Dynamics of the order parameter of superfluid phases of helium-3

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The spin dynamics and the orbital dynamics in the A and B phases of superfluid ^{3}He are analyzed. Attention is focused on solitons and instantons: nonsingular configurations of the field of the order parameter of the ^{3}He superfluid phases. A qualitative explanation in terms of solitons and instantons is offered for several experiments involving the A and B phases of ^{3}He. The analysis of these questions is prefaced by some general information on the order parameter and the free energy in superfluid ^{3}He.

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1. INTRODUCTION

Nature has two liquids which do not solidify as the temperature is lowered all the way to absolute zero (at standard pressure). These are the quantum liquids helium-4 and helium-3. The bahavior of these liquids is quantum-mechanical because in the region in which they exist (at temperatures of the order of 1 K) the de Broglie wavelength of the thermal motion of the He^4 and He^3 atoms is comparable to the distance between the atoms of the liquid. The quantum-mechanical properties of helium-4 and helium-3 are quite different: The spin-zero He^4 atoms form a Bose liquid, while the spin-1/2 He^3 atoms condense into a Fermi liquid.

A characteristic property of Bose systems is a Bose condensation: the accumulation of a finite fraction of the particles in the ground state (with zero momentum). Associated with the property of Bose condensation is the phenomenon of superfluidity, which was discovered in helium-4 (at temperatures below 2.17 K) by Kapitsa in 1938. Superfluidity is the property of a liquid which allows it to flow without friction through capillaries, but the term "superfluidity" itself combines many properties of quantum liquids at temperatures below the temperature of the transition to the superfluid state. The liquids themselves in this state are called "superfluids."

Another phenomenon associated with Bose condensation is superconductivity (the property that an electric current can flow through a conductor without experiencing a resistance). In this case the quantum fluid is formed by electrons, which are spin-1/2 particles. For electrons, condensation is forbidden by Fermi statistics. At low temperatures, however, the conditions become such (an attraction arises between electrons) that the system of electrons is unstable with respect to the formation of Cooper electron pairs. Since an electron has a spin of 1/2, a Cooper pair has an integer spin and obeys Bose statistics. A Bose condensation of Cooper pairs therefore occurs in a superconductor. An explanation for superconductivity as superfluidity of Cooper pairs was offered by Bardeen, Cooper, and Schrieffer (BCS) in 1957.

After the mechanism for superconductivity became clear, it was expected that there was the further possibility of a transition of helium-3 to a superfluid state, associated with Copper pairing of He^3 atoms. The mechanism for this pairing was found by Pitaevskiĭ[1] in 1959. Pitaevskiĭ showed that the van der Waals attraction causes He^3 atoms to form a Cooper pair with a nonzero orbital angular momentum l. By virtue of the Pauli principle, the wave function of the pair is antisymmetric with respect to the interchange of particles, so that a triplet pairing (a total spin $S=1$) corresponds to odd values of l, while a singlet pairing ($S=0$) corresponds to even values of l.

The superfluidity of helium-3 was therefore predicted. This prediction launched an experimental search for the effect. This search was rewarded with success when cryogenic techniques made it possible to reach temperatures of the order of a thousandth of a kelvin.

In 1972, Osheroff, Richardson, and Lee[2] discovered two phase transitions ("A" and "B") in helium-3 at temperatures below 3 mK. It soon become clear that Osheroff,

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Richardson, and Lee had discovered two superfluid phases of helium-3: A and B.

Phase transitions from normal He^3 to the A phase and from the A phase to the B phase were detected on the melting curve of solid He^3 (at a pressure $P = 35$ atm) at temperatures $T_c = 2.6$ mK and $T_{AB} = 2.07$ mK, respectively. As the pressure is lowered, the transition temperature (T_c) between the normal state of He^3 and the superfluid state decreases (from $T_c = 0.9$ mK at $P = 0$), while T_{AB} increases (to $T_{AB} = 2.4$ mK at $P = 20$ atm); i.e., there is a multicritical point. The phase transition between the normal state and the superfluid state (the A or B phase) in a second-order transition, while the AB transition (between He^3-A and He^3-B) is of first order. Theoretical and experimental research on the superfluidity of helium-3 has shown that a triplet Cooper pairing ($S = 1$) with an orbital angular momentum $l = 1$ occurs in superfluid He^3.

Superfluid states of a Fermi liquid with $l = 1$ were first studied by Anderson and Morel[3] and Balian and Werthamer[4] in the early 1960s. These Anderson-Morel[3] and Balian-Werthamer[4] states—the two states of the many possible states which were the first two to be studied—were in fact realized in nature as the A and B phases, respectively, of superfluid helium-3.

The transition of helium-3 to a superfluid state was predicted,[1] as we have already mentioned, but the existence of two superfluid phases of He^3 was an unexpected result. It is interesting to note in this regard that Balian and Werthamer[4] were not actually trying to describe the superfluidity of He^3. Regarding the Anderson-Morel state, they wrote that Anderson and Morel had predicted that a state with a p-wave attraction would have many strange features such as an energy spectrum with an anisotropic gap, which vanishes in certain directions, a nonexponential heat capacity, and surface currents. Balian and Werthamer concluded that the net effect is to make that state physically improbable and in sharp contradiction of the results of the BCS theory and experiment.[4] Those comments show that anything in the way of a quantitative prediction regarding the superfluid properties of He^3 (the temperature of the phase transition, the orbital angular momentum, etc.) was impossible before the experiments by Osheroff, Richardson, and Lee.[2]

The Anderson-Morel and Balian-Werthamer theories are BCS theories for an interaction potential which depends on the angles between the momenta of the interacting particles. This interaction allows different states of the condensate of Cooper pairs of He^3 atoms. The Anderson-Morel state corresponds to the pairing of two spins, directed "up" or "down," while the Balian-Werthamer state corresponds in addition to a pairing between the "up" and "down"spins, i.e., among all the components of the triplet $S_z = +1, 0, -1$. The properties of the Anderson-Morel and Balian-Werthamer states are determined by the structure of the pairing amplitude or of the order parameter and also by additional interactions. The properties of the superfluid phases of He^3 are remarkable because of their variety because they combine the properties of superconductors, liquid crystals, and magnetic substances. From the theoretical standpoint, superfluid He^3 has won popularity because it has presented some fundamental new problems in the physics of the condensed state.

Superfluid helium-3 has been the subject of several reviews.[5–9] Most noteworthy is the early and frequently cited review of Leggett,[5] which has detailed discussions of general questions of the theory of a Fermi liquid, the BCS theory, and the Ginzburg-Landau theory as they pertain to helium-3. The review by Brinkman and Cross[6] is devoted to the spin dynamics and orbital dynamics of superfluid He^3. The rather complicated structure of the order parameter of the superfluid phases of He^3 has forced the theoreticians to appeal to elements of topology (homotopy theory) in order to study the superfluidity in helium-3. This circumstance and the particular features and textures of the field of the order parameter in the A phase of He^3 are examined in the popular review by Volovik and Mineev.[7] Another review by Mineev[8] introduces the reader to the basic experimental methods for studying superfluid He^3 and its properties. That review[8] also gives a detailed description of the structure of the order parameter and of the free energy of the superfluid phases of He^3. It also gives a theory for the spin dynamics of the A and B phases of He^3. Another review by Volovik[9] introduces the reader to the variety of superfluid properties of the A phase of He^3.

Perhaps the most interesting properties of the superfluid phases of He^3 stem from the spatially inhomogeneous configurations of the fields of the order parameter: disclinations, vortices, solitons, and so forth. All these entities play important roles in the spin dynamics and orbital dynamics of superfluid helium-3.

The present review supplements the existing reviews with an examination of the spin dynamics and orbital dynamics in the superfluid phases of He^3 in the presence of textures (spatial inhomogeneities) in the field of the order parameter. Most of the effort in research on the dynamics has been devoted to solitons and instantons, which are nonsingular configurations of the order-parameter fields. Our examination of the dynamics of textures of the order parameter is preceded by a general discussion of the order parameter and the free energy of the A and B phases of He^3 (Sections 2 and 3). Section 4 sets forth Leggett's theory of spin dynamics, which, along with experiments on nuclear magnetic resonance, has made it possible to identify the A and B phases of He^3 as Anderson-Morel and Balian-Werthamer states. Section 5 is devoted to the dynamics of domain walls or solitons in the fields of the order parameter. The method of the inverse scattering problem is used to describe the creation of solitons when a magnetic field is turned off. The research on He^3-A and He^3-B deals with solitons whose existence stems from a dipole-dipole interaction of the magnetic moment of He^3 nuclei and the presence of an external magnetic field. A qualitative description of experiments with the A and B phases of He^3 is offered in terms of dipole and magnetic solitons. Section 6 deals with the persistent orbital motion in the A phase of He^3, which is a result of a dissipative flow of a superfluid liquid which is maintained by virtue of the formation of an array of instantons: space-time oscillations of the orbital part of the order parameter of the A phase of He^3. A phenomenological description of this motion is offered.

2. ORDER PARAMETER

The superfluid phases of He^3, like other ordered systems (superconductors, liquid crystals, and magnetic substances), are conveniently described in terms of an order parameter. In superfluid liquids the order parameter represents the wave function of a condensate. In the case of He^4 below the λ point or for superconductors, for example, the order parameter is

$$\theta(\mathbf{r}, t) = |\theta(\mathbf{r}, t)| e^{i\Phi(\mathbf{r}, t)}, \tag{1}$$

where $|\theta(\mathbf{r},t)|^2 = \rho_s(\mathbf{r},t)$ is the superfluid density, and Φ is the phase of the wave function. The condensate wave function is a macroscopic quantity; the most important parameter is the phase, whose variations in space and time are related to the superfluid motion of the liquid. The superfluid flux density is

$$\mathbf{j}_s = \frac{i\hbar}{2m}(\theta\nabla\theta^* - \theta^*\nabla\theta) = \frac{\hbar}{m}\rho_s\nabla\Phi \tag{2}$$

(m is the mass of a particle), from which we find

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\Phi, \tag{3}$$

since $\mathbf{j}_s \equiv \rho_s \mathbf{v}_s$. It follows from (3) that the phase of the wave function agrees with a constant factor with the velocity potential of the macroscopic superfluid motion. We might also note here that for superfluid liquids and for ordered systems in general an important concept is the space of the order parameter or the region of degeneracy of states of the system.[7,8,10,11] The "space of the order parameter" is that subspace of the region of values of the order parameter which corresponds to various equilibrium states of the system with an identical free energy. In the example under consideration here, the region of values of the order parameter $\theta(\mathbf{r},t)$ is a complex plane, while the degeneracy space is a circle of radius $|\theta| =$ const on the complex plane. The phase Φ is a degeneracy parameter: The free energy does not depend on the phase, but each value of Φ corresponds to a particular equilibrium state of the system.

We turn now to He^3. In superfluid He^3, the order parameter is the wave function of a pair or the anomalous Green's function

$$\Psi_{\alpha\beta}(\mathbf{k}) = \langle a_\alpha(\mathbf{k})\, a_\beta(\mathbf{k})\rangle. \tag{4}$$

From (4) we find

$$\Psi_{\alpha\beta}(\mathbf{k}) = -\Psi_{\beta\alpha}(-\mathbf{k}). \tag{5}$$

In the case of pairing with an odd value of l, $\Psi_{\alpha\beta}$ is an odd function of $\mathbf{k}$ and is a symmetric second-rank spinor corresponding to a particle with a spin $S = 1$. Assuming that the order parameter $\Psi_{\alpha\beta}$ depends on only the direction in momentum space, $\mathbf{n} = \mathbf{k}/k$ (this is a good approximation), we can write $\Psi_{\alpha\beta}$ as a linear combination of spherical harmonics with the given value of l. For $l = 1$, we can choose the components of the vector $\mathbf{n}$ as these functions. We know that a symmetric spinor can be associated with a complex vector, which we denote by $\mathbf{d}(\mathbf{n})$. Here are the direct and inverse relations between $\Psi_{\alpha\beta}$ and $\mathbf{d}$:

$$\Psi_{\alpha\beta}(\mathbf{n}) = i\,\mathbf{d}(\mathbf{n})\,(\hat{\sigma}\hat{\sigma}^y)_{\alpha\beta},$$
$$\mathbf{d}(\mathbf{n}) = -\frac{i}{2}(\hat{\sigma}^y\hat{\sigma})_{\alpha\beta}\,\Psi_{\alpha\beta}(\mathbf{n}), \tag{6}$$

where $\hat{\sigma} = \{\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z\}$ are Pauli matrices. The quantities $(\hat{\sigma}\hat{\sigma}^y)_{\alpha\beta}$ constitute a basis for the expansion of the spinor $\Psi_{\alpha\beta}$ in spin space, and the order parameter itself is the quantity[1)] $\mathbf{d}(\mathbf{n})$. Another form of the order parameter can be found after an expansion of $\mathbf{d}(\mathbf{n})$ is spherical harmonics, as mentioned above. In the case $l = 1$, this expansion is

$$d_i(\mathbf{n}) = A_{ij} n_j, \tag{7}$$

where A_{ij} is a complex 3×3 matrix, which is the order parameter in the form which we will be using below. In the case of triplet pairing the space of the order parameter is therefore 18-dimensional; in other words, there are nine complex or 18 real degeneracy parameters. Obviously, this circmstance seriously complicates the study of He^3, but at the same time it leads to the large variety of properties of the superfluid phases of He^3.

Barton and Moore[12] found 11 phases or 11 types of order parameters by minimizing the Ginzburg-Landau free-energy functional. Each of the phases which they found is stable in a certain region of the parameters of the functional. Among these phases are the Anderson-Morel and Balian-Werthamer states, which describe the A and B phases of superfluid He^3.

We see that there is some arbitrariness in the choice of the order parameter for the two superfluid phases of He^3 which have been seen experimentally. If we also note that we have not eliminated other types of pairing, e.g., in the d-state ($l = 2, S = 0$), this choice cannot be made without analyzing the experimentally observed properties of the superfluid phases. For example, some of the phases found theoretically have a spontenaous magnetic moment,[12] but such a moment is not observed experimentally. Accordingly, such phases (order parameters) must be discarded (there are other arguments of this sort). Consequently, it was not immediately obvious that the first two of all the states with $l \geqslant 1$ which were considered would turn out to be the only suitable states having all the properties of the two superfluid phases of He^3. As we have already mentioned, these states are the Anderson-Morel and Balian-Werthamer states.

A state of a pair of He^3 atoms is characterized by projections of the orbital angular momentum and the spin onto certain axes. In the Anderson-Morel state, all the pairs have a projection $l = 1$ onto some axis, whose direction we denote by the unit vector $\mathbf{l}$, and a zero spin projection onto some axis $\mathbf{V}$. The directions $\mathbf{V}$ and $-\mathbf{V}$ are equivalent ($\mathbf{V}^2 = 1$). Neither the directions of the axes $\mathbf{l}$ and $\mathbf{V}$ nor the relative orientation of these axes is fixed; i.e., there is a degeneracy in terms of the directions of the quantization axes for the spin and the orbit angular momentum.

The orbital part of the wave function of a pair in the A phase can be written in the form $\Psi(\mathbf{r}) = R(r)\sin\vartheta\cdot\exp(i\varphi)$, where $|\mathbf{r}|$ is the distance between the atoms of the pair, ϑ is the angle between $\mathbf{l}$ and $\mathbf{r}$, and φ is the azimuthal angle in the plane perpendicular to $\mathbf{l}$, reckoned from an arbitrary direction $\mathbf{\Delta}'$. We see from this discussion that the degeneracy in terms of the directions of **l** and **V** is supplemented with a degeneracy in terms of the phase φ. The situation could alternatively by represented as a degeneracy in terms of the orientations of the triad of mutually perpendicular vectors **l**, $\mathbf{\Delta}'$, $\mathbf{\Delta}''$ and in terms of the directions of the spin axis **V** (Refs. 7 and 9).

DeGennes[13] introduced the following expression for the order parameters in the A phase:

$$A^{\rm A}_{ij} = \Delta_{\rm A}(T)\,V_i\Delta_j, \tag{8}$$

where

$$\mathbf{\Delta} = \mathbf{\Delta}' + i\mathbf{\Delta}'', \quad \mathbf{l} = [\mathbf{\Delta}'\mathbf{\Delta}''].$$

The expression for the superfluid velocity is of the form in (3) only if the vector **l** is constant over space. In general, when **l** depends on the coordinates, the expression for $\mathbf{v}_s$ is (Ref. 9, for example)

$$v^i_{\rm s} = \frac{\hbar}{2m}\,\mathbf{\Delta}'\nabla^i\mathbf{\Delta}'',$$

where m is the mass of the He^3 atom.

In the Balian-Werthamer state, the projections of the spin and the orbial angular momentum onto any axis are equiprobable, and the spin coordinates rotate in an arbitrary way with respect to the orbital coordinates. In this case the order parameter is proportional to a real orthogonal matrix and is given by[14]

$$A^{\rm B}_{ik} = \Delta_{\rm B}(T)\,e^{i\varphi}R_{ik}(\mathbf{n},\ \vartheta), \tag{9}$$

where R_{ik} is a matrix describing a rotation through some angle ϑ around some direction **n**. The degeneracy parameters here are **n**, ϑ, and the phase φ. In (8) and (9), $\Delta_{\rm A}$ and $\Delta_{\rm B}$ are phenomenological parameters of the gap in the quasiparticle spectrum.

3. FREE ENERGY

The free energy of ordered systems near the phase-transition temperature is constructed as an expansion in powers of the order parameter. The free energy must be real and must have certain symmetry properties, associated with the degeneracy of the system. In the superfluid phases of He^3, in the absence of a dipole-dipole interaction of the magnetic moments of the He^3 nuclei, the directions and relative orientation of the quantization axes for the spin and the orbital angular momentum are arbitrary. Consequently, at equilibrium the free energy must be invariant under independent rotations of the spin and coordinate space.[2)] The Ginzburg-Landau free energy, which meets these requirements and which is constructed from invariants of second and fourth orders, is

$$\begin{aligned} F_{\rm c} = &-\alpha\,{\rm Sp}\,(AA^+) + \beta_1\,|\,{\rm Sp}\,(A\widetilde{A})\,|^2 + \beta_2\,[{\rm Sp}\,(AA^+)]^2 \\ &+ \beta_3\,{\rm Sp}\,[(A^+A)\,(A^+A)^*] + \beta_4{\rm Sp}\,[(AA^+)^2] \\ &+ \beta_5{\rm Sp}\,[(AA^+)\,(AA^+)^*], \end{aligned} \tag{10}$$

where α and β_i are phenomenological parameters of the theory, which can be calculated at a microscopic level in the so-called weak-coupling approximation $\Delta \ll \varepsilon_{\rm F}$ ($\varepsilon_{\rm F}$ is the Fermi energy). The condensate energy (10) and also the additional interactions (discussed below) describe spatially homogeneous states of the A and B phases of superfluid He^3, for which the matrix A is determined by (8) and (9), respectively.

If the order parameter A depends on the coordinates, it is necessary to take into account the gradient terms in the expansion of the Ginzburg-Landau free energy.[13] The gradient energy is written in most general form as[5]

$$F_{\rm grad} = K_1\partial_i A_{\mu i}\partial_j A^*_{\mu j} + K_2\partial_i A_{\mu j}\partial_i A^*_{\mu j} + K_3\partial_i A_{\mu j}\partial_j A^*_{\mu i}. \tag{11}$$

For the simple BCS model with weak coupling we would have $K_1 = K_2 = K_3 \equiv K/2$, and K would be

$$K = \frac{3}{5}\,\frac{7\zeta(3)\,\hbar^2}{(2\pi k_{\rm B}T_{\rm c})^2}\,\frac{N}{m^*} \equiv \frac{3}{5}\,N_{\rm F}\xi_0^2,$$

$$N = \frac{k_{\rm F}^3}{3\pi^2}, \quad N_{\rm F} = \frac{m^*k_{\rm F}}{\pi^2\hbar^2}, \quad \xi_0^2 = \frac{7\zeta(3)}{48\pi^2}\,\frac{\hbar^2v_{\rm F}^2}{k_{\rm B}^2T_{\rm c}^2},$$

where N is the He^3 density, m^* is the effective mass, ξ_0 is the radius of the pair, and $N_{\rm F}$ is the state density of the Fermi surface.

The spaces of the order parameter are five-dimensional and four-dimensional for the Anderson-Morel and Balian-Werthamer states, respectively [as follows from (8) and (9)]. The degeneracy is partially lifted when the spin-orbit interaction, the boundary conditions, and a magnetic field are taken into account. The role of the spin-orbit interaction is played by the dipole-dipole interaction of the magnetic moments of the nuclei of the helium atoms making up a pair. We write the dipole energies and magnetic energies for the A and B phases as follows[5]:

$$F^{\rm A}_{\rm dip} = -\frac{3}{5}\,g_{\rm D}\,(\mathbf{lV})^2, \tag{12}$$

$$F^{\rm B}_{\rm dip} = \frac{8}{5}\,g_{\rm D}\left(\cos\vartheta + \frac{1}{4}\right)^2, \tag{13}$$

$$F^{\rm A}_{\rm magn} = \lambda\,(\mathbf{HV})^2, \quad \lambda \sim \chi_{\rm A} - \chi_{\rm B} > 0, \tag{14}$$

$$F^{\rm B}_{\rm magn} = -a\,(\mathbf{Hn})^2, \qquad a \sim g_{\rm D}\left(\frac{\mu_0}{\Delta}\right)^2. \tag{15}$$

In (12)–(15), $g_{\rm D}$ is the dipole-dipole interaction constant, $\chi_{\rm A}$ and $\chi_{\rm B}$ are the susceptibilities of the A and B phases of He^3, μ_0 is the nuclear magnetic moment of the He^3 atom, and Δ is the gap in the quasiparticle spectrum.

If we ignore the boundary conditions, we would write the free energy density as the sum of the energies listed above:

$$F = F_{\rm c} + F_{\rm grad} + F_{\rm dip} + F_{\rm magn}. \tag{16}$$

The energies in (16) are given in order of magnitude by

$$F_{\rm c} \sim \alpha\Delta^2, \quad F_{\rm grad} \sim K\frac{\Delta^2}{r^2}, \quad F_{\rm dip} \sim g_{\rm D},$$

$$F^{\rm A}_{\rm magn} \sim \lambda H^2, \quad F^{\rm B}_{\rm magn} \sim g_{\rm D}\left(\frac{\mu_0 H}{\Delta}\right)^2, \tag{17}$$

where r is a characteristic length over which the gradient energy becomes comparable to the energies $F_{\rm c}$, $F_{\rm dip}$, and $F_{\rm magn}$. Consequently, we can find a correlation length, a dipole length, and a magnetic length by working from the equalities $F_{\rm grad} = F_{\rm c}$, $F_{\rm grad} - F_{\rm dip}$, and $F_{\rm grad} = F_{\rm magn}$:

$$\xi = \left(\frac{K}{\alpha}\right)^{1/2}, \quad \xi_{\rm dip} = \Delta\left(\frac{K}{g_{\rm D}}\right)^{1/2}, \quad \xi^{\rm A}_{\rm magn} = \left(\frac{K\Delta^2}{\lambda H^2}\right)^{1/2},$$

$$\xi^{\rm B}_{\rm magn} = \xi_{\rm dip}\,\frac{\Delta}{\mu_0 H}. \tag{18}$$

These scale length satisfy the inequalities

$$\xi \ll \xi_{dip}, \ \xi^{A}_{magn}; \quad \xi \ll \xi_{dip} \ll \xi^{B}_{magn}, \tag{19}$$

where $\xi_{dip} \sim \xi^{A}_{magn}$ is magnetic fields $H \sim 20$ G.

The dipole length and the magnetic length tell us the distances over which the spin-orbital symmetry is lost in the superfluid phases of He^3. For example, we see from (12) that in the A phase the vectors **l** and **V**, which determine the quantization axes for the orbital angular momentum and the spin, are oriented parallel to each other over distances $r > \xi_{dip}$; i.e., the dipole energy eliminates the arbitrariness in the relative orientation of the **l** and **V** axes. The magnetic energy in (14) causes the vector **V** to beome oriented perpendicular to the magnetic field at distances $r > \xi^{B}_{magn}$. In the case of the B phase, the dipole energy reaches a minimum at $\cos\vartheta_0 = -1/4$, but the direction of **n** remains arbitrary. This direction can be fixed by a magnetic field. It can be seen from (15) that at distances $r > \xi^{B}_{magn}$ we have $\mathbf{n} \| \pm \mathbf{H}$. Finally, at short range the description of the A and B phases by means of order parameters (8) and (9) is limited by the correlation length ξ. This analysis shows that the order parameters are different at different distances in the superfluid phases of He^3 (Refs. 8, 10, and 11).

4. SPIN DYNAMICS

4.1. A phase. Spatially homogeneous case

The experiments by Osheroff, Richardson, and Lee[2] in which the superfluid phases of He^3 were discovered in the course of measurements of the pressure on the melting curve, were quickly followed by an NMR study of He^3, by the same group of investigators.[15] They found that in normal He^3 there was, as usual, a signal at the Larmor frequency ω_L, but as the temperature was lowered below T_c the resonant frequency was observed to become continuously higher until the transition from the A phase to the B phase, at which point the resonant frequency abruptly returned to the value ω_L. It was these experiments and Leggett's theory,[16] which explains these experiments in terms of triplet pairing, which made it possible to identify the A and B phases as Anderson-Morel and Balian-Werthamer states.

The experiments of Ref. 15 showed that the shift of the resonant frequency in the A phase is described by

$$\omega^2 = \omega_A^2(T) + \omega_L^2, \tag{20}$$

$\omega_A^2 \propto 1 - \dfrac{T}{T_c}$. According to Leggett's theory of spin dynamics,[16] this shift is caused by a nuclear dipole-dipole interaction.

Let us use Leggett's phenomenological Hamiltonian[16]

$$H = \frac{\gamma_0}{2\chi} S^2 - \gamma_0 \mathbf{S}\mathbf{H} + F_{dip} \tag{21}$$

to derive equations of motion for the spin (**S**) of the system and for the order parameter. In (21), γ_0 is the gyromagnetic ratio of the He^3 nuclei, χ is the susceptibility, and **H** is the magnetic field.

We consider the Heisenberg equations of motion for the spin **S** and the order parameter $A_{\mu i}$:

$$i\hbar \frac{\partial \mathbf{S}}{\partial t} = [\mathbf{S}, H], \tag{22}$$

$$i\hbar \frac{\partial A_{\mu i}}{\partial t} = [A_{\mu i}, H], \tag{23}$$

where the following commutation relations[3)] hold for **S** and $A_{\mu i}$ (Ref. 16):

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k, \tag{24}$$

$$[A_{\mu i}, S_j] = i\hbar\varepsilon_{\mu jk}A_{ki}. \tag{25}$$

Following Leggett,[16] we assume that the orbital motion is slower than the spin motion, and we replace the equation for $A_{\mu i}$ by an equation for **V**, assuming that the vector **l** is given. After evaluating the commutators, we then find a system of equations for **S** and **V** (Ref. 17):

$$\frac{d\mathbf{S}}{dt} = \gamma_0[\mathbf{S}\mathbf{H}] + \frac{6}{5} g_D [\mathbf{V}\mathbf{l}](\mathbf{V}\mathbf{l}), \tag{26}$$

$$\frac{d\mathbf{V}}{dt} = \gamma_0[\mathbf{V}(\mathbf{H} - \gamma_0\chi_A^{-1}\mathbf{S})]. \tag{27}$$

Now linearizing (26) and (27) in small deviations of **S** and **V** from their equilibrium values $\mathbf{S}_0 = \chi_A\gamma_0^{-1}\mathbf{H}_0$ and $\mathbf{V}_0 \| \mathbf{l}$ ($\mathbf{H}_0$ is a static magnetic field) and also in terms of the alternating external magnetic field $\delta\mathbf{H}$, we find equations for $\delta\mathbf{S} = \mathbf{S} - \mathbf{S}_0$ and $\delta\mathbf{V} = \mathbf{V} - \mathbf{V}_0$:

$$\delta\dot{\mathbf{S}} = [\chi_A\gamma_0^{-1}\omega_L(\delta\mathbf{H} - \gamma_0\chi_A^{-1}\delta\mathbf{S})] + \Omega_A^2[\delta\mathbf{V}\mathbf{V}_0]\chi_A\gamma_0^{-2}, \tag{28}$$

$$\delta\dot{\mathbf{V}} = [\gamma_0\mathbf{V}_0(\delta\mathbf{H} - \gamma_0\chi_A^{-1}\delta\mathbf{S})], \tag{29}$$

where $\omega_L = \gamma_0 H_0$ and $\Omega_A^2 = \frac{6}{5} g_D \gamma_0^2 \chi_A^{-1}$.

Equations (28) and (29) can be solved easily in the cases $\delta\mathbf{H} \| \mathbf{H}_0$ and $\delta\mathbf{H} \perp \mathbf{H}_0$. For $\delta\mathbf{H} \| \mathbf{H}_0 \| Oz$ ($\mathbf{V}_0 \| Ox$), for example, we have

$$\delta S_z = \frac{\gamma_0^{-1}\chi_A\Omega_A^2}{\Omega_A^2 - \omega^2}\delta H_z \equiv \chi_z(\omega)\,\delta H_z. \tag{30}$$

From Eq. (30) we see that there is a longitudinal resonance at the frequency Ω_A.

In the case of a transverse resonance the dispersion relation is

$$\omega^2 = \Omega_A^2 + \omega_L^2, \tag{31}$$

in agreement with the experimental expression (30) for the frequency of a transverse resonance. Experimentally, a longitudinal resonance was discovered by Osheroff and Brinkman[18]; they also established the relationship between the frequencies of the longitudinal and transverse resonances, (31).

We wish to stress that the longitudinal resonance has nothing in common with the ordinary resonance at the Larmor precession frequency. The longitudinal resonance must occur even in a zero magnetic field. In this case the equation for $\delta\mathbf{S}$ takes the particularly simple form

$$\delta\ddot{\mathbf{S}} + \Omega_A^2\delta\mathbf{S} = 0; \tag{32}$$

i.e., the spin oscillates around its equilibrium position because of the spin-orbit interaction. The same events occur in the case of a transverse resonance, but in this case against the background of the ordinary classical precession.

4.2. B phase. Presence of textures

To study the magnetic resonance in the B phase we use Leggett's equations for the variables **S**, **n**, and ϑ, which were derived by Brinkman.[19] We restrict the analysis of the case $\mathbf{n} = \text{const}$; the equations for $\mathbf{S}$ and ϑ linearized in terms of small deviations from equilibrium are then

$$\delta\dot{\mathbf{S}} = [\chi_B\gamma_0^{-1}\omega_L(\delta\mathbf{H} - \chi_B^{-1}\gamma_0\delta\mathbf{S})] - n\chi_B\gamma_0^{-2}\Omega_B^2\delta\vartheta, \tag{33}$$

$$\delta\dot{\vartheta} = \gamma_0\mathbf{n}(\delta\mathbf{H} - \chi_B^{-1}\gamma_0\delta\mathbf{S}), \tag{34}$$

where $\Omega_B^2 = 3g_D\gamma_0^2\chi_B^{-1}$.

It follows from (33) and (34) that here, as in the A phase, we should observe a longitudinal resonance at the frequency $\omega = \Omega_B$. In the case of a transverse resonance, however, there is an interesting feature: The shift of the resonant frequency depends on the angle (α) between $\mathbf{n}$ and $\mathbf{H}_0$:

$$\omega^2 = \omega_L^2 + \Omega_B^2 \sin^2\alpha. \tag{35}$$

As mentioned above, the minimum of the magnetic energy, (15), is reached in the case $\mathbf{n}\|\mathbf{H}$, so that in the spatially homogeneous case there is no shift of the transverse resonance in the B phase, in agreement with the observations of Ref. 15. It therefore follows from (35) that a shift of the transverse-resonance frequency should be evidence of the presence of spatially inhomogeneous configurations or textures in the field of the vector $\mathbf{n}$.

Certain NMR experiments with the B phase have revealed a dip on the absorption profile (Ref. 20, for example). This dip can be explained quite easily under the assumption that the field of the vector $\mathbf{n}$ has a domain wall in which the parallel orientation of $\mathbf{n}$ with respect to $\mathbf{H}$ gives way to an antiparallel orientation.[20] Such domain walls or solitons are analogous to the walls in nematic liquid crystals.[21,22]

An $\mathbf{n}$-texture is implicitly present in this example. In the discussion below we will directly study textures of the order parameter in the superfluid phases of He^3.

5. DOMAIN WALLS OR SOLITONS

From the topological standpoint, solitons are nonsingular stable configurations of the order-parameters fields. Such configurations may form because the order parameter may be different at different ranges. The scale lengths determining the structure of the order parameter at different ranges are related to the dipole-dipole interaction and the presence of a magnetic field. The same lengths determine the sizes of the solitons. In the present section of the paper we examine dipole and magnetic planar solitons in those very simple cases in which the solitons can be described by sine-Gordon equations. The solutions of these equations correspond to domain walls in the $\mathbf{V}$, $\mathbf{l}$, $\mathbf{n}$, and ϑ fields, which form order parameters of the He^3 superfluid phases. The basic topics of this section are the creation and dynamics of solitons.

5.1. Creation of solitons

Before we take up specific examples, let us examine the creation of solitons and find the characteristics of solitons in situations describable by a sine-Gordon equation $[\psi = \psi(\xi,\tau)]$

$$\psi_{\tau\tau} - \psi_{\xi\xi} + \sin\psi = 0 \tag{36}$$

with the initial conditions

$$\tau = 0: \quad \psi = 0, \quad \psi_\tau = 2\omega(\xi). \tag{37}$$

In the A phase of He^3, for example, $\psi/2$ would be the angle between $\mathbf{l}$ and $\mathbf{V}$, and the initial conditions would correspond to the switching off of a nonuniform magnetic field.[23]

We apply to (36), (37) the method of the inverse scattering problem, developed for the sine-Gordon equation by Ablowitz *et al.*[24] (see also Ref. 25). This method can be summarized as follows: We introduce the auxilliary system of equations

$$\varphi_\xi = U(\lambda, \psi)\varphi, \tag{38}$$

$$\varphi_\tau = V(\lambda, \psi)\varphi, \tag{39}$$

where $\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$, and the matrixes U and V depend on the parameter λ and the unknown function ψ. The matrices U and V are chosen in such a way that Eq. (36) follows from the condition for the compatibility of the system (38), (39):

$$U_\tau - V_\xi + [U, V] = 0. \tag{40}$$

Treating the eigenvalue problem for Eq. (38) as a scattering problem in which the potential is given in terms of ψ and ψ_τ at $\tau = 0$, we can then find the scattering data: φ_N and λ_N at $\tau = 0$. The time evolution of the scattering data is determined by Eq. (39). The method of the inverse scattering problem makes it possible to use the scattering data to find the scattering potential, i.e., the function ψ which satisfies Eq. (36).

For the purposes of the present calculation it is sufficient to solve an eigenvlaue problem, so that we can determine, e.g., the velocity of the Nth soliton.

Equation (38) is a system of equations for φ_1 and φ_2. For the case of initial conditions (37), this system of equations takes the form[23]

$$\varphi_{1\xi} = -\frac{1}{2} ik\varphi_1 - \frac{1}{2}\omega(\xi)\varphi_2, \tag{41}$$

$$\varphi_{2\xi} = \frac{1}{2}\omega(\xi)\varphi_1 + \frac{1}{2} ik\varphi_2, \tag{42}$$

where $k = \lambda - 1/4\lambda$.

Maki and Kumar[23] showed that for a Π-shaped function,

$$\omega(\xi) = \omega\theta(l^2 - \xi^2), \tag{43}$$

the problem of finding the eigenvalues λ_N reduces to the problem of solving the equations

$$\sin(lp + q) = 0, \tag{44}$$

where $p^2 = \omega^2 - \varkappa^2$, $\sin q = \dfrac{p}{\omega}$, $\cos q = \dfrac{\varkappa}{\omega}$, and $\varkappa = -ik$.

Purely imaginary values of λ correspond to solitons, while complex-conjugate pairs $(\lambda, -\lambda^*)$ correspond to breather (pulsating) modes or doublet solitons.[24]

It follows from the solution of (44) that the following inequality must hold for the formation of a pair of solitons or breather modes[23]:

$$I > I_N; \tag{45}$$

where $I = l\omega$, $I_N = \pi(N - \frac{1}{2})$, $N = 1,2,\ldots$. Pairs of solitons are created under the condition[23]

$$\omega > \omega_N \tag{46}$$

(the opposite inequality corresponds to breather modes). The dependence of ω_N on I was found numerically in Ref. 23, but at values of $|\varkappa/\omega|$, which are not too close to unity it is simple to derive the following expression[26] for ω_N:

$$\omega_N = \frac{1+I^2}{I(1+I^2-I_N^2)-I_N}. \tag{47}$$

This expression remains qualitatively correct even as we let $|\varkappa/\omega|\to 1$. The velocity of the Nth soliton and the frequency of the Nth breather mode are, respectively,

$$v_N = \left(1-\frac{\omega_N^2}{\omega^2}\right)^{1/2}, \tag{48}$$

$$\Omega_N = \left(1-\frac{\omega^2}{\omega_N^2}\right)^{1/2}. \tag{49}$$

The results found here make it possible to describe experiments in which a magnetic field is switched off in terms of an inverse scattering problem (the quantity ω is proportional to the field which is switched off).

5.2. A phase. V-solitons

Let us examine the solitons which exist in the A phase of He^3 by virtue of the dipole energy in (12). We begin by writing the energies which we will need in the Lagrangian, taking into account the form of the order parameter in the A phase, (8). The gradient part of the potential energy is[27] (in the absence of surface terms)

$$F_{\text{grad}} = \frac{K\Delta_A^2}{2}\int d^3r\,[3\,|\mathrm{div}\,\Delta|^2+|\mathrm{rot}\,\Delta|^2+2\,|(\Delta\nabla)\,\mathbf{V}|^2 + (\mathrm{div}\,\mathbf{V})^2+(\mathrm{rot}\,\mathbf{V})^2]. \tag{50}$$

The kinetic energy describing the rotation of the vector **V** is[28]

$$T = \frac{\chi_A}{2\gamma_0^2}\int d^3r\,[(\omega-\omega_L)^2-\omega_L^2], \tag{51}$$

where ω is expressed in terms of the Euler angles α, β, and γ:

$$\begin{aligned} \omega_x &= -\beta_t \sin\gamma + \alpha_t \cos\delta\sin\beta, \\ \omega_y &= \beta_t\cos\gamma + \alpha_t\sin\delta\sin\beta, \\ \omega_z &= \gamma_t + \alpha_t\cos\beta. \end{aligned} \tag{52}$$

We impose a static magnetic field **H** along the z axis. At equilibrium, the vectors **V** and **l** are therefore in the xy plane. We assume that the distribution of **l** is uniform, and that the V-texture changes only along the z direction. In this case the Lagrangian takes the simple form

$$\mathcal{L} = \frac{\chi_A}{2\gamma_0^2}(\gamma_t^2-2\omega_L\gamma_t-c_\perp^2\gamma_z^2+\Omega_A^2\cos^2\gamma), \tag{53}$$

where γ is the angle between **l** and **V**, $c_\perp = \gamma_0(K\Delta_A^2/\chi_A)^{1/2}$ is the velocity of spin waves propagating perpendicular to **l**, and the last term in (53) is the dipole energy in (12).

From (53) we easily find an equation for the angle between **l** and **V**:

$$\gamma_{tt}-c_\perp^2\gamma_{zz}+\Omega_A^2\sin\gamma\cos\gamma = 0. \tag{54}$$

This equation has the soliton solution

$$\gamma = 2\,\mathrm{arctg}\exp\left[\pm\frac{z-ut}{\xi_{\text{dip}}(1-v^2)^{1/2}}\right], \tag{55}$$

which describes a perturbation which is moving at a velocity $u = vc_\perp$ and whose dimension is $\xi_{\text{dip}} = c_\perp/\Omega_A$ (at $v=0$). The parameter v is determined by the initial conditions.

Maki and Kumar[23] have studied the experimental situation in which a nonuniform magnetic field $H_0(z)$ is switched off. In this case, we must add the following initial condition to Eq. (54):

$$t=0:\quad \gamma = 0,\quad \gamma_t = \delta\gamma_0 H_0(z), \tag{56}$$

where the constant δ is determined empirically (in Ref. 23, $\delta = 1$).

Using the substitution $2\gamma = \psi$, and transforming to dimensionless variables $\tau = \Omega_A t$ and $\xi = \Omega_A z/c_\perp$, we find a standard sine-Gordon equation, (36), with the initial conditions in (37). In the case of a Π-shaped pulse, (43), we would have $l = L/\xi_{\text{dip}}$, $\omega = \delta\gamma_0 H_0/\Omega_A$ here, where $2L$ is the "length" of the pulse, H_0 is the strength of the field which is switched off, and the velocity of the Nth soliton is $u_N = v_N c_\perp$. If we assume, in accordance with Ref. 23, the value $\delta = 1$, we find that at $\omega\sim 1$ hundreds of solitons are produced.[29]

A V-soliton is a domain wall between two orientations of the vector **V**: parallel and antiparallel to **l**. A V-solition can be observed if the homogeneous distribution of is imposed by, for example, boundary conditions. In an open system, a homogeneous distribution of **l** is unstable, and a mixed structure forms in which the vectors **l** and **V** rotate in the plane perpendicular to the magnetic field.[27,29]

Recent experiments[30] in which He^3-A was studied by a pulsed NMR method demonstrated effects of domain walls in the field of the vector **V**. In these experiments, V-solitons (apparently of the splay type[21,22,29]) were produced by an rf pulse which deflected the magnetization through a large angle ($>140°$). Fomin[31] has offered a theoretical explanation for the creation of solitons in such experiments on the basis that the spatially uniform precession of a magnetization deflected through a finite angle is unstable.[32] Fomin showed[31] that periodic structures in which the vector **V** can rotate through 2π with respect to **l**, forming a domain wall, cannot form in the A phase of He^3. A domain wall of this sort would be unstable and should either disappear or break in two. A breakup of domain walls can explain the time evolution of the shift of the NMR frequency which was observed in Ref. 30. Furthermore, a transition from uniform precession to a soliton precession[31] provides a qualitative explanation for the existence of two precession frequencies of a magnetization deflected through a large angle.[30]

5.3. B phase. n-solitons

As we mentioned earlier, a magnetic field gives rise to **n**-textures in the B phase of He^3 which are analogous to magnetic walls in nematic liquid crystals.[21] In this subsection of the paper we examine the creation and propagation of **n**-solitons when a nonuniform magnetic field is switched off. In a situation of this sort, Webb, Sager, and Wheatley[33] observed in He^3-B a propagation of slow magnetic perturbations at a velocity which was a complicated function of the exciting magnetic field. In order to identify the magnetic pertubations observed in Ref. 33 with **n**-solitons, it is necessary to carry out more-detailed measurements of the dependence of the wave velocity on the magnetic field. However, a qualitative explanation for the results of Ref. 3 can be offered in terms of **n**-solitons.[26]

The potential energy in the B phase of He^3 in a magnetic field is the sum of the orientational energy in the magnetic field,[34] (15), the dipole energy (13), and the gradient energy (11), the last of which is written in the following way[35] in terms of the variables **n** and ϑ:

$$F_{\text{grad}} = \frac{K\Delta_B^2}{2} \int d^3r \left\{ (\nabla\vartheta)^2 + 2(1-\cos\vartheta) \times [(\operatorname{div}\mathbf{n})^2 + (\operatorname{rot}\mathbf{n})^2] - \frac{1}{2}[(\mathbf{n}\nabla)\vartheta + \sin\vartheta \operatorname{div}\mathbf{n} + (1-\cos\vartheta)(\mathbf{n}\operatorname{rot}\mathbf{n})]^2 \right\}. \quad (57)$$

The kinetic energy is given by (51), where χ_A must be replaced by χ_B, and it is convenient to express $\boldsymbol{\omega}$ in terms of **n** and ϑ (Ref. 35):

$$\boldsymbol{\omega} = \mathbf{n}\vartheta_t - (1-\cos\vartheta)[\mathbf{n}\mathbf{n}_t] + \sin\vartheta\,\mathbf{n}_t. \quad (58)$$

It is now a straightforward matter to derive an equation of motion for the vector **n**, under the assumption that we are dealing with a Leggett configuration $\cos\vartheta_0 = -1/4$, since the dipole energy is considerably greater than the magnetic energy for a broad range of magnetic fields. We furthermore restrict the analysis to planar structures, ignoring excursions of the vector **n** from the plane because of a mixing[35] of bend and twist deformation.[22] We accordingly assume that the magnetic field is directed along the x axis and we have $\mathbf{n} = \{\cos\varphi(x), \sin\varphi(x), 0\}$. This case corresponds to splay-bend deformations and is described by the equation[26]

$$\varphi_{tt} - c_3^2\varphi_{xx} + \frac{c_3^2}{2\xi_{H3}\varphi_x}\frac{\partial}{\partial x}[\sin^2\varphi(1+\alpha^2\xi_{H3}^2\varphi_x^2)] = 0; \quad (59)$$

Here

$$c_i^2 = \frac{4}{5}\frac{\gamma_0^2}{\chi_B}K_i, \quad \xi_{Hi} = \frac{1}{H}\left(\frac{K_i}{a}\right)^{1/2},$$

$$K_i = \frac{5}{64}b_iK\Delta_B^2, \quad \alpha^2 = 1 - \frac{b_1}{b_3},$$

where $b_1 = 13$, $b_2 = 11$, $b_3 = 16$, K_i are the "Frank constants,"[22] and a is given by expression (6) in Ref. 36 $[(a \sim g_D(\mu_0/\Delta_B)^2]$.

If $K_1 = K_3$ ($\alpha^2 = 0$), static solution (59) represents a wall perpendicular to the magnetic field.[21] An anisotropy ($K_1 \neq K_3$) has the consequence that the wall becomes asymmetric with respect to the $x = 0$ plane.

Seeking a traveling-wave solution (59), $\varphi = \varphi(x - ut)$, we find the condition[26]

$$\frac{u^2}{c_3^2} \ll \frac{b_1}{b_3} \approx 1. \quad (60)$$

Under this condition, we can ignore the difference between K_1 and K_3, and we can solve the following equation instead of Eq. (59):

$$\varphi_{tt} - c^2\varphi_{xx} + \Omega_H^2 \sin\varphi\cos\varphi = 0, \quad (61)$$

where

$$c_1 = c_3 = c, \ \Omega_H^2 = \gamma^2 H, \ \gamma^2 = \frac{4}{5}\frac{a\gamma_0^2}{\chi_B}.$$

The situation with a magnetic field being switched off is now described precisely as in the preceeding subsection, except that now we have

$$l = \frac{L}{\xi_H}, \quad \omega = \frac{\delta\gamma_0 H_0}{\gamma H},$$

and the velocity of the Nth soliton is

$$u_N = c\left(1 - \frac{\omega_N^2}{\omega^2}\right)^{1/2}. \quad (62)$$

Let us briefly discuss the behavior of the soliton velocity as a function of the temperature T and of the fields H_0 and H. The temperature dependence of the limiting soliton velocity $c \propto \Delta_B \propto [1 - T/T_c]^{1/2}$ (we are ignoring the temperature dependence of χ_B here), is the same as the temperature dependence of the velocity of the magnetic perturbations observed in Ref. 33. This velocity is roughly three or four times smaller than c; the velocity of the magnetic perturbations increased with decreasing field H_M and with increasing field H_R (Ref. 33). According to (62), the velocity of an **n**-soliton depends in a similar way on the fields H and H_0, which correspond to H_M and H_R. That this is true can be seen easily when we have $\omega_N \sim 1$, and the dependence of u_N on H and H_0 is determined primarily by the quantity $\omega \propto H_0/H$. We should also note that the temperature dependence of the soliton velocity given by (62) is stronger than that observed in Ref. 33 if we take into account the "experimental" temperature dependence of the quantity a, in accordance with Ref. 36.

5.4. B phase. ϑ-solitons

In the B phase of He^3, the ϑ-solitons stem from a dipole-dipole interaction, as do the V-solitons in the A phase. The energies which we need have already been written—in (13), (51), and (57)—so that we can immediately write an equation for $\mathbf{n}\|\mathbf{H}\|Oz$, under the assumption that the field of the vector **n** is constant and that we have $\vartheta = \vartheta(t,z)$ (Refs. 37 and 23):

$$\vartheta_{tt} - c_\parallel^2\vartheta_{zz} = \frac{16}{15}\Omega_B^2\left(\cos\vartheta + \frac{1}{4}\right)\sin\vartheta, \quad (63)$$

where $c_\parallel = \gamma_0(K\Delta_B^2/2\chi_B)^{1/2}$ is the velocity of the spin waves along the **n** direction.

We have two types of soliton solutions of Eq. (63) (Ref. 37):

$$\operatorname{tg}\frac{\vartheta}{2} = \pm\left(\frac{5}{3}\right)^{1/2} \operatorname{cth}\frac{z-ut}{2\xi_{\text{dip}}(1-v^2)^{1/2}}, \quad (64)$$

$$\operatorname{tg}\frac{\vartheta}{2} = \pm\left(\frac{5}{3}\right)^{1/2} \operatorname{th}\frac{z-ut}{2\xi_{\text{dip}}(1-v^2)^{1/2}}, \quad (65)$$

where $\xi_{\text{dip}} = c_\parallel/\Omega_B$ is the dipole length, $u = vc_\parallel$ is the soliton velocity, and v is a parameter.

In a soliton of the first type, in (64), the angle ϑ varies from ϑ_0 to $2\pi - \vartheta_0$; in a soliton of the second type, a domain wall, (65), ϑ varies from $-\vartheta_0$ to ϑ_0, where $\cos\vartheta_0 = -1/4$. Mineyev and Volovik[11] have shown that a soliton of this second type undergoes a continuous transformation into an **n**-soliton with decreasing energy.

Solutions (64) and (65) are particular solutions of Eq. (63). In contrast with the case of the sine-Gordon equation, we cannot analytically derive N-soliton solutions in this case. We simply note that, as in the case of the sine-Gordon equation, states analogous to breather modes have been found for Eq. (63) by numerical integration.[38]

6. ORBITAL DYNAMICS IN THE PRESENCE OF A SUPERFLUID FLOW IN He^3-A

Up to this point we have been concerned primarily with the spin subsystem of the superfluid phases of He^3. We have simply mentioned briefly that mixed spin-orbital textures may form in the A phase of He^3 by virtue of a dipole-dipole interaction.[27,29] There is yet another important case in which a wall appears between two opposite orientations of **l**. Such a wall arises when there is a superfluid flow in a liquid. This situation was studied by Hall and Hook,[39,40] who proposed an explanation for the experiments by Paulson, Krusius, and Wheatley,[41,42] which had revealed a persistent orbital motion in He^3-A. Paulson *et al.* observed persistent oscillations of the intensity of sound transmitted through the A phase after peliminary changes in the direction and strength of a magnetic field. These oscillations could continue for hours, and the very finest details of the shape of the oscillations were reproduced.[43]

Hook and Hall[40] showed that these oscillations can be explained in terms of a precession of a domain wall in the field of the vector **l**, to whose orientation the absorption of sound is very sensitive. The period of the oscillations is $3\pi\mu_l/\rho_{s\parallel} v_S^2$, according to Ref. 40 (where μ_l is the orbital viscosity coefficent, and $\rho_{s\parallel}$ is the superfluid density along **l**). As a result we find a rather good description of the period and temperature dependence of the oscillations of the sound intensity in the experiments by Paulson *et al.*[41]

Volovik[44] and Hall[45] have proposed another explanation for these experiments. According to this alternative explanation,[44,45] a precession of the vector **l** under dissipative conditions, by analogy with the time-varying Josephson effect in an ordinary superfluid liquid. This effect stems from a so-called phase-slippage mechanism.[46] Before we take up the description of the phase-slippage mechanism in He^3-A, which is called an "instanton mechanism,"[47] we consider the dissipative superfluid motion and the phase-slippage mechanism in superfluid He^4 and a superconductor.

6.1. Dissipative superfluid motion

As we have already mentioned, superfluidity stems from ordering in a system or from coherent behavior of a macroscopic fraction of the particles of a superfluid liquid. From the topological standpoint, this phenomenon stems from a degeneracy in the phase of the wave function of a condensate or an order parameter. Another, and no less interesting, consequence of the topological properties of the space of the order parameter is a different type of superfluid motion: dissipative. In this case, there is a periodic and abrupt change in phase (the "phase slippage") of the order parameter due to the formation of vortices in the liquid flow when the velocity of the superfluid flow exceeds a certain critical level. This is the case of the time-varying Josephson effect.[46,7,9] In a superconductor, this process is accompanied by the emission of electromagnetic radiation.

A vortex in superfluid He^4 or in a superconductor is a singular line. As this line is circumvented along a closed contour, the phase of the order parameter, (1), changes by $2\pi n$ (where n is an integer), while the phase is not defined on the line itself. Far from the vortex line, the order parameter is described by

$$\theta(\mathbf{r}) = \theta_\infty \exp(in\varphi),\ n = 0,\ \pm 1,\ \pm 2,\ \ldots,$$

where θ_∞ = const, φ is the polar angle in a cylindrical coordinate system $\{z,\rho,\varphi\}$, and the vortex line runs along the x axis. The form of the order parameter is found from the corresponding Euler-Lagrange equations. Far from the vortex axis, the equation for the phase $\Phi = \Phi(\varphi)$ takes the simple form

$$\frac{d^2\Phi}{d\varphi^2} = 0,$$

from which we find $\Phi = C\varphi$, where C is a constant. From the requirement that the order parameters $\theta(\mathbf{r})$ be continuous upon a change of 2π in the angle φ we conclude that the constant C must be equal to an integer n ($n = 0, \pm 1, \pm 2,\ldots$). The distribution of the order-parameter field can be described in a graphic way as the distribution of a two-dimensional vector (of length θ_∞) which can be introduced in place of $\theta_\infty \exp\{in\varphi\}$. For example, in the case $n = 1$ a vortex can be depicted by the field of vectors of this sort, tangent to concentric circles. The center of the circles corresponds to the vortex line and is a singular point of the vector field (the direction of the vector field is not defined at this point). Actually, $|\theta|$ (and thus the superfluid density) vanishes on the vortex line. This circumstance corresponds to a destruction of the condensate and to a transition of the liquid into the normal state. The latter assertion means that energy must be expended on the formation of a vortex, since the ordered state corresponds to a minimum of the energy (at temperatures below T_c).

We turn now to the dissipative case of superfluid motion, associated with the formation of vortices in a superfluid flow. We write the equation of motion of an ordinary superfluid liquid:

$$\frac{d\mathbf{v}_s}{dt} = -\nabla\mu^0, \tag{66}$$

where μ^0 is the chemical potential in the coordinate system in which the velocity of the normal motion is $\mathbf{v}_n = 0$ (here and below, we are asssuming $\mathbf{v}_n \equiv 0$).[6] Equation (66) describes an acceleration of the superfluid liquid by an external force.

Since $\mathbf{v}_s = (\hbar/m)\Delta\Phi$, Eq. (66) can be integrated along some path connecting points 1 and 2:

$$\frac{d}{dt}(\varphi_2 - \varphi_1) = -\frac{m}{\hbar}(\mu_2^0 - \mu_1^0), \tag{67}$$

where m is the mass of the particle (the He^4 atom or the Cooper pair). According to Eq. (67), the phase difference between certain points increases linearly with the time. Anderson[46] showed that vortices prevent an unbounded acceleration of a superfluid liquid; as they intersect the line connecting points 1 and 2, the vortices reduce the phase difference by 2π (in the case $n = 1$). The result is the establishment of a steady state, which can be described by averaging Eq. (67) over a long time interval τ. The number of vortices which intersect the line connecting points 1 and 2 in a unit time interval is therefore[46]

$$\left\langle \frac{dN_V}{dt} \right\rangle = \frac{1}{2\pi\tau} \int_0^{\tau} \frac{d(\varphi_2 - \varphi_1)}{dt}\, dt = -\frac{m}{2\pi\hbar} \langle \mu_2^0 - \mu_1^0 \rangle. \quad (68)$$

Equation (68) determines the steady-state dissipative flow of a superfluid liquid which is the result of phase slippage, i.e., a change in the phase difference between certain points due to the formation of vortices.

6.2. Instanton mechanism for phase slippage

Anderson and Toulouse[48] have shown that phase slippage can occur in the A phase of superfluid He^3 as a result of the motion of nonsingular vortices (the order-parameter fields are continuous on the lines of these vortices; in other words, the condensate does not break up[7,9]). In He^3-A, however, we are apparently dealing with the instanton mechanism of phase slippage proposed by Volovik[44,7,9] and Hall.[45] Let us examine this mechanism in more detail.

In the case of the He^3 A phase we would replace Eq. (66) by[6] ($\mathbf{v}_n = 0$)

$$\frac{\partial v_s^i}{\partial t} = -\frac{\partial \mu^0}{\partial x_i} - \frac{\hbar}{2m} \mathbf{l} \left[\frac{\partial \mathbf{l}}{\partial x_i} \frac{\partial \mathbf{l}}{\partial t} \right]. \quad (69)$$

We consider a one-dimensional motion, assuming that all the variables depend on only the single coordinate (z) along the flow ($v_s \equiv v_s^z$). We rewrite (69) as

$$\frac{\partial v_s}{\partial t} = -\frac{\partial \mu^0}{\partial z} - n, \quad (70)$$

where

$$n = \frac{\hbar}{2m} \mathbf{l} \left[\frac{\partial \mathbf{l}}{\partial z} \frac{\partial \mathbf{l}}{\partial t} \right]. \quad (71)$$

It is easy to see that a steady state is reached in the system if[44]

$$-\left\langle \frac{\partial \mu^0}{\partial z} \right\rangle = \langle n \rangle \neq 0,$$

i.e., if oscillations of the vector $\mathbf{l}$ maintain a nonzero time average of the gradient of the chemical potential.

The quantity n has the meaning of the density of the instanton charge. The integral of n over some space-time region ΔS (in the case at hand, the dimensions of ΔS are determined by the periods of the oscillations along z and t) with a homogeneous distribution of $\mathbf{l}$ at the boundary of ΔS is a topological invariant (a charge), i.e., a quantity which does not depend on continuous deformations of the $\mathbf{l}$ field. Specifically, the integer[44,9]

$$N = \frac{1}{4\pi} \int_{\Delta S} dz\, dt\, \mathbf{l} \left[\frac{\partial \mathbf{l}}{\partial z} \frac{\partial \mathbf{l}}{\partial t} \right]$$

is the "order" of the mapping of ΔS on to the unit sphere $\mathbf{l}^2 = 1$. The order of the mapping is the number of times the sphere is circumvented by the vector $\mathbf{l}$ during motion over the region ΔS. Configurations of the $\mathbf{l}$ field with $N \neq 0$ are "instantons" (Ref. 49, for example).

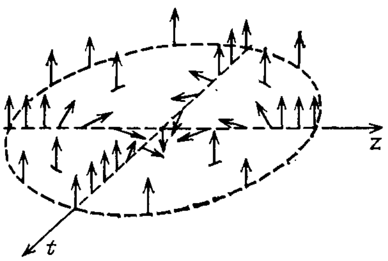

FIG. 1.

Figure 1 shows an example of a configuration of the $\mathbf{l}$ field with $N = 1$. We see from this figure that the inhomogeneous distribution of the $\mathbf{l}$ field has finite dimensions in both space and time; hence the name "instanton." Such configurations were apparently first found for the field of the director $\mathbf{n}$ ($\mathbf{n}^2 = 1$; the directions $\mathbf{n}$ and $-\mathbf{n}$ are equivalent) in a nematic liquid crystal.[22] Figure 2 shows the distribution of the $\mathbf{n}$ field, whose intersection with the y, z plane has the configuration shown in Fig. 1 (with a replacement of the time t by a second spatial coordinate, y). Such a distribution of a unit vector field is also called a "nonsingular vortex" or "linear soliton."[51,11] These are precisely the vortices in the field of the vector $\mathbf{l}$ which Anderson and Toulouse[48] studied in the paper in which they proposed a phase-slippage mechanism. It should also be noted that configurations characterized by the order of a mapping of a sphere onto a sphere (instantons or nonsingular vortices or linear solitons) assumed a general significance after the appearance of Belavin and Polyakov's paper[50] on a planar Heisenberg magnetic material, which is simultaneously a representative of a wide range of field-theory models with an interaction of a geometric type. Belavin and Polyakov[50] derived exact multi-instanton solutions of the equations in the model in which they were working. One of the many examples of the application of the instanton concept was described above: Space-time oscillations of the vector $\mathbf{l}$ in the form of an array of instantons in the space z,t effect a phase slippage in He^3-A. Questions associated with Belavin-Polyakov instantons were studied in detail in the review by Perelomov.[49]

We turn now to the hydrodynamics of the He^3A phase.

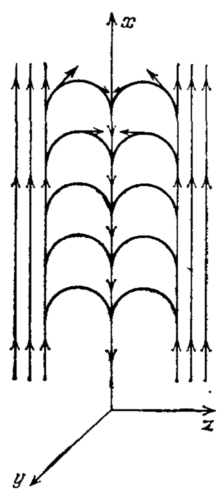

FIG. 2.

6.3. Phenomenological equations and trajectories of the system

A persistent orbital motion in He^3-A occurs when special initial conditions are arranged. Furthermore, a persistent motion will not be observed in practice at pressures below the melting pressure.[43] The numerical solutions of the complicated nonlinear equation for the vector **l** which have been carried out[40] do not tell us which regimes are possible in a superfluid liquid and in which regime will a liquid be under some particular set of conditions. To generate a picture of possible regimes in the behavior of He^3-A containing superfluid flows, Volovik[47] proposed a phenomenological description of the hydrodynamics of He^3-A on the basis of an instanton phase-slippage mechanism.[7,9,44,45] Let us examine that model.

We supplement Eq. (70) for $\mathbf{v}_s$ with an equation for **l** (Refs. 47,6),

$$\mu_l\frac{\partial \mathbf{l}}{\partial t}=-\frac{\delta E}{\delta \mathbf{l}}+\mathbf{l}\left(\mathbf{l}\frac{\delta E}{\delta \mathbf{l}}\right)+\frac{\hbar}{2m}\frac{\delta E}{\delta v_s}\left[\mathbf{l}\frac{\partial \mathbf{l}}{\partial z}\right], \quad (72)$$

where

$$E=\frac{\rho_s v_s^2}{2}+\varepsilon(\mathbf{l}), \quad (73)$$

$\rho_s \sim K\Delta_A^2 m^2/\hbar^2$ is the superfluid density, and $\varepsilon(\mathbf{l})$ is the orbital part of the gradient energy in (50) (Ref. 47).

We now transform to Volovik's "crude" model,[47] in which system (70), (72), a system of hydrodynamic equations for $\mathbf{v}_s$ and **l**, reduces to a system of ordinary differential equations for variables averaged over regions larger than the characteristic region z_0t_0 of the space-time variations of the vector **l**, the density of the gradient energy ε, the density of the superfluid flow v_s, and the density of the instanton charge n. Formally, this approach is made by replacing the derivatives $\partial/\partial z$ and $\partial/\partial t$ by $1/z_0$ and $1/t_0$, respectively, where z_0 and t_0 are given by[47]

$$n\sim\frac{\hbar}{m}\frac{1}{t_0z_0}, \quad (74)$$

$$\varepsilon\sim\frac{K\Delta_A^2}{z_0^2}\sim\rho_s\frac{\hbar^2}{m^2}\frac{1}{z_0^2}. \quad (75)$$

For example, an equation for ε can be found from Eq. (72), multiplied by $\partial \mathbf{l}/\partial t$,

$$\frac{\partial\varepsilon}{\partial t}=\frac{\delta\varepsilon}{\delta\mathbf{l}}\frac{\partial\mathbf{l}}{\partial t}=-\mu_l\left(\frac{\partial\mathbf{l}}{\partial t}\right)^2+\rho_s v_s n, \quad (76)$$

and the estimate

$$\left(\frac{\partial\mathbf{l}}{\partial t}\right)^2\sim\frac{1}{t_0^2}\sim\rho_s\frac{n^2}{\varepsilon}. \quad (77)$$

Substituting (77) into (76), we find

$$\frac{\partial\varepsilon}{\partial t}=\rho_s v_s n-a_1\rho_s\mu_l\frac{n^2}{\varepsilon}, \quad (78)$$

where $a_1>0$ is a parameter of the model, which is of order unity. An equation for n is derived in an analogous way; it contains four phenomenological parameters.[47]

An equation for v_s can be found by replacing the quantities in (70) by their average values. The gradient of the chemical potential either is given or in the case of a given current, is found from (70).

We thus have a system of equations for the large-scale variables v_s, ε, and n. This system of equations contains five dimensionless parameters, one of which can be eliminated by a gauge transformation. In the case of given current, ($\partial v_s/\partial t=0$), we are left with two equations, which take the following form[47] in the system of units $v_s=\rho_s=\mu_l=1$:

$$\frac{\dot{\varepsilon}}{\alpha_1}=n-\frac{n^2}{\varepsilon},\quad \alpha_1>0, \quad (79)$$

$$\dot{n}=n\varepsilon-\frac{n^3}{\varepsilon^2}+\frac{\alpha_2n^2}{\varepsilon}-\alpha_3n, \quad (80)$$

where α_1, α_2, and α_3 are parameters of the model.

A study of the dependence of the trajectories of the system, (79), (80), on the parameters α_i yields information on the states of the A phase in the presence of superfluid flows and on transitions of the system from one state to another induced by temperature changes and external fields. Such transitions correspond to changes in the topology of trajectories (79), (80) or bifurcations.

We see from (79), (80) that two types of steady state are possible:

1) Without dissipation, i.e., in the absence of instantons,

$$n=0,\quad \varepsilon=\varepsilon_0, \quad (81)$$

where ε_0 is arbitrary.

2) An oscillatory dissipative steady state,

$$\varepsilon=(1+\alpha_3-\alpha_2)\rho_s v_s^2,\quad n=\frac{\varepsilon v_s}{\mu_l}, \quad (82)$$

under the condition

$$1+\alpha_3>\alpha_2. \quad (83)$$

In the case without dissipation, with $\varepsilon_0\neq o$, there is a superfluid flow with an inhomogeneous distribution of **l**.

In the dissipative case, as follows from (74), (75), and (82), the time and space scales of the oscillations are[47]

$$t_0\sim\frac{\mu_l}{\rho_s v_s^2},\quad z_0\sim\frac{\hbar}{mv_s}. \quad (84)$$

In other words, we have the oscillatory situation which was observed experimentally in Ref. 41 and 42. This situation is stable if

$$\alpha_1+\alpha_2<2. \quad (85)$$

In addition to the cases which we have just listed, there may be a current state without dissipation in the system.[47] One of the possible regimes in which such a state is stable is[47]

$$\alpha_1>1,\ \alpha_3>0. \quad (86)$$

It is easy to see that the ranges of the parameters α_i which are determined by inequalities (83), (85), and (86) overlap, so that there may be either a homogeneous state or a dissipative state, depending on the initial conditions. This is apparently the result which has been found experimentally[41] (see the discussion above).

7. CONCLUSION

Solitons and instantons are found in many physical systems. Several of these systems were described above. We can add a few related examples: In magnetic substances, domain walls or solitons can exist. In rotating He^3-A, nonsingular vortices form.[52–54] In thin superconductors, there is a phase-slippage mechanism analogous to that discussed in the preceeding section of the present paper.[55] This list of examples could easily be continued, but we turn instead to some general comments.

Superfluid helium-3 is interesting in many regards. On the one hand, the superfluid phases of He^3 constitute rather exotic physical entities which exist at low temperatures. On the other hand, these phases combine the properties of many physical systems, such as magnetic substances, liquid crystals, and superconductors; they are also a real embodiment of certain field-theory models. Furthermore, the realization of triplet pairing in He^3 makes it extremely likely that the superfluid phases of He^3 are not the only entities in nature whose Bose condensates exhibit anisotropic properties. For example, triplet pairing may occur in neutron stars and superconductors.[9]

Yet another important point in research on superfluid helium-3 is the circumstance that a description of the properties of the superfluid phases of helium-3 has required appealing to topological methods. The reason is that the spaces of the order parameter of the He^3 superfluid phases have a nontrivial topology, which allows the existence of a variety of topologically stable extended entities (vortices, disclinations, solitons, instantons, etc.) in the field of the order parameter. Furthermore, as was mentioned above, the spaces of the order parameter are different at different length scales, so that the field configurations which are topologically stable at short range may be unstable at long range. The net result is that we must abandon the ordinary methods which are used, for example, in research on disclinations in liquid crystals.[22] The methods of the theory of homotopy[10] have made it possible to classify extended entities in superfluid phases of helium-3.

Finally, it is also important to note that topology not only gives us methods for studying the superfluidity of helium-3 (and of other physical systems) but also a language for describing physical phenomena which involve extended entities. In other words, the topological consequences of the theory are amenable to direct experimental observation.

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[1)]The order parameter is proportional to the quantity $\Delta(T)$ (T is the temperature), which has the meaning of a gap in the quasiparticle spectrum. The quantity $\Delta(T)$ may be regarded as a phenomenological parameter of the theory. Near the phase-transition temperature T_c, the behavior of this parameter is $\Delta(T) \propto (T_c - T)^{1/2}$, and in the limit $T \to 0$ we have $\Delta \to$const. See Ref. 5 for a detailed interpretation of the quantity $\mathbf{d}(\mathbf{n})$ and the choice of normalization (see also Ref. 8).

[2)]The matrix element of the order parameter, A_{ij} (of column j and row i), transform as a vector under a rotation of the coordinate space or of the orbital space and also as a vector under a rotation of the spin space.

[3)]Commutation relations (24) and (25) can be derived by a variety of methods. The quantities $\mathbf{S}$ and $A_{\mu i}$ may be regarded as quantum-mechanical operators, and the corresponding commutators can be calculated by specifying $\mathbf{S}$ and $A_{\mu i}$ in terms of the creation and annihilation operators of the BCS theory.[16] It should also be kept in mind that $\mathbf{S}$ and $A_{\mu i}$ are macroscopic quantities, and for them we can calculate Poisson brackets and write classical equations of motion. Alternatively, we could switch from Poisson brackets to commutators in accordance with

$$\{X, Y\} \to \frac{i}{\hbar}[\hat{X}, \hat{Y}].$$

The simplest approach, however, is to regard the components of the spin operator $\mathbf{S}$ as the generators of the group SO(3) of three-dimensional spin rotations [or the generators of an associated representation of the SU(2) group]. Relations (24) are then simply commutation relations for the generators of the SO(3) group, and relations (25) follow from (24) if we note that the spin index of the matrix $A_{\mu i}$ specifies the components of a vector which takes on values in a Lie algebra of the SO(3) group (the generators S_i form the basis of this algebra).

[1]L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **37**, 1794 (1959) [Sov. Phys. JETP **10**, 1267 (1960)].
[2]D. D. Osheroff, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. **28**, 885 (1972).
[3]P. W. Anderson and P. Morel, Phys. Rev. **123**, 1911 (1961).
[4]R. Balian and N. R. Werthamer, Phys. Rev. **131**, 1553 (1963).
[5]A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975).
[6]W. F. Brinkman and M. C. Cross, in: Progress in Low Temperature Physics (Ed. D. F. Brewer), Vol. VII A, North-Holland, Amsterdam (1978), p. 105.
[7]G. E. Volovik and V. P. Mineev, Fizika i topologiya (Physics and Topology), Znanie, M., (1980).
[8]V. P. Mineev, Usp. Fiz. Nauk **139**, 303 (1983) [Sov. Phys. Usp. **26**, 160 (1983)].
[9]G. E. Volovik, Usp. Fiz. Nauk **143**, 73 (1984) [Sov. Phys. Usp. **27**, 363 (1984)].
[10]G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **72**, 2256 (1977) [Sov. Phys. JETP **45**, 1186 (1977)].
[11]V. P. Mineev and G. E. Volovik, Phys. Rev. **B18**, 3197 (1978).
[12]G. Barton and M. A. Moore, J. Phys **C7**, 4220 (1974).
[13]P. G. DeGennes, Phys. Lett. **A44**, 271 (1973).
[14]W. F. Brinkman, H. Smith, D. D. Osheroff, and E. I. Blount, Phys. Rev. Lett. **33**, 624 (1974).
[15]D. D. Osheroff, W. J. Gully, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. **29**, 920 (1972).
[16]A. J. Leggett, Ann. Phys. (N.Y.) **85**, 11 (1974).
[17]W. F. Brinkman and H. Smith, Phys. Lett. **A51**, 449 (1975).
[18]D. D. Osheroff and W. F. Brinkman, Phys. Rev. Lett. **32**, 584 (1974).
[19]W. F. Brinkman, Phys. Lett. **A49**, 411 (1974).
[20]D. D. Osheroff, in: Quantum Fluids and Solids (Ed. S. B. Trickey, E. D. Adams, and J. W. Dufty), Plenum, N.Y., 1978, p. 161 (Russ. Transl., Mir, M., 1979, p. 164).
[21]W. Helfrich, Phys. Rev. Lett. **21**, 1518 (1968).
[22]P. G. deGennes, The Physics of Liquid Crystals, Clarendon Press, Oxford, 1974 (Russ. Transl. Mir, M., 1977).
[23]K. Maki and P. Kumar, Phys. Rev. **B14**, 3920 (1976).
[24]M. J. Ablowitz, D. J. Kaup, A. S. Newell, and H. Segur, Phys. Rev. Lett. **30**, 1262 (1973).
[25]V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, Teoriya solitonov: Metod obratnoĭ zadachi (Theory of Solitons: Method of the Inverse Problem), Nauka, M., 1980).
[26]S. S. Rozhkov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 44 (1981) [JETP Lett. **33**, 41 (1981)].
[27]K. Maki and P. Kumar, Phys. Rev. **B16**, 182 (1977).
[28]K. Maki, Phys. Rev. **B11**, 4264 (1975).
[29]K. Maki, in: Quantum Fluids and Solids (Ed. S. B. Trickey, E. D. Adams, and J. W. Dufty), Plenum, N.Y., 1978, p. 65 (Russ. Transl. Mir, M., 1979, p. 56).
[30]C. M. Gould, T. J. Bartolac, and H. M. Bozler, J. Low Temp. Phys. **39**, 291 (1980).
[31]I. A. Fomin, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 317 (1981) [JETP Lett. **33**, 301 (1981)].
[32]I. A. Fomin, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 179 (1979) [JETP Lett. **30**, 164 (1979)].
[33]R. A. Webb, R. E. Sager, and J. C. Wheatley, Phys. Lett. **A54**, 243 (1975).
[34]S. Engelsberg, W. F. Brinkman, and P. W. Anderson, Phys. Rev. **A9**, 2592 (1975).
[35]K. Maki and P. Kumar, Phys. Rev. **B16**, 4805 (1977).
[36]H. Smith, W. F. Brinkman, and S. Engelsberg, Phys. Rev. **B15**, 199 (1977).
[37]K. Maki and P. Kumar, Phys. Rev. **B14**, 118 (1976).
[38]P. W. Kitchenside, P. J. Caudrey, and R. K. Bullough, Phys. Scripta **20**, 673 (1979).
[39]H. E. Hall and J. R. Hook, J. Phys. C **10**, L91 (1977).
[40]J. R. Hook and H. E. Hall, J. Phys. **C 12**, 783 (1979).
[41]D. N. Paulson, M. Krusius, and J. C. Wheatley, Phys. Rev. Lett. **36**, 1322 (1976).
[42]M. Krusius, D. N. Paulson, and J. C. Wheatley, J. Low Temp. Phys. **33**, 255 (1978).

[43]J. C. Wheatley, in: Quantum Fluids and Solids (Ed. S. B. Trickey, E. D. Adams, and J. W. Dufty), Plenum, N.Y., 1978, p. 85 (Russ. Transl. Mir, M., 1979, p. 109).
[44]G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 605 (1978) [JETP Lett. **27**, 573 (1978)].
[45]H. E. Hall, J. Phys. (Paris) **39**, No. 8, Colloq. No. 6/1, 15 (1978).
[46]P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).
[47]G. E. Volovik, Zh. Eksp. Teor. Fiz. **79**, 309 (1980) [Sov. Phys. JETP **52**, 156 (1980)].
[48]P. W. Anderson and G. Toulouse, Phys. Rev. Lett. **38**, 508 (1977).
[49]A. M. Perelomov, Usp. Fiz. Nauk **134**, 577 (1981) [Sov. Phys. Usp. **24**, 645 (1981)].
[50]A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975).
[51]V. R. Chechetkin, Zh. Eksp. Teor. Fiz. **71**, 1463 (1976) [Sov. Phys. JETP **44**, 766 (1976)].
[52]G. E. Volovik and N. B. Kopnin, Pis'ma Zh. Eksp.Teor. Fiz. **25**, 26 (1977) [JETP. Lett. **25**, 22 (1977)].
[53]G. E. Volovik and P. J. Hakonen, J. Low Temp. Phys. **42**, 503 (1981).
[54]I. A. Fomin and V. G. Kamenskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 241 (1982) [JETP Lett. **35**, 302 (1982)].
[55]B. I. Ivlev and N. B. Kopnin, Pis'ma Zh. Eksp. Teor. Fiz. **28**, 640 (1978) [JETP Lett. **28**, 592 (1978)].

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