Electromagnetic field lines of a point charge moving arbitrarily in vacuum

S. G. Arutyunyan

Erevan Physics Institute
Usp. Fiz. Nauk **150**, 445–452 (November 1986)

The Lienard-Wiechert field of an arbitrarily moving charge can be visualized with the aid of electric and magnetic field lines. The system of such lines is Lorentz-invariant if a specified motion along Poynting vectors is ascribed to them. In this note the field lines are parametrized by light signals emitted from the trajectory. Ordinary differential equations are obtained describing the instantaneous configuration of electric and magnetic field lines. These equations have been solved in a general form for the magnetic field, and for the electric field in the case when the quantity $\tau/\varkappa\gamma$ is constant where γ is the Lorentz factor of the charge, and $\varkappa$ and τ are the curvature and torsion of the trajectory.

This study of the geometric field structure of a charge in curvilinear motion was prompted by the practical question of stability of motion of intense, ultrarelativistic particle beams in accelerators and storage rings. It appeared impractical to calculate the Lorentz forces between the beam particles via the infinitely broad Fourier-decomposition of the field of a single charge. The first approximation to the space-time field pattern of a particle in curvilinear motion was obtained in Refs. 1 and 2 by solving the retardation equation in Lienard-Wiechert field equations for distances comparable to the radius of curvature of the trajectory. Perhaps, further investigation of the field geometry was not practically essential, but the spatially mosaic structure of the retardation equation solution—and hence of the field itself—encouraged the attempt to "view" the entire field of the particle, long distances included.

This study does so by focussing on the electromagnetic field lines as well as the Poynting vector field. The field lines were conveniently parametrized by light signals emitted from the trajectory at retarded times. The resulting geometric information proved useful in a number of practical problems; in particular, it became possible to evaluate the coherent superposition of the fields of many particles moving along curvilinear trajectories.

1. The intensity of the Lienard-Wiechert field at the observation point is determined by the characteristics of motion of the field-creating charge at a single trajectory point. Therefore it is convenient to apply the concept of a point light signal. At the observation time t (the same for all points in space) the totality of light signals emitted at a retarded time t' forms a sphere of radius $D = c(t - t')$ centered at the point $\mathbf{r}_0(t')$, where $\mathbf{r}_0(t')$ is the position vector of the charge. The system of such mutually enveloping spheres, hereafter referred to as light spheres, fills all space (Fig. 1) and serves as a convenient coordinate system for this problem. A point on a light sphere is specified by the direction of a unit vector $\mathbf{n}$ from the centre of the light sphere. The quantities $(D,\mathbf{n})$ determine a unique point in space. The inverse problem, that is the determination of $(D,\mathbf{n})$ for a given point $\mathbf{r}$ requires the solution of the retardation expression $|\mathbf{r} - \mathbf{r}_0(t - D/c)| = D$, but such a procedure is unnecessary as the Lienard-Wiechert field equations can be naturally expressed in terms of parameters D and $\mathbf{n}(D)$.

An arbitrary line at a fixed moment t may be specified by the parametrization $(D(\sigma),\ \mathbf{n}(\sigma))$, such that as σ changes the radius vector $\mathbf{R}$ "unfolds" in space according to $\mathbf{R}(\sigma) = \mathbf{r}_0(t - D(\sigma)/c) + D(\sigma)\mathbf{n}(\sigma)$. The tangent direction to a given line is obtained by differentiating with respect to σ. By comparing this direction with the prescribed vector field $\mathbf{F}(D,\mathbf{n})$ it is possible to find equations for $(D(\sigma), \mathbf{n}(\sigma))$ that determine the line integrals of the $\mathbf{F}(D,\mathbf{n})$ field.

Omitting the intermediate steps (which are found in Refs. 3 and 4), let us consider the principal results for the case when $\mathbf{F}$ stands for the electromagnetic field of a charge

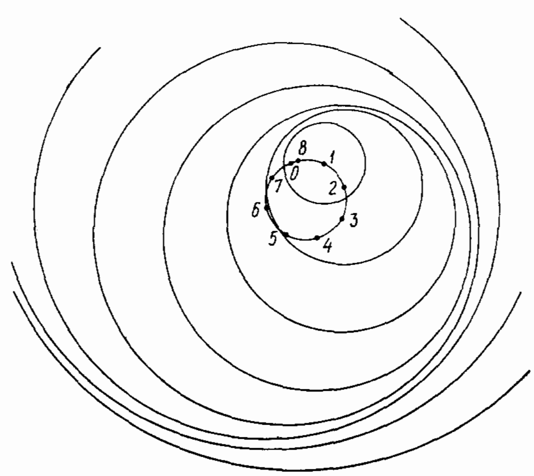

FIG. 1. The system of light spheres of a particle moving in a circle with $\beta \simeq 0.9$. Shown are the intersections of light spheres with the plane of the orbit. The numbers denote the light sphere centers corresponding to the particle positions at times $t - t_0 k$ where $k = 1,2,\ldots$. The region of minimum spacing between the spheres is characteristically spiral-like; in this so-called γ-region the field attains extremal values.

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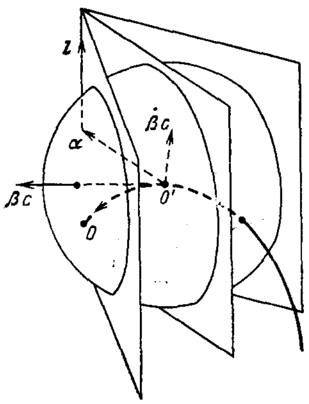

FIG. 2. The circular magnetic field lines of a charge in arbitrary motion. The algorithm of constructing a family of lines on one of the light spheres of radius R is illustrated. Points O and O' lie on the trajectory: O represents the particle's position at the observation time t, O' represents its position at the retarded time $t' = t - (D/c)$. Magnetic lines are constructed by slicing the light sphere with a bundle of planes with a common axis $\mathbf{l}$, where $\mathbf{l}$ is perpendicular to the velocity $\boldsymbol{\beta}c$ and acceleration $\dot{\boldsymbol{\beta}}c$ at point O'. The position of the axis is specified by vector $\mathbf{a}$ originating at O', such that the scalar product of $\mathbf{a}$ with $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ respectively equals: $\mathbf{a}\cdot\boldsymbol{\beta} = D, \mathbf{a}\cdot\dot{\boldsymbol{\beta}} = \gamma^{-2}c$. By varying D the system of lines can be constructed over all space.

in arbitrary motion, and also for the Poynting vector field of synchrotron radiation.

2. The magnetic lines lie on light spheres (obviously, since $\mathbf{H} = \mathbf{n}\times\mathbf{E}$). On the light sphere of radius D they are circles "sliced" by a bundle of planes that intersect on an axis perpendicular to the vectors $\boldsymbol{\beta}(t - D/c)$ and $\dot{\boldsymbol{\beta}}(t - D/c)$ (where $\boldsymbol{\beta}c$ is the particle velocity, and $\dot{\boldsymbol{\beta}}(t) = d\boldsymbol{\beta}(t)/dt$). The position of this axis in space is defined by the vector $\mathbf{a}$ originating from the centre of the sphere, such that the projections of $\boldsymbol{\beta}$ and $\gamma^2 D\dot{\boldsymbol{\beta}}/c$ onto $\mathbf{a}$ equal D (where $\gamma = (1 - \beta^2)^{-1/2}$ is the Lorentz factor of the particle). The bundle axis always lies outside the sphere since $\beta < 1$ Fig. 2).

Let us consider the limiting cases of the arrangement of the spheres. As the particle approaches uniform, rectilinear motion ($\dot{\boldsymbol{\beta}} \to 0$) the bundle axis recedes to infinity in the direction perpendicular to the particle velocity (Fig. 3b). In this case all magnetic lines become perpendicular to $\boldsymbol{\beta}$ (the same happens as $D \to 0$, i.e. close to the charge).

The bundle axis also recedes to infinity in the case when the angle between $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ goes to zero (rectilinear accelerated motion). There is an indeterminacy at $\boldsymbol{\beta} = \gamma^2 D\dot{\boldsymbol{\beta}}/c$ but in this exceptional case it can be shown that the magnetic field is zero everywhere on the light sphere.

The lines differ significantly from the rectilinear limit when the distance of the bundle axis from the centre of the light sphere is of the order of the radius of the sphere (Fig. 3a). For steady-state circular motion of radius R this corresponds to the condition $D \gg R(\beta\gamma)^{-2}$. In the ultrarelativistic case the quantity $R(\beta\gamma)^{-2} \sim R\gamma^{-2}$ can become small compared to the average distance l between particle beams in modern electron accelerators and storage rings. Thus for the PETRA storage ring $l \sim 10^{-4}$ cm, whereas $R\gamma^{-2} \sim 10^{-5}$ cm. When calculating the effect of the beam's field on its own stability in such a case, the rectilinear trajectory approximation may become unacceptable.[2]

3. Let us now take the vector field $\mathbf{F}$ to represent the total electric field, including the Coulomb term whose omission would destroy the topology of the field lines.

The scalar product $\mathbf{n}\cdot\mathbf{E}$ never becomes zero, i.e., the electric field vector always "penetrates" the light sphere, so the retarded time can be used for the line parameter σ.

Substituting[1)]

$$\mathbf{n} = \frac{\boldsymbol{\beta}\,(1 + (\boldsymbol{\beta}\mathbf{v}\,(1 - \gamma^{-1})/\beta^2)) + \mathbf{v}\gamma^{-1}}{1 + \boldsymbol{\beta}\mathbf{v}} \tag{1}$$

we obtain for vector $\mathbf{v}$ the following equation[2)]

$$\frac{d\mathbf{v}}{dt'} = \frac{\gamma - 1}{\beta^2}\,\mathbf{v}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}). \tag{2}$$

Formally equation (2) describes the rotation of vector $\mathbf{v}$ with instantaneous angular velocity $\boldsymbol{\Omega} = (\gamma - \ +)\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}/\beta^2$.

An analytic solution of equation (2) is obtained in Refs. 3 and 4 for the case when the quantity $b = \tau|\varkappa\gamma$ is constant along the trajectory, where τ and $\varkappa$ are the torsion and the curvature of the trajectory.

Let us illustrate the electric field lines for the "simplest" kind of curvilinear motion—uniform curvilinear motion of radius R. The lines drawn in Figs. 4–6 lie in the plane of the orbit (denoted by the arrow). At $\gamma = 1.08$ (Fig. 4) the field is of the dipole type: the concentration of lines is uniform on the average with no preferred directions. At two other values $\gamma = 2.5$ and $\gamma = 6$ (Figs. 5 and 6) these properties hold no longer. Characteristic kinks in the lines demarcate the so-

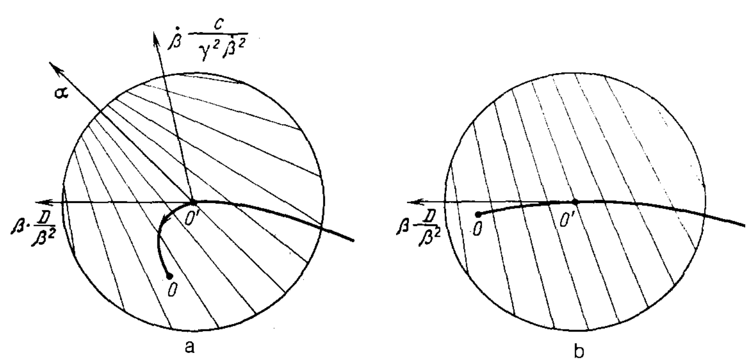

FIG. 3. The characteristic magnetic line patterns for the limiting cases of the bundle axis position. Figure 3a corresponds to the distance from the axis to point O' being of the order of the light sphere radius (as in Fig. 2); whereas in Fig. 3b this distance is much larger than D, so that the light sphere is "sliced" by planes that are almost perpendicular to the direction of motion. The field in Fig. 3a is markedly different from the rectilinear limit shown in Fig. 3b.

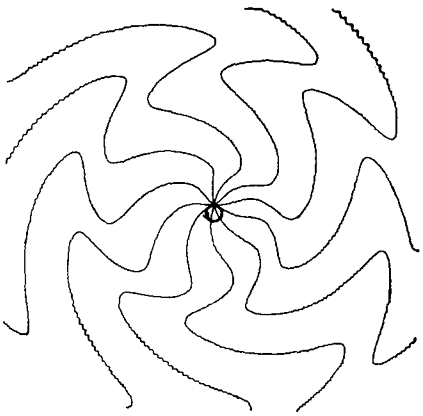

FIG. 4. The "spider" illustrates the electric field lines of a charge with the Lorentz factor $\gamma = 1.08$. The charge moves in a circle in the direction shown by the arrow. The kinks in the lines lie on a spiral, but as yet there is no spatial redistribution of the synchrotron radiation field characteristic of relativistic motion.

called γ-region[2] trailing the charge. In this so-called γ-region the field depends on γ and takes on significantly larger values than in the neighboring regions, where the field is entirely independent of γ. Solving the retardation equation[1] yields for the field amplitude the quantity $E \sim 2\sqrt{2}\, e\gamma^4 R^{-2}\delta^{-1/2}$ where δR is the distance along the γ-region. The dimensions of *γ-region* are $\sim R\gamma^{-3}$ in the plane of the orbit and $\sim R\sqrt{\delta}\gamma^{-1}$ in the perpendicular direction. The neutral magnetic field line that defines the γ-region in the relativistic case is also shown in Fig. 5,—it is also useful to construct such a line for more complicated trajectories, for example with alternating curvature[3] (Fig. 7).

The γ-region field incident on a stationary observer changes sign twice—it is, in fact, synchrotron radiation or, more precisely, its high-energy component (the transverse

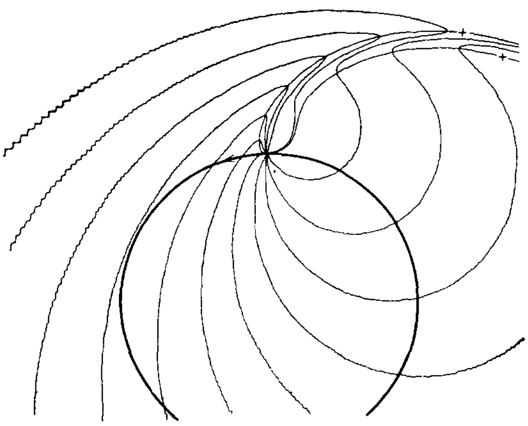

FIG. 5. Synchrotron radiation electric field of a charge with a Lorentz factor $\gamma = 2.5$. This figure requires a greater scale than Fig. 4 because the transverse dimensions of the γ-region characterized by the kinks in the lines are smaller according to the formula $R\gamma^{-3}$, where R is the radius of the orbit. In the γ-region the field attains its maximum values and is "radiative" in character (field components acting on a stationary observer change sign at the $R\gamma^{-3}$ wavelength). Crosses denote the neutral magnetic field line which demarcates the γ-region, as shown in the figure.

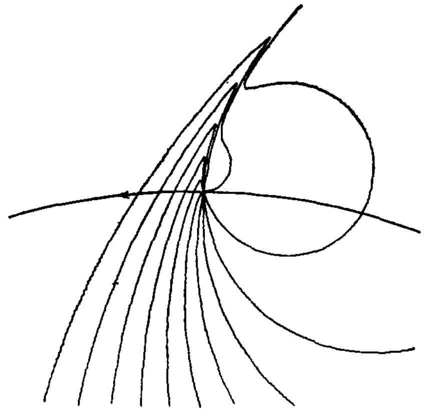

FIG. 6. A Coulomb field region can be defined in the neighborhood of a charge in arbitrary motion: there the electric lines approximate straight lines originating at the charge. The forward extent of the synchrotron radiation Coulomb region reaches $\sim R\gamma^{-3}$ from the particle in the direction of motion. Behind the particle the Coulomb field extends to greater distances $\sim R\gamma^{-2}$. The field pattern of a charge with a Lorentz factor $\gamma = 6$ graphically demonstrates where one may use the approximation of rectilinear motion of the particle.

size of the γ-region $R\gamma^{-3}$ is of the order of the synchrotron radiation wavelength limit). The detailed pattern of the γ-region field lines is clearly visible in Fig. 6($\gamma = 6$). In that same figure we see the Coulomb field region: its forward extent ranges out to $\sim R\gamma^{-3}$ from the particle, whereas the size of the region is $\sim R\gamma^{-2}$.

4. The mathematical apparatus used to construct the system of electric and magnetic lines yields several new characteristics of the field. One such quantity is the phase of the electric field lines

$$\Phi = c\,(1 + b^2)^{1/2} \int_{t'}^{t} \beta\varkappa\gamma\, dt''. \tag{3}$$

In the b = const case the direction cosines of vector $\mathbf{n}$ can be

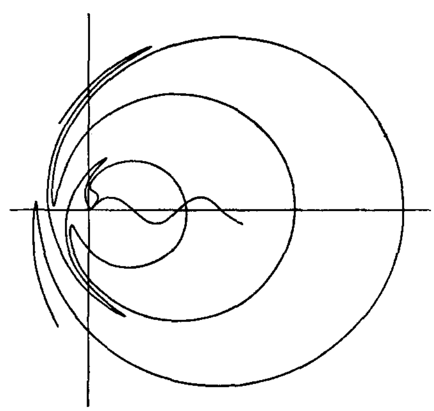

FIG. 7. The neutral magnetic line of a charge ($\gamma = 2$) in sinusoidal motion. The angle of rotation along a curve with curvature of a given sign is chosen large enough to form a piece of the γ-region. Such pieces are joined near the inflection points with field components changing sign. In fact, an understanding of the regions in which the field of a charge in arbitrary motion attains extremal values may be obtained by constructing the line of zero magnetic field only.

expressed as trigonometric functions of the phase, i.e., they are periodic in Φ. For uniform circular motion of the charge the passage of the electric line through the γ-region corresponds to a change in t'' by the quantity $Rc^{-1}\beta^{-1}\gamma^{-1}$. The corresponding trajectory length $R\gamma^{-1}$ coincides in order of magnitude with the introduced synchrotron radiation formation zone length (see, for instance, Ref. 6). If the trajectory changes smoothly this quantity retains its meaning, which permits one to introduce a formally exact definition of the formation zone length as the trajectory length that corresponds to a phase change of exactly 2π.

In order to characterize roughly the field of a charge moving along a trajectory with alternating curvature it is sufficient to evaluate the phase integral

$$\Phi_0 = \int_{t'=-\infty}^{t=+\infty} d\Phi \tag{4}$$

(the observation time is chosen to be $t = +\infty$). If $\Phi_0 \gg 2\pi$ the radiation has the properties of synchrotron radiation (in certain regions of space the field is identical to a synchrotron radiation field). All electric lines complete many ($\sim\Phi_0/2\pi$) revolutions about the charge. If $2\pi \gg \Phi_0 \gg \gamma^{-1}$ an almost full revolution about the charge is completed by several electric lines -- these lines number many more than one and the situation corresponds to the bremsstrahlung regime. If $\gamma^{-1} \gg \Phi_0$ not a single electric field line changes its direction after scattering of the particle, and the pattern remains as that of a Coulomb field.

In fact, the integrated phase is determined by the angle of rotation θ of the trajectory: $\Phi \sim \gamma\theta$. The range of the angles of rotation is broken up by the critical values $\theta_1 \sim \gamma^{-1}$, $\theta_2 \sim \gamma^{-2}$; in particular, if $\theta \ll \theta_2$ the curvature of the trajectory does not alter the Coulomb nature of the field (transition through the value θ_1 which marks the onset of bremsstrahlung is well-understood.

Using the field geometry of a single charge it is possible to find the interference maxima of the fields of many charges. One considers a coherent superposition of fields in bounded spatial regions during a finite time interval for particular trajectories or trajectories arranged in a certain pattern. In Ref. 7, for example, such a superposition is studied for synchrotron radiation fields.

5. Let us consider the field lines of the Poynting vector (FLPV for short) which define the local energy flow of the electromagnetic field. FLPV are orthogonal to the electric and magnetic fields.

In order to construct FLPV one employs the methods discussed above. The $(D,\mathbf{n})$ equations become complex, however, and we shall consider only a qualitative description of FLPV for synchrotron radiation.

The neutral magnetic field line plays a special role in FLPV, for on it one finds the poles of FLPV. In the case of a charge in uniform rectilinear motion the lines originating from one pole fall onto a sphere with the center at $\mathbf{r}_0(t)$ (where t is the observation time). This is not the case in synchrotron radiation. Close to the charge, where the neutral magnetic field line is almost straight, FLPV are spirally wound about the charge in the plane of the orbit, with the

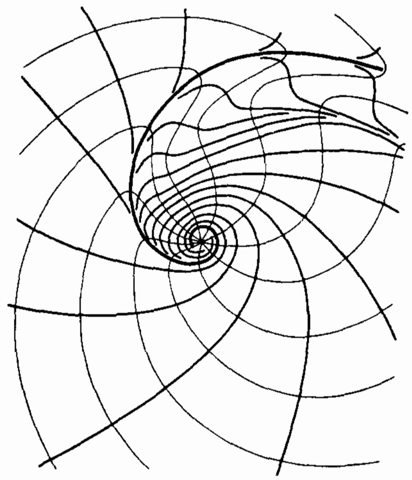

FIG. 8. A mutually orthogonal net of electric field lines and Poynting vector lines for synchrotron radiation of a charge with Lorentz factor $\gamma = 2$. This figure illustrates the behavior of the field lines of the Poynting vector (FLPV) close to the charge. The bold line is the unstable FLPV. The directions of FLPV are not shown.

spacing of the winding tending to zero (the result is a line composed of smoothly-joined pieces of FLPV pointing in various directions and ending up at the poles). At certain, sufficiently large distances, each FLPV no longer crosses the neutral line, but uncoils in its vicinity along a curve that resembles the Archimedean spiral. There exists an unstable FLPV which radiates FLPV that fill the γ-region at arbitrarily large distances.

The fact the FLPV do not close upon themselves in the plane of the orbit implies that the area of the surface "made up" of FLPV originating from one of the poles is infinite. The energy flow integrated over any closed surface enclosing the charge is non-zero, i.e., there is radiation.

The structure FLPV for a charge with the Lorentz factor of $\gamma = 2$ is schematically depicted in Fig. 8. Figure 9

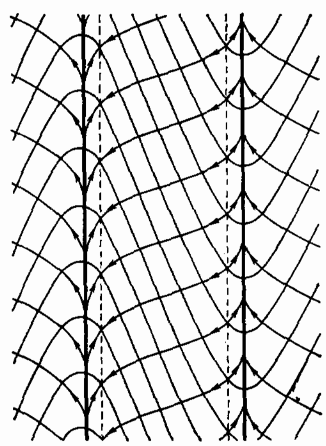

FIG. 9. The γ-region is filled by the field lines of the Poynting vector far from the orbit. The value of γ is not specified. The dashed lines mark the branches of the neutral magnetic field line. The bold lines mark the unstable FLPV, the branches emanating from which form the γ-region.

illustrates the filling of the γ-region at large distances.

6. The mutually orthogonal system of electric and magnetic lines, as well as Poynting vector lines, was considered at a single fixed observation time. For a given trajectory it is a simple matter to obtain a "snapshot" of the field at any moment in time (for example, in synchrotron radiation the whole field undergoes uniform circular motion together with the charge). The motion of the lines on different "snapshots" can be followed by considering that certain marked lines move one onto another. For instance, in synchrotron radiation such moving lines can be marked according to their location with respect to the orbit. However, beginning at a certain distance these lines would move faster than light.

The following method of following the motion of electric and magnetic lines as long as they are orthogonal (the Lienard-Wiechert field obviously satisfies this condition) is discussed in Ref. 8. Each line element is taken to move along the Poynting vector with a certain "drift" velocity. For the electric field this velocity is $W_E = c(\mathbf{E}\times\mathbf{H})/E^2$ (always smaller than c in the Lienard-Wiechert field), for the magnetic field it is $W_H = c(\mathbf{E}\times\mathbf{H})/H^2 (W_H > c)$. It is proven that such local motion does not affect the physical meaning of the lines, and that after such a procedure a system of lines moving slower than light becomes Lorentz-invariant (equations describing the element $dx^k = (dt, d\mathbf{r})$ of a magnetic or electric line are written in the form $F_{ik}dx^k = 0$ and $F^*_{ik}dx^k = 0$, where F_{ik} is the electromagnetic field tensor, and F^*_{ik} is its dual tensor).

Applying this mechanism to the Lienard-Wiechert field one obtains that if the moving electric lines are Lorentz-transformed as material objects, in the new coordinate system they will still be tangent to the transformed electric field. Possibly this "materialization" of electric field lines of a charge in arbitrary motion will satisfy the "string" concept of the development of electrodynamics (see, for instance, Ref. 9).

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[1)] This substitution represents the Lorentz transformation of the light signal velocity $\mathbf{n}c$ from the laboratory frame to the comoving inertial frame associated with the trajectory at the retarded point $\mathbf{r}_0(t')$.

[2)] By expressing $\boldsymbol{\beta}$ in (2) in terms of external fields $\mathbf{E}_0$ and $\mathbf{H}_0$ that determine the trajectory, we can rewrite equation (2) in the form

$$\frac{d\mathbf{v}}{dt'} = -\frac{e}{mc}\left\{\frac{\gamma-1}{\gamma}\mathbf{v}\times\mathbf{H}_0 + \frac{\gamma}{\gamma+1}\boldsymbol{\beta}\times\mathbf{v}(\boldsymbol{\beta}\cdot\mathbf{H}_0) + \frac{\gamma}{\gamma+1}\mathbf{v}\times\mathbf{E}_0\times\boldsymbol{\beta}\right\}. \quad (2)$$

This equation coincides with the equations of motion of a spin in external fields $\mathbf{E}_0$ and $\mathbf{H}_0$,[5] if the intrinsic magnetic moment is taken as $\mu = 0$, and the anomalous magnetic moment is taken as $\mu' = -e\hbar/(2mc)$. Possibly this coincidence is due to the same Lorentz invariance considerations imposed on the equation of the a^μ 4-vector, whose components in the comoving coordinate system are equal to $(0,\mathbf{v})$: linearity and homogeneity with respect to the external field tensor and dependence on the 4-velocity of the charge. The Lorentz invariance of the electric field line system will be discussed below.

[1] S. G. Arutyunyan, A one-time near-field pattern of an ultrarelativistic, charged particle moving in a circle: EPI preprint (in Russian) 387-(45)-79, Erevan, 1979.

[2] S. G. Arutyunyan and G. A. Nagorskiĭ, intensity effects in accelerators and storage rings related to the Lienard-Wiechert fields of particles: EPI preprint (in Russian) 477-(20)-81, Erevan, 1981.

[3] M. A. Aginian and S. G. Arutunian, Force lines of electric and magnetic field of a charge in arbitrary motion: EPI preprint (in English) 684-(74)-83, Erevan, 1983.

[4] S. G. Arutyunyan, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **28**, 896 (1985) [Radiophys. Quantum Electron. **28**, 619 (1985)].

[5] V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya V. 1, Nauka, M., 1968, p. 41 [English Transl. Relativistic Quantum Theory, Pergamon Press, New York, 1971].

[6] B. M. Bolotovskiĭ, Tr. Fiz. Inst. Akad. Nauk SSSR **140**, 95 (1982) [Proc. (Tr.) P.N. Levedev Phys. Inst. Acad. Sci. USSR **140** (1982)].

[7] S. G. Arutyunyan and G. A. Nagorskiĭ, Zh. Tekh. Fiz. **55**, 1494 (1982) [Sov. Phys. Tekh. Phys., **30**, 864 (1985)].

[8] M. A. Leontovich, Usp. Fiz. Nauk, **84**, 715 (1964) [Sov. Phys. Usp. **7**, 877 (1964)].

[9] P. A. M. Dirac, The Evolution of the Physicist's Picture of Nature, Sci. Am. **208**(5), 45 (May 1963) [Russ. Transl. in: Élementarnye chastitsy (Elementary particles), ed. B. V. Medvedev, Nauka, M., 1965, p. 123].

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