Negative energy waves in hydrodynamics

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Usp. Fiz. Nauk **150**, 417–437 (November 1986)

The utility of the concept of negative energy waves (NEW) in hydrodynamics is discussed. Examples are given of the excitation of waves by flow past elastic membranes, and of the amplification and generation of capillary-gravity and internal waves in liquids in the presence of vertically inhomogeneous flows. The concepts of "linear" and "nonlinear" energy are introduced, and it is shown that energy defined as the first integral of the equations of motion linearized against the flow background can be negative, whereas the inclusion of all the quadratic terms in the expression for the energy can give a positive value. Nonlinear processes associated with NEW are also discussed, as is the radiation instability of oscillators in hydrodynamics. The review is largely based on the authors' own work.

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INTRODUCTION

The readers of this journal have frequently had the opportunity of familiarizing themselves with the idea of "current" instability in electronics and plasma physics (see, for example, Ref. 1). At the same time, it seems to us that familiarity with studies of hydrodynamic instability is no less instructive to a physicist. This extensive range of problems is related, in the first instance, to the stability of shear flows. Although early work in this field was done in the nineteenth century, new trends have been noticed during the last 10–15 years and, in many respects, they have much in common with electrodynamics.

However, hydrodynamic problems are more complicated if only because much can be achieved in electrodynamics in the one-dimensional approximation (beam-plasma, interacting beams, one-dimensional description of a beam in a slowing-down system), whereas hydrodynamic problems are essentially two-dimensional because the synchronism between the wave and the beam occurs only in a bounded "layer" of the flow (critical layer), or on an abrupt velocity jump (tangential discontinuity).

In this review, we discuss certain aspects of the theory of hydrodynamic instability that are related to the concept of negative energy waves (NEW). It appears that this concept was first used in 1951 in relation to waves in electron beams (see Ref. 2 in this connection). Kadomtsev *et al.*[3] were the first to note the possibility of negative energy waves in plasmas. As far as hydrodynamics is concerned, negative energy waves did not attract particular attention until very recently, and physics literature showed evidence of the opinion that negative energy waves could not exist in fluid mechanics. However, the first paper in which negative energy waves were examined in hydrodynamics, although without using this phrase, was published by Benjamin as far back as 1963,[4] but such waves were not discussed for about fifteen years after that time.

1. NEGATIVE ENERGY WAVES

The phrase "negative energy wave" is usually taken to mean a wave whose excitation is accompanied by a reduction in the total energy of the system (as we shall see, this definition is not entirely adequate in relation to the properties of such waves). We note, at once, that the removal of energy from a wave of this type (by dissipation or coupling to another wave of positive energy) results in the growth of the wave, i.e., in an instability of the system. Actually, a reduction in the wave energy E ($\dot{E}<0$) when $E<0$ signifies an increase in the modulus of E and, hence, in the wave amplitude. It is clear that such waves are possible only in active or, more precisely nonequilibrium systems and, in particular, "current" systems containing charged-particle beams or shear flows of a neutral fluid.

The properties of negative energy waves are related to the dispersion properties of the system. Indeed, this relation becomes particularly clear when we use the averaged variational principle, put forward by Whitham,[5] which applies to waves with slowly-varying amplitude, frequency, and wave number. In particular, suppose that a wave variable ψ

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(or a set of such variables) takes the form of the quasiharmonic progressive wave

$$\psi = a(x, t) \cos\theta(x, t), \quad (1)$$

where the amplitude a, instantaneous frequency $\omega(x,t) = \theta_t$, and wave number $k(x,t) = -\theta_x$, are slowly-varying functions, i.e.,

$$\left|\frac{a_x}{a}\right| \ll k, \quad \left|\frac{a_t}{a}\right| \ll \omega.$$

When this system is described by the Lagrangian $L(\psi,\psi_t,\psi_x)$ (the Lagrangian may also contain an explicit but relatively slow dependence on x and t), the dynamics of waves such as (1) can be described by the average Lagrangian

$$\mathscr{L}(a, \omega, k) = \frac{1}{2\pi}\int_0^{2\pi} L\, d\theta,$$

where $\mathscr{L}$ can be represented by an expansion in even powers of a:

$$\mathscr{L} = Z_0(\omega, k)\, a^2 + Z_1(\omega, k)\, a^4 + \ldots . \quad (2)$$

The equations of motion are obtained by taking the variation of $\mathscr{L}$ with respect to a and θ:

$$\frac{\partial \mathscr{L}}{\partial a} = 2a\,(Z_0 + 2Z_1 a^2 + \ldots) = 0, \quad (3)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial k}\right) = 0, \quad (4)$$

where

$$\mathscr{L}_\omega = \frac{1}{2\pi}\int_0^{2\pi} \psi_\theta \frac{\partial L}{\partial \psi_t}\, d\theta, \quad \mathscr{L}_k = -\frac{1}{2\pi}\int_0^{2\pi} \psi_\theta \frac{\partial L}{\partial \psi_x}\, d\theta.$$

(This is clear from the expressions $\psi_t = \omega\psi_\theta$ and $L_{\theta t} \equiv L_\omega = L_{\psi t}\psi_\theta$, and similarly for L_k.) This must be supplemented by the obvious relation

$$\frac{\partial \omega}{\partial k} + \frac{\partial k}{\partial t} = 0. \quad (5)$$

The expression given by (3) is a nonlinear dispersion relation and, if we neglect the quadratic and higher powers in the Lagrangian (linear approximation), it follows from (3) that

$$Z_0(\omega, k) = 0, \quad (6)$$

so that

$$\mathscr{L}(a, \omega, k) = 0. \quad (7)$$

The average Hamiltonian, i.e., the wave energy density, is given by

$$\overline{H} = \overline{p\dot{q}} - \overline{L} = \omega\mathscr{L}_\omega - \mathscr{L}, \quad (8)$$

where

$$p = L_{\psi_t}, \quad \dot{q} = \psi_t,$$

and the energy flux density is given by

$$S = \overline{\dot{q}_x \frac{\partial L}{\partial \dot{q}_x}} = -\omega\mathscr{L}_k.$$

It follows from (4) that the quantity

$$I = \int_{-\infty}^{\infty} \mathscr{L}_\omega\, dx$$

is conserved for any perturbation localized in space (wave packet), i.e., this quantity is an adiabatic invariant, so that $\mathscr{L}_\omega$ is the density of the adiabatic invariant. This is also valid for media with parameters slowly-varying (with x and t). Similarly, the adiabatic invariant in the x direction

$$\Pi = \int_{-\infty}^{\infty} \frac{\partial \mathscr{L}}{\partial k_x}\, dt,$$

whose density is $\partial\mathscr{L}/\partial k_x$, is also conserved.

In the linear case, these general relationships yield

$$E = \omega \frac{\partial Z_0}{\partial \omega} a^2, \quad S = -\omega \frac{\partial Z_0}{\partial k} a^2, \quad (9)$$

where $S = v_{gr}E$ and $v_{gr} = -Z_{0k}/Z_{0\omega}$ is the group velocity of the wave. The density of the adiabatic invariant and of its flux is, respectively, given by

$$I = \frac{E}{\omega}, \quad \Pi = \frac{S}{\omega}. \quad (10)$$

In our case, however, this type of motion is usually non-one-dimensional and is characterized by a layered structure, so that the parameters of the medium are functions of the lateral coordinate z. The analysis is then usually confined to the eigenmodes (normal waves) of the system and the perturbations retain the form described by (1), but the wave amplitude a depends on z, in accordance with the mode structure. All that we have said remains valid in this case if we replace L with the Lagrangian integrated with respect to z, i.e.,

$$L_0 = \int_a^b L\, dz,$$

where a and b are the boundaries of the system in z (which can actually be located at infinity).

We now return to the question of negative energy. Consider an arbitrary dispersion relation, such as (6), in the domain of real k and ω. According to (9), the energy can change sign at the points on the k axis at which ω or $\partial z_0/\partial\omega$ changes sign. The first case signifies a change in the sign of the phase velocity relative to the group velocity (the "forward" wave becomes a "backward" wave, or vice versa). In general, the second condition ($\mathscr{L}_k \neq 0$) corresponds to infinite group velocity $v_{gr} \equiv d\omega/dk = -\mathscr{L}_k/\mathscr{L}_\omega$. A typical NEW segment on the dispersion curve $\omega = \omega(k)$ is therefore of the form shown in Fig. 1. We now note a further important feature: the two branches of the dispersion curve, one of which corresponds to the NEW and the other to a positive energy curve, are "joined" at the point at which $v_{gr} = \infty$. The coupling between these waves leads to the instability: the frequency becomes complex after the branching point. This instability of coupled waves is well-known in electrodynamics (see, for example, Ref. 6). In hydrodynamics, it corresponds, in particular, to the Kelvin-Helmholtz instability (this will be further elucidated below).

However, at this point, we are mostly interested in the

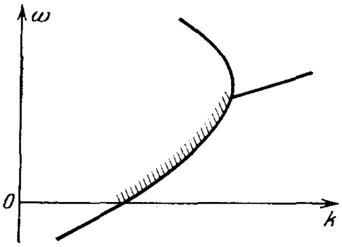

FIG. 1. The region of negative energy waves is bounded by the points $\omega=0$ and $v_{gr}=\infty$.

domain of existence of neutrally stable, negative energy waves. As already noted, dissipative instability is possible in this region: the introduction of losses leads to a growth of the NEW in time. This is clear from the dispersion relation which now has a small imaginary component:

$$Z_0(\omega, k) + i\nu(\omega, k) = 0.$$

This immediately yields a small imaginary component in the unperturbed frequency of the wave:

$$\omega'' = \frac{\nu(\omega, k)}{Z_{0\omega}} = \frac{\omega\nu\,|a|^2}{E}. \tag{11}$$

Of course, the sign of the wave energy depends on the chosen reference frame. Under the Galilean transformation, the wave energy transforms simultaneously with the Doppler frequency shift. However, $\mathscr{L}_\omega = Z_{0\omega}a^2$ is invariant under the Galilean transformation, so that only one of the ends of the wave number range ($\omega=0$) is shifted, which corresponds to the NEW, whereas the other end ($\mathscr{L}_\omega = 0$) remains unaffected. At the same time, it is equally clear that the very stability or instability is unaffected, and the corresponding time growth rate is invariant. Thus, in accordance with (11), the dissipative function $\omega' \nu a^2$ must change sign together with the wave frequency. Of course, "true dissipation" is necessary for the stability of the negative energy wave, i.e., there must be positive losses by the NEW in its local reference frame. In the moving frame, the signs of the energy and of the losses change at the same time. The energies of the two coupled waves continue to differ in sign in all reference frames.

Another formulation of the problem is also possible: a harmonic source that is at rest in a given reference frame excites a wave of frequency ω in the medium. Here, we can have spatial amplification of the wave. To find the spatial growth rate k'', the simplest procedure is to use the relation

$$\frac{\partial Z_0}{\partial\omega}\omega'' + \frac{\partial Z_0}{\partial k}k'' = 0,$$

from which it follows that

$$k'' = -\omega''\frac{Z_{0k}}{Z_{0\omega}} = \frac{\omega''}{v_{gr}},$$

i.e., the spatial growth rate is related to the time growth rate through the group velocity. We note that the change in the sign of k'' in the region where $\omega'' < 0$ does not lead to amplification because v_{gr} changes sign at the same time as k'' and, consequently, there is a change in the direction of the wave energy flux, i.e., the wave is damped in the direction of energy transport. Spatial instability, on the other hand, corresponds to a simultaneous change in the sign of k'' and of ω'' (at constant sign of the group velocity v_{gr}). Near the point where $\mathscr{L}_k = 0$, we must take the next term in the expansion for $Z_0(\omega,k)$ in powers of k, in which case

$$k'' = \left(-\frac{2Z_{0\omega}\omega''}{Z_{0kk}}\right)^{1/2}$$

and the spatial decay (growth) rate k'' becomes proportional to the square root of the time rate ω''.

Let us now examine some examples of negative energy waves and the associated dissipative instability.

2. SOME EXAMPLES OF NEGATIVE ENERGY WAVES IN HYDRODYNAMICS

2.1. Flow past elastic membranes[7]

Benjamin[4] was the first to consider negative energy waves in hydrodynamics, and illustrated the phenomenon by examining the example of an elastic membrane moving with velocity U in contact with a homogeneous and incompressible fluid occupying the half-space $Z<0$. We shall generalize the formulation of the problem to some extent by assuming that the other half-space is occupied by a compressible medium of density $\rho_2 \ll \rho_1$ (ρ_1 is the density of the incompressible liquid), and by taking losses in the membrane into account. This will enable us to demonstrate the effect of dissipation and emission of acoustic waves on the development of the NEW.

The equation for the deflection ξ of the membrane from the equilibrium position is

$$m\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\xi - T\xi_{xx} + \beta\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\xi = p_1 - p_2, \tag{12}$$

where m, T, β are, respectively, the density, elasticity, and viscosity of the membrane, and p_1 and p_2 are the pressures on the membrane due to the incompressible liquid and the compressible gas, respectively.

The equations of motion of the liquid in both half-spaces are

$$\begin{aligned} &\text{I.}\ \Delta\varphi_1 = 0, && \rho=\rho_1. \\ &\text{II.}\ \Delta\varphi_2 = \frac{1}{c^2}\varphi_{2tt}, && \rho=\rho_2. \end{aligned} \tag{13}$$

If we write the progressive-wave potential in the form

$$\begin{aligned} \varphi_1 &= \varphi_{10}\exp(-i\omega t + ikx + kz), \\ \varphi_2 &= \varphi_{20}\exp(-i\omega t + ikx + iqz), \end{aligned} \tag{14}$$

where $q^2 = \omega^2/c^2 - k^2$, and use the relationships

$$\left.\begin{aligned} p_{1,2} &= -\rho_{1,2}\frac{\partial\varphi_{1,2}}{\partial t}, \\ \varphi_{10} &= -\frac{i\omega}{k}\xi, \\ \varphi_{20} &= -\frac{\omega}{q}\xi, \end{aligned}\right\} \tag{15}$$

we find that the dispersion relation for the waves in the membrane is

$$Z(\omega, k) \equiv \frac{1}{2}\Big[m(\omega-Uk)^2 - Tk^2 + \frac{\rho_1}{k}\omega^2 - \frac{\rho_2}{iq}\omega^2 + i\beta(\omega - Uk)\Big] = 0. \qquad (16)$$

Since absorption in the membrane, β, and the density of the compressible medium, ρ_2, are small, the last two terms on the right-hand side of (16) are small in comparison with the other terms. We now write the dispersion relation in the form

$$Z_0(\omega, k) \equiv \frac{1}{2}\Big[m(\omega-Uk)^2 + \frac{\rho_1\omega^2}{k} - Tk^2\Big] = 0 \qquad (17)$$

which is valid for the conservative model (without losses or radiation). The corresponding dispersion curves are shown in Fig. 2 and it is clear that, for $U > c_0$, where $c_0 = (T/m)^{1/2}$ is the velocity of the elastic wave in the free membrane, there is a branching point $k = k_m$ at which $v_{gr} \to \infty$, so that the system is unstable for $k < k_m$ and one of the branches of the dispersion relation, namely, the lower branch, corresponds to the NEW for $k > k_m$. It is readily seen from (17) that, for this branch,

$$\frac{\partial Z_0}{\partial \omega} = m(\omega - Uk) + \frac{\rho_1\omega}{k} < 0$$

when $\omega > 0$. The critical value k_m and the corresponding frequency ω_m are given by

$$k_m = \frac{\rho_1}{m}\Big[\Big(\frac{U}{c_0}\Big)^2 - 1\Big], \quad \omega_m = \frac{\rho_1}{mU}\frac{(U^2 - c_0^2)^2}{c_0^2}. \qquad (18)$$

Let us now take into account the emission of waves and viscous friction in the membrane. When the small imaginary terms in (17) are taken into account, the frequency ω acquires a small imaginary correction ω'' for $k > k_m$. This correction can be found by the perturbation method:

$$\omega'' = -\frac{\beta(\omega - Uk) + (\rho_2\omega^2/q)}{2\,\partial Z_0/\partial\omega}, \qquad (19)$$

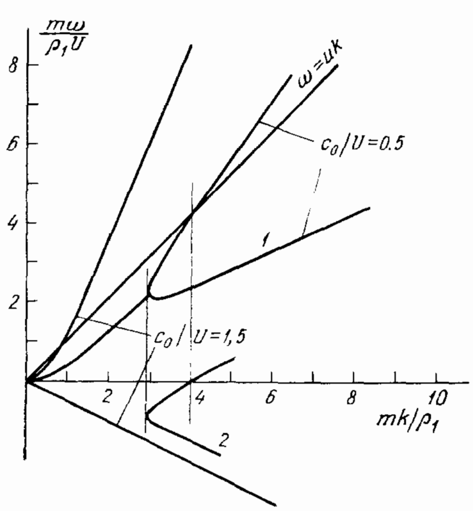

FIG. 2. Dispersion curves for waves in a membrane moving relative to a stationary liquid (1) and in the neighborhood of the point of bifurcation in the coordinate frame in which the membrane is at rest (2).

$$\frac{\omega''}{\omega} = -\frac{\rho_2}{\rho_1 q}\Big[\frac{mk}{\rho_1}\Big(1 - \frac{U}{c}\Big) + 1\Big]^{-1} - \Big(1 - \frac{U}{c}\Big)\frac{\beta k}{2\rho_1}\Big[\frac{mk}{\rho_1}\Big(1 - \frac{U}{c}\Big) + 1\Big]^{-1}, \qquad (19')$$

where $c = \omega/k$.

Let us begin by considering the emission of acoustic waves [first term in (19')]. It is readily seen that the instability ($\omega'' > 0$) sets in for $\partial Z_0/\partial\omega < 0$ or $U/c > 1 + (\rho_1/mk)$, i.e., for all negative energy waves, since positive energy waves are being radiated. We emphasize that the energy is negative in the reference frame relative to the half-space II into which sound is radiated. For $\partial Z_0/\partial\omega > 0$, (19') gives damping, which corresponds to leaky waves, well-known in electrodynamics. We note in passing that analysis of the eigenmodes is not directly physically significant because the wave field does not fall but, on the contrary, increases as exp(Im qz) along the normal to the membrane, and the mode is not localized. On the other hand, in the case of the negative energy waves, the wave is localized near the membrane (Im $q < 0$) and, according to (19), grows in amplitude.

This "radiative instability" will be encountered again when we consider internal waves in stratified liquids. Essentially, this instability may be looked upon as a coupled-wave instability in which one of the waves is a membrane wave and has a negative energy, whereas the other is a wave radiated into the medium, and has positive energy. Another way of treating the instability is to look upon it as an analog of the anomalous Doppler effect, i.e., emission by an oscillator into the Cherenkov cone. This problem is examined separately below.

The second term in (19') produces a correction to the frequency of the "membrane wave" due to viscous friction in the membrane. The instability is possible in a relatively narrow range of U, namely, $1 < U/c < 1 + (\rho_1/mk)$, in which the energy and the losses have opposite signs (the instability vanishes for $\rho_1 \to 0$ because the sign of the losses changes at the same time as the sign of the wave energy). It is precisely in this range of U that the wave energy is negative in the coordinate frame that is at rest relative to the membrane (Fig. 2b). This example illustrates the point that wave energy dissipation leads to instability in a layer if the sign of the wave energy is negative in the reference frame attached to this particular layer.

The phenomenon of "superreflection," in which the reflection coefficient becomes greater than one, is possible when a wave is reflected from this layer (as the latter moves with velocity U relative to the remainder of the medium). This occurs when the velocity of the layer exceeds the phase velocity of the wave and dissipation takes place in the layer. This effect was examined in Ref. 38 in relation to plasma waves.

An analogous amplification of waves is possible when a plane wave is reflected from a rotating cylinder in the case of those harmonics in the expansion of the plane wave in terms of cylindrical waves for which the angular phase velocity is less than the angular velocity of the cylinder in which there is

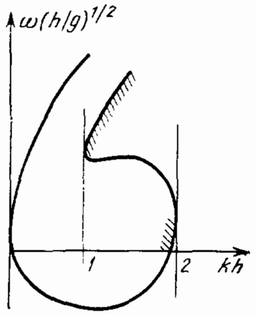

FIG. 3. Dispersion curve for capillary-gravity waves on a thin moving layer.[8]

also dissipation. This effect was examined by Ya. B. Zel'dovich for electromagnetic and gravity waves,[39,40] and the relevant bibliography is given in Ref. 41 (p. 171). Amplification of sound by reflection from a rotating vortex is discussed in Ref. 42.

2.2. Capillary-gravity waves on the surface of water[8]

Suppose that a layer of thickness h on the surface of a liquid moves with velocity U relative to the resting half-space of the same liquid. In the absence of losses, surface waves in this flow are described by the dispersion relation

$$Z_0(\omega, k) = (\omega_1^2 - \omega_0^2)(\omega_1^2 + \omega_0^2) - e^{-2kh}(\omega_1^2 + \omega_0^2)(\omega_1^2 - \omega^2) = 0, \qquad (20)$$

where $\omega_1 = \omega - Uk, \omega_0 = gk + \alpha k^3$, g is the acceleration due to gravity, and α is the surface tension. The dispersion curve corresponding to (20) is shown in Fig. 3. We note that, when the motion of the top layer is taken into account, the order of the dispersion relation becomes higher and, in general, there are four normal branches (in contrast to two such branches when $U = 0$). Negative energy waves correspond to the shaded segments of the curve. Points 1 and 2 are points of bifurcation, which separate regions of Kelvin-Helmholtz-type instability from the stable region.

We now take dissipation into account. First, we introduce a small imaginary correction to the surface tension:

$$\alpha = \alpha_0 - i\alpha_1(\omega - Uk).$$

This correction describes absorption in the surface film due to relaxation mechanisms accompanying the bending of the film.[9] Once again, we have introduced a frequency shift due to the motion of the liquids. Second, we take into account viscosity in the lower half-space. When the losses are small, the dispersion relation (20) remains valid provided ω^2 is replaced with $\omega_2^2 = \omega^2 + 4i\nu\omega k^2$. The imaginary correction to the frequency is then

$$\mathrm{Im}\,\omega = -i\,\frac{Z_\nu \nu + Z_\alpha \alpha_1(\omega - Uk)}{Z_\omega}, \qquad (21)$$

where

$$Z_\nu = (\omega - Uk)\,k^3\,[\omega_1^2(1 + e^{-2kh}) + \omega^2(1 - e^{-2kh})],$$
$$Z_\alpha = 8\omega\omega_1 k^2(\omega_1^2 + \omega_0^2)\,e^{-2kh}(\omega_1^2 + \omega^2)^{-1}.$$

It is clear from this that, when $Z_\omega > 0$, the sign of the contribution of each of the terms in (21) is determined by the sign of the frequency (energy) in the local (for the given layer) reference frame. This is in complete correspondence with the discussion given above. Instability can arise on negative energy segments because of viscous losses in the liquid half-space, and, for positive energy, because of surface dissipation. We note that, when we transform to the reference frame in which the top layer is at rest, the sign of the wave energy is, in general, reversed.

2.3. Internal waves in a layered liquid

An internal wave is generally understood to be wave motion in a liquid whose density increases in the direction of the force of gravity (stable stratification). This type of stratification is always present in the ocean. Internal waves are also considered in the case of two homogeneous layers of different density ρ_1 and ρ_2 and, in particular, surface waves on water can be looked upon as internal waves for $\rho_1 = 0$. Instability (Kelvin-Helmholtz instability) becomes possible when the individual layers are in relative motion, while negative energy waves can occur in stable regions.

We begin with a classical example, i.e., two liquids of different density, of which the lower, denser liquid occupies the region $z < 0$ and is stationary, whereas the upper moves with velocity U. In the case of an ideal liquid, the dispersion relation for this system is well-known:

$$\omega^2 + s(\omega - Uk)^2 - gk(1 - s) = 0, \qquad (22)$$

where $s = \rho_1/\rho_2 < 1$. It is also well-known[10] that this system contains a region of Kelvin-Helmholtz instability. More careful analysis shows that, according to the general discus-

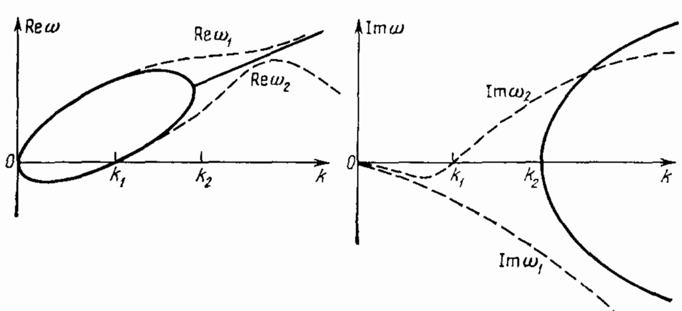

FIG. 4. Dispersion curves for waves in the Kelvin-Helmholtz model: solid curve—both liquids ideal; broken curve—lower liquid viscous.

sion given above, this region is adjacent to the NEW region[11] lying between the usual critical points $\omega = 0$ and $v_{gr} = \infty$ (Fig. 4). We note that $k_1 \simeq \frac{1}{2}k_2$ when ρ_1 and ρ_2 are not too different. We now suppose that the lower, resting liquid has a viscosity ν. When ν is small, so that the losses over one wavelength are small, the right-hand side of (22) acquires the term $-4i\nu\omega k^2$. The corresponding dispersion curves are shown in Fig. 4 by the broken lines. In the region of the Kelvin-Helmholtz instability, the viscosity simply reduces instability and, in particular, as $k \to \infty$, we find that Im ω tends to the finite value[12] $sU^2/2\nu$. At the same time, an instability can arise in the NEW region: Im $\omega > 0$ for $k > k_1$. The NEW region has, typically, low values of growth rate (provided we are not too near the point $k = k_2$, where the formula given below is invalid):

$$\mathrm{Im}\,\omega_{1,2} \simeq \mp 4\nu k^2 \frac{1}{1+s}\frac{\omega_{1,2}}{\omega_1-\omega_2}\,; \tag{23}$$

where ω_1, ω_2 are the values of ω at $\nu = 0$ for the upper and lower branches of the dispersion curve for given k. The negative sign in this expression corresponds to the upper branch of the curves of Fig. 4 and the positive sign to the lower branch, i.e., to the negative energy waves. We can readily obtain for this region the energy balance equation $\dot W = F$, where W is the total average energy and F is the dissipative function, where, in this case,[13]

$$W = \pm\omega_{1,2}\,|\omega_1-\omega_2|\,\frac{\rho_1+\rho_2}{k}\,|A|^2, \qquad F = -4\nu\rho_2 k\omega_{1,2}, \tag{24}$$

and A is the amplitude of the displacement of the boundary. This, or course, yields the same value for Im ω but, in addition, we see that W and F are proportional to ω. It is precisely for this reason that the transformation to a new Galilean frame will result only in the replacement of ω with $\omega + Uk$, but will not affect the growth rate. Simultaneously, with the change in the wave energy, there is a change in the magnitude and sign of dissipation because the moving medium becomes viscous. An analogous solution can be given for layers of finite thickness between rigid walls.[14]

As noted above in connection with the example of a moving membrane, the NEW instability may be due not only to absorption in the medium, but also to emission of positive energy waves.

A typical example is provided by the following problem,[15,16] which is relevant for the excitation of internal waves in an ocean. For $z > 0$, the liquid is homogeneous and is dragged with velocity U relative to the lower region, where the liquid is stationary and stratified in density, i.e.,

$$\rho = \rho_1,\ z > 0,$$
$$\rho = \rho_2\ (1 - \alpha z) \approx \rho_2 e^{-\alpha z}, \qquad z < 0,$$

where $\rho_2 > \rho_1$, $\alpha > 0$; the density discontinuity occurs at $z = 0$. The solution of this problem may be sought in the form of a wave traveling along the x axis, i.e., $\exp[i(kx + mz - \omega t)]$, where $m = ik$ for $z > 0$ (exponential attenuation) and $m = k[(N^2/\omega^2) - 1]^{1/2}$ for $z < 0$ [$N = (g\alpha/\rho_2)^{1/2}$ is the Brent-Väisälä frequency in the lower half-space. In general, this quantity is, of course, complex.

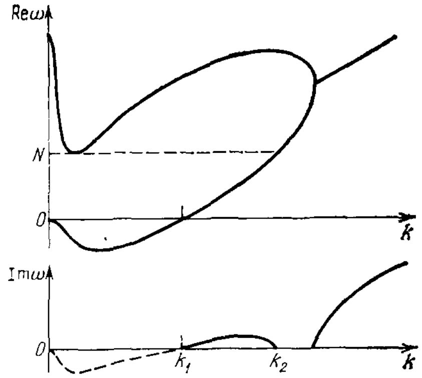

FIG. 5. Dispersion curves for radiative instability in the Kelvin-Helmholtz model with a smoothly stratified lower layer.

The dispersion relation for the problem is

$$\rho_1\,(\omega - Uk)^2 + \rho_2\omega\,(\omega^2 - N^2)^{1/2} - gk\,(\rho_2 - \rho_1) = 0. \tag{25}$$

We must also add the radiation condition: the vertical component of the energy flux S_z must point downward in the lower region and must be zero for $\omega > N$. It may be shown that this requirement is equivalent to Re $m\omega \equiv \mathrm{Re}(N^2 - \omega^2)^{1/2}k > 0$. The solution of (25) is, in general, complex (Fig. 5) and, in addition to the Kelvin-Helmholtz instability (for $k > k_0$), there is also the radiative instability in the NEW region ($k_1 < k < k_2$), where $\omega < N$, and energy is radiated downward. This region is separated from the Kelvin-Helmholtz region by a stable interval in which $\omega > N$, i.e., internal waves cannot propagate downward, and there is no mechanism for removing energy.

For long waves ($k < k_1$), the wave is damped owing to radiation. We note, however, that Im $m < 0$ in this region, i.e., the amplitude of the wave increases (and does not decrease!) in the direction into the liquid, and the field is not localized in z. As noted above, these waves are not the eigensolutions of the field equations (i.e., in the absence of forces), and the boundary and initial conditions must be specified for their correct analysis. On the other hand, in the NEW region ($k > k_1$), a z-localized mode exists everywhere.

We note that the phenomenon of radiative instability in relation to flow systems in plasmas has been studied earlier than in hydrodynamics.[37]

2.4. Continuous flow profiles

In the case of a smooth velocity profile, instability may be due to the presence of the so-called critical layer, i.e., a level at which the flow velocity is equal to the phase velocity of perturbations. The general theory of the instability of such flows in an ideal liquid is now well established "at the level" of theorems formulating sufficient conditions for such instabilities.[17] Thus, the Miles theorem[17-19] demands that the inequality Ri $< \frac{1}{4}$, where Ri $= N^2/U_z'^2$ is the Richardson number for a flow of a stratified liquid, should be a necessary condition for stability. Actually, this theorem describes the conditions necessary for the appearance of negative energy. This is already clear if we consider the properties of the criti-

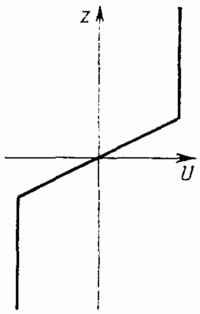

FIG. 6. Approximation to a sheared velocity profile by a broken line.

cal layer: the wave phase velocity on one side of this layer is greater than the flow velocity but, on the other side, it is lower. The sign of the frequency and, hence, the sign of the wave energy are therefore determined by its localization above or below the critical layer. The coupling between the NEW and PEW (positive energy waves), i.e., the joining of the dispersion branches, defines the bifurcation point, and the instability region lies on one side of this point. This discussion may be made more specific by representing the smooth profile, approximately, by a broken line.[20] In this approximation (as in Fig. 6), there are only two wave modes in the homogeneous liquid. They are localized near the two kinks on the line. When the mode coupling is weak, the mode energy is given by[20]

$$W=(U_2'-U_1')\,c\,|\eta|^2,$$

where U_2', U_1' are the derivatives of the flow velocity above and below the kink on the line, c is the velocity of the perturbation, and η represents the position of the kink above the horizon. We thus see that waves localized near the lower kink have positive energy, while those near the upper kink have negative energy. When the separation between the two kinks is large enough, these perturbations do not interact, and their dispersion curves do not cross. However, if we increase the number of kinks on the profile, we pass, in the limit, to the original smooth profile, for which there is a continuum of modes with positive and negative energy. The convergence of this approach was demonstrated in Ref. 22.

This approach was used, in particular, in Ref. 20 to interpret the wind instability of surface waves on water. A family of negative energy waves appears in the air flow when the wind profile is approximated by a piecewise linear function. When capillarity is taken into account, the instability sets in at the critical wind velocity of 23 cm/s, at which for the first time a reconnection occurs of the dispersion curves corresponding to positive and negative energy waves.

It is interesting to note that these theorems can be derived by recalling the above proposition that the instability region must be in contact with the NEW region at the point where $Z_{0\omega}=0$ (Ref. 43). Actually, the problem of the stability of a layer of stratified liquid with shear flow can be reduced to the problem of the eigenvalues of the Taylor-Goldstein equation[17]

$$\frac{d}{dz}\left[\rho\left(U-\frac{\omega}{k}\right)^2F_z\right]+\bar\rho\left[\frac{\rho'}{\rho}g-k^2\left(U-\frac{\omega}{k}\right)^2\right]F=0, \quad (26)$$

where $\eta=F(z)\exp(-i\omega t+ikx)$ is the vertical deviation of the constant-density line. Let us suppose, for simplicity, that a layer of the liquid of thickness H is bounded by solid walls, i.e., $F(0)=F(-H)=0$. If we then substitute $G=[U-(\omega/k)]^{1/2}F$, multiply (26) by G^*, and integrate with respect to z, we obtain the integral dispersion relation[19]

$$Z_0(\omega,k)=\int_{-H}^{0}\Big[\bar\rho\left(U-\frac{\omega}{k}\right)(|G'|^2+k^2|G|^2)+\frac{1}{2}(\bar\rho U')'|G|^2+\bar\rho\left(\frac{1}{4}U'^2-g\frac{\rho'}{\rho}\right)\frac{|G|^2}{U-(\omega/k)}\Big]dz=0. \quad (27)$$

Let us consider, for this equation, the bifurcation point at which $Z_\omega=0$. We note that, in the case of complex frequencies $\omega=\omega_0+i\delta$, the quantity δ appears in the expressions for $|G|^2$ and $|G'|^2$ only in even powers,[43] so that the imaginary part of $Z_0(\omega,k)$ for small δ is equal to $i\delta Z_{0\omega}$ and does not produce derivatives of $|G|^2$ and $|G'|^2$ with respect to ω. In view of this, we find that

$$Z_{0\omega}=\int_{-H}^{0}\bar\rho(|G'|^2+k^2|G|^2)\,dz+\int_{-H}^{0}\bar\rho\left(g\frac{\rho'}{\rho}-\frac{1}{4}U'^2\right)dz=0.$$

Since the first term in this is always positive, this equation is possible only for $U'^2>4g(\rho'/\rho)$, i.e., when the Miles criterion is satisfied. We note that, when there is no stratification ($\rho'=0$), the well-known Rayleigh equation follows from (26), and (27) assumes the form

$$Z_0(\omega,k)=\int_{-H}^{0}\left\{-|W'|^2+2\left[\frac{U''}{(\omega/k)-U}-k^2\right]|W|^2\right\}dz, \quad (28)$$

where $W=(\omega-kU)F$. The point of bifurcation then corresponds to the condition

$$Z_{0\omega}=\frac{2}{k}\int_{-H}^{0}\frac{U''|W|^2}{[(\omega/k)-U]^2}\,dz=0.$$

From this, we have the requirement that $U''=0$ for any point in the layer, which is a necessary condition for the instability. This is the well-known Rayleigh theorem.

We note further that it was shown in a recent paper[23] that instability could appear in stable shear flow when stable stratification was introduced. This effect is related to the appearance of internal positive energy waves which remove energy from the NEW in the flow.

Thus, well-known instability theorems are essentially related to the presence of the negative energy waves that appear on the boundary of the region of "ideal" Kelvin-Helmholtz instability. It follows that the introduction of dissipation or radiation into the system will, as a rule, extend the instability region.

As far as specific models with a continuous profile are concerned, there has been very little work on negative energy waves in such cases. We merely note that computer calculations of stability diagrams for profiles of the form $\tanh(z/L)$ (Ref. 24) have demonstrated the presence of the NEW branch (see Ref. 15).

3. WHAT IS "WAVE ENERGY"?

We have often referred to "wave energy." We shall now show that this refers not to the "true" energy of the wave, but to a part of it that is essentially defined in the sense indicated above.

How are we to interpret the meaning of "wave energy" for weak waves? A common approach is to start with the linearized equations of hydrodynamics, multiply them by the complex conjugate quantities, and reduce them to the divergence form:

$$\frac{\partial E_l}{\partial t}+\operatorname{div} S_l=0. \quad (29)$$

Since E_l and S_l have, respectively, the dimensions of energy and flux, we shall regard them as such (we shall not be concerned with "gage" ambiguities in the definition of E_l and S_l because this is of no direct significance for our problem). For example, for acoustic waves, this leads to the well-known expressions

$$E_l=\frac{\rho_0 v^2}{2}+\frac{p'}{2\rho_0 c^2}, \quad S_l=p'v,$$

which are coupled by (29).

The equation $E=\omega\mathscr{L}_\omega$ is obtained in precisely this way since it employs $\mathscr{L}=0$, which is true only when the dispersion relation $Z(\omega,k)=0$ is taken into account, and this relation follows from the linearized equations of motion.

However, strictly speaking, this approach is physically incorrect because E_l is not the total energy associated with the wave. Actually, energy variables are of the second order in the wave amplitude, and the quadratic terms that have been discarded in the course of the linearization procedure may contribute to the energy, momentum, etc., in the presence of flows (we note that this is true even in the absence of flow in a compressible medium[25,26]). For surface and internal waves in a liquid, the situation is somewhat more complicated because, fundamentally, the motion is not one-dimensional.

We begin with an elementary explanation. Suppose that some periodic wave motion is propagating in the liquid along the x-axis in the presence of a constant shear flow $U(z)$. When the wave amplitude is small enough, the resultant velocity field can be represented by the series

$$\begin{aligned} u &= U(z)+A\,\{f_1(z)\exp[i(\omega t-kx)]+\text{c.c.}\}\\ &\quad +A^2\{f_0(z)+f_2(z)\\ &\quad \times\exp[2i(\omega t-kx)]+\text{cc.}\}+\ldots, \end{aligned} \quad (30)$$

where A is the wave amplitude and $A^2 f_0$ describes the quadratic correction to the mean flow velocity. By definition,[11] the mean kinetic energy density associated with the wave is given by

$$E_k=\frac{\rho}{2}(\overline{U}^2-U^2)=\rho\,|A|^2(f_1^2+Uf_0). \quad (31)$$

Thus, the energy consists of the two terms $E_l\sim f_1^2$ and $E_{nl}\sim Uf_0$, which are of the same order as $|A|^2$. It is clear that the linearized equations do not contain E_{nl}. When free boundaries are present, the total kinetic energy of the wave, $E_k=\int E_k(z)dz$, may also depend on nonlinear corrections to the displacement of the boundary.

Thus, to calculate the "true" energy of the wave, we must know the nonlinear solution to within terms of the second order, inclusive. We note that this solution is not universal, since it depends not only on the "stimulating force," i.e., on the amplitude and configuration of the main wave, but also on the boundary and initial conditions, i.e., on the method of excitation of the field.

Waves excited without the transfer of additional momentum to the system form an important special class. This situation occurs when the waves are excited by a wave generator that is stationary on average, or when a perturbation that is homogeneous in space is found to grow as a result of instability, and so one. In such cases, the total mass transfer associated with the wave is also zero because it is equal to the total momentum. This condition is sufficient for the unique determination of the mean energy. Moreover, in such zero-momentum systems, the total wave energy is invariant under transformations between moving reference frames that do not alter the potential energy of the system (for example, horizontal motion of a vertically stratified medium). Actually, transport with constant velocity U produces the additional term $\rho|A|^2Uf_{0x}$ in the expression for the energy (31), and the absence of mass transport means that $\int f_{0x}dz=0$ (we recall that this result is valid in the region with $\rho=$ const although it is readily generalized to a variable-density liquid). Of course, this applies only to the total energy E_l+E_{nl} because the "linear" energy is $E=\omega\mathscr{L}_\omega$ and, of course, depends on the motion of this system, which provides us with a graphic demonstration of the fact that the "linear" and "nonlinear" energies must be treated on a par.

The energy conservation equation (29) is thus seen to be simultaneously valid for both the linear and total energies.

Similar considerations apply to the dissipative function F, so that the equation

$$\frac{\partial E}{\partial t}+\operatorname{div} S=F \quad (32)$$

is also valid for E_l and F_{nl}. Since all these quantities are proportional to $|A|^2$, it follows that, for homogeneous plane waves whose amplitude does not depend on the coordinates, we have $F_{nl}/F_l=E_{nl}/E_l$ in any reference frame.[26]

It is now clear that, when we speak of negative energy, we mean only the "linear" part E_l that is associated with the linearized equations, whereas the total wave energy may, in general, have a different value and even different sign. We note, in particular, that, in the model of an infinite two-layer liquid with a viscous lower layer, the field grows on the NEW segment, and so does the mean flow, which is given by

$$U_{\text{mean}}=-4k^2\,|A|^2(\varkappa\omega)^{-1}\exp(2\,\text{Im}\,\omega t+\varkappa z),$$

where $\varkappa^2=2\nu^{-1}\text{Im}\,\omega$, ν is the viscosity, and A is the amplitude of the displacement of the boundary. Obviously, this flow declines exponentially with depth in the viscous liquid but, in general, more slowly than the oscillatory part of the field. It accurately compensates the mean momentum of the wave associated with the oscillations of the separation boundary (this is the well-known Stokes flow). The total wave energy is

$$E = \pm\omega\,|\omega_1 - \omega_2|\,\frac{\rho_1+\rho_2}{k}\,|A|^2, \tag{33}$$

where $\omega_{1,2}$ and the signs $\pm$ again correspond to different branches of the dispersion curve for given k. This result is independent of the reference frame, whereas E_l, which differs from (33) only by the replacement of ω with $\omega-kU$, can change both in magnitude and in sign.

4. NONLINEAR PROCESSES ASSOCIATED WITH NEGATIVE ENERGY WAVES

4.1. We now turn to specific effects associated with the nonlinearity of the negative energy waves. These effects may have a variety of properties: for example, it is known that, if the highest-frequency member of a resonance triplet obeying the conditions $\omega_1 + \omega_2 = \omega_3$, $k_1 + k_2 = k_3$ has a negative energy, this may result in a growing amplitude of all three waves, and the instability is then explosive, i.e., in this approximation, the amplitude blows up in a finite interval of time. For a layered liquid, this problem was first examined in Ref. 27 in relation to flows of the form

$$U(z) = \begin{cases} 0, & |z| > h, \\ U, & |z| < h, \end{cases}$$

$$\rho(z) = \begin{cases} \rho_1, & z > 0, \\ \rho_2, & z < 0, \end{cases}$$

i.e., two tangential discontinuities separated by a density jump between them. The corresponding dispersion curve is shown in Fig. 7. The shaded segment AB corresponds to negative energy waves, and points A and B are bifurcation points bounding the Kelvin-Helmholtz instability regions. The above resonance conditions can be satisfied in this flow, and in Fig. 7 the triplet corresponds to the waves at C, D, and E. The highest frequency is that of the negative energy wave (vector $\vec{OC}$), this being necessary for explosive instability.[28] In this case, the explosion time is of the order of $h/A_0\Omega$, where A_0 is the initial amplitude, Ω is the wave frequency, and h is the separation between the tangential discontinuities. We emphasize that, despite the nonlinear growth of the field, the ratio of the linear to nonlinear part of the energy of each of the waves remains unaltered in this approximation (i.e., when only the quadratic nonlinearity is taken into account) during the amplification process. We note, in passing, that the total "linear" energy of the triplet is zero in this case, as expected.

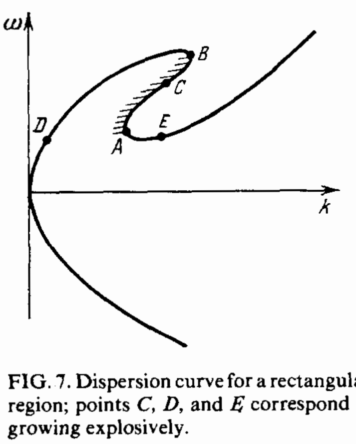

FIG. 7. Dispersion curve for a rectangular jet. Points A, B define the NEW region; points C, D, and E correspond to the resonance triplet of waves growing explosively.

We note that an important special case of negative energy waves is that of zero-energy waves near the threshold of linearly unstable systems. Nonlinear processes associated with them are also explosive in character, and the "explosion time" is, in this case, the shortest in comparison with other "explosion times." These processes were discussed in Ref. 36 for stratified flows with discontinuous and continuous velocity profiles.

4.2. Another class of problems involves the nonlinear stage of the dissipative instability of negative energy waves. It is clear that, in the instability region, only nonlinear effects can restrict the growth of the wave (they can actually accelerate it, as we have just seen). In the case of the Kelvin-Helmholtz instability, it is usually considered that this growth is limited by "breakdown," i.e., the turbulization of wave crests. However, for a "weak" dissipative instability associated with negative energy waves, nonlinear effects may play the leading part even at the laminar stage. Most frequently, this is due to the dependence of wave velocity on wave amplitude, and leads to breakdown in the synchronous supply of energy by the flow to the wave. In the region of weak dispersion, the instability may then be accompanied by an effective growth of a wave harmonic right up to the damping of essentially nonlinear waves and, in particular, solitons. The simplest technique for obtaining such solutions can be summarized as follows.[15] When the velocity shear in the region of sufficiently long waves is large enough, the dispersion relation for the waves can be written in the form

$$\omega = Vk + \omega'(k), \tag{34}$$

where V is a constant velocity and ω' is a small complex correction. This means that the dispersion, dissipation, and amplification effects are small. As far as nonlinearity is concerned, it is local in this approximation in the same sense as in, for example, gas dynamics: the wave velocity depends on the instantaneous value of the perturbation. This means that the well-known methods of nonlinear wave theory can be used to obtain the evolution equation for one of the variables, Φ, characterizing the wave:

$$\phi_t + V\phi_x + \alpha\phi\phi_x + \beta\phi_{xxx} = L'(\phi), \tag{35}$$

where L' is a linear character corresponding to ω' in (34). Roughly speaking, the method of obtaining the linear terms in (35) involves the replacement of ω with $-i\partial/\partial t$ and k with $i\partial/\partial x$, and so on. Although it is true that, depending on the form of $\omega'(k)$ in (35), there may also be integral operators. For example, there may be a term of the form ik describing wave damping or growth, which cannot be written in a simple, real, differential form and contributes the following integral to (35):

$$\int \frac{\phi \, d\xi}{x-\xi}.$$

When the right-hand side of (35) is small enough, its solution can be sought in quasistationary form, i.e., we can assume that it is close to one of the stationary solutions of the K-dV equation, i.e., cnoidal waves or solitons, with slowly-varying parameters. Asymptotic methods for finding such solutions are now well established (see, for example, Refs. 29 and 31). In the first approximation, they are equivalent to using the energy balance equation obtained by multiplying (35) by ϕ and then integrating over the spatial period of the wave (for a soliton, integrating between infinite limits):

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \phi^2 \rangle = \langle \phi L'(\phi) \rangle, \tag{36}$$

where the angle brackets represent this integration. This finally takes us to an ordinary differential equation for the wave parameters. A discussion of this question can be found in Ref. 31.

A number of examples of this kind of description has recently been discussed in relation to negative energy waves in the hydrodynamics of a stratified liquid, using dispersion relations of the kind discussed in Section 2. On of them refers to the above radiative instability[15,30] in a model in which a layer of homogeneous liquid of thickness h moves with velocity U above an infinite stratified half-space with Väisälä frequency N. Here we must also take into account the viscosity ν_T of the upper layer, assuming that the lower layer is an ideal liquid. These assumptions reflect qualitatively the true structure of the upper layer of the ocean and lead to the following dispersion relation:

$$s(\omega - kU)^2 + \omega(\omega^2 - N^2)^{1/2} \tanh kh - (1-s)gk \tanh kh = -4i\nu_T k^2 (\omega - kU) \tanh k_\nu h, \tag{37}$$

where $k_\nu^2 = k^2 - i[(\omega - kU)/\nu_T]$, $s = \rho_1/\rho_2$. This, of course, leads to (25) for $\nu_T \to 0$, $h \to \infty$.

Figure 8 shows the dispersion curves (37) for $U > U_c = [(1-s)gh/s]^{1/2}$. In this case, we have long-wave instability within the interval $0 < k < k_N$, where Re $\omega < N$, and the waves are radiated downward. For small values of k,

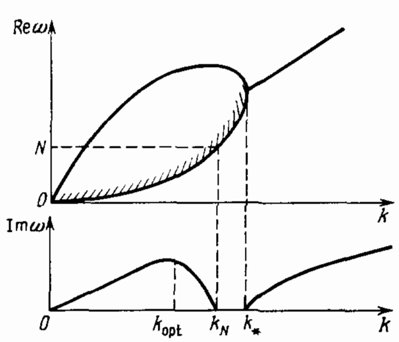

FIG. 8. Dispersion curves for radiative instability in the model with a thin moving upper layer and an infinite, smoothly stratified, stationary lower layer.

the expansion given by (34) is valid:

$$\omega = \varphi k + \psi k^3 - 2i\nu_T k^2,$$

where

$$\varphi = U_c(c-1) - i\frac{Nh}{2}(c-1),$$

$$\psi = U_c h^2 \left[\frac{1}{6} + \frac{(c-1)^2}{2} - \frac{(c-1)^3}{4}\right] - i\frac{U_c^2 h}{2N}(c-1)^3.$$

This leads to the evolution equation of the form of (35):

$$\eta_t + v\eta_x + \frac{3}{2}\frac{U_c}{h}\eta\eta_x - \beta\eta_{xxx} = \frac{\mu}{\pi}\int_{-\infty}^{\infty} \eta_{x'} \frac{dx'}{x-x'} + 2\nu_T \eta_{xx} + \frac{\nu}{\pi}\int_{-\infty}^{\infty} \eta_{x'x'x'} \frac{dx'}{x-x'}, \tag{38}$$

where $v = \text{Re}\,\varphi, \beta = \text{Re}\,\psi, \mu = \text{Im}\,\varphi, \nu = \text{Im}\,\psi$ and the principal value is taken for each of the integrals. When the right-hand side of (38) is small, this equation becomes very similar to the Korteweg–de Vries (K-dV) equation, and its solution can be sought in a form close to the steady-state solutions of the latter. In particulăr, writing the soliton in the usual form

$$\eta = A\,\text{sech}^2 \frac{x - v_s t}{\Delta}, \quad \Delta = \left(\frac{12\beta}{\alpha A}\right)^{1/2}, \quad v_s = v\left(1 + \frac{\alpha A}{3}\right), \tag{39}$$

and assuming that A is a variable, we obtain the following equation for the soliton amplitude:

$$\frac{dA}{dt} = \frac{2{,}92}{\pi}\mu\left(\frac{\alpha}{12\beta}\right)^{1/2} A^{3/2} - \frac{16}{15}\frac{\alpha\nu_T}{12\beta} \times A^2 - \frac{5{,}11}{\pi}\nu\left(\frac{\alpha}{12\beta}\right)^{3/2} A^{5/2}. \tag{40}$$

Qualitative analysis of this equation presents no difficulty. If the soliton amplitude A_0 is small at the initial time, the first term on the right-hand side of (40) (the lowest power of A) is the leading term, and the solution is

$$A = \frac{A_0}{[1 - (t/T_{exp}]^2}, \tag{41}$$

where

$$T_{exp} = \frac{\pi}{1{,}46\mu\,(12\beta/\alpha A_0)},$$

i.e., we have explosive growth in the soliton amplitude which becomes infinite in a time T_{exp}. The physical explanation of this effect then relies on the reduction in the length of the soliton with increasing A, which leads to accelerated growth in amplitude. However, as A increases, other terms in (40) come into play, and the solution tends to the steady-state value A_{st} (Fig. 9). On the other hand, when $A_0 > A_{st}$, the soliton is monotonically damped down to the steady-state level.

Let us now consider some estimates of what happens in the ocean. If $\Delta\rho/\rho = 0.001$, $N = 0.005\ s^{-1}$, and $U(\Delta\rho/\rho g h)^{-1/2} = 1.1$, the radiative instability manifests itself most clearly at $h = 10$–50 m for wavelengths of 100–1000 m and time scales of 20–200 min. When $h = 25$ m, the time-independent amplitude of the soliton is 7 m and its width is $\Delta = 600$ m. These parameters are typical for the uppermost

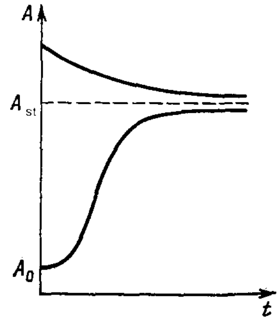

FIG. 9. Amplitude of internal wave solitons as a function of time for different initial conditions.

layer of the ocean, in which there is a density "discontinuity" (seasonal thermocline).

Another example of a weakly nonlinear theory of hydrodynamic NEW instability concerns the relative motion of two thin viscous layers of a liquid.[14]

It is important to note that the validity of such models is subject to a number of restrictions (i.e., to suppress the stronger short-wave Kelvin-Helmholtz-type instability, we must include the finite thickness of the layer in which the abrupt change in density takes place, or assume rapid turbulization of the thin transition layer without a change in the flow outside this layer, and so on). Moreover, on the one hand, these models provide us with the possibility of a purely dynamic theory of hydrodynamic instability in the NEW region while, on the other, they are probably valid for the description of a number of real oceanic situations.

5. RADIATIVE INSTABILITY OF OSCILLATORS IN HYDRODYNAMICS

We have already noted the connection between the concept of the negative energy wave and the so-called anomalous Doppler effect, i.e., the change in the sign of the field frequency radiated into the Cherenkov cone as compared with the field radiated outside this cone. This question was specially examined in Ref. 1. For the one-dimensional or, more precisely, plane two-dimensional waves discussed above, both the NEW and the anomalous Doppler effect correspond simply to waves with phase velocity lower than the flow velocity and wave vector pointing in the same direction.

So far, we have confined our attention to "current" instabilities associated with the corresponding phasing for the collective response of particles in the medium. Another class of wave instabilities, which is well-known in electrodynamics but has not been extensively studied in hydrodynamics, involves the growth of the oscillations of moving oscillators due to the emission by them of negative energy waves into the region of the anomalous Doppler effect, i.e., within the angle θ given by the Cherenkov condition $\cos\theta = v_{ph}/v$, where v is the velocity of the oscillator and v_{ph} the phase velocity of the wave at the given frequency. This is also a variant of radiative instability: the NEW emission acts as a pump, and if its contribution is much greater than losses by emission of positive energy waves, the oscillations grow

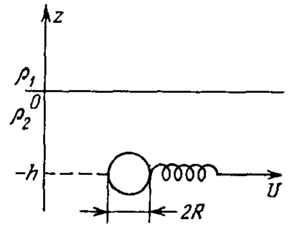

FIG. 10. Oscillator moving below the separation boundary between two liquids of different density.

Similar effects have been known in electrodynamics for a relatively long time.[32]

The analogous hydrodynamic problem was recently discussed in Ref. 33 for a simple model in which a small sphere is pulled by an elastic spring parallel to the separation boundary between two incompressible liquids of different density at a distance h from the boundary (Fig. 10). The rate of pull is constant and the motion of the sphere consists of a translational component with velocity U and a vibrational component along the line of motion with natural frequency Ω. Waves excited on the separation boundary produce a radiation resistance F with a constant component F_0 (balancing the pull of the string) and an oscillating component $\widetilde{F}$, so that the equation of motion for the oscillations of the ball is

$$\ddot{\xi} + \omega_0^2\xi = \frac{\widetilde{F}}{m}, \qquad (42)$$

where m is its mass (including the associated mass of the medium), ω_0 is the resonance frequency, and ξ the oscillatory part of the displacement.

When the radiation force is small enough, the oscillation frequency is close to ω_0, the function $\widetilde{F}$ is almost harmonic for small oscillations, and its amplitude is of the form $iG(\omega_0, U)a$, where a is the oscillator amplitude. The instability then occurs for $G>0$ (its growth rate is $G/2m\Omega$), the sign of G being due to the integrated effect of emission at all the Doppler frequencies (it is clear that the radiation frequency depends on direction because of the motion of the oscillator). The case of low-frequency oscillations is particularly simple and illustrative. The frequency ω_0 is then so low that the radiation force at each instant of time is the same as for motion with constant velocity, i.e., it is equal to $F_0(U-\dot{\xi})$, and the oscillatory part of the force is given by $\widetilde{F} = -(\partial F_0/\partial U)a$, where a is the oscillation amplitude. Hence it is clear that the pump action occurs for $F_{0U}<0$, i.e., on the falling part of the quasistatic curve $F_0(U)$.

However, in general, the problem must be solved rigorously by writing down the radiation field and determining its reaction on the body. This force has both a constant component S_0 (the usual wave resistance) and an oscillatory component with amplitude $\widetilde{F}_0 = \widetilde{F}_+ + \widetilde{F}_-$, where[33]

$$\widetilde{F}_\pm = \mp \frac{R^3\rho_2^2 a}{U^3(\rho_1+\rho_2)} \int \frac{\omega^3(\omega \mp \omega_0)^2 \exp(-2\omega^2 h/g)}{\{\omega^2 - (g'^2/U^2)[1 \mp (\omega_0/\omega)^2]\}^{1/2}}\, d\omega. \qquad (43)$$

($g' = g\Delta\rho/2\rho$) and the integral is evaluated within frequency intervals satisfying the condition $\omega^2 \geqslant (g'^2/U^2)(1 \mp \omega_0/\omega)^2$, $\omega>0$. It is clear that $\widetilde{F}_+$ corresponds, in this case, to

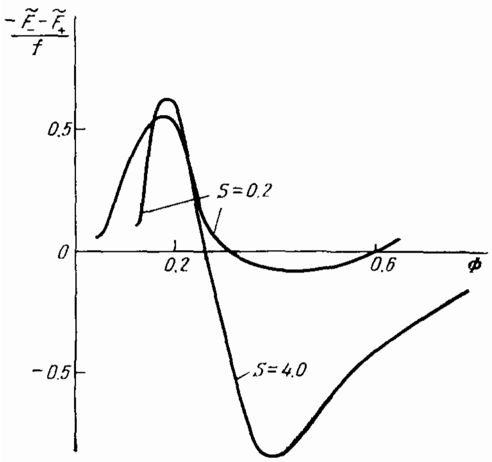

FIG. 11. Amplitude of the oscillatory resistance force $\tilde F$ [normalized to $f = (2\pi R^{-3})^2\rho_2^2 a g'/2\pi h^{-4}(\rho_1+\rho_2)$] as a function of the Froude number Fr.

"normal" waves of positive energy, radiated outside the Cherenkov cone, whereas $\tilde F_-$ corresponds to "anomalous" NEW propagating within this cone. The resultant effect depends on the velocity of motion and the frequency of the oscillations.

Figure 11 shows $\tilde F$ as a function of the Froude number $\mathrm{Fr} = U/(g'h)^{1/2}$ for two values of the parameter $S = 4\omega_0 U/g'$. When $S = 0.2$ (low frequencies), the result is close to the "quasistatic" case whereas for $S = 4.0$ (high frequencies), the pump effect is much more clearly defined and occurs even for $F_{0U} > 0$.

The viscosity of the liquid again leads to effects of opposite sign for "normal" and "anomalous" waves. For problems such as that examined above, the viscosity ν of the upper liquid was taken into account in Refs. 33 and 34 and the result was found to depend on the three parameters $\beta = U^3/g'\nu$, Fr, and $\gamma = \rho_2/\rho_1$. In particular, when $\beta \gg 1$ (low viscosity), the viscosity effect can be neglected for $[\beta(\gamma+1)]^{-1} \ll \mathrm{Fr}^2 \ll \beta$. The instability vanishes outside this interval. Consequently, the instability manifests itself in a finite range of depth h, and vanishes for both very small and very large values of h. In highly viscous media ($\beta \ll 1$), the oscillations are always damped.

Finally, there is the very interesting question of the nonlinear stage of this instability, when the resistance force is no longer proportional to a. Of course, the higher harmonics then become significant in $\tilde F$ but, if, as before, this component is small in comparison with the other terms in (40), the oscillations of the oscillator become nearly harmonic and the quasilinear approximation is valid for them. We can then calculate $\tilde F = iG(\omega_0, U, a)$ relatively simply in the quasistatic case. This case is discussed in detail in Refs. 33 and 34 in relation to the two-dimensional variant of this problem (motion of an infinite cylinder along the normal to its axis). Nonlinear corrections appear both in the amplitude of the oscillatory force $\tilde F$ and in the expression for the mean force (self-action). The expression for the complex frequency of the oscillations is

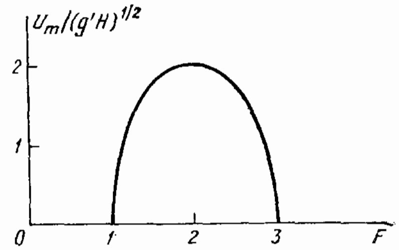

FIG. 12. Amplitude of oscillatory velocity as a function of wave resistance.

$$\Omega = \omega_0 + \frac{i}{2m}\left(F'_{0U} + \frac{F'''_{0U}U_m^2}{8}\right), \tag{44}$$

where U_m is the amplitude of the oscillatory velocity and F_0 is the wave resistance without taking into account the nonlinearity. The derivatives are taken at $U = U_0$, where U_0 is the pull velocity. Hence, it is clear that, when $F'' < 0$ (which is the case here), the equilibrium amplitude is $U_m = (8F'/F''')^{1/2}$, which is a maximum at the point of inflection of the curve $F(U)$ and reaches values comparable with U_0 (Fig. 12).

The expression for the mean retarding force in this steady-state motion is

$$\bar F = F_0 - \frac{2F'F''}{F'''}. \tag{45}$$

Other models have also been recently examined, including motion of an oscillator in a continuously stratified liquid.[35] In our view, anomalous Doppler instability of oscillators in hydrodynamics deserves further investigation.

CONCLUSION

We have shown that, although the concept of the negative energy waves is relative in the sense that the sign of the energy depends on the adopted reference frame, it can be very useful because it enables us to predict a relatively wide class of instabilities. In hydrodynamics, the negative energy wave is not at all an exotic object. Whenever shear instability is possible, we may expect the presence of a negative energy wave, which usually leads to an extension of the range of instability and, occasionally, is the only instability factor. Unfortunately, the dissipative instability associated with the NEW has not been specially investigated experimentally, but there is at least one experiment[21] in which this association was readily demonstrated. In this experiment, the upper layer of a two-layer liquid was made to move relative to the lower layer, and instabilities excited by disturbances with different wave numbers were recorded. In Fig. 13, segment 2 corresponds to the Kelvin-Helmholtz instability and segment 3 to negative energy waves. Estimates show that this type of instability can play a significant part in, for example, the ocean. The structure of flows and density stratification examined above (abrupt shear of velocity and a jump in density, separating the upper homogeneous layer from the lower in a smoothly stratified liquid) is typical for the upper layer of the ocean in which the stratification of the lower layer assures the possibility of radiative instability. If we

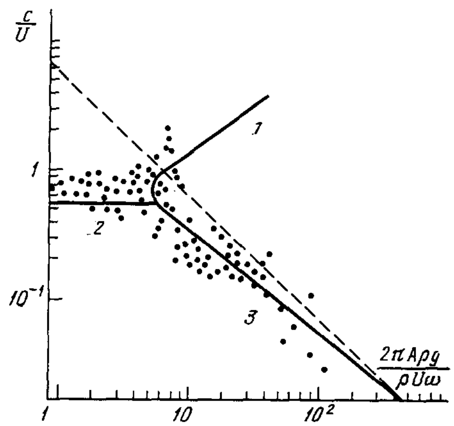

FIG. 13. Dispersion relation for waves on the separation boundary between two layers in relative motion, and the corresponding experimental points[21] ($A\rho/\rho \equiv \Delta\rho/\bar{\rho}$). The broken lines bound the region of negative energy waves.

adopt the realistic values $\Delta\rho/\rho = 0.001$, $N = 0.005$ rad/s, and $U = 1.1(g\Delta\rho h/2\rho)^{1/2}$ (the notation was explained in Section 4), we find that the radiative instability is most effective at h = 10–50 m for wavelengths of 100–1000 m and periods of 20–200 min. It follows from the solution of the nonlinear problem (Section 4) that, at h = 25 m, the amplitude of the resulting solitary internal wave under time-independent conditions is about 7 m and its length is about 600 m. This justifies the assumption that radiative instability may be the cause of the soliton-like internal waves frequently observed in different parts of the Pacific Ocean. Apparently the complexity and range of problems in hydrodynamics is at least comparable with those in electrodynamics, and it may be expected that the problem will continue to be intensively investigated.

[1)]For simplicity, we are assuming that ρ = const in the neighborhood of the given point since, otherwise, other nonlinear terms may well appear.

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Translated by S. Chomet