## Anomalies in gauge theories

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In a quantum system with an infinite number of degrees of freedom, loop corrections may break symmetries of the original Lagrangian. This phenomenon, a "quantum anomaly," arises from the need for a "regularization": a supplemental definition of the theory in the ultraviolet region. A supplemental definition of this sort unavoidably runs into a contradiction with certain symmetries of the classical theory. In particular, it causes a nonconservation of corresponding Noether currents. Reasons for the appearance of anomalies and their place in the structure of modern field-theory models are discussed in this review. An emphasis is placed on anomalies in the internal currents of gauge theories. These anomalies may disrupt the invariance under infinitesimal or global gauge transformations, with the result that the theory is no longer self-consistent. The condition which must be met for the cancellation of internal anomalies severely restricts the composition of fields and the choice of interaction in realistic models. Methods for calculating anomalies are discussed in detail. Emphasis is placed on the nonconservation of axial and chiral fermion currents. The hierarchy of anomalies is introduced. A special section is devoted to global anomalies, in particular, Witten's SU(2) anomaly and a corresponding phenomenon in odd-dimensional Yang-Mills theories.

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This review is devoted to quantum anomalies. Anomalies were first observed by Steinberger' and Schwinger ${ }^{2}$ but did not attract widespread interest until the 1969 papers by Adler, Bell, ${ }^{3}$ and Jackiw. ${ }^{4}$ Many new results on anomalies have recently been obtained, ${ }^{5-47}$ and the interest in this field is constantly growing. Although anomalies have not yet found a wide variety of applications, one might suspect that such a profound phenomenon will eventually assume a more prominent place both in the structure of a future fundamental theory and in the dynamics of specific complex physical systems.

## 1. INTRODUCTION

In the first few sections of this introduction we would like to review some elementary aspects of anomalous symmetries. We will first state just what such a symmetry represents in general in a field theory and why conserved Noether currents are linked with all global symmetries. Second, we will point out the role played by Noether currents in a transition to a gauge theory which is invariant under local transformations, and we will explain why an unbroken gauge invariance requires a (covariant) conservation of these
currents. Third, we will show what can cause a breaking of classical (tree) symmetries at the quantum (loop) level and how a true breaking of symmetry (anomalies and current nonconservation) differs from a spontaneous breaking (in which case the Noether currents are conserved). All these questions are discussed in Subsections 1.1 and 1.2. In Subsection 1.3 we offer a general classification of anomalies. We conclude this introduction with a brief outline of the contents of the remaining sections (2-4) of this review, which are devoted to calculations of anomalies.

### 1.1. Symmetries in general

By "anomalies" we mean nonspontaneous breakings of classical symmetries by quantum effects. To explain this statement, we would naturally begin by recalling just what we mean by

## Symmetries and conserved currents

Any local field theory is described by a Lagrangian $L\left(\phi, \phi_{\mu}\right)$, which depends on the fields $\phi(x)$ and their derivatives $\partial_{\mu} \phi \equiv \phi,_{\mu}$ with respect to the coordinates and the time. For most substantive theories the Lagrangian contains no derivatives higher than the first (otherwise, the situation regarding unitarity would not be clear), and we will consider only such theories here. There are always several fields $\phi$, but for the most part we will omit the index $i$, which specifies the particular field $\boldsymbol{\phi}^{i}$, and also the summation over this index. A fundamental concept is that of the action $S=\int L(\phi) \mathrm{d}^{D} x$, which is found by integrating the Lagrangian over the entire $D$-dimensional space-time. By varying the action $S$ over the fields, we find equations of motion

$$
\begin{equation*}
0=\frac{\delta S}{\delta \phi(x)}=-\partial_{\mu}\left(\frac{\partial L}{\partial \phi_{, \mu}}\right)+\frac{\partial L}{\partial \phi} . \tag{1.1}
\end{equation*}
$$

Symmetry transformations $\phi \rightarrow \phi+\delta_{\varepsilon} \phi$ do not alter the equations of motion; they accordingly leave the action invariant. The Lagrangian, on the other hand, can, despite the invariance of the action, change by a total derivative:

$$
\begin{equation*}
L(\phi) \rightarrow L_{\varepsilon}\left(\phi+\delta_{\varepsilon} \phi\right) \equiv L(\phi)+\partial_{\mu} \Lambda_{\mu}^{(\varepsilon)}+O\left(\varepsilon^{\imath}\right) \tag{1.2}
\end{equation*}
$$

The equations of motion do indeed remain the same as before. The equality
$0=\frac{\partial L}{\partial \phi}-\partial_{\mu} \frac{\partial L}{\partial \phi_{, \mu}}=\frac{\partial L}{\partial \phi}-\frac{\partial^{2} L}{\partial \phi \partial \phi_{, \mu}} \phi_{, \mu}-\frac{\partial^{2} L}{\partial \phi_{, \mu} \partial \phi_{, \nu}} \phi_{\cdot \mu \nu}$
which is linear in $L$, becomes the following when we replace $L$ by $\partial_{,} \Lambda_{v}=\left(\partial \Lambda_{v} / \partial \phi\right) \phi,_{r}:$

$$
\frac{\partial^{2} I_{v}}{\partial \phi^{2}} \phi_{, v}-\frac{\hat{\partial}^{2} \cdot I_{v}}{\partial \phi^{2}} g_{\mu v} \phi, \mu \equiv 0
$$

(We have assumed here that $\Lambda_{r}$. does not contain derivatives of the fields $\phi$; otherwise, we would have to use equations of motion with terms $\partial L / \partial \phi,{ }_{\mu}$, , etc.)

An important point is that the change in Lagrangian (1.2) must be a total derivative without the use of the equations of motion. Taking the equation of motion into account, we see that the action is invariant under any changes in the fields,

$$
S\{\phi+\delta \phi\}-S\{\phi\}=\frac{\delta S}{\delta \bar{\phi}} \delta \phi=0
$$

while the change in the Lagrangian is always equal to a total derivative:

$$
\begin{align*}
& L(\phi+\delta \phi)-L(\phi)=\frac{\partial L}{\partial \phi} \delta \dot{\phi}+\frac{\partial L}{\partial \phi, \mu}(\delta \dot{\rho})_{\mu} \\
& =\left(\partial_{\mu} \frac{\partial L}{\partial \phi, \mu}\right) \delta \phi+\frac{\delta L}{\partial \theta, \mu} \partial_{\mu} \delta \phi=\hat{\partial}_{\mu}\left(\frac{\pi L}{\partial \phi, \mu} \delta \phi\right) \tag{1.3}
\end{align*}
$$

If we are instead dealing with a symmetry transformation, then (1.2) holds along with (1.3), and from these two relations we find the equation

$$
\partial_{\mu}\left(\frac{\partial L}{\partial O, \mu} \delta_{\varepsilon} \phi-\Lambda_{\mu}^{(\varepsilon)}\right)=C
$$

In other words, taking the equations of motion into account, we see that the current is conserved:

$$
\begin{equation*}
J_{\mu}^{(\varepsilon)}=\frac{\partial L}{\partial \varphi, \mu} \delta_{\varepsilon} \phi-I_{\mu}^{(\varepsilon)} . \tag{1.4}
\end{equation*}
$$

This assertion is known as the first Noether theorem: Invariance of an action under a global transformation of fields is equivalent to the existence of a current whose divergence is equal to a linear combination of the equations of motion.

It was in the formulation of this theorem that the distinction between global and local symmetries was first pointed out. The two differ in terms of the arbitrariness regarding the choice of the transformation parameter $\varepsilon$. If $\varepsilon=$ const, the transformation is global, while if $\varepsilon$ can depend on the coordinates $x$ in an arbitrary way we speak in terms of a local or gauge transformation. A global transformation may itself affect the coordinates. For example, the displacement $x_{\mu} \rightarrow x_{\mu}+\varepsilon_{\mu}$ corresponds to the field transformation $\phi \rightarrow \phi+\varepsilon_{\mu} \partial_{\mu} \phi$. If we have $\varepsilon_{\mu}=$ const here, this is a global displacement, while if $\varepsilon_{\mu}(x)$ is a variable this is a local coor-dinate-independent transformation. Why is it specifically a global transformation which appears in the Noether theorem? We single out from $\delta_{\varepsilon \phi}$ the parameter $\varepsilon$ : $\delta_{\epsilon} \phi=\varepsilon \delta \phi$. Correspondingly, we have $\Lambda_{\mu}^{(r)}=\varepsilon \Lambda_{\mu}$. What happens if we assume that (1.2) holds for arbitrary $\varepsilon$, including an $\varepsilon$ which depends on the coordinates in an arbitrary way? From $\delta_{\epsilon} L=\partial_{\mu}\left(\varepsilon \Lambda_{\mu}\right)$ we find in this case

$$
\begin{align*}
& \varepsilon\left[\frac{\partial L}{\partial \phi} \delta \phi+\frac{\partial L}{\partial \sigma_{, \mu}}(\delta \phi), \mu\right]+\partial_{\mu} \varepsilon\left(\frac{\partial L}{\partial \sigma, \mu} \delta \phi\right) \\
& =\varepsilon \partial_{\mu} A_{\mu}+\Lambda_{\mu} \partial_{\mu} \varepsilon . \tag{1.5}
\end{align*}
$$

Equating the two expressions after multiplication by $\partial_{\mu} \varepsilon$ on the left and right sides, we find $\Lambda_{\mu}=\delta \phi \partial L / \partial \phi_{\mu}$. Comparing the coefficients of $\varepsilon$, we then find

$$
\delta \phi\left(\frac{\partial L}{\partial O}-\partial_{\mu} \frac{\partial L}{\sigma \phi \cdot \mu}\right)=0 .
$$

We now recall that equality (1.2) must hold identically; without the use of equations of motion. We therefore see that if the action is invariant under local transformation then some linear combination of Lagrangian derivatives ( $\partial L$ / $\partial \phi)-\partial_{\mu}\left(\partial L / \partial \phi_{\mu}\right)$ is identically zero. The Noether current (1.4), is not present (it is zero) in this case, as can be seen from (1.5). The identical vanishing of certain Lagrangian derivatives in a gauge-invariant theory is the content of the second Noether theorem. \{In the more general case with $\Lambda_{\mu}^{(e)}=\varepsilon \Lambda_{\mu}+\partial_{,} \varepsilon \Lambda_{\mu},+\ldots$, what vanishes is a linear combi-
nation of Lagrangian derivatives and their derivatives: $\left.\partial_{\alpha}\left[(\partial L / \partial \phi)-\partial_{\mu}\left(\partial L / \partial \phi_{\mu}\right)\right], \ldots\right\}$

We turn now to global symmetries, which correspond to Noether currents (1.4). An important class of such symmetries is associated with field transformations which do not contain derivatives, i.e., $\delta \phi=f(\phi)$ but $\partial f / \partial \phi_{, \mu}=0$. We call such transformations "internal" transformations. The appearance of derivatives of the fields in $\delta \phi$ would mean that the coordinates transform along with the fields, as we have just seen in the example of a displacement. Most internal transformations leave not only the action but also the Lagrangian itself invariant; i.e., for them we have $\Lambda_{\mu}=0$. In such cases the Noether current is $J_{\mu}=\left(\partial L / \partial \phi_{\mu}\right) \delta \phi$. It is easy to see that, after quantization, the integral of the timevarying component of this current over space is the generator of a global transformation $\phi \rightarrow \phi+\varepsilon \delta \phi$. The component $J_{0}(\mathbf{x})$ itself generates local transformations $\phi \rightarrow \phi+\varepsilon(x) \delta \phi$. The commutator (in a quantum theory; in a classical theory, we would talk in terms of Poisson brackets) of the canonical momentum $\pi=\partial L / \partial \phi_{0}$ and the field $\phi$ is a $\delta$-function: $\left[\pi(\mathbf{x}), \phi(\mathbf{y})=i \delta^{(D-1)}(\mathbf{x}-\mathbf{y})\right.$. Here we have

$$
\begin{aligned}
& i\left[\int \mathrm{~d}^{D-1} \mathbf{x} J_{1}^{(\varepsilon)}(\mathbf{x}), \phi(\mathbf{y})\right] \\
& =i \int \mathrm{~d}^{D-1} \mathbf{x}\left[\pi(\mathbf{x}) \delta_{\varepsilon} \phi(\mathbf{x}), \phi(\mathbf{y})\right]=+\delta_{\varepsilon} \phi(\mathbf{y})
\end{aligned}
$$

On the other hand, there exist internal symmetries which change the Lagrangian and for which we have $\Lambda_{\mu} \neq 0$. In such cases we will say that the Lagrangian is implicitly invariant, and those parts of it which change under internal transformations are "Wess-Zumino terms." " Let us assume, for example, $L=c_{\mu}^{i j} \phi^{i} \partial_{\mu} \phi^{j}$, where $c_{\mu}^{i j}$ is a constant which is antisymmetric with respect to the indices $i, j$, so that $L$ is not a total derivative. This Lagrangian changes by a total derivative under the transformation $\delta_{i} \phi^{i}==\varepsilon^{i}=$ const: $\delta L=c_{\mu}^{j j} \varepsilon^{\prime} \partial_{\mu} \phi^{j}=\partial_{\mu}\left(c_{\mu}^{i j} \varepsilon^{i} \phi_{j}\right)$. A Noether current exists and is $\varepsilon_{i} J_{\mu}^{i}=\left(\partial L / \partial \phi_{\mu}\right) \delta_{i} \phi-\Lambda_{\mu}^{(\theta)}=-2 c_{\mu}^{j} \varepsilon^{i} \phi^{j}$. The condition of current conservation is the same as the equation of motion: $\partial_{\mu} c_{\mu}^{i j} \phi^{\prime}=0$. A more substantive example is an abelian electrodynamics of odd dimensionality. The simplest case of such an electrodynamics is in three dimensions $(D=3): \quad L=-(1 / 4) F_{\mu,}^{2}+c \varepsilon_{\mu \nu \lambda} A_{\mu} F_{v i}, F_{j u}=\partial_{\mu} A_{v}$ $-\partial_{\|} A_{\mu}$. This theory is invariant under the transformation $\delta_{i} A_{\mu}=\varepsilon_{\mu}=$ const: $\delta L=\partial_{v}\left(\varepsilon_{\mu} 2 c \varepsilon_{\mu \mathrm{w}} A_{\lambda}\right)$. The Noether current is $\quad \varepsilon_{v} J_{\mu \nu}=\left(\partial L / \partial A_{,, \mu t}\right) \delta_{i} A_{v}-\Lambda_{v}^{(c)}$ $=\varepsilon_{r}\left(F_{\mu \nu}+4 c \varepsilon_{\mu, \lambda} A_{\lambda}\right)$. The conservation law $\partial_{\mu} J_{\mu,}=0$ is again the same as the equation of motion: $\partial_{\mu}\left(F_{\mu,}+4 c \varepsilon_{\mu, \lambda} A_{i}\right)=0$.

The increments in the Noether currents due to the Wess-Zumino terms are proportional to antisymmetric $\varepsilon$ symbols in a multidimensional field theory, so that the 0 components $\Lambda_{0}$ do not contain time derivatives of the fields and thus do not contain canonical momenta. For this reason, the addition of $-\Lambda_{o}^{(\epsilon)}$ to $\pi \delta_{\epsilon} \phi$ in the 0 -component of the current, $J_{0}^{(c)}$, does not alter the computation relations between $J_{0}^{(c)}$ and $\phi$, and $\int \mathrm{d}^{D-1} x J_{0}^{(c)}$ remains a generator of a transformation. We note that $\int \mathrm{d}^{D-1} x J_{0}^{(\epsilon)}$ remains a generator of this transformation even if it is not a symmetry
transformation. In this case, however, the variations of the Lagrangian and the action are nonzero and are totally unremarkable.

Before we take up gauge-invariance theories, we would like to say a few words about global transformations which contain derivatives of fields. As we have already mentioned, the corresponding symmetries are associated with replacements of coordinates, so that they are called "space-time" symmetries. The most important for our purposes are the displacement transformation $\quad \delta_{c} \phi=\varepsilon_{\mu} \partial_{\mu} \phi, \delta_{i} L$ $=\varepsilon_{\mu} \partial_{\mu} L \Lambda_{\mu}^{(e)}=\varepsilon_{\mu} L \quad$ and $\quad$ the $\quad$ dilatation $\delta_{f} \phi=\varepsilon\left(x_{\mu} \partial_{\mu}+d_{(\phi)}\right) \phi, \delta_{t} L \varepsilon\left(x_{\mu} \partial_{\mu}+D\right) L=\varepsilon \partial_{\mu}\left(x_{\mu} L\right) ;$ here $d_{(\phi)}$ is a number, the "conformal dimensionality" of field $\phi$. In classical field theory, this number is usually the same as the physical dimensionality of the field. (In the quantum arena, the fields acquire anomalous dimensionalities, and $d_{(\phi)}$ changes-it is represented as a power series in the coupling constant and the Planck constant.)

The Noether current corresponding to displacements is the energy-momentum tensor

$$
\begin{equation*}
\varepsilon_{\alpha} T_{\mu \alpha}^{\mathrm{C}}=\varepsilon_{\alpha}\left(\frac{\partial L}{\partial \phi_{, \mu}} \phi_{, \alpha}-g_{\alpha \mu} L\right) \tag{1.6}
\end{equation*}
$$

The superscript C in $T_{\mu \alpha}^{\mathrm{C}}$ means "canonical." At this point we should recall that the conservation of any current $J_{f /}$ is not disrupted if we add to $J_{\mu}$ an arbitrary expression of the type $\partial_{\nu} C_{\mu}$, with antisymmetric $C_{\mu \nu}=-C_{v \mu}$. Exploiting this arbitrariness, we can always convert a canonical energymomentum tensor $T_{\mu \mu}^{\mathrm{C}}$ into a symmetric one: $T_{\mu \mu}^{\mathrm{S}}=T_{\alpha \mu}^{\mathrm{S}}$. One of the advantages of $T_{\mu \alpha}^{\mathrm{s}}$ is its simple relationship with the rotation generator: $M_{\mu \mu \alpha \beta}=T_{\mu \alpha}^{\mathrm{S}} x_{\beta}-T_{\mu \beta}^{\mathrm{S}} x_{\alpha r}$. Furthermore, it is specifically a symmetric energy-momentum tensor which is associated with a graviton in the (classical) general theory of relativity. It is that tensor which is found from the Lagrangian by a variation with respect to the metric:

$$
\begin{equation*}
T_{\mu \nu}^{\dot{s}}=\frac{2}{|g|^{1 / 2}}\left(\frac{\partial|g|^{1 / 2} L}{\partial \mu^{\mu \nu}}-\partial_{2} \frac{\partial|g|^{1 / 2} L}{, g_{\mu}^{\mu v}}\right) . \tag{1.7}
\end{equation*}
$$

The dilatation Noether current, on the other hand, is

$$
\begin{align*}
\mathrm{D}_{\mu}^{\mathrm{S}} & =\frac{\partial L}{\partial \phi_{, \mu}}\left(x_{\alpha} \partial_{\alpha} \phi+d_{(\sigma)} \phi\right)-x_{1!} L \\
& =x_{\alpha}\left(\frac{\partial L}{\partial \phi, \mu} \theta_{\alpha} \phi-g_{\mu \alpha} L\right)+d_{(\phi)} \frac{\mu L}{\partial \phi, \mu} \phi \\
& =x_{\alpha} T_{\mu \alpha}^{\prime} ;-\frac{\partial L}{\partial \phi, \mu} d_{(\sigma) \phi} . \tag{1.8}
\end{align*}
$$

It turns out that by making use of the arbitrariness in the choice of the conserved current one can choose the dilatation current in the form $D_{\mu}^{\mathrm{conf}}=x_{\alpha} T_{\mu \mu r}^{\mathrm{comf}}$, where $T_{\mu(z}^{\mathrm{conf}}$ is the socalled conformal energy-momentum tensor. In four-dimensional theories of vector and spinor fields, this tensor is the same as the metric tensor (1.7), and the difference between $T^{\mathrm{s}}$ and $T^{\text {C }}$ completely "absorbs" the contribution with the conformal dimensionality $d_{(d)}$ to $D_{\mu}^{C}$ In spaces of other dimensionalities or for scalar fields, this is not the case. For scalars, for example, we have $T_{\mu \mathrm{v}}^{\mathrm{S}}=T_{\mu,}^{\mathrm{C}}$, and ${ }^{2)}$

$$
\begin{equation*}
\left.T_{\mu \nu}^{\mathrm{onf}} T_{\mu \nu}^{(i}-\frac{l-\underline{2}}{4(I)-1)}\left(\partial_{\mu} j_{\nu}-g_{\mu v} d^{2}\right) \xi^{2 \quad}\right) . \tag{1.9}
\end{equation*}
$$

[Possibly we should point out that $\left(\partial_{\mu} \partial_{v}-g_{\mu}, \partial^{2}\right) \phi^{2}$ $=\partial_{\alpha}\left(g_{\alpha v} \partial_{\mu}-g_{\mu \nu} \partial_{\alpha}\right) \phi^{2}$, and a change of this sort in the energy-momentum tensor would not disrupt its conservation. The increment in the dilatation current, on the other hand, is $\frac{D-2}{4(D-1)} x_{\mu} \times\left(\partial_{\mu} \partial_{\mu}-g_{\mu}, \partial^{2}\right) \phi^{2}$ $=\frac{D-2}{4(D-1)} \partial_{\mu}\left(x_{\mu} \partial_{\nu}-x_{\nu} \partial_{\mu}\right) \phi^{2}-\frac{D-2}{2}\left(\partial_{\mu} \phi\right) \phi$. The first term on the right side is unimportant, while the second completely cancels the contribution

$$
\frac{\partial L}{\partial \varphi_{\cdot} \mu} \mathrm{d}_{(\varphi)} \phi=\phi \cdot \mu \frac{D-2}{2} \phi
$$

to $D_{\mu}^{C}$ in (1.8).]
The redefinitions of the Noether currents carried out above affect neither the conservation of these currents nor the commutation relations of the integrals of the 0 -components. To illustrate this point, we assume $J_{0} \rightarrow J_{0}+\partial_{v} C_{0 v}$. The index $v$ here cannot take on the value 0 because of the antisymmetry of $C_{\mu \nu}$. The integral $\int J_{0} \mathrm{~d}^{D-1} \mathbf{x}$ is therefore unchanged by this redefinition. \{The local commutator $\left[J_{0}(\mathbf{x}), \phi(\mathbf{y})\right]$, however, changes by a total spatial derivative.\}

The 0 -components $T_{0 \alpha}$ and $D_{0}$ of course generate translation and extension transformations. In these cases, the presence of $\Lambda_{0}$ in the definition of the current $J_{0}=\pi \delta \phi-\Lambda_{0}$ is important. For a scalar field, for example, we would have $L=(1 / 2)\left(\partial_{\mu} \phi\right)^{2}$, and the energy-momentum tensor would be $T_{\mu \alpha}^{\mathrm{C}}=T_{\mu \alpha}^{\mathrm{s}}=\partial_{\mu} \phi \partial_{\alpha} \phi-g_{\mu \alpha} \cdot(1 / 2)(\partial \phi)^{2}$. For $\alpha \neq 0$ we would have $i\left[T_{0 \alpha}^{C}(\mathbf{x}), \phi(\mathbf{y})\right]=i\left[\pi \partial_{\alpha} \phi(\mathbf{x}), \phi(\mathbf{y})\right]$ $=\delta(\mathbf{x}-\mathbf{y}) \partial_{\alpha} \phi(\mathbf{x})$. If, on the other hand, we had $\alpha=0$, then we would have $\left(\partial L / \partial \phi_{0}\right) \phi_{.0}=\pi^{2}$, and the commutator would be twice as large as necessary except for the term $-\Lambda_{0}=-g_{00} \cdot(1 / 2)(\partial \phi)^{2}$, which contains $-(1 / 2) \pi^{2}$. Taking this contribution into account, we find $i\left[T_{00}^{\mathrm{C}}(\mathbf{x}), \phi(\mathbf{y})\right]=i\left[(1 / 2) \pi^{2}(\mathbf{x}), \phi(\mathbf{y})\right]=\delta(\mathbf{x}-\mathbf{y}) \pi(\mathbf{x})$. The commutators of the dilatation current can be discussed in a corresponding way:
$i \int \mathrm{~d}^{D-1} \mathbf{x}\left[D_{i}^{\prime}(\mathrm{x}), \phi(\mathbf{y})\right]=i \int \mathrm{~d}^{D-1} \mathrm{x}\left\{x_{l}\left[T_{0 i}^{\mathrm{C}}, \phi\right]\right.$
$\left.+x_{0}\left[T_{00}^{\mathrm{C}}, \phi\right]+\left[\pi \frac{D-2}{2} \phi, \phi\right]\right\}=\left(y_{\mu} \partial_{\mu}+\frac{D-2}{2}\right) \phi(\mathbf{y})$.
(We are dealing here with single-time commutators, so that we have $x_{0}=y_{0}$.) In exactly the same way we find
$i \int \mathrm{~d}^{D-1} \mathbf{x}\left[D_{0}^{\text {conf }}(\mathbf{x}), \phi(\mathbf{y})\right]$
$=i \int \mathrm{~d}^{\mathrm{D}-1} \mathbf{x}\left\{x_{i}\left[T_{0 i}^{\text {cont }}, \phi\right\}+x_{v}\left[\mathcal{T}_{10 \mathrm{i}}^{\text {conf }}, \phi\right]\right\}$
$=\int \mathrm{d}^{D-1} \mathbf{x}\left[x_{i} \partial_{i} \phi+\frac{D-2}{2(D-1)}(D-1) \phi+x_{6} \pi\right](\mathbf{x}) \delta(\mathbf{x}-\mathbf{y})$
$=\left(y_{\mu} \partial_{\mu}+\frac{D-2}{2}\right) \phi(\mathbf{y})$.
Having dealt with global symmetries, we can move on to local symmetries. Before we get intimately involved with gauge transformations, we wish to point out the following important circumstance: If a theory has a global symmetry,
then a change in its action under a corresponding local transformation with a parameter $\varepsilon$ which depends on $x$ is, if the equations of motion are ignored,

$$
\begin{gather*}
\delta_{\varepsilon} S:=\int d^{D} x\left[\varepsilon\left(\frac{\partial L}{\partial \phi, \mu}(\delta \phi)_{, \mu}+\frac{\partial L}{\partial \phi} \partial \phi\right)\right. \\
\left.+\partial_{\mu} \varepsilon\left(\frac{\partial L}{\partial \phi, \mu} \delta \phi\right)\right] . \tag{1.10}
\end{gather*}
$$

The expression in the first set of square brackets in the integrand "knows nothing" about the $x$ dependence of $\varepsilon$, so that it is equal to $\varepsilon \partial_{\mu} \Lambda_{\mu}$, as it would be for a constant value of $\varepsilon$. As a result we find

$$
\begin{align*}
\delta_{\mathbf{e}} S & =\int \mathrm{d}^{D} x\left[\varepsilon \partial_{\mu} \Lambda_{\mu}+\partial_{\mu} \varepsilon\left(\frac{\partial L}{\partial \phi, \mu} \delta \phi\right)\right] \\
& =-\int \mathrm{d}^{D} x \varepsilon \partial_{\mu} J_{\mu}+\oint \mathrm{d}^{D-1} x \varepsilon \frac{\partial L}{, \phi, \mu} \delta \phi \tag{1.11}
\end{align*}
$$

When the equations of motion are taken into account, we have $\partial_{\mu} J_{\mu}=0$, and the action turns out to be invariant (within surface contributions) under local transformations also. We wish to stress that, in contrast with local invariance, the relation $\delta_{\varepsilon} S=0$ holds not identically in this case, but only on the equations of motion (or, as is often said, on the mass shell).

Gauge symmetries do not correspond to any Noether currents, as we have already mentioned. Nevertheless, it has been established quite well that gauge invariance requires a covariant conservation of matter currents. What is the origin of this requirement? Why do we need precisely a covariant conservation instead of an ordinary conservation, as for Noether currents? For definiteness we will talk about YangMills theories, although all the arguments hold both for antisymmetric tensor fields and for gravity.

The Lagrangian of a free Yang-Mills theory, ${ }^{3 \prime}$ $L=\operatorname{Tr}(1 / 4) F_{\mu v}^{2}, F_{\mu v}=\partial_{\mu} A_{v}-\partial_{v} A_{\mu}+\left[A_{\mu} A_{v}\right]$, is invariant under local field transformations $\delta A_{\mu}=\partial_{\mu} \varepsilon+\left[A_{\mu} \varepsilon\right]$ $\equiv \mathrm{D}_{\mu} \varepsilon$, so that we have $\Lambda_{\mu} \equiv 0$. Equation (1.5) is not valid for the variation of the Lagrangian in this case, since the transformation of $\delta_{\varepsilon} A_{\mu}$ contains a derivative of $\varepsilon$. However, there is still no nontrivial Noether current, of course. Indeed, on the equations of motion, the quantity

$$
\operatorname{Tr} \frac{\partial L}{\partial A_{\alpha, \mu}} \delta A_{\alpha}=\operatorname{Tr} F_{\alpha \mu} D_{\alpha} \varepsilon
$$

is conserved. To find the current from this result, we need to transfer the derivative for $\varepsilon$ to $F_{\alpha \mu}: \operatorname{Tr} F_{\alpha \mu} \mathrm{D}_{\alpha} \varepsilon$ $=+\operatorname{Tr} \varepsilon \mathrm{D}_{\alpha} F_{\mu \alpha}+\partial_{\alpha}\left(\operatorname{Tr} F_{\alpha \mu} \varepsilon\right)$. The total divergence can be omitted, since it does not affect the conservation of the current, because of the antisymmetry of $F_{\alpha \mu}$. However, the "Noether current" $\mathrm{D}_{\alpha} F_{\mu \alpha}$ which results in none other than an equation of motion; i.e., when the equations of motion are taken into account, it is not simply conserved but is actually equal to zero, in complete agreement with the second Noether theorem. This conclusion does not depend on the particular form of the Lagrangian. For example, the identity condition

$$
\begin{aligned}
0 & =\delta L=\operatorname{Tr}\left[\frac{\partial L}{\partial A_{\alpha}} \mathrm{D}_{\alpha} \varepsilon+\frac{\partial L}{\partial A_{\alpha \cdot \beta}}\left(\partial_{\beta} \mathrm{D}_{\alpha} \varepsilon\right)\right] \\
& =\operatorname{Tr} \frac{\partial L}{\partial A_{\alpha, \beta}} \partial_{\alpha \beta}^{2} \varepsilon+\operatorname{Tr}\left(\frac{\partial L}{\partial A_{\beta}}+\left[\frac{\partial L}{\partial A_{\alpha, \beta}}, A_{\alpha}\right]\right) \partial_{\beta} \varepsilon \\
& +\operatorname{Tr}\left(\left[\frac{\partial L}{\partial A_{\alpha}}, A_{\alpha}\right]+\left[\frac{\partial L}{\partial A_{\alpha, \beta}}, A_{\alpha, \beta}\right]\right) \varepsilon
\end{aligned}
$$

implies (first) antisymmetry of the derivative $\partial L / \partial A_{a, \beta}$ with respect to the indices $\alpha$ and $\beta$ and (second) two identity relations for the matrix commutators:

$$
\begin{align*}
& \frac{\partial L}{\partial A_{\beta}} \equiv\left[\frac{\partial L}{\partial A_{\alpha}, \beta}, A_{\alpha}\right], {\left[\frac{\partial L}{\partial A_{\alpha}, \beta}, A_{\alpha, \beta}\right] } \\
&+\left[\frac{\partial L}{\partial A_{\alpha}}, A_{\alpha}\right] \equiv 0 \tag{1.12}
\end{align*}
$$

Together, these results imply

$$
\left[\frac{\partial L}{\partial A_{\alpha, \beta}}, F_{\alpha \beta}\right]=0
$$

i.e., that the Lagrangian may depend on the derivative $A_{\alpha, \beta}$ exclusively through $F_{\alpha \beta}$. The "Noether current," on the other hand, is found to be

$$
\begin{aligned}
\operatorname{Tr} \frac{\partial L}{\partial A_{\alpha, \mu}} \delta A_{\alpha} & =\operatorname{Tr} \frac{\partial L}{\partial A_{\alpha, \mu}} \mathrm{D}_{\alpha} \varepsilon \\
& =\operatorname{Tr} \varepsilon \mathrm{D}_{\alpha} \frac{\partial L}{\partial A_{\mu, \alpha}}+\partial_{\alpha}\left(\operatorname{Tr} \varepsilon \frac{\partial L}{\partial A_{\alpha, \mu}}\right)
\end{aligned}
$$

The total derivative is inconsequential because of the antisymmetry of $\partial L / \partial A_{\alpha, \mu}$, and we have

$$
\mathrm{D}_{\alpha} \frac{\partial L}{\partial A_{\mu, \alpha}}=\partial_{\alpha} \frac{\partial L}{\partial A_{\mu, \alpha}}+\left[A_{\alpha}, \frac{\partial L}{\partial A_{\mu, \alpha}}\right]
$$

which, according to the first equation in (1.12), is equal to the Lagrangian derivative

$$
\partial_{\alpha} \frac{\partial L}{\partial A_{\mu, \alpha}}-\frac{\partial L}{\partial A_{\mu}}
$$

The Noether current thus vanishes on the equations of motion.

Gauge invariance is necessary in Yang-Mills theories if they are to have a physical meaning: No other way is known to achieve unitarity (and, in the case $D=4$, renormalizability also) in theories with vector fields. ${ }^{48}$ Consequently, a Yang-Mills theory which is interacting with scalar and spinor fields (collectively called "matter fields") must therefore also have gauge invariance. The interaction of gauge fields with spinor fields ${ }^{4}$ is constructed in accordance with

$$
\begin{equation*}
L(A, \psi)=L_{0}(A)+L_{0}(\psi)+\operatorname{Tr} A_{\mu} J_{\mu}(\psi) \tag{1.13}
\end{equation*}
$$

We are interested in invariance under transformations $A_{\mu} \rightarrow A_{\mu}+\mathrm{D}_{\mu} \varepsilon(x), \psi \rightarrow \psi$, for which the Yang-Mills action $L_{0}(A)$ and the spinor action $L_{0}(\psi)$ do not change, while we have $\delta \operatorname{Tr} A_{\mu^{\prime}} J_{\mu}(\psi)=\operatorname{Tr} \varepsilon\left(\mathrm{D}_{\mu} \varepsilon\right) J_{\mu}(\psi)=\partial_{\mu^{\prime}}\left(\operatorname{Tr} \varepsilon J_{\mu^{\prime}}(\psi)\right)$ - $\operatorname{Tr} \varepsilon \mathrm{D}_{\mu} J_{\mu}(\psi)$. In other words, a theory of vector (gauge) fields will be meaningful only under the condition $\mathrm{D}_{\mu} J_{\mu}(\psi)=0$. It is sufficient that this equation hold on the equations of motion of matter fields. The need for a covariant conservation of the matter current in a gauge theory can be seen even in the equations of motion of a Yang-Mills field
$\mathrm{D}_{\mu} F_{\mu}=J_{v}$. Acting on this equation with the covariant derivative $\mathrm{D}_{v}$, we find $\mathrm{D}_{v} \mathrm{D}_{\mu} F_{\mu v}=-(1 / 2)\left[F_{\mu v}, F_{\mu v}\right]=0$ on the left side and $\mathrm{D}_{1}, J_{1}$, on the right.

The current $J_{\mu}(\psi)$, which determines the interaction in the Lagrangian (1.13), must thus be conserved covariantly. An important point is that it is often possible to choose this current to be a Noether current corresponding to a global symmetry of the action of the matter, $L_{0}(\psi)$. This symmetry must of course be described by the same group as the gauge symmetry of a Yang-Mills field. In the theory of $L_{0}(\psi)$ matter fields without Yang-Mills vector bosons, the Noether current is conserved, but not in a covariant fashion: $\partial_{\mu} J_{\mu}(\psi)=0$. This equation, however, holds only when the equations of motion are taken into account. When we switch to the theory in (1.13), the equations of motion of fermions change and lead to the covariant conservation law $\mathrm{D}_{\mu} J_{\mu}(\psi)=0$. This point can be demonstrated in a very general form by means of simple but comparatively lengthy calculations. The current $J_{\mu}(\psi)$ is determined from the matter Lagrangian $L_{0}(\psi): J_{\mu}^{a}=\partial L_{0} / \partial \psi_{\mu}^{b} \delta_{a} \psi^{b}$. The invariance of Lagrangian $L_{0}$ under global transformations $\psi^{b} \rightarrow \psi^{b}$ $+\varepsilon^{a} \delta_{a} \psi^{b}$ means

$$
\frac{\partial L_{0}}{\partial \psi^{b}} \delta_{a} \psi^{b}+\frac{\partial L_{0}}{\partial \psi_{, \mu}^{b}}\left(\delta_{a} \psi^{b}\right) \cdot \mu=0
$$

The divergence of the current $J_{\mu}^{a}$ in a theory (1.13) which is interacting with a vector field $A_{\mu}$ is found from this relation and the equation of motion

$$
\partial_{\mu} \frac{\partial L}{\partial \psi_{, \mu}^{b}}==\frac{\partial L}{\partial \psi^{b}}+A_{\mu}^{c} \frac{\partial J_{\mu}^{c}}{\partial \psi^{b}}
$$

to be

$$
\begin{aligned}
\partial_{\mu} J_{\mu}^{a} & =\partial_{\mu}\left(\frac{\partial L}{\partial \psi_{, \mu}^{b}} \delta_{a} \psi^{b}\right) \\
& \therefore \frac{\partial L_{0}}{\partial \psi_{, \mu}^{b}}\left(\delta_{a} \psi^{b}\right)_{\mu}+\left(\partial_{\mu} \frac{\partial L_{0}}{\partial \psi_{, \mu}^{b}}\right) \delta_{a} \psi^{b}=A_{\mu}^{c} \frac{\partial J_{\mu}^{c}}{\partial \psi^{b}} \delta_{\alpha} \psi^{b}
\end{aligned}
$$

We can now show that this combination is equal to $-i f^{a b c} A_{\mu}^{b} J_{\mu}^{c}$. Hence there is a covariant conservation of $J_{\mu}: \mathrm{D}_{\mu} J_{\mu}^{a}=\partial_{\mu} J_{\mu}^{a}+\left[A_{\mu} J_{\mu}\right]^{a}=\partial_{\mu} J_{\mu}^{a}+i f^{a b c} A_{\mu}^{b} J_{\mu}^{c}=0$. Here we need to make use of the group structure of the transformations $\delta_{\varepsilon} \psi^{b}=\varepsilon^{a} \delta_{a} \psi^{b}$. We first recall that a variation of the Lagrangian $L_{0}$ under a transformation with a variable parameter $\varepsilon$ is equal to $\delta_{\varepsilon} L_{0}=\partial_{\mu} \varepsilon^{c} J_{\mu}^{c}[\operatorname{see}(1.11) ; \Lambda=0]$. We can write the group law $\left(\delta_{\varepsilon_{1}} \delta_{\varepsilon_{2}}-\delta_{\varepsilon_{2}} \delta_{\varepsilon_{1}}\right) L_{0}$ $=-\delta_{\left\{\varepsilon_{1}, e_{2}\right\}} L_{0}$ or, in terms of currents,

$$
\begin{aligned}
& \left(\partial_{\mu} \varepsilon_{2}^{c}\right) \varepsilon_{1}^{a} \frac{\partial J_{\mu}^{c}}{\partial \psi^{b}} \delta_{a} \psi^{b}-(1 \leftrightarrow 2) \\
& =-\left[\partial_{\mu}\left(i j^{a b c} \varepsilon_{1}^{a} \varepsilon_{2}^{b}\right)\right] J_{\mu}^{c}=+\left(\partial_{\mu} \varepsilon_{2}^{c}\right) \varepsilon_{1}^{a} i f^{a b c} J_{\mu}^{b}-(1 \leftrightarrow 2)
\end{aligned}
$$

Making use of the arbitrariness in the parameters $\varepsilon_{1}$ and $\varepsilon_{2}$, we then find the relation which we need:

$$
\frac{\partial J_{\mu}^{c}}{\partial \psi^{b}} \delta_{a} \psi^{b}=i f^{a b c} J_{\mu}^{b}
$$

We have thus shown that the symmetries of a theory are
unambiguously related to the conservation of currents. Conserved Noether currents correspond to global symmetries. Local invariance requires a covariant conservation of the matter currents describing the interaction with gauge fields. These currents may in turn, by virtue of the Noether theorem, correspond to some global symmetry of the Lagrangian of matter fields.

There are some interesting classes of covariantly conserved currents which are not associated with either a global invariance or a local invariance. An example is a non-abelian Yang-Mills theory with massless fermions:

$$
\begin{equation*}
L(\psi, A)=L_{0}(A)+\bar{\psi} \hat{D} \psi \tag{1.14}
\end{equation*}
$$

The color axial current $J_{\mu}^{a 5}=\bar{\psi} \gamma_{\mu} \gamma^{5} t^{a} \psi$ is covariantly conserved in it. The Lagrangian of the theory is noninvariant under the global transformation $\psi \rightarrow \varepsilon^{i \varepsilon^{u_{1} u^{\prime}} \gamma^{5}} \psi$ because a commutator term $\psi[\hat{A} \varepsilon] \gamma^{5} \psi$ which arises here cannot be eliminated by the field transformation $A_{\mu}=-i A_{\mu}^{a} t^{a}$, since $A_{\mu}^{a}$ is related exclusively to a vector (color) current $\psi \gamma_{\mu} t^{a} \psi$. (This theory has the standard $\gamma^{5}$ invariance, according to which $\psi$ is multiplied by a color-singlet factor $e^{i a \gamma^{`}}$.) Nevertheless, it is easy to see that the equation $\mathrm{D}_{\mu} J_{\mu}^{5 a}=0$ holds on the equations of motion. From the standpoint of anomalies, this current is just as interesting as a Noether current, and we will discuss it on the same basis.

### 1.2. Quantum anomalies

In discussing symmetries we have so far drawn no distinction between classical physics and quantum physics. There is a good reason for this approach: The quantum operators satisfy the same equations of motion as are satisfied by the classical variables, and the symmetries of the theory correspond to conserved currents in both classical mechanics and quantum mechanics.

We find a different situation when we turn to quantum field theory. When there are an infinite number of degrees of freedom, regularization becomes necessary. There are various (equivalent) ways to interpret this operation; the following way is particularly convenient for our purposes. In the quantization of a field theory it may be necessary to introduce some new-"regulator"-fields and to alter the classical action and the equations of motion in a corresponding manner. We will discuss here only an ultraviolet regularization, in which case the rule for introducing regulator fields has a very simple formulation. For each physical field we introduce a certain number of regulator fields. The number depends on the dimensionality of the space and on the index of the quantum correction (i.e., on the number of loops in the corresponding Feynman diagram; if the number of regulators required does not increase with the index of the loop, the theory is renormalizable). The interaction of the regulators with each other and with the original fields is constructed in exactly the same way as the interaction of the corresponding physical fields, with one exception: The regulators have a large mass $M_{\text {reg }}$. Furthermore, an additional minus sign is assigned to each regulator loop. Such a regularization is called a "Pauli-Villars regularization." Remarkably, the symmetry of the resulting action may be narrower
than the symmetry of the original action. The symmetries which are characteristic of massless particles-conformal, axial, and gauge symmetries and supersymmetry-cannot be generalized to the case of massive regulator fields. Because of the quantum corrections, the noninvariance of the regulator part of an action may change the interactions of physical fields; furthermore, the noninvariance may persist even in the limit $M_{\text {reg }} \rightarrow \infty$ when the regularization is "removed." In this case we speak in terms of a quantum anomaly which breaks a classical symmetry. An anomaly in a global symmetry is manifested as a nonconservation of the regularized Noether current $J_{\mu}^{\text {reg }}=J_{\mu}(\psi)-J_{\mu}(\Psi) \quad(\Psi$ are regulator fields): $\partial_{\mu} J_{\mu}^{\text {reg }}=-\partial_{\mu} J_{\mu}(\Psi)=\hbar A\{\phi\}$. An anomaly in a local symmetry is associated with a nonzero covariant divergence of a regularized matter current which is interacting with gauge bosons. In other cases, the divergences of the currents are expressed in terms of physical fields because of quantum effects, so that they are proportional to the Planck constant; in terms of diagrams, we would say that they are associated with single-loop diagrams. We will discuss a very simple example, which demonstrates a mechanism for the appearance of an anomaly, slightly further on.

The anomaly in chiral symmetry has been studied most thoroughly in the theory of elementary particles. This invariance is exhibited by the classical Lagrangian of massless fermions in any even-dimensional space-time, in which it is possible to separate left-hand and right-hand spinor fields by means of projection operators ${ }^{5}\left(1 \pm \gamma^{5}\right) / 2$. The general chiral transformations rotate independently the phases of left-hand and right-hand fermions: $\psi_{\mathrm{L}}=(1 / 2)\left(1-\gamma^{5}\right) \psi_{\mathrm{L}}$ $\rightarrow e^{i \alpha} \psi_{\mathrm{L}}, \psi_{\mathrm{R}}=(1 / 2) \times\left(1+\gamma^{5}\right) \psi_{\mathrm{R}} \rightarrow e^{i \beta} \psi_{\mathrm{R}}$. In the case $\alpha=\beta$ we call these "vector transformations," and in this case the entire bispinor

$$
\psi=\binom{\psi_{\mathrm{L}}}{\psi_{\mathrm{R}}}
$$

rotates as a whole: $\psi \rightarrow e^{i c} \psi$. In the case $\alpha=-\beta$, the transformations are "axial," and we have $\psi \rightarrow \mathrm{e}^{\mathrm{i} \alpha \gamma^{*}} \psi$. Finally, the condition $\alpha=0$ or $\beta=0$ selects respectively right-hand and left-hand chiral transformations:

$$
\psi \rightarrow \exp \left(i \alpha \frac{1+\gamma^{5}}{2}\right) \psi, \quad \psi \rightarrow \exp \left(i \beta \frac{1-\gamma^{5}}{2}\right) \psi
$$

The simplest example which we will discuss here is an anomaly in the axial symmetry of a two-dimensional theory: $L(\psi)=\bar{\psi} \hat{D} \psi=\bar{\psi}(\hat{\partial}+\hat{A}) \psi$. It is regularized by adding a Pauli-Villars field $L(\Psi)=\Psi \hat{\mathrm{D}} \Psi+M \bar{\Psi} \Psi$. The regularized axial current corresponding to the transformation $\psi \rightarrow e^{i a \gamma}{ }^{4} \psi, \Psi \rightarrow e^{i \alpha \gamma^{~}}{ }^{*} \Psi$ is $J_{\mu}^{5 \mathrm{reg}}=\bar{\psi} \gamma_{\mu} \gamma^{5} \psi-\Psi \gamma_{\mu} \gamma^{5} \Psi$, and its divergence is $\partial_{\mu} J_{\mu}^{5 \mathrm{reg}}=-\partial_{\mu} J_{\mu}^{S}(\Psi)=-2 M \bar{\Psi} \gamma^{5} \Psi$. Finally, the product $M \frac{\mu}{\Psi} \gamma^{5} \Psi$ is related to the field $A_{\mu}$ by the diagram in Fig. 1, which is described by the expression ${ }^{6 \prime}$


FIG. 1. Divergence of an axial current in an external field. Regulator fermions propagate in the loop.


FIG. 2. Correlation function of two vector fermion currents. By virtue of the relation $J_{a}^{\mathrm{s}}=\epsilon_{\text {ato }} J_{b}$ in two dimensions, the same diagram represents an axial current in an external field. Physical fermions propagate in the loop.

$$
\begin{equation*}
M \operatorname{Tr} \gamma^{5} \int \frac{1}{\hat{p}+i M} \hat{A} \frac{1}{\hat{p}-\hat{q}+i M} \frac{\mathrm{~d}^{2} p}{(2 \pi)^{2}}=\frac{1}{\pi} \varepsilon_{\mu v} g_{\mu} A_{v}^{* *)} . \tag{1.15}
\end{equation*}
$$

Since an anomaly is associated with a regularization of the theory, however, one could be forgiven for suspecting that the results might be ambiguous. Is it impossible to avoid an anomaly through an appropriate choice of regularization? We have already mentioned that a symmetric regularization may not exist. However, this is only part of the answer. To reach clearer conclusions, we seek an expression not for the divergence $\partial_{\mu} J_{\mu}^{5}$ but for the current itself, $J_{\mu}^{5}$. In two dimensions we have $J_{\mu}^{5}=\varepsilon_{\mu \mu \alpha} J_{\alpha \alpha}$, and we can restrict the analysis to an evaluation of the diagram in Fig. 2 for the vector current $J_{\alpha}=\bar{\psi} \gamma_{\sigma} \psi$ :

$$
\begin{align*}
&\left\langle J_{\alpha}\right\rangle= \int \frac{d^{2} p}{\left(2^{2}\right)^{2}} \operatorname{Tr}\left(\gamma_{\alpha} \frac{1}{\hat{p}} \gamma_{\beta} \frac{1}{\hat{p}-\hat{q}}\right. \\
&\left.-\gamma_{\alpha} \frac{1}{\hat{p}+i M} \gamma_{\beta} \frac{1}{\hat{p}-\hat{q}+i M}\right) A_{\beta} \\
&= \frac{1}{2 \pi}\left[\int_{0}^{1} d \xi \frac{\xi(1-\xi)}{\xi(1-\xi) q^{2}}\left(2 q_{\alpha} q_{p}-g_{\alpha \beta} q^{2}\right)\right. \\
&\left.-\int_{0}^{1} d \xi \frac{\xi(1-\xi)\left(2 q_{\alpha} q_{\beta}-g_{\alpha \beta} q^{2}\right)-g_{\alpha \beta} M^{2}}{\xi(1-\xi) q^{2}+M^{2}}\right] A_{\beta} . \tag{1.16}
\end{align*}
$$

Letting the regulator mass $M$ go to infinity, we then find

$$
\begin{equation*}
\left\langle J_{\alpha}\right\rangle=\frac{1}{\pi}\left(\frac{q_{\alpha} q_{\hat{\beta}}}{q^{2}}-g_{\alpha \beta}\right) A_{\beta} . \tag{1.17}
\end{equation*}
$$

[When we multiply this expression by $q_{\mu} \varepsilon_{\mu \alpha x}$, i.e., take the divergence of the axial current, we find (1.16).] Looking at this expression, it is easy to see that by changing the regularization (e.g., by using a separation of points instead of the Pauli-Villars procedure) it is possible to vary the coefficient of $g_{\alpha \beta}$ in an arbitrary way, but the coefficient of $q_{\alpha} q_{\beta}$ is singular in $q^{2}$ and therefore universal-independent of the choice of regularization. The existence of a structure $q_{\alpha} q_{\beta} /$ $q^{2}$ is totally objective: It results from a nontrivial imaginary part of the diagram in Fig. 2. An evaluation of the diagram in terms of its imaginary part relates it to the amplitudes for the production of real particles by virtue of the unitarity relation

$$
\operatorname{Im} \int y_{i i} \sim \sum_{k}\left|\int y_{i k}\right|^{2} .
$$

In the evaluation of the diagram in Fig. 2 in terms of the imaginary part, we need an infrared regularization; it can be specified by, for example, introducing a small mass $m$ of a physical fermion. We then write

$$
\begin{align*}
& \operatorname{Im}\left\langle J_{\alpha}\right\rangle=\frac{1}{2} \int \frac{d^{2} p}{(2 \pi)^{2}} \cdot 2 \pi i \delta\left(p^{2}-m^{2}\right) \cdot 2 \pi i \delta\left((p-q)^{2}-m^{2}\right) \\
& \times \operatorname{Tr} \gamma_{\alpha}(\hat{p}+m) \gamma_{B}(\hat{p}-\hat{q}+m) A_{\beta} \\
&=\left\{2 \frac{m^{2}}{q^{4}} \frac{0\left(q^{2}-4 m^{2}\right)}{\left[1-\left(4 m^{2} / q^{2}\right)\right]^{1 / 2}} q_{\alpha} q_{\beta}\right. \\
&\left.\quad-2 \frac{m^{2}}{q^{2}} \frac{\theta\left(q^{2}-4 m^{2}\right)}{\left[1-\left(4 m^{2} / q^{2}\right)\right]^{1 / 2}} \cdot g_{\alpha \beta}\right\} A_{\beta} . \tag{1.18}
\end{align*}
$$

To remove the regularization, we need to let $m$ go to zero. In this case, the right side of (1.18) does not vanish; instead we have

$$
\lim _{m \rightarrow 1} \frac{2 m^{2}}{q^{4}} \frac{\theta\left(q^{2}-4 m^{2}\right)}{\left[1-\left(4 m^{2} ; q^{2}\right)\right]^{1 / 2}}=\delta\left(q^{2}\right),
$$

as can be seen by taking the integral of this expression over $d q^{2}$. As for the coefficient of $g_{\alpha \beta}$ in (1.18), we note that it is less singular at small values of $q^{2}$ and vanishes in the limit $m \rightarrow 0$. For a massless fermion in two dimensions we thus have

$$
\begin{equation*}
\operatorname{Im} J_{\alpha}=\delta\left(q^{2}\right) q_{\alpha} q_{\beta} A_{\beta} . \tag{1.19}
\end{equation*}
$$

The real part of the polarization operator is reconstructed from the imaginary part in an ambiguous way. The coefficient of the structure $q_{\alpha} q_{\beta}$ is

$$
\frac{1}{\pi} \int \frac{\delta\left(s^{\prime}\right) d s^{\prime}}{q^{2}-s}==\frac{1}{\pi} \frac{1}{q^{2}} .
$$

There is no particular interest in the arbitrary constant which might be added here, since it would be dimensional. The coefficient of the structure $g_{\alpha \beta}$, which has a zero imaginary part, can be an arbitrary constant $c$, so we can write

$$
\begin{equation*}
\operatorname{Re} J_{\alpha}=\frac{1}{\pi}\left(\frac{q_{\alpha} q_{\beta}}{q^{2}}+c g_{\alpha \beta}\right) A_{\beta} . \tag{1.20}
\end{equation*}
$$

For an axial current we find

$$
\begin{equation*}
\operatorname{Re} J_{\alpha}^{\mathrm{b}}=\varepsilon_{\alpha \mu} \operatorname{Re} J_{\mu}=\frac{1}{\pi}\left(\frac{\varepsilon_{\alpha \mu} q_{\mu} q_{\beta}}{q^{2}}+c \varepsilon_{\alpha \beta}\right) A_{\beta} . \tag{1.21}
\end{equation*}
$$

The arbitrariness in the sole parameter $c$ is not sufficient to achieve the simultaneous conservation of the vector and axial currents:

$$
\begin{align*}
& q_{\alpha} J_{\alpha}=\frac{1}{\pi}(c+1) q_{\alpha} A_{\alpha}, \\
& q_{\alpha} J_{\alpha}^{5}=\frac{1}{\pi} c \varepsilon_{\alpha \beta} q_{\alpha} A_{\beta} . \tag{1.22}
\end{align*}
$$

The Pauli-Villars regularization guarantees conservation of the vector current, so that we find the completely definite value $c=-1$ for it, instead of an arbitrary $c$.

Three conclusions which follow from this discussion are important for our purposes.
a) An anomaly is usually linked with a pair of symmetries and contains a definite arbitrariness, which makes it possible to break either of these two symmetries without affecting the other.
b) If these symmetries are continuous (i.e., if infinitesimal transformations exist), the "objectiveness" of the anomaly stems from the nonvanishing imaginary part of some correlation function. The anomaly is actually not removable.
c) A Pauli-Villars regularization always conserves invariance under "vector" transformations; in particular, it conserves all vector currents.

These three points require further discussion, to which we now turn.

### 1.2.1. Anomalous symmetries

Five pairs of anomalous symmetries are known.

1) Vector phase transformations of fermions, $\psi \rightarrow \exp (i \varepsilon) \psi:$ axial, $\quad \psi \rightarrow \exp \left(i \varepsilon \gamma^{5}\right) \psi, \quad$ or chiral, $\quad \psi \rightarrow \exp$ $\times\left[i \varepsilon\left(1-\gamma^{5}\right) / 2\right] \psi$, transformations. Here $\varepsilon$ can be either a number (abelian transformations) or a matrix (non-abelian, color transformations). The example of the $\gamma^{5}$ anomaly , which we discussed above, is of this type.
2) Gauge invariance (covariant conservation of a matter current in a gauge theory): Bose symmetry. This anomaly arises if Yang-Mills fields interact with, not a vector current, but with a chiral (left-hand or right-hand) fermion current. In this case, several single-loop fermion diagrams at whose vertices these currents are found (Fig. 3) disrupt the conservation. More precisely, if we require symmetry of the correlation function $\left(J_{a}^{L} J_{\beta}^{L} J_{\gamma}^{L} \ldots\right\rangle$ under an interchange of currents, then we have $\left\langle\mathrm{D}_{\alpha} J_{\alpha}^{L}, J_{\mu}^{L}, J_{\gamma}^{L}, \ldots\right\rangle \neq 0$. We might also note that a Pauli-Villars regularization guarantees Bose symmetry of the correlation function. In Sections 2 and 3 of this review we will discuss this class of anomalies in great detail. Yet another important anomaly of this class is the gravitational anomaly. It stems from the correlation function not of chiral currents but of energy-momentum tensors. It describes a violation of the overall covariance of gravitational theories due to the nonconservation of the energymomentum tensor. (The original paper by Alvarez-Gaume and Witten ${ }^{35}$ on gravitational anomalies is also a splendid review of this subject, so we will not discuss gravitational anomalies in the present review.)
3) Gauge invariance: a discrete inversion transformation. This anomaly, which is associated with breaking of a discrete symmetry, occurs in odd-dimensional Yang-Mills theories, e.g., in three-dimensional electrodynamics, where, after an integration over the fermions in the effective action, a Wess-Zumino term $\varepsilon_{\mu \nu \lambda} A_{\mu} \partial_{v} A_{\lambda}$ appears. This term violates the $P_{1}$ and $T$ invariance. On the other hand, in the nonabelian case a Wess-Zumino term is necessary for invariance of the theory under topological nontrivial ("major") gauge transformations. This anomaly will be discussed in Section 4 of this review.
4) Displacement transformations: dilatation. This is a well-known conformal anomaly. While the classical theory is gauge-invariant, i.e., the dilatation current $D_{\mu}=T_{\mu \nu}^{\text {conf }} x_{v}$ is conserved ( $\partial_{\mu} D_{\mu}=T_{\mu \mu}^{\mathrm{conf}}=0$ ), in a regularized quantum


FIG. 3. Correlation function of chiral currents. If there is an anomaly, this contribution to the effective action either is not gauge invariant or does not have Bose symmetry.
theory $T_{i m}^{\mathrm{conr}}$ is nonzero provided that we require conservation of the energy-momentum tensor: $\partial_{\mu} T_{\mu r}^{c o n r}=0$. A wellknown example is the conformal anomaly in a Yang-Mills theory with $L=(1 / 4 \alpha) F_{\mu}^{2}$ :

$$
T_{\mu \mu}=-\frac{\beta(\alpha)}{2 \alpha^{2}} F_{\mu \nu}^{2}
$$

The appearance of anomalous dimensionalities for the operators in a regularized theory may also be thought of as a consequence of a conformal anomaly. We will return to anomalies of this type in the following section.
5) Gauge invariance: supersymmetry. We see no reason why an anomaly would not appear in the supercurrent, $\partial_{n} S_{\mu} \neq 0$, in certain supersymmetry theories (an anomaly of such a nature that it could not be converted into a superconformal anomaly). Such an anomaly, if it exists, could apparently be removed at the cost of violating gauge invariance (?). Aside from this brief comment, we will not discuss anomalies in a supercurrent here; the reader is referred to Refs. 49 and 50.

### 1.2.2. Imaginary parts and anomalies

In this regard we should point out that anomalies have been recognized which are not associated with nonzero imaginary parts of any correlation functions. The best example is Witten's global $\operatorname{SU}(2)$ anomaly (Ref. 46; see also Section 4 of the present review). This anomaly consists of a noninvariance of a fermion determinant under topologically nontrivial gauge transformations which cannot be reduced to infinitesimal transformations. Anomalies of this sort (their general label is "global"; see Subsection 1.3) are not related to divergences of currents or in general with perturbative Green's functions of any sort.

Another example is a Redlich anomaly in an odd-dimensional gauge theory. ${ }^{35,41}$ On the one hand, this anomaly is similar in meaning to Witten's $\mathrm{SU}(2)$ anomaly, and on the other it is directly related to a fermion determinant itself, rather than its change caused by some field transformations. In the simplest case-three-dimensional electrodynamics-an anomaly is found from the same diagram as that from which the two-dimensional Schwinger anomaly is found: the diagram in Fig. 2 for the vector current $J_{\sigma}$. Calculating the loop-now in a three-dimensional space-with the help of a Pauli-Villars regularization, we find that the regulator fields make a finite contribution, which is equal to $\left\langle J_{c x}\right\rangle$ minus the contribution of the physical fermion, i.e.,

$$
\begin{aligned}
& =\int \frac{d^{3} p}{(2 \pi)^{3}} \operatorname{Tr} \gamma_{\alpha} \frac{1}{\hat{p}-i M} \\
& \gamma_{\beta} \frac{1}{\hat{p}-\hat{q}-i M} A_{\beta}=\frac{i}{2 \pi} \operatorname{sgn}|M| \varepsilon_{\alpha \beta \gamma} q_{\beta} A_{\gamma}
\end{aligned}
$$

This result corresponds to the appearance of an anomalous term $(i / 8 \pi) \varepsilon_{\alpha \beta \gamma} A_{\alpha} F_{\beta \gamma}$ in the action. In calculating the same diagram in terms of its imaginary part, however, we would find the following result for the coefficient of the anomalous structure $\varepsilon_{\alpha \beta \gamma} q_{\beta} \alpha_{\gamma}$;

$$
\frac{n}{\left(q^{2}-m^{2}\right)^{1,2}} \theta\left(q^{2}-4 m^{2}\right)
$$

Letting the mass of the physical fermion, $m$ (introduced for
the infrared regularization), go to zero, we find that the imaginary part is zero. Correspondingly, the real part of the diagram is proportional to an arbitrary constant. In other words, Redlich's anomalies do not correspond to any imaginary part of a polarization operator, so that they cannot be removed by a suitable choice of regularization. Nevertheless, the Pauli-Villars regularization leads to a nonzero result for this anomaly. This is an important circumstance, because there may be physical requirements which select a Pauli-Villars regularization.

### 1.2.3. Pauli-Villars regularization

The primary advantage of this regularization for fermion fields is the conservation of vector currents, which we have already mentioned. This is a very important property since all the vector currents with which we have had to deal so far are conserved (while the conservation of chiral currents in the Glashow-Weinberg-Salam model would have to be deliberately arranged by choosing an appropriate quarklepton composition of the matter fields in order to cancel the corresponding anomalies). Furthermore, Bose symmetry is automatically ensured for all the correlation functions in this regularization. This Bose symmetry is again a physical requirement which is imposed on the choice of regularization. We will state without proof that the results for the anomalies are the same in all other regularizations which conserve vector currents and the Bose symmetry. One such regularization is a dimensional one. It is a particularly instructive regularization, because in this case the appearance of the $\varepsilon$-symbol in the expressions for all the anomalies becomes obvious immediately. In a dimensional regularization the only source of symmetry breaking (other than dilatation symmetry and supersymmetry) is an uncertainty regarding the analytic continuation of the $\varepsilon$-symbol and of the $\gamma^{5}$ matrix into spaces of arbitrary dimensionality. ${ }^{7}$ Unfortunately, I do not know of any simple and understandable method for calculating anomalies in a dimensional regularization, and it will not be used here.

The last point which will be discussed in this subsection 1.2 is the difference between an anomaly and a spontaneously broken symmetry. After a spontaneous breaking, the dynamics of a theory retains the corresponding invariance: Only the choice of basis states is asymmetric, while the Ward identities-the relations between the Green's functions which follow from the symmetry of the Lagrangian-remain valid. In particular, the Noether currents are again conserved. In the Glashow-Weinberg-Salam theory, for example, spontaneous breaking changes the equations of motion of fermions; they acquire a mass, and as a result the current $\bar{\psi} \gamma_{\mu}\left[\left(1-\gamma^{5}\right) / 2\right] \psi$ ceases to be conserved: $\partial_{\mu} \bar{\psi} \gamma_{\mu}\left[\left(1-\gamma^{5}\right) / 2\right] \psi=m \bar{\psi} \gamma^{5} \psi$. The chiral current with which the Z-bosons interact, however, incorporates not only fermions but also the phase of a Higgs field (goldstone): $J_{\mu}^{\mathrm{L}}=\bar{\psi} \gamma_{\mu}\left[\left(1-\gamma^{5}\right) / 2\right] \psi+\partial_{\mu} \underline{\chi}$. The goldstone itself is related to fermions: $\partial^{2} \chi=+m \bar{\psi} \gamma^{5} \psi$ (this is an equation of motion). We thus have $\partial_{\mu} J_{\mu}^{L}=0$, providing gauge invariance.

The situation is different with anomalous symmetries. Here it is the dynamics itself which is invariant-not the states. Let us say that $\gamma^{5}$ invariance in a chiral model ${ }^{3,4}$ implies conservation of the number of pseudoscalar $\pi$ mesons ( $G$-parity). Because of the anomaly $\partial_{\mu} J_{\mu}^{5}$ $+\left(i / 8 \pi^{2}\right) F_{\mu v} \tilde{F}_{\mu v}$, the decay $\pi^{0} \rightarrow 2 \gamma$ becomes possible and violates this selection rule.

In a spontaneous symmetry breaking, massless goldstone particles necessarily arise. This effect is an unambiguous consequence of the existence of asymmetric vacuum expectation values in a theory with a conserved Noether current. Interestingly, goldstones do not necessarily acquire a mass in a theory with an anomaly, in which we have $\partial_{\mu} J_{\mu} \neq 0$. The point here is that the anomalous divergence of the Noether current is equal to a total derivative (Section 3): $\partial_{\mu} J_{\mu}=\partial_{\mu} K_{\mu}$. (In the case of the $\gamma^{5}$ anomaly with $D=2$, for example, we would have $K_{\mu} \sim \varepsilon_{\mu v} A_{v}$.) Because of this circumstance, there is again a conserved current ( $J_{\mu} \rightarrow J_{\mu}-K_{\mu}$ ), although this is no longer a Noether current; i.e., it is not of the form ( $\left.\partial L / \partial \phi_{\mu}\right) \delta \phi$. To determine whether the goldstone has become heavy we need to examine the eigenenergy diagram in Fig. 4. The contribution of an anomaly has the structure

$$
\begin{aligned}
\int & \langle 0| \partial_{\mu} K_{\mu}(x) \partial_{\nu} K_{v}(0)|0\rangle e^{i q x_{\mathrm{d}}} \mathbf{D}_{x} \\
& \sim q_{\mu} q_{\nu} \int\langle 0| K_{\mu}(x) K_{v}(0)|0\rangle e^{i q x} \mathrm{~d}^{D} x
\end{aligned}
$$

Consequently, if the correlation function $\langle 0| K K|0\rangle$ has no pole in $q^{2}$, the anomaly will not contribute a mass term to the eigenenergy diagram of the goldstone. In particular, the anomalous decay $\pi^{0} \rightarrow 2 \gamma$ generates no correction of any sort to the mass of a pion. However, there are correlation functions $\langle 0| K K|0\rangle$ which do not contain a pole. The best-known example is the Veneziano ghost in quantum chromodynamics, which makes the ninth pseudoscalar goldstone $\eta^{\prime}$ meson heavy. ${ }^{51}$ The pole in the correlator in this case stems from instanton fluctuations of gauge fields. In the abelian case, there are no such fluctuations, so that the anomalous decay $\pi^{0} \rightarrow 2 \gamma$ generates no correction of any sort to the mass of a pion. The virtual process $\pi^{0} \rightarrow 2 W \rightarrow \pi^{0}$ involving $W$ bosons, however, adds to the mass of $\pi^{0}$ a vanishingly small quantity $\sim M_{w} \exp \left[-\right.$ const/ $\left.\alpha\left(M_{w}\right)\right]$, by virtue of instanton effects. Instantons in quantum chromodynamics have a strong effect not on a $\pi^{0}$ but on $U_{f}(3)$ flavor-singlet $\eta^{\prime}$ mesons.

### 1.3. Types of anomalies and their physical consequences

We begin this section with a few specific examples of physical theories with anomalies.


FIG. 4. Eigenenergy diagram for a Goldstone field. The appearance of a mass for this field corresponds to a momentum-independent contribution to the expression for this diagram.

### 1.3.1. Quantum chromodynamics in the chiral limit

Here we have $L=\bar{\psi} i\left(\hat{\partial}+\hat{A}_{\mathrm{c}}\right) \psi$;

$$
\psi=\left(\begin{array}{l}
u \\
d \\
\mathbf{s}
\end{array}\right)
$$

is a triad of quark fields, and $A_{\mathrm{c}}$ is a gluon field. For the time being we will not discuss the color degrees of freedom [in terms of the color group $\mathrm{U}_{\mathrm{C}}$ (3), u,d,s are singlets, while $\boldsymbol{A}_{\mathrm{c}}$ is an octet ], so will not write the corresponding indices. We are interested instead in the flavor symmetry group $\mathrm{U}_{\mathrm{f}}(3)$ among $\mathbf{u}, \mathrm{d}$, and s quarks. According to this group, $\psi$ is a triplet, while $A_{\mathrm{c}}$ is a singlet. In the chiral limit, in which there are no quark masses in the Lagrangian (this is in many cases a reasonable approximation of real quantum chromodynamics) the classical theory in fact has the symmetry $\mathrm{U}_{\mathrm{f}}(3) \times \mathrm{U}_{\mathrm{f}}(3)$, determined by the transformations $\psi \rightarrow e^{i \alpha} \psi, \psi \rightarrow e^{i \alpha \gamma^{*}} \psi$ with matrices $\alpha$ from a $\mathrm{U}_{\mathrm{f}}(3)$ algebra. The group $U_{f}(3) \times U_{f}(3)$ is global in quantum chromodynamics; this statement means that no gauge bosons of any sort are related with Noether currents $\bar{\psi} \gamma_{\mu} \tau^{a} \psi$ and $\bar{\psi} \gamma_{\mu} \gamma^{5} \tau^{a} \psi$ [ $\tau^{a}$ are the generators of $\mathrm{U}_{\mathrm{f}}(3)$ ].

In quantum chromodynamics there is a spontaneous breaking of the symmetry, $\mathrm{U}_{\mathrm{f}}(3) \times \mathrm{U}_{\mathrm{f}}(3) \rightarrow \mathrm{U}_{\mathrm{f}}(3)$, since a vacuum condensate $\left\langle\bar{\psi}_{i} \psi_{j}\right\rangle=(1 / 3) \delta_{i j}\langle\bar{\psi} \psi\rangle$ forms which is not invariant under the replacement $\psi \rightarrow e^{i \alpha \gamma^{\circ}} \psi$. The dynamics of the theory, of course, still has the complete $U_{f}(3) \times U_{f}(3)$ symmetry, but its axial part is realized in a nonlinear way on fields which describe elementary particles: Among these fields there is a nonet of pseudoscalar goldstone mesons $\Pi$ ( $\pi^{ \pm}, \pi^{0}, \mathbf{K} \pm, \overline{\mathbf{K}} \pm, \eta, \eta^{\prime}$ ), which transform under the action $\psi \rightarrow e^{i a \gamma^{`}} \psi$ in accordance with the rule $\Pi \rightarrow \Pi+\varepsilon$. The fields $\Pi$ are usually called "pseudogoldstone" fields, because when the masses of quarks are taken into account these fields also acquire a small mass $\sim\left(m_{\mathrm{q}}\langle\bar{\psi} \psi\rangle / f_{\pi}^{2}\right)^{1 / 2}$ [while the masses of all the other non(pseudo)-goldstone mesons are determined exclusively by the condensates of gluon and quark fields and are essentially independent of $m_{\mathrm{q}}$ ].

A spontaneous breaking of an axial flavor symmetry has entirely perceptible physical consequences. In addition to each meson and baryon, the spectrum of physical states has some other mesons and baryons which are degenerate with the former in terms of the vector $U_{f}(3)$ symmetry and which have the same masses (more precisely, they have nearly the same masses, since there are mass corrections $\sim m_{\mathrm{q}}$ and electromagnetic corrections which explicitly break this symmetry). They also have the same quantum numbers. For example, associated with a proton in terms of $U_{f}(3)$ is the octet of baryons ( $p, n, \Lambda, \Sigma^{ \pm 0}, \Xi^{0}, \Xi^{-}$). However, there are no such particles with the other parity. For example, the resonance with spin and parity $1 / 2^{-}$has a mass which is 600 MeV greater than that of $\mathrm{p}\left(1 / 2^{+}\right)$. Instead, the spontaneously broken axial $U_{f}(3)$ symmetry relates a particle with "a particle plus a pseudogoldstone meson with zero momentum." For example, in the chiral limit a proton is degenerate with the pairs $\mathrm{p}+\pi, \mathrm{p}+\mathrm{K}, \ldots, \Xi+\eta, \ldots$, and also with the triples $p+\pi+\pi$, etc. An axial symmetry which actually is broken spontaneously is manifested not in
the particle mass spectrum but in the existence and properties of (pseudo) goldstone mesons. In particular, any interactions of these particles should vanish at zero momentum. This circumstance guarantees that no forces will lift the degeneracy of the proton with the $p+\pi$ state. The assertion that all the interactions of pions are of a gradient nature [i.e., that the potential $V(q)$ is proportional to the momentum $q_{\mu}$ ] is important for all nuclear physics.

Still, all this is not an anomaly. Anomalous in chiral quantum chromodynamics is a flavor axial $U_{f}(1)$ symmetry, determined by the transformations $\psi \rightarrow e^{i \varepsilon \gamma^{*}} \psi$ with a unit matrix $\varepsilon$. The corresponding Noether current, $J_{\mu}^{s}=\bar{\psi} \gamma_{\mu} \gamma^{5} \psi=\bar{u} \gamma_{\mu} \gamma^{5} u+\bar{d} \gamma_{\mu} \gamma^{5} d+\bar{s} \gamma_{\mu} \gamma^{5} s$, is not conserved:

$$
\partial_{\mu} J_{\mu}^{s}=\frac{3}{4 \pi} \mathrm{~T}_{\mathrm{c}} F_{\mathrm{c}}^{\mu \nu} \widetilde{F}_{\mathrm{c}}^{\mu v}
$$

[ $\mathrm{Tr}_{\mathrm{c}}$ is the trace in the $\mathrm{SU}_{\mathrm{c}}$ (3) color algebra]. Because of the anomaly there is no axial symmetry, not only in the spectrum of one-particle states but in general in the dynamics of the theory. The $\mathrm{SU}(3)$-singlet $\eta^{\prime}$ meson, which has the quantum numbers of an anomalous $\mathrm{U}_{\mathrm{f}}(3)$-singlet current $J_{\mu}^{\mathrm{s}}$, is no longer a goldstone boson or even a pseudogoldstone boson. Its mass is determined not by the quark masses $m_{q}$ but, as in the case of all non-goldstone mesons, by nonperturbative vacuum condensates of the fields of quantum chromodynamics. Actually, we have $m_{\eta^{\prime}} \approx 958 \mathrm{MeV}$; this figure is considerably larger than the masses of the octet of pseudogoldstones: $m_{\pi} \approx 140 \mathrm{MeV}, m_{\mathrm{K}} \approx 498 \mathrm{MeV}, m_{\eta} \approx 549 \mathrm{MeV}$. For the same reason, there is no degeneracy among the states p and $\mathrm{p}+\eta^{\prime}, \rho$ and $\rho+\eta^{\prime}$, etc.

From the standpoint of the spectroscopy of elementary particles, the role played by an axial anomaly in quantum chromodynamics is actually not very important. The reason is that the mass of an $s$ quark is actually extremely large, $m_{\mathrm{s}} \approx 150 \mathrm{MeV}$, and the $\mathrm{U}_{\mathrm{f}}(3)$ flavor symmetry is broken quite strongly without any anomaly. Essentially the only place where it is possible to distinguish the effects of an anomaly and $m_{\mathrm{s}} \neq 0$ is in the properties of $\eta^{\prime}$. [At one time, the question of the large value of the ratio $m_{\eta^{\prime}} / m_{\eta}$ was called the " $\mathrm{U}(1)$ problem" ${ }^{54}$ : the problem of the prominence of the $U_{f}(1)$ subgroup in axial $U_{f}(3)$. This problem can be resolved only by taking an anomaly into account.] This is, of course, a very fundamental point. Furthermore, an analysis of this problem will help us reach an understanding of how the consequences of spontaneous and anomalous breakings of symmetry will differ in a realistic theory incorporating both of these effects. We turn now to another anomaly, again in quantum chromodynamics, with some experimental consequences which are much more apparent. For this purpose we wish to discuss

### 1.3.2. Chiral quantum chromodynamics in an external electromagnetic field

The Lagrangian here is $L=\sum_{\mathcal{f}} \bar{\psi}_{\mathrm{f}} i\left(\hat{\partial}+\hat{A}_{\mathrm{c}}+e_{\mathrm{f}} \hat{A}_{\mathrm{e}_{\mathrm{m}}}\right) \psi_{\mathrm{f}}$. Incorporating an electromagnetic field results in new anom-
alies. In particular, anomalies appear in currents which do not contain s-quark fields. ${ }^{8)}$ The masses of the $u$ and $d$ quarks, in contrast, are very small ( of the order of 10 MeV ), and in this case the anomalies stand out in stark contrast with the background of mass corrections.

The electromagnetic interaction breaks the flavor symmetry: Different quarks now have different electric charges. We will discuss here the two-quark theory:

$$
\begin{aligned}
L=\bar{u} i\left(\hat{\partial}+\hat{A}_{\mathrm{c}}+\frac{2}{3} \hat{A}_{\mathrm{em}}\right) u & +\bar{d} i\left(\hat{\partial}+\hat{A}_{\mathrm{c}}-\frac{1}{3} \hat{A}_{\mathrm{em}}\right) d \\
& =\bar{\psi} i\left(\hat{\partial}+\hat{A}_{\mathrm{c}}+Q \hat{A}_{\mathrm{em}}\right) \psi
\end{aligned}
$$

here
$\psi$ is the doublet ( $\left.\begin{array}{l}\mathrm{u} \\ \mathrm{d}\end{array}\right)$,

$$
A_{\mathrm{c}} \text { is a } \mathrm{U}_{\mathrm{f}}(2) \text { singlet, and }
$$

$$
Q=\left(\begin{array}{cc}
+\frac{2}{3} & 0 \\
0 & -\frac{1}{3}
\end{array}\right)+\frac{1}{6} I+\frac{1}{2} J_{3}
$$

The Lagrangian has not the complete $\mathrm{U}_{\mathrm{f}}(2) \times \mathrm{U}_{\mathrm{f}}(2)$ symmetry but only $\left(\mathrm{U}(1) \times \mathrm{U}(1)_{\mathrm{f}}\right) \times(\mathrm{U}(1) \times \mathrm{U}(1))_{\mathrm{f}}$. As before, the axial symmetry is broken spontaneously by the $\langle\bar{u} u\rangle \approx\langle\bar{d} d\rangle$ condensates, and (pseudo) goldstone mesons $\pi^{0} \sim(1 / \sqrt{2})(\bar{u} u-\bar{d} d), \pi^{ \pm} \sim u \bar{d}, \bar{u} d$ form. There is no point in discussing the fourth meson $\sim(1 / \sqrt{2})$ ( $\bar{u} u+\bar{d} d$ ) in a twoquark model. The solution of the $\mathrm{U}(1)$ problem shows that it is completely mixed with $\bar{s} s$ and forms a heavy $\eta^{\prime}$ meson. Because of the electromagnetic interaction, which breaks $\mathrm{U}_{\mathrm{f}}(2)$ to $(\mathrm{U}(1) \times \mathrm{U}(1))_{\mathrm{f}}$, the masses of $\pi^{0}$ and $\pi^{ \pm}$, which are proportional to $\left[\left(m_{\mathrm{u}}+m_{\mathrm{d}}\right)\langle\bar{u} \mathrm{u}\rangle\right]^{1 / 2} / f_{\pi}$, are slightly different: $m_{\pi_{,},} \approx 135.0 \mathrm{MeV}, m_{\pi} \approx 139.6 \mathrm{MeV}$. The anoma$l y$ which we wish to discuss here disrupts the conservation of one of the axial $\mathrm{U}(1)$ currents $J_{\mu(3)}^{5}=\bar{\psi} \gamma_{\mu} \gamma^{5} \tau_{3} \psi$ $=\bar{u} \gamma_{\mu} \gamma^{5} u-\bar{d} \gamma_{\mu} \gamma^{5} d$, which is associated with the transformation $\psi \rightarrow e^{i \varepsilon \tau, \gamma^{\gamma}} \psi: u \rightarrow e^{+i \varepsilon \gamma^{*}} u, d \rightarrow e^{-i \varepsilon \gamma^{\prime}} d$. This current is orthogonal to the singlet axial current which we discussed in Subsection 1.3.1. In pure quantum chromodynamics without electromagnetic interactions, this current is conserved exactly. It is with this current that the $\pi^{0}$ meson is associated by the hypothesis of partial conservation of axial-vector current (PCAC):

$$
\left\langle\pi^{0}\right| \ldots=\frac{1}{i f_{\pi}} \int d^{3} y\langle 0|\left[J_{0(3)}^{5}(y), \ldots\right] .
$$

The anomaly in the current, $\partial_{\mu} J_{\mu(3)}^{s} \sim F_{\mathrm{em}}^{\mu \nu} \widetilde{F}_{\mathrm{em}}^{\mu \nu}$, gives rise to a nonvanishing amplitude for the transition of one $\pi$ meson to zero $\pi$ mesons ( +2 photons):

$$
\begin{aligned}
\langle\pi| \mathscr{H}|2 \gamma\rangle & =\frac{1}{i f_{\boldsymbol{\pi}}} \int \mathrm{d}^{3} \mathbf{y}\langle 0|\left[J_{0(3)}^{5}(\mathbf{y}), \mathscr{H}\right]|2 \gamma\rangle \\
= & -\frac{1}{f_{\pi}} \int \mathrm{d}^{3} \mathbf{y}\langle 0| \partial_{0} J_{0(3)}^{s}(\mathbf{y})|2 \gamma\rangle \\
= & -\frac{1}{f_{\pi}} \int \mathrm{d}^{3} \mathbf{y}\langle 0| \partial_{\mu} J_{\mu(3)}^{5}(\mathbf{y})|2 \gamma\rangle \\
& \sim \frac{1}{f_{\pi}} \int \mathrm{d}^{3} \mathbf{y}\langle 0| F_{\mu \nu} \widetilde{F}_{\mu \nu}(\mathbf{y})|2 \gamma\rangle \\
& \sim \frac{1}{f_{\pi}\left(\omega^{(1)} \omega^{(2)}\right)^{1 / 2}} \varepsilon_{\mu v \alpha \beta} p_{\mu}^{(1)} p_{v}^{(2)} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)} .
\end{aligned}
$$

Here $\mathscr{H}$ is the Hamiltonian of the interaction of the pions with the photons, and $p^{(1,2)}, \omega^{(1,2)}$, and $\varepsilon^{(1,2)}$ are the momenta, frequencies, and polarizations of the photons. We have used the equation $\partial_{0} J_{0}=i\left[J_{0} \mathscr{H}\right]$ and the identity $\int \mathrm{d}^{3} \mathbf{y}\langle | \vec{\partial} \mathbf{J}| \rangle=0$, because of which we have $\int \mathrm{d}^{3} \mathbf{y}\langle | \partial_{0} J_{0}| \rangle$ $=\int \mathrm{d}^{3} \mathbf{y}\langle | d_{\mu} J_{\mu}| \rangle$. In principle, the decay $\pi^{0} \rightarrow 2 \gamma$ might occur even without an anomaly, because of the mass terms of the quarks in the Lagrangian, which also violate conservation of the current $J_{\mu(3)}^{S}$. However, the contribution of those terms to the decay amplitude differs from the contribution of the anomaly by a factor $\sim m_{\mathrm{q}}^{4} / \omega^{4}$ (cf. Subsection 2.5 ). The typical value of the photon frequencies is $\omega \sim m_{\pi} / 2$, and this factor is very small. We can thus assert that the width of the decay $\pi^{0} \rightarrow 2 \gamma$ is determined entirely by the anomaly in the axial current.

We should perhaps point out that an anomaly in the current $J_{\mu(3)}^{5}$ does not violate spatial parity. As before, there is even a one-parameter transformation which generalizes a discrete parity transformation. The only difference is that in the present case the generator of this transformation is not $\int \mathrm{d}^{3} \mathbf{y} J_{\mathrm{c}(3)}^{5}$ but $\int \mathrm{d}^{3} \mathbf{y}\left(J_{0(3)}^{5}-c \varepsilon_{0 i j k} A_{\mathrm{em}}^{i} \partial_{j} A_{\mathrm{em}}^{k}\right)$. However, while this law also required conservation of the parity of the number of $\pi$ mesons ( $G$-parity) in the case without an anomaly, in the present case there is no requirement of this sort.

The next example of a theory with anomalies is the Gla-show-Weinberg-Salam (GWS) model, This model has two interesting anomalies. We will begin with that which leads to the decay of the proton.

### 1.3.3. Proton decay in the Glashow-Weinberg-Salam model (Anomaly in the baryon current)

The fermion part of the GWS Lagrangian, $L=\sum_{i} \bar{\psi}_{i} i\left(\hat{\partial}+\hat{W}^{a} \tau_{i}^{a}+\widehat{B} Y_{i}\right) \psi_{i}+\sum_{i j} \phi_{i j} \bar{\psi}_{i} \psi_{j}$, contains fields $\psi_{i}$ which belong to singlet and doublet representations of the $S U(2)$ group and which have definite values of the $U(1)$ hypercharge, $Y_{i}$. The left-hand and right-hand components of ordinary particles have different quantum numbers in the GWS model:
$\binom{u}{d}_{\mathrm{L}}$
which is an $\operatorname{SU}(2)$ doublet with hypercharge $Y=1 / 6$, and $\mathrm{u}_{\mathrm{R}}$ and $\mathrm{d}_{\mathrm{R}}$, are $\mathrm{SU}(2)$ singlets with respective hypercharges $+2 / 3$ and $-1 / 3$. The electric charges of the particles are $Q=\tau^{3}+Y$. Here $\phi_{i j} \bar{\psi}_{i} \psi_{j}$ represents Yukawa terms of the Lagrangian, which describe the interaction of fermions with Higgs fields. The matrix $\phi_{i j}$ is, of course, not arbitrary. The requirements of $\mathrm{SU}(2) \times \mathrm{U}_{Y}(1)$ covariance are imposed on it. In addition, the Yukawa terms change the chirality of the fermions. Of the many properties of the GWS model only one is of interest to us at present. The Lagrangian has not only the gauge "flavor" $\mathrm{SU}(2) \times \mathrm{U}_{Y}$ (1) symmetry but also a global symmetry $\mathrm{U}_{\mathrm{r}}(1)$, specified by the transformations $\psi_{i} \rightarrow e^{i \varepsilon} \psi_{i}$ : multiplication of all the fermion fields by the same factor. The Higgs fields are not changed by this transformation [in contrast with $\mathrm{U}_{Y}(1)$ ]: $\phi_{i j} \rightarrow \phi_{i j}$. More precisely, the GWS model has not one but two global $U(1)$ symmetries. The reason is that the Yukawa terms do not mix quarks
with leptons (because the Higgs fields are colorless). The quarks and leptons can therefore be transformed in accordance with the rule $\psi_{i} \rightarrow e^{i \varepsilon} \psi_{i}$ separately. These two $\mathrm{U}(1)$ symmetries correspond to the conservation of baryon and lepton charges. For example, the quark $\mathrm{U}(1)$ symmetry corresponds to the Noether current $J_{\mu}^{B}=\bar{u}_{L} \gamma_{\mu} u_{L}$ $+\bar{u}_{\mathrm{R}} \gamma_{\mu} u_{\mathrm{R}}+\bar{d}_{\mathrm{L}} \gamma_{\mu} d_{\mathrm{L}}+\bar{d}_{\mathrm{R}} \gamma_{\mu} \bar{d}_{\mathrm{R}}+$ terms with other quarks $=u \gamma_{\mu} u+d \gamma_{\mu} d+\ldots$ The conservation of baryon current ensures the stability of the lightest of the baryonsthe proton - and forces all the other baryons to decay into precisely one proton plus an even number of leptons $(+p \bar{p}$ pairs). The even number of leptons is in turn provided by conservation of the lepton current $J_{\mu}^{\text {lep }}$. In the GWS model, however, the conservation of both currents $J_{\mu}^{\mathrm{B}}$ and $J_{\mu}^{\text {lep }}$ is disrupted by an anomaly:

$$
\begin{aligned}
\partial_{\mu} J_{\mu}^{\mathrm{B}} & =a \operatorname{Tr} W_{\mu \nu} \widetilde{W}_{\mu \nu}+b B_{\mu \nu} \widetilde{B}_{\mu \nu} \\
\partial_{\mu} J_{\mu}^{\text {lep }} & =a \operatorname{Tr} W_{\mu v} \tilde{W}_{\mu \nu}+b B_{\mu v} \widetilde{B}_{\mu v}
\end{aligned}
$$

Only the difference $J_{\mu}^{\mathrm{B}}-J_{\mu}^{\text {lep }}$ is conserved (this is the so called B-L law). As a result the proton in the GWS model is in principle unstable: It should, even if only very slowly, decay into $\pi^{0} \mathrm{e}^{+}$, etc.

We wish to emphasize that in most of the grand unification theories the instability of the proton is related to nonconservation of the baryon current in the classical Lagrangian. In the $S U(5)$ theory, for example, we do not have invariance under rotation of the phases of exclusively the quarks in the 5-plet

$$
\left(\begin{array}{l}
d \\
d \\
d \\
e \\
v
\end{array}\right)
$$

In contrast, the classical Lagrangian of the GWS model has this invariance, but it is anomalous: It is violated at the quantum level. This distinction not only is of fundamental importance but also leads to very different physical predictions. In the GWS model the proton decays only in the presence of the topologically nontrivial gauge field

$$
\begin{gathered}
\Delta B \equiv \int \partial_{\mu} J_{\mu}^{\mathrm{B}} \mathrm{~d}^{4} x:=a \int \operatorname{Tr} W_{\mu \nu} \tilde{W}_{\mu \nu} \mathrm{d}^{4} x+b \int B_{\mu \nu} \widetilde{B}_{\mu \nu} \mathrm{d}^{4} x \\
=c \int F_{\mu \nu}^{(\mathrm{em})} \widetilde{F}_{\mu \nu}^{(\mathrm{em})} \mathrm{d}^{4} x+\text { contributions } \\
\text { of } \mathrm{Z} \text { and } \mathrm{W}^{ \pm} \text {bosons. }
\end{gathered}
$$

In order to achieve the relation $\Delta B \neq 0$ we need a field configuration with a nonzero topological charge. In a grand unification theory, nothing of the sort is required: The proton decays spontaneously, through virtual $X$ and $Y$ bosons.

The decay of the proton in the GWS model has no "practical" importance. It might be caused by static parallel magnetic and electric fields. These fields, however, would have to be unattainably strong: $E, H \gg m_{\pi}^{2}$. (We have omitted from our equations all terms which stem from the masses of quarks, so that these equations are reliable only in this limit. For weaker fields, the baryon current is conserved, and the $F \widetilde{F}$ contribution is cancelled by the formation of an induced fermion condensate $m(\bar{\psi} \psi\rangle \sim F \widetilde{F}$.) Furthermore, the fields
would have to be classical: The decay $\mathrm{p} \rightarrow\left(\pi_{0} e^{+}\right) \gamma \gamma$, for example, also would not occur. Although $F_{\mu} \widetilde{F}_{\mu}$, may not vanish for a pair of photons, the integral $\int F_{\mu}, \widetilde{F}_{\mu}, \mathrm{d}^{4} x$ will always vanish because of oscillations of the photon fields over time. An external gauge field with a topological charge can be produced by, for example, a dyon: a hypothetical particle which carries magnetic and electric charges simultaneously. Near a dyon, a proton would be unstable. Actually, we would not need even a dyon-just a magnetic monopole. An electric field which contributes to the integral $\int F_{\mu}, \widetilde{F}_{\mu}$ d $\mathrm{d}^{4} x$ is then produced by the proton itself. The decay of a proton in the field of a monopole (monopole catalysis) is known as the "Callan-Rubakov effect" it is usually discussed in the wider context of grand unified theories, but all that is required of them is a monopole; the decay process itself can also be explained in the GWS model).

In principle, there is also a spontaneous decay of the proton in the absence of external fields in the GWS model. This decay is caused by instanton fluctuations. However, the probability of such fluctuations and the contributions of electroweak instantons to the amplitudes of physical processes are usually vanishingly small:

$$
\exp \left(-\frac{4 \pi}{g^{2}}\right) \sim \exp \left(-\frac{\sin ^{2} \theta_{w}}{\alpha_{e m}}\right)<10^{-10}
$$

It may be that instanton effects and the proton decay which they induce become important at high temperatures.

These three examples are unified by a common property: anomalies violate the conservation of "external" Noether currents, with which no gauge bosons of any sort are interacting. As a result, the anomalies lead to perceptible physical consequences, but not pathological events. Because of these anamolies, there are breakings of global symmetries: Mass spectra become deformed, a degeneracy of states is lifted, forbidden decay channels open up, and reaction amplitudes change. However, no new physical states arise, unitarity is not disrupted, and the ultraviolet properties of the theory $d o$ not change.

There are, however, anomalies which radically change the physical content of a theory. These are the "internal" anomalies, which violate gauge invariance. In gauge theories the space of states of vector lines is obtained from an infinitedimensional space of fields $A_{"}^{a}$ by identifying the gaugeequivalent fields $A$ and $G A$ which are found from each other by transformations from the group $G=\prod_{x} G(\mathbf{x})$ of all gauge transformations. The topology of this "space of orbits of the gauge group" is extremely complex in the general case: Not all the transformations from $G$ reduce to superpositions of infinitely small-infinitesimal-transformations. Discrete sets of topologically nontrivial transformations may also exist which are not reducible to infinitesimal (uncontractible) gauge transformations. Anomalies may violate invariance under transformations of both types. The consequences of these violations, however, differ.

If a theory is noninvariant under uncontractible gauge transformations (in which case one would say that there is a global anomaly), this circumstance may present some insurmountable difficulties in a quantum description of the the-
ory. For example, if the action $S$ varies by $\pi$ under uncontractible transformations, then we would have
(over gauge transformations)

$$
\sum \quad e^{i S} \sim e^{\mathbf{S}}+e^{i(\mathrm{~S}+\pi)}=0 .
$$

As a result, a generating functional is not determined for such a theory and it is quite likely that an $S$-matrix is also undetermined: For it we find an expression of the 0/0 type. Another global anomaly, which arises frequently, appears in the case $\pi_{d}(G)=\mathbb{Z}$, where $d$ specifies the dimensionality of the space-time, and $G$ is the gauge group. The effective action then usually changes by $n \alpha$ under uncontractible gauge transformations with a topological charge $n \in \mathbb{Z}$. The theory can be formulated in a noncontradictory way only in cases in which $\alpha$ is a multiple of $2 \pi$. Global anomalies therefore frequently impose severe restrictions on the form of any permissible theory; for example, they may require quantization of certain charges in the Lagrangian. In many ways, global anomalies present a situation analogous to that regarding Wess-Zumino terms in odd-dimensional Yang-Mills theories:

$$
\begin{aligned}
L- & -\frac{1}{4 \alpha} \operatorname{Tr} F_{\mu v}^{2} \\
& +c \varepsilon_{\alpha_{1}} \ldots \alpha_{d} \operatorname{Tr} A_{\alpha_{1}}\left(\partial_{\alpha_{2}} A_{\alpha_{3}} \ldots \partial_{\alpha_{d-1}} A_{\alpha_{d}}+\ldots\right) .
\end{aligned}
$$

The condition that such theories be gauge-invariant requires quantization of the coefficient of the Wess-Zumino term. We will return to this analogy for a more detailed discussion in Section 4.

The loss of invariance under infinitesimal gauge transformations stems from the violation of the covariant conservation of the fermion currents which are interacting with gauge bosons. This situation could not occur in the case with vector currents, and nonvector gauge interactions arise in chiral theories where the left- and right-hand matter fields have different quantum numbers. The simplest example of the theory in which there is a potential internal anomaly is the Glashow-Weinberg-Salam model. In example c) above we ran into an anomaly in an external baryon current which breaks the global symmetry responsible for the conservation of baryon charge. At this point we are interested in an internal anomaly.

### 1.3.4. Internal anomaly in the Glashow-Weinberg-Salam model

We can write the action in terms of left-hand fermion fields (a right-hand particle is the same thing as a left-hand antiparticle). We denote by $\chi_{\mathrm{L}}$ the right-hand fermions, i.e., the left-hand antifermions, which are $\mathrm{SU}(2)$-singlets, $\overline{\mathrm{e}}_{\mathrm{L}}$, $\bar{u}_{L}, \bar{d}_{L}, \ldots$; and we denote by $\psi_{L}$ the left-hand doublets $\left(\mathrm{e}^{-}, v\right)_{L},(i, d)_{L}, \ldots$ Furthermore, as everywhere else in this review, we will add to the action some "sterile" right-hand fields $\chi_{\mathrm{R}}$ and $\psi_{\mathrm{R}}$, which do not interact with gauge bosons. This approach makes it possible to formally write a Lagrangian in terms of Dirac fields $\chi=\left(\chi_{\mathrm{L}}, \chi_{\mathrm{R}}\right)$ and $\psi=\left(\psi_{\mathrm{L}}, \psi_{\mathrm{R}}\right)$. We choose the Higgs sector in accordance with the so-called standard GWS model: the only scalar $\phi$ which is an SU(2)doublet. The Lagrangian of the fermions and scalars is then

$$
\begin{aligned}
L=\sum_{i} \bar{\psi}_{i} i & {\left[\hat{\partial}-\left(\hat{W}+Y_{i} \hat{B}\right) \frac{1-\gamma_{5}}{2}\right] \psi_{i} } \\
& +\sum_{j} \bar{\chi}_{j} i\left(\hat{\partial}+\tilde{Y}_{j} \hat{B} \frac{1-\gamma^{5}}{2}\right) \chi_{j} \\
& +\sum_{i j} c_{i j} \phi \bar{C}_{i} \frac{1-\gamma_{5}}{2} \chi_{j}+\text { c.c. } \\
& \left.\left.+\frac{1}{2} \right\rvert\, \hat{\partial}+W+Y_{\phi} B\right)\left.\phi\right|^{2}+V\left(|\phi|^{2}\right) .
\end{aligned}
$$

For our purposes, the particular choice of Yukawa constants $c_{i j}$ is unimportant. The matter currents which interact with the gauge bosons are

$$
\begin{gathered}
J_{\mu}^{\mathrm{W}^{a}}=i \sum_{i} \bar{\psi}_{i} \gamma_{\mu} \frac{1-\gamma_{5}}{2} t^{a} \psi_{i}+\phi^{+} t^{a} \mathrm{D}_{\mu} \phi-\text { c.c. } \\
\quad\left(\mathrm{D}_{\mu}=\partial+W_{\mu}+Y_{\phi} B, W_{\mu}=-i W_{\mu}^{a} t^{a}\right), \\
J_{\mu=}^{\mathrm{B}}=i \sum_{i} Y_{i} \bar{\psi}_{i} \gamma_{\mu} \frac{1-\gamma_{5}}{2} \psi_{i}+\sum_{j} \tilde{Y}_{j} \bar{\chi}_{j} \gamma_{\mu} \frac{1-\gamma_{5}}{2} \chi_{j} \\
\quad+Y_{\phi} \phi^{+} \mathrm{D}_{\mu} \phi-\text { c.c. }
\end{gathered}
$$

We see that these currents are not purely fermion currents; they also contain Higgs fields. This circumstance is crucial to the Higgs effect in general and to the GWS model in particular. Because of it, a spontaneous symmetry breaking due to the formation of a condensate $\langle\phi\rangle$ and the generation of masses for fermions and gauge bosons does not violate the conservation of the currents $J^{\mathrm{W}}$ and $J^{\mathrm{B}}$ and therefore does not disrupt the gauge invariance of the theory.

On the other hand, this modification of the currents does not affect the anomaly. The reason is that for scalars the mass terms do not disrupt the conservation of $J^{\mathrm{w}, \mathrm{B}}$ [an arbitrary potential $V\left(|\phi|^{2}\right)$ already plays a role in the Lagrangian], so that scalar regulator fields to not contribute to an anomaly. The mass terms, on the other hand, for fermion regulator fields, $M \bar{\Psi}_{i} \Psi_{i}+M \bar{X}_{i} \bar{X}_{i}$, give rise to an anomaly. A very important point here is that the large masses of the regulators cannot arise in the same way as the masses of ordinary physical fermions, i.e., as a result of Yukawa couplings with Higgs particles. In fact, it is impossible to write expressions of the type $C_{i j} \phi \bar{\Psi}_{i} X_{j}+$ c.c. with very large constants $C_{i j}$ in the Lagrangian of the regulators. After the field $\phi$ is separated out into a condensate, such terms would lead to the formation of masses $M \sim C\langle\phi\rangle$, of the regulator fields, but this event would not be a regularization. The simplest way to understand this situation is to recall that the ultraviolet properties of the theory do not depend on the choice of a vacuum for the perturbation theory, and near an unstable vacuum, $\phi=0$, the fields $\Psi$ and $X$ would have no masses at all. An explicit infinity is present in the case of this "regularization" in, for example, the effective potential of scalars, $\varphi=\phi^{-}-\langle\phi\rangle$ which is generated by single-loop diagrams with fields $\Psi$ and $X$ which propagate in a loop:

$$
\begin{aligned}
V_{\mathrm{eff}}(\varphi) & \sim \ln \operatorname{det}(i \hat{\partial}+C\langle\phi\rangle+C \varphi) \\
& \sim \sum_{n} \frac{C^{n} \varphi^{n}}{n} \operatorname{Tr} \int \frac{\mathrm{~d}^{4} p}{(\hat{p}+C\langle\phi\rangle)^{n}} \\
& \sim \sum_{n=0}^{n=4} \Lambda_{n} \varphi^{n}+\sum_{n=5} \gamma_{n} \frac{C^{4}}{\langle\phi\rangle^{n-4}} \varphi^{n} ;
\end{aligned}
$$

here $\Lambda_{0, \ldots, 4}$ are the ordinary divergent parameters of the effective potential, which are free parameters of the theory (the vacuum energy and the mass and the self-effect constant of the scalar field). All the difficulty stems from the circumstance that the other terms in the expansion of $V_{\text {eff }}$ also diverge in the limit $C \rightarrow \infty$. It is this circumstance which permits us to say that the theory would be unregularized.

The regularization rules require that all the interaction constants of the regulator fields (including the Yukawa constant $C$ ) must be the same as for physical fields (i.e., $C_{i j}$ $=c_{i j}$ ), and the masses of the regulators-and they alonemust be large. In the case at hand, this requirement means that terms $M \bar{\Psi}_{i} \Psi_{i}, M \bar{X}_{i} X_{i}$ must be incorporated in the Lagrangian of the regulators. Mass terms of this sort violate the conservation of the chiral currents $J_{\mu}^{\mathrm{w} . \mathrm{B}}$ and may, generally speaking, lead to a quantum anomaly which violates the gauge invariance of the theory at the single-loop level. According to the general rules (Sections 2 and 3), this anomaly is of the form

$$
\begin{aligned}
& \left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{W}}\right)^{a} \sim \varepsilon_{\alpha \beta \gamma \delta} \operatorname{Tr} t^{\mathrm{a}} \partial_{\alpha}\left(A_{\beta} \partial_{\gamma} A_{\delta}+\frac{1}{2} A_{\beta} A_{\gamma} A_{\delta}\right), \\
& \left(\partial_{\mu} J_{\mu}^{\mathrm{B}}\right) \sim \sum_{i} Y_{i \alpha \beta \gamma \delta} \operatorname{Tr} \partial_{\alpha}\left(A_{\beta} \partial_{\gamma} A_{\delta}+\frac{1}{2} A_{\beta} A_{\gamma} A_{\delta}\right) ;
\end{aligned}
$$

here $A$ represents the field $A_{\mu}=-i\left(W_{\mu}^{a} t^{a}+Y_{i} B_{\mu}\right)$, which corresponds to the gauge algebra $\mathrm{SU}(2) \times \mathrm{U}(1) 1$, and the trace is taken with respect to this algebra. The $t^{a}$ are the generators of an $\mathbf{S U}(2)$ subalgebra. In terms of the fields $W^{\prime \prime}$ and $B$ we would write

$$
\begin{aligned}
\partial_{\mu} J_{\mu}^{\mathrm{B}} \sim \varepsilon_{\alpha \beta \gamma \delta} \partial_{\alpha}\left[( \sum _ { i } Y _ { i } ) \left(\frac{1}{2} W_{\beta}^{a} \partial_{\gamma} W_{\delta}^{a}\right.\right. & \left.-\frac{1}{4} f^{a b c} W_{\beta}^{a} W_{\gamma}^{b} W_{\delta}^{c}\right) \\
& \left.+\left(\sum_{i} Y_{i}^{3}\right) B_{\beta} \partial_{\gamma} B_{\delta}\right] .
\end{aligned}
$$

As regards $\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{w}}\right)^{a}$, this divergence contains two terms:

$$
\begin{gathered}
\varepsilon_{\alpha \beta \gamma \delta} \operatorname{Tr} t^{a} \partial_{\alpha}\left(W_{\beta} \partial_{\gamma} W_{\delta}+\frac{1}{2} W_{\rho} W_{\gamma} W_{\delta}\right), \\
\left(\sum_{i} Y_{i}\right) \varepsilon_{\alpha \beta \gamma \delta} \partial_{\alpha}\left[B_{\beta} \operatorname{Tr} t^{a}\left(\partial_{\gamma} W_{\delta}+\frac{1}{4} W_{\gamma} W_{\delta}\right)\right] .
\end{gathered}
$$

The first is proportional to $d^{a b c}$ and therefore vanishes for the $\operatorname{SU}(2)$ algebra. Consequently, if there is to be no anomaly in the current $J_{\mu}^{\mathrm{W}}$ due to $\mathrm{SU}(2)$-gauge bosons in four dimensions, the sum of the hypercharges must vanish. The conservation of the $\mathrm{U}(1)$ current $J_{\mu}^{\mathrm{B}}$ requires that the hypercharges satisfy one more condition: $\sum_{i} Y_{i}=0, \sum Y_{i}^{3}=0$. These so-called anomaly cancellation conditions are required for the validity of the ordinary treatment of the GWS theories with a single Higgs doublet. In the standard GWS model these conditions hold separately for each generation of fermions when the three-color nature of the quarks is taken into account:

$$
\begin{aligned}
& Y_{\mathrm{e}}=-\frac{1}{2}, \quad Y_{\widetilde{\mathrm{e}}}=+1, \quad Y_{v}=-\frac{1}{2} \\
& Y_{\mathrm{u}}=+\frac{1}{6}, \quad Y_{\widetilde{\mathrm{u}}}=-\frac{2}{3} \\
& Y_{\mathrm{d}}=+\frac{1}{6}, \quad Y_{\widetilde{\mathrm{d}}}=+\frac{1}{3},
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i} Y_{i}=0+3 \cdot 0=0 \\
& \sum Y_{i}^{3}=\left[1^{3}+2\left(-\frac{1}{2}\right)^{3}\right] \\
& +3\left[2\left(+\frac{1}{6}\right)^{3}+\left(-\frac{2}{3}\right)^{3}+\left(+\frac{1}{3}\right)^{3}\right]=\frac{3}{4}-\frac{3}{4}=0
\end{aligned}
$$

A more complex example of a theory with a cancellation of anomalies is

### 1.3.5. Four-dimensional grand unified $S U(5)$ model

Each generation of fermions in this model belongs to a 5 - or $\overline{10}$-plet according to the $\mathrm{SU}(5)$ gauge group. An anomaly is found by summing expressions

$$
\begin{aligned}
& \varepsilon_{\alpha \beta \gamma \delta} \partial_{\alpha} \operatorname{Tr} t^{a}\left(A_{\beta} \partial_{\gamma} A_{\delta}+\frac{1}{2} A_{\beta} A_{\gamma} A_{\delta}\right) \\
& \sim \Sigma_{\alpha \beta \gamma \delta} \partial_{\alpha}\left(d^{a b c} A_{\beta}^{\delta} \partial_{\gamma} A_{\delta}^{c}+\frac{1}{4} d^{a b m} f^{c d m} A_{\rho}^{b} A_{\gamma}^{c} A_{\delta}^{d}\right)
\end{aligned}
$$

for fermions in all representations of the group; here

$$
\begin{gathered}
t^{a} t^{b}=\text { const } \cdot \delta^{a b}+\frac{1}{2}\left(d^{a b c}+i f^{a b c}\right) t^{c}, \\
f^{a b c}=-f^{b a c}, \quad d^{a b c}=+d^{b a c}
\end{gathered}
$$

Since the structure constants $f^{c d m}$ do not depend on the representation, the condition for a cancellation of anomalies in a four-dimensional theory is

$$
\sum_{\text {(over all representations }} \sum_{\text {which contain chiral fermions }} d_{R}^{a b c}=0
$$

This condition is satisfied for the $\operatorname{SU}(5)$ model, by virtue of the relation $d_{s}^{a b c}+d_{10}^{a b c}=0$.

A violation of the anomaly cancellation conditions would have many unpleasant consequences and might even make the theory internally inconsistent. The most important of these unpleasant consequences occurs because the interaction with an unconserved current would "give life" to gauge degrees of freedom of vector bosons. As a result, at the quantum level the theory would have more degrees of freedom than it would in the classical limit. This situation would contradict the standard understanding of the unitarity of a quantum field theory. We will return to unitarity later on, but at this point we wish to explain that there is a simpler way to build in a mechanism for removing anomalies and to restore the unitarity of the theory in its ordinary sense than the approach of the GWS and SU(5) models. Instead of choosing the composition of fermion fields in such a way that their contributions to the anomaly cancel out, one could compensate for the noninvariance of the effective action which stems from the anomaly in the fermion determinant by means of explicitly noninvariant terms in the original action. For example, in the case of abelian theory one could add to the action of a four-dimensional theory an expression of the type

$$
\begin{aligned}
& \hbar \int \partial_{\alpha} A_{\alpha} \frac{1}{\partial^{2}} F_{\mu v} \check{F}_{\mu v} \mathrm{~d}^{4} x \\
& \quad \sim \hbar \int \mathrm{~d}^{4} x \mathrm{~d}^{4} y \partial_{\alpha} A_{\alpha}(x) \frac{1}{(x-y)^{2}} F_{\mu v} \widetilde{F}_{\mu v}(y)
\end{aligned}
$$

Under gauge transformations $A_{\alpha} \rightarrow A_{\alpha}+\partial_{\alpha} \varepsilon$ the variation of this increment is $\hbar \int \varepsilon F_{\mu \nu} \widetilde{F}_{\mu \nu} \mathrm{d}^{4} x$, and it can cancel the variation in the fermion determinant. This is a quantum modification of the action, since the invariance of the fermion determinant is manifested in only first order (not zeroth order) in $\hbar$, and the compensating term in the action must be proportional to $\hbar$. The difficulty is, however, that it is necessary to add to the action a nonlocal expression, which is not a counterterm in the standard sense of this word. A nonlocal situation can be avoided if there is a scalar field which varies in an inhomogeneous way under gauge transformations $A_{\alpha} \rightarrow A_{\alpha}+\partial_{\alpha} \varepsilon: \chi \rightarrow \chi+\varepsilon$. The compensating term will then depend on the fields in a local way: $\int \chi F_{\mu v} \widetilde{F}_{\mu v} \mathrm{~d}^{4} x$. The role of the scalar $\chi$ can be played by the phase of an ordinary $\mathrm{U}(1)$-charged scalar $\phi=|\phi| e^{i x}$. In this case, the transformation $\phi \rightarrow e^{i \varepsilon} \phi$ will not change $\{\phi\rangle$, and we have $\chi \rightarrow \chi+\varepsilon$. The only problem is in writing the kinetic term for the field $\chi$. There are two possibilities of interest. First, one could write a gauge-invariant expression $\left(\partial_{\mu} \chi-A_{\mu}\right)^{2}$. A kinetic term of this type has the shortcoming that it leads to an unrenormalizable four-dimensional theory: It is simple to recognize in this expression a (gauge-invariant) mass term for the vector boson. Nevertheless, in theories on which the requirement of renormalizability is not imposed a mechanism of this sort for removing anomalies is very useful. This comment applies, for example, to the ten-dimensional field theory which is found from certain versions of superstring theory. In this case the field theory describes only the lowenergy limit of the complete theory and may be unrenormalizable. The anomaly cancellation mechanism which was discovered in these models by Green and Schwarz ${ }^{52}$ is a direct generalization of this example, except that the role of the field $\chi$ is played there by an antisymmetric field $B_{\mu \nu}$.

The second possibility for generating a kinetic term for the field arises in non-abelian theories. A non-abelian generalization of the expression $\chi \widetilde{F}$-the so-called Wess-Zumino-Witten Lagrangian $L_{\text {wZw }}(\chi, A)$-is not linear in $\chi$, and it contains derivatives of $\chi$, including contributions of the type $\varepsilon_{\alpha \beta \gamma \delta} \operatorname{Tr} \chi \partial_{\alpha} \chi \partial_{\beta} \chi \partial_{\gamma} \chi \partial_{\delta} \chi$, $\varepsilon_{a \beta \gamma \delta} \operatorname{Tr} \chi \partial_{\alpha} \chi A_{B} \partial_{\gamma} A_{\delta}, \ldots$ Accordingly, $L_{\mathrm{wzw}}$ describes a certain dynamics of the field $\chi$ even at the single-loop level. Attempts have been undertaken-so far without successto prove that there are no pathologies in a model of this sort. If such a proof were to be found one might also suspect that renormalizability is conserved since $L_{\text {wZw }}$, in contrast with $\left(\partial_{\mu} \chi-A_{\mu}\right)^{2}$ contains no dimensional parameters.

At any rate, even if a noncontradictory theory with an internal anomaly is constructed through the introduction of auxiliary fields $\chi$, which are not present in the classical limit, models without anomalies would remain quite special. A particular case of this situation is the celebrated problem of the critical dimensionality in string theory. A way from the critical dimensionality, the theory is either meaningless ( $d>d_{\text {cr }}$ ) or contains an additional degree of freedom ( $d<d_{c r}$ ). At the same time, the massless excitations disappear: A gap appears in the spectrum. Other dynamic characteristics become much more complicated. In the discussion below we will require cancellation of internal anomalies.

Now, summarizing this brief review of the most important applications of anomalies in the physics of elementary particles (although we have not taken up string theory, where the anomalies are simultaneously the most important and the most interesting entities), we wish to offer something in the way of a classification of anomalies. We see three possible principles for such a classification.
a) Anomalies: local and global. Examples of global anomalies are the Witten $\operatorname{SU}(2)$ anomaly ${ }^{46}$ and the corresponding Redlich anomaly in an odd-dimensional non-abelian Yang-Mills theory. ${ }^{35,41}$ These anomalies correspond to a noninvariance of an action under topologically nontrivial gauge transformations and are not manifested by a nonconservation of currents of any sort. All other anomalies which describe a violation of invariance under infinitesimal transformations and which are associated with the nonconservation of Noether currents are "local." There should be no global anomalies in gauge theories. This statement means that if some field contributes to a global anomaly, then it is necessary to choose the complete set of fields in such a way that this contribution is cancelled out. In the case at hand, the change in the action under topologically nontrivial gauge transformations [which satisfy the condition $g(x) \rightarrow 1$ at infinity] must be a multiple of $2 \pi i$. Otherwise, the generating functional of the theory, proportional to $e^{-S}$, would not be determined.

The entire classification below pertains to local anomalies alone.
b) Anomalies inglobal and local (gauge) symmetries. We discussed the difference between these two classes of anomalies in detail in Subsection 1.1. At this point we note that an anomalous divergence of the same current can be described as a breaking of either a global or local symmetry. The situation depends on whether gauge bosons are associated with this current in the Lagrangian or not. If not, then global invariance is violated; if yes, then it is local invariance. The violation of a local (gauge) invariance would seem to be forbidden, and the contributions of different particles to an anomaly would have to cancel out in a consistent theory. The condition that the anomalies of currents which are interacting with gauge bosons must cancel out imposes some severely restrictive selection rules on realistic theories. Here we will mention only the two leading examples: the prediction of the existence of a c-quark and, more generally, the prediction that the numbers of quarks and leptons are equal, on the basis of the cancellation of anomalies in the Glashow-Weinberg-Salam theory; and the unambiguous fixing of a chiral ( $D=10$ ) supergravity without matter fields as a consequence of the cancellation of gravitational anomalies. ${ }^{35}$ An even more important example is the fixing of the gauge group in superstring theories. ${ }^{52}$

A violation of a global symmetry, in contrast, poses absolutely no danger to the self-consistency of a theory, and it is not necessary to require that such anomalies cancel out. An important application of global anomalies was proposed by 't Hooft. ${ }^{26}$ His comments are based on two facts. First, an anomaly (axial or chiral) is determined by a single-loop diagram and does not depend on the subsequent corrections
associated with the interaction between fermions; this is the Adler-Bardeen theorem. ${ }^{35}$ Second, an anomaly is associated with the existence of massless excitation; this is a result established by Dolgov and Zakharov. ${ }^{8}$ The Adler-Bardeen theorem follows immediately from the circumstance that a regularization which breaks no symmetries other than dilatation and supersymmetry exists in two loops and above. This regularization might be, for example, a regularization by higher derivatives. ${ }^{53}$ Adler and Bardeen themselves discussed the situation in electrodynamics and used a regularization which introduces a mass for the photon but not for a fermion. (See also Ref. 50 regarding this theorem.) This relationship between anomalies and the existence of massless excitations can be seen simply from the circumstance that the anomalous symmetries are precisely those which are characteristic of massless fields, and they do not survive when a mass is introduced. Otherwise the Pauli-Villars regularization would not break the symmetry, and there would be no anomaly. Dolgov and Zakharov generated a clearer assertion: An anomaly is associated with a nonzero imaginary part of some correlation function (Subsection 1.2), and, furthermore, this imaginary part is proportional to $\delta(s)$, i.e., is due entirely to massless excitations [see, for example, Eq. (1.19) in the preceding section]. The conclusion which 't Hooft reached from these assertions is that the anomalies in a theory with an interaction must be the same as when the interaction is excluded (i.e., the anomalies must be the same as the anomalies of the corresponding fundamental fields, which appear directly in the Lagrangian). Consequently, even in the strong-coupling region we know something about the spectrum of the theory. If the anomalies for fundamental particles are nonzero, the spectrum will always have massless excitations. These excitations must have quantum numbers such that they reproduce the results for the fundamental anomalies. An important point is that if a theory has a symmetry of some sort, but the Lagrangian does not contain the corresponding Noether currents, the consistency condition still holds: It is sufficient to introduce an infinitely weak interaction of these currents with nonphysical ("spectator") gauge bosons. As an example we might cite quantum chromodynamics. The fundamental fermions in this theory are quarks. If they are assumed to be massless (the chiral limit) then the theory at the classical level will have an axial symmetry, but the symmetry will be broken by an anomaly. The physical hadrons are constructed from strongly interacting quarks and are generally massive. Nevertheless, the consistency condition guarantees that massless hadrons will exist in the chiral limit. These are the pseudoscalar mesons $\pi, \mathrm{K}, v$, whose masses are proportional to the masses of the quarks. (The masses of the baryons, in contrast, are essentially independent of the quark masses.) Pseudoscalar mesons interact directly with the axial current $J_{\mu}^{5}=\partial_{\mu} \pi$, and they reproduce (Fig. 5) an axial anomaly. In quantum chromodynamics we have a good understanding of the actual mechanism for the appearance of the masses of hadrons: In the chiral limit, this mechanism is due entirely to the spontaneous breaking of axial invariance; the masses of all the hadrons are proportional to the breaking parameter,


FIG. 5. Example of the 't Hooft consistency relation. The axial anomaly in terms of fundamental quark fields (a) coincides with the axial anomaly which is expressed at the anomalous vertex $\pi \gamma$ (b). The results for the anomalous divergence of an axial current calculated for fundamental fields (quarks) and physical fields ( $\pi$ mesons) are the same.
which is the asymmetric fermion condensate $\langle\bar{\psi} \psi\rangle$; and the pseudoscalar mesons are goldstone particles which arise in a spontaneous breaking of symmetry. The 't Hooft condition, however, makes it possible to draw the conclusion that massless bosons exist even without this understanding of the dynamics: They exist exclusively because of the form of the Lagrangian and the massive nature of the physical fermions, in this case, baryons. For a more detailed discussion of the consistency conditions we refer the reader to the original paper by 't Hooft ${ }^{26}$ and the many subsequent papers. ${ }^{27}$
c) Basic classification of anomalies. This classification of course involves an enumeration of the various anomalous symmetries. We already mentioned, back in Subsection 1.2, that we will not be taking up supersymmetric or gravitational anomalies. We should still say a few words about a conformal anomaly. Dilatational invariance usually stems from the absence of dimensional parameters in the classical Lagrangian. In a regularization of a divergent theory, a parameter of this sort-a normalization point-unavoidably appears (this phenomenon is known as a "dimensional transmutation"). There is accordingly no dilatational invariance in theories with divergences. Bearing this source of symmetry breaking in mind, we can easily calculate the anomalous trace of the energy-momentum tensor, which is equal to the divergence of the dilatation current: $T_{\mu \mu}=\partial_{\mu} \mathrm{D}_{\mu}$. For example, a Yang-Mills theory $L=(1 / 4 \alpha) \operatorname{Tr} F_{\mu}^{2}$, becomes the following when quantum corrections are taken into account:

$$
L\left(q^{2}\right)=\frac{1}{4 \alpha\left(q^{2} / \mu^{2}\right)} \operatorname{Tr} F_{\mu v}^{2} .
$$

The breaking of dilatational invariance results entirely from the appearance of the dependence of the effective charge $\alpha\left(q^{2} / \mu^{2}\right)$ on the momentum transfer which arises here. Correspondingly, we have

$$
T_{\mu}^{\mu}=\frac{2 g^{\mu v}}{|g|^{1 / 2}} \frac{\partial|g|^{1 / 2} L}{\partial g^{\mu \nu}}=-\frac{2 g^{\mu v}}{\alpha^{2}} \frac{\partial \alpha}{\partial g^{\mu \nu}} L
$$

the dependence of $\alpha$ on $g^{4 u^{\prime}}$ is very simple:

$$
\alpha\left(g^{\mu \nu} \frac{q_{\mu} q_{\nu}}{\mu^{2}}\right), \quad \frac{g^{\mu \nu} \partial \alpha}{\partial g^{\mu \nu}}=\frac{q^{2}}{\mu^{2}} \frac{\partial \alpha}{\partial\left(q^{2} / \mu^{2}\right)}=\frac{\partial \alpha}{\partial \ln q^{2}}
$$

Here $T_{\mu}^{\mu}$ is expressed in terms of the $\beta$-function $\beta(\alpha)=\partial \alpha /$ $\partial \ln q^{2}$ :

$$
T_{\mu}^{\mu}=-\frac{1}{2} \frac{\beta(\alpha)}{\alpha^{2}} \operatorname{Tr} F_{\mu v}^{2} .
$$

The broadest class of anomalies stems from axial and chiral currents. These currents are usually fermion currents, but we also know of some anomalies which stem from chiral bosons: self-dual tensor fields. ${ }^{35}$ In the present review we will discuss only spin-1/2 fermions. The anomalies which arise in this case differ in two other characteristics: the nature of the interaction of the fermions with gauge fields and the choice of current whose divergence is to be calculated. On the basis of the first of these characteristics-the form of the Lagrangian of the theory-we distinguish between a Dirac anomaly, with

$$
L_{\mathrm{int}}=\vec{i} \vec{\psi} \psi_{\mu} t^{a} \psi A_{\mu}^{a}=\bar{\psi} \hat{A} \psi
$$

and a Weyl anomaly, with

$$
L_{\mathrm{int}}=i \bar{\psi} \gamma_{\mu} \frac{1-\gamma^{5}}{2} t^{\alpha} \psi A_{\mu}^{\alpha}=\bar{\psi} \hat{A} \frac{1-\gamma^{5}}{2} \psi
$$

Each is a particular case of the general Bardeen anomaly

$$
L_{1 \mathrm{n} \mathbf{t}}=i \bar{\psi} \gamma_{\mu} t^{a} \psi V_{\mu}^{a}+i \bar{\psi} \gamma_{\mu} \gamma^{5} t^{a} \psi A_{\mu}^{a}=\bar{\psi}\left(\hat{V}+\hat{A} \gamma^{5}\right) \psi
$$

The Bardeen anomaly will be discussed only briefly here, since I do not have a clear picture of its place in the overall formalism of the hierarchy of anomalies (Section 3). As usual, if $t^{a}$ is replaced by unity in a Lagrangian, we will speak in terms of an abelian theory.

In these theories we may be interested in the divergences of various currents, primarily vector, axial, and chiral currents. The currents themselves may or may not contain color generators, depending on how the Lagrangian of the theory is constructed. ${ }^{9 /}$ We will call the corresponding currents and the anomalies associated with them "non-abelian" and "abelian."

Two sections of the present review (the second and third) are devoted to calculations of anomalies, i.e., calculations of the expressions on the right sides of equalities of the form $\mathrm{D}_{\mu} J_{\mu}=. \not 2$. Here $\mathscr{A}$ is some function of the gauge fields, $A_{\mu}=i t^{a} A_{\mu}^{a}$. We will derive expressions for $\mathscr{A}(A)$ in three different cases.

1) Abelian axial anomaly in a Dirac theory (abelian or non-abelian) in a $2 n$-dimensional space. The expression for this anomaly is very simple in form:

$$
\begin{equation*}
\partial_{\mu} J_{\mu}^{\delta}=\frac{2}{(2 \pi)^{n} n!} \varepsilon_{\alpha_{1} \ldots \alpha_{2 n}} \operatorname{Tr} F_{\alpha_{1} \alpha_{2},,} F_{\alpha_{2 n-1} \alpha_{2 n}} \tag{1.23}
\end{equation*}
$$

2) Non-abelian axial anomaly in the same Dirac theory (abelian or non-abelian). The entire difference from the preceding case is the appearance of a covariant derivative on the left and of a color generator on the right
$\left(\mathrm{D}_{\mu} J_{\mu}^{5}\right)^{a}=\frac{2}{(2 \pi)^{n} n \mid} \varepsilon_{\alpha_{1} \ldots \alpha_{2 n}} \operatorname{Tr} t^{a} F_{\alpha_{1} \alpha_{2}} \quad F_{\alpha_{2 n-1} \alpha_{2 n}}$.
We have stipulated that we will use a Pauli-Villars regularization which conserves vector currents everywhere. Consequently, the expressions for the axial and chiral anomalies in the same theory are the same. To avoid the appearance of a difference in the normalization, we adopt the convention that we will always calculate the divergence of the current $J_{\mu}^{L a}=\bar{\psi} \gamma_{\mu}\left(1-\gamma^{5}\right) t^{a} \psi$ (while the Lagrangian of Weyl theory contains

$$
\left.\vec{\psi} \hat{A} \frac{1-\gamma^{5}}{2} \psi=\frac{1}{2} J_{\mu}^{L a} A_{\mu}^{a}\right)
$$

The chiral anomaly in a Dirac theory is thus the same as the axial anomaly in (1.24):
$\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{L}}\right)_{\text {Dir }}^{a}=-\frac{2}{(2 \pi)^{n} n!} \varepsilon_{\alpha_{1}} \ldots \alpha_{2 n} \operatorname{Tr} t^{a} F_{\alpha_{1} \alpha_{2}} \quad F_{\alpha_{2 n-1} \alpha_{2 n}}$.
3) Non-abelian chiral anomaly not in a Dirac but in a Weyl theory. This case, in contrast, is the most interesting for modern theories. It is in this case that gauge invariance is violated: The "internal" current with which the gauge bosons interact is not conserved. In a transformation to a Weyl theory, the expression for the chiral anomaly changes markedly. In particular, it ceases to be gauge-covariant, and it is not written in terms of the intensities $F_{\alpha \beta}$. The derivation of a general expression for the chiral anomaly in a Weyl theory is based on the Wess-Zumino relation, ${ }^{5}$ which arises specifically because we are considering an anomaly in an internal current. We consequently have $\mathscr{A}_{u}(A) \equiv \operatorname{Tr} u \mathrm{D}_{\mu} J_{\mu}^{L}$ $=\delta_{u} S_{\text {eff }}\{A\}$; i.e., this quantity is equal to the variation of the effective action (which incorporates single-loop corrections) for a gauge transformation of the fields $A$ : $\delta_{u} A_{\mu}=\left[A_{\mu} u\right]+\partial_{\mu} u=\mathrm{D}_{\mu} u$. Since the gauge transformations form a group, it is easy to see that the relation $\delta_{u} \cdot \alpha_{t}-\delta_{u} \cdot \delta_{u}=\alpha_{\{u c \mid}$ holds. This is a consistency condition. There are various ways to make use of this relation. Wess and Zumino themselves used it to reconstruct $\mathscr{A}_{u}(A)$ from the well-known "leading term" $\operatorname{Tr} u \varepsilon_{\alpha_{1} \ldots \alpha_{n},} \partial_{\alpha_{1}} \partial_{\alpha_{2}} \ldots \partial_{\alpha_{i, \prime}, A_{2}} A_{\alpha_{2 n}}$. The consistency condition does not determine $\mathscr{A}_{u}(A)$ unambiguously, but only within a variation of some arbitrary local functional of the fields $A$. The addition of such a functional to the effective action, however, is allowed by the renormalization procedure; this step is a so-called addition of local counterterms. The consistency condition therefore unambiguously fixes only that part of the anomaly which is independent of the choice of local counterterms, i.e., independent of the choice of regularization. However, it is precisely this "truly anomalous" contribution to $\mathscr{A}_{u}(A)$ which is of physical interest. It is easy to see that all structures which contain the $\varepsilon$-symbol, and only such structures, are "truly anomalous."

A method for deriving a general expression for $\alpha_{u}(A)$ on the basis of a so-called hierarchy of anomalies, rather than through an iterative solution of the consistency solution (the approach taken by Wess and Zumino), is now available. This question is examined in Subsection 2.4 and Section 3; at this point we will simply outline the basic idea. Instead of constructing nonlocal functionals which generate $\mathscr{A}_{u}(A)$ one can make use of a local functional, but one which depends on one additional field. If the dependence on this field drops out under the gauge transformation, we obtain a possible candidate for the role of $\mathscr{A}_{u}(A)$. It turns out that when we take this approach we can find a structure with an $\varepsilon$-symbol in $\mathscr{A}_{u}(A)$. The functional which is found, however, cannot be accepted as a counterterm, since it depends on an additional nonphysical field. (It may turn out to be a counterterm in a theory in which there are scalars in an asso-
ciated representation of the gauge group. In such a case, the additional field can be identified with a gauge degree of freedom: the phase of these scalars. In such a case, the anomaly turns out to be removable.) In general, the functional acquires a natural meaning in a space with a dimensionality one unit larger, where the additional field is interpreted as the $(2 n+1)$ st component of $A_{\mu}$. In this case this functional is the same as the Wess-Zumino term, which, as we have already mentioned, itself arises in the fermion determinant of a $(2 n+1)$-dimensional theory. The relationship between a $(2 n+1)$-dimensional Wess-Zumino term and a $2 n$-dimensional function $\mathscr{A}_{u}(A)$ could naturally be called the "relation among anomalies" of their "hierarchy." Furthermore, the hierarchy of anomalies continues even further: upward, to $2 n+2$ and downward, to $2 n-1$, etc., dimensions.

We have one more comment regarding chiral anomalies in Weyl theories. One might ask why we use a Pauli-Villars regularization in a discussion of such anomalies. One might recall the example of an axial anomaly [see (1.22)], in which an appropriate choice of regularization would make it possible to conserve an axial current by virtue of the nonconservation of a vector current. In a Weyl theory, we should strive to conserve chiral currents, while there is no need to conserve vector currents. The situation is, however, that it is not possible to achieve the conservation of chiral currents and simultaneously achieve Bose symmetry. Under these conditions, the conservation of vector current is a totally harmless additional condition which has no effect of any sort on the existence of an anomaly. We will clarify this assertion in the by now familiar example of a two-dimensional field theory. From (1.20) we easily find the following equation, in addition to (1.22):

$$
\begin{align*}
q_{\alpha} J_{\alpha}^{L} & \cdots \eta_{\alpha}\left(\delta_{\alpha \mu}-i \varepsilon_{\alpha \mu}\right) \frac{1}{\pi}\left(\frac{q_{\mu} q_{v}}{q^{2}}+c g_{\mu v}\right)\left(\delta_{\beta v}-i \varepsilon_{\beta v}\right) A_{\beta} \\
& =\frac{1}{\tau}\left(q_{\alpha} A_{\alpha}+i \varepsilon_{\alpha \beta} q_{\alpha} A_{\beta}\right) . \tag{1.26}
\end{align*}
$$

As promised, the dependence on the arbitrary parameter $c$ has dropped out of the expression for the Weyl anomaly.

Let us outline the rest of this review. Section 2 is devoted to a calculation of anomalies. In Subsection 2.1 we use diagrams in a two-dimensional field theory to demonstrate the difference between Dirac and Weyl anomalies. In contrast with the calculations which we already presented in the Introduction, we will also discuss non-abelian contributions to anomalies in that subsection. In Subsection 2.2. we describe the calculation of anomalies by the Vergeles-Fujikawa method. This method is a particular case of the operator formalism, now used widely (e.g., Ref. 54), which dramatically simplifies the calculation of diagrams in gauge theories. The Vergeles-Fujikawa method immediately points out the relationship between the nonconservation of an axial current and the index of the Dirac operator, ind $(i \hat{\mathrm{D}})=\mathrm{Sp} \gamma^{5}$. It allows us to find expressions for the Dirac anomalies in their most general form. In the case of a Weyl anomaly, this method can be used to calculate the anomaly in each individual case, but it is difficult to find results of a general nature (which hold in all dimensionalities). We will list the general
principles for calculations by the Vergeles-Fujikawa method. We hope that where necessary the reader will be able to use this method without any difficulties to calculate any anomalies. In Section 2.3 we studied the relationship between Dirac anomalies in even-dimensional theories and Wess-Zumino terms in odd-dimensional gauge theories. Unfortunately, it has not yet been possible to put the corresponding relationship between Wess-Zumino terms and Weyl anomalies in a simple form. (Alvarez-Gaumé and Ginsparg ${ }^{23}$ have proposed a very instructive method; unfortunately, that method is too long to be included in the present review while in a summary it would require appealing to new methods.) In the brief Subsection 2.4 we will accordingly discuss a different method for deriving a Weyl anomaly: on the basis of Wess-Zumino consistency relations. Finally, in Subsection 2.5 we calculate an axial anomaly on the basis of the imaginary part in a four-dimensional Dirac theory. Together with the example of this calculation for $D=2$ which we already discussed in Subsection 1.2, this yields a complete idea of the relationship between anomalies with massless excitations, and more generally, how an anomaly is constructed at the level of currents, rather than at the level of their divergences.

The following section, Section 3, is devoted to the hierarchy of anomalies. In that section we will not refer to fermions; we will discuss only boson expressions or, more precisely, differential forms which stand on the right sides of anomalies. The differential forms here are simply a laconic language for writing convolutions of fields $A_{\mu}$ with $\varepsilon$-symbols. The relationship between the right sides of the anomalies is of course the same as that between the corresponding fermion determinants (Subsections 2.3 and 2.4). Zumino ${ }^{21}$ can apparently be credited with the greatest contribution to the discussion in the physics literature of the relations between the right sides of anomalies. Corresponding ideas date back to papers by Gabriélov, Gel'fand, and Losik. ${ }^{55}$ We should also mention the important papers by Novikov et al., ${ }^{28}$ Faddeev et al., ${ }^{33}$ and Witten and Alvarez-Gaumé. ${ }^{35}$ In Subsection 3.1 we will briefly recall the definition of the differential forms and the operation of external differentiation, d. Subsection 3.2 is devoted to the general structure of the "inverse" operation $d^{-1}$. More important for the hierarchy of anomalies is a modification of this operation which is applied to a bounded set of differential forms, which we call $k_{z}$. Zumino ${ }^{21}$ appears to have been the first to use this operation in the literature on anomalies. In the present review, this operation is introduced in Subsection 3.3. The hierarchy of anomalies itself is described in Subsection 3.4. In the same place, we derive a general result for a chiral Weyl anomaly.

In Section 4 we return to the calculation of the fermion determinant in odd-dimensional gauge theories. The appearance of a Wess-Zumino term is treated here as a phenomenon analogous to Witten's global $\operatorname{SU}(2)$ anomaly. The meaning of this analogy is explained in a brief introduction to this section of the review. In Subsection 4.1 we then take up global anomalies, and in Subsection 4.2 we examine the origin of the Wess-Zumino term. At this point we simply note that global anomalies exist in many theories, including
gravitational theories ${ }^{35,47}$ So far, we have no general classification of them.

We do not have space in this review to include applications of anomalies. At present we may distinguish among three types of such applications. First, there is the question of the cancellation of internal anomalies or their removal by local counterterms in specific physical theories. Second, there is the question of extracting "dynamic" information from anomalies by means of the 't Hooft conditions, ${ }^{6}$ which we mentioned in Subsection 1.3. Third, there is the question of calculating the anomalous contributions to fermion determinants (i.e., the anomalous contributions to $e^{-S_{\mathrm{cf}}}$ ). The latter problem is particularly useful in cases in which the determinant has no nonanomalous part. Such cases usually arise in two dimensions, but they may also occur at $D=4$, in, e.g., a study of the chiral Lagrangians which describe the interaction of pseudoscalar goldstone particles.

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## 2. CALCULATION OF ANOMALIES

### 2.1. Calculation of anomalies from diagrams in twodimensional theories

The simplest way to derive anomalies in currents which are conserved at the classical level is to calculate diagrams for (covariant) divergences of currents through the use of a Pauli-Villars regularization. We write the current in the form $J=j-j_{\text {reg }}, \mathrm{D}_{j}=0, \mathrm{DJ}=-\mathrm{D} j_{\text {reg }} \sim M_{\text {reg }} \neq 0$; from the technical standpoint the anomaly is associated with a contribution of regulators which survives in the limit $M_{\text {reg }} \rightarrow \infty$. We will make use of this approach frequently. We will use the capital letter $M$ in this section of the review to mean the regulator mass: $M=M_{\text {reg }}$.

In a Pauli-Villars regularization, vector currents are always conserved (covariantly):

$$
\begin{aligned}
& J_{\mu}^{a}=\bar{\psi} \gamma_{\mu} t^{a} \psi-\bar{\Psi} \gamma_{\mu} t^{a} \Psi \\
& \begin{aligned}
\left(\mathrm{D}_{\mu} J_{\mu}\right)^{a} & =\left(\partial_{\mu} J_{\mu}+\left[A_{\mu} J_{\mu}\right)^{a}\right. \\
& \left.=\left(\mathrm{D}_{\mu} j_{\mu}\right)^{a}-\left(\left(\mathrm{D}_{\mu} j\right)\right)_{\mu}^{\mathrm{reg}}\right)^{a}=0-0=0 .
\end{aligned} \\
& \quad=0=0
\end{aligned}
$$

On the other hand, the divergence of an axial current $\quad J_{\mu}^{a 5}=\bar{\psi} \gamma_{\mu} \gamma^{5} t^{a} \psi-\bar{\Psi} \gamma_{\mu} \gamma^{5} t^{a} \Psi \quad$ is nonzero: $\left(\mathrm{D}_{\mu} J_{\gamma}^{5}\right)^{a}=2 i m \bar{\psi} \gamma^{5} t^{a} \psi-2 i M \bar{\Psi} \gamma^{5} t^{a} \Psi \rightarrow{ }_{m \rightarrow 0}-2 i M \bar{\Psi} \gamma^{5} t^{a} \Psi$.


FIG. 6. Divergence of the current in terms of diagrams.

In terms of diagrams, these assertions are arrived at in the following manner. For simplicity we consider an abelian theory, in which the covariant derivative is the same as the ordinary derivative. The expression for any diagram (Fig. 6) which contributes to a divergence of the vector current then contains the combination

$$
\begin{equation*}
\frac{1}{\hat{p}+\hat{q}-i m} \hat{q} \frac{1}{\hat{p}-i m}=\frac{1}{\hat{p}-i m}-\frac{1}{\hat{p}+\hat{q}-i m} ; \tag{2.1}
\end{equation*}
$$

here $\hat{p}$ is the integration variable in the loop. We now consider the particular case in Fig. 7. If we shift the integration variable ( $p \rightarrow p+k_{1}$ ) in diagram b , this diagram cancels out with diagram c . Similarly, the shift $p \rightarrow p+k_{2}$ causes diagram d to cancel out with diagram a. In the case of a diagram with an arbitrary number of external ends it is necessary to sum over all permutations of the external momenta and to introduce shifts of the integration variable to all possible linear combinations of them. It is sufficient to consider only single-loop diagrams, since the other loops are fixed by photon lines, whose virtuality has no effect of any sort on the cancellation of diagrams. An important point is that we are working with regularized currents. It is this circumstance which allows us to shift the integration momenta freely, although on the other hand we need to take account of the regulator contributions explicitly. In the case of a vector current, the cancellation of diagrams does not depend on the mass of the fermion, so that in the Pauli-Villars regularization vector currents are always conserved (covariantly). For the vector current in (1.1) the matrix $\gamma^{5}$ also appears:


FIG. 7. Example of the calculation of a divergence of the current (see the text proper for an explanation).


FIG. 8. Divergence of an axial current in an external field ( $D=2$ ). The contribution is linear in the external field.

$$
\begin{align*}
& \frac{1}{\hat{p}+\hat{q}-i m} \hat{q} \gamma^{5} \frac{1}{\hat{p}-i m}=-\frac{1}{\hat{p}+\hat{q}-i m} \hat{q} \frac{1!}{\hat{p}+i m} \gamma^{5} \\
& =-\left(\frac{1}{\hat{p}+i m}-\frac{1}{\hat{p}+\hat{q}-i m}\right) \gamma^{5}-\frac{1}{\hat{p}+\hat{q}-i m} 2 i m \frac{1}{\hat{p}+i m} \gamma^{5} \\
& =\frac{1}{\hat{p}+\hat{q}-i m} 2 i m \gamma^{5} \frac{1}{\hat{p}-l m}+\left[\gamma^{5} \frac{1}{\hat{p}-i m}+\frac{1}{\hat{p}+\hat{q}-i m} \gamma^{5}\right] . \tag{2.2}
\end{align*}
$$

The expression in square brackets drops out of the result for the same reasons as in the discussion of the vector current; the first term stays. Consequently, the diagrams for the divergence of the axial current are the same as the diagrams for $2 i m \bar{\psi} \gamma^{5} \psi$.

In order to calculate the axial anomaly we thus need to find the average of $2 i M \bar{\Psi} \gamma^{5} t^{a} \Psi$ in the external field. The result will of course depend on the nature of the interaction of the fermions with the external field, and it will be different in the Weyl and Dirac theories. We can demonstrate the situation in the very simple example of two-dimensional fermions. At $D=2$ we need to consider only biangular and triangular diagrams. The expression for a four-tail is

$$
M \int \frac{d^{2} p}{(p+M)^{4}}=O\left(\frac{1}{M}\right)
$$

i.e., it vanishes in the limit $M \rightarrow \infty$. The two diagrams which we need to calculate are shown in Figs. 8 and 9. We can write expressions for these diagrams.

Dirac theory:

$$
\begin{aligned}
2 i \frac{\delta^{\prime} \hbar}{2} \int & \frac{\operatorname{Tr} M \gamma^{5}(\hat{p}+i M) \gamma_{\alpha}(\hat{p}+\hat{q}+i M)}{\left[p^{2}+M^{2}\right]\left[(p+q)^{2}+M^{2}\right]} \frac{\mathrm{d}^{2} p}{(2 \pi)^{2}} \\
& =-M^{2} \delta^{a b}\left(\operatorname{Tr} \gamma^{5} \gamma_{\alpha} \hat{q}\right) \int \frac{(2 \pi)^{-2} \mathrm{~d}^{2} p}{\left(p^{2}+M^{2}\right)^{2}}+O\left(\frac{1}{M}\right) \\
& =+\frac{\delta^{a b}}{2 \pi} \varepsilon_{\alpha \beta} i q_{\beta}+O\left(\frac{1}{M}\right)
\end{aligned}
$$

[All the equations are written in Euclidean space, so that the fermion propagator is $(\hat{p}-i M)^{-1}$, and we have $\gamma_{\alpha} \gamma^{5}=i \varepsilon_{\alpha \beta} \gamma_{\beta}$.] The contribution of this diagram corresponds to the term

$$
\begin{equation*}
\frac{1}{2 \pi} \varepsilon_{\alpha \beta} \partial_{\alpha} A_{\beta}^{a}=\frac{1}{\pi} \varepsilon_{\alpha \beta} \operatorname{Tr} i^{\alpha} \partial_{\alpha} A_{\beta} \tag{2.3}
\end{equation*}
$$



FIG. 9. Divergence of an axial current in an external field ( $D=2$ ). The contribution is quadratic in the external field.
in the divergence of the axial current $\left(\mathrm{D}_{\mu} J_{\mu}^{5}\right)^{a}$. We also note that in order to go over to an abelian theory we would have to not simply discard the indices $a, b, c, \ldots$, but also multiply all the expressions by 2 . The reason is the normalization of the generators $t: \operatorname{Sp} t^{a} t^{b}=\delta^{a b} / 2$.

Weyl theory. All that we need to do in order to derive an expression for the diagram in Fig. 7 is to replace $\gamma_{c}$ by $\gamma_{\alpha}\left(1-\gamma^{5}\right) / 2$. We again recall that the Pauli-Villars regularization explicitly conserves a vector current, so that in this regularization an anomalous divergence of a left-hand current

$$
J_{\mu}^{L a}=\bar{\psi} \gamma_{\mu} \frac{1-\gamma^{5}}{2} t^{a} \psi
$$

is the same (aside from the sign) as the divergence of an axial current,

$$
\begin{aligned}
& -\operatorname{Tr} \gamma^{5}(\hat{p}+i M) \gamma_{\alpha} \frac{1-\gamma^{b}}{2}(\hat{p}+\hat{q}+i M) \\
& =-\frac{1}{2}\left[i M \operatorname{Tr} \gamma^{5} \gamma_{\alpha} \hat{q}-\operatorname{Tr}(\hat{p}-i M) \gamma_{\alpha}(\hat{p}+\hat{q}+i M)\right]
\end{aligned}
$$

and the result for the overall diagram is

$$
\begin{equation*}
-\frac{\delta^{a b}}{4 \pi}\left(\varepsilon_{\alpha \beta} i q_{\beta}-q_{\alpha}\right)=\frac{\delta^{a b}}{4 \pi}\left(\delta_{\alpha \beta}-i \varepsilon_{\alpha \beta}\right) q_{\beta} . \tag{2.4}
\end{equation*}
$$

There is a difference in the normalization of the Weyl and Dirac anomalies. In $D=2 n$ dimensions, the leading diagram contains an extra factor of $-1 /(n+1)$ in a Weyl theory. The term without $\varepsilon$ in (2.4) is not truly anomalous but in two dimensions, by virtue of the relation $J_{\alpha}^{5}=i \varepsilon_{\alpha \beta} J_{\beta}$, the field $A$ satisfies the relation $A_{\beta}=\left(\delta_{\alpha \beta}-i \varepsilon_{\alpha \beta}\right) A_{\alpha}$, so that there is no point in discarding this term. In the multidimensional case we do not write out such contributions [they are characterized by the circumstance that one obtains through a gauge transformation an expression which is local in terms of the fields $A$, e.g., $u \partial_{\mu} A_{\mu}=-\delta_{u}(1 / 2) A_{\mu}^{2}$.

We turn now to the calculation of three-tails.
Dirac theory. The external momenta can be immediately set equal to zero. Furthermore, the result for the diagram turns out to be symmetric under an interchange of external lines, and we do not have to be concerned about this point. The expression for the diagram is

$$
\begin{aligned}
& -2 i \frac{j^{a b c}}{4} \int \frac{\operatorname{Tr} M \gamma^{5}(\hat{p}+i M) \gamma_{\alpha}(\hat{p}+i M) \gamma_{\beta}(\hat{p}+i M)}{\left(p^{2}+M^{2}\right)^{3}} \frac{\mathrm{~d}^{2} p}{(2 \pi)^{2}}=
\end{aligned}
$$

We now need to take an average over the directions of the vector $p$, i.e., to make the substitution $p_{\mu} p_{v} \rightarrow p^{2} \delta_{\mu \nu} / 2$. We then need to use the identities

$$
\begin{gather*}
\gamma_{\mu} \gamma_{\alpha} \gamma_{\mu}=(2-D) \gamma_{\alpha}=0, \\
\gamma_{\mu} \gamma_{\alpha} \gamma_{\beta} \gamma_{\mu}=D \delta_{\alpha \beta}+\left(\frac{D}{2}-2\right) \sigma_{\alpha \beta}=+2 \gamma_{\beta} \gamma_{\alpha} . \tag{2.5}
\end{gather*}
$$

We find
$-\frac{f a b c}{2} \operatorname{Tr} \gamma^{5} \gamma_{\alpha} \gamma_{\beta} \int \frac{\mathrm{d}^{2} p}{(2 \pi)^{2}} \frac{M^{4}-M^{2} p^{2}}{\left(p^{2}+M^{2}\right)^{3}}=+i \cdot \frac{1}{4 \pi} f^{a b c} \varepsilon_{\alpha \beta}$.

Weyl theory. Here we need to evaluate the expression
$-2 i \frac{f^{a b c}}{4}$
$\times \int \frac{\operatorname{Tr}\left[M \gamma^{\mathrm{b}}(\hat{p}+i M) \gamma_{\alpha} \frac{1-\gamma^{5}}{2}(\hat{p}-i M) \gamma_{\beta} \frac{1-\gamma^{5}}{2}(\hat{p}--i . M)\right]}{\left(p^{2}+M^{2}\right)^{3}}$
$\times \frac{\mathrm{d}^{2} r^{2}}{(2 \pi)^{2}}$.
We begin by transforming the central structure within the trace: $\quad\left(1-\gamma^{5}\right) \times(\hat{p}+i M) \gamma_{\beta}\left(1-\gamma^{5}\right)=2\left(1-\gamma^{5}\right) \hat{p} \gamma_{\beta}$ carrying ( $1-\gamma^{5}$ ) even further through to the left, we find $\gamma^{5}(\hat{p}+i M) \gamma_{\alpha}\left(1-\gamma^{5}\right)=\gamma^{5}(\hat{p}+i M) \gamma_{\alpha}-(\hat{p}-i M) \gamma_{\alpha}$. As a result, the original expression is proportional to $\left[\gamma^{5}(\hat{p}+i \boldsymbol{M}) \gamma_{\alpha}-(\hat{p}-i \boldsymbol{M}) \gamma_{\alpha}\right] \hat{p} \gamma_{\beta} \hat{p}(\hat{p}+i \boldsymbol{M})$. Using relation (2.5), we easily see that this expression vanishes.

At $D=2$ the anomaly in the Dirac theory is thus
$\left(\mathrm{D}_{\mu} J_{\mu}^{5}\right)^{a}=\frac{1}{2 \pi} \varepsilon_{\alpha \beta}\left(\partial_{\alpha} A_{\beta}^{n}+\frac{i}{2} f^{a b c} A_{\alpha}^{b} A_{\beta}^{c}\right)=\frac{1}{\pi} \varepsilon_{\mu \nu} \operatorname{Tr} t^{a} \frac{F_{\mu v}}{2}$,
while that in the Weyl theory is

$$
\begin{align*}
\left(\mathrm{D}_{\mu} J_{\mu}^{L}\right)^{L} & =\frac{1}{4 \pi}\left(\varepsilon_{\alpha \beta} \partial_{\alpha} A_{\beta}^{a}+\partial_{\alpha} A_{\alpha}^{a}\right) \\
& =\frac{1}{2 \pi}\left(\varepsilon_{\mu \nu} \operatorname{Tr} t^{a} \partial_{\mu} A_{\nu}+\operatorname{Tr} t^{a} \partial_{\mu} A_{\mu}\right) \tag{2.8}
\end{align*}
$$

Diagrams for an anomaly in spaces of any dimensionality can be calculated in a similar way, but such calculations are exceedingly tedious and hardly worth the trouble (although four-dimensional anomalies were originally found in precisely this way). We turn now to a method for calculating anomalies directly. It was proposed by Vergeles (but not published; see Ref. 16) and then developed in a series of papers by Fujikawa. ${ }^{17}$ Romanov and Shvarts ${ }^{15}$ took a similar approach. The use of this method to calculate non-abelian Weyl ${ }^{17-19}$ and gravitational anomalies has been the subject of many papers.

### 2.2. Calculation of anomalies by the Vergeles-Fujikawa method

The original idea of Vergeles was that an anomaly could be interpreted as the result of an invariance of the measure in a path integral. Specifically, if in the expression

$$
\left.I \int D \bar{\psi} D \psi \exp \left(i \int \bar{\psi} \hat{D} \psi \mathrm{~d}^{2 n} x\right)\right|_{\mathrm{reg}}
$$

we make the change of variables

$$
\begin{equation*}
\psi \rightarrow e^{i \varepsilon \gamma^{5}} \psi \tag{2.9}
\end{equation*}
$$

then we find in the integral first $\exp \left(i \operatorname{Tr} \varepsilon \mathrm{D}_{\mu} J_{\mu}^{5}\right)$, and second the Jacobian ( $Y^{(\varepsilon)}$ ) of transformation (2.9). If we require that the integral not change upon a change in integration variable, we find the relation

$$
\begin{equation*}
\mathrm{D}_{\mu} J_{\mu}^{\mathrm{\llcorner }}=\left.\frac{\delta Y^{(\varepsilon)}}{\delta \varepsilon}\right|_{\varepsilon=0}, \tag{2.10}
\end{equation*}
$$

In other words, an anomalous divergence of a current is unambiguously related to the nontrivial nature of the Jacobian $Y^{(\epsilon)}$. "Naively" we have $Y^{(\varepsilon)}=1$, but strictly speaking we need first to regularize the Jacobian:

$$
\begin{aligned}
\left(\ln Y^{(\mathrm{E})}\right)_{\mathrm{reg}} & =\left(\ln \operatorname{det} e^{2 i \varepsilon \gamma^{5}}\right)_{\mathrm{reg}} \\
& =\left(\mathrm{Sp} 2 i \varepsilon \gamma^{5}\right)_{\mathrm{reg}}=\lim _{M \rightarrow \infty} \mathrm{Sp}\left(2 i \varepsilon \gamma^{5}\right) e^{\hat{\mathrm{D}^{2} / M^{2}} ;}
\end{aligned}
$$

here we have used a "proper-time" regularization. The 2 results from the fact that both fields $\psi$ and $\bar{\psi}$ change under transformation (2.9). We turn now to a calculation of the regularized determinant. We first consider the $\gamma$-matrix trace. For this purpose we note that we have

$$
\begin{aligned}
\hat{\mathrm{D}}^{2} & =\mathrm{D}_{\mu} \mathrm{D}_{v}\left(\delta_{\mu \nu}+\sigma_{\mu v}\right) \\
& =\mathrm{D}^{2}+\frac{1}{2} F_{\mu v} \sigma_{\mu v} \quad\left(\sigma_{\mu v}=\frac{1}{2}\left(\gamma_{\mu} \gamma_{v}-\gamma_{\nu} \gamma_{\mu}\right)\right) .
\end{aligned}
$$

If the trace with the matrix $\gamma^{5} \sim \gamma_{0} \gamma_{1} \ldots \gamma_{2 n-1}$ in $2 n$ dimensions is to be nonzero, we need to take at least the $n$th term of the expansion of the exponential function in a Maclaurin series:

$$
\begin{align*}
& 2 \mathrm{Sp} i \varepsilon \gamma^{5} \exp \left(\mathrm{D}^{2}+\frac{1}{2} F_{\mu \nu} \sigma_{\mu \nu}\right) M^{2} \\
& =\ldots+\frac{2}{n!} \operatorname{Sp}\left[i \varepsilon \gamma^{5}\left(\frac{F_{\mu \nu} \sigma_{\mu \nu}}{2 M^{2}}\right)^{n}\right] e^{\mathrm{D}^{2} / M^{2}}+O\left(\frac{1}{M^{2 / L+2}}\right) . \tag{2.11}
\end{align*}
$$

The remaining functional trace in momentum space incorporates an integration over $\mathrm{d}^{2 n} p /(2 \pi)^{2 n}$ :

$$
\int \frac{d^{2 n} p}{(2 \pi)^{3 n}} e^{-p^{2 / M^{2}}}=\frac{M^{2 n} \Omega_{2 n-1}(n-1)!}{2(2 \pi)^{2 n}}
$$

The $M^{2 n}$ which appears in the numerator cancels the $1 / M^{2 n}$ in (2.11), but this cancellation is not sufficient to make the contributions $O\left(1 / M^{2 n+2}\right)$, which stem from the higherorder terms in the expansion of the exponential function, finite in the limit $M \rightarrow \infty$. Furthermore, we need to know the area of a unitsphere $S^{2 n-1}: \Omega_{2 n-1}=(2 \pi)^{n} / 2^{n-1}(n-1)$ ! [for $S^{2 n} \Omega_{2 n}=2(2 \pi)^{n} /(2 n-1)!!$ ]. As a result we find

$$
\begin{align*}
\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{b}}\right)^{a} & =\left.\frac{\delta Y^{(\varepsilon)}}{\delta \varepsilon}\right|_{\varepsilon=0} \\
& =\frac{2^{n} \Omega_{2 n-1}}{n(2 \pi)^{2^{n}}} \operatorname{Tr} t^{a} \frac{\varepsilon_{\mu_{1} v_{1} \ldots \mu_{n} v_{n}} F_{\mu_{1} v_{1}} \ldots F_{\mu_{n} v_{n}}}{2^{n}} \\
& =\frac{2^{2 n} \Omega_{2 n-1}}{n(2 \pi)^{2 n}} \operatorname{Tr} t^{1} F^{n}=\frac{2}{(2 \pi)^{n} n!} \operatorname{Sp} t^{a} F^{n} . \tag{2.12}
\end{align*}
$$

The last two equalities are written in terms of differential forms (see Subsection 3.1 below); here $F_{\mu}=\partial_{\mu} A_{\text {, }}$ $-\partial_{,} A_{\mu}+\left[A_{\mu} A_{v}\right]$, while the form $F$ is $F=d A+A^{2}$. An additional factor of $2^{\prime \prime}$ is associated with this circumstance. Yet another factor of $2^{\prime \prime}$ occurs because the dimensionality of the $\gamma$ matrices in a $2 n$-dimensional space is $2^{\prime \prime} \times 2^{\prime \prime}$.

Two comments are in order here.
a) It is not absolutely necessary to use specifically an exponential function in the regularization. We could write any arbitrary function $f\left(-\hat{\mathrm{D}}^{2} / M^{2}\right)$ which falls off more rapidly than a polynomial at infinity. The result includes an integral of the $n$th derivative of this function:

$$
\begin{aligned}
\frac{1}{n!} \int f^{(n)}\left(\frac{p^{2}}{M^{2}}\right) \frac{\mathrm{d}^{2 n} p}{M^{2 n}} & =\frac{1}{n!} \frac{\Omega_{2 n-1}}{2} \int f^{(n)}(x) x^{n-1} \mathrm{~d} x \\
& =\frac{\Omega_{2 n-1}}{2 n} f(0)
\end{aligned}
$$

Consequently, all that we require of the function $f$ is that its derivatives fall off sufficiently rapidly at infinity and also the condition $f(0)=1$ : This function becomes equal to unity at an infinite regulator mass, i.e., when the regularization is removed. Actually, it is sufficient that the condition $f^{(n)}(x) \leqslant$ const $/(1+x)^{n+\eta}$ hold in the limit $x \rightarrow \infty(\eta>0)$.
b) The derivation of (2.12) turned out to be as simple as it was because only a single term in the expansion of the regularizing function was important. This situation is explained in turn by the absence of an explicit dependence of the coefficients of the matrices $\sigma_{\mu v}$ on the momenta. Here they have turned out to be equal to $c$-numbers $(1 / 2) F_{\mu r}$. In a Weyl theory this is no longer the case: The $\sigma_{\mu \nu}$ are multiplied by not only $c$-number expressions but also by, say, the $q$ number $i A_{\mu} \partial_{v} \rightarrow A_{\mu} p_{v}$. The appearance of a momentum here makes it possible for the following terms in the expansion of the regularizing function to survive as $M$ goes to infinity; the result is to complicate the calculations in an exceedingly severe way. In other words, while the Vergeles-Fujikawa method has an indisputable advantage over a simple calculation of diagrams in an application to Dirac anomalies, in the case of Weyl anomalies this advantage is less obvious. We will discuss two examples in this Subsection 2.2: the calculation of a two-dimensional Weyl anomaly and the calculation of the coefficient of the leading term in the expression for the Weyl anomaly in an arbitrary dimensionality. ("Leading term" here is understood as the structure $\varepsilon_{\alpha_{1}, \ldots \alpha_{\ldots n}} \operatorname{Tr} t^{a} \partial_{\alpha_{1}} A_{\alpha_{2}} \ldots \partial_{\alpha_{2},}, A_{\alpha_{2},}$, which contains no matrix commutators.) While the first of these two examples will give the reader the chance to evaluate the complexity of the method in application to Weyl anomalies, the second example has a more "practical" purpose: A calculation of the Weyl anomaly on the basis of the hierarchy of anomalies in Subsection 2.4 and Section 3 does not make it possible to determine the overall normalization of the anomaly. We find this normalization in our second example.

### 2.2.1. Two-dimensional Weyl anomaly

The difference between the Dirac and Weyl theories stems from the replacement of the operator $\hat{\mathrm{D}}=\hat{\partial}+\hat{A}$ in the Lagrangian by $\overline{\mathrm{D}} \equiv \hat{\partial}+\left[\hat{A}\left(1-\gamma^{5}\right) / 2\right]$. Accordingly, in a calculation of a Weyl anomaly by the Vergeles-Fujikawa method we find an expression

$$
\begin{align*}
\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{L}}\right)^{a} & =2\left[\mathrm{D}_{\mu}\left(\psi \gamma_{\mu} \frac{1-\gamma^{5}}{2} t \psi\right)\right]^{a}=\left.2 \frac{\delta Y^{(\varepsilon)}}{\delta \varepsilon}\right|_{\varepsilon=0} \\
& =\left.2 \frac{\delta}{\delta \varepsilon} \operatorname{Sp}\left[i \varepsilon\left(1-\gamma^{5}\right)\right]_{\mathrm{reg}}\right|_{\varepsilon=0} \tag{2.13}
\end{align*}
$$

Using the same regularization as in the Dirac case, we find

$$
\begin{equation*}
\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{L}}\right)^{a}=2 \lim _{M \rightarrow \infty} \mathrm{Sp} t^{a}\left(1-\gamma^{5}\right) e^{\widetilde{\mathrm{D}^{2}} / M^{2}} \tag{2.14}
\end{equation*}
$$

We now consider the operator $\widetilde{\mathrm{D}}^{2}$ :

$$
\widetilde{D}^{2} \equiv \frac{1-\gamma^{5}}{2} \hat{\partial} \hat{D}+\frac{1+\gamma^{5}}{2} \hat{\mathrm{D}} \hat{\partial}
$$

where $\hat{\mathrm{D}}=\hat{\partial}+\hat{A}$ is the ordinary Dirac operator. The trace which must be calculated is therefore

$$
2 \mathrm{Sp} t^{a}\left(1-\gamma^{5}\right) e^{\hat{a} \hat{\mathrm{D}} / \mathrm{M}^{2}}
$$

As we have already mentioned, the appearance of a structure $A_{\mu} \partial_{\mu}-\sigma_{\mu v} A_{\mu} \partial_{v} \quad$ in $\quad \hat{\partial} \widehat{\mathrm{D}}=\partial \mathrm{D}+\left[\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{v}\right]$ $\sigma_{\mu v}\left(\partial \mathrm{D}=\partial_{\mu} \mathrm{D}_{\mu}=\partial^{2}+(\partial A)+A \partial\right)$ is an important point. We turn now to the consequences of this modification.

We begin by calculating only the "truly anomalous" part of the trace, which contains a $\gamma^{5}$ matrix:

$$
\begin{equation*}
2 \mathrm{Sp} t^{a}\left(1-\gamma^{5}\right) e^{\hat{\partial} \hat{D} / M^{2}} \rightarrow-2 \mathrm{Sp} t^{a} \gamma^{5} e^{\hat{\partial} \hat{D}_{j} / M^{2}} \tag{2.15}
\end{equation*}
$$

As before, the $\gamma$-matrix trace with $\gamma^{5}$ is nonzero only if at least $n$ of the matrices $\sigma_{\mu v}$ drop out of the exponential function. The trace with the larger number $\sigma_{\mu v}$, however, is also nonzero. In the Dirac case, the higher-order terms in the expansion vanish in the limit $M \rightarrow \infty$. The reason is that the exponential function contains an expression $-F \sigma / 2 M^{2}$ and integrals

$$
\int \mathrm{d}^{2 n} p e^{-p^{2} / M^{2}}\left(-\frac{F \sigma / 2}{M^{2}}\right)^{h} \sim O\left(\frac{1}{M^{2}(\bar{k}-\bar{n})}\right)
$$

In the Weyl case, the exponential function has terms which are linear in $p$ :

$$
-\frac{i(A p-\sigma A p)+(\partial A+\sigma \partial A)}{M^{2}}
$$

The behavior of the $k$ th term of the expansion is now

$$
\begin{aligned}
& \int^{2 n} p e^{-p^{2} / M^{2}}\left[-\frac{i(A p-\sigma A p)+(\partial A+\sigma \partial A)}{M^{2}}\right]^{k} \\
& \sim O\left(\int \mathrm{~d}^{2 n} p e^{-p^{2} / \mathrm{M}^{2}} \frac{p^{k}}{M^{2 h}}\right) \sim O\left(\frac{1}{M^{k-2 n}}\right)
\end{aligned}
$$

so that terms of the expansion with integer values of $n$, from $n$ to $2 n$, can survive in the limit $M \rightarrow \infty$. In a specific calculation we need to consider two other points: the accurate arrangement of the differentiation operators (which do not commute with the fields $A$ ) and the $\gamma$-matrix traces. In particular, the structure of these traces is such that after an average is taken over the directions of the vector $p$ the terms of the expansion from $n$ to $2 n-1$ lead to integrals $\sim O(1)$, while the contribution of the $2 n$th term vanishes. As for the ordering of the derivatives, we first write the entire exponential function, including the highest-order term $\partial^{2} / M^{2}$, in a series. We then carry all the operators $\partial$ through to the right, and we then recollect factors $e^{\partial 2} / M^{2}$. The calculation for $D=2$, for example, is as follows:

$$
\begin{align*}
& 2 \operatorname{Sp} t^{a} \gamma^{\mathrm{s}} e^{\hat{\partial} \hat{D} \hat{\rho} / M^{2}}=2 \operatorname{Sp} t^{a} \gamma^{5}\left[\frac{\hat{\partial} \hat{\mathrm{D}}}{M^{2}}+\frac{1}{2!}\left(\frac{\hat{\partial} \hat{\mathrm{D}}}{M^{2}}\right)^{2}+\ldots\right] \\
& \begin{aligned}
&=2 \operatorname{Sp} t^{a} \gamma^{5}\left\{\frac{\sigma_{\mu \nu}\left(\partial_{\mu} A_{v}\right)}{M^{2}}+\frac{1}{2!M^{4}}\left[\frac{\partial^{2} \sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{\nu}\right)+\sigma_{\mu v}\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{v}\right) \partial^{2}}{+\sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{\nu}\right)}\left(\left(\partial_{\alpha} A_{\alpha}\right)+A_{\alpha} \partial_{\alpha}\right)+\left(\left(\partial_{\alpha} A_{\alpha}\right)+A_{\alpha} \partial_{\alpha}\right) \sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{\nu}\right)\right.\right. \\
&\left.\left.+\sigma_{\mu v}\left(\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{\nu}\right) \sigma_{\alpha \beta}\left(\left(\partial_{\alpha} A_{\beta}\right)-A_{\alpha} \partial_{\beta}\right)\right]+\cdots\right\} .
\end{aligned}
\end{align*}
$$

We begin with the expression which we have underscored. We put it in the form
$\frac{1}{2!M^{4}}\left\{2 \sigma_{\mu v}\left[\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{v}\right\} \partial^{2}+\left\{\partial^{2} ; \sigma_{\mu v}\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{v}\right)\right]\right\}$.

In exactly the same way, we run into a combination
$\frac{1}{3!M^{6}}\left\{3 \sigma_{\mu \nu}\left\lfloor\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{\nu}\right\rfloor \partial^{1}+\right.$ commutator terms $\}$
in the third term of the expansion of the exponential function, etc. All these structures are summed in the expression

$$
\begin{equation*}
\left\{\frac{\sigma_{\mu v} \partial_{\mu} A_{v}}{M^{2}} e^{\partial z / M^{2}}+\text { commutator terms }\right\} \tag{2.19}
\end{equation*}
$$

(The terms $\sigma_{\mu \nu}, A_{\mu} \partial_{v} \cdot \partial^{2}$, etc., have been omitted; they disappear when an average is taken over the directions of the vector $p: \sigma_{\mu v} A_{\mu} \partial_{v} \partial^{2} \rightarrow-i \sigma_{\mu}, A_{\mu} p_{v} p^{2} \rightarrow 0$.) Let us ignore the commutator terms for a moment and return to (2.16). The first term of the expansion, part of the underscored expression, etc., are written in the form in (2.19):

$$
2 \mathrm{Sp} \gamma^{5} t^{2} \frac{\sigma_{\mu v} \partial_{\mu} A_{v}}{M^{2}} e^{\sigma^{2} / M^{2}}
$$

In the momentum representation this expression becomes

$$
\begin{equation*}
2 \int \frac{\mathrm{~d}^{2} p}{(2 \pi)^{2}} e^{-p^{2} / M^{2}} \operatorname{Tr} t^{a} \gamma^{5} \frac{\sigma_{\mu v} \dot{j}_{\mu} A_{v}}{M^{2}}==\frac{1}{\pi}-\varepsilon_{\mu v} \operatorname{Tr} t^{a} \partial_{\mu} A_{v} \tag{2.20}
\end{equation*}
$$

We now turn to the commutator terms in (2.17) and (2.18). They also make a finite contribution to the anomaly. Here, however, we need to carry the maximum number of operators $\partial$ through to the right. We then have

$$
\frac{\partial_{\alpha} \partial_{\beta}}{M^{2}} \rightarrow-\frac{p_{\alpha} p_{\dot{\beta}}}{M^{2}} \rightarrow-\frac{1}{2 M^{2}} p^{2} \delta_{\alpha \beta},
$$

and after an integration over $\mathrm{d}^{2} p /(2 \pi)^{2}$ with a weight $e^{-\rho^{2} / M^{2}}$ we find the final result in the limit $M \rightarrow \infty$ :

$$
-\int \frac{p^{2} \delta_{\alpha \beta}}{2 M^{4}} e^{-p^{2} / M^{2}} \frac{d^{2} p}{(2 \pi)^{2}}=-\frac{1}{8 \pi} \delta_{\alpha \beta} .
$$

In this manner, we find from $\left[\partial^{2} ; \sigma_{\mu v}\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{v}\right)\right]$ in (2.17)

$$
\begin{align*}
& 2 \frac{1}{2!M^{4}}
\end{align*} \quad \operatorname{Sp} \gamma^{5} t^{\alpha}\left(-2 \sigma_{\mu \nu}\left(\partial_{\alpha} A_{\mu}\right) \partial_{v} \partial_{\alpha}\right) .
$$

The commutator terms in (2.18) and in the following terms of the expansion lead simply to the appearance of the factor
$e^{\partial^{2} / M^{2}}$ in the expressions which are written out explicitly in (2.16); all the other contributions vanish in the limit $M \rightarrow \infty$. The factors $e^{\partial^{2} / M^{2}}$ have already been taken into account in our calculations. We thus find two contributions to the twodimensional Weyl anomaly, (2.20) and (2.21):

$$
\begin{align*}
\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{L}}\right)^{a} & =\frac{1}{\pi} \varepsilon_{\mu v} \operatorname{Tr} t^{a} \partial_{\mu} A_{v}-\frac{1}{2 \pi} \varepsilon_{\mu v} \operatorname{Tr} t^{a} \partial_{\mu} A_{v} \\
& =\frac{1}{2 \pi} \varepsilon_{\mu v} \operatorname{Tr} t^{a} \partial_{\mu} A_{v} \tag{2.22}
\end{align*}
$$

As a result, the normalization of the right side differs from that in the Dirac case. We shall see below that in $2 n$ dimensions the normalization differs by a factor of $n+1$.

However, we have not yet completed our calculation of the two-dimensional anomaly. Why do we need the other terms (not underscored) in (2.16)? As we have already mentioned, the only terms which are important in the limit $M \rightarrow \infty$ are those in which the derivatives are carried through the fields $A$ without acting on the latter. In the remaining part of (2.16) there are several terms of this type:

$$
\begin{aligned}
& -\gamma^{5} \sigma_{\mu \nu}\left(A_{\mu} \partial_{\nu} A_{\alpha} \partial_{\alpha}+A_{\alpha} \partial_{\alpha} A_{\mu} \partial_{\psi}\right)+\gamma^{5} \sigma_{\mu \nu} \sigma_{\alpha \beta} A_{\mu} \partial_{\nu} A_{\alpha} \partial_{\beta} \\
& \quad \rightarrow-\gamma^{5} \sigma_{\mu \nu}\left(A_{\mu} A_{\alpha} \partial_{\nu} \partial_{\alpha}+A_{\alpha} A_{\mu} \partial_{\alpha} \partial_{\nu}\right) \\
& \quad+\gamma^{5} \sigma_{\mu v} \sigma_{\mu \beta} A_{\mu} A_{\alpha} \partial_{\nu} \partial_{\beta} .
\end{aligned}
$$

Their contributions to the anomaly cancel each other out.
$\operatorname{Tr} t^{a}\left[-\gamma^{5} \sigma_{\mu \nu}\left(A_{\mu} A_{\nu}+A_{\nu} A_{\mu}\right)+\gamma^{5} \sigma_{\mu \nu} \sigma_{\alpha \nu} A_{\mu} A_{\alpha}\right]$
$=2 \operatorname{Tr} t^{a}\left[-\varepsilon_{\mu \nu}\left(A_{\mu} A_{\nu}+A_{\nu} A_{\mu}\right)+0\right]=0$.
As a result, structures of the type $\varepsilon_{\mu v} A_{\mu^{\prime}} A_{v}$ do not occur in the case of a two-dimensional Weyl anomaly, and (2.22) is the complete result. Nevertheless, in higher dimensionalities contributions of similar origin do survive and do contribute to the Weyl anomaly. In all cases, the only structures which cancel out are those which are constructed from the same fields $A$, without derivatives $\partial A$.

It is a very simple matter to understand the latter assertion. Terms without derivatives arise when all the differentiation operators in $\hat{\partial} \widehat{\mathrm{D}}$ are replaced by momenta: $\hat{\partial} \widehat{\mathrm{D}} \rightarrow-p^{2}+i A_{\alpha} p_{\alpha}+i \sigma_{\alpha \beta} A_{\beta} p_{\alpha}$. In $2 n$ dimensions, the $2 n$th term of the expansion of the exponential function in a Maclaurin series is important here:

$$
\begin{aligned}
& \sim \int \frac{d^{2 n} p}{(2 \pi)^{2 n}} e^{-p^{2} / M^{2}} \operatorname{Tr} \gamma^{5} t^{a} \frac{\left(A_{\alpha} p_{\alpha}+\sigma_{\alpha \beta} A_{\beta} p_{\alpha}\right)^{2 n}}{M^{12 n}} \\
& =\int \frac{d^{2 n} p}{(2 \pi)^{2 n}} \frac{e^{-p^{2} / M^{2}}}{M^{4 n}} \operatorname{Tr} t^{\hat{a}} \hat{p} \gamma_{\beta_{1}} \ldots \hat{p} \gamma_{\beta_{2 n}} A_{\beta_{1}} \ldots A_{\beta_{2 n}}
\end{aligned}
$$

The $\gamma$-matrix trace is proportional to

$$
\begin{aligned}
\operatorname{Tr} \gamma^{5} \hat{p} \gamma_{\beta_{1}} \ldots \hat{p} \gamma_{\beta_{2 n}} \sim & \varepsilon_{\beta_{1} \ldots \beta_{2 n}} \operatorname{Tr} \hat{p^{2 n}} \\
& -2 n \varepsilon_{\alpha \beta_{2} \ldots \beta_{2 n}} p_{\alpha} \operatorname{Tr} \dot{p^{2 n-1}} \gamma_{\beta_{1}} \\
= & 2^{n}\left(p^{2 n} \varepsilon_{\beta_{1} \ldots \beta_{2 n}}-2 n \varepsilon_{\alpha \beta_{2} \ldots \beta_{2 n}} p_{\alpha} p^{2 n-2} p_{\beta_{1}}\right)
\end{aligned}
$$

and this difference vanishes after an average is taken over the directions of the vector $p: p_{\alpha} p_{\beta} \rightarrow(1 / 2 n) p^{2} \delta_{\alpha \beta}$. The absence of terms $\varepsilon_{\alpha_{1} \ldots \alpha_{2} n} \operatorname{Tr} A_{\alpha_{1}} \ldots A_{\alpha_{2} n}$ in the expression for the Weyl anomaly is an extremely important point: It serves as an indication that this expression is a "total derivative," and the operation $k_{z}$ can be applied to it (Subsection 3.3).

### 2.2.2. Leading term in the expression for a Weyl anomaly

We can now show how the term

$$
\frac{1}{n!} \varepsilon_{\alpha_{1} \ldots \alpha_{2 n}} \operatorname{Tr} t^{a} \partial_{\alpha_{1}} A_{\alpha_{2}} \ldots \partial_{\alpha_{2 n-1}} A_{\alpha_{2 n}}
$$

which arises in the expression for a Dirac anomaly is replaced by

$$
\frac{1}{(n+1)!} \varepsilon_{\alpha_{1} \ldots \alpha_{2 n}} \operatorname{Tr} t^{a} \partial_{\alpha_{1}} A_{\alpha_{2}} \ldots \partial_{\alpha_{2 n-1}} A_{\alpha_{2 n}}
$$

in the case of Weyl anomaly. We have already demonstrated the appearance of $1 / 2$ in two dimensions ( $n=D / 2=1$ ). We will not discuss the normalization of the other non-abelian terms $\left(\sim \varepsilon_{\alpha_{1}, \ldots \alpha_{2} n} \operatorname{Tr} t^{a} A_{\alpha_{1}} A_{\alpha_{2}} \partial_{\alpha_{3}} A_{\alpha_{4}} \ldots \partial_{\alpha_{2} n,} A_{\alpha_{2} n}\right.$, etc.) at $n \neq 1$ in this section.

All the important aspects of the calculation were already discussed in the preceding example. We now need to formalize them slightly. A difference between Dirac and Weyl anomalies is that the expression $\hat{D}^{2}=D^{2}$ $+(1 / 2) F_{\mu \nu} \sigma_{\mu \nu}$ in the former case is replaced by $\hat{\partial} \hat{\mathrm{D}}=\partial^{2}+(\partial A)+A \partial+\sigma_{\mu \nu} \times\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{v}\right)$ in the latter case. Since we have decided to consider only the leading abelian term in the expression for the anomaly, we can replace $F_{\mu \nu} \sigma_{\mu \nu} / 2$ by $\sigma_{\mu \nu}\left(\partial_{\mu} A_{\nu}\right)$. The difference between the two anomalies is then related to the presence of "free" derivatives in the Weyl case. It is convenient to work with these derivatives, writing $\partial_{\mu}+i p_{\mu}$ in place of $\partial_{\mu}$. An average is carried out over the vector $p_{\mu}$, while the $\partial_{\mu}$, carried through all the $A$ to the right, are assumed to be zero for this choice of actions. We then have

$$
\begin{align*}
\hat{\partial} \hat{\supset} & +p^{2} \rightarrow\left(2 i p \partial+\partial^{2}+(\partial A)+A_{\alpha} \partial_{\alpha}+i A_{\alpha} p_{\alpha}\right) \\
& +\sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{\nu}+i p_{\mu} A_{\nu}\right) . \tag{2.23}
\end{align*}
$$

In deriving a Weyl anomaly, this operator is raised to the $k$ th power, divided by $M^{2 i} k!$, and integrated over $d^{2 n} p$ with a weight $\exp \left(-p^{2} / M^{2}\right)$. If the trace with the $\gamma^{5}$ matrix is to be nonvanishing, the power of $k$ must be no smaller than $n$. If the result is not to vanish in the limit $M \rightarrow \infty$, precisely $2(k-n)$ momenta $p$ must "operate" in $\left[\left(2 i p_{\mu} \partial_{\mu}+\partial^{2}+(\partial A)+A_{\mu} \partial_{\mu}+i A_{\mu} p_{\mu}\right)+\sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{v}\right)\right.\right.$ $\left.\left.-A_{\mu} \partial_{\nu}+i p_{\mu} A_{v}\right)\right]$. Since we are following the contribution $\sim(d A)^{n}$, and a field $A$ is necessarily associated with each matrix $\sigma_{\mu \nu}$ in (2.23), we need to consider only those terms which contain $\sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{v}+i p_{\mu} A_{v}\right)$ precisely $n$
times. We now recall that after we take the $\gamma$-matrix trace the quantity $\sigma_{\mu}$, converts into an $\varepsilon$-symbol. We can thus take $i \sigma_{\mu \nu} p_{\mu} A_{\nu}$ no more than once, for otherwise the convolution of the momenta $p$ with the $\varepsilon$-symbol will give us zero. Consequently, the remaining $k-n$ cofactors without $\sigma_{\mu v}$ matrices must contain at least $2(k-n)-1$ momenta $p$. On the other hand, each of these cofactors contains $p$ raised to at least the first power. From $2(k-n)-1 \leqslant k-n$ we find $k=n$ or $k=n+1$. In other words, a contribution $\sim(\partial A)^{n}$ to a Weyl anomaly is found in the Vergeles-Fujikawa method from only two terms of the expansion of the exponential function:
$\frac{1}{n!} \operatorname{Tr} 2 \gamma^{5} t^{a}\left(\sigma_{\mu \nu} \partial_{\mu} A_{\nu}\right)^{n} \int \frac{\mathrm{~d}^{2 n} p}{(2 \pi)^{2 n}} \frac{e^{-\mathrm{p}^{2} / M^{2}}}{M^{2 n}}$
$\not-\frac{1}{(n+1)!} \operatorname{Tr} 2 \gamma^{5} t^{n} \int\left[2 i_{p_{\mu}} \partial_{\mu}+\sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{\nu}+i p_{\mu} A_{v}\right)\right]$
$\times \frac{\mathrm{d}^{2 n} p}{(2 \pi)^{2 n}} \frac{e^{-p^{2} / M^{2}}}{M^{2 n+2}}$.
In the Dirac case we have only the first of these two terms, $\left[2 /(2 \pi)^{n}!\right] \operatorname{Tr} t^{a} \mathrm{~d} A^{n}$. In the second term, only the two factors with $p$ are important, as we have already mentioned; the other $n-1$ factors do not contain $p$. After an average is taken over the directions of $p$ we find $2 i p_{\alpha} i p_{\beta} \rightarrow-2 \delta_{\alpha \beta} p^{2} /$ $2 n$, and the integral over $d^{2 n} p$ is, when we take the factor of $2 /(n+1)$ ! into account,

$$
-\frac{2 \delta_{\alpha \beta}}{2 n} \frac{2 n}{(2 \pi)^{n}(n+1)!}=-\frac{2}{(2 \pi)^{n}(n+1)!} .
$$

We will show below that a combination of fields $A$ which is equal to $n \operatorname{Tr} t^{a} \mathrm{~d} A^{n}$ also arises when, in collecting all these factors and terms, we find the following expression for the leading contribution to the Weyl anomaly:

$$
\begin{aligned}
& \frac{2}{(2 \pi)^{n} n!} \operatorname{Tr} t^{a} \mathrm{~d} A^{n}-\frac{2}{(2 \pi)^{n}(n+1)!} n \operatorname{Tr} t^{\alpha} \mathrm{d} A^{n} \\
& =\frac{2}{(2 \pi)^{n}(n+1)!} \operatorname{Tr} t^{a} \mathrm{~d} A^{n} .
\end{aligned}
$$

The origin of $n \operatorname{Tr} t^{a} \mathrm{~d} A^{n}$ can be understood most simply by looking at some examples. We will present these examples without repeating the explanations given earlier:

$$
\begin{aligned}
& n=1: \\
& \operatorname{Tr} t^{a} \gamma^{5}\left[2 i p_{\alpha} \partial_{\alpha}+\sigma_{\mu \nu}\left(\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{\nu}+i p_{\mu} A_{\nu}\right)\right]^{2} \\
& \rightarrow \operatorname{Tr} t^{a} \gamma^{5}\left(2 i p_{\alpha} \partial_{\alpha} \sigma_{\mu v} i p_{\mu} A_{\nu}+\sigma_{\mu \nu} i p_{\mu} A_{v} \cdot 2 i p_{\alpha} \partial_{\alpha}\right) \\
& \rightarrow\left(2 i p_{\alpha} i p_{\mu}\right) \operatorname{Tr} t^{a} \gamma^{5}\left[\sigma_{\mu \nu}\left(\partial_{\alpha} A_{\nu}\right)+0\right] \\
& \rightarrow-\frac{2 p^{2}}{2}\left[\operatorname{Tr} t^{a} \mathrm{~d} A\right] \rightarrow \operatorname{Tr} t^{a} \mathrm{~d} A
\end{aligned}
$$

(the 2 from the $\gamma$-matrix trace was taken into account earlier);

$$
\begin{aligned}
& n=2: \\
& \operatorname{Tr} t^{a} \gamma^{5}\left[2 i p_{\alpha} \partial_{\alpha}+\sigma_{\mu v}\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{\nu}+i p_{\mu} A_{\nu}\right)\right]^{3} \\
& \rightarrow \operatorname{Tr} t^{a} \gamma^{5}\left[2 i_{p_{\alpha}} \partial_{\alpha} \sigma_{\mu v}\left(\left(\partial_{\mu} A_{\nu}\right)-A_{\mu} \partial_{\nu}+i p_{\mu} A_{\nu}\right)\right. \\
& \times \sigma_{\rho v}\left(\left(\partial_{\rho} A_{\sigma}\right)+i p_{\rho} A_{\sigma}\right) \\
& \left.+\sigma_{\mu v}\left(\partial_{\mu} A_{\nu}\right) \cdot 2 i p_{\alpha} \partial_{\alpha} \sigma_{\rho \sigma}\left(\left(\partial_{\rho} A_{\sigma}\right)+i p_{\rho} A_{\sigma}\right)\right] .
\end{aligned}
$$

there is nothing new regarding the second term in the trace: $2 i p_{c r} i p_{\mu} \rightarrow \delta_{c \mu \mu} p^{2}$. The quantity $\partial_{\alpha}$ acts on $\delta_{\alpha \mu} A_{\nu}$. The first term, on the other hand, gives us, in addition to two contributions of the same type, yet another contribution, with a minus sign: $\left.p_{\alpha} \partial_{\alpha}\left(-A_{\mu} \partial_{v}\right) p_{\rho} A_{g}\right) \rightarrow-\left(\partial_{\rho} A_{\mu}\right)\left(\partial_{v} A_{\sigma}\right)$ $\rightarrow-\mathrm{d} A^{2}$. As a result, the overall coefficient of $\operatorname{Tr} t^{a} \mathrm{~d} A^{2}$ is $[(2-1)+1]=2=n$;
$n=3$ :
Here there are $3+2+1$ contributions of the type $p_{\alpha} \partial_{\alpha} p_{\mu} A_{v} \rightarrow\left(\partial_{\mu} A_{v}\right)$,
$(2+1)+1$ contributions of the type $p_{\alpha} \partial_{\alpha}\left(-A_{\mu} \partial_{v}\right)\left(p_{\rho} A_{\sigma}\right) \rightarrow-\left(\partial_{\rho} A_{\mu}\right)\left(\partial_{v} A_{\sigma}\right)$
and 1 contribution $p_{\alpha} \partial_{\alpha}\left(-A_{\mu} \partial_{v}\right)\left(-A_{\rho} \partial_{\sigma}\right) p_{\tau} A_{\pi}$ $\rightarrow+\left(\partial_{\tau} A_{\mu}\right) \times\left(\partial_{t} A_{\mu}\right)\left(\partial_{\sigma} A_{\pi}\right)$.
The complete coefficient of $\operatorname{Tr} t^{a} \mathrm{~d} A^{3}$ is $[(3+2+1)$ $-((2+1)+1)+1]=3=n$.

The combination $(2+1)+1$ has the following origin. We multiply the four identical parenthetical expressions $\left(2 i p \partial+\left(\left(\partial_{\mu} A_{v}\right)-A_{\mu} \partial_{v}-i A_{\mu} p_{v}\right) \sigma_{\mu v}\right)$. In one of these expressions we take a structure $\left(\partial_{\mu} A_{v}\right)$, and in the three others we take, from left to right, $-2 i p \partial,-A_{\mu} \partial_{v}$, and $-i A_{\mu} p_{v}$. The only question here is which three of the expressions of the four are to be selected.

Let us assume that we have taken $2 i p \partial$ from the first parenthetical expression. If we now take $-A_{\mu} \partial_{\nu}$ from the second, we can take $-i A_{\mu} p_{v}$ from the third or the fourth in two ways. If we take $-A_{\mu} \partial_{v}$ from the third, then we can take $-i A_{\mu} p_{v}$, only from the fourth, hence we have $(2+1)$.

We now assume that we have taken $2 i p d$ from the second parenthetical expression. We then have no variants: We can take only $-A_{\mu} \partial_{v}$, from the third expression and $-i A_{\mu} p_{v}$ from the fourth. Again we have +1 , and the total number of variants for the combination $2 i p \partial\left(-A_{\mu} \partial_{v}\right)\left(-i A_{\rho} p_{\sigma}\right)$ is $[(2+1)+1]$.

Working by the same principles, we can calculate the combinatorial factor in the case

$$
\begin{aligned}
& n=4: \\
& 2 i p \partial\left(-i A_{\mu} p_{v}\right):(+)[4+3+2+1], \\
& 2 \operatorname{ip\partial }\left(-A_{\mu} \partial_{\nu}\right)\left(-i A_{\mu} p_{v}\right): \\
& (-)[(3+2+1)+(2+1)+1], \\
& 2 i p \partial\left(-A_{\mu} \partial_{v}\right)\left(-A_{\mu} \partial_{\nu}\right)\left(-i A_{\mu} p_{v}\right): \\
& (+)[((2+1)+1)+1], \\
& 2 i p \partial\left(-A_{\mu} \partial_{v}\right)\left(-A_{\mu} \partial_{v}\right)\left(-A_{\mu} \partial_{\nu}\right)\left(-A_{\mu} p_{v}\right):(-) 1 .
\end{aligned}
$$

Combining all these terms, we find $+4=n$.
It is now simple to see how we find the result for any $n$. Denoting by $S_{n}^{k}$ the coefficient in the $k$ th row, we find the recurrence relations

$$
\begin{aligned}
2 i p \partial\left(-i A_{\mu} p_{v}\right): & S_{n+1}^{1}=S_{n}^{1}+(n+1) \\
2 i p \partial\left(-A_{\mu} \partial_{v}\right)\left(-i A_{\mu} p_{v}\right): & (-)\left\{S_{n+1}^{2}=S_{n}^{1}+S_{n}^{2}\right\}
\end{aligned}
$$

If we take $2 i p \partial$ from the first expression, we can take the two other structures, $-A_{\mu} \partial_{r^{\prime}}$ and $-i A_{\mu} p_{v}$, from the $n$ expressions in $S_{, j}^{1}$ ways. If we instead take $2 i p \partial$ not from the first
expression but from any of the $n$ remaining others, the number of versions is by definition $S_{n}^{2}$. In exactly the same way we find
$2 i p \partial\left(-A_{\mu} \partial_{v}\right)\left(-A_{\mu} \partial_{v}\right)\left(-i A_{\mu} p_{v}\right):(+)\left[S_{n+1}^{3}=S_{n}^{2}+S_{n}^{3}\right] ;$
in general,

$$
S_{n+1}^{k+1}=S_{n}^{k}+S_{n}^{k+1}, \text { with } S_{n}^{0}=(n+1)
$$

We now see that $S_{n}^{k}$ are the binomial coefficients,

$$
S_{n}^{k}=C_{n+1}^{k+1}=\frac{(n+1)!}{(k+-1)!(n-k)!},
$$

and the sum which is to be calculated is

$$
\begin{align*}
\sum_{k=1}^{n} S_{n}^{k}(-)^{k+1} & =\sum_{k=2}^{n+1} C_{n+1}^{k}(-)^{k} \\
& =(1-1)^{n+1}-C_{n+1}^{0}+C_{n+1}^{1}=n . \tag{2.24}
\end{align*}
$$

### 2.3. Relations among the anomalies of fermion currents

An anomaly corresponds to the contribution of regulators to fermion loops which describe the interaction of external currents and gauge fields. All can thus be written in terms of determinants or propagators of regulator fermions.

For an abelian Dirac anomaly in $2 n$ dimensions we have

$$
\begin{equation*}
\left\langle\partial_{\mu} J_{\mu}^{5}\right\rangle=2 \mathrm{Sp} M \gamma^{5} \frac{1}{i \hat{\mathrm{D}}-i M}=\int W^{\left(2^{n i}\right)}(A) \mathrm{d}^{2 n} x . \tag{2.25}
\end{equation*}
$$

For a color Dirac anomaly in $2 n$ dimensions we have
$\left\langle\mathrm{D}_{\mu} J_{\mu}^{5}\right\rangle^{n}=2 \operatorname{Sp} M \gamma^{5} t^{a} \frac{1}{i \hat{\mathrm{D}}-i M}=\int W^{a\left(2^{n}\right)}(A) \mathrm{d}^{2 n} x$.
For a Weyl anomaly in a left-hand current in $2 n$ dimensions we have

$$
\begin{align*}
\left\langle\mathrm{D}_{\mu} J_{\mu}^{L}\right\rangle^{a} & =2 \mathrm{Sp} M \gamma^{5} t^{\prime \prime} \frac{1}{i \widetilde{\mathrm{D}}-i M}=\int W_{1}^{a(2 n)}(A) \mathrm{d}^{2 n} x, \\
\widetilde{\mathrm{D}} & =\hat{\partial}+\hat{A} \frac{1-\gamma^{5}}{2} . \tag{2.27}
\end{align*}
$$

For the anomalous part of a fermion determinant in $2 n+1$ dimensions we have

$$
\begin{equation*}
\{\ln \operatorname{det}(i \hat{\mathrm{D}}-i M)\}_{\mathrm{anom}}=\int W_{0}^{(2 n+1)}(A) \mathrm{d}^{2 n+1} x . \tag{2.28}
\end{equation*}
$$

We already calculated the expressions on the right sides of (2.25)-(2.27) in the preceding subsection. Here we will derive relations between the left-hand sides of the various anomalies, and at the same time we will show that the following holds:

$$
\begin{align*}
& W_{0}^{(2 n+1)}(A) \\
& =\frac{2 \varepsilon_{\alpha_{1}} \ldots \alpha_{2 n+1}}{(2 \pi)^{n}(n+1)!} \operatorname{Tr}\left(A_{\alpha_{1}} \partial_{\alpha_{2}} A_{\alpha_{3}} \ldots \partial_{\alpha_{2 n}} A_{\alpha_{2 n+1}}+\ldots\right) \\
& \cdots \frac{2}{(2 \pi)^{n}(n+1)!} \operatorname{Tr}\left(A \mathrm{~d} A^{n}+\ldots\right) . \tag{2.29}
\end{align*}
$$

The relations between the right sides of the various anomalies in (2.25)-(2.29) are discussed in Subsection 3.4.

We begin with the relations between divergences of the currents and traces. We first consider the law of conservation of an abelian vector current:
$\left\langle\partial_{\mu} J_{\mu}\right\rangle \equiv \operatorname{Sp} \stackrel{+}{i \hat{\partial}} \frac{1}{i \hat{\mathrm{D}}-i m}$-the contribution of regulators.

We should now use the obvious identities

$$
\begin{align*}
& i \hat{\partial} \frac{1}{\hat{i} \hat{\mathrm{D}}-i m}=-(i \hat{A}-i m) \frac{1}{i \hat{\mathrm{D}}-i m}+1,  \tag{2.31}\\
& \frac{1}{i \hat{\mathrm{D}}-i m} i \stackrel{\overleftarrow{\partial}}{ }=+\frac{1}{i \hat{\mathrm{D}}-i m}(i \hat{A}-i m)-1 . \tag{2.32}
\end{align*}
$$

We then have

$$
\begin{aligned}
& \stackrel{+}{+} \mathrm{Sp}_{i \hat{\partial}} \frac{1}{i \hat{\mathrm{D}}-i m}=\mathrm{Sp} i \overrightarrow{\hat{\partial}} \frac{1}{i \hat{\mathrm{D}}-i m}+\mathrm{Sp} \frac{1}{i \hat{\mathrm{D}}-i m} i \stackrel{\stackrel{\hat{\partial}}{ }}{ } \\
& =-\mathrm{Sp}(i \hat{A}-i m) \frac{1}{i \hat{\mathrm{D}}-i m}+\mathrm{Sp} \frac{1}{i \hat{\mathrm{D}}-i m}(i \hat{A}-i m\rangle \\
& =\operatorname{Sp}[-(i \hat{A}-i m)+(i \hat{A}-i m)] \frac{1}{i \hat{\mathrm{D}}-i m}=0 .
\end{aligned}
$$

The contribution of the regulators is also zero.
We now consider an axial anomaly:

$=\operatorname{Sp}\left[-\gamma^{5} i \overrightarrow{\hat{\partial}} \frac{1}{i \hat{\mathrm{D}}-i m}+i \gamma^{5} \frac{1}{i \hat{\mathrm{D}}-i m} \overline{\hat{\partial}}\right]-$ reg.
$=\operatorname{Sp}\left[-2 \gamma^{5}+\gamma^{5}(i \hat{A}-i m) \frac{1}{i \hat{\mathrm{D}}-i m}+\gamma^{5} \frac{1}{i \hat{\mathrm{O}}-i m}(\hat{A}-i m)\right]$
$=-2 i m \operatorname{Sp} \gamma^{5} \frac{1}{i \hat{\mathrm{D}}-i m}$-the contribution of regulators -

$$
\begin{equation*}
\underset{m \rightarrow 0}{\rightarrow}+2 i M \mathrm{Sp} \gamma^{5} \frac{1}{i \hat{\mathrm{D}}-i M}, \tag{2.33}
\end{equation*}
$$

as is confirmed by the first of equalities (2.25).
For non-abelian currents the equations become a bit more complicated:

$$
\left\langle\left(\mathrm{D}_{\mu} J_{\mu}\right)^{a}\right\rangle=\left\langle\partial_{\mu} J_{\mu}^{a}+\left[A_{\mu} J_{\mu}\right]^{a}\right\rangle .
$$

For a vector current we thus have

$$
\left\langle\left(\mathrm{D}_{\mu} J_{\mu}\right)^{a}\right\rangle=\stackrel{+}{\stackrel{+}{\hat{\partial}}} \mathrm{Sp} \frac{1}{i \hat{\mathrm{D}}-i m}+\mathrm{Sp}\left(i f^{a b r} \hat{A}^{b} f^{c}\right) \frac{1}{i \hat{\mathrm{D}}-i m}
$$

We can now write

$$
\begin{gathered}
\stackrel{+}{\hat{\partial}}=\overrightarrow{\hat{\mathrm{D}}}+\stackrel{\leftarrow}{\hat{\mathrm{D}}}, \quad i \overrightarrow{\hat{\mathrm{D}}} \frac{1}{i \hat{\mathrm{D}}-i m}=1+i m \frac{1}{i \hat{\mathrm{D}}-i m} \\
\frac{1}{i \hat{\mathrm{D}}-i m} i \stackrel{\stackrel{\rightharpoonup}{\mathrm{D}}}{ }=-1-i m \frac{1}{i \hat{\mathrm{D}}-i m}
\end{gathered}
$$

As a result, everything is very similar to the situation in the abelian case:

$$
\begin{aligned}
\left\{\left(\mathrm{D}_{\mu} J_{\mu}\right)^{a}\right\rangle= & \mathrm{Sp}\left(\overrightarrow{\left.\hat{\mathrm{D}} t^{\prime \prime} \frac{1}{i \hat{\mathrm{D}}-i m}+t^{a} \frac{1}{i \hat{\mathrm{D}}-i m} \overline{\hat{\mathrm{D}}}\right)}\right. \\
& \div \mathrm{Sp} i f^{i b c} \hat{A}^{b} t^{f} \frac{1}{i \hat{\mathrm{D}}-i m} \\
= & 0+\mathrm{Sp}\left[\hat{\mathrm{D}}, t^{t}\right] \frac{1}{i \hat{\mathrm{D}}-i m}+\mathrm{Sp} i i^{a b c} \hat{A}^{l} t^{c} \frac{1}{i \hat{\mathrm{D}}-i m}
\end{aligned}
$$

The commutator [ $\hat{\mathrm{D}}, t^{a}$ ] is if ${ }^{c b a} t^{c} \hat{A}^{b}=-i f^{a b c} \hat{A}^{b} t^{c}$, and the other two contributions to $\left\langle\left(\mathrm{D}_{\mu} J_{\mu}\right)^{a}\right\rangle$ cancel each other out: $\left\langle\left(\mathrm{D}_{\mu} J_{\mu}\right)^{a}\right\rangle=0$. In exactly the same way, we can verify the first of equalities (2.26) for massless fermions. On the other hand, noting that the appearance of the projection operator $\left(1-\gamma^{5}\right) / 2$ in front of $A$ does not alter the equations used above, we see that an analogous relation (2.27), holds for a Weyl anomaly.

The functions $W, W^{a}$ and $W_{1}^{a}$, which are functions of the fields $A$, and which appear on the right side of (2.25)(2.27), can now be calculated in precisely the same way as in the Vergeles-Fujikawa method in the preceding section. Specifically, we write

$$
i \operatorname{Sp} M \gamma^{5} \frac{1}{i \hat{\mathrm{D}}-i . M}=i M \operatorname{Sp} \gamma^{5} \frac{\hat{\mathrm{D}}-\hat{\mathrm{D}}^{2} M}{-\hat{\mathrm{D}}^{2}-M^{2}} .
$$

The term with $\widehat{\mathbf{D}}$ in the numerator drops out after we take the $\gamma$-matrix trace (since $\hat{\mathrm{D}}^{2}$ contains an even number of $\gamma$ matrices in any integer power). We are thus left with the expression

$$
\operatorname{Sp} \gamma^{5} \frac{1}{1-\left(\hat{\mathrm{D}}^{2} / M^{2}\right)} .
$$

In other words, in the equations in Subsection 2.2 we now need to calculate traces not with a weight of $e^{\hat{D}^{-} / M^{2}}$ but with the weight of $1 /\left[1-\left(\hat{D}^{2} / M^{2}\right)\right]$. We know, however [see the comment after Eq. (2.12)], that when an exponential function is replaced by some other regularizing function no change is caused in the result of calculations by the VergelesFujikawa method.

We turn now to relation (2.28), which holds for gauge theories in an odd dimensionality. The simplest way to discuss the relationship between this anomaly and even-dimensional anomalies is to put the left side of ( 2.28 ) in the same form as in (2.25)-(2.27). We can do this by differentiating (2.28):

$$
\begin{align*}
\delta \ln \operatorname{det}(i \dot{\mathrm{D}}-i .1 I) & =\delta \operatorname{sp} \ln (i \hat{\mathrm{D}}-i M) \\
& =\Delta \mathrm{p} \frac{1}{i \hat{\mathrm{D}}-i . M} \delta(i \hat{\mathrm{D}}-i .1 /) \tag{2.34}
\end{align*}
$$

(The trace symbol makes it possible to arrange the factors in the indicated order.) However, with respect to what quantity should we carry out the differentiation?

A variation with respect to the regulator mass $M$ of any physical quantity, in particular of the anomalous contribution to an effective Lagrangian will of course disappear. [To avoid any misunderstanding, we point out that a variation of nonanomalous parts of an effective Lagrangian may be nonvanishing; e.g., $L_{\text {eff }}(\mu)$ may contain contributions $\sim \ln (\mu)$ $M$ ) which stem from ultraviolet divergences. In such cases,
it is not the effective Lagrangian itself which is physically meaningful but its change under a displacement of the normalization point. We are interested here, however, in only the anomalous terms in $L_{\text {eff }}$, which do not depend on $M$.]

A variation in terms of the mass of a physical fermion, $m$, cannot depend on the presence of regulators. The theorem on the splitting off of heavy fermions, ${ }^{56}$ on the one hand, and the infrared finiteness of perturbation theory on the other, rule out any dependence of physical quantities on the parameter $m / M$. It is thus not surprising that in differentiating with respect to $m$ we lose all the information on the anomalies.

We are left with two other ways to vary $\ln$ det:
-variation with respect to the field $A_{\mu}$;
-variation with respect to some additional parameter on which the field $A_{\mu}$ depends.

Each of these methods turns out to be substantive. The first allows us to relate $W_{0}^{(2 n+1)}$ to $W^{a(2 n)}$, and the second allows us to relate $W_{0}^{(2 n+1)}$ to $W^{(2 n+2)}$. In principle, there is yet another possibility:
-a gauge transformation of the field $A_{\mu}$.
We can thus find the relationship between $W_{0}^{(2 n+1)}$ and $W_{1}^{\alpha(2 n)}$. However, we will not pursue that possibility here.

### 2.3.1. Relationship between $W_{o}^{(2 n+1)}$ and $W^{a(2 n)}$.

A variation of the left side of (2.28) with respect to $A_{0}^{a}$ gives us

$$
\begin{equation*}
\frac{\delta}{\delta A_{0}^{a}} \ln \operatorname{det}(i \hat{\mathrm{D}}-i M)=\operatorname{Sp} \gamma_{0} t^{\prime \prime} \frac{1}{i \hat{\mathrm{D}}-i M} . \tag{2.35}
\end{equation*}
$$

We now choose the gauge $A_{0}=0$, and we note that the matrix $\gamma_{0}$ in $2 n+1$ dimensions is the same as the $\gamma^{5}$ matrix in a $2 n$-dimensional theory. There is the possibility of comparing (2.35) with (2.26) [but not with (2.27), where $\hat{\mathrm{D}}$ is replaced by the different operator $\widetilde{\mathrm{D}}]$. The entire difference between the expressions in these equations lies in the additional factor $M$ in (2.26) and the different value of the symbol Sp : In (2.35) the trace contains an average over a $(2 n+1)$-dimensional space, while the average in (2.26) is over a $2 n$-dimensional space. It turns out that under certain conditions, corresponding to the singling out of the anomalous contribution from (2.35), these two differences cancel each other out:

$$
\begin{align*}
& \mathrm{Sp}_{2 n+1} \gamma_{0} t^{a} \frac{1}{i \partial_{0} \gamma_{0}+i \hat{\mathrm{D}}-i M} \\
& =\mathrm{Sp}_{2 n+1} \gamma_{0} t^{n} \frac{i \partial_{0} \gamma_{0}+i \hat{\mathrm{D}}+i M}{-\partial_{0}^{2}-\mathrm{D}^{2}+M^{2}+\left(F_{\mu \nu} \sigma_{\mu \nu} / 2\right)} \tag{2.36}
\end{align*}
$$

We have explicitly singled out the component $\partial_{0}$ of the derivative $\mathrm{D}_{\mu}\left(A_{0}=0\right)$; D contains the $2 n$ other components. The fermion determinant describes diagrams with an arbitrary external lines, e.g., corrections to the Lagrangian $F_{\mu v}^{2}$, etc. We are interested here not in such corrections but in the anomalous contributions containing an $\varepsilon$-symbol. To distinguish them in (2.36), we need to ignore the dependence of the fields $A_{i}$ on $x_{0}$ and assume $F_{0 i}=0$. The averaging over the coordinate $x_{0}$ can then be carried out in a manner inde-
pendent of the averaging over the other coordinates:

$$
\begin{align*}
& \mathrm{Sp}_{2 n+1} \gamma_{0} t^{a}=\frac{i \partial_{0} \gamma_{0}+i \hat{\mathrm{D}}+i M}{-\partial_{0}^{2}-\mathrm{D}^{2}+M^{2}+\left[\left(F_{\mu v} \sigma_{\mu \nu}\right) / 2\right]} \\
& =\int \frac{\mathrm{d} p_{0}}{2 \pi} \mathrm{Sp}_{2 n} \gamma_{0} t^{t} \frac{p_{0} \gamma_{0}+i \hat{\mathrm{D}}+i M}{p_{0}^{2}-\mathrm{D}^{2}+M^{2}+\left[\left(F_{\mu v} \sigma_{\mu \nu}\right) / 2\right]} \\
& =\int \frac{\mathrm{d} p_{0}}{2 \pi} \mathrm{Sp}_{3 n} \gamma_{i} t^{\pi} \frac{i M}{-\mathrm{D}^{2}+\left(M^{2}+p_{0}^{2}\right)+\left[\left(F_{\mu \nu} \sigma_{\mu \nu}\right) / 2\right]} \tag{2.37}
\end{align*}
$$

The term $p_{0} \gamma_{0}$ is the numerator drops out after the integration over $d p_{0}$, while $\widehat{\mathrm{D}}=\mathrm{D}_{i} \gamma_{i}$ drops out when we take the $\gamma$ matrix trace. The resulting expression is

$$
\begin{align*}
& \int \frac{\mathrm{d} p_{0}}{2 \pi} \frac{M}{2\left(M^{2}+p_{0}^{2}\right)} S p_{2 n} \gamma_{0} t^{a} \frac{2 i\left(M^{2}+p_{0}^{2}\right)}{-\hat{\mathrm{D}}^{2}+\left(M^{2}+p_{0}^{2}\right)} \\
& =\int \frac{\mathrm{d} p_{0}}{2 \pi} \frac{M}{2\left(M^{2}-p_{0}^{2}\right)} \operatorname{Sp} \gamma_{5} t^{a} \frac{2\left(M^{2}+p_{0}^{2}\right)^{1 / 2}}{i \hat{\mathrm{D}}-i\left(M^{2}+p_{0}^{2}\right)^{1 / 2}} . \tag{2.38}
\end{align*}
$$

The remaining trace describes a color Dirac anomaly (2.26), with $M$ replaced by $\left(M^{2}+p_{0}^{2}\right)^{1 / 2}$. However, $M$ is the mass of a regulator on which $W^{a}$ of course does not depend. Accordingly, the trace is simply equal to $W^{u(2 n)}$ and the integral over $d p_{0}$ is the coefficient in front of it. This coefficient is $1 / 4$, and we find the relation

$$
\begin{equation*}
\frac{\delta}{\delta A^{a}} W_{0}^{(2 n+1)}=\frac{1}{4} W^{a(2 n)} \tag{2.39}
\end{equation*}
$$

(we recall that $W_{0}^{(2 n+1)}$ is the anomalous part of the logarithm of a fermion determinant, which is determined entirely by the regulators).

The quantities $\boldsymbol{W}_{0}^{(2 n+1)}$ are none other than the WessZumino terms which arise in odd-dimensional Yang-Mills theories after an integration over fermions. They are implicitly invariant under infinitesimal gauge transformations: the gauge variation of $W$ is nonzero, but it is a total derivative, so that a fermion determinant which includes an integral of $W_{0}^{(2 n+1)}$ is gauge-invariant. As for global gauge transformations which do not reduce to infinitesimal transformations, we note that they can lead to a change $\int W_{0}^{(2 n+1)} d^{2 n+1} x$ see Refs. 41 and 45 and Section 4 of the present review. Gauge invariance is restored when nonlocal contributions to the fermion determinant from light fermions are taken into account. These additional contributions are unimportant in any case in which the scale of the process under study is smaller than the reciprocal of a physical fermion, $1 / \mathrm{m}$. Relation (2.39) makes it a simple matter to write expressions for Wess-Zumino terms by working from an expression which we already have for $W^{a(2 n)}$, i.e., expression (2.12):

$$
\begin{aligned}
W_{0}^{(1)} & =\frac{1}{2} \operatorname{Tr} A, \\
W_{0}^{(3)} & =\frac{1}{4 \pi \cdot 2}\left(A \mathrm{~d} A+\frac{2}{3} A^{3}\right) \\
& =\frac{1}{4 \pi \cdot 2} \varepsilon_{\alpha \beta \gamma} \operatorname{Tr}\left(A_{\alpha} \partial_{\beta} A_{\gamma}+\frac{2}{3} A_{\alpha} A_{\beta} A_{\gamma}\right), \\
W_{0}^{(5)} & =\frac{1}{16 \pi^{2} \cdot 3} \operatorname{Tr}\left(A d A^{2}+\frac{3}{2} A^{3} \mathrm{~d} A+\frac{3}{5} A^{5}\right) \ldots
\end{aligned}
$$

The factors $2,3, \ldots$ in the denominators arise from the integrals $\int x^{n} \mathrm{~d} x=x^{n+1} /(n+1)$.

The derivation of relation (2.39) given above was first proposed (in a slightly different form) by Niemi and Semenoff. ${ }^{42}$

### 2.3.2. Relationship between $W_{o}^{(2 n+1)}$ and $W^{(2 n+2)}$

We now wish to determine what happens when we use the second method for dealing with the anomaly in (2.28). We assume that the fields $A_{\mu}$ depend on not only the $2 n+1$ coordinates $x_{\mu}$ but also the one additional parameter $t$. We differentiate ( 2.28 ) with respect to this parameter (we will use a superior dot to denote $d / d t$ ):

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \ln \operatorname{det}(i \hat{\mathrm{D}}-i M)=\mathrm{Sp}_{2 n+1} \dot{A}_{\mu} \gamma_{\mu} \frac{1}{i \hat{\mathrm{D}}-i M} . \tag{2.40}
\end{equation*}
$$

The $(2 n+1)$-dimensional $\gamma$ matrices which appear here are of dimensionality $2^{n}+2^{n}$.

On the other hand, let us carry out a few elementary transformations on the left side of the expression for the abelian anomaly (2.25). We first go over to the gauge $A_{0}=0$. Second, we choose the $2^{n+1} \times 2^{n+1} \Gamma$ matrices in $2 n+2$ dimensions as follows:

$$
\left.\begin{array}{c}
\Gamma_{\mu}=\left(\begin{array}{ll}
\gamma_{\mu} & \\
& -\gamma_{\mu}
\end{array}\right), \quad \mu=1 \div 2 n+1, \quad \Gamma_{0}=\left(\begin{array}{ll}
1_{1} & 1
\end{array}\right) \\
\Gamma_{5}=\left(\text {-i }^{i}\right. \tag{2.41}
\end{array}\right),
$$

(here we have made the obvious change in the notation of the $2 n$-dimensional matrices: $\gamma_{0} \rightarrow \gamma_{1} ; \gamma_{1} \rightarrow \gamma_{2}, \ldots, \gamma_{2 n-1} \rightarrow \gamma_{2 n}$, $\gamma_{5} \rightarrow \gamma_{2 n+1} ; \gamma_{1} \ldots \gamma_{2 n} \gamma_{2 n+1} \sim \gamma_{2 n+1}^{2}=1$ ). We then have

$$
\begin{align*}
& 2 \mathrm{Sp}_{2 n+2} M \Gamma^{5} \frac{1}{\hat{\mathrm{D}}-i M} \\
& =2 M \mathrm{Sp}_{2 \pi+2} \Gamma^{5} \frac{i M}{--1)^{2}+M^{2}+\left(F_{\mu v} \sigma_{\mu v} / 2\right)+F_{0 \mu} \sigma_{0 \mu}} \\
& =2 i M^{2} \mathrm{Sp}_{2 n+2} \frac{\Gamma^{5} \dot{A}_{\mu} \sigma_{\mu \mu}}{[-1)^{2}+M^{2}+\left(F_{\mu \nu} \sigma_{\mu v} / 2\right]^{2}} \\
& =4 M^{2} \mathrm{Sp}_{2 n+2} \dot{A}_{\mu} \gamma_{\mu} \frac{1}{\left[-D^{2}+M^{2}+\left(F_{\mu \nu} \sigma_{\mu v} 2\right)\right]^{2}} . \tag{2.42}
\end{align*}
$$

Here we have made use of the fact that the relation $F_{0 i}=\dot{A}_{i}$ holds in the gauge $A_{0}=0$, and we have written a 2 which arises from the transition from the trace of the $\Gamma$ matices to $\gamma$, which has a dimensionality smaller by a factor of 2 . Expression (2.42) can be written

$$
\begin{equation*}
-4 M^{2} \frac{\partial}{\partial M^{2}} \operatorname{Sp}_{2 n+2} \dot{A}_{\mu} \gamma_{\mu} \frac{1}{-D^{2}+M^{2}+\left(F_{\mu \nu} \sigma_{\mu v} / 2\right)} \tag{2.43}
\end{equation*}
$$

and the average over the null coordinate $t$ can be carried out explicitly. (Here we do not need to consider the dependence of $F_{\mu \nu}$ and $\dot{A}_{\mu}$ on $t$. In contrast with the transition between $W_{0}^{(2 n+1)}$ and $W^{a(2 n)}$, the situation at hand does not involve the discrimination of an anomalous contribution. It is easy to see that in this case the changes which arise when this dependence is taken into account vanish in the limit $M \rightarrow \infty$.) Taking an average over $t$, we find
$-4 M^{2} \frac{\partial}{\partial M^{2}} \mathrm{Sp}_{2 n+1} \dot{A}_{\mu} \gamma_{\mu} \int \frac{\mathrm{d} p_{0}}{2 \pi} \frac{1}{p_{0}^{2}-D^{2}+M^{2}+\left(F_{\mu v} \sigma_{\mu v} / 2\right)}$
$=-4 M^{2} \frac{\partial}{\partial M^{2}}$
$\times \int \frac{\mathrm{d} p_{0}}{2 \pi\left(p_{0}^{2}+M\right)^{1 / 2}}\left[\mathrm{Sp}_{2 n+1} \dot{A}_{\mu} \gamma_{\mu} \frac{1}{i \hat{\mathrm{D}}-i\left(M^{2}+p_{0}^{2}\right)^{1 / 2}}\right]_{\text {anom }}$.

The trace $S p_{2 n+1}$ which appears here should be compared with (2.40). We wish to call attention to the fact that (2.44) contains only the anomalous part of the trace $\mathrm{Sp}_{2 n+1}$; the reason is the elimination of the term $\widehat{\mathrm{D}}$ from the numerator. (In the odd-dimensional case, in contrast with the even-dimensional case, the $\gamma$-matrix trace may be nonzero for either an even or odd number of $\gamma$ matrices.) Since the anomalous part of trace $\mathbf{S p}_{2 n+1}$ does not depend on $M$, the integration over $d p_{0}$ and the differentiation with respect to $M^{2}$ can be carried out explicitly:
$-4 M^{2} \frac{\partial}{\partial M^{2}} \int \frac{\mathrm{~d} p_{0}}{2 \pi} \frac{1}{\left(p_{0}^{2}+M^{2}\right)^{1 / 2}} \quad \frac{1}{\pi} \int \mathrm{~d} p_{0} \frac{M^{2}}{\left(p_{0}^{2}+M^{2}\right)^{1 / 2}} \quad \frac{2}{\pi}$.
The result is

$$
\begin{equation*}
W^{\left(2^{n}+2\right)}=\frac{2}{\pi} \frac{\mathrm{~d} W_{0}^{(2 n+1)}}{\mathrm{d} t} . \tag{2.45}
\end{equation*}
$$

The derivative $d / d t$ here can be replaced by a formal external differentiation (Subsection 3.1). Some examples are

$$
\begin{aligned}
& \begin{aligned}
& W^{(2)}==\frac{2}{2 \pi} \operatorname{Tr} F=\frac{1}{\pi} \operatorname{Tr} \mathrm{~d} A=\frac{2}{\pi} \mathrm{~d} \frac{1}{2} \operatorname{Tr} A=-\frac{2}{\pi} \mathrm{~d} W_{0}^{(1)}, \\
& W^{(4)}=\frac{2}{(2 \pi)^{2} 2!} \operatorname{Tr} F^{2}=\frac{1}{4 \pi^{2}} \operatorname{Tr} \mathrm{~d}\left(A \mathrm{~d} A+\frac{2}{3} A^{3}\right) \\
&=\frac{2}{\pi} d \frac{1}{4 \pi \cdot 2} \operatorname{Tr}\left(A d A+\frac{2}{3} A^{3}\right)=\frac{2}{\pi} \mathrm{~d} W_{0}^{(3)},
\end{aligned} \\
& \begin{aligned}
W^{(6)} & =\frac{2}{(2 \pi)^{3} 3!} \operatorname{Tr} F^{3} \\
& =\frac{2}{\pi} \mathrm{~d} \frac{1}{16 \pi^{2} \cdot 3} \operatorname{Tr}\left(A \mathrm{~d} A^{2}+\frac{3}{2} A^{3} \mathrm{~d} A+\frac{3}{5} A^{5}\right)=\frac{2}{\pi} \mathrm{~d} W_{0}^{(5)} .
\end{aligned}
\end{aligned}
$$

We have discussed the relationship among the left sides of Dirac anomalies (2.25), (2.26), and (2.28). The relationship between the Weyl anomaly in (2.27) and the odd-dimensionality anomaly in (2.28) is found by means of WessZumino consistency conditions in the following subsection.

### 2.4. Wess-Zumino consistency conditions and relationship between Dirac and Weyl anomalies

The change in $W_{o}^{(2 n+1)}$ under a gauge transformation is a total derivative. We will begin this subsection by proving this assertion, which may appear to be totally unrelated to the title of this subheading. Nevertheless, the relationship between the Dirac and Weyl anomalies is based on this assertion. The proof is very simple. A Dirac fermion determinant is a gauge-invariant quantity, so that it is clear at the outset that under infinitesimal gauge transformations which are not singular anywhere, including at infinity, the quantity $\int W_{0}^{(2 n+1)} \mathrm{d}^{2 n+1} x$ does not change, so that if $W_{0}^{(2 n+1)}$ changes at all it does so by a total derivative. [One could also make use of (2.45).] Since $W^{(2 n+2)} \sim \operatorname{Tr} F^{n+1}$ is an explicitly gauge-invariant quantity, we find $0=\delta_{\mathrm{u}} W^{(2 n+2)}=\mathrm{d} \delta_{\mathrm{u}} W_{0}^{(2 n+1)}$ immediately from it. In any singly connected region away from singularities, $\delta_{\mathrm{u}} W_{n}^{(2 n+1)}$ is an external derivative of some expression.] A less trivial point is that this total derivative is not identically zero, as can be seen most simply by looking at the specific expression in (2.29) for

$$
W_{0}^{(2 n+1)}=\frac{1}{2(2 \pi)^{n}(n+1)!} \operatorname{Tr}\left(A d A^{n}+\ldots\right) .
$$

We now take the integral of $W_{0}^{(2 n+1)}$ not over the entire space $R^{2 n+1}$ but only over the lower half-space $R^{2 n+1}$. We assume that all the fields decay at infinity, so that this is actually an integral over a hemisphere $S^{(2 n+1)}$. We denote it by $U^{(2 n+1)}$. This integral itself is not remarkable in any way, but its variation under a gauge transformation,

$$
U^{a\left(_{2} 2 n\right.}=\frac{\delta}{\delta u^{a}} U^{\left(2^{h+1}\right)}
$$

depends only the value of the fields on this sphere $S^{2 n}$, which is the upper boundary of $S^{(2 n+1)}$. We will make use of the functional $U^{a(2 n)}$ in the discussion below.

We turn now to a Weyl anomaly. The Lagrangian
changes under the gauge transformation

$$
\psi \rightarrow \exp \left(-i u^{a} t^{a} \frac{1-\gamma^{5}}{2}\right) \psi
$$

by

$$
\begin{aligned}
& -\bar{\psi} \gamma_{\mu} t^{a} \frac{1-\gamma^{s}}{2} \psi\left(\mathrm{D}_{\mu} u\right)^{a} \\
& =-\frac{1}{2} J_{\mu}^{\mathrm{La}}\left(\mathrm{D}_{\mu} u\right)^{a} \quad\left(\mathrm{D}_{\mu} u=\partial_{\mu} u+\left[A_{\mu} u\right]\right)
\end{aligned}
$$

This change is cancelled exactly by the gauge variation of the external field $A: A_{\mu} \rightarrow A_{\mu}+\partial_{\mu} u+\left[A_{\mu} u\right]=A_{\mu}+D_{\mu} u$. Accordingly, the average of $\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{L}}\right)^{a}$ over the vacuum is a gauge variation of an effective action which arises after an integration over fermions:

$$
\left\langle\mathrm{D}_{\mu} J_{\mu}^{\mathrm{L}}\right\rangle^{a} \equiv W_{1}^{a(2 n)}=2 \frac{\delta_{u} S_{\mathrm{eff}}\{A\}}{\delta u^{a}}
$$

We can now use the Jacobi identity $\delta_{u} \delta_{u}-\delta_{u} \delta_{u}$ $+\delta_{\left.\mid u^{\prime}\right\rfloor}=0$, which means that the gauge transformations form a group. In particular, we have $\delta_{r} \delta_{u} S_{\text {eff }}$ $-\delta_{u} \delta_{\mathrm{l}} S_{\mathrm{cff}}+\delta_{\text {lup }} S_{\mathrm{eff}}=0$, i.e.,
$\delta_{v} \operatorname{Tr} u W_{1}^{(2 n)}-\delta_{u} \operatorname{Tr} v W_{1}^{(2 n)}+\operatorname{Tr}[u v] W_{1}^{(2 n)}=$ a total derivative.

This is the Wess-Zumino consistency condition for a Weyl anomaly. The total derivative arises because $S_{\text {eff }}$, in contrast with $W_{1}^{(2 n)}$, contains an integration over $\mathrm{d}^{2 n} x$. This total derivative is extremely important. Expressions for Weyl anomalies satisfy equation (2.46) only if this total derivative is taken into account. For example, in the two-dimensional case (the simplest) we have

$$
\begin{aligned}
\operatorname{Tr} W_{1}^{2 j} \sim & \operatorname{Tr} u \mathrm{~d} A \\
\delta_{v} \operatorname{Tr} u W_{1}^{(2)}=\operatorname{Tr} u([\mathrm{~d} A, v] & -[A, \mathrm{~d} v]) \\
& =-\operatorname{Tr}[u v] d A-\operatorname{Tr}[u \mathrm{~d} v] A, \\
-\delta_{u} \operatorname{Tr} v W_{1}^{(2)} & =-\operatorname{Tr}[u v] \mathrm{d} A-\operatorname{Tr}[\mathrm{d} u v] A, \\
+\operatorname{Tr}[u v] W_{1}^{(2)} & =+\operatorname{Tr}[u v] \mathrm{d} A .
\end{aligned}
$$

Adding these three expressions, we find
$(-1-1+1) \operatorname{Tr}[u v] \mathrm{d} A-\operatorname{Tr}([u \mathrm{~d} v]+[\mathrm{d} u v]) A$
$=-\mathrm{d} \operatorname{Tr}[u v] A=$ a total derivative.
We note that $\operatorname{Tr} u A^{2}$ docs not satisfy the consis-
tency condition: In this case, we have $-\operatorname{Tr}[u v] A^{2}$ $+\operatorname{Tr}(d[u v]) A$, on the left side of (2.46), and that quantity is not equal to a total derivative. Consequently, a Weyl anomaly $\mathrm{W}_{1}^{2}$ contains a contribution $\sim d A$ but not $\sim A^{2}$. In general, it is not difficult to see that one can work from the given "senior" term of the anomaly $W_{1}^{a(2 n)}$, $\operatorname{Tr} t^{a} d A^{\prime \prime}$, to find all the other terms with the help of (2.46). The arbitrariness here reduces to possible structures without an $\varepsilon$-symbol, which can be obtained by means of a gauge variation of local functionals and which we have agreed to discard for this reason. (Truly anomalous contributions to an anomaly correspond to a nonlocal $S_{\text {eff }}$, e.g., if non-abelian contributions are ignored,

$$
\begin{aligned}
& S_{\mathrm{eff}} \sim \partial_{\alpha} A_{\alpha} \frac{1}{\partial^{2}} \varepsilon_{\mu_{1}} \ldots \mu_{2 n} F_{\mu_{1} \mu_{2}} \ldots F_{\mu_{2 n-1} \mu_{2 n}} \\
& \delta_{u} S_{\text {eff }}\left.\sim u \varepsilon_{\mu_{2}} \ldots \mu_{2 n} F_{\mu_{1} \mu_{2}} \ldots F_{\mu_{2 n-1} \mu_{2 n}} .\right)
\end{aligned}
$$

Accordingly, any nontrivial solution of Eq. (2.46), beginning with

$$
\frac{2}{(2 \pi)^{n}(n+1)!} \operatorname{Tr} t^{a} \mathrm{~d} A^{n}
$$

will certainly lead to the correct result for a Weyl anomaly in a $2 n$-dimensional theory. It is now time to recall the expression $U^{a(2 n)}$ which was determined at the beginning of Subsection 2.4 from $W_{0}^{(2 n+1)}$. By definition, $U^{a(2 n)}$ is a gauge variation of a functional $U^{(2 n+1)}$, and it depends on only the values of the fields in the $2 n$-dimensional space $S^{2 n}$. Consequently, $U^{a(2 n)}$ necessarily satisfies condition (2.46). On the other hand, $U^{a(2 n)}$ contains an $\varepsilon$-symbol, so that (with a correct normalization) it leads to the result for a Weyl anomaly (see the examples in Section 3): $W_{1}^{a(2 n)} \sim U^{a(2 n)}$.

We will conclude this Subsection with a few words about the applicability of the Wess-Zumino condition in other situations. Since it is simply the Jacobi identity applied to an effective action, this condition is applicable whenever the action of a group of gauge transformations on fermions and vector fields is defined, and one is examining the covariant divergence of an internal current with which gauge bosons are interacting. In reality, both conditions hold in only two cases: for the divergence of a vector current in the Dirac theory (in which case the anomaly vanishes, and the WessZumino condition is satisfied trivially) and for the divergence of a left-hand (or right-hand) current in the Weyl theory. The reason is that in a fermion theory the action of the group $G_{L} \times G_{\mathbf{R}}$, for which there are subgroups of only two types, $G_{\mathrm{V}}$ and $G_{\mathrm{L}}\left(G_{\mathrm{R}}\right)$, is defined. One could of course write consistency conditions for the total group $G_{\mathrm{L}} \times G_{\mathrm{R}}$, and precisely those conditions were derived in the classic paper by Wess and Zumino. ${ }^{7}$ These conditions can be written as two relations (2.46), separately for right-hand and left-hand transformations, or they can be written in a mixed form: in terms of vector (V) transformations $\left\|\rightarrow e^{i a}\right\|^{\prime}$ and axial (A) transformations $\psi \rightarrow e^{i d l^{*}} \psi$ :

$$
\begin{aligned}
& \delta_{u}^{V} \delta_{v}^{V}-\delta_{v}^{V} \delta_{u}^{V}+\delta_{u}^{\mathrm{A}} \delta_{v}^{\mathrm{A}}-\delta_{v}^{\mathrm{A}} \delta_{u}^{\mathrm{A}}=\delta_{[u v]}^{\mathrm{V}}, \\
& \delta_{u}^{\mathrm{A}} \delta_{v}^{\mathrm{V}}-\delta_{v}^{\mathrm{V}} \delta_{u}^{\mathrm{A}}+\delta_{u}^{\mathrm{V}} \delta_{v}^{\mathrm{A}}-\delta_{v}^{\mathrm{A}} \delta_{u}^{\mathrm{V}}=\delta_{[u v]}^{\mathrm{A}}
\end{aligned}
$$

These relations are satisfied by, for example. the four-dimen-
sional Bardeen anomaly ${ }^{6}$ in a theory with a Lagrangian $\bar{\psi} \widehat{\boldsymbol{V}} \psi+\bar{\psi} \hat{\boldsymbol{A}} \boldsymbol{\gamma}^{5} \psi:$

$$
\begin{align*}
&\left(D_{\mu} J_{\mu}^{s}\right)^{a} \equiv \partial_{\mu} J_{\mu}^{s a}+\left[V_{\mu} J_{\mu}^{\mathrm{s}}\right]^{a}+\left[A_{\mu} J_{\mu}\right]^{a} \\
&=\frac{1}{4 \pi^{2}} \operatorname{Tr} t^{a}\left[F_{\mathrm{V}}^{\mathrm{a}}+\frac{1}{3} F_{\mathrm{A}}^{2}-\frac{4}{3}\left(A^{2} F_{\mathrm{v}}^{\mathbf{q}}+A F_{\mathrm{V}} A\right.\right.\left.+F_{\mathrm{V}} A^{2}\right) \\
& \quad\left.+\frac{8}{3} A^{4}\right] ; \tag{2.47}
\end{align*}
$$

here $F_{\mathrm{V}}=d V+V^{2}+A^{2}$, and $F_{\mathrm{A}}=d A+V A+A V$. The gauge transformations act on fields in accordance with the obvious rules

$$
\begin{aligned}
\delta_{\mathbf{u}}^{\mathbf{v}} V & =\mathrm{d} u+[V u], & \delta_{\mathbf{u}}^{\mathbf{V}} A & =[A u], \\
\delta_{\mathbf{u}}^{\mathbf{A}} V & =[A u], & \delta_{\mathbf{u}}^{\mathbf{A}} A & =\mathrm{d} u+[V u], \\
\delta_{\mathbf{u}}^{\mathbf{V}} F_{\mathbf{v}} & =\left[F_{\mathbf{v}} u\right], & \delta_{\mathbf{u}}^{\mathbf{V}} F_{\mathbf{A}} & =\left[F_{\mathbf{A}} u\right], \\
\delta_{\mathbf{u}}^{\mathbf{A}} F_{\mathbf{v}} & =\left[F_{\mathbf{A}} u\right], & \delta_{\mathbf{u}}^{\mathbf{v}} F_{\mathbf{A}} & =\left[F_{\mathbf{v}} u\right] .
\end{aligned}
$$

As we are doing everywhere else in this review, we are assuming in (2.47) that the vector current is conserved: $\mathrm{D}_{\mu} J_{\mu}=\partial_{\mu} J_{\mu}+\left[V_{\mu} J_{\mu}\right]+\left[A_{\mu} J_{\mu}^{5}\right]=0$. With $A=0$ we find from (2.47) a Dirac anomaly, $W^{a(4)}$ $=\left(1 / 4 \pi^{2}\right) \operatorname{Tr} t^{a} F_{\mathrm{v}}^{2}$; with $V= \pm A$ we find a Weyl anomaly (in order to find the correct Lagrangian

$$
\bar{\psi} \hat{V} \frac{1-\gamma^{b}}{2} \psi
$$

it is necessary to also make the substitutions $V \rightarrow V / 2$, $A \rightarrow A / 2$ ):
$W_{1}^{a(4)}=\frac{1}{4 \pi^{2}} \operatorname{Tr} t^{a}\left\{\left(\frac{\mathrm{~d} V}{2}+2 \frac{V^{2}}{4}\right)^{2}+\frac{1}{3}\left(\frac{\mathrm{~d} V}{2}+2 \frac{V^{2}}{4}\right)^{2}\right.$.
$-\frac{4}{3}\left[\frac{V^{2}}{4}\left(\frac{\mathrm{~d} V}{2}+2 \frac{V^{2}}{4}\right)+\frac{V}{2}\left(\frac{\mathrm{~d} V}{2}+2 \frac{V^{2}}{4}\right) \frac{V}{2}\right.$
$\left.\left.+\left(\frac{\mathrm{d} V}{2}+2 \frac{V^{2}}{4}\right) \frac{V^{2}}{4}\right]+\frac{8}{3} \frac{V^{4}}{16}\right\}$
$=\frac{1}{12 \pi^{2}} \operatorname{Tr} t^{a}\left[\mathrm{~d} V^{2}+\frac{1}{2}\left(V^{2} \mathrm{~d} V-V \mathrm{~d} V V+\mathrm{d} V V^{2}\right)\right]$
$=\frac{1}{12 \pi^{2}} \operatorname{Tr} t^{a} \mathrm{~d}\left(V \mathrm{~d} V+\frac{1}{2} V^{3}\right)$.
This is indeed a Weyl anomaly, since we have $\mathrm{D}_{\mu} J_{\mu}^{\mathrm{L}}=-\mathrm{D}_{\mu} J_{\mu}^{5}, \mathrm{D}_{\mu} J_{\mu}^{\mathrm{R}}=+\mathrm{D}_{\mu} J_{\mu}^{5}$ from the conservation of the vector current. Yet another important particular case is found with $V=0$, in which case the theory has only an axial gauge boson:

$$
\begin{equation*}
\left(\mathrm{D}_{\mu} J_{\mu}^{\mathrm{B}}\right)_{A}^{a}=\frac{1}{12 \pi^{2}} \operatorname{Tr} t^{a}\left(\mathrm{~d} A^{2}-A^{4}\right) \tag{2.48}
\end{equation*}
$$

At first glance, this anomaly would appear to satisfy the Wess-Zumino relation: One calculates the divergence of the same current with which the fields $A$ are interacting. However, the second condition does not hold: The axial transformations $\psi \rightarrow e^{i \alpha \gamma^{*}} \psi$ do not form groups: $G_{\mathrm{L}} \times G_{\mathrm{R}}$ has no axi-al-transformation subgroups $G_{\mathrm{A}}$. The commutator of two axial transformations $\psi \rightarrow e^{i \alpha \gamma^{i}} \psi$ is equal to a vector transformation $\psi \rightarrow e^{i \alpha} \psi$, not an axial transformation. Consequently, (2.48) does not satisfy relation (2.46).

Unfortunately, the limitations imposed on the Bardeen anomaly by the Wess-Zumino relations are not great in an
arbitrary number of dimensions, so that this anomaly cannot be incorporated in the hierarchy of anomalies.

### 2.5. Calculation of anomalies by means of dispersion relations

This approach to anomalies was proposed by Dolgov and Zakharov. ${ }^{8}$ We have already mentioned it in the Introduction, in the example of a two-dimensional theory. A calculation of anomalies on the basis of the imaginary part is not only of fundamental importance but also technically much simpler than other methods in certain cases. One such case is the calculation of anomalies of antisymmetric tensor fields, since it is immeasurably simpler to work with these fields on the mass shell (and nothing more is required in the Dolgov-Zakharov method) than off it. We will reproduce here a passage from Ref. 8 on the calculation on the basis of the imaginary part of an ordinary triangle $\gamma^{5}$ anomaly (i.e., an abelian Dirac four-dimensional axial anomaly) (Fig. 10).

Instead of calculating the regulator diagram with a vertex $2 \gamma^{5} M_{\text {reg }}$, the idea is to find an answer for the current $J_{\mu}^{5}$ itself, initially for its imaginary part. The imaginary part is determined by the behavior of the fields on the mass shell; it is ultraviolet-finite; and it has no contribution from regulators. On the other hand, an infrared regularization is necessary in this case. There are two convenient ways to choose this regularization:
a) a nonzero fermion mass $m$;
b) a nonzero photon mass $p_{(1)}^{2}=p_{(2)}^{2}=\mu^{2}$.

Regularization b) is technically simpler. Furthermore, it never requires leaving the mass shell, so that it should be used in a discussion of antisymmetric fields and other complicated examples. Regularization a) was used by Dolgov and Zakharov in their own calculations. A simple example of the use of this regularization is given in Subsection 1.2. Below we will use regularization $b$ ).

We denote the difference between the momenta of the photons by $p_{\mu}=p_{\mu}^{(1)}-p_{\mu}^{(2)}$; the sum of these momenta is $q_{\mu}=p_{\mu}^{(1)}+p_{\mu}^{(2)}$. Under the condition $p_{(1)}^{2}=p_{(2)}^{2}=\mu^{2}$ we have the scalar product $q p=0$. The calculation for $\operatorname{Im}\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta}$ must be symmetric with respect to the interchange of photons; i.e., it must be an invariant of the transformation $\alpha \leftrightarrow \beta, p \leftrightarrow-p$. This condition is satisfied by the four following structures:


FIG. 10. Four-dimensional electrodynamics. Axial current in an external photon field. The dashed line is the cutoff for calculating the imaginary part of the diagram.

$$
\left.\begin{array}{cc}
\mid q_{\mu} \varepsilon_{\alpha \beta \xi \eta} p_{\xi} q_{\eta}, & q^{2} \varepsilon_{\mu \alpha \beta \xi} p_{\xi},  \tag{2.49}\\
p_{\alpha} \varepsilon_{\mu \beta \xi \eta} p_{\xi} q_{\eta}+p_{\beta} \varepsilon_{\mu \alpha \xi \eta} p_{\xi} q_{\eta}, \\
q_{\alpha} \varepsilon_{\mu \beta \xi \eta} p_{\varepsilon} q_{\eta}-q_{\beta} \varepsilon_{\mu \alpha \xi \eta} p_{\xi} q_{\eta} .
\end{array}\right\}
$$

Other possible structures，e．g．，$p_{\alpha} \varepsilon_{\mu B \xi \% \eta} p_{\xi} q_{\eta}$ $-p_{\beta} \varepsilon_{\mu \alpha \xi \eta} p_{\xi} q_{\eta}$ ，are themselves asymmetric，and they could appear in the result for the diagrams in Fig． 10 only with a factor（ $q p$ ）．That factor，however，is zero in our kine－ matics（and this is precisely the reason why it is convenient to choose the masses of the photons to be identical）．

The last structure in（2．49）is actually expressed in terms of the first two．This situation follows from the specific four－dimensional relation among $\varepsilon$－symbols，which is found when the product $\delta_{\lambda \mu} \varepsilon_{\alpha \beta \xi \eta}$ is rendered antisymmetric in terms of the indices $\mu, \alpha, \beta, \xi, \eta$ ．Since each of these five indices can take on only four values，the result of the conver－ sion to an antisymmetric form should be zero：

$$
\begin{align*}
0 \equiv \delta_{\lambda \mu} \varepsilon_{\alpha \beta \eta} & -\delta_{\lambda \alpha} \varepsilon_{\mu \beta \xi \eta}+\delta_{\lambda \beta} \varepsilon_{\mu \alpha \xi \eta} \\
& -\delta_{\lambda \xi} \varepsilon_{\mu \alpha \beta \eta}+\delta_{\lambda, \eta} \varepsilon_{\mu \alpha \beta \xi} . \tag{2.50}
\end{align*}
$$

Contracting this identity with $q_{\lambda} p_{\xi} q_{\eta}$ ，we find

The last term on the right drops out in the case $p q=0$ ．
When the fermions in the loop are on the mass shell，as they are in a calculation of the diagram on the basis of is imaginary part，the axial current is conserved；i．e．， $q_{u} \operatorname{Im}\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta}=0$ ．［If we were using regularization a）， which gives the fermion a mass，this would not be correct．］ Furthermore，vector currents are conserved as usual：

$$
p_{\alpha}^{(1)} \operatorname{Im}\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta}=p_{\beta}^{(2)} \operatorname{Im}\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta}=0 .
$$

The general expression for the imaginary part here is

$$
\begin{align*}
& \operatorname{Im}\left\langle J_{\mu}^{b}\right\rangle_{\alpha \beta}=A g_{\mu} \varepsilon_{\alpha \beta \xi} p_{\xi} q_{\eta}+B q^{2} \varepsilon_{\mu \alpha \beta} F_{\xi} \tag{2.52}
\end{align*}
$$

where the coefficients $A, B$ ，and $C$－arbitrary at this point－ are functions of $q^{2}$ ．The conditions for the conservation of three currents then make it possible to express $B$ and $C$ in terms of $A$ ．For example，if we multiply（2．52）by $q_{\mu}$ ，we find $B=A$ ；if we multiply（2．52）by $p_{\alpha}^{(1)}=(1 / 2)(p+q)_{\alpha}$ ，we find the relation $B q^{2}+C p^{2}=0$ ．Precisely the same relation arises if we multiply $\operatorname{Im}\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta}$ by $p_{\beta}^{(2)}=(1 / 2)$ $\times(-p+q)_{\beta}$ ．At this point we note that we have $p^{2}=\left(p^{(1)}-p^{(2)}\right)^{2}=4 \mu^{2}-q^{2}$ ，and thus

$$
C=-\frac{B q^{2}}{p^{2}}=\left(1-\frac{4 u^{2}}{q^{2}}\right)^{-1} A
$$

As a result we find

$$
\begin{align*}
\operatorname{Im} .\left\langle J_{\mu}^{\dot{j}}\right\rangle_{\alpha \beta}= & A\left[q_{\mu} \varepsilon_{\alpha \beta \xi \eta} p_{\xi} q_{\eta}+q^{2} \varepsilon_{\mu \alpha \beta \xi} p_{\xi}\right. \\
& \left.+\left(1-\frac{4 \mu^{2}}{q^{2}}\right)^{-1}\left(p_{\alpha} \varepsilon_{\mu \beta \xi \eta} p_{亏}^{\prime} q_{\eta}+p_{\beta} \varepsilon_{\alpha \times \xi} \eta f_{\xi} q_{\eta}\right)\right] \tag{2.53}
\end{align*}
$$

The coefficient can be found by direct calculation ${ }^{10}$ ：

$$
\begin{equation*}
A=\frac{\theta\left(q^{2}-4 \mu^{2}\right)}{8 \pi q^{2}}\left(1+O\left(\frac{\mu^{2}}{q^{2}}\right)\right) \tag{2.54}
\end{equation*}
$$

In Subsection 1.2 ［see（1．18）－（1．22）］we have already dis－ cussed the calculation of anomalies on the basis of the imagi－ nary part．We saw there that an imaginary part $\sim \delta\left(q^{2}\right)$ corresponds to an anomaly．In an infrared－regularized the－ ory there is of course no $\delta$－function；in its place there is an expression of the type $\mu^{2} / q^{4}$ ．In other words，an anomalous imaginary part must be proportional to the square of the regularizing mass．This condition has not yet been satisfied in（2．53）．What is going on here？From the expression which we have found we still need to single out the anomaly proper： the structure which describes a transition to specifically two photons．The amplitude differs from（2．53）in the presence of photon polarization vectors $\varepsilon_{\alpha}^{(1)}$ and $\varepsilon_{\beta}^{(2)}$ ： $\operatorname{Im}\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}$ ．A very important point is that the polar－ ization vectors are transverse：$p_{\alpha}^{(1)} \varepsilon_{\alpha}^{(1)}=p_{\beta}^{(2)} \varepsilon_{\beta}^{(2)}=0$ ．It turns out that after we multiply by $\varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}$ we are left in （2．53）with only a contribution proportional to $\left(\mu^{2} / q^{2}\right) A$ ． Again using identity（2．51），we find

$=q_{\alpha} \varepsilon_{\mu \beta \xi \eta} p_{\xi} q_{\eta}-q_{\beta} \varepsilon_{\mu \alpha \xi \eta} p_{\zeta} q_{\eta}+p_{\alpha} \varepsilon_{\mu \beta \xi \eta} p_{\xi} q_{\eta}+p_{\beta} \varepsilon_{\mu \alpha \xi \eta} p_{\dot{\delta}} q_{\eta}$
$=2\left(p_{\alpha}^{(1)} \varepsilon_{\mu \rho \xi \eta} p_{\xi} q_{\eta}-p_{\beta}^{(2)} \varepsilon_{\mu \alpha_{\xi} \eta} p_{\bar{\xi}} q_{\eta}\right)$,
which vanishes upon multiplication by $\varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}$ ．According－ ly，something remains in the amplitude found from（2．53） only because of the difference between $\left[1-\left(4 \mu^{2} / q^{2}\right)\right]^{-1}$ and unity：

$$
\begin{align*}
& \operatorname{Im}\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)} \\
& =\frac{4 \mu^{2}}{q^{2}} A\left(q_{\mu} \varepsilon_{\alpha \beta \xi \eta} p_{亏} q_{\eta}+q^{2} \varepsilon_{\mu \alpha \beta \bar{\xi}} p_{\xi}\right) \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}\left(1+O\left(\frac{\mu^{2}}{q^{2}}\right)\right) \\
& =\frac{\theta\left(q^{2}-4 \mu^{2}\right)}{\pi}\left[\frac{\mu^{2}}{q^{4}} q_{\mu} \varepsilon_{\alpha \beta_{亏}^{2} \eta} p \xi_{\xi}^{(1)} p_{\eta}^{(2)} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}\right. \\
& \left.\quad+\frac{\mu^{2}}{2 q^{2}} \varepsilon_{\mu \alpha \beta}\left(p_{\xi}^{(1)} \rightarrow p_{\xi}^{(2)}\right) \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}\right]\left(1+O\left(\frac{\mu^{2}}{q^{2}}\right)\right) \tag{2.55}
\end{align*}
$$

As $\mu$ tends toward zero，the coefficient of the first structure becomes $\delta\left(q^{2}\right) / 4 \pi$ ，while that of the second structure van－ ishes．We thus finally find

$$
\begin{equation*}
\operatorname{Im}\left\langle J_{\mu}^{b}\right\rangle_{\alpha \beta} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}=\frac{1}{4 \pi} \delta\left(q^{2}\right) q_{\mu} \varepsilon_{\alpha \beta \xi \eta} p \xi^{(1)} p_{\eta}^{(2)} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)} . \tag{2.56}
\end{equation*}
$$

From this expression we can easily find the real part of the amplitude：

$$
\begin{align*}
\left\langle J_{\mu}^{5}\right\rangle_{\alpha \beta} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)}= & \frac{1}{4 \pi^{2}} \frac{q_{\mu}}{q^{2}} \varepsilon_{\alpha \beta 5 \eta} p_{\xi}^{(1)} p_{\eta}^{(2)} \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(2)} \\
& +c \varepsilon_{\mu \alpha \beta 亏}\left(p_{5}^{(1)}-p_{\underset{3}{(2)})}\right) \varepsilon_{\alpha}^{(1)} \varepsilon_{\beta}^{(3)} ; \tag{2.57}
\end{align*}
$$

here $c$ is an arbitrary constant which arises in accordance with the dispersion relations from a zero imaginary part of the corresponding form factor．Vector currents are con－ served if we choose $z=0$ ，and in this case we have ${ }^{8}$

$$
\begin{aligned}
\left\langle J_{\mu}^{s}\right\rangle & =\frac{q_{\mu}}{8 \pi^{2} q^{2}} F_{\alpha \beta} \widetilde{F}_{\alpha \beta}, \\
\left\langle\partial_{\mu} J_{\mu}^{j}\right\rangle & =\frac{1}{8 \pi^{2}} F_{\alpha \beta} \widetilde{F}_{\alpha \beta} .
\end{aligned}
$$

## 3. HIERARCHY OF ANOMALIES

### 3.1. Differential forms

The expressions for the anomalies contain an $\varepsilon$-sym-bol-they are rendered antisymmetric in terms of all vector indices-so that they can be written most compactly in terms of differential forms. Differential forms are constructed in $D$ dimensions by means of $D$ anticommuting differentials $\mathrm{d} x_{1} \ldots \mathrm{~d} x_{D}, \mathrm{~d} x_{\mu} \mathrm{d} x_{v}=-\mathrm{d} x_{\nu} \mathrm{d} x_{\mu}$ and all possible products and linear combinations of these differentials. A formal sum each of whose terms contains the product of $k$ differentials with coefficients which depend on $x$, $\Phi=\Phi_{\mu_{1} \ldots \mu_{k}}(x) \mathrm{d} x_{\mu}, \ldots \mathrm{d} x_{\mu_{k}}$ is called a "form of rank $k$ " or a " $k$-form." In terms of differential forms, the field $A_{\mu}^{a}$ corresponds to the 1 -form $A=i A_{\mu}^{a} t^{a} \mathrm{~d} x_{\mu}$, while the antisymmetric tensor $F_{\mu \nu}^{a}$ corresponds to the 2 -form $F=i F_{\mu \nu}^{a},{ }^{a} \mathrm{~d} x_{\mu} \mathrm{d} x_{v} / 2!$.

Two important operations are defined for differential forms: the external product and the external differentiation. They are found from the tensor product and from ordinary differentiation through a conversion to an antisymmetric form with respect to all indices. The external product brings two forms of ranks $k_{1}$ and $k_{2}$ into a ( $k_{1}+k_{2}$ )-form:

$$
\left(\Phi^{(1)} \Phi^{(2)}\right)_{\mu_{1}} \ldots \mu_{k_{1}+k_{2}}=\Phi_{\left[\mu_{1}\right.}^{(1)} \ldots u_{k_{1}} \Phi_{\mu_{k_{1}+1}^{(2)} \ldots \mu_{\left.k_{1}+k_{2}\right]} .}
$$

The operation of external differentiation transforms a $k$ form into a $(k+1)$-form:
$\left.\left.(\mathrm{d} \Phi)_{\mu_{1}} \ldots \mu_{k+1}=\partial_{\left[\mu_{1}\right.} \Phi_{\mu_{2}} \ldots \mu_{k+1}\right]=(-)^{k} \partial_{\left[\mu_{k+1}\right.} \Phi_{\mu_{1}} \ldots \mu_{k}\right]$.
Using these operations, we can write $F=\mathrm{d} A+A^{2}$. [In more detail: $F=(1 / 2) F_{\mu \nu} \mathrm{d} x_{\mu} \mathrm{d} x_{v}=(1 / 2)\left(\partial_{\mu} A_{v}\right.$ $\left.-\partial_{v} A_{\mu}+\left[A_{\mu} A_{v}\right]\right) d x_{\mu} d x_{v}=\left(\partial_{\mu} A_{v}+A_{\mu} A_{v}\right) \mathrm{d} x_{\mu} \mathrm{d} x_{v}$ $=\mathrm{d} A+A^{2}$.]

Finally, a divergence using the language of forms can be written with the help of yet another operation, *, which transforms a $k$-form into a $(D-k)$-form: $\left({ }^{*} \Phi\right)_{\mu_{1} \ldots \mu_{d-k}}$ $=\varepsilon_{\mu_{1} \ldots \mu_{D}} \Phi_{\mu_{D-k+1} \ldots \mu_{D}}, \partial_{\mu_{1}} \Phi_{\mu_{1} \mu_{2} \ldots \mu_{k}} \sim\left(\mathrm{~d}^{*} \Phi\right)_{\mu_{2} \ldots \mu_{k}}$. For example, if $J^{5}=J_{\mu}^{5} \mathrm{~d} x_{\mu}$ is a 1 -form corresponding to an axial current, an abelian Dirac anomaly (2.25) can be put in the form

$$
\mathrm{d} * J^{5}=\frac{2}{(2 \pi)^{n} n!} \operatorname{Tr} F^{n} .
$$

We have already written out some particularly complicated expressions in terms of differential forms. This way of writing the expressions makes it possible to avoid writing a large number of vector indices.

### 3.2. The operation which is the inverse of external differentiation

Any closed differential form $\Phi$ of rank $k+1$, with $d \Phi=0$, is locally integrable, i.e., can be represented as an external derivative of some $k$-form $B: \Phi=d B$. Form $B$ can be expressed in terms of $\Phi$ in a nonlocal way, in the form of an integral. For example, the closed 1 -form $\Phi_{\mu} \mathrm{d} x_{\mu}, \Phi_{\mu, r}=\Phi_{\nu, \mu}$, is a differential zero-form:

$$
\begin{equation*}
B=\int^{x} \Phi_{\mu} \mathrm{d} x_{\mu} \tag{3.1}
\end{equation*}
$$

Our first task in this subsection is to generalize expression (3.1) to the case of an arbitrary rank $k \neq 0$.

### 3.2.1. Integration of differential forms

We can attempt to seek a solution of the equation

$$
\begin{equation*}
\left.\Phi_{\mu_{1} \ldots \mu_{k+1}}=(-)^{k} \partial_{\left[\mu_{k+1}\right.} B_{\mu_{1}} \ldots \mu_{k}\right] \tag{3.2}
\end{equation*}
$$

in the case of a closed form $\Phi$ in the same way as in (3.1):

$$
\begin{equation*}
B_{\mu_{1}} \ldots \mu_{k}(x)=\int_{C}^{x} \Phi_{\mu_{1}} \ldots \mu_{k+1} \mathrm{~d} x^{\mu_{k+1}}, \tag{3.3}
\end{equation*}
$$

where the integral is taken along a contour $C$ which terminates at point $x$. Unfortunately, the integral depends on the choice of contour, so that (3.3) is meaningless. The closed nature of the $(k+1)$-form $\Phi$ guarantees independence from the choice of integration surface only for the integral $\int \Phi_{\mu, \ldots \mu_{k+1}} \mathrm{~d} x^{\mu_{1}} \ldots \mathrm{~d} x^{\mu_{k+1}}$-this is by no means the case for integrals of lower order. The validity of representation (3.1) in this sense is simply fortuitous: It is correct for closed 1forms, and only for such forms.

However, this is not the end of the story regarding representation (3.3). Since the dependence on the path of integration is a complicating factor, we can attempt to fix this path. The simplest way to choose the contour $C$ is as a straight line with a linear parametrization which emerges from some given point, e.g., zero, and goes to point $x$ :

$$
\begin{equation*}
C=\{x t ; t \in[0,1]\} \tag{3.4}
\end{equation*}
$$

We immediately note that there is no guarantee that this choice of contour in (3.3) will lead to a solution of Eq. (3.2). When the point $x$ is changed (Fig. 11), contour $C$ also changes: $C+\delta C \neq C$. If the integral were independent of the choice of contour, the difference

$$
\int_{c+\delta c}-\int_{C}
$$

would then be equal to an integral over the interval $[x, x+\delta x]$, and everything would be fine. Actually, however, the integral depends on the path and this difference is not necessarily determined by the neighborhood of the point $x$; it may turn out to be a nonlocal expression which depends on the values of the integrand at all points on contour $C$. We will see below that this suspicion is warranted, but we will also see that yet another simple modification of (3.3) will make it possible to obtain a correct representation of $B(x)$.

Let us attempt to use contour (3.4) in (3.3), and let us see whether the resulting expression satisfies Eq. (3.2):


FIG. 11. Variation of the contour $C$ in expression (3.3).
$\left.\partial_{[\lambda} \int_{0}^{1} \Phi_{\mu_{1}} \ldots \mu_{k}\right] \mu_{k+1}(x t) x^{\mu_{k+1}} \mathrm{~d} t$
$=\int_{0}^{1}\left[\Phi_{\mu_{1} \ldots \mu_{k} \lambda}(x t)+\Phi_{\left[\mu_{1} \ldots \mu_{k} \check{\mu}_{k+1}, \lambda\right]}(x t) t x^{\left.\mu_{k+1}\right] d t}\right.$.
What are we to do with the second term in this equation? Making use of the closed nature of the $(k+1)$-form $\Phi$, we can interchange the indices $\mu_{k+1}$ and $\lambda$ :

$$
\Phi_{\left[\mu_{1} \ldots \mu_{k} \check{\mu}_{k+1}, i\right]}=\frac{1}{k+1} \Phi_{\mu_{1} \ldots \mu_{k} \lambda, \mu_{k+1}}
$$

It is now simple to see that the square brackets in (3.5) contain a total derivative with respect to $t$ :

$$
\begin{aligned}
& \left(\Phi_{\mu_{1} \ldots \mu_{k} \lambda}(x t)+\frac{t x^{\mu_{k+1}}}{k+1} \Phi_{\mu_{1}} \ldots \mu_{k^{\lambda}, \mu_{k+1}}\right) \\
& =\frac{1}{(k+1) t^{k}} \frac{\mathrm{~d}}{\mathrm{~d} t}\left(t^{k+1} \Phi_{\mu_{2}} \ldots \mu_{k} \lambda(x t)\right)
\end{aligned}
$$

Clearly, if we integrate this expression with a weight of $t^{k}$, rather than with a unit weight, as in (3.3) and (3.5), we would find

$$
\begin{equation*}
\Phi_{\mu_{1} \ldots \mu_{k^{\lambda}}}(x)-\lim _{t \rightarrow+0} t^{k+1} \Phi_{\mu_{1}} \ldots \mu_{k^{\lambda}}(t x)=\Phi_{\mu_{1} \ldots \mu_{k^{\lambda}}}(x) \tag{3.6}
\end{equation*}
$$

In other words, the solution of Eq. (3.2) becomes

$$
\begin{equation*}
B_{\mu_{1} \ldots \mu_{k}}(x)=(-)^{k} \int_{0}^{1} \Phi_{\mu_{1} \ldots \mu_{k+1}}(x t) x^{\mu_{k+1} t^{k}} \mathrm{~d} t \tag{3.7}
\end{equation*}
$$

We have deliberately retained the limit $t \rightarrow 0$ in (3.6). There are some important limits of singular forms $\Phi$ for which this limit is nonzero (Subsection 3.2.3).

### 3.2.2. Fixed-point gauge

Expression (3.7) should be familiar to anyone who is familiar with the fixed-point gauge, which is actively used in quantum chromodynamics. ${ }^{54.57}$ Indeed, there exists a representation of the gluon field $A_{\mu}$ in terms of the intensity $F_{\mu}=\partial_{\mu} A_{v}-\partial_{r} A_{\mu}+\left[A_{\mu} A_{v}\right]:$

$$
\begin{equation*}
A_{\mu}(x)=-\int F_{\mu v}(t x) x_{\nu} t \mathrm{~d} t \tag{3.8}
\end{equation*}
$$

[This is not a literal analog of (3.7), since in the non-abelian case we would have $d F \neq 0$ and $F=\mathrm{d} A+A^{2} \neq \mathrm{d} A$. This circumstance is unimportant here, and we will ignore it, especially since we offer a derivation below which applies to both (3.7) and (3.8).]

Expression (3.8) holds in the fixed-point gauge, which is specified by the condition

$$
\begin{equation*}
x_{\mu} A_{\mu}(x)=0 \tag{3.9}
\end{equation*}
$$

We note at this point that the $k$-form $B$ is also determined ambiguously from the equation $d B=\Phi$ : There exists a "gauge arbitrariness," and (3.7) corresponds to a definite fixing of the gauge in accordance with, again, condition (3.9):

$$
\begin{equation*}
x_{\mu_{\mathrm{s}}} B_{\mu_{1} \ldots \mu_{k}}(x)=0 \quad \text { (i.e., } x * B=0 \text { ) } \tag{3.10}
\end{equation*}
$$

Conversely, condition (3.10) is by itself a sufficient condi-
tion for finding (3.7). Specifically, we have

$$
\begin{align*}
B_{\mu_{1} \ldots \mu_{k}}(y) & =\frac{\partial}{\partial y_{\mu_{2}}}\left(y_{\lambda} B_{\lambda \mu_{1}} \ldots \mu_{k}\right)-y^{2} \frac{\partial}{\partial y_{\mu_{2}}} B_{\lambda \mu_{2}} \ldots \mu_{k} \\
& =-y_{\lambda} \frac{\partial}{\partial y_{\mu_{1}}} B_{\lambda \mu_{2}} \ldots \mu_{k} \tag{3.11}
\end{align*}
$$

It would be convenient to have a derivative with respect to $Y_{\lambda}$, rather than $Y_{\mu_{1}}$. We can arrange this by making use of the relation

$$
\begin{align*}
& \frac{\partial}{\partial y_{\mu_{1}}} B_{\lambda_{1}} \ldots \mu_{k}-\frac{\partial}{\partial y_{\mu_{2}}} B_{\lambda \mu_{1} \mu_{2}} \ldots \mu_{k}+\ldots \\
& =\Phi_{\mu_{s} \lambda \mu_{1}} \ldots \mu_{k}+\frac{\partial}{\partial y_{\lambda}} B_{\mu_{1}} \ldots \mu_{k} \tag{3.12}
\end{align*}
$$

In order to find a linear combination of this sort, it is sufficient to note that in addition to (3.11) we have

$$
B_{\mu_{1}} \ldots \mu_{k}(y)=-y_{\lambda} \frac{\partial}{\partial y_{\mu_{z}}} B_{\mu_{1} \lambda_{1}} \ldots \mu_{k}
$$

etc. We can thus write
$k B_{\mu_{1}} \ldots \mu_{k}(y)+y_{\lambda} \frac{\partial}{\partial y_{\lambda}} B_{\mu_{1}} \ldots \mu_{k}(y)=-y_{\lambda} \Phi_{\mu_{1} \lambda \mu}, \ldots \mu_{k}$
$=(-)^{k} \Phi_{\mu_{1}} \ldots \mu_{k+1}(y) y^{\mu_{k+1}}$.
Now substituting $y$ in the form $y=t x$, we find a total derivative on the left,

$$
\frac{1}{t^{k-1}} \frac{\mathrm{~d}}{\mathrm{~d} t}\left[t^{k} B_{\mu_{1}} \ldots \mu_{k}(t x)\right]
$$

and on the right we find $(-)^{k} \Phi_{\mu_{1} \ldots \mu_{k},}(t x) x^{\mu_{k} \cdot{ }_{t} t \text {. Integrat- }}$ ing this expression over $d t$ with a weight of $t^{k-1}$, we again find (3.7). Returning to (3.8) we note that in the derivation given here the replacement of the short derivative $\partial$ by the long derivative $\delta+A$ might be dangerous in (3.12). In the fixed-point gauge, however, the necessary equation

$$
y_{\mu} \frac{\partial A_{\mu}}{\partial y_{v}}-y_{\mu} \frac{\partial A_{v}}{\partial y_{\mu}}==-y_{\mu} F_{\mu v}
$$

holds.

### 3.2.3. Integration of singular forms

This subsection is purely technical, concerned with the violation of (3.7) due to the singularity of form $\Phi$ at zero, mentioned at the end of Subsection 3.2.1. We will discuss here a simple example, but one which is important and representative. We assume that $\Phi_{\mu}$ has the form $q_{\mu} \delta\left(q^{2}\right)$ in momentum space (and the form $\frac{x^{\prime \prime}}{x^{2 n}}$ in coordinate space, in $2 n$ dimensions). We then have $q_{\mu} \Phi_{\mu}(q)=q^{2} \delta\left(q^{2}\right)=0$; i.e., the divergence of $\Phi_{\mu}(x)$ is zero. This result means that the $(2 n-1)$-form $\Phi=* \widetilde{\Phi}, \Phi_{\mu_{1} \ldots \mu_{1,} \quad}=\varepsilon_{\mu_{1, \ldots}, \mu_{2 n}} \widetilde{\Phi}_{\mu_{2 n}}$, is closed. Some questions arise here: Of just what is this the external product? What is $B_{\mu_{1} \ldots \mu_{2 n}}$ equal to? The problem here is that $B$ has $2 n-2$ indices and necessarily contains an $\varepsilon$-symbol with $2 n$ indices. The two extra indices have to be contracted with something. At first glance it would appear that we have no vectors other than $x^{4 t}$ at our disposal. But it is just one. Let us see what we can find from (3.7):
$B_{\mu_{1}} \ldots \mu_{2 n-2}(x)=\int_{0}^{1}\left[\varepsilon_{\mu_{1}} \ldots \mu_{2 n} \frac{(x t)^{\mu_{2 n}}}{|x t|^{2^{n}}}\right] x^{\mu_{2 n-1} t^{2 n-2}} \mathrm{~d} t=0$.

What has happened? The answer is relation (3.6). The product

$$
i^{2 n-1} \Phi_{\mu_{1} \ldots . \mu_{2 n=1}}(t x)=\varepsilon_{\mu_{1}} \ldots \mu_{2 n} \frac{x^{\mu_{2 n}}}{x^{2 n}}
$$

is completely independent of $t$; in particular, it does not vanish in the limit $t \rightarrow 0$, so that (3.7) is inapplicable. [The derivation given in Subsection 3.2.2. does not apply, for a similar reason: It follows from that derivation that (3.7) actually determines not $B(x)$ but $\left.B(x)-\lim _{t \rightarrow+0} t^{k} B(t x)\right]$.

If (3.7) is to be used in the present situation, it must be regularized. The limit $\lim _{t \rightarrow 0} t^{2 n-1} \Phi(t x)$ becomes vanishing if we replace the function $\Phi(y)$ by $\Phi(y+\mathbf{n})$, where $\mathbf{n}_{\mu}$ is any nonzero vector. Removing the regularization corresponds to the limit $|\mathbf{n}| \rightarrow 0$. After this replacement, the integral in (3.13) becomes
$B_{\mu_{1} \ldots \mu_{2 n-2}}(x)=\int_{0}^{1} \varepsilon_{\mu_{1}} \ldots \mu_{2 n} \frac{(x t+\mathbf{n})^{\mu_{2 n}}}{|x t+\mathbf{n}|^{2 n}} x^{\mu_{2 n-1} t^{2 n-2}} \mathrm{~d} t$
$=\varepsilon_{\mu_{1}} \ldots \mu_{2 n} \frac{\mathbf{n}^{\mu_{2 n}}{ }^{\mu_{2 n-1}}}{x^{2 n}} \int_{0}^{1} \frac{t^{2 n-2} \mathrm{~d} t i}{\left[t+\left(2(x \mathrm{n}) / x^{2}\right)\right]^{2 n}}+O(\mathrm{n})$
$=\varepsilon_{\mu_{1} \ldots \mu_{2 n}} \frac{x^{\mu_{2 n-1} \mu_{2 n}}}{2 x^{2 n-2}(x n)}+O(\mathbf{n})$,
or, in momentum space,

$$
\begin{equation*}
B_{\mu_{1}} \ldots \mu_{2 n-2}(q) \sim \varepsilon_{\mu_{1}} \ldots \mu_{2 n} \frac{q^{\mu_{2 n-1} \mathbf{n}^{\mu_{2 n}}}}{(q n)} \delta\left(q^{2}\right) \tag{3.15}
\end{equation*}
$$

In two dimensions ( $n=1$ ), for example, we have

$$
\varepsilon_{\alpha \mu} q_{\mu} \delta\left(q^{2}\right)=q_{\alpha}\left[\varepsilon_{\mu \nu} \frac{q^{\mu} \mathbf{n}^{v}}{(q \mathbf{n})} \delta\left(q^{2}\right)\right]
$$

The dependence of $B$ on the direction of the regularizing vector $\mathbf{n}_{\mu}$ is not surprising: It corresponds to the arbitrariness in the determination of $B$ from (3.2). An important point is that there exist singular forms $\Phi$ for which $B$ cannot be chosen in an "isotropic" form: depending on only the vector $x^{\mu}$ and the parameters in $\Phi$. In such cases, Eq. (3.7) requires regularization, which gives rise to an arbitrary unit vector.

### 3.2.4. Inexact forms and the K operation

The next question concerning (3.7) arises in an examination of unclosed forms $\Phi$. Equation (3.7) determines a linear operation which acts on any forms $\Phi$. What is its meaning in the case $d \Phi \neq 0$ ? For brevity, we denote this operation by $K$. The $K$ operation lowers the rank of the form by one:

$$
\begin{align*}
& K\left\{\Phi_{v_{1}} \ldots v_{k+1}\right\}_{\mu_{1}} \ldots \mu_{k}(x) \\
& \equiv(-)^{k} \int_{0}^{1} \Phi_{\mu_{1}} \ldots \mu_{k+1}(t x) x^{\mu_{k+1}} t^{k} \mathrm{~d} t . \tag{3.16}
\end{align*}
$$

It follows from the derivation in Subsection 3.2.1. that the $(k+1)$-form $d K\{\Phi\}$ is
$\mathrm{d} K\{\Phi\}_{\mu_{1}} \ldots \mu_{\mu_{k+1}}(x)$
$=\Phi_{\mu_{1} \ldots \mu_{k+1}}(x)-(-)^{k+1} \int_{0}^{1}(\mathrm{~d} \Phi)_{\mu_{1}} \ldots \mu_{k+1} \mu_{k+2} x^{\mu_{k+2} t^{k+1}} \mathrm{~d} t$.
If $\Phi$ is closed, we return to our old assertion, $d K\{\Phi\}=\Phi$. In the case $d \Phi \neq 0$, however, we find the new relation

$$
\begin{equation*}
\mathrm{d} K\{\Phi\}+K\{\mathrm{~d} \Phi\}=\Phi \tag{3.17}
\end{equation*}
$$

or, in "operator form,"

$$
\begin{equation*}
\mathrm{d} K+K \mathrm{~d}=1 \tag{3.18}
\end{equation*}
$$

Identity (3.18) holds when applied to any nonsingular forms (Subsection 3.2.3.) of arbitrary nonzero rank. If the rank of $\Phi$ is zero, it becomes necessary to recall relation (3.6), in which we have, in this case

$$
\lim _{t \rightarrow 0} \Phi(t x)
$$

When applied to a 0 -form, (3.18) is valid only if this limit is zero, i.e., only if the 0 -form vanishes at the origin of coordinates.

Equation (3.16) determines a linear operation which is the inverse of external differentiation: $K=d^{-1}$. Since the operator $d$ is nilpotent $\left(d^{2}=0\right)$, the operator which is its inverse must be determined specifically by rule (3.18), not by the more customary relations $K d=d K=1$ (such an operator would not be defined on inexact forms). The operator $K$ is often called a "homotopic" operator. It is defined ambiguously by (3.18); the choice in (3.16) corresponds to the fixed-point gauge , $x^{*} K\{\Phi\}=0$. The operator $K$ is also nilpotent, and its square satisfies $K^{2}=0$, as follows immediately from (3.16) and from the antisymmetry of the differential forms:
$K^{2}\{\Phi\}_{\mu_{1}} \ldots \mu_{k-2}$
$=(-)^{k-2}(-)^{k-1} \int_{0}^{1} \int_{0}^{1} \Phi_{\mu_{1}} \ldots \mu_{k}(\tau t x)(\tau x)^{\mu_{k} t^{k-1}} \mathrm{~d} t x^{\mu_{k-1}} \tau^{k-2} \mathrm{~d} \tau$ $=0$.

At this point we have a few words regarding the action of the $K$ operation on an external product of differential forms. External differentiation is an odd operation; i.e., it acts on a product in accordance with the rule

$$
\mathrm{d}(A B)=\mathrm{d} A \cdot B+(-)^{R_{A}} A \mathrm{~d} B,
$$

where $R_{\mathrm{A}}$ is the rank of form $A$, and (-) ${ }^{R_{A}}$ is its parity. The inverse $K$ operation does not have this property: When it acts on a product, it is completely impossible to distinguish the effect on only one of the factors (just as the Leibnitz formula, valid for differentiation, does not apply to ordinary integrals).

### 3.2.5. Can $d^{-1}$ be a local operator?

The general answer is of course no. In general, $d^{-1}$ is an integral, nonlocal operation. Nevertheless, there may exist a
certain set of forms on which the operation $d^{-1}$ which does not explicitly contain an integration, acts. Taking a bit of liberty, we will classify such an operation as local. Below we will discuss the important class of forms which are constructed from the 1 -form $A(x)$ and its external derivative $d A(x)$, which do not contain an explicit dependence on $x$. These are precisely the forms which arise in the study of anomalies. Zumino ${ }^{21}$ introduced an operation (which we call $k_{Z}$; it is printed as $k_{z}$ here) on this class of forms which is local and which, under certain stipulations, satisfies the relation $\mathrm{d} k_{z}+k_{z} \mathrm{~d}=1$. We will discuss this operation in Subsection 3.3. In the present subsection we attempt to determine how $K$, given by (3.16), can be converted into a local operation. As a result, the definition of $k_{z}$ becomes clearer, as does its range of applicability. The reader may, if he wishes, move on directly to Subsection 3.3 , whose contents do not depend on the remainder of the present subsection.

We apply the $K$ operation to an exact 2 -form $d A$ :

$$
\begin{align*}
K\{\mathrm{~d} A\}_{\mu}(x) & =-\int(d A)_{\mu v}(x t) x^{v} t \mathrm{~d} t \\
& =\int\left[-\partial_{\mu} A_{v}(x t)+\partial_{v} A_{\mu}(x t)\right] x^{v} t \mathrm{~d} t \\
& =A_{\mu}(x)-\frac{\partial}{\partial x^{\mu}} \int_{0}^{1} A_{v}(x t) x^{v} \mathrm{~d} t \\
& =\left[A-\mathrm{d} \int_{0}^{1} A_{v}(x t) x^{v} \mathrm{~d} t\right]_{\mu} \tag{3.19}
\end{align*}
$$

As a result, we find not the original form $A$ but its "gauge transformation." After what we saw in Subsection 3.2.2., this result should not be surprising. We already know that the application of the $K$ operation leads to a result in the fixed-point gauge.

We stipulate that we are choosing the field $A$ in the fixed-point gauge. The relation $K\{d A\}=A$ then holds. Furthemore, in this case we have

$$
K\{A\}=\int_{0}^{1} A_{\mu}(x t) x^{\mu} \mathrm{d} t=0
$$

We now consider a form which is an external product of any number of $A$ 's and $d A$ 's, and we apply the $K$ operation to it.

In applying the $K$ operator to the external product of forms $A$ and $d A$, we obtain in the integrand a sum of several terms which correspond to a convolution of $x^{\lambda}$ with various factors. Each time $x^{\lambda}$ is convolved with $A^{\lambda}$, we obtain a zero in the fixed-point gauge, while when $x^{\lambda}$ is convolved with $(\mathrm{d} A)_{\mu \lambda}=\left[\partial_{\mu} A_{\lambda}-\partial_{\lambda} A_{\mu}\right](x t)$, we obtain an expression in which

$$
\mathrm{d} A_{\mu \lambda}(x t) x^{\lambda}
$$

is replaced by

$$
-\frac{1}{t} \frac{\mathrm{~d}}{\mathrm{~d} t}\left(t A_{\mu}(x t)\right)
$$

For example, when $K$ is applied to the product of $p$ forms $d A$ and $k+1-2 p$ forms $A$ [this is a $(k+1)$-form], we obtain the sum

$$
\begin{align*}
& K\{\ldots A \ldots \mathrm{~d} A \ldots \mathrm{~d} A \ldots A\} \\
& \equiv(-1)^{k} \int_{0}^{1}(\ldots A \ldots \mathrm{~d} A \ldots \mathrm{~d} A \ldots A \ldots)(x t) \ldots \lambda x^{\lambda t^{k}} \mathrm{~d} t \\
& =(-)^{h} \int_{0}^{1}\left\{\ldots+0+\ldots+\ldots A \ldots\left[-\frac{1}{t} \frac{\mathrm{~d}}{\mathrm{~d} t}(t A(x t))\right]\right. \\
& \ldots \mathrm{d} A \ldots A \ldots+\ldots A \ldots \mathrm{~d} A \ldots\left[-\frac{1}{t} \frac{\mathrm{~d}}{\mathrm{~d} t} t A(x t)\right] \\
& \ldots A \ldots+\ldots+0+\ldots\} t^{h} \mathrm{~d} t . \tag{3.20}
\end{align*}
$$

In each of the nonzero terms, we now have to "distribute" $t^{k}$ as follows: one $t$ to each of $k+1-2 p$ factors $A$ and two $t$ 's to $p-1$ factors $d A$. Yet another $t$ is expended on cancelling $1 / t$ in

$$
-\frac{1}{t} \frac{\mathrm{~d}}{\mathrm{~d} t}(t A)
$$

As a result, all of the $A(x t)$ are replaced by $t A(x t)$, and the $d A(x t)$ are replaced by

$$
t^{2} d A_{\mu v}(x t)=\frac{\partial}{\partial x^{\mu}}\left[t A_{v}(x t)\right]
$$

[we recall that

$$
\left.(\partial A)(x t) \equiv \frac{\partial}{\partial y} A(y)\right|_{y=x t}=\frac{1}{t} \frac{\partial}{\partial x}(A(x t)]
$$

We can now set $x=0$ and take the integral over $t$. The operation found by this formal approach looks the same as if we had replaced one of the factors by $A$ and written an additional factor
$\int_{0}^{1} t^{p+(k+1-2 p)-1} \mathrm{~d} t=\int_{j}^{1} t^{k-p} \mathrm{~d} t=\frac{1}{k+1-p}$
$=$ (the number of times the letter $A$ appears in the product $)^{-1}$.

Finally, the sign is determined by the circumstance that $K$ is an odd operation; i.e., a minus sign appears when we pass through each 1 -form $A$ or external differentiation. This is the $k_{\mathrm{z}}$ operation. In the following subsection we discuss this operation in greater detail, since it is extremely important in dealing with anomalies.

### 3.3. The $k_{z}$ operation ${ }^{21}$

This operation is defined only on forms constructed from a 1 -form $A$ and its external derivative $d A$. The $k_{\mathrm{z}}$ operation lowers the rank of the form by one, and on the specified class of forms it satisfies the relation

$$
\begin{equation*}
k_{z} \mathrm{~d}+\mathrm{d} k_{z}=1 \tag{3.21}
\end{equation*}
$$

i.e., it acts as $d^{-1}$ on this class. We begin with actions on the "elementary" forms:

$$
\begin{equation*}
k_{z} A=0, \quad k_{z}(\mathrm{~d} A)=A \tag{3.22}
\end{equation*}
$$

Clearly, (3.21) holds:

$$
\begin{aligned}
& \left(k_{z} \mathrm{~d}+\mathrm{d} k_{z}\right) A=k_{z}(\mathrm{~d} A)=A \\
& \left(k_{z} \mathrm{~d}+\mathrm{d} k_{\mathrm{z}}\right) \mathrm{d} A=\mathrm{d}\left(k_{\mathrm{z}}(\mathrm{~d} A)\right)=\mathrm{d} A
\end{aligned}
$$

We now consider the product of several forms $A$ and $d A$, e.g., $A d A$. We assume that $k_{z}$ acts on a product as a local odd operation:

$$
\begin{aligned}
k_{x} A & =-A k_{z}+\left(k_{z} A\right)=-A k_{z} \\
k_{x} \mathrm{~d} A & =+(\mathrm{d} A) k_{x}+\left(k_{z}(\mathrm{~d} A)\right)=+(\mathrm{d} A) k_{z}+A
\end{aligned}
$$

We then have $k_{z}(A \mathrm{~d} A)=-A k_{z} \mathrm{~d} A=-A A=-A^{2}$. Does (3.21) hold?

$$
\begin{aligned}
d k_{z}(A \mathrm{~d} A)=d\left(-A^{2}\right) & =-\mathrm{d} A A+A \mathrm{~d} A \\
k_{z} \mathrm{~d}(A \mathrm{~d} A)=k_{z}\left(\mathrm{~d} A^{2}\right) & =\left(A+(\mathrm{d} A) k_{z}\right) \mathrm{d} A \\
& =A \mathrm{~d} A+\mathrm{d} A A
\end{aligned}
$$

We find $\left(d k_{z}+k_{z} d\right)(A d A)=2 A d A$. An extra factor of 2 has appeared. We see from this result that a local $k_{z}$ operation does not exist. However, let us see what happens in the general case. We take the product of $p 1$-forms $A$ and of $q 2$ forms $d A$, written in any order. We have
$\mathrm{d} k_{z}\{. . . A . . A .$.
$=\ldots+\ldots\left(\mathrm{d} k_{z} A\right) \ldots A \ldots+\ldots A \ldots\left(\mathrm{~d} k_{z} A\right) \ldots$
$+\left[\ldots(\mathrm{d} A) \ldots\left(k_{z} A\right) \ldots-\ldots\left(k_{z} A\right) \ldots(\mathrm{d} A) \ldots\right]+\ldots$
On the other hand, we have

```
\(k_{z} d\{\). . A. . . \(A .\). . \(\}\)
\(=\ldots+\ldots\left(k_{z} \mathrm{~d} A\right) \ldots A \ldots+\ldots A \ldots\left(k_{2} \mathrm{~d} A\right) \ldots\)
\(+\left[-\ldots(\mathrm{d} A) \ldots\left(k_{2} A\right) \ldots+\ldots\left(k_{x} A\right) \ldots(\mathrm{d} A) \ldots\right]+\ldots\).
```

An important point is that the expressions in square brackets in these two equations differ only in their sign. We will change nothing by replacing $A$ by $d A$. This observation means that when the sum $k_{z} d+d k_{z}$ is applied to a product it is not necessary to consider cross terms, in which $d$ and $k_{z}$ act on different elementary forms:

$$
\begin{aligned}
& \left(\mathrm{d} k_{z}+k_{z} \mathrm{~d}\right)\{\ldots A \ldots \mathrm{~d} A\} \\
& =\ldots+\ldots\left[\left(\mathrm{d} k_{z}+k_{z} \mathrm{~d}\right) A\right] \ldots \mathrm{d} A \ldots \\
& +\ldots A \ldots\left[\left(k_{z} \mathrm{~d}+\mathrm{d} k_{z}\right) \mathrm{d} A\right] \ldots+\ldots \\
& =(p+q) \ldots . \ldots \mathrm{d} A \ldots
\end{aligned}
$$

All the signs are plus signs here since $k_{z} d+d k_{z}$ acts as an even form.

We thus see that, under the assumption that $k_{z}$ is local, during the application of $d k_{z}+k_{z} d$ to the product of $p$ forms $A$ and of $q$ forms $d A$ it is multiplied not by unity but by $p+q$. In order to satisfy relation (3.21), we should introduce an additional rule: We assign a factor of $1 /(p+q)$ to each such product. We wish to emphasize that after we have done this $k_{z}$ is no longer a local operation in the sense that its action on a product is not determined as an action on the factors one after another; we also need a "global"' characteristic: the total number of forms $p+q$. Zumino ${ }^{21}$ suggested the following mnemonic rule for dealing with this "nonlocal nature": assign a factor $t$ to each $A$ and $d A$ and take the integral

$$
\int_{0}^{1} \frac{\mathrm{~d} t}{t}
$$

of the resulting expression. Clearly, a factor of $1 /(p+q)$ is reproduced in this manner. Less clear are the origin and meaning of this rule. In the preceding subsection we attempted to show that the nonlocal nature of the $k_{z}$ operation and the appearance of a parameter $t$ are consequences of the general structure of the operation $d^{-1}$.

To avoid any misunderstanding, we need to point out that $k_{z}$ is defined "algebraically": The operation of integration here essentially reduces to the replacement of $d A$ by $A$. Consequently, $k_{z}$ "does not know about" the important property that the $D$-forms are closed in a $D$-dimensional space. To see the situation, we consider the 2 -form $A^{2}$ in a two-dimensional space. On the one hand, we have $k_{z} A^{2}=0$, while on the other $d\left(A^{2}\right)$ also vanishes in this situation. However, this result is an explicit contradiction of (3.21)! How could this have happened? Returning to the derivation of the properties of the $k_{z}$ operation, we find $k_{z} \mathrm{~d}\left(A^{2}\right)=\left[\left(k_{z} \mathrm{~d} A\right) A+A\left(k_{z} \mathrm{~d} A\right)\right]=2 A^{2} \neq 0$. In other words $k_{z}$ "does not know" that it is forbidden to raise the rank of a $D$-form and that the product $k_{z}$ does not change the rank. The result found by this algebraic approach contains no hint that the derivation is erroneous. We could look at this example from a slightly different standpoint: The $k_{z}$ operation does not always generate from a closed form the original of this form [as it should, according to (3.21)]:

$$
\left.\mathrm{d}\left(k_{z} \Phi\right)=\Phi-k_{z} \mathrm{~d} \Phi=\Phi\right)
$$

For example, from a closed 2 -form $A^{2}$ in two dimensions we find zero instead of the correct result

$$
\int_{0}^{1} A_{\mu}(t x) A_{v}(t x) x^{v} t \mathrm{~d} t
$$

In order to obtain the original correctly, we need algebraic closure, i.e., $d \Phi=0$, regardless of the dimensionality of the space [while in the example above we algebraically have $\left.\mathrm{d}\left(A^{2}\right)=\mathrm{d} A A-A \mathrm{~d} A \neq 0\right]$. Pathological behavior of this type naturally does not arise for the original $K$ operation [see (3.16)]. When we went from $K$ to $k_{z}$ at the end of the preceding subsection, we needed the fixed-point gauge. Consequently, we need to be particularly careful in applying the $k_{z}$ operation to gauge-invariant forms.

As some important examples we consider two $2 n$-forms in a $2 n$-dimensional space, which arise in a discussion of Dirac anomalies: $W=\operatorname{Tr} F^{n}$ and $W^{a}=\operatorname{Tr} t^{a} F^{n}$ ( $F=\mathrm{d} A+A^{2}$ ). Both these forms are closed, and they can be written in the form $d(K W)$. However, we can replace $k$ by $k_{z}$ only in the case of $W$. At this point we first need to recall that $W$ is a gauge-invariant, while $W^{a}$ is not: If $A \rightarrow U^{-1} A U+U^{-1} d U$ we find $W^{a} \rightarrow \operatorname{Tr} U^{-1} t^{a} U F^{n}$ $\neq W^{a}$. Furthermore, $W$ vanishes during a formal (algebraic) external differentiation, while $W^{a}$ does not ${ }^{11}$ ):

$$
\begin{array}{r}
\mathrm{d} W=n \operatorname{Tr}(\mathrm{~d} F) F^{n-1}=n \operatorname{Tr}(\mathrm{~d} A A-A \mathrm{~d} A) F^{n-1} \\
=n \operatorname{Tr}(F A-A F) F^{n-1}=0, \\
d W^{a}=\operatorname{Tr} t^{a}\left[(F A-A F) F^{n-1}+F(F A-A F) F^{n-2}+\right. \\
\left.\ldots+F^{n-1}(F A-A F)\right]=\operatorname{Tr}\left(A t^{a}-t^{a} A\right) F^{n} \neq 0
\end{array}
$$

[ the last expression is a $(2 n+1)$-form and of course vanishes in a $2 n$-dimensional space, but the $k_{z}$ operation "cannot
know" about this]. The closed form $W^{a}=\operatorname{Tr} t^{a} F^{n}$ can thus be written as the external derivative of only an expression which is nonlocal in terms of $x$, while we have $W=\operatorname{Tr} F^{n}=\mathrm{d}\left\{k_{z} \operatorname{Tr} F^{n}\right\}$. For the discussion below it is convenient to introduce a symmetrized trace, by which we mean a trace which has been rendered symmetric with respect to all possible permutations of the color matrices before the trace is taken:

$$
\operatorname{STr}(A, B)=\frac{1}{2} \operatorname{Tr}\left(A B+(-)^{\left.R_{A}+R_{B} B A\right)}\right.
$$

etc. We then have

$$
\begin{aligned}
\operatorname{Tr} F^{n} & =S \operatorname{Tr} F^{n}(\equiv \operatorname{STr}(F, F, \ldots, F))=\mathrm{d}\left(k_{\boldsymbol{z}} \mathrm{S} \operatorname{Tr} F^{n}\right) \\
& =\mathrm{d}\left[k_{z} \sum_{k=0}^{n} C_{n}^{k} \operatorname{STr}\left(\mathrm{~d} A^{k},\left(A^{2}\right)^{n-k}\right)\right] \\
& =d \sum_{n=0}^{n} \frac{k C_{n}^{h}}{2 n-k} \operatorname{STr}\left(\mathrm{~d} A^{k-1}, A,\left(A^{2}\right)^{n-k}\right)
\end{aligned}
$$

Here $C_{n}^{k}=n!/ k!(n-k)!$ are the binomial coefficients, and the factor $2 n-k=2(n-k)+k$ results from the integration over $t$ in the definition of the $k_{z}$ operation.

We have one more comment regarding the application of the $k_{z}$ operation to $\operatorname{Tr} F^{n}$. It is clear that when $k_{z}$ is applied to a closed form it is necessary (but not sufficient) that it not contain terms of the type $A^{k}$, which do not contain $d A$. For a closed form we have $d\left\{k_{z} \Phi\right\}=\Phi$, and since $k_{z} \Phi$ does not contain integrals, when we differentiate it we necessarily find an expression proportional to $d A$. It is thus clear that a term without $d A$ in $\operatorname{Tr} F^{n}$, i.e., $\operatorname{Tr} A^{2 n}$, is zero. This is indeed the case; e.g.,

$$
\begin{aligned}
\operatorname{Tr} A^{2} & =\frac{1}{2!} \varepsilon_{\mu \nu} \operatorname{Tr} A_{\mu} A_{\nu}=\frac{1}{2!\cdot 2} \varepsilon_{\mu \nu} A_{\mu}^{a} A_{\nu}^{a} \approx 0 \\
\operatorname{Tr} A^{4} & =\frac{1}{4!} \varepsilon_{\mu v \alpha \beta} \operatorname{Tr} A_{\mu} A_{\nu} A_{\alpha} A_{\beta} \\
& =\frac{1}{4!\cdot 8} \varepsilon_{\mu v \alpha \beta} f^{a b m} f^{c d m} A_{\mu}^{a} A_{v}^{b} A_{\alpha}^{c} A_{\beta}^{u}=0
\end{aligned}
$$

(the latter equality stems from the Jacobi identity for structure constants: $f^{a \mid b m} f^{c d] m}=0$ ). The general proof looks much simpler:
$\operatorname{Tr} A^{2 n}=\operatorname{Tr} A A^{2 n-1}=(-)^{2 n-1} \operatorname{Tr} A^{2 n-1} A=-\operatorname{Tr} A^{2 n}=0$
[we have moved $A$ from left to right within the trace, and the factor $(-)^{2 n-1}$ has arisen because of the interchange of the 1 -form $A$ with the ( $2 n-1$ )-form $A^{2 n-1}$ ). It should not be surprising that $\operatorname{Tr} t^{a} F^{n}$ does not have an analogous property: $\operatorname{Tr} t^{a} d A^{2 n} \neq 0$, since even at $n=1$ we have

$$
\operatorname{Tr} t^{a} A^{2}=\frac{1}{8} f^{a b c} \varepsilon_{\mu v} A_{\mu}^{b} A_{v}^{c} \neq 0
$$

### 3.4. Relations among anomalies ${ }^{20-23.33}$

In this section we discuss the relations among the expressions on the right sides of the four simplest anomalies.

A Dirac abelian anomaly in $2 n$ dimensions,

$$
\begin{align*}
& \mathrm{d} * J^{5} \equiv W^{(2 n)}=\frac{2}{(2 \pi)^{n} n!} \operatorname{Tr} F^{n} \\
& F=\mathrm{d} A+A^{2}, \quad J_{\mathbf{u}}^{\mathbf{s}}=\bar{\psi} \gamma_{\mu} \gamma^{5} \psi . \tag{3.23}
\end{align*}
$$

A Dirac color anomaly in $2 n$ dimensions,
$\left(\mathrm{D} * J^{5}\right)^{a} \equiv W^{a\left(2^{n}\right)}=\frac{2}{(2 \pi)^{n} n} \operatorname{Tr} t^{a} F^{n}, \quad J_{\mu}^{5 a}=\bar{\psi} \gamma_{\mu} \psi^{5} t^{a} \psi$.

A Wess-Zumino term (Chern-Simons class) in a $(2 n+1)$-dimensional gauge theory,

$$
\begin{align*}
& \ln \operatorname{det} \mathrm{D}_{\mathrm{anom}}=\int_{0} W_{0}^{(2 n+1)}, \quad W_{0}^{(2 n+1)} \\
& \quad=\frac{1}{2(2 \pi)^{n}(n+1)!} \operatorname{Tr}\left[A(\mathrm{~d} A)^{n-1}+\ldots\right] \tag{3.25}
\end{align*}
$$

A Weyl anomaly in $2 n$ dimensions,

$$
\begin{equation*}
\left(\mathrm{D} * J^{L}\right)^{a} \equiv W_{1}^{a(2 n)}=\frac{2}{\left.(2 \pi)^{n}(n+1)\right]} \operatorname{Tr} t^{a}\left[(\mathrm{~d} A)^{n}+\ldots \mathrm{l}\right. \tag{3.26}
\end{equation*}
$$

We can show that these structures are related to each other by the $k_{z}$ operation in the following way (Fig. 12):

$$
\begin{align*}
& W_{0}^{(2 n+1)}=(\pi / 2) k_{z} W^{\left(2^{2 n+2)}\right.} \\
& \text { or }(\pi / 2) W^{\left(2^{n+2}\right)}=\mathrm{d} W_{0}^{(2 n+1)} \tag{3.27}
\end{align*}
$$

For $n=1$, for example, we have

$$
\frac{\pi}{2} \cdot \frac{1}{4 \pi^{2}} \operatorname{Tr} F^{2}=\mathrm{d} \frac{1}{8 \mathrm{\pi}} \operatorname{Tr}\left(A \mathrm{~d} A+\frac{2}{3} A^{3}\right)
$$

The left branch of the diagram corresponds to the relation

$$
\begin{equation*}
W^{a\left(2^{n}\right)}=4 \frac{\delta W_{1}^{\left(2^{n+1}\right)}}{\delta A^{a}} . \tag{3.28}
\end{equation*}
$$

For example,

$$
\begin{aligned}
\frac{2}{2 \pi} \operatorname{Tr} t^{a} F & =4 \frac{\delta}{\delta A^{a}} \frac{1}{8 \pi} \operatorname{Tr}\left(A \mathrm{~d} A+\frac{2}{3} A^{3}\right) \\
& =\frac{2}{2 \pi} \operatorname{Tr} t^{a}\left(\mathrm{~d} A+A^{2}\right)
\end{aligned}
$$

The second branch of the diagram stems from more-complicated transformations, specifically gauge variations of fields. They are $\delta_{u}: A \rightarrow A+[A u]+\mathrm{d} u ; \delta_{u}: F=\mathrm{d} A+A^{2} \rightarrow \mathrm{~d} A$ $+[\mathrm{d} A, u]-[A, \mathrm{~d} u]+A^{2}+A[A u]+[A u] A+\mathrm{d} u A$ $+A \mathrm{~d} u=F+[F u)$. The arrow in Fig. 12 corresponds to the relation


FIG. 12. Hierarchy of anomalies. The transitions indicated by the dashed arrows are not discussed in the text.

$$
\begin{gather*}
u^{a} W_{1}^{a(2 n)}=4 k_{2} \delta_{u} W_{0}^{(2 n+1)} \\
\text { or } \mathrm{d}\left(u^{a} W_{1}^{a(2 n)}\right)=4 \delta_{u} W_{0}^{(2 n+1)} \tag{3.29}
\end{gather*}
$$

For $n=1$, for example, we have

$$
\mathrm{d}\left(u^{a} W_{1}^{a(2)}\right)=\mathrm{d} \frac{1}{2 \pi} \operatorname{Tr} u \mathrm{~d} A=\frac{1}{2 \pi} \operatorname{Tr} \mathrm{~d} u \mathrm{~d} A .
$$

On the other hand, we have

$$
\begin{aligned}
4 \delta_{u} W_{0}^{(3)} & =4 \delta_{u} \frac{1}{8 \pi} \operatorname{Tr}\left(A \mathrm{~d} A+\frac{2}{3} A^{3}\right) \\
& =4 \frac{1}{8 \pi} \operatorname{Tr}\{\mathrm{dud} A-A[\dot{A} \mathrm{~d} u] \\
& \left.+\frac{2}{3}\left(A^{2} \mathrm{~d} u+A \mathrm{~d} u A+\mathrm{d} u A^{2}\right)\right\}=\frac{1}{2 \pi} \operatorname{Tr} \mathrm{~d} u \mathrm{~d} A .
\end{aligned}
$$

The reader will easily recognize in these relations between different anomalies the coupling equations which we have already discussed in Subsection 2.3. There, however, we were dealing with left-hand fermion parts of equalities (3.23)-(3.26). Here we are concerned with right-hand, boson parts; in particular, we wish to derive general equations for $W_{0}^{(2 n+1)}$ and $W_{1}^{a(2 n)}$. The hierarchy of anomalies given by (3.27)-(3.29) is a purely algebraic construction, which relates different cohomological characteristics of operations involving a variation with respect to the gauge field (connectivity) ${ }^{55}$ and gauge transformations. ${ }^{33}$ From the practical standpoint, Eqs. (3.27)-(3.29) make it possible to find automatically complicated expressions for the Weyl anomaly in spaces with more dimensions by working from the simple expression (3.23) for the Dirac anomaly in a space with a dimensionality two units higher. We also note that the hierarchy of anomalies is not exhausted by the diagram (Fig. 12): We could go down even further, into spaces of $2 n-1,2 n-2$, etc., dimensions. (A method for generating new cohomological characteristics in this way is sometimes called the "descent method.") However, the physical interpretation of the equations which appear as a result is still in dispute, ${ }^{33,50-60}$ and we will not discuss those questions here.

The remainder of Subsection 3.4 is purely technical. We will write explicit expressions for all the anomalies.

We begin with an abelian Dirac anomaly:

$$
\begin{align*}
\frac{(2 \pi)^{n} n!}{2} W^{(2 n)} & =\operatorname{Tr} F^{n}=\mathrm{S} \operatorname{Tr} F^{n}=\sum_{k=0}^{n} C_{n}^{k} \mathrm{~S} \operatorname{Tr}\left(\mathrm{~d} A^{k},\left(A^{2}\right)^{n-k}\right) \\
& =\mathrm{d} \sum_{k=1}^{n} \frac{C_{n}^{k} k}{2 n-k} \mathrm{~S} \operatorname{Tr}\left(\mathrm{~d} A^{k-1}, A,\left(A^{2}\right)^{n-k}\right) \tag{3.30}
\end{align*}
$$

The last equality follows from the properties of the $k_{z}$ operation; the reader is referred to Subsection 3.3, where all the nuances of this transition were analyzed. It follows from (3.30) that we have

$$
\begin{align*}
W_{0}^{(2 n-1)} & =\frac{1}{2(2 \pi)^{n-1} n!} \sum_{k=1}^{n} \frac{C_{n}^{k} k}{2 n-k} \operatorname{STr}\left(\mathrm{~d} A^{k-1}, A,\left(A^{2}\right)^{n-k}\right), \\
C_{n}^{k} & =\frac{n!}{k!(n-k)!} . \tag{3.31}
\end{align*}
$$

To go over to a color Dirac anomaly, we need to vary this expression with respect to the field $A$. It is convenient to
replace (3.31) by an integral representation generated by the $k_{z}$ operation:

$$
\begin{equation*}
2(2 \pi)^{n-1} n!W_{0}^{(2 n-1)}=n \mathrm{~S} \operatorname{Tr} \int_{0}^{1} A F_{\xi}^{n-1} \mathrm{~d} \xi, \quad F_{\xi}=\xi \mathrm{d} A+\xi^{2} A^{2} . \tag{3.32}
\end{equation*}
$$

We thus vary (3.32) with respect to the field $A$. For this purpose we use the replacements $A \rightarrow A+\alpha$ and

$$
\begin{align*}
& \mathrm{STr} A F_{\xi}^{n-1} \rightarrow \operatorname{STr} A F_{\xi}^{n-1}+\operatorname{STr}\left(a, F_{\bar{\xi}}^{n-1}\right) \\
& +(n-1) \operatorname{STr}\left\{A, F_{\xi}^{n-2}, \xi \mathrm{~d} a+\xi^{2}(A a+a A)\right\} \tag{3.33}
\end{align*}
$$

At this point we note two circumstances. First we have $A a+a A=[A, a]$, and thus $\xi \mathrm{d} a+\xi^{2}(A a+a A)=\xi \mathrm{D}_{\xi} a$, where $\mathrm{D}_{\xi}=\mathrm{d}+[\xi A$,$] is a covariant derivative with respect$ to the field $\xi A$. Second, we have

$$
\begin{equation*}
\mathrm{D}_{\xi} F_{\xi}=0 \tag{3.34}
\end{equation*}
$$

This relation can be found from the Bianchi identity $\mathrm{DF}=0$ by replacing $A$ by $\xi A$. As a result, the last trace in (3.33) becomes equal, within a total derivative, to $+\xi(n-1) S \operatorname{Tr}\left(a, \mathrm{D}_{\xi} A, F_{\xi}^{n-2}\right)$. The sign is a plus sign here, since the odd parity of the 1 -form tells us that we have $a\left(\mathrm{D}_{\xi} a\right) B-a\left(D_{\xi} B\right)=\mathrm{D} \xi(a B)$. Furthermore, we have $\mathrm{D}_{\xi} A=\mathrm{d} A+2 \xi \dot{A}^{2}$, and from (3.33) we find

$$
\begin{aligned}
& \frac{\delta}{\delta A^{a}} 2(2 \pi)^{n-1} n!W_{0}^{(2 n-1)} \\
& =n \operatorname{STr}\left\{t^{a}, \int_{0}^{1} \mathrm{~d} \xi\left[F_{\xi}^{n-1}+(n-1)\left(\xi \mathrm{d} A+2 \xi^{2} A^{2}, F_{\xi}^{n-2}\right)\right]\right\} \\
& =n \operatorname{STr}\left\{t^{a}, \int_{0}^{1} \mathrm{~d} \xi\left[\left(n F_{\xi}+(n-1) \xi^{2} A^{2}\right), F_{\xi}^{n-2}\right]\right\} \\
& =n \sum_{k}\left\{n C_{k-1}^{n-1} \frac{S \operatorname{Tr}\left\{t^{a}, \mathrm{~d} A^{k-1},\left(A^{2}\right)^{n-k}\right]}{2 n-k}\right. \\
& \left.\quad+(n-1) C_{k-1}^{n-2} \frac{\operatorname{STr}\left[t^{a}, \mathrm{~d} A^{k-1},\left(A^{2}\right)^{n-k}\right]}{2 n-k}\right\} \\
& =\sum_{k} \frac{n C_{k-1}^{n-1}}{2 n-k} \operatorname{STr}\left[t^{a}, \mathrm{~d} A^{k-1},\left(A^{2}\right)^{n-k}\right][n+(n-k)] \\
& =n \sum_{k} C_{k-1}^{n-1} \operatorname{STr}\left[t^{a}, \mathrm{~d} A^{k-1},\left(A^{2}\right)^{n-k}\right] \\
& =n \operatorname{Tr}\left(t^{a} F^{n-1}\right)=\frac{(2 \pi)^{n-1}}{2} n!W^{o(2 n-2)} .
\end{aligned}
$$

The relation $\delta W_{0}^{(2 n-1)} / \delta A^{a}=W^{a(2 n-2)} / 4$ has thus been proved.

We now consider the relationship between $W_{0}^{(2 n-1)}$ and $W_{1}^{a(2 n)}$. Here we need a gauge variation of expression (3.32):

$$
\begin{aligned}
& \delta_{u}\left\{2(2 \pi)^{n}(n+1)!W_{0}^{(2 n+1)}\right\}=(n+1) \delta_{u} S \operatorname{Tr} \int_{0}^{1} A F_{\xi}^{n} \mathrm{~d} \xi \\
& \begin{array}{r}
(n+1) S \operatorname{Tr} \int_{0}^{1} \mathrm{~d} \xi\left\{\left(\mathrm{~d} u, F_{\xi}^{n}\right)+n\left[A, F_{\xi}^{n-1},-\xi[A, \mathrm{~d} u]\right.\right. \\
\\
\left.\left.\quad+\xi^{2}(A \mathrm{~d} u+\mathrm{d} u A)\right]\right\}
\end{array}
\end{aligned}
$$

$$
\begin{align*}
=(n+1) S \operatorname{Tr} \int_{0}^{1} \mathrm{~d} \xi & \left\{\left(\mathrm{~d} u, F_{\xi}^{n}\right)-n \xi(1-\xi)\right. \\
& \left.\times\left([\mathrm{d} u, A], A, F_{\xi}^{n-1}\right)\right\} . \tag{3.36}
\end{align*}
$$

Understandably, $u$ appears without a derivative only in the commutators $[A, u]$, and $\left[F_{\xi}, u\right]$, and it drops out after the trace is taken. We will accordingly ignore from the outset terms which do not contain $d u$. To prove (3.29) we must first show that (3.36) contains a total derivative. Since this derivative is proportional to $d u$, however, the entire expression (3.36) must be of the form $\operatorname{STr}\{d u, d V\}$. As a starting point, to indicate where we are headed, we will show that (3.36) contains a term $\operatorname{STr}\left\{d u, A^{2 n}\right\}$, which cannot appear exactly in $\operatorname{STr}\{d u, d V\}$ :
$\int_{0}^{1} \mathrm{~d} \xi \mathrm{~S} \operatorname{Tr}\left\{\mathrm{~d} u, \xi^{2 n} A^{2 n}-n \xi(1-\xi)[\mathrm{d} u, A], A, \xi^{(n-1)} A^{2(n-1)}\right\}$ $=\mathrm{S} \operatorname{Tr}\left\{\mathrm{d} u, A^{2 n}\right\} \int_{0}^{1} \mathrm{~d} \xi\left[\xi^{2 n}-2 n \dot{\xi}(1-\xi) \xi^{2(n-1)}\right]$.

The integral is

$$
\frac{1}{2 n+1}-2 n\left(\frac{1}{2 n}-\frac{1}{2 n+1}\right)=0
$$

Before we derive a general expression for $W_{1}^{(2 n-1)}$, we will repeat the entire passage from a Dirac anomaly to a Weyl anomaly. The original expression is

$$
W^{(2 n+2)}=\frac{2}{(2 \pi)^{n+1}(n+1)!} \operatorname{Tr} F^{n+1}
$$

which is an explicitly gauge-invariant expression. We then write $W^{(2 n+2)}$ in the form

$$
\frac{2}{\pi} d W_{0}^{(2 n+1)}
$$

where

$$
W_{0}^{(2 n+1)}=\frac{1}{2(2 \pi)^{n} n!} \operatorname{Tr} \int_{0}^{1} \mathrm{~d} \xi A F_{\xi}^{n}
$$

is no longer necessarily gauge-invariant. However, the change in $W_{0}^{(2 n+1)}$ under gauge transformations of fields is a total derivative: $\delta_{u} W_{0}^{(2 n \cdots)^{1)}}=\mathrm{d} U_{u}$, by virtue of the relation $\mathrm{d} \delta_{u} W_{0}^{(2 n+1)}=(\pi / 2) \delta_{u} W_{0}^{(2 n+2)}=0$. The variation $\delta_{u} W_{0}^{(2 n+1)}$ is linear in $u$, but since $W_{0}^{(2 n+1)}$ is colorless this variation may contain $u$ only in the form $d u$ ( $u$ without a derivative drops out when the trace is taken). We thus have $\delta_{u} W_{0}^{(2 n+1)}=\mathrm{d} U_{u}=\operatorname{Tr} \mathrm{d} u U \quad$ and $\quad 0=\mathrm{d} \delta_{u} W_{0}^{(2 n+1)}$ $=\operatorname{Tr} d u \mathrm{~d} U$. This equation holds for arbitrary $u$, so we have $\mathrm{d} U \equiv 0$ and thus $U=d V$ and $\delta_{u} W_{0}^{(2 n+1)}=\operatorname{Tr} \mathrm{d} u \mathrm{~d} V$. This circumstance means that we can apply the $k_{z}$ operation to (3.36); that operation acts only on $A$ and $d A$, not on $d u$. As a result we find precisely $S \operatorname{Tr} \mathrm{~d} u V:-\mathrm{dS} \operatorname{Tr} \mathrm{d} u V$ $=\operatorname{STr} \mathrm{d} u \mathrm{~d} V=W_{0}^{(2 n+1)}$. Remarkably, the application of $k_{z}$ causes the second term in

$$
\begin{array}{rl}
\delta_{u} W_{0}^{(2 n+1)}=\frac{1}{2(2 \pi)^{n} n!} & S T r \\
\int_{0}^{1} & \mathrm{~d} \xi\{
\end{array} \begin{aligned}
& \mathrm{d} u, F_{\xi}^{n}-n \xi(1-\xi)  \tag{3.37}\\
&\left.\times[\mathrm{d} u, A], A, F_{\xi}^{n-1}\right\}
\end{aligned}
$$

to vanish. The reasons are the odd parity of $k_{z}$ and the taking of a symmetrized trace. We recall (Subsection 3.3) that we have $k_{z} A=0$ and $k_{z} d A=A$. Since $k_{z}$ transforms the 2 -form $d A$ into the 1 -form $A$, it must be an odd operation. As a result of the application of $k_{z}$ to the symmetrized trace $S \operatorname{Tr}\left([\mathrm{~d} u, A], A, F^{n-1}\right)$ we find an expression $\xi^{(n-1)} S \operatorname{Tr}\left([\mathrm{~d} u, A], A, A, F_{\xi}^{n-2}\right)$, which vanishes becasue of the symmetrization with respect to the two 1 -forms $A$. Accordingly, the role of the second term in (3.37) is exclusively one of modifying $\operatorname{Trd} u F_{\xi}^{n}$ to the exact form to permit use of the $k_{z}$ operation. This operation itself acts in a nontrivial way on only $F_{\xi}^{n}$ :

$$
\mathrm{STr}\left(\mathrm{~d} u, F_{\mathrm{E}}^{n}\right) \xrightarrow{k_{z}} n \operatorname{STr}\left(\mathrm{~d} u, \xi A, F_{\bar{\xi}}^{n-1}\right),
$$

since $k_{z} F_{\xi}=k_{z}\left(\xi \mathrm{~d} A+\xi^{2} A^{2}\right)=\xi A$. We recall, however, that by virtue of the definition of the $k_{z}$ operation we should still multiply all the fields $A$ by $t$ and take the integral

$$
\int_{0}^{1} \frac{\mathrm{~d} t}{t} .
$$

Exactly the same role, however, is played by the parameter $\xi$ in the transformation from $W^{(2 n+2)}$ to $W_{o}^{(2 n+1)}$, so that it is sufficient to replace $\xi$ by $\xi t$ and to integrate as follows:

$$
\int_{0}^{1} \frac{\mathrm{~d} t}{t} \int_{0}^{1} \mathrm{~d} \xi .
$$

We then find

$$
k_{z} \operatorname{STr} \int_{0}^{1} \mathrm{~d} \xi\left(\mathrm{~d} u, F_{\xi}^{n}\right)=n \int_{0}^{1} \frac{\mathrm{~d} t}{t} \int_{0}^{1} \mathrm{~d} \xi \mathrm{STr}\left(\mathrm{~d} u, \xi t A, F_{\xi t}^{n}\right)
$$

We switch to the new integration variables $(\xi t)$ and $t$ :

$$
\begin{aligned}
\int_{0}^{1} \frac{\mathrm{~d} t}{t} \int_{0}^{1} \mathrm{~d} \xi \ldots & =\int_{0}^{1} d(\xi t) \int_{(\xi t)}^{1} \frac{\mathrm{~d} t}{t} \frac{1}{t} \cdots \\
& =\int_{0}^{1} \mathrm{~d}(\xi t) \frac{1-(\xi t)}{(\xi t)} \cdots
\end{aligned}
$$

Denoting $(\xi t)$ by $\xi^{\prime}$ and then in terms of $\xi$ again, we find

$$
\begin{aligned}
k_{z} \operatorname{STr} \int_{0}^{1} \mathrm{~d} \xi\left(\mathrm{~d} u, F_{\xi}^{n}\right) & =n \int_{0}^{1} \mathrm{~d} \xi^{\prime} \frac{1-\xi^{\prime}}{\xi^{\prime}} \mathrm{STr}\left(\mathrm{~d} u, \xi^{\prime} A, F_{\xi^{\prime}}^{n-1}\right) \\
& =n \int_{0}^{1} \mathrm{~d} \xi(1-\xi) \mathrm{STr}\left(\mathrm{~d} u, A, F_{\xi^{n-1}}^{n-1}\right)
\end{aligned}
$$

As a result we have

$$
\begin{align*}
& \delta_{u} W_{0}^{(2 n+1)}=\frac{1}{4} \operatorname{Tr} \mathrm{~d} u d V^{(2 n-1)} \\
& =\frac{1}{2(2 \pi)^{n}(n-1)!} d \int_{0}^{1} \mathrm{~d} \xi(1-\xi) \mathrm{STr}\left(\mathrm{~d} u, A, F_{\xi}^{n-1}\right) \tag{3.38}
\end{align*}
$$

The quantity $V^{(2 n-1)}$ itself can be expressed in terms of a symmetrized product $\mathbf{S P}$ (again, a conversion to symmetry in terms of color indices):

$$
\begin{align*}
V^{(2 n-1)} & =\frac{2}{(2 \pi)^{n}(n-1)!} \int_{0}^{1} \mathrm{~d} \xi(1-\xi) \mathrm{SP}\left(A_{1}^{!} F_{\xi}^{n-1}\right) \\
& =\frac{2}{(2 \pi)^{n}(n+1)!}\left(\mathrm{SP} A \mathrm{~d} A^{n-1}+\ldots\right) . \tag{3.39}
\end{align*}
$$

The trace $W_{1}^{a(2 n)}=\operatorname{Tr} t^{a} \mathrm{~d} V^{(2 n-1)}$ determines a Weyl anomaly.

### 3.5. Anomalies and the coboundary operator ${ }^{33}$

The calculations in the preceding subsection can be interpreted in an elegant and meaningful way. We again return to the Wess-Zumino consistency condition. The anomaly $w^{1}(u \mid A) \equiv \int W_{D}^{1}(u \mid A) \mathrm{d} x$ satisfies the relation $\delta_{u} w^{\prime}(v \mid A)$ $-\delta_{v} w^{-1}(u \mid A)-w^{1}([u v] \mid A)=0$. We are interested in nontrivial solutions of this equation which are not gauge transformations of an integral of any local expression, $w^{0}(A) \equiv \int W^{0}(A) \mathrm{d}^{D} x: w^{\prime}(u \mid A) \neq \delta_{u} w^{0}(A)$, for any $w^{0}(A)$. Otherwise, $w^{0}(A)$ could be added to the action as a counterterm, which eliminates the anomaly. In other words, an anomaly exists only if there are solutions $w^{1}(u \mid A)$ of the equation

$$
\begin{align*}
\left(\Delta w^{1}\right)(u, v \mid A) \equiv & \delta_{u} w^{1}(v, A) \\
& -\delta_{v} u^{1}(u \mid A)-w^{1}([u v] \mid A)=0 \tag{3.40}
\end{align*}
$$

which cannot be put in the form $\left(\Delta w^{0}\right)(u \mid A) \equiv \delta_{u} w^{0}(A)$. At this point we need to recall that any variation $\delta_{u} w^{0}(A)$ automatically satisfies condition (3.40): $\Delta\left(\Delta w^{0}\right) \equiv 0$. There is accordingly an operation $\Delta$ whose square is zero ( $\Delta^{2}=0$ ), and we are interested in solutions of the equation $\Delta w^{1}=0$ which are nontrivial in the sense that we have $w^{1} \neq \Delta w^{0}$. (Solutions of the type $w^{1}=\Delta w^{0}$ are trivial in the sense that for them we have $\Delta w^{1}=0$ because of the properties of the operation $\Delta$ not because of the solutions themselves.) The operation $\Delta$ transforms expressions of the type $w^{\circ}(A)$, which depend on only the fields $A$, into expressions $\left(\Delta w^{\circ}\right)(u \mid A)$, which contain an additional dependence on the parameter $u$ of the gauge transformation (or, stated more simply, a dependence on the element of the Lie algebra, $G$ ). In turn, the expressions $w^{1}(u \mid A)$ which already contain a dependence on a single element of the Lie algebra are transformed by this operation into ( $\left.\Delta w^{\prime}\right)(u, v \mid A)$, which depend on the two elements $u$ and $v$. This construction can of course be generalized quite easily. The only point to be careful about is the conservation of the property $\Delta^{2}=0$. For example, for $w^{2}(u$, $v \mid A)=-w^{2}(v, u \mid A)$ one can define
$\left(\Delta w^{2}\right)(u, v, w \mid A)$
$\equiv \delta_{u} w^{2} j(v, w \mid A)+\delta_{v} w^{2}(w, u \mid A)+\delta_{w} w^{2}(u, v \mid A)$
$-w^{2}([u v], w \mid A)-w^{2}([v w], u \mid A)-w^{2}([w u], v \mid A)$.
It is easy to verify that we have
$\left(\Delta^{2} w^{1}\right)(u, v, w \mid A)=\left(\Delta\left(\Delta w^{2}\right)\right)(u, v, w \mid A)$
$=w^{1}([[u v] w] \mid A)$
$+w^{1}([[v w] u] \mid A)+w^{1}([[w u] v] \mid A)=0$
by virtue of the Jacobi identity for the Lie algebra, which can easily be applied if $w^{\prime}(u \mid A)$ is linear in the variable $u$.

In general, for any $n$ we can introduce a linear space $L^{n}$ of expressions $w^{n}\left(u_{1}, \ldots, u_{n} \mid A\right)$ which are completely antisymmetric with respect to all the $u_{1}, \ldots, u_{n}$ and linear with respect to each of them. Such $w^{n}$ are called " $n$-cochains," and the action of a "coboundary operator" on them is defined: $\Delta: L^{n} \rightarrow L^{n+1}$, with $\Delta^{2}=0$. On the $n$-cochain $w^{n}, \Delta$ acts in accordance with the rule

$$
\begin{align*}
& \left(\Delta w^{n}\right)\left(u_{1}, \ldots, u_{n+1} \mid A\right) \\
& =\sum_{i}(-)^{P_{i}} \delta_{u_{i}} w^{n}\left(u_{1} \ldots \dot{u}_{i} \ldots u_{n+1} \mid A\right) \\
& -\sum_{i<j}(-)^{P_{i j}} w^{n}\left(\left[u_{i} u_{j}\right], u_{1} \ldots \dot{u}_{i} \ldots \dot{u}_{3} \ldots u_{n+1} \mid A\right) ; \tag{3.41}
\end{align*}
$$

where $P_{i}$ and $P_{i j}$ are the parities of the interchanges $(i, 1, \ldots$, $i, \ldots, n+l)$ and ( $i, j, 1, \ldots, i, \ldots, \check{j}, \ldots n+1$ ), respectively. The carets mean that the marked letters are omitted. The sequence of mappings

$$
L^{0} \stackrel{\Delta}{\rightarrow} L^{1} \xrightarrow{\Delta} \ldots \xrightarrow{\Delta} L^{n-1} \xrightarrow{\Delta} L^{n} \xrightarrow{\Delta} L^{n+1} \xrightarrow{\Delta} \ldots
$$

has the following property:

the kernel $\operatorname{Ker}_{n} \Delta=\left\{\begin{array}{l}\text { the set of all "coclosed" elements } \\ L^{n} \text { which vanish under the mapping } \\ \Delta \text { from } L^{n} \text { to } L^{n+1},\end{array}\right.$

$$
\operatorname{Im}_{n-1} \Delta \subset \operatorname{Ker}_{n} \Delta
$$

These mappings are not necessarily the same, however: There may exist "cocyclic" elements which are coclosed but not coboundary. For this purpose, the "group of cohomologies" $H^{n}(\Delta) \equiv \operatorname{Ker}_{n} \Delta / \operatorname{Im}_{n-1} \Delta$ must be nontrivial. $\left(\operatorname{Im}_{n-1} \Delta\right.$ and $\operatorname{Ker}_{n} \Delta$ are linear subspaces in $L^{n}$, and therefore
$H^{n} \Delta=\left\{\begin{array}{l}\text { the set of equivalence classes of the elements of } \\ \operatorname{Ker}_{n} \Delta \text { which differ by elements } \operatorname{Im}_{n-1} \Delta\end{array}\right.$
is also a linear space. In particular, it has the structure of an abelian summation group.)

Returning to the anomalies, we can conclude that the anomaly $w^{\prime}(u \mid A)$ is a 1 -cocycle: an element of the group of cohomologies $H^{1}(\Delta)$.

What have we accomplished through this reformulation of the consistency conditions? The first and foremost accomplishment is the determination of the algebraic meaning of this condition. The question of an anomaly has been related to many other (mathematical) questions. There are also some
"more practical" results: An anomaly turns out to be one of the elements of an infinite series: In addition to the first cohomology $H^{1}$ there are also $H^{2}, H^{3}$, etc.; a hierarchy of anomalies has arisen. We will return to this series a bit later.

Finally, we have obtained a new computational method. To find the anomaly $w^{1}(u \mid A)$ we need to write down all the elements of $L^{\prime}$ and to map those linear combinations of these elements which vanish under the application of $\Delta$. We obtain $\mathrm{Ker}_{1} \Delta$. We then need to write down all the elements of $L^{0}$ and to act on them with $\Delta$ : We find $\operatorname{Im}_{0} \Delta$. The difference between $\operatorname{Ker}_{1} \Delta$ and $\operatorname{Im}_{0} \Delta$ is the anomaly (within numerical factors). Let us see how this method works in a specific case. The spaces $L^{0}, L^{1}, \ldots$ themselves depend on the dimensionality of the space-time $D$. This circumstance determines the dependence of the cochains which are integrals of $D$-forms on the fields $A$. We consider the simplest case, $D=2$. The basis in the linear space $L_{D=1}^{\prime}$ consists of two elements $\operatorname{Tr} \int u \mathrm{~d} A$ and $\operatorname{Tr} \int u A^{2}$, which are transformed by the operation $\Delta$ into
$\int \mathrm{d}[\operatorname{Tr}[u v] A)=0$ (an integral of a total derivative) and

$$
\operatorname{Tr} \int A d[u v]+\operatorname{Tr} \int[u v] A^{2} \neq 0
$$

Accordingly, $\operatorname{Ker}_{D=2}^{1} \Delta$ consists of 2 -forms $\operatorname{Tr} \int u \mathrm{~d} A$ with an arbitrary numerical factor. The space $L_{D=2}^{0}$ turns out to be empty: We have both $\operatorname{Tr} \int \mathrm{d} A=0$ and $\operatorname{Tr} \int A^{2}=0$ so that $\mathrm{Im}_{D}^{0}=0$. We thus have $H_{D=2}^{1}(\Delta)=\left\{\right.$ const $\left.\cdot \operatorname{Tr} \int u \mathrm{~d} A\right\}$. We know that a Weyl anomaly is indeed of this form in two dimensions. The numerical factor is of course not fixed by a calculation of the group of cohomologies, just as it is not fixed by the Wess-Zumino consistency condition itself.

We should point out that a purely algebraic method of this sort for calculating anomalies is extremely useful in complicated multidimentional problems. Furthermore, it is not restricted to Weyl anomalies: All that is required is to relate an anomaly to cohomologies of some complex. Interestingly, this turns out to be possible even for conformal anomalies. ${ }^{24}$ In order to convert the corresponding operator $\Delta^{\prime}=g^{\mu \prime} \delta / \delta g^{\mu v}$ into a boundary operator (i.e., one satisfying the condition $\Delta^{2}=0$ ), we simply multiply it by the Grassmann parameter $\theta: \Delta=\theta \Delta^{\prime}=\theta g^{\mu v \delta} / \delta g^{\mu \nu}, \theta^{2}=0$. This operation, which seems at first glance to be somewhat pointless, will make it possible to simplify dramatically the calculation of conformal anomalies in the presence of external gravitational and Yang-Mills fields.

We return to a hierarchy of anomalies. We would first like to see what the descent method (used in the preceding subsection) looks like from the standpoint of a complex:

$$
L^{0} \xrightarrow{\Delta} L^{1} \xrightarrow{\Delta} \ldots \stackrel{\Delta}{\rightarrow} L^{n} \xrightarrow[\rightarrow]{\Delta} \ldots .
$$

Since the descent method uses a transition from one dimensionality of space-time to another by means of the operation $d$ and $d^{-1}$, it of course applies not to spaces $L_{D}^{\prime \prime}$ of integrals of $D$-forms but to spaces of $A_{D}^{n}$ of the $D$-forms themselves: $L_{D}^{\prime \prime}=\int A_{D}^{\prime \prime}$. The operation $\Delta$ also acts in accordance with rule (3.41) on the forms themselves, and in this case we again have $\Delta^{2}=0$. To avoid any misunderstanding, we will
use a different letter, $\delta$, to denote this operation on the forms: $\Delta \int W_{D}^{n}=\int \delta W_{D}^{n}$. Since $\delta^{2}=0$, we have the complex

$$
A_{D}^{0} \xrightarrow{\delta} A_{D}^{1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} A_{D}^{n} \xrightarrow{\delta} \ldots \xrightarrow{\delta} A_{D}^{D} \xrightarrow{\delta} 0 .
$$

Furthermore, we now also have the operation of external differentiation, which transforms $A_{D}^{n}$ into $A_{D+1}^{n}$, at our disposal. That operation also satisfies the condition $\mathrm{d}^{2}=0$. We thus have even a "double complex"


The operations $\delta$ and $d$ commute with each other, as follows from the equality $\mathrm{d} \delta_{u} A=\mathrm{d}([A u]+\mathrm{d} u)=[\mathrm{d} A, u]-[A$, $\mathrm{d} u]=\delta_{u} \mathrm{~d} A$. Consequently, diagram (3.42) is commutative. What is the anomaly $W_{D}^{1}$ in terms of a double complex of this sort? The Wess-Zumino condition requires $\Delta \int W_{D}^{1}=\int \delta W_{D}^{1}=0$, and thus $\delta W_{D}^{1}=\mathrm{d} W_{D-1}^{2}$ for some $W_{D-1}^{2} \in A_{D-1}^{2}: W_{D}^{1}$ does not necessarily have to belong to the kernel of $\delta$, but it must be transformed by this operator into a total derivative. On the other hand, the integral of the (D-1)-form $\int W_{D-1}^{2} \in L_{D-1}^{2}$ which arises as a result itself satisfies the relation $\Delta \int W_{D-1}^{2}=0$. Consequently, $\delta W_{D-1}^{2}$ is an exact form, since we have $\mathrm{d}\left(\delta W_{D-1}^{2}\right)=\delta\left(\mathrm{d} W_{D-1}^{2}\right)=\delta\left(\delta W_{D}^{1}\right)=\delta^{2} K_{D}^{1}=0, \quad$ and $\Delta \int W_{D-1}^{2}=\int \delta W_{D-1}^{2}=$ an integral of an exact form $=0$. More briefly, $\Delta$-closed elements from $L_{D}^{1}$ are mapped into $\Delta$-closed elements of $L_{D-1}^{2}$. Furthermore, $\Delta$ exact elements from $L_{D}^{1}$ are transformed into $\Delta$-exact elements of $L_{D-1}^{2}$. In fact, if we have $\int W_{D}^{1}=\Delta \varsigma W_{D}^{0}$, then we have $W_{D}^{1}=\delta W_{D}^{0}+\mathrm{d} W_{D-1}^{1}$ for some $W_{D-1}^{1} \in A_{D-1}^{1}$. We then have $\delta \boldsymbol{W}_{D}^{1}=\delta \mathrm{d} \boldsymbol{W}_{D-1}^{1}$ and $\mathrm{d} \delta \boldsymbol{W}_{D}^{1}$ corresponds in $A_{D-1}^{2}$ to an element $\delta W_{D-1}^{2}+\mathrm{d} W_{D-2}^{2}$ with some $W_{D-2}^{2}$. The integral of this element, $\int\left(\delta W_{D-1}^{1}\right.$ $\left.+\mathrm{d} W_{D-2}^{2}\right)=\int \delta W_{D-1}^{1}=\Delta \int / W_{D-1}^{1}$, is $\Delta$-exact in $L_{D-1}^{2}$.

All this means that the properties of the double complex (3.42) can be used to construct a mapping $P: H_{D}^{1}(\Delta) \rightarrow H_{D-1}^{2}(\Delta)$ of groups of cohomologies of the complexes: $L_{D}^{0} \xrightarrow{\Delta} L_{D}^{1} \xrightarrow{\Delta} \ldots \xrightarrow{\Delta} L_{D}^{n} \stackrel{\Delta}{\rightarrow} \ldots$ and $L_{D-1}^{0} \xrightarrow{\Delta} L_{D-1}^{1}$ $\stackrel{\Delta}{\rightarrow} \stackrel{\Delta}{\rightarrow} L_{D-1}^{n} \stackrel{\Delta}{\rightarrow}$... In exactly the same way we could of course construct the operation $P: H_{D}^{n}(\Delta) \rightarrow H_{D-1}^{n+1}(\Delta)$ for arbitrary $n$ and $D$. [The rule is that for $\Delta \int W_{D}^{n}=0$ we have $\delta W_{D}^{n}=\mathrm{d} W_{D-1}^{n+1}$ and $\Delta \int W_{D-1}^{n+1}=0$, and if $\int W_{D}^{n}$ $=\Delta \varsigma W_{D}^{n-1}$ then we have $W_{D}^{\prime \prime}=\delta W_{D}^{\prime-1}+\mathrm{d} W_{D-1}^{\prime \prime}$, $\delta W_{D}^{\prime \prime}=\mathrm{d}\left(\delta W_{D}^{n}+\mathrm{d} W_{D-2}^{n}\right)$
and $\left.\int\left(\delta W_{D}^{\prime \prime} \quad 1+\mathrm{d} W_{D}^{\prime \prime}{ }_{2}\right)=\Delta \rho W_{D}^{\prime \prime} \quad 1.\right]$

A hierarchy of anomalies is found by applying the operation $\mathrm{P}_{\mathrm{p}}$ successively to $H_{\mathrm{p}}^{\mathrm{o}}(\Delta)$ : $H_{D}^{0}(\Delta) \xrightarrow{\mathrm{P}} H_{D-1}^{1}(\Delta) \xrightarrow{\mathrm{P}} H_{D-2}^{2}(\Delta) \xrightarrow{\mathrm{P}} \ldots$ In this sequence, $\mathrm{P}^{2}$ is generally not zero, and $P$ itself usually performs a surjective mapping: Any cocyle from $H_{D-k-1}^{k+1}$ can be found through the application of P from some cocycle in $H_{D-k}^{k}$. The original $H_{D}^{0}$-the space of gauge-invariant integrals of $D$-forms-is constructed in different ways in the cases of even and odd $D$. For $D=2 n$, the basis element in $H_{D}^{0}$ is $\operatorname{Tr} \int F^{n}$. In this case the $2 n$-form itself, $W_{2 n}^{0}=\operatorname{Tr} F^{n}$ is gauge-invariant, $\delta W_{2 n}=0$, and the first application of the operation P leads to a vanishing result. For an odd dimensionality $D=2 n-1$, this situation is different. In this case we have $W_{2 n}^{0}=\operatorname{Tr} F^{n}=d W_{2 n-1}^{0}$, and $\int W_{2 n-1}^{0}$ is a basis element of $H_{2 n-1}^{0}$. The quantity $\delta W_{2 n-1}^{0}$ no longer vanishes, and the operation P makes it possible to find from $\int \boldsymbol{W}_{2 n-1}^{0}$ the expression $\int \boldsymbol{W}_{2 n-2}^{1}$ for a Weyl anomaly in $2 n-2$ dimensions. An example of a calculation of this type was given in the preceding subsection.

## 4. GLOBAL ANOMALIES

In 1982 Witten ${ }^{46}$ studied the behavior of a fermion determinant under topologically nontrivial gauge transformations. Of interest in this case are Weyl fermions, since the Pauli-Villars regularization guarantees the gauge invariance of the determinant of Dirac fermions. The action of Weyl fermions is invariant simply by virtue of the Weyl anomaly associated with the determinant of regulators. In the case at hand, in contrast, we are talking about an additional noninvariance. [A global anomaly will of course be more important in cases in which there is no local Weyl anomaly. One such case is that of the $\mathrm{SU}(2)$ group at $D=4$, since we have $\operatorname{Tr} t^{a} F F \sim d^{a b c} F^{b} F^{c}=0$. This case will be focused on below.] It turns out that at $D=4$ the Weyl fermions in the fundamental representation of the group $G=\operatorname{Sp}(n)$, in particular $S U(2)=S p(1)$ necessarily generate an action which changes by exactly $i \pi$ (while the exponential function of the action changes sign) under an uncontractable gauge transformation, whose existence stems from the nontrivial nature of the homotopic group $\pi_{D=4}(G)$. This gauge transformation is global in a substantial way and does not reduce to a composition of infinitesimal transformations. The corresponding noninvariance of the action is consequently not manifested in the nonconservation of some current.

Somewhat later, Redlich ${ }^{41}$ (see also Ref. 35) discovered an analogous phenomenon in odd-dimensional theories. In this case, a regularized determinant of course cannot disrupt the gauge invariance (Dirac fermions), but the determinants of the physical fermion and of the regulator, are separately noninvariant. More precisely, we need to examine the contributions to the effective action which are proportional to an $\varepsilon$-symbol. The corresponding anomalous structure in the Lagrangian of a $(2 n+1)$-dimensional theory is

$$
\begin{equation*}
W_{0}^{(2 n+1)}=\frac{1}{2(2 \pi)^{n}(n+1)!} \operatorname{Tr}\left(A_{-} \mathrm{d} A^{n}+\ldots\right) \tag{4.1}
\end{equation*}
$$

and its coefficient in momentum space is constructed in accordance with ${ }^{45}$

$$
\begin{equation*}
1-\frac{2 m}{|p|} \arcsin \frac{|p|}{\left(|p|^{2}+4 m^{2}\right)^{1 / 2}} \tag{4.2}
\end{equation*}
$$

The unity is the contribution of the regulator, and the second (nonlocal) term is the contribution of a physical fermion with mass $m$. The limit $m=0$ is not always meaningful: The theory contains infrared divergences. For a very light fermion, at all energies $|p|>m$, the second term in parantheses is inconsequential, so that all the physical properties of the theory (the spectrum and the scattering characteristics) are determined by the action with a unit coefficient for the structure $\int W_{0}^{(2 n+1)}$. If we instead are interested in global gauge transformations which fall off slowly at infinity, the nonlocal term "comes into play" and cancels the noninvariance of the action with $\int W_{0}^{(2 n+1)}$ under such transformations. For $D=2 n+1=3$, for example, variation of $\int W_{0}^{(2 n+1)}$ under the transformation $A \rightarrow g^{-1}(A g+\mathrm{dg})$ is equal to $\pi \operatorname{Tr} \int\left(24 \pi^{2}\right)^{-1} \operatorname{Tr}\left(g^{-1} \mathrm{~d} g\right)$ and is a multiple of $\pi$, so that there are transformations [which correspond to an odd topological charge of an ordinary 4-dimensional (BPTS) instanton] under which $\exp \left(i \int W_{0}^{(3)}\right)$ changes sign. The action of a physical fermion, however, does not change by $\pi$ here, and the theory turns out to be invariant.

We have already calculated the anomalous contribution to the determinant of a regulator fermion, in Subsection 2.3. Interestingly, the noninvariance of this determinant can be seen by using the same method as was used by Witten in his study of the $\operatorname{SU}(2)$ anomaly. We should again emphasize the profound distinction between the theories studied by Witten and Redlich. In the former case, a regularization which makes the generatoring functional invariant does not exist; i.e., the theory is nonself-consistent. In the latter case, everything is fine theoretically, but not every regularization can be used. (In particular, there is an unavoidable violation of the $P_{1}$ and $T$ invariances, which are present in the classical theory of massless fermions in an odd number of dimensions.) The derivation of the $\operatorname{SU}(2)$ anomaly is ultimately based on theorems regarding the index of the Dirac operator. The indices in the four-dimensional theory derived by Witten and in the three-dimensional theory by Redlich are different. It is not our purpose here to derive the corresponding theorems; we will attempt, for the most part following Ref. 45 , to explain how the difference in indices leads to a difference in physical results: the possibility or impossibility of regularizing the theory.

### 4.1. Properties of a fermion determinant under topologically nontrivial gauge transformations

The method for studying this question which was used in the paper by Witten is as follows.
a) We first note that the determinant of a Dirac operator for Weyl fermions is not defined, since $\hat{\mathrm{D}}=\hat{\partial}+\hat{A}$ transforms left-hand fermions into right-hand fermions, i.e., removes the left-hand fermions from the space. In the following subsections we resolve this difficulty by switching to the operator

$$
\mathrm{D}=\hat{\partial}+\hat{A} \frac{1-\gamma^{5}}{2}
$$

This operator is non-Hermitian, however, and thus inconvenient for a study of topological effects. In addition, we could define a Weyl determinant as the square root of a Dirac determinant, making use of the circumstance that the eigenvalues of the latter are "degenerate":

$$
\begin{aligned}
i \hat{\mathrm{D}}^{(4)} \psi_{\lambda} & =\lambda \psi_{\lambda}, \\
i \hat{\mathrm{D}}^{(4)}\left(\gamma^{5} \psi_{\lambda}\right) & =-\lambda\left(\gamma^{5} \psi_{\lambda}\right),
\end{aligned}
$$

i.e., $\gamma^{5} \psi_{\lambda}=\psi_{-\lambda}$ (for brevity, we will put quotation marks around the word "degeneracy" to specify this degeneracy within a sign). We wish to emphasize that the massless nature of the Dirac operator $i \hat{\mathrm{D}}^{(4)}$ is important here; the question of regularization is still open. In a regularized theory, an ordinary Weyl anomaly generally arises. On the other hand, a Witten anomaly does not necessarily occur in the determinant of regulators (more on this below). In particular, it does not occur in the $D=4$ case, so that at this point it is sufficient to consider an unregularized determinant.

Taking a root is defined up to a sign. Strictly speaking, all the information about the phase of a Weyl determinant is lost when we switch to an ordinary Dirac operator: de$t\left(i \hat{D}^{(4)}\right)$ is the product of determinants for right-hand and left-hand fermions, which are complex conjugates of each other; the phase for each of them is cancelled out completely by the phase of the other when a product is formed. Even more remarkably, a change in this phase which is unrelated to regulators leaves its trace in the characteristics of the operator $i \widehat{\mathrm{D}}^{(4)}$.
b) This trace is observed when a study is made not of the entire determinant $\operatorname{det}\left(i \hat{\mathrm{D}}^{(4)}\right)$ but of the evolution of the individual eigenvalues $i \widehat{\mathrm{D}}^{(4)}$, upon a change in the field $A$. Let us use the field $A$ to perform the topologically nontrivial gauge transformation $A \rightarrow \Omega A$ (which is of such a nature that the mapping $\Omega: S^{4} \rightarrow G$ is not homotopic with a unit mapping). The spectra of the operators $i \widehat{\mathrm{D}}^{(4)}(A)$ and $i \widehat{\mathrm{D}}^{(4)}(\Omega A)$, are of course the same. The eigenfunctions of the second of these operators are found from the eigenfunctions of the first by multiplying by the matrix $\Omega^{-1}$. Any interpolation between $A$ and $\Omega A$, e.g.,

$$
A_{\mathrm{s}}=A \frac{1-\mathrm{th} \xi}{2}+(\Omega A) \frac{1+\mathrm{th} \xi}{2}
$$

is not a gauge-invariant field $A$ by virtue of the definition of $\Omega$. Accordingly, for finite values of $\xi$ the spectrum of the operator $i \widehat{\mathrm{D}}^{(4)}\left(A_{\xi}\right)$ differs from that of $i \hat{\mathrm{D}}^{(4)}(A)$, and we can raise the question of the evolution of the eigenvalues of the operator $i \hat{\mathrm{D}}_{\xi}^{(4)}\left(A_{\xi}\right)$ as $\xi$ changes. Figure 13 shows a picture of this evolution. The picture is symmetric with respect to the abscissa, since for arbitrary values of $\xi$ the operator $i \hat{\mathrm{D}}^{(4)}\left(A_{\xi}\right)$ anticommutes with $\gamma^{5}$, and its spectrum is "degenerate." That result, however, is not what is important here. What is important is that there are two eigenvalues, which have exchanged places. It might appear that at the point of intersection it is difficult to say whether an eigenvalue has moved from top to bottom or has "been reflected" from the abscissa and has moved back up. Actually, a Weyl determinant must be determined as an alytic function of the fields, and near its zero, $\operatorname{det}\left[i \hat{\mathrm{D}}^{(4)}\left(A_{\xi=\xi_{1}}\right)\right]=0$, it must be of the


FIG. 13. Evolution of the eigenvalues of an even-dimensional Dirac operator $i \hat{\mathrm{D}}_{s}^{(4)}=i\left(\hat{\partial}+\hat{A}_{\xi}\right)^{(4)}$ in the case of an uncontractable gauge transformation $\Omega$.

$$
A_{\xi}=\frac{1-\operatorname{th} \xi}{2} A+\frac{1+t^{2} \xi}{2} \Omega A
$$

form $\operatorname{det}\left(i \hat{\mathrm{D}}^{(4)}\left(A_{\xi}\right)\right) \sim\left(A_{\xi}-A_{\xi_{0}}\right) \sim \xi-\xi_{0}$. In other words, it must change sign upon the crossing of a simple zero. Consequently, there can be no jog of any sort on the path of an eigenvalue.

The question of how the picture of the evolution of eigenvalues shown in Fig. 13 arises is the subject of the following subsection. At this point we would also like to take up the odd-dimensional Yang-Mills theory, which was studied by Redlich.
a) An odd-dimensional Dirac operator $i \widehat{\mathrm{D}}^{(2 n+1)}$ has well-defined eigenvalues, and the spectrum of this operator has no "degeneracy" of any sort (in this case, there are no matrices which anticommute with all the $\gamma$ matrices). If the homotopic group has the property $\pi_{2 n+1}(G) \neq 0$, there are still some uncontractable gauge transformations $\Omega$ here, and we can examine the evolution of an eigenvalue of the operator $i \hat{\mathrm{D}}_{\xi}^{(2 n+1)}\left(A_{\xi}\right)$. However, the picture of this evolution turns out to be completely different (why will be explained in the following subsection) (Fig. 14). The asymmetry of the picture stems from the absence of a "degeneracy" in the spectrum. If we wish to have the eigenvalues moved downward, rather than upward, we need to examine either the operator $i \hat{\mathrm{D}}_{\xi}^{(2 n+1)}$, or the transformation $\Omega^{\prime}$, with the opposite topological charge (the Pontryagin index).

What can we learn from these figures? In order to determine a determinant we need to choose some subset of the eigenvalues. First, we need to discard all eigenvalues which exceed the ultraviolet cutoff $\lambda$ in modulus. Second-in the


FIG. 14. Evolution of the eigenvalues of an odd-dimensional Dirac operator $i \hat{\mathbf{D}}_{\xi}^{(3)}=i\left(\hat{\partial}+\hat{A}_{s}\right)^{(3)}$ in the case of an uncontractable gauge transformation $\Omega$.

$$
A_{\xi}=\frac{1-\operatorname{th} \xi}{2} A+\frac{1+\operatorname{th} \xi}{2} \Omega A
$$

Weyl case (Fig. 13) -we need retain only one of each pair of eigenvalues $\pm \lambda$. The product of the remaining eigenvalues changes sign for both Fig. 13 and 14, as is easily seen. The theory is not yet regularized, however: The ultraviolet cutoff $\Lambda$ is by itself a poor regularization for $\operatorname{det}\left(i \widehat{\mathbf{D}}^{(2 n+1)}\right)$, since one of the eigenvalues in the region ( $-\Lambda,+\Lambda$ ) necessarily drops out of this region as $\xi$ varies: The regularizations with $\xi=-\infty$ and $\xi=+\infty$ are different. It is thus necessary to take the determinant of the regulators into account. An important point is that by virtue of its very nature regularization affects only the higher eigenvalues. In the Weyl case (Fig. 13), there is nothing remarkable in the behavior of the higher eigenvalues; they are mapped into themselves upon a change, so that the determinant of the regulators cannot compensate for the change in the sign of the "low-energy" determinant. In Fig. 14, this is not the case. Now all the eigenvalues no matter how high, are raised one step upon a change in $\xi$. We can always find an eigenvalue which intersects the horizontal line drawn at the height $-M$, so that the determinant $\operatorname{det}\left(i \widehat{\mathrm{D}}^{(2 n+1)}+\boldsymbol{M}\right)$ changes sign at the same time as $\operatorname{det}\left(i \hat{\mathrm{D}}^{(2 n+1)}\right)$. This determinant is still not a real determinant of regulators, in which the operator $i \widehat{\mathrm{D}}^{(2 n+1)}=i M$ (non-Hermitian) would occur, but it is easy to see that during the motion of the real eigenvalues of the operator $i \widehat{\mathrm{D}}^{(2 n+1)}$ in accordance with Fig. 14 the phase $\operatorname{det}\left(i \hat{\mathrm{D}}^{(2 n+1)}+i M\right)$ changes by exactly $\pi$, and the regulator determinant also changes sign. As a result, the regularized determinant is gauge-invariant.

In short, a Weyl theory in which the eigenvalues behave as in Fig. 13 [in the $D=4$ case, this comment applies to gauge groups $G=\operatorname{Sp}(n)$ ] cannot be regularized with invariance under topologically nontrivial gauge transformations being retained. As a result, a generating functional is not determined. In an odd-dimensional Yang-Mills theory, in which the eigenvalues behave in accordance with Fig. 14 [ this is the situation whenever the group has the property $\pi_{2 n+1}(G) \neq 0$ ], the determinants of physical and regulator fermions are not separately gauge-invariant, but their ra-tio-a regularized determinant-is a well-defined quantity.

### 4.2. Difference between the $\boldsymbol{\gamma}^{5}$ index and the $\boldsymbol{C}$ index

In this subsection we prove that the evolution of the eigenvalues of the operators $i \hat{\mathrm{D}}^{(D)}\left(A_{\xi}\right)$ is indeed as depicted in Figs. 13 and 14, depending on the dimensionality of spacetime.

The evolution of an eigenvalue of the Dirac operator $i \hat{\mathrm{D}}_{\underline{\xi}}^{(D)}$ can be studied by making use of its relationship with the zero modes of operators with a dimensionality one unit larger, e.g.,

$$
\mathrm{D}_{\mathrm{i}}^{(D+1)}=\frac{\partial}{\partial \xi}+i \hat{\mathrm{D}}_{\mathrm{\xi}}^{(D)}-M
$$

Actually, the eigenvalues of a Hermitian operator $i \widehat{\mathrm{D}}_{5}^{(D)}$ are real, and if for some eigenvalue we have $\lambda(\xi=-\infty)<M$ and $\lambda(\xi=+\infty)>M$, then the operator $\mathrm{D}_{1}^{(D+1)}$ must have a zero mode,

$$
\psi(\xi) \sim \exp \left[-\int^{\xi}(\lambda(\xi)-M) \mathrm{d} \xi\right]
$$

This function is normalizable; i.e., it is one of the functions
which must be taken into account in the measure of the path integration, and it satisfies the equation $D_{1}^{(D+1)} \psi=0$. More precisely, if $i \hat{\mathrm{D}}_{\xi}^{(D)} \psi_{\lambda, \xi}=\lambda(\xi) \psi_{\lambda, \xi}$ then

$$
\psi(\xi)=\psi_{\lambda, \xi} \exp \left[-\int^{\xi}(\lambda(\xi)-M) \mathrm{d} \xi\right]
$$

The derivative of $\psi(\xi)$ with respect to $\xi$ is

$$
\left(-\lambda(\xi)+M+\frac{\partial}{\partial \xi} \ln \psi_{\lambda, \xi}\right) \psi(\xi) .
$$

The magnitude of the first term is determined by the quanti$\operatorname{ty} \lambda(\xi)$ (which is in turn related to the index of the eigenvalue and the "size of the box" holding the system). The magnitude of the second term can be made arbitrarily small through a sufficiently slow variation of the field $A_{\xi}$. We can choose the interpolation of $A(\xi)$ between $A$ and $\Omega A$ in any way we wish, so that we can ignore the second term. The inverse is also true: Each zero mode of the operator $D_{1}^{(D+1)}$ corresponds to some eigenvalue of the operator $i \hat{\mathrm{D}}_{\xi}^{(D)}$ which intersects the line $\lambda=M$ as it moves upward. Analogously, the eigenvalues which go downward are in a mutually one-to-one correspondence with the zero modes of another operator

$$
\mathrm{D}_{2}^{(D+1)}=-\frac{\partial}{\partial \xi}+i \hat{\mathrm{D}}_{\xi}^{(D)}-\boldsymbol{M}
$$

which differs from $D_{1}^{(D+1)}$ in the sign of $\partial / \partial \xi$. The evolution of an eigenvalue can also be related to the zero modes of other $(D+1)$-dimensional operators. The meaning of these manipulations is that if an index has been defined for the operator $D$ then the problem of the zero modes of $D$ is quite simple. For the existence of an index, it is necessary also to find one more operator $P$ which anticommutes with $D$ : $\mathrm{DP}=-\mathrm{PD}$. The existence of P ensures the "degeneracy" of the spectrum of $\mathrm{D}: \mathrm{D} \psi=\lambda \psi \Leftrightarrow \mathrm{D}(\mathrm{P} \psi)=-\lambda(\mathrm{P} \psi)$. If, as the parameters of the operator D are varied continuously it remains anticommutative with $P$, then the number of zero modes of $\mathbf{D}$ can be varied only as a result of the arrival or departure of a pair of eigenvalues. Consequently, the even parity of the number of zero modes remains constant under a continuous variation of the parameters of $D$. We usually have $P^{2}=1$, but the only necessary conditions are that $P$ be nondegenerate and uniformly bounded. (These conditions are necessary so that $\psi_{-\lambda}=\mathbf{P} \psi_{\lambda}$ will not vanish identically and will be a normalizable function and also so that these conditions hold when the pair of functions $\psi_{ \pm \lambda}$ converts into two zero modes.) In the case $\mathbf{P}^{2}=1$, a more precise assertion is that the number $n_{+}$of zero modes $\mathbf{P} \psi=+\psi$ minus the number $n_{-}$of zero modes $\mathrm{P} \psi=-\psi$ is invariant: $n_{+}-n_{-} \equiv \operatorname{ind}_{\mathrm{P}} \mathrm{D}$. This difference is called the "P-index" of the operator D . As we have already explained, the index of an operator is a topological invariant (i.e., it does not change under continuous deformations), and it can be determined by topological means or found for some convenient choice of parameters.

Returning to the problem of the evolution of the eigenvalues, we must construct a $(D+1)$-dimensional operator which is associated with $i \widehat{\mathrm{D}}_{\xi}^{(D)}$ and which has an index. For the simplest operators, $\mathrm{D}_{1,2}^{(D+1)}$, there are no operators $\mathbf{P}$
which anticommute with them, so we need to move on to a more complicated construction.

It is well known that even-dimensional massless Dirac operators have an index associated with $P=\Gamma^{5}$. Accordingly, in discussing the evolution of the eigenvalues of an odddimensional $i \widehat{\mathrm{D}}_{\substack{(2 n}}{ }^{1)}$ (the Redlich case) we can attempt to continue the construction of this operator to a ( $D+1=2 n-1+1=2 n$ )-dimensional Dirac operator. In this process we are forced primarily to change the dimensionality of the $\Gamma$ matrices. Specifically, $2 n$-dimensional matrices $\Gamma$ are found from ( $2 n-1$ )-dimensional matrices $\gamma_{i}$, $i=1,2, \ldots, 2 n$, in accordance with the rule

$$
\Gamma_{i}=\left(\begin{array}{ll}
\gamma_{i} & -\gamma_{i}
\end{array}\right), \quad \Gamma_{0}=\left({ }_{+i}^{-i}\right), \quad \Gamma_{5}=\left(1_{1}^{1}\right) .
$$

The operator

$$
i \hat{\mathrm{D}}^{(2 n)}=\left(\begin{array}{cc}
\left.i \hat{\mathrm{D}}^{(2)} n-1\right) & \frac{\partial}{\partial \xi} \\
-\frac{\partial}{\theta \xi} & -i \hat{\mathrm{D}} \hat{\xi}_{\xi}^{(2 n-1)}
\end{array}\right)
$$

is completely suitable for our purposes. Its left-hand zero modes,

$$
\Psi_{L}=\binom{\psi}{\psi}
$$

satisfy the equation

$$
\mathrm{D}_{1}^{(2 n)}(M=0) \psi=\frac{\partial}{v \xi} \psi+i \hat{D}_{\xi}^{(2 n-1)} \psi=0
$$

and correspond to eigenvalues of the operator $i \widehat{\mathrm{D}}_{\underline{S}}^{(2 n-1)}$ which intersect zero moving upward. The right-hand modes,

$$
\Psi_{\mathrm{R}}=\binom{+\Psi}{-\psi}
$$

satisfy the equation

$$
\mathrm{D}_{2}^{(2 n)}(M=0) \psi=-\frac{\partial}{\partial \xi} \psi+i \hat{\mathrm{D}}_{\xi}^{(2 n-1)} \psi=0
$$

and determine eigenvalues which cross zero moving downward. We wish to emphasize that, in contrast with the operators $D_{1,2}^{(2 n}$ themselves, the quantity $i \widehat{\mathrm{D}}^{(2 n)}$ has an index, so that the number of its zero modes can be found. It is determined by the Dirac anomaly and is given by

$$
\text { ind } i \hat{\mathrm{D}}^{(2 n)}\left\{A_{\xi}\right\}=\frac{1}{2} W^{(2 n)}=\frac{1}{(2 \pi)^{n} n!} \int \operatorname{Tr} F^{n}
$$

the topological charge of the field $A_{\xi}$.
We have now seen that for topologically nontrivial gauge transformations $\Omega$ with a Pontryagin index +1 (under the boundary conditions $A_{\xi} \rightarrow A$ as $\xi \rightarrow-\infty$ and $A_{\xi} \rightarrow \Omega A$ as $\xi \rightarrow+\infty$ the operator $i \hat{\mathrm{D}}^{(2 n)}$ has one more of left-hand zero modes than the number of right-hand zero modes) the number of eigenvalues of the operator $i \hat{\mathrm{D}}_{\xi}^{(2 n-1)}$ which cross zero moving upward is one greater than the number of eigenvalues which cross zero moving downward. To prove the validity of Fig. 14, we still need to explain why the other eigenvalues, which do not cross zero, rise. It is intuitively clear that for them "there is no other way out": the picture near the $\lambda=0$ predetermines the behavior of the other eigenvalues. The proof, on the other hand, is based on the circumstance that the operator $i \hat{\mathrm{D}}^{(2 n)}-i M \Gamma_{0} \Gamma_{5}$, as before, anticommutes with $\Gamma$, so that its index does not depend
on the value of the parameter $M$. In other words, it is the same as the known index for $M=0$. On the other hand, the left-hand and right-hand modes of this operator satisfy the equations

$$
\begin{aligned}
& \mathrm{D}_{1}^{(2 n)} \psi:=\frac{\partial}{\hat{\sigma}_{5}^{\xi}} \psi+i \hat{\mathrm{D}}_{\xi}^{(2 n-1)} \psi-M \psi=0, \\
& \mathrm{D}_{2}^{(2 n)} \psi=-\frac{\partial}{\partial_{\xi}^{(2}} \psi+i \hat{\mathrm{D}}_{\tilde{E}}^{(2 n-1)} \psi-M \psi=0
\end{aligned}
$$

respectively. These results are sufficient to prove the validity of Fig. 14. (See Refs. 41 and 45 for a more detailed analysis.)

It is slightly more complicated to find the Witten picture in Fig. 13, since the version of the theorem regarding the index in this case is less familiar in the physics literature. Instead of proving this theorem (Ref. 11), we will end this subsection with a simple example which illustrates it.

Going from the operator $i \mathrm{D}^{(+)}$to a ( $D+1=5$ )-dimensional Dirac operator $i \widehat{\mathrm{D}}^{(5)}$ does not require changing the dimensionality of the $\gamma$ matrices:

$$
i \hat{\mathrm{D}}^{(5)}=i \gamma^{5}\left(\frac{\partial}{\partial \mathrm{~s}}+\gamma^{5} \hat{\mathrm{D}}_{\mathrm{E}^{(4)}}\right)
$$

The zero modes of this operator are completely suitable for our purposes. It is sufficient to note that the spectra of the operators $\gamma^{5} \hat{\mathbf{D}}_{5}^{(4)}$ and $i \hat{\mathbf{D}}_{5}^{(4)}$ are the same:

$$
\begin{gathered}
i \hat{\mathrm{D}}_{\mathrm{S}}^{(\omega)} \psi=\lambda \psi \Rightarrow \gamma^{5} \hat{\mathrm{D}}^{(4)}\left(\psi \mp i \gamma^{5} \psi\right)= \pm \lambda\left(\psi \mp i \gamma^{5} \psi\right) \\
\gamma^{5} \hat{\mathrm{D}}^{(4)} \chi=\mu \chi \Rightarrow i \hat{\mathrm{D}}^{(4)}\left(\chi \pm i \gamma^{5} \chi\right)= \pm \mu\left(\chi \pm i \gamma^{5} \chi\right)
\end{gathered}
$$

The only difficulty is that for odd-dimensional operators there is no analog of the $\gamma^{5}$ matrix, and in general no index exists. In several cases, however, an operator $\mathbf{P}$ which anticommutes with $i \hat{\mathbf{D}}^{(5)}$ can be constructed by working from the operation of complex conjugation, $C$. In the basis

$$
\left.\begin{array}{rl}
\gamma_{i}=\left(\begin{array}{ll}
\sigma_{i} & \\
& -\sigma_{i}
\end{array}\right) \quad(i=1,2,3), \quad \gamma_{0} & =\left(1_{-i}\right.
\end{array}\right),
$$

the operation $C P_{1} P_{3}$, consisting of complex conjugation and inversion of the first and third coordinates (i.e., rotation through $\pi$ in the plane of Fig. 13), anticommutes with the Hermitian operator

$$
i \gamma^{5}\left(\frac{\partial}{\partial \xi}+\gamma^{5} \hat{\partial}^{(4)}\right)=i \hat{D}^{(5)} \quad(A=0)
$$

[The Hermitian nature is particularly important here. Otherwise, the operator $P$, containing a complex conjugation, would not commute with the eigenvalues $i \widehat{\mathrm{D}}^{(5)}$, and we would find $i \hat{\mathrm{D}}(\mathrm{P} \psi)=-\mathrm{P} i \hat{\mathrm{D}} \psi=-\mathrm{P} \lambda \psi=-\hat{\lambda}(\mathrm{P} \psi)$.] It turns out that this property can be retained even when we turn on the fields $A_{\mu}=i A_{\mu}^{a} t^{a}$. For this purpose, there must exist an involution $\Sigma C$ for the algebra $G$ which relates com-plex-conjugate representations: $\Sigma^{-1} t^{a} \Sigma=-t^{a}$. We then have $\left(C \Sigma^{-1}\right)(\partial+A)(C \Sigma)=\partial+A$, and the operator P can be written in the form $\mathrm{P}=\mathrm{C} \Sigma P_{1} P_{3}$. Such an involution exists for the $\mathbf{S U}(2)$ group: $\Sigma=\tau_{2}, t^{1,2,3}=\tau_{1,2,3}$. An important point is that for a Dirac mass operator $i \widehat{\mathbf{D}}^{(4)}+M$ there is
no five-dimensional operator of any sort which has an index:

$$
i \frac{\partial}{\partial \xi}+\gamma^{5} \hat{\partial}^{(4)}+M
$$

does not anticommute with $C P_{1} P_{3}$, while the operator

$$
i \frac{\partial 1}{\partial \xi}+\gamma^{3}\left(\hat{\partial^{(4)}}-i M\right)
$$

is non-Hermitian, so that its anticommutivity with $\mathrm{P}=C P_{1} P_{3}$, which contains complex conjugation, is insufficient for invariance of the index. This result shows that for an even-dimensional Dirac operator the eigenvalues cross only the line $\lambda=0$ (and lines which are close to it in a layer with a thickness of the order of the reciprocal of the size of the box), as illustrated in Fig. 13. In order to prove that there is nevertheless an interchange of one pair of eigenvalues, we need the index theorem ${ }^{\prime \prime}$ mentioned above, which relates the parity of the number of zero modes of the operator $i \widehat{\mathrm{D}}^{(5)}$ to the topological nontriviality of the field $A$. We have already stated that a global Witten anomaly is unrelated to the nonconservation of any current. This situation is also manifested in the absence of an integral representation for the $C$ index of the operator $i \widehat{\mathrm{D}}^{(5)}$.

We conclude with the promised example that illustrates the situation regarding the operators $i \widehat{\mathrm{D}}_{\xi}^{(4)}$ and $i \widehat{\mathrm{D}}^{(5)}$ and the index theorem. ${ }^{11}$ The pair of operators in this example is $\Delta_{\xi}^{(1)}=\sigma_{1} i \partial_{x}+\sigma_{2} A_{\xi}, \Delta^{(2)}=\sigma_{3} i \partial_{\xi}+\sigma_{1} i \partial_{x}+\sigma_{2} A_{\xi}$. Each is Hermitian and purely imaginary, so that each has an index ( $\mathbf{P}=C$ ). We will discuss the evolution of the eigenvalues of the operator $\Delta^{(1)}\left(A_{\xi}\right)$, and we will need only the index of the operator $\Delta^{(2)}$. In a topologically nontrivial field $A_{\xi}(x), \Delta^{(2)}$ will have a zero mode which corresponds to an eigenvalue of the operator $\Delta_{\xi}^{(1)}$ which intersects the abscissa moving upward [ the eigenvalue which is "degenerate" with it (such an eigenvalue exists, by virtue of $\left.\Delta^{(1)} \sigma_{3}=-\sigma_{3} \Delta^{(1)}\right)$, and which is moving downward, is described by a zero mode of the operator $\left.\widehat{\Delta}^{(2)}=-\sigma_{3} i \partial+\sigma_{1} i \partial_{x}+\sigma_{2} A_{\xi}\right]$. The term topologically nontrivial" here refers to a classification in accordance with the homotopic group $\pi_{1}(G)$. We take $G=\mathrm{U}(1)$ in the field $A_{5}$ with a unit topological charge, determined on $S^{1}$ for each $\xi$ on a circle of perimeter $L$ : $A_{\xi}=(\pi / L)$ th $\xi$. [The topological charge is

$$
\begin{aligned}
\frac{1}{2} W^{(2)}=\frac{1}{2 \pi} \varepsilon_{\mu \nu} \int \partial_{\mu} A_{\nu} \mathrm{d}^{2} x & =\frac{1}{2 \pi} L \int \mathrm{~d} \xi \frac{1 \partial}{\partial \xi} A_{\xi} \\
& =\frac{L}{2 \pi}\left(A_{+\infty}-A_{-\infty}\right)=1
\end{aligned}
$$

We cannot add a constant to $\pi / L$ th $\xi$, because if we did the spectra of the operators $\Delta_{-\infty}^{(1)}$ and $\Delta_{+\infty}^{(1)}$ would not coincide: We do not have perfect gauge invariance for the operator $\Delta^{(1)}(A)$ (this is the price we pay for the simplicity and the clarity of this example)]. As is predicted by the index theorem for the operator $\Delta^{(2)}$, of all the eigenvalues of the operator $\Delta^{(1)}\left(A_{\xi}=(\pi / L)\right.$ th $\left.\xi\right)$, which are equal to $\lambda_{n}^{2}=(2 \pi n / L)^{2}+A_{\xi}^{2}$ there is an interchange of only one pair; the others are mapped into themselves, in complete accordance with Fig. 13.

[^0](the gauge version of which reproduces an anomaly). In the 1980s, Lagrangians of this sort were introduced explicitly and used by Witten $^{29}$; Schönfeld ${ }^{3 \kappa}$; Deser, Jackiw, and Templeton ${ }^{40}$; Novikov ${ }^{2 \kappa}$; and, later, by many other authors.
${ }^{2}$ 'This distinction between $T_{\mu \nu}^{c o n r}$ and $T_{\mu u}^{\mathrm{C}}$. actually stems from the conformal invariance of the theory:
$$
g^{1 / 2} L=g^{1 / 2}\left[\frac{1}{2}(\partial \phi)^{2}+\frac{D-2}{8(D-1)} R \Phi^{2}\right] .
$$
${ }^{3}$ The fields $A_{\mu}$ are anti-Hermitian matrices from a Lie algebra $G$ corresponding to gauge group $G$. They are related to the fields $\bar{A}_{\mu}{ }^{\prime}$, which are frequently used and for which we have $\tilde{F}_{\mu,}^{u}=\partial_{\mu} \tilde{A}_{r}^{u}-\partial_{v} \bar{A}_{\mu}^{u}$
 $-\partial_{\lambda} A_{\mu}+\left[A_{\mu} A_{v}\right]$. Here the $t^{u}$ are Hermitian generators of the algebra $\left.\hat{G} ; t^{a}, t^{b}\right]=i f^{a b c} t^{c}, f^{t b c}$ are structure constants of the algebra; and $\operatorname{Tr} t^{a} t^{b}=\delta^{a b} / 2$.
${ }^{4}$ We will not go through a special discussion of scalar matter fields. In the first place, all the basic assertions concerning unbroken, local symmetries are identical in the scalar and spinor cases, although the results for the scalars are more complicated. The reason is that in the scalar case the interaction has a term not present in (1.13): $\operatorname{Tr} A_{\mu} J_{\mu}(\phi)$ $+\operatorname{Tr} A_{\mu} J_{\mu}(\phi) A_{,} .\left[\right.$For example, $\operatorname{Tr}(\mathrm{D} \phi)^{2}=\operatorname{Tr}(\partial \phi)^{2}+2 \operatorname{Tr} A \phi \partial \phi$ $=\operatorname{Tr} A \phi A \phi$.$] Second, the scalar matter fields are nonchiral and cannot$ lead to anomalies (more on this below).
${ }^{5}$ In a $(D=2 n)$-dimensional space we understand $\gamma$ ' as a matrix which is proportional to the product $\gamma_{0} \gamma_{1} \cdots \gamma_{2 n-1}$ and which satisfies the conditions $\gamma^{5}=+\gamma^{5},\left(\gamma^{5}\right)^{2}=+1$.
${ }^{61}$ At his point we set Planck's constant equal to unity. We recall that each loop corresponds to one power of $\hbar$. Actually, in a path integral the action is divided by $\hbar$, so that $1 / \hbar$ corresponds to each vertex $V$, and an $\hbar$ corresponds to each propagator $P$. Since we have $P-V=L-1$, the effective action which arises from a diagram with $L$ loops is proportional to $\hbar \hbar^{P} / \hbar^{V}=\hbar^{1+P-V}=\hbar^{L}$.
${ }^{7}$ The more accurate reason for the appearance of the $\varepsilon$-symbol is cosmological. In particular, integrals of structures with the $\varepsilon$-symbol are topological invariants.
${ }^{\text {8/ We }}$ We should perhaps explain that all the quark fields with masses below the characteristic hadronic scale $m_{\rho}$, i.e., precisely $\mathrm{u}, \mathrm{d}$, and s , are important for the $U(1)$ problem. The heavy $c, b$, etc., quarks do not contribute to the anomaly at hadronic energies. It is for this reason that we discussed the $U_{\Gamma}(3)$ group, rather than $U_{r}(2)$ or $U_{\Gamma}(4)$, in Subsection I.3.1).
${ }^{9}$ We will assume, however, that the $t^{a}$ in a non-abelian current are the same as the generators of the gauge group which describes the interaction with vector bosons. In principle, it would be meaningful to study the nonconservation of currents which are associated with other groups, e.g., the breaking of the flavor group $\mathrm{SU}_{\mathrm{L}}\left(n_{\mathrm{F}}\right) \times \mathrm{SU}_{\mathrm{R}}\left(n_{F}\right)$ in quantum chromodynamics. The generalization of all the equations to such cases is almost obvious and we will not burden the calculations by striving for unnecessary generality.
${ }^{101}$ When regularization a) is carried out through the introduction of a fermion mass, the coefficient of the structure $q_{\mu} \varepsilon_{\alpha \beta \xi \xi \prime \prime} p_{\xi} q_{\eta}$ contains an additional logarithmic factor
$$
\ln \frac{1-\left(4 m^{2} / q^{2}\right)}{1+\left(4 m^{2} / q^{2}\right)}
$$
which is related to threshold effects. There is no factor of this type when the threshold is related to external particles, which do not propagate in the loop.
${ }^{1 "}$ Here we could also use the Bianchi identity $\mathrm{D} F=0$, where $\mathrm{D}=\mathrm{d}+[A,] ; \mathrm{D}\left(\Phi_{1}, \Phi_{2}\right)=\left(\mathrm{D} \Phi_{1}\right) \Phi_{2}+(-) R_{\Phi} \Phi_{1} \mathrm{D} \Phi_{2}$. Since $W$ and $W^{u}$ are traces and do not have free color indices, we have $d W=\mathrm{D} W$ and $d W^{a}=\mathrm{D} W^{u}, d \operatorname{Tr} F^{n}=\mathrm{D} \operatorname{Tr} F^{n}=0$, but $\operatorname{Tr} t^{a} F^{n}$ $=\mathrm{D} \operatorname{Tr} t^{a} F^{n}=\operatorname{Tr}\left(\mathrm{D} t^{a}\right) F^{n}+\operatorname{Tr} t^{a} \mathrm{D} F^{n}=\operatorname{Tr}\left[A, t^{a}\right] F^{n}=\operatorname{Tr}\left(\mathbf{A} \mathrm{t}^{a}\right.$ $\left.-t^{\Delta} A\right) F^{n}$.

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Translated by Dave Parsons


[^0]:    ${ }^{1}$ This name derives from Wess and Zumino's 1971 paper, ${ }^{7}$ in which we find the first mention of the idea of implicitly symmetric Lagrangians

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