1 - G)^{-1} \{1 + (kd_e)^{-2} + (kd_1)^{-2} \\ &\quad - G[1 + (kd_1)^{-2}]\} = 1 + (kD)^{-2}. \end{aligned} \quad (5.16)$$

The solution of the equation $\varepsilon(0, \mathbf{k}) = 0$ is in complete agreement with the thermodynamic formulas given by (5.11)–(5.13). When (5.14) is satisfied, this equation has a real solution $k^2 > 0$, $k^2 = -D^{-2}$, which corresponds to the structural phase transition to a state with charge-density waves. Of course, this is accompanied by a change in the character of the Debye screening. Thus, the field due to an extraneous test charge e_1 in a weakly-ionized gas is given by

$$\varphi(r) = \int \frac{4\pi e_1 e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2 \varepsilon(0, \mathbf{k})} \frac{d^3 k}{(2\pi)^3} = \frac{e_1}{r} \cdot \begin{cases} e^{-r|D|}, & D^2 > 0, \\ \cos \frac{r}{|D|}, & D^2 < 0. \end{cases} \quad (5.17)$$

In spin-polarized plasma at $T \ll \hbar^2 / m_e r_0^2$, the propagation of weakly-damped waves is possible in principle as in any other

quantum gas. Here, we shall confine our attention to the solution for the homogeneous paramagnetic resonance with $k = 0$ in an external field:^{1,14}

$$\omega_1 = \frac{2\beta H}{\hbar}, \quad \omega_2 = \omega_1 - \frac{2\pi a_d \hbar}{m_e} (N_e + N_d) \operatorname{th} \frac{\beta H}{T}. \quad (5.18)$$

In the case of weakly-ionized gaseous Cs at $T \sim 1000$ K, $N_1 \sim N_2 \sim 10^{16} - 10^{17} \text{ cm}^{-3}$, $|a_2| \sim 10 \text{ \AA}$, the additional resonance frequency ω_2 satisfies the relation $|\omega_2 - \omega_1|/\omega_1 \sim 10^{-5} - 10^{-4}$, which is definitely within the range of experimental possibility.

As we have seen, there is a great variety of collective quantum-mechanical phenomena in binary gases containing a light electron component, and such phenomena occur in a much broader temperature range than in the case of single-component systems. Nevertheless, experimentally, they are much more difficult to identify because of the presence of various masking effects at the higher temperatures. Moreover, a number of specific difficulties is encountered, for example, in the production of sufficient densities of electrons and neutrals in the ionized gas, or electrons and point defects in semiconductors. However, it is worth overcoming these difficulties in order to be able to observe macroscopic quantum-mechanical effects at room temperature.

5.2. Semimagnetic semiconductors

The effects discussed above are probably much easier to detect experimentally in semimagnetic semiconductors. Semiconductors in which the nonmagnetic host lattice contains magnetic point defects, with low enough concentration for the dipole-dipole interaction between the spins of the defects to be neglected, are often referred to as semimagnetic semiconductors. Because the separation between the defects is much greater than the lattice constant, the behavior of the defects (from the magnetic point of view) is then similar to that of atoms in an ideal gas with susceptibility obeying the Curie law. The magnetic defects can take the form of unionized impurity levels or impurity atoms of rare-earth, actinide, and certain transition elements with uncompensated electron spins in the f- and d-shells. No special assumptions are necessary to calculate the contribution of collisions between electrons and defects to all the thermodynamic functions. The corresponding terms in the total free energy of the system can be expressed in terms of the exact electron-defect scattering amplitude. The concentrations of electrons and defects are the natural small parameters of the problem. It follows that, although the interaction between electrons and defects can in no way be regarded as weak, all the thermodynamic quantities can be obtained in the form of virial expansions in powers of the small concentrations, i.e., in the form of a thermodynamic perturbation-theory series. Owing to exchange effects, the amplitude for the scattering of an electron by a paramagnetic impurity depends on the resultant spin of the electron and defect. Exchange effects in the scattering of conduction electrons (or holes) by localized point defects lead to the possibility of magnetic phase transitions and a variety of high-frequency phenomena in the classical temperature range.¹⁴

We shall now consider a nondegenerate semiconductor in which the host lattice has cubic symmetry, so that the electron energy spectrum has the simple form $\varepsilon = p^2/2m^*$, where m^* is the effective scalar mass. If we are not interested in the structure of the disposition of defects in the lattice, they can be described by a stochastic distribution of particles with infinite masses. Averaging over the positions of the impurities is then equivalent to taking the limit of the usual gas formulas as the mass of the defect tends to infinity. The electron-electron exchange interaction in the nondegenerate semiconductor will also be neglected. Of course, at low enough temperatures, and when the concentration of paramagnetic impurities is low, an indirect effective interaction may take place between them and may continue up to the formation of a spin glass, so that the model involving noninteracting magnetic defects will definitely become unacceptable. At still lower temperatures, the semimagnetic semiconductor will exhibit observable phenomena such as the Kondo effect. The essential point is, however, that all these phenomena can occur only in the region of the Fermi degeneracy of electrons, so that, for $T \gg \varepsilon_d$, we can be sure that these effects are absent, and the scheme adopted above is valid.

Minimizing the free energy (4.1) with $\mathbf{H} = 0$, we obtain the stability criterion for the disordered paramagnetic state of a semiconductor containing defects:

$$\chi_{11}\chi_{12} \geq \chi_{1s}^2 \equiv \chi_{int}^2, \quad (5.19)$$

where χ_{11} , χ_{22} are the paramagnetic susceptibilities of the electron and defect subsystems. The equality sign in (5.19) corresponds to the equation for the magnetic phase transition temperature T_c . When $T < T_c$, a ferromagnetic structure with $\mathbf{M}_e \uparrow \mathbf{M}_d$ is formed for $\chi_{int} < 0$, whereas a ferrimagnetic order with $\mathbf{M}_e \uparrow \mathbf{M}_d$ is formed for $\chi_{int} > 0$ (this state may turn out to be unstable against a transition to the spatially inhomogeneous state). When $T \ll \hbar^2/m^*r_0^2 \sim 10^5 - 10^7$ K, which is known to be the case in practice, we can use (4.9) and (5.19) or (5.2) and (5.4) to determine the critical temperature for the phase transition¹⁴

$$T_c = \frac{\pi \hbar^2}{3m^*a_{\pm}^3} [3S(S+1)]^{1/2} (N_e a_e^2)^{1/2} (N_d a_d^2)^{1/2}, \\ a_{\pm} = \frac{2(a_+ - a_-)}{2S + 1}, \quad (5.20)$$

where S is the spin of the paramagnetic impurity, N_e , N_d are the concentrations of the electrons and defects, respectively, and a_{\pm} are the scattering lengths for the resultant electron and defect spins of $S \pm 1/2$. When $S = 1/2$, the equilibrium polarizations α_e and α_d , i.e., the order parameters, are given by (5.3) with m_e and α_n replaced with m^* and α_d , respectively:

$$\alpha_e = (6\tau)^{1/2}, \quad \alpha_d = \left(\frac{N_e}{N_d}\right)^{1/2} \alpha_e, \quad \tau \equiv \frac{T_c - T}{T_c} \ll 1; \\ \alpha_e = 1 - \exp\left[-2\left(\frac{T_c}{T}\right)^2\right] \approx 1, \\ \alpha_d = \frac{T_c}{T} \left(\frac{N_e}{N_d}\right)^{1/2} \ll 1, \quad \varepsilon_d \ll T \ll T_c. \quad (5.21)$$

The condition $T_c \gg \varepsilon_d$ is equivalent to the requirement that the electron density is not too high, i.e., $N_e/N_d \ll N_d^2 a_d^6 \ll 1$.

Hence, in a nondegenerate, magnetically ordered semiconductor, we always have $\alpha_c \gg \alpha_d$ although $N_e \alpha_c \ll N_d \alpha_d$, i.e., $M_d \gg M_c$. Two types of collective spin wave can propagate in a spin-polarized semiconductor containing defects. The first type is the zero-gap Goldstone mode corresponding to the inhomogeneous precession of the total magnetization with spectrum given by¹⁴

$$\omega_1 = \frac{N_e \alpha_c}{N_d \alpha_d \pm N_e \alpha_c} \frac{(kv_T)^2}{\Omega_{\text{int}}^{(d)}}, \quad \Omega_{\text{int}}^{(d)} = \frac{2\pi a_2 \hbar}{m^*} N_d \alpha_d, \quad v_T^2 = \frac{T}{m^*}. \quad (5.22)$$

The second type corresponds to the precession of the magnetizations of both components occurring in such a way that the total magnetization vector remains constant ($\mathbf{M}_c + \mathbf{M}_d = \text{const}$), although the individual magnetizations \mathbf{M}_c and \mathbf{M}_d execute oscillations of frequency ω_2 given by

$$\omega_2 = \mp \Omega_{\text{int}}^{(d)} - \Omega_{\text{int}}^{(e)} = \frac{N_d \alpha_d}{N_d \alpha_d - N_e \alpha_c} \frac{(kv_T)^2}{\Omega_{\text{int}}^{(e)}}, \quad \Omega_{\text{int}}^{(e)} = \frac{2\pi \hbar a_3}{m^*} N_e \alpha_c. \quad (5.23)$$

The expressions given by (5.22) and (5.23) are written for $S = 1/2$, where the upper sign in (5.23) corresponds to the ferromagnetic structure ($\mathbf{M}_d \uparrow \uparrow \mathbf{M}_c$) and the lower to the ferrimagnetic order ($\mathbf{M}_d \uparrow \downarrow \mathbf{M}_c$). It is readily verified with the aid of (5.21) that the condition for weak collisionless absorption $\alpha_d \gg |a_2|/\Lambda$ is not satisfied in a spontaneously magnetized semiconductor in the classical temperature range, so that spin waves are highly damped in this case. Higher values of α_d and α_c than those shown by (5.21) are necessary for the existence of weakly-damped spin waves. They can be attained by dynamic polarization or by applying a strong enough external magnetic field. A further term, representing the Lorentz force acting on charged particles, must be included in the transport equations (2.1)–(2.2) when an external magnetic field is present. The dispersion relation then assumes a relatively complicated form,¹⁴ but becomes much simpler for spin waves propagating parallel to the magnetic field ($\mathbf{k} \parallel \mathbf{H}$). The spectrum of the two branches of spin waves for $T > T_c$ is then described by

$$\omega_{1H} = \Omega_H + \frac{N_e}{N_e + N_d} \frac{(kv_T)^2}{\Omega_{\text{int}}^{(d)}}, \quad \omega_{2H} = \Omega_H - \frac{2\pi \hbar a_3}{m^*} (N_e + N_d) \text{th} \frac{\beta H}{T} - \frac{N_d}{N_d + N_e} \frac{(kv_T)^2}{\Omega_{\text{int}}^{(e)}}, \quad (5.24)$$

and the condition for weak damping $\alpha \gg |a_2|/\Lambda$, where $\alpha_c = \alpha_d \equiv \alpha = \tanh(\beta H/T)$, is satisfied for $H \sim 100$ kOe even at high temperatures $T \ll 10^3$ K. When $k = 0$, (5.24) gives us the two ESR frequencies, whose values in strong enough magnetic fields and at low enough concentrations N_e , N_d may be very different. For the typical values $N_e \sim 10^{16}$ – 10^{17} cm⁻³, $N_d \sim 10^{17}$ – 10^{18} cm⁻³, $T \sim 10^2$ K, $|a_2| \sim 1$ Å, $m^* \sim (10^{-2}$ – $10^{-1})m_e$, the quantitative measure of the effect $|\Omega_H - \omega_{2H}(k=0)|/\Omega_H$ turns out to be of the order of 10^{-3} – 10^{-1} .

The spin-wave damping is proportional to k^2 , which means that, although there are no weakly-damped spatially inhomogeneous magnetization waves in a spontaneously magnetized semiconductor when $T < T_c$ and $H = 0$, the homogeneous magnetic resonance $\omega_2(k=0)$ is nevertheless observable, where ω_2 is given by (5.23). Since, in the case of spontaneous polarization with $\varepsilon_d \ll T < T_c$, we always have $N_d \alpha_d \gg N_e \alpha_c$, the resonance frequency is finally given by

$$\omega_2(k=0) = \mp \frac{2\pi a_2 \hbar}{m^*} N_d \alpha_d, \quad (5.25)$$

where α_d is given by (5.21).

So far, we have examined the interaction between free electrons and a system of neutral scattering centers. Quantum-mechanical refraction effects can lead to interesting phenomena even when electrons are in bound states. The spectroscopy of localized electron states in condensed media is fundamentally dependent on the properties of these media. Even a tenuous gas medium will very appreciably modify the energy levels of an electron as compared with the spectrum in vacuum, and this may extend to the total vanishing of localized discrete states. If, for some reason, the medium is magnetically polarized, the interaction between an electron and the medium will remove the spin degeneracy, and the energy levels will split (even in the absence of an external magnetic field!). Quantum-mechanical refraction of a bound electron by particles of the medium in an external magnetic field will change the mutual disposition of the Zeeman lines in the spectrum.

A clear example of the influence of the ambient medium on a localized electron state is the change in the energy of shallow levels of highly-excited atoms (Rydberg atoms) in an atmosphere of an extraneous gas,^{14,39-41} or the analogous phenomena in a system of impurity levels and large-radius excitons in semiconductors containing point defects.¹⁴ Thus, the level splitting of Rydberg atoms in a spin-polarized gas and the spectral line intensity ratio in a doublet are given by the following simple expressions:

$$\Delta \mathcal{E} = \frac{\pi \hbar^2}{m_e} N |a_2| \alpha, \quad \frac{I_+}{I_-} = \frac{1+\alpha}{1-\alpha}. \quad (5.26)$$

Methods of producing relatively high degrees of polarization in alkali-metal vapor by optical pumping are now well established. In the system of electrons localized above the surface of liquid helium,⁴² quantum-mechanical refraction by atoms of the saturated vapor can also be used to explain a number of observed effects.¹⁴

6. CONCLUSION

The properties of quantum gases have attracted increasing attention during the last twelve months because of the experimental confirmation of the clear and nontrivial predictions of the theory. The theory presented in the present review provides a rigorous description of tenuous systems such as gaseous helium or hydrogen, without introducing any particular models. The theory is also satisfactory in the case of the weakly nonideal gas of quasiparticles in condensed media such as ³He–⁴He solutions, semimagnetic semiconductors, and so on. And while a surprising phenomenon such as collective spin oscillations has already been

discovered in gaseous $H\uparrow$ and ${}^3He\uparrow$, and in ${}^3He-{}^4He$ solutions, the analogous effect in semiconductors, and the no less surprising magnetic and structural phase transitions in the classical temperature range, still await their experimental discovery.

¹We note that, even earlier, Aronov⁴³ used a model to examine undamped spin-wave oscillations in a nondegenerate gas of electrons in semiconductors with nonequilibrium spin orientation.

²This is why numerical calculations²⁰ of the shape of the NMR lines in the spectrum of gaseous $H\uparrow$ under the conditions of the experiment reported in Ref. 19 appear to be essentially qualitative.⁴⁴ In the phenomenological sense, the inclusion of $I_2(\lambda_1)$ means that it is possible to introduce two relaxation times⁴⁵ into the macroscopic equations.

³A further object in which the weakly-damped spin modes were detected is the quasi-two-dimensional $H\uparrow$ adsorbed on a helium-covered surface.⁴⁶

⁴Recently, Laloë *et al.*⁴⁸ carried out an experiment in which they used a radio-frequency field gradient and a uniform static magnetic field. They were able to identify reliably spin waves in gaseous ${}^3He\uparrow$ for $T = 2-5$ K, $\alpha \approx 30\%$, $N \approx 10^{18} \text{ cm}^{-3}$.

⁵A physically similar phenomenon in a liquid electrolyte near the critical point was considered in Ref. 47.

¹E. P. Bashkin, Zh. Eksp. Teor. Fiz. **86**, 937 (1984) [Sov. Phys. JETP **59**, 547 (1984)].

²Yu. Kagan, I. A. Vartan'yants, and G. V. Shlyapnikov, Zh. Eksp. Teor. Fiz. **81**, 1113 (1981) [Sov. Phys. JETP **54**, 590 (1981)].

³Yu. Kagan, G. V. Shlyapnikov, I. A. Vartan'yants, and N. A. Glukhov, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 386 (1982) [JETP Lett. **35**, 477 (1982)].

⁴I. F. Silvera and J. T. M. Walraven, Scientific American, **246** (1), 66 (1982) [Russ. transl., Usp. Fiz. Nauk **139**, 701 (1983)].

⁵I. F. Silvera, Physica B **109/110**, 1499 (1982).

⁶R. Sprik, J. T. M. Walraven, and I. F. Silvera, Phys. Rev. Lett. **51**, 479, 942 (1983).

⁷H. F. Hess, D. A. Bell, G. P. Kochanski, D. Kleppner, and T. J. Greytak, *ibid.* **52**, 1520 (1984).

⁸J. Phys. (Paris) C-7, "Spin polarized quantum systems," **41** (1980).

⁹P. J. Nacher, M. Leduc, G. Trenc, and F. Laloë, J. Phys. Lett. (Paris) **43**, L-525 (1982).

¹⁰M. Leduc, P. J. Nacher, S. B. Crampton, and F. Laloë, in: Quantum Fluids and Solids, ed. by E. D. Adams and G. G. Ihas, 1983, p. 179.

¹¹R. Barbe, F. Laloë, and J. Brossel, Phys. Rev. Lett. **34**, 1488 (1975).

¹²I. M. Khalatnikov, Teoriya sverkhtekuchesti (Theory of Superfluidity), Nauka, M., 1971, p. 211.

¹³E. P. Bashkin, Zh. Eksp. Teor. Fiz. **73**, 1849 (1977) [Sov. Phys. JETP **46**, 972 (1977)].

¹⁴E. P. Bashkin, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 86 (1981) [JETP Lett. **34**, 81 (1981)]; Zh. Eksp. Teor. Fiz. **82**, 254, 1868 (1982) [Sov. Phys. JETP **55**, 152, 1076 (1982)].

¹⁵E. P. Bashkin, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 11 (1981) [JETP Lett. **33**, 8 (1981)].

¹⁶C. Lhuillier and F. Laloë, J. Phys. (Paris) **43**, 197, 225, 833 (1982).

¹⁷A. J. Leggett, J. Phys. C **3**, 448 (1970).

¹⁸A. E. Meyerovich, J. Low Temp. Phys. **53**, 487 (1983).

¹⁹B. R. Johnson, J. S. Denker, N. Bigelow, L. P. Levy, J. H. Freed, and D. M. Lee, Phys. Rev. Lett. **52**, 1508 (1984); **53**, 302.

²⁰L. P. Levy and A. F. Ruckenstein, *ibid.* **52**, 1512; **53**, 302.

²¹P. J. Nacher, G. Tastevin, M. Leduc, S. B. Crampton, and F. Laloë, J. Phys. Lett. (Paris) **45**, L-441 (1984).

²²E. P. Bashkin, Zh. Eksp. Teor. Fiz. **87**, 1948 (1984) [Sov. Phys. JETP **60**, 1143 (1984)].

²³E. P. Bashkin and A. E. Meierovich, Zh. Eksp. Teor. Fiz. **77**, 383 (1979) [Sov. Phys. JETP **50**, 196 (1979)].

²⁴E. P. Bashkin and A. E. Meyerovich, Adv. Phys. **30**, 11 (1981).

²⁵L. D. Landau and E. N. Lifshitz, Kvantovaya mekhanika, Nauka, M., 1974, p. 678 [Eng. transl., Quantum Mechanics, Pergamon Press, Oxford, 1977].

²⁶A. I. Akhiezer and S. V. Peletminskii, Metody statisticheskoi fiziki, Nauka, M., 1977, p. 137 [Engl. transl., Methods of Statistical Physics, Pergamon Press, N.Y., 1981].

²⁷V. M. Galitskii, Zh. Eksp. Teor. Fiz. **34**, 151 (1958) [Sov. Phys. JETP **7**, 104 (1958)].

²⁸V. P. Silin, Vvedenie v kineticheskuyu teoriyu gazov (Introduction to the Kinetic Theory of Gases), Nauka, M., 1971, p. 221.

²⁹L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika, Nauka, M., 1976, Part 1, p. 384 [Eng. transl., Statistical Physics, Pergamon Press, Oxford, 1980].

³⁰C. Lhuillier, J. Phys. (Paris) **44**, 1 (1983).

³¹W. J. Gully and W. J. Mullin, Phys. Rev. Lett. **52**, 1810 (1984).

³²E. P. Bashkin and A. E. Meierovich, Pis'ma Zh. Eksp. Teor. Fiz. **26**, 696 (1977) [JETP Lett. **26**, 534 (1977)]; Zh. Eksp. Teor. Fiz. **74**, 1904 (1978) [Sov. Phys. JETP **47**, 992 (1978)].

³³E. P. Bashkin and A. E. Meierovich, Usp. Fiz. Nauk **130**, 279 (1980) [Sov. Phys. Usp. **23**, 156 (1980)].

³⁴A. E. Meyerovich, J. Low Temp. Phys. **47**, 271 (1982).

³⁵E. M. Lifshitz and L. P. Pitaevskii, Fizicheskaya kinetika, Nauka, M., 1979, pp. 53, 89 [Engl. transl., Physical Kinetics, Pergamon Press, Oxford, 1981].

³⁶E. P. Bashkin, Phys. Lett. A **101**, 164 (1984).

³⁷V. P. Silin, Zh. Eksp. Teor. Fiz. **6**, 945 (1957) [Sov. Phys. JETP **6**, 726 (1958)].

³⁸G. H. Van Yperen, I. F. Silvera, J. T. M. Walraven, J. Berkhout, and J. G. Brisson, Phys. Rev. Lett. **50**, 53 (1983).

³⁹E. Fermi, Nuovo Cimento **11**, 157 (1934).

⁴⁰V. A. Alekseev and I. I. Sobel'man, Zh. Eksp. Teor. Fiz. **49**, 1274 (1965) [Sov. Phys. JETP **22**, 882 (1966)].

⁴¹V. G. Baryshevskii, Yadernaya optika polarizovaniykh sred (Nuclear Optics of Polarized Media), Beloruss. Univ., Minsk, 1976, p. 32.

⁴²V. S. Edel'man, Usp. Fiz. Nauk **130**, 675 (1980) [Sov. Phys. Usp. **23**, 227 (1980)].

⁴³A. G. Aronov, Zh. Eksp. Teor. Fiz. **73**, 577 (1977) [Sov. Phys. JETP **46**, 301 (1977)].

⁴⁴E. P. Bashkin, Phys. Rev. Lett. **55**, 1426 (1985).

⁴⁵A. E. Meyerovich, Phys. Lett. A **107**, 177 (1985).

⁴⁶E. P. Bashkin, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 383 (1984) [JETP Lett. **40**, 1197 (1984)].

⁴⁷V. M. Nabutovskii, N. A. Nemov, and Yu. G. Peisakhovich, Zh. Eksp. Teor. Fiz. **79**, 2196 (1980) [Sov. Phys. JETP **52**, 1111 (1980)].

⁴⁸G. Tastevin, P. J. Nacher, M. Leduc, and F. J. Laloë, J. Phys. Lett. (Paris) **45**, L-249 (1985).

Translated by S. Chomet