

# Dimensionalities and other geometric critical exponents in percolation theory

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A review is given of the studies of the dimensionality characteristics of percolation clusters. The purely geometric nature of a percolation phase transition and the great variety of the quantities exhibiting critical behavior make this geometric approach both informative and useful. In addition to the fractal dimensionality of a cluster and its subsets (such as the backbone, hull, and other dimensionalities), it is necessary to introduce additional characteristics. For example, the maximum velocity of propagation of excitations is determined by the chemical dimensionality of a cluster, and the critical behavior of the conductivity, diffusion coefficient, etc., is determined by spectral (or other related to it) dimensionalities. Scaling relationships between different dimensionalities, as well as relationships between dimensionalities and conventional critical exponents are discussed.

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## 1. INTRODUCTION

The concept of dimensionality (or dimensions) has recently become popular in theoretical physics. This concept is understood to mean the Hausdorff dimensionality (or its analogs, such as the Kolmogorov and other capacities or dimensionalities, and also information capacity or dimensionality), which may assume fractional values and is a characteristic of the self-similar properties of objects. In addition to the "intrinsic" dimensionalities, other properties of self-similar objects are described by quantities which are in many respects analogous to dimensionalities. Some of these quantities can be regarded as the dimensionalities of special sets and some have a more complicated meaning. Introduction of such quantities has been found to be useful in studies of the

properties of a great variety of physical objects and processes.

The concept of dimensionality (or dimensions), investigated actively by mathematicians from the beginning of the present century, has been brought to the attention of physicists thanks to a monograph of Mandelbrot<sup>1</sup> which appeared in 1975 in French and then in 1977 in English. This book consists mainly of illustrations and examples, but it is provided also with a rigorous mathematical appendix, and demonstrates the usefulness of the concept of dimensionality in the description of the properties of the very great variety of self-similar objects and structures in hydrodynamics, statistical physics, astrophysics, and other fields. This monograph can be regarded as an excellent example of scientific advertising or popularization, in this case naturally of new concepts and models.

To the best knowledge of the present author, the first paper treating the dimensionalities of percolation clusters was that of Stanley,<sup>2</sup> which appeared in 1977, and a review by Stauffer<sup>3</sup> which appeared in 1979 has a section dealing with this topic. In recent years the number of communications dealing with these dimensionalities has reached several hundred.

It has gradually become clear that there are two groups of physical problems in which the concept of dimensionality can be and is useful. On the one hand, these are the problems of dynamic stochasticity, turbulence, and generally chaotic behavior, and on the other, a series of problems in statistical physics.

In the first case the concept of dimensionality makes it possible, for example, to provide a more or less satisfactory classification of strange attractors and the associated chaotic behavior. In this case the dimensionalities are not independent characteristics: they are related to the Lyapunov exponents, asymptotic behavior of the spectra, etc. It is important to note that these dimensionalities represent "coarse-grained" and measurable characteristics of the system. These problems are discussed in the review of Farmer, Ott, and Yorke,<sup>4</sup> and we shall not deal with them any further.

Our review is concerned with the second group of problems, or more exactly with one such problem which is the percolation theory. We shall use this problem as an example to show how the dimensionalities and other geometric exponents are introduced, what are the scaling relationships between them, and how they are related to other more usual critical exponents. For each critical exponent (apart from  $\nu$  describing the divergence of the correlation length) we can identify a set with the dimensionality to which this exponent is related. Studies of the structure of these sets helps greatly in the understanding of the critical behavior of a system and of the relationships between the exponents. We shall also consider the general relationship between the intermediate asymptotic behavior of a system and its fractal geometry. Since we shall confine ourselves to just the one problem of percolation, we shall be able to discuss all these topics in detail. The exponents introduced by us will be largely universal and useful in studies of other systems. These systems include in particular linear and branched polymers and epoxy resins,<sup>5-7</sup> protein structures,<sup>8,9</sup> porous materials and substances with highly developed surfaces,<sup>10-12</sup> and aggregates formed in the course of diffusion-limited reactions (dendrites). An extensive literature is available on the last topic (see, for example, Refs. 13-27).

The "fractal boom" in percolation theory, which reached its peak in 1983-84, is now subsiding somewhat. The main concepts have been formulated and have become familiar, the results which could be obtained by rewriting the familiar relationships in terms of new notation have been already obtained. We can now draw some conclusions. However, we shall show below that many topics are still unresolved or even untouched, whereas the solution of others will require much computational or analytic work. The continuing interest in the topic should ensure that the gaps will be filled in the near future.

## 2. PERCOLATION THEORY. DEFINITIONS AND CRITICAL BEHAVIOR

From the moment of appearance in 1957 of the first paper of Broadbent and Hammersley,<sup>28</sup> interest in percolation theory has been maintained for over 30 years now. There are several reasons for this interest. We shall mention just two of them.

1. This theory describes satisfactorily many systems exhibiting a purely geometric phase transition, such as the conductor-insulator transition in mixtures of conducting and insulating particles,<sup>29</sup> fracture of rocks after formation of a sufficient number of cracks,<sup>30</sup> etc. The theory is used to describe the elasticity of polymer gels,<sup>31</sup> hopping conduction in doped semiconductors,<sup>32</sup> Anderson localization in disordered systems,<sup>33,34</sup> and in many other applications. An extensive literature is available on the formulation and applications of percolation problems: in the Russian language we have the review of Shklovskii and Éfros,<sup>32</sup> Chap. 5 in the monograph of Ref. 35 by the same authors, and a popular treatment by Éfros,<sup>36</sup> whereas in English there are two complementary reviews of Stauffer<sup>3</sup> and Essam.<sup>37</sup> In the years since the writing of the last of these reviews many new problems have been treated using percolation models. Modifications of these models are numerous and it is not possible to enumerate them all.

2. The percolation theory models are simple and informative. By simplicity we mean here both the simplicity of the initial formulation of the problem and the ability to go far toward its solution. Simple models soon become standard and analysis of these models then attracts theoretical physicists for years. The standard models are useful because they represent a testing ground for various methods. The uniformity of the initial problems makes it possible to compare and integrate the results of various methods.

In addition to these two aspects of simplicity, which make the approach so attractive, there is one other facet. Many percolation theory models (or at least the lattice models) are readily accessible to numerical simulation or modeling on computers. The simplicity of the initial formulation and the high effectiveness of the algorithms (particularly those involving sorting and search, and also solutions of systems of linear equations), make it possible to investigate these models by the Monte Carlo, finite scaling, and other methods. The existence of extensive and extremely precise "experimental" data on many models and the ability to formulate practically any thought (gedanken) experiment makes these problems even more attractive.

This discussion should be sufficient to demonstrate that the percolation theory problems are extremely interesting topics for research. This review will be focused on the geometric nature of phase transitions and on a great variety of quantities exhibiting critical behavior, all of which make the approach informative and useful. This great variety of the quantities makes our concepts and relationships important means for the solution of other problems, particularly those mentioned in the introduction.

## 2.1. Principal definitions and models

In the present section we shall briefly recall the formulation of the main percolation theory problems and discuss the most important features of critical behavior of the models. This information will be needed later.

The simplest lattice problems in the percolation theory can be formulated as follows. Let us consider a lattice regarded usually as a set of sites and bonds. A given site can be labeled (for example, it can be of black color) with a probability  $x$ . A set of black sites connected to one another will be called a black cluster. If  $x = 0$ , the system has no black clusters, whereas for  $x \ll 1$  the black clusters represent mainly sets with small numbers of sites (single sites, pairs, triplets, etc.). On the other hand, if  $x = 1$ , then all the sites are black: for  $1 - x \ll 1$  there is an infinite black cluster in the system. There should be a critical concentration  $x_c$  at which a transition takes place from one regime to the other, i.e., when an infinite cluster first appears. A similar formulation can be used to tackle the problem of bonds, in which we regard as bond as labeled with a probability  $x$  and define a cluster as a set of sites linked by labeled bonds. Sometimes in the case of problems of electrical conductivity and random walk we shall regard a labeled site as empty, and a labeled bond as whole (unbroken).

The formation of an infinite cluster represents a phase transition and the order parameter of this transition is the power of an infinite cluster, defined as the probability that a site belongs to such a cluster. The critical behavior of this quantity in the limit  $x \rightarrow x_c$  for  $x > x_c$  is described by

$$P_\infty \propto (x - x_c)^\beta. \quad (2.1)$$

The behavior of  $P_\infty(x)$  is demonstrated for different lattices in Fig. 1. We shall use  $\tau = (x - x_c)/x_c$  to denote later the "dimensionless" deviation of the concentration from its critical value. A detailed discussion of the critical behavior of percolation models can be found in Refs. 3 and 35 and we shall simply quote here that information which will be absolutely essential in the subsequent treatment.

The average number of sites in a finite cluster (which is an analog of susceptibility) behaves as follows in the limit  $\tau \rightarrow 0$ :

$$s \propto |\tau|^{-\gamma}. \quad (2.2)$$

Introducing an analog of the specific heat  $c$ , we obtain

$$c \propto |\tau|^{-\alpha}, \quad (2.3)$$

where  $c = \frac{\partial^2 F}{\partial \tau^2} \Big|_{h=0}$ ,  $F(p, h) = \sum_s n_s e^{-sh}$  ( $s$  is the number of sites,  $n_s$  is the fraction of clusters with this number of sites, and  $h$  is the "external field" which has a very specific meaning in percolation theory<sup>38</sup>).

The characteristic spatial scale of the system is the correlation length  $\xi$ , the rigorous meaning of which we shall establish later. At this stage we can assume that  $\xi$  is the characteristic size of a black cluster when  $x < x_c$  and the characteristic size of the voids in the cluster when  $x > x_c$ . The critical behavior of this quantity is described by

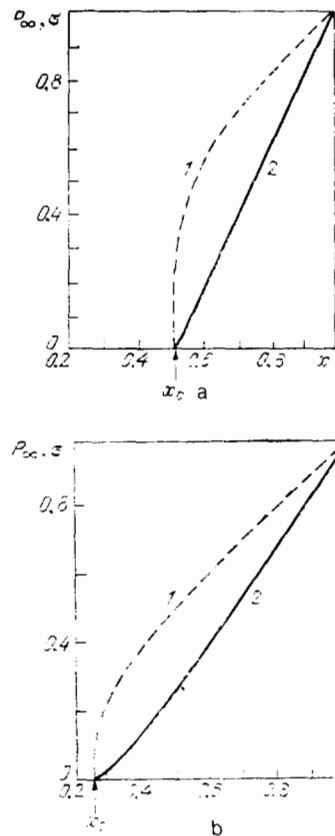


FIG. 1. Dependences of the power of an infinite cluster (1) and of the conductivity of a percolation net (2) on the concentration.<sup>40</sup> The bond problem: a) square lattice; b) simple cubic lattice.

$$\xi \propto |\tau|^{-\nu}. \quad (2.4)$$

When the  $h \neq 0$  case is considered, we can introduce a further exponent  $\delta$ :

$$P_\infty(\tau = 0, h) = h^{1/\delta}. \quad (2.5)$$

All the critical exponents introduced above (which we shall call thermodynamic) are linked by the usual scaling relationships<sup>3,35,39</sup>:

$$\left. \begin{aligned} \alpha &= 2 - d\nu = 2 - 2\beta - \gamma, \\ \delta &= \frac{\gamma}{\beta} + 1, \\ \eta &= 2 - \frac{\gamma}{\nu}, \\ d\nu &= 2\beta + \gamma. \end{aligned} \right\} \quad (2.6)$$

This system of relationships corresponds to the two-exponent scaling: the values of all the exponents given above can be found if we know the values of two of them. We shall assume that we know the exponents  $\nu$  and  $\beta$ . Some of the above relationships are valid for any dimensions of space  $d$  and the others (those which contain  $d$ ) are valid in the dimensions of space less than the critical value  $d \leq 6$ . The latter are frequently called the hyperscaling relationships. In the classical range  $d \geq 6$  the values for all the exponents are given

TABLE I. Principal critical exponents in percolation theory.

$d$	2	3	4	5	6
$\nu$	4/3	0.88	0.7	0.6	1/2
$\beta$	5/36	0.40	0.5	0.7	1
$t$	1.29	1.7	2.4	2.7	3

by the system (2.6) for  $d = 6$ ,  $\beta = 1$ , and  $\nu = 1/2$ , and are independent of  $d$ . The values of the exponents  $\beta$ ,  $\nu$ , and  $t$  (the last one will be introduced later) are given in Table I, which is taken basically from the review of Stauffer.<sup>3</sup> For  $d = 2$  the exponents  $\nu$  and  $\beta$  are rational numbers: they can be obtained in the model which is exactly soluble<sup>41,42</sup> (see also Chap. 12 of Baxter's monograph<sup>43</sup>); the values for  $d \geq 6$  are given by the mean-field approximation or by the exactly soluble model based on a Bethe lattice.

We shall need another critical exponent. If  $\tau \neq 0$ , then

$$n_s(\tau) = f(s) \exp(-As|\tau|^\Delta), \quad (2.7)$$

where  $f(s)$  is a certain function which arises no more rapidly than proportionally to a quantity with a power exponent  $s$ . In this case the exponent  $\Delta$  determines the characteristic largest size of a finite cluster. Clusters with  $s_c \propto |\tau|^{-\Delta}$  are known as critical; the probability of the occurrence of larger clusters decreases exponentially. It should be noted that the characteristic spatial scale of a critical cluster is equal to the correlation length, but in the sense defined above. The exponent  $\Delta$  obeys the hyperscaling relationship<sup>35</sup>

$$\Delta = d\nu - \beta. \quad (2.8)$$

It should be noted that the notation used for the critical exponents in the present review is standard, with the exception of  $\Delta$ .

We shall terminate here the review of thermodynamic critical exponents. All of them are fully analogous to the exponents introduced in the theory of other phase transitions. However, it should be pointed out that because the external field in percolation theory can be introduced only formally, the meaning of some of these critical exponents is not quite clear.

An important feature of percolation theory is the existence of nontrivial critical exponents which are meaningless in the conventional models of thermodynamic phase transitions. One of the applications of percolation theory is the problem of electrical conductivity of random nets (conductivity of mixtures of conducting and insulating particles). In the simplest case such a system can be modeled in the bond problem: each whole bond has the resistance  $r = 1$ , whereas each broken bond has the resistance  $r = \infty$ . The resistance of a large part of such a net located between two perfectly conducting plates behaves as follows:

$$\bar{R} = \begin{cases} \infty, & \tau \leq 0, \\ R_0 \tau^{-t}, & \tau > 0. \end{cases} \quad (2.9)$$

The resistance of a random net composed of normal and

superconducting conductors in concentrations  $x$  and  $(1-x)$ , respectively, obeys  $\bar{R}(x-x_c)^s$  (Refs. 44-46) where in the two-dimensional case we have  $S = t$  (Ref. 44). The values of  $t$  are listed in Table I. Similar relationships determine the behavior of the elastic constants of a random net (broken bonds are assumed to have no elastic moduli). The behavior of the elasticity of such a net depends strongly on the symmetry properties of the elastic forces acting between the atoms (see Ref. 47). In particular, within the framework of the Born model, we find that the cases of isotropic and central interactions differ greatly. In the former case, the critical behavior is the same as that of the electrical conductivity, whereas in the latter case we have

$$E = \begin{cases} 0, & \tau' \leq 0, \\ E_0 \tau'^T, & \tau' > 0, \end{cases} \quad (2.10)$$

where  $T \neq t$ ; this problem will be discussed in detail later (see Sec. 6.1). If  $d = 2$ , then  $T = 2.4 \pm 0.4$ , whereas for  $d = 3$ , we have  $T = 4.4 \pm 0.6$ . The problem of the elasticity of a random network is considered in Ref. 48 and it is assumed that bonds can bend and have an appropriate elastic modulus. There is a suspicion that the index  $T_1$  encountered in this problem differs from  $T$  and  $t$ .

It should be pointed out that the model considered in Ref. 49 and characterized by the purely central interaction with just the nearest neighbors is oversimplified. According to this model, the square and simple cubic lattices have, for example, no shear elasticity. Therefore, the triangular and fcc cubic lattices are considered in Ref. 49. In this case the critical concentration  $x_c$ , at which an infinite cluster assumes elastic properties, differs from the percolation concentration:  $x_c > x_c$ , so that in Eq. (2.10) we have  $\tau' = x - x_c \neq \tau$ . This circumstance is related to the fact that if  $x_c < x < x_c$ , an infinite cluster contains many one-dimensional twisted chains capable of extension without a change in the length of the component bonds, but involving only changes in the angles between the bonds. A cluster becomes elastic only when such an extension mechanism is impossible. For a triangular lattice we have  $x_c = 0.58$  ( $x_c = 0.3473$ ), whereas for an fcc lattice we have  $x_c = 0.42$  ( $x_c = 0.119$ ).

Considerable effort has been spent (see Ref. 35) so as to be able to express the exponent  $t$  in terms of  $\beta$  and  $\nu$  within the framework of the two-exponent scaling. The existence of many different critical exponents  $t$ ,  $T$ , and  $T_1$  makes it necessary to formulate a more general (and less interesting) problem: how the nature of the relevant equation (see Sec. 6) can be used to determine the critical exponent of the behavior of the quantity of interest. It may be that such a general expression does not exist at all.

## 2.2. Universality. Scaling and self-similarity

The exceptional interest in the critical exponents is due to their universality, i.e., due to the fact that they are practically completely independent of many coarse features of the selected model and yet are sensitive to such fine characteristics as the symmetry, the existence of long-range correlations, etc. In studies of the problem of the motion of a parti-

cle in a random potential we of necessity meet the continuous percolation problem, i.e., the problem of a random distribution of two colors in space. The color of regions is correlated over a distance  $\lambda_0$ , whereas the color of small regions separated by larger distances is independent. This formulation is extremely useful in the theoretical sense: the boundary of a cluster (i.e., the boundary of a connected region with a given color) is a continuous smooth curve; moreover, the topological structure of clusters is informative.

If we consider, for example, a polycrystalline material or some other similar system, which appears as a result of crystallization, we must first consider the process of crystallization itself. In the model developed by Meijering<sup>50</sup> this reduces to splitting the space into polyhedra of Voronoi, corresponding to the initial distribution of the crystallization centers. Then, the polyhedra are assigned colors at random: some of them are regarded as conducting and the others as nonconducting.<sup>51</sup> There are also other nonlattice modifications of percolation problems, particularly various problems of random sites.

The postulate of universality means that all (listed in Sec. 2.1) critical exponents are independent of the selection of the model and are governed only by the dimensions of space. This postulate, checked in a very large number of numerical experiments, is the foundation of the theory.

The behavior of a system is governed by the ratio of two spatial scales, which are the minimum length  $a_0$  (lattice constant, scaling length  $\lambda_0$ , average distance between crystallization centers) and the correlation length  $\xi$ . If  $\tau \ll 1$ , then  $\xi \gg a_0$  and there is also a range of intermediate asymptotic behavior corresponding to  $a_0 \ll l \ll \xi$ . In this range all the characteristics of the clusters (determined on a scale shorter than  $\xi$ ) are similar to the characteristics of the critical point itself when  $\tau = 0$  and  $\xi = \infty$ . Their properties in this region are characterized by self-similarity (scaling invariance). For scaling lengths greater than  $\xi$  the system is homogeneous: it can be represented as consisting of blocks of  $\xi \times \xi \times \xi \dots$  dimensions. The properties of blocks are characterized by the behavior in the intermediate asymptotic range. If  $\tau = 0$ , the scaling invariance applies for all scaling lengths  $l > a_0$ . All this has been known for some time and it forms the basis of the theory of scaling.

We can describe the behavior of a system near  $\tau = 0$  provided we know only the correlation length  $\xi$  (i.e., one thermodynamic exponent  $\nu$ ), and the geometric properties of an infinite cluster or of very large clusters. The reason for similarity of the critical phenomena is the similarity (self-similarity) of geometric objects. The fractal dimensionality is a characteristic of this self-similarity. Introduction of this concept and a careful study of the dimensionalities of clusters and their subsets (hulls, backbone, etc.) and of the relationships between them has not only extended the vocabulary of the theory, but has led also to the understanding of new relationships.

Moreover, a study of the geometric similarity or scaling represents a study of a single object at  $\tau = 0$ . There is no need to discuss the asymptotic behavior of the relevant quantities in the limit  $\tau \rightarrow 0$  and  $N \rightarrow \infty$ , where  $N$  is the number of sites in

a sample. Instead of going to the limit twice, we can simply consider only the thermodynamic limit  $N \rightarrow \infty$ . There is no need to justify the convenience of such an approach from the point of view of effectiveness and precision of numerical experiments.

### 3. DIMENSIONALITIES. PRINCIPAL DEFINITIONS

The basic information on the dimensionalities can be found in Mandelbrot's book<sup>1</sup> and in the review by Zel'dovich and Sokolov.<sup>52</sup> The review of Farmer, Ott, and Yorke<sup>4</sup> also contains all the necessary definitions and also deals with the problems encountered in the calculation of the dimensionalities of various sets. However, as in the preceding section, it is reasonable to summarize all the main concepts and to introduce some standard models which are not mentioned in Ref. 52. In this section we shall not cite the original papers, but direct the reader to the reviews just quoted.

All the objects under discussion are sets of points in a  $d$ -dimensional Euclidean space. A topological dimensionality can be introduced for any such set. This quantity is introduced as follows. The dimensionality of any finite or denumerable set of points is  $d_T = 0$ . The dimensionality of any connected set is  $d_T + 1$ , if it can be cut into two unconnected parts by excluding at least a  $d_T$ -dimensional set of points (by a  $d_T$ -dimensional cut). From the very definition of the topological dimensionality it follows that it can be only an integer. The topological dimensionality of a straight line is 1, that of a plane or spherical surface is 2, for a sphere it is 3, etc.

Neither the dimensionality of the Euclidean space nor the topological dimensionality have anything to do with the self-similarity of the sets under discussion. A characteristic of this self-similarity is the fractal dimensionality or, more exactly, the fractal dimensionalities introduced in different ways. We shall begin with an example.

#### 3.1. Dimensionality of self-similarity. Ramification and phase transitions

We shall consider a square (Fig. 2). Two cuts make it possible to separate it into four similar squares but with the size half as large. The self-similar dimensionality is

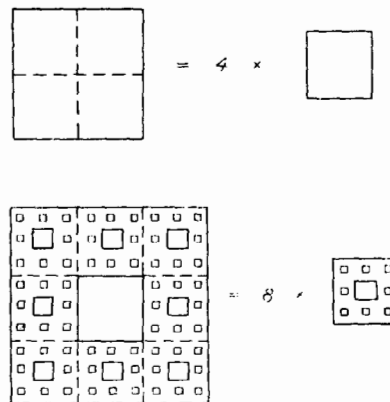


FIG. 2. Square ( $D = 2$ ) and Sierpiński carpet ( $D = 1.8817 \dots$ ).

$D = \ln N / \ln n$ . Here,  $N$  is the number of objects similar to the object in question and characterized by a spatial scale which is  $n$ -times less from which we can compose the object in question. In this case the self-similar dimensionality is  $D = \ln 4 / \ln 2 = 2$ .

We shall now consider the following set. We shall take a square and by four cuts we shall divide it into nine parts. We shall remove the central part. We shall then deal similarly with each of the eight remaining parts. The result is a set known as the Sierpiński carpet (or the universal Sierpiński curve) and it has the self-similar dimensionality  $D = \ln 8 / \ln 3 = 1.8817 \dots$  Such objects with a fractional dimensionality will be called fractals, following Mandelbrot.

In the construction of a Sierpiński carpet the process of cutting out the center of the square can be continued indefinitely. However, in a physical system there is a minimum scale  $a_0$  such that for lower scales the self-similarity no longer applies. We can say that a system consists of atoms of size  $a_0$ . Then, if the size of the system is large, the total number of atoms in the system (the mass of atoms) rises on increase in the linear size as follows:

$$M \propto l^D, \quad (3.1)$$

so that the self-similar dimensionality has the simple meaning of a certain critical (because in real systems the self-similarity occurs only in the vicinity of a critical point) exponent.

By way of illustration, we shall give several other examples of regular fractals. Nevertheless, it should be mentioned that these objects (constructed very simply) have been investigated very thoroughly as models of clusters and other self-similar systems. These fractals are shown in Fig. 3. This figure should be regarded as follows: it shows "unit cells" of fractals where each shaded region is a fractal similar to a given cell. We can easily show that a fractal with any speci-

fied self-similar dimensionality can be constructed in this way.

It is worth noting one difference between the objects in Figs. 3a–3c and those in Figs. 3d–3h, which is quite obvious. The first three objects consist of unit cells connected by vertices, whereas in the other objects the cells are connected along their sides. This difference can be described mathematically. From the topological point of view all the objects in Fig. 3 are one-dimensional: each of them can be cut into two unconnected parts by excluding at most a denumerable set of points. The minimum number of the excluded points needed to cut the objects in Figs. 3d–3h is finite (for a triangular Sierpiński curve it amounts to two: these points are identified by arrows), whereas for the objects in Figs. 3a–3c this minimum is infinite. Such a minimum number of points is known as the ramification of a fractal  $r$ . The specific value of the ramification is unimportant, but some properties of fractals with finite and infinite ramifications are fundamentally different.

In the case of fractals with an infinite ramification we can introduce a ramification order. Let us consider a fractal lattice of size  $R$ . Such a lattice is understood to be a system constructed on a large scale in the same way as a regular fractal, whereas on the minimum scale each of the shaded regions in Fig. 3 represents one site of the lattice connected by bonds to sites corresponding to the neighboring shaded regions. Such a lattice can be cut into two parts by breaking at least  $N_{\min}$  bonds. The ramification order is then the exponent in the following expression:

$$N_{\min} \propto R^p. \quad (3.2)$$

The most interesting property (from our point of view) of infinitely ramified fractal lattices is that, in contrast to lattices with a finite ramification on which a percolation path breaks down when a finite number of sites is removed, on

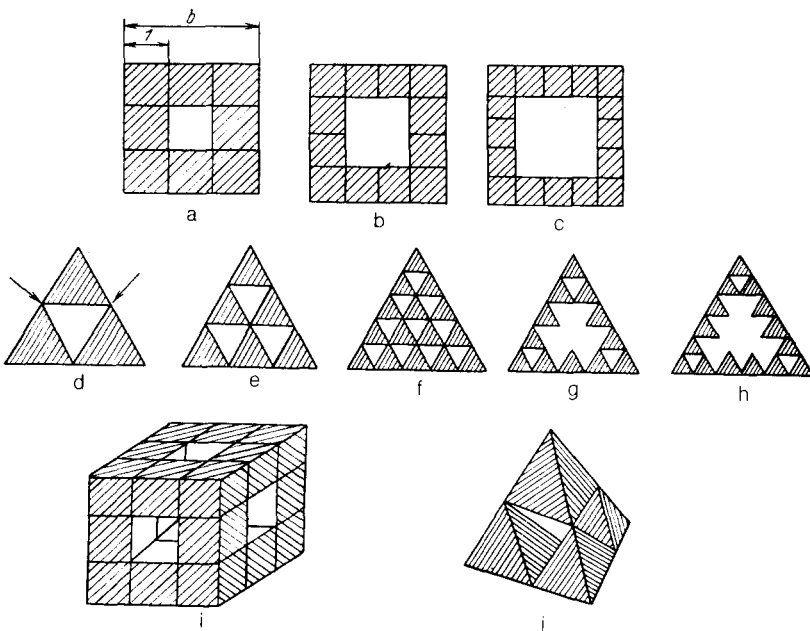


FIG. 3. Models of regular fractals. a)–c) Sierpiński carpet and other fractals considered in Ref. 55 (see Table II);  $r = \infty$ . d) Triangular Sierpiński curve. e)–h) Other "chessboard triangles";  $r < \infty$ . i), j) Generalizations of a and d to the three-dimensional case.

TABLE II. Critical concentration and critical exponents for model of percolation on an infinitely ramified fractal.

$b$	$D$	$\rho$	$x_c$	$\nu$	$\beta$
2	2	1	0.5	1.83	0.63
3	1.89	0.63	0.85	2.13	0.27
4	1.79	0.5	0.92	2.43	—
5	1.72	0.43	0.95	2.69	—
10	1.56	0.30	0.989	3.60	—
100	1.30	0.15	0.9999	6.79	—
$b \rightarrow \infty$	$1 + \frac{\ln 4}{\ln b}$	$\frac{\ln 2}{\ln b}$	$1 - \frac{1}{b^2}$	$\frac{\ln b}{\ln 2}$	$\frac{\ln b}{b^2 \ln 2}$

these lattices a very real percolation transition exists. A family of regular fractals of the type shown in Figs. 3a–3c was considered in Ref. 55. These fractals differ in respect of the value of  $b$ , which is the number of “unit cells” along one side of a square. Table II gives the fractal dimensionality  $D$ , the ramification order (exponent)  $\rho$ , and the critical percolation concentration  $x_c$ , as well as the exponents  $\beta$  and  $\nu$  calculated by the standard renormalization group approach.<sup>56</sup>

The absence of phase transitions on lattices with finite ramification and the possibility of such transitions in the case of infinitely ramified lattices is a fairly general property<sup>57</sup>: it is known that in the case of Euclidean one-dimensional systems ( $r = 1$ ) we cannot expect phase transitions (§ 163 in Ref. 58), but phase transitions can occur in Euclidean systems with  $d \geq 2$  ( $r = \infty$ ).

Since for  $\tau = 0$  the removal of a finite number of sites in an infinite cluster suppresses percolation, we have to assume that its ramification is finite. Therefore, a cluster (or, more exactly, its backbone) can be simulated by a triangular Sierpiński curve and not a Sierpiński carpet.

These definitions of the self-similar dimensionalities are valid generally speaking only for regular geometric objects exhibiting self-similarity in its rigorous sense. Objects such as percolation clusters, polymer molecules, etc., exhibit only a statistical self-similarity.

### 3.2. More rigorous definitions

Rigorous mathematical definition of analogs of the self-similar dimensionality lead us to the concepts of the Kolmogorov capacity and of the Hausdorff-Besicovitch dimensionality.

We shall consider a set of points in a  $d$ -dimensional Euclidean space and see how they can be covered with  $d$ -dimensional spheres. We shall define  $N(\epsilon)$  as the minimum number of spheres of radius  $\epsilon$  needed to cover this set. If  $N(\epsilon)$  rises on reduction in  $\epsilon$  as  $\epsilon^{-D_c}$ , then  $D_c$  is the Kolmogorov capacity of the set.

Strictly speaking, this quantity should be defined as follows: the capacity is the number  $D$  for which the following limit differs from zero:

$$\lim_{\epsilon \rightarrow 0} N(\epsilon) \frac{\Gamma(D+1/2)}{\Gamma(1/2)^D} \epsilon^D. \quad (3.3)$$

In this definition we simply multiply  $N(\epsilon)$  by a quantity which is a generalization of the formula for the volume of a

$d$ -dimensional sphere to the case of fractional values of  $D$ . The limit discussed here is simply an upper bound for the  $D$ -dimensional volume of our set.

The definition of the Hausdorff dimensionality differs because the covering is by spheres of radius not exceeding  $\epsilon$ , so that the Hausdorff measure is

$$\lim_{\epsilon \rightarrow 0} \sum_{\Omega_i} \frac{\Gamma(D+1/2)}{\Gamma(1/2)^D} r_i^D, \quad (3.4)$$

where the sum is taken over all the spheres of radius  $r_i \leq \epsilon$  covering the lattice. The value of  $D$  at which this limit differs from 0 and  $\infty$  is the Hausdorff dimensionality of a set.

Spheres are used to deal with the coverage only in order to avoid the problem of orientation. We can equally use  $d$ -dimensional cubes; the term with the  $\Gamma$  functions in Eqs. (3.3) and (3.4) should then be omitted. In the experimental determination of  $D$  (see Sec. 4.1) it is preferable to use the latter method.

The Hausdorff dimensionality and the Kolmogorov capacity may differ even for very simple sets. For example, for a set of points on a straight line with the coordinates  $x_n = 1/n$  the former is 0 and the latter is 1/2. In contrast to the dimensionalities, particularly in the case of the Hausdorff dimensionality, the capacities do not remain invariant in the case of piecewise-smooth (but with possible singularities) transformation of the coordinates<sup>59</sup>: in the case of the quantities that lay claim to being dimensionalities such an invariance is an essential feature.

In all the physical applications the dimensionality is defined as the exponent of Eq. (3.1)<sup>60</sup> and it is not possible to say whether this represents a Hausdorff dimensionality or a capacity. This is due to the fact that the dimensionality describes the properties of intermediate asymptotic behavior and it is not possible to go to the limit required by the formal definition. On a small scale the system is not fractal: its behavior is described by the minimum scaling length  $a_0$ .

### 3.3. Self-similar curves

We shall consider a self-similar curve without recurrences (without return to the initial point) and self-intersections. We shall assume that the dependence of the average distance between the ends of this curve on its length is given by the relationship  $\langle r^2 \rangle^{1/2} \propto (l/a_0)^{\nu}$ , where  $l$  is the length of the curve (for a long polymer molecule we find that, for





FIG. 4. Determination of the length of a curve with a scale length  $\lambda$ .

example,  $\varphi \approx 4/3$  corresponds to  $d = 2$  and  $\varphi \approx 7/6$  corresponds to  $d = 3$ —see Ref. 31). Bearing in mind that the “mass” of this curve is proportional to  $l$  (in the case of a molecule this mass is proportional to the number of monomers, whereas in random walk problems it is proportional to the number of steps, etc.), we find that the fractal dimensionality of the curve<sup>5</sup> is

$$D = \varphi^{-1}. \quad (3.5)$$

We can approach the definition of the dimensionality of the curve in a different way. We shall consider the behavior of the curve on a scale considerably greater than the minimal (the existence of the minimal scale is not essential: we can consider also a “mathematical” fractal). We shall determine the length of a curve with a spatial resolution  $\lambda_0$  (Fig. 4). The measured length depends on  $\lambda_0$  as follows:

$$\mathcal{L} \propto \lambda_0^{1-D}. \quad (3.6)$$

In fact, over the whole length of the curve it can be divided into  $N \propto \lambda_0^{-D}$  pieces (which are arcs subtended by segments of length  $\lambda_0$ ), because a change of the scale  $\lambda_0$  by a factor  $a$  changes this number (by definition) by a factor  $b$  such that  $\ln b / \ln a = D$ . The total measured length of the curve  $\mathcal{L} \propto \lambda N$  is then given by Eq. (3.6).

In the case of curves without recurrences and self-intersections there are no problems in using both definitions of  $D$ . However, if recurrences are present, we have to consider not only the Hausdorff dimensionality, but also the dimensionality defined in a somewhat different way. If a curve exhibits recurrences or self-intersections (i.e., if there are pieces traversed several times), we are then faced with the question of how many times such a piece should be allowed for in the determination of the length. The standard definition of the capacity or dimensionality says that this should be done only once: each point is covered by just one sphere for any passage across it. We can modify the definition (3.4) by taking a point into account as many times as there have been passages across it. In this case the measure introduced by us differs from the Hausdorff dimensionality.

We shall now consider some parametrization of a curve, for example, its natural parametrization when the parameter is  $l$  representing the length (including recurrences) from the initial point or  $t$  which is the time taken by a point moving at a constant velocity along the curve. We shall assign a weighting factor  $p(\Omega_i)$  to each sphere and assume that this factor is equal to the fraction of time that a given point spends inside this sphere or the probability of finding a moving point in-

side a given sphere. This makes it possible to define the dimensionality  $D'$  as the number which obeys

$$0 < \lim_{\varepsilon \rightarrow 0} \sum_{\Omega_i} \frac{\Gamma(D' + 1/2)}{\Gamma(1/2)^{D'}} p(\Omega_i) r_i^{D'} < \infty. \quad (3.7)$$

We shall now obtain the corresponding capacity by replacing in Eq. (3.7) the summation over spheres  $\Omega_i$  of radius  $r_i \leq \varepsilon$  with the summation over spheres of the same radius. The dimensionality obtained subject to such weighting is a quantity similar to that called the internal (interior) dimensionality in Ref. 52. The dimensionality  $D'$  is called the dimensionality of a natural measure in Ref. 4. This dimensionality describes, for example, the dependence of the “mass” of a random walk on time if each site is ascribed a mass equal to the number of times that the site is crossed. The quantity  $1/\varphi$  for such random walk is  $D'$  and not  $D$ .

Different weighting factors can be used in other applications: for example, in the theory of strange attractors it is frequently possible to introduce the concept of information dimensionality  $D_I$  for which the weighting factor is  $\mu(\Omega_i) = -p(\Omega_i) \ln p(\Omega_i)$ .

The problem of determination of the length of a curve in finding the length of a coastline using maps of increasing scale (with greater detail) was evidently the first practical problem in which the fractal curves have been encountered. The history of this problem, numerical results, and excellent illustrations can be found in Mandelbrot's book.<sup>1</sup> The boundary of a percolation cluster in a two-dimensional continuous model is also a fractal curve. We shall return to this problem later.

For any curve defined on a two-dimensional set (which may be a plane or a spherical surface) the fractal dimensionality satisfies the inequality  $1 < D \leq 2$ . The fractal dimensionality of the western coast of Great Britain is  $D \approx 1.24$ , whereas the coastline of Australia is characterized by  $D \approx 1.13$ ; the dimensionalities of the majority of coastlines are close to unity. On the other hand, the fractal dimensionality of the boundary of a percolation cluster which can be regarded as a line with a random relief is  $D_h \approx 1.74$ , i.e., it is much closer to 2. The fractal dimensionality of the outer boundary or hull of a cluster with  $s \gg s_c$  (also called lattice animal) is approximately 1.5 (Ref. 3). The coastlines of islands and continents are not random lines at sea level: they have been formed as a result of a complex interaction between sea and land.

## 4. DIMENSIONALITIES OF PERCOLATION CLUSTERS

### 4.1. Homogeneous fractals and dimensionality of a cluster

We shall now consider the problems of percolation and calculate the fractal dimensionality of a percolation cluster (which is either very large or infinite). This dimensionality is easily expressed in terms of thermodynamic critical exponents.

As already pointed out, near a critical point a system can be regarded as fractal and self-similar on a scale  $a_0 \ll l < \xi$ , where  $\xi$  is the correlation length, and as homogeneous on a larger scale. An idea of how such a system looks



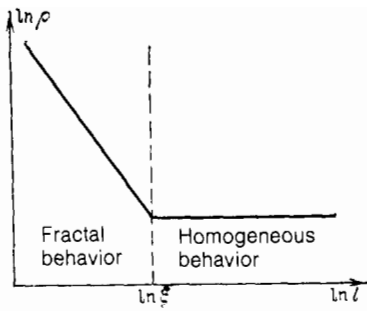


FIG. 5. Behavior of the density of a homogeneous fractal.

can be gained if a plane is covered with Sierpiński carpets with a side of length  $\xi$ : on a scale less than  $\xi$  the system is self-similar, whereas on a larger scale it is homogeneous. If a system has a minimum scale, the dependence of its mass on the scale  $l$  is given by Eq. (3.1) for  $l < \xi$  and by  $M \propto l^d$  for  $l > \xi$ . Introducing the density

$$\rho = \frac{M}{l^d} = \begin{cases} l^{D-d}, & l < \xi, \\ \text{const.} & l > \xi, \end{cases} \quad (4.1)$$

we find that  $\xi$  is the scale at which the density becomes constant. Figure 5 shows this behavior of the density on a double logarithmic scale. Similar behavior is expected for any intensive quantity characterizing the behavior of such a system. In the fractal regime such a quantity  $Y$  obeys  $Y \propto l^\lambda$ , where  $\lambda$  is a certain power exponent; in the homogeneous case it becomes constant.

Structures which behave as fractal on a small scale and as homogeneous on a large scale are frequently called homogeneous fractals. In particular, percolation clusters characterized by  $x \neq x_c$  are objects of this kind. The power of an infinite cluster, i.e., the fraction of the sites or the volume belonging to a cluster, varies with the concentration as  $P_\infty \propto \tau^\beta$ . This quantity is the density of a cluster. The scale for which this quantity is defined is as follows: an infinite cluster is a fractal object if  $l < \xi \propto \tau^{-\nu}$  and a homogeneous one for larger scales. When the characteristic scale  $\xi$  is altered, i.e., when the concentration  $x$  is modified, the density varies as follows:  $\rho(\xi) \propto \xi^{-\beta/\nu}$ . Bearing in mind that  $\rho(\xi) \propto \xi^{D-d}$ , we obtain from Refs. 2, 3, and 61 the expression

$$D = d - \frac{\beta}{\nu}. \quad (4.2)$$

We shall return to Eq. (4.2) later, but at this stage we note that the above analysis is standard: this is the approach used in the establishment of the relationship with thermodynamic and other geometric exponents.

Let us consider the behavior of some quantity  $Y$  related to our homogeneous fractal; for example, the quantity may be the electrical conductivity considered in Sec. 5.1. Above all, we have to introduce an intensive quantity  $y$  related to  $Y$ . In the case of conduction problems this quantity is of course the electrical conductivity. Such an intensive quantity is independent of the scale  $l$  in the homogeneous regime. In the fractal regime when  $l < \xi$ , it behaves as  $l^\lambda$ . Consequently, the

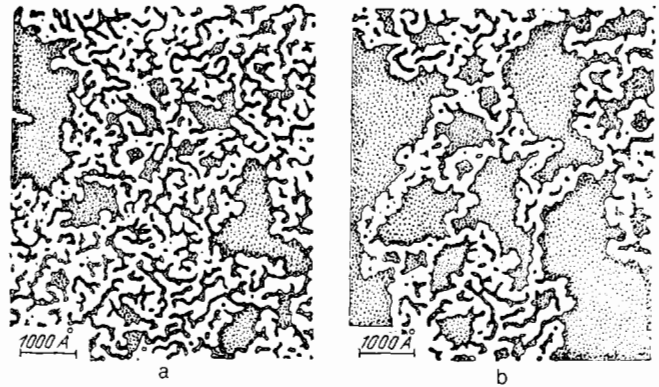


FIG. 6. Photomicrographs of a metal island film with  $x = x_c$  (Ref. 63): a) infinite cluster; b) backbone of an infinite cluster.

value of  $y$  for the homogeneous regime obeys  $y \propto \xi^\lambda$ . The dependence of  $y$  on the control parameter  $\tau$  is then manifested because of the dependence on  $\tau$  of the length  $\xi$ , so that  $y(\tau) \propto \xi(\tau)^\lambda \propto \tau^{-\nu\lambda}$ . It means that the thermodynamic exponent  $\lambda'$  ( $y \propto \tau^{\lambda'}$ ) is related to  $\lambda$  by  $\lambda' = -\nu\lambda$ . This approach will be used throughout the present review.

There are also other approaches which give Eq. (4.2). The result (4.2) has been confirmed in the two-dimensional case by independent measurements of  $D$ ,  $\beta$ , and  $\nu$  (Ref. 61). Among the experiments confirming Eq. (4.2) we will mention particularly the physical experiments on metal island films. A study of lead films was reported in Ref. 62 and of gold films in Ref. 63. The values  $D \approx 1.9$  (Figs. 6–8) were obtained. The standard method for the determination of the fractal dimensionalities of real objects involves the use of their photomicrographs obtained using different magnifica-

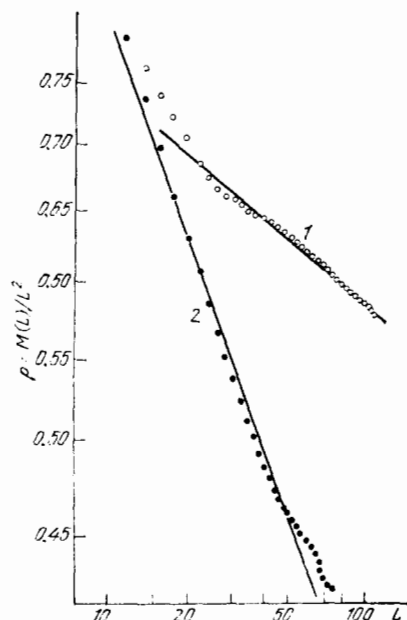


FIG. 7. Dependences of the density of an infinite cluster (1) and of its backbone (2) on the linear size of the investigated sample with  $x = x_c$  (Ref. 63). Double logarithmic scale.

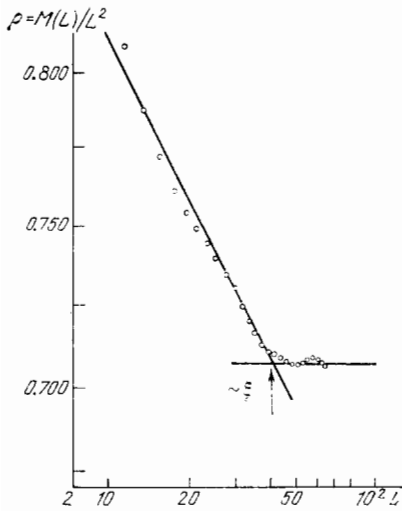


FIG. 8. Dependence  $\rho(L)$  for  $x > x_c$  (Ref. 63). This curve is an experimental analog of Fig. 5.

tions, drawing of a square grid on such photomicrographs, and calculating the number of squares within which the points of the object occur. Then, the fractal dimensionality is the slope of the straight line representing the dependence of the number of such squares on the magnification, plotted on a double logarithmic scale. For example, this method was used in Ref. 64 to determine the fractal dimensionality of the boundaries of clouds governed by turbulent diffusion of water vapor in the atmosphere.<sup>65</sup>

The relationship (4.2) includes  $d$  and it is valid in the hyperscaling regime, i.e., when  $d \leq 6$ . If  $d > 6$ , the quantity  $D$  assumes a constant value, like many other critical exponents in the classical range. This value is  $D = 4$ . The formula which gives  $D$  for any dimensionality of space is<sup>66</sup>

$$D = \frac{\beta + \gamma}{\nu}. \quad (4.3)$$

Eliminating from Eq. (4.3) the value of  $\gamma$  by the hyperscaling relationship  $d\nu = 2\beta + \gamma$ , we obtain Eq. (4.2).

In discussing the problem of the number of infinite percolation clusters, it has been shown rigorously that this number is  $n_\infty = 0, 1, \text{ or } \infty$ . The zero value naturally corresponds to the absence of percolation, whereas 1 or  $\infty$  correspond to percolation.<sup>67,68</sup> If  $d < 6$ , the first two possibilities are realized, whereas for  $d = \infty$  (Bethe lattice), we find that the first and the third possibilities occur. It has been frequently suggested that if  $d > 6$ , then we can have  $n_\infty = \infty$  (Refs. 67 and 68). If this is indeed true then the divergence between Eqs. (4.2) and (4.3), and any breakdown of the hyperscaling relationships in general can be understood: these equations give the dimensionalities of various sets. Equation (4.2) gives the dimensionality of a set of all the infinite clusters, whereas Eq. (4.3) gives the dimensionality of one cluster. If  $d < 6$ , when  $n_\infty = 1$  for  $x > x_c$ , these sets are naturally identical. The author of the present review is not aware of any treatments proving or disproving the hypothesis that  $n_\infty = \infty$  when  $d > 6$ .

In addition to the dimensionality of a cluster, it is interesting to consider the dimensionality of its current-carrying part which is called the backbone. The backbone of an infinite cluster is a set of sites belonging to paths in a cluster infinite in both directions. It is assumed that the knowledge of this dimensionality will help in dealing with the electrical conductivity of a percolation net. The dimensionality of the backbone is described by

$$D_b = d - \frac{\beta_b}{\nu}, \quad (4.4)$$

which is analogous to Eq. (4.2), except that  $\beta_b$  is the critical exponent of the strength of the backbone of a cluster: the probability that a given site belongs to the backbone is  $P_b \propto |\tau|^{\beta_b}$ . This dimensionality was determined in Ref. 69 and it amounts to  $1.67 \pm 0.17$  for  $d = 2$  and  $1.68 \pm 0.14$  for  $d = 3$ . It does not differ too greatly from the dimensionality of a triangular Sierpiński curve, which is therefore often used as the model of the backbone.<sup>70,71</sup>

#### 4.2. Chemical dimensionality and velocity of propagation of excitations

In addition to the fractal dimensionality of a cluster, it is of interest to consider its chemical dimensionality.<sup>72,73</sup> Let us consider, for example, the site problem. We shall define the chemical distance  $R_c$  between the sites  $i$  and  $j$  as the minimum number of steps in which we can reach  $j$  from  $i$  passing only between the empty sites (Fig. 9). If a sphere  $B_c$  is regarded as a set of sites for which  $R_c \leq n$ , we can define the chemical dimensionality as the exponent  $D_c$  such that a number  $N$  of sites belonging to  $B_c$  increases on increase in  $n$  in accordance with

$$N \propto n^{D_c}. \quad (4.5)$$

This quantity has been determined numerically for the two-dimensional case and it is  $D_c \approx 1.72$ . The value of  $D_c$  is the ratio of two dimensionalities, that of a cluster  $D$  and  $D_R$  of a curve of length which gives the chemical distance. The dimensionality of this "geodesic" is  $D_R = D/D_c \approx 1.10$ , which shows that the line is not broken too much. This terminology is due to the following application of the percolation theory. We shall consider a percolation cluster as a lat-

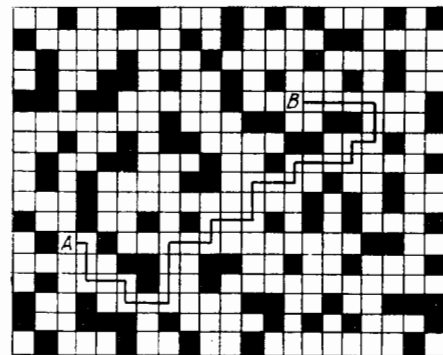


FIG. 9. Chemical distance for a percolation cluster.

tice model of a branched polymer molecule. The sites of the lattice then correspond to monomers and the bonds to chemical bonds between the monomers. The number of steps in a cluster is the number of chemical bonds along the path from the  $i$ th to the  $j$ th monomer and it is quite natural to call this the chemical distance and the associated dimensionality to be called chemical. A different name for this dimensionality is the dimensionality of the connectivity.

We shall consider in detail the meaning of a thermodynamic exponent  $\psi = (D_R - 1)\nu$ , associated with  $D_R$  (Ref. 74).

Let us assume that there is a perturbation which travels as follows. If at a moment  $t$  a site  $i$  is excited, then at a moment  $t + 1$  all the sites which are available next to  $i$  are excited. A site excited at any time remains in that state during all subsequent moments. Let us assume that at  $t = 0$  we excite the site 0. The excitation front at a time  $t$  is a set of sites for which the chemical distance from the site 0 is  $t$ .

Such a model describes, for example, propagation of a disease between trees<sup>36</sup> or of a forest fire, and it is also used (within the framework of the model discussed in Ref. 75) to describe propagation of neural pulses in myocardium.<sup>74</sup>

If  $x < x_c$ , there is no propagating perturbation in the system. For any value of  $x > x_c$ , the asymptotic behavior of the front is described in the limit of long times by a certain  $x$ -dependent velocity. As  $x$  approaches  $x_c$  from above, this velocity decreases: critical retardation takes place. At  $x = x_c$ , the perturbation velocity vanishes.

Let us assume that the concentration of empty sites is  $x > x_c$ . The correlation length of such a system is  $\xi = \tau^{-\nu}$ . In the fractal regime the time in which the perturbation front travels a distance  $R$  is  $t \propto R^{D_R}$  and, in particular, the distance  $\xi$  is traveled in a time  $t_\xi \propto \xi^{D_R} \propto \tau^{-D_R\nu}$ . Over long distances the behavior of the system becomes homogeneous. Then, we have  $t/t_\xi = R/\xi$ . Therefore, in a time  $t \gg t_\xi$  the perturbation front travels a distance  $R \propto (\xi/t_\xi)t$ ; the effective velocity is  $v \propto \frac{\xi}{t_\xi} \propto \tau^{(D_R - 1)\nu}$ . The exponent  $\psi = (D_R - 1)\nu$  for the effective velocity governs the critical retardation. For  $d = 2$ , we have  $\psi = 0.13 \pm 0.06$ , whereas for  $d = 3$ , we find that  $\psi = 0.31 \pm 0.06$ . In the classical range, we obtain  $\psi = 0.5$ .

### 4.3. Boundaries (hulls) of percolation clusters

Investigations have also been made of the dimensionality of the outer boundary of a critical percolation cluster. The dimensionality of the whole boundary (outer and inner) is the same as the dimensionality of the cluster itself, since the number of sites on the boundary (perimeter) is proportional to the total number of sites in the cluster:  $l \propto s^p$ , where  $p = 1$  (Refs. 3 and 76). The dimensionality of the outer boundary or hull of a large two-dimensional cluster has been considered<sup>77</sup> and found to be  $D_h = 1.72-1.74$  (Fig. 10). We shall consider the relationship between this quantity and other critical exponents.<sup>78</sup> We shall discuss a critical cluster. Its boundary is a self-similar curve; when the length of the curve

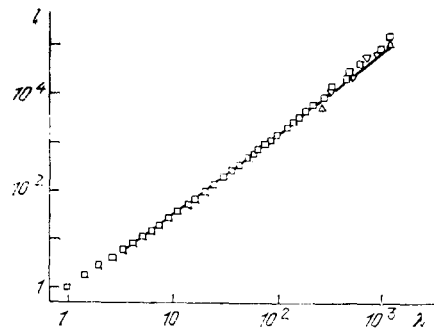


FIG. 10. Dependence of the perimeter of a percolation cluster on its transverse size.<sup>77</sup> Double logarithmic scale.

is of the order of the total length of the boundary of the cluster (for example, of the order of  $l/2$ ), its distance from the initial point is of the order of the cluster size, i.e., it is of the order of the correlation length  $\xi$ . A characteristic number of sites in a critical cluster is  $s_c \propto \tau^{d\nu - \beta}$ , whereas the relationship between the length of the outer boundary or hull of a cluster and the number of sites is given by  $l \propto s_c^{p + \kappa}$ , where  $\kappa \leq 0$  is an exponent governing the critical behavior of the ratio of the outer and inner perimeters. In the two-dimensional case the quantity  $\kappa$  has been determined numerically in Ref. 76 and in Ref. 79 an expression for this quantity (again in the two-dimensional case) is obtained in terms of  $d = 2$ ,  $D$ , and thermodynamic exponents:

$$\kappa = \frac{d}{D} \left\{ - \left[ \frac{1 - (D/d)}{d\nu} \right]^{1/2} + 1 - \frac{D}{d} p \right\}. \quad (4.6)$$

The quantity  $\kappa$  can be very small:  $\kappa \approx -0.1$ . If we define  $D_h$  using  $D_h = 1/\varphi$ , where  $\varphi$  is the exponent in  $\xi \propto \tau^\varphi$ , and recall that  $\xi \propto \tau^{-\nu}$  and  $l \propto \tau^{(d\nu - \beta)(p + \kappa)}$ , we obtain

$$D_h = \left( d - \frac{\beta}{\nu} \right) (p + \kappa) = D(p + \kappa). \quad (4.7)$$

The formula (4.7) remains valid also if  $d > 2$ , if the perimeter or boundary of a cluster is understood to be the corresponding surface or hypersurface.

The problem of a special type of diffusion of particles emitted by a linear source is considered in Ref. 80 where it is asserted that there is an analogy between the diffusion front which is then formed and the boundary of a percolation cluster. On the basis of this analogy [which, like Eq. (4.6), is invalid in the three-dimensional case] it has been found that  $D_h = 1 + (1/\nu)$ . A numerical modeling of the boundary of a cluster by a random walk<sup>81</sup> gives a similar estimate amounting to  $D_h = 1.751 \pm 0.002$ .

It is concluded in Ref. 78 that if  $d \geq 3$ , then a finite fraction of the perimeter of a cluster belongs to the outer boundary or hull, i.e.,  $\kappa = 0$ . The dimensionality of the hull of a cluster and of the cluster itself are then identical. The reason for the difference in the cases  $d = 2$  and  $d \geq 3$  is the considerable difference between the topology of the boundaries of clusters: in the two-dimensional case the outer boundary is singly connected, whereas in the  $d \geq 3$  case it is not singly connected.

## 5. DIMENSIONALITIES AND TRANSPORT COEFFICIENTS

### 5.1. Conductivity and elasticity of fractal nets

We shall now consider the geometric critical exponents associated with the thermodynamic exponent  $t$ . We shall assume that  $\tau \neq 0$ . In the homogeneous range of scales a system can be regarded as consisting of blocks of size  $\xi$ . We shall assume that the characteristic spatial size of the system is  $L > \xi$ ; we shall discuss the specific case when the sample is in the form of a  $d$ -dimensional cube. We shall determine its resistance by connecting two perfectly conducting  $(d-1)$ -dimensional electrodes to the opposite sides of the cube.

If the resistance between the two faces of a cube of side  $\xi$  is  $\mathcal{R}_0$ , then the resistance of the whole system is

$$\mathcal{R} = \mathcal{R}_0 \left( \frac{L}{\xi} \right)^{-d+2}. \quad (5.1)$$

Indeed, in view of the identity of the cubes, the currents do not flow through their side surfaces and the resistance of one column of cubes connecting the electrodes is  $R_0(L/\xi)$ ; the total number of such parallel-connected columns is  $(L/\xi)^{d-1}$ , so that we obtain Eq. (5.1). This is a familiar relationship showing that the resistance of a wire rises proportionally to its length and the resistances of similar plane figures are identical, and so on.

If  $\mathcal{R} \propto \tau^{-t}$ , it then follows that  $\mathcal{R}_0(\xi) = (L/\xi)^{-d+2} / \mathcal{R}[\tau(\xi)] \propto \xi^{d-2+(t/\nu)}$ . Assuming that the dependence  $\mathcal{R}_0(l)$  applies for any scale in the self-similar regime, we reached the conclusion that the resistance of a percolation cluster with  $l \leq \xi$  depends on the distance between the electrodes as follows:

$$\mathcal{R}(l) \propto l^{\zeta}, \quad \text{where } \zeta = \frac{t}{\nu} - d + 2. \quad (5.2)$$

This exponent was first introduced in Ref. 82. The exponent  $\zeta$  is also a geometric one representing the properties of clusters in the self-similar regime. This exponent describes the rate of growth of the resistance of a fractal on increase in its spatial scale:  $\zeta = \ln c / \ln a$ , where  $c$  is the ratio of the resistances measured between similar points of fractals which have spatial scales differing by a factor  $a$ . In an earlier study<sup>83</sup> representing a calculation of  $t$  by the scaling method it has been postulated that  $\zeta = 1$ . This is not a self-evident assumption, although it may be valid in the  $d = 2$  case.<sup>84</sup>

The value of  $\zeta$  can easily be calculated for the simplest families of regular fractals. We shall carry out such a calculation for a triangular Sierpiński curve.<sup>63</sup> This fractal consists of three similar fractals of half the size and, consequent-

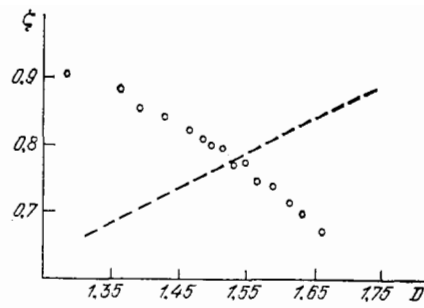


FIG. 12. Values of the exponent  $\zeta$  for fractals of the "chessboard triangle" type. The dashed line represents  $\zeta = D/2$ .

ly, we have  $D = \ln 3 / \ln 2 = 1.585$ . In this calculation we shall assume that the resistance between the vertices of each of the small triangles is 1 and replace each of them by a star in accordance with the usual rules (Fig. 11); this shows that the resistance between the vertices of the large triangle is  $R = 4/3$ , so that  $\zeta = \ln(4/3) / \ln 2 = 0.7370$ . This fractal is selected because its fractal dimensionality is close to the dimensionality of the backbone of an infinite cluster, so that to some extent it models its current-carrying part. The values of  $\zeta$  for objects of this kind also do not differ appreciably. If we consider fractals of the "chessboard triangle" type and plot the exponent  $\zeta$  as a function of the fractal dimensionality, we find that the points obtained are grouped around a certain curve, although they do not lie exactly on this curve (Fig. 12). However, we can construct fractals for which the deviation from this curve is large: there is no universal dependence of  $\zeta$  on  $D$ . We shall return later to this important topic.

It is also of interest to consider the behavior of the elastic modulus  $T$ . By analogy with  $\zeta$ , we can once again introduce a geometric exponent  $Z = (T/\nu) + d - 2$ . In modeling the elasticity of a random net we should, as already pointed out, consider separately the cases of isotropic and central elastic forces. For the isotropic forces the potential energy due to the displacement of lattice sites is simply the sum of the contributions made by the displacements along the axes:

$$U = \sum_{\alpha=1}^d \frac{k}{2} (\Delta x^\alpha)^2.$$

The corresponding equilibrium conditions are of the form<sup>31</sup>

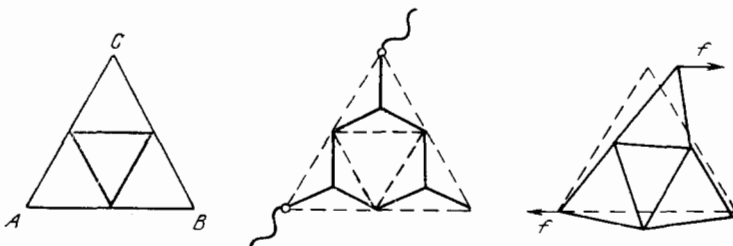


FIG. 11. Calculation of the conductivity and elasticity of a fractal lattice.

$$\left. \begin{aligned} \sum_m k(x_m - x_m, 0) &= 0, \\ \sum_m k(y_m - y_m, 0) &= 0. \\ \dots \dots \dots \end{aligned} \right\} \quad (5.3)$$

(the summation is carried out over the whole bonds  $m$  in a percolation lattice and each of these bonds is assigned an elastic constant  $k$ ). Such a system represents a system of Kirchhoff equations which are repeated  $d$  times (for each of the axes) and describe the electric currents in a net and, consequently, in the homogeneous case when  $T = t$ .

When the forces are central, the potential energy is

$$U = \sum_{i,j} \frac{k_{ij}}{2} (\mathbf{r}_{ij} - \mathbf{r}_{ij,0})^2,$$

and the equilibrium conditions are different than in the case of the Kirchhoff equations: the terms corresponding to displacements along different axes are now mixed. In the case of the simplest fractals (such as a triangular Sierpiński curve and its generalization to spaces with a larger number of dimensions) this exponent is calculated in Ref. 85. It is assumed that each of the parts of a fractal represents a rigid figure with known bulk and shear moduli whereas at the point of connection the figures are linked by ballbearings (Fig. 11). If in the first order in respect of displacement of the vertices of a large fractal, we determine the forces acting on them, we find that the bulk and shear moduli behave in the same way:  $K \propto \mu \propto l^z$ , where  $z = d - 1$ , and their ratio is  $K/\mu = 4d$ .

It therefore follows that problems of elasticity of a polymer net may belong to different classes of universality in the case of isotropic and central elasticity forces. This has been confirmed by a numerical experiment.<sup>49</sup> This experiment was concerned with the case in which there are homogeneous and central contributions, and a crossover takes place from one type of behavior to another.

### 5.2. Conductivity exponent, anomalous diffusion, and internal dimensionality of random walks on a cluster

The value of the exponent  $\zeta$  is related directly to some dimensional characteristics of the fractals or, more precisely, of the sets created by random walks on the initial fractal.

Let us consider a point which wanders randomly on an ordered  $d$ -dimensional lattice. The dependence of the average square of the distance reached by this point in  $N$  steps (after a discrete time  $t$ ) is then

$$\langle r^2 \rangle \propto N, \quad t_0 \quad (5.4)$$

where this asymptotic law in no way depends on the dimensions of space. For a random walk on a fractal lattice the dependence  $\langle r^2(t) \rangle$  is different<sup>86-91</sup>:

$$\langle r^2 \rangle \propto N^{2\theta} = N^{2/(2+\theta)}, \quad (5.5)$$

where the quantity  $\theta \neq 0$  is sometimes called the anomalous diffusion exponent. Equation (5.5) can be regarded as describing diffusion in a medium with a variable coordinate-dependent diffusion coefficient:

$$\mathcal{K}(r) \propto r^{-\theta}. \quad (5.6)$$

The fractal properties of the medium appear because such a dependence  $\mathcal{K}(r)$  is realized for any initial point of a random walk.

It seems quite surprising that the fractal structure of a set disobeys such a universal law as that given by Eq. (5.4). Therefore, it seems that although the distance measured in the external space does not obey Eq. (5.4), the average distance inside the system (i.e., the chemical distance) may obey this law. This is true only of fractal lines. An analysis of such a line which represents a very involved one-dimensional lattice<sup>86</sup> shows that after  $N$  steps a random walker travels on the average  $N_{av} \propto N^{1/2}$ . In the Euclidean space this distance corresponds to  $r \propto N_{av}^{1/D}$ , so that  $\langle r^2 \rangle \propto N^{1/D}$ . We then have

$$\theta = 2D - 2. \quad (5.7)$$

However, Eq. (5.7) is not valid in the case with a ramification order greater than 1. The reason for this is that the effect of the fractal structure of a cluster is not just that after a long walk on a fractal the net displacement in the surrounding Euclidean space is small. A complex structure of the path and the inhomogeneity of the distribution of its points of intersection violates even the relationship  $N_{av} \propto N^{1/2}$ .

We shall consider a random net. On the smallest scale we can regard it as formed by ordinary conductors, each with a resistance  $\mathcal{R}$ , and assume that the probabilities of electron jumps between the sites are given:  $w_{ij} = w$  for any whole bond and  $w_{ij} = 0$  for a broken bond. When  $\tau > 0$ , in either of these cases the large-scale behavior of the particles can be described as diffusion. The conductivity of the system is proportional to the carrier mobility which in its turn is proportional to the diffusion coefficient because of the fundamental relationship to Einstein. Since the diffusion coefficient  $\mathcal{K}(\tau)$  is defined only for an infinite cluster to which the fraction  $P_\infty(\tau)$  of the sites belongs, the relationship can be written in the form<sup>86,91</sup> (see also Ref. 92):

$$\sigma(\tau) \propto \mathcal{K}(\tau) P_\infty(\tau). \quad (5.8)$$

We shall calculate the asymptotic behavior of  $\mathcal{K}(\tau)$  using our standard approach. The diffusion coefficient is an intensive quantity. Its value is  $\mathcal{K}(\tau) = \mathcal{K}(r = \xi) \propto \tau^{-\theta}$ . Substituting it in Eq. (5.8), we find that

$$\theta = \frac{t - \beta}{\nu}. \quad (5.9)$$

A relationship similar to Eq. (5.9) was first derived in Ref. 93 and began to be used more frequently after publication of Refs. 86 and 91. A confirmation of this relationship by a more rigorous approach can be found in Ref. 94. We shall rewrite Eq. (5.2) using Eq. (5.9):

$$\zeta = (\theta + 2) - \left( d - \frac{\beta}{\nu} \right). \quad (5.10)$$

We shall consider the relationship between the exponent  $\zeta$  and the dimensional characteristics. The meaning of the second term in Eq. (5.10) is already known: this is the fractal dimensionality of the cluster itself.

We shall consider a line (with recurrence, i.e., return to the starting point, and self-intersections) which describes a random walk of a point on a cluster. The length  $l$  of the curve is proportional to the number of steps  $N$ . The exponent  $\varphi$  for this curve defined by the formula  $\langle r^2 \rangle^{1/2} \propto l^\varphi$  is  $\varphi = 1/(2 + \theta)$  and, consequently, its dimensionality is  $D'_w = 2 + \theta$ . Since  $\theta > 0$ , in the case of a random walk on any fractal, we have  $D'_w > 2$ . Therefore, the expression for the exponent  $\zeta$  can be rewritten in the form

$$\zeta = D'_w - D, \quad (5.11)$$

where  $D$  is the dimensionality of a set and  $D'_w$  is the internal dimensionality of a random walk on this set. For any regular lattice we have  $\theta = 0$ ,  $D'_w = 2$ , and  $D = d$ , so that  $\zeta = 2 - d$ : this is the known exponent for the relationship between the resistance of a system and its electrical resistivity.

The fact that  $D'_w > 2$  means that a random walker returns many times to the starting point. This can be seen directly by examination of Fig. 13 representing the result of a numerical modeling of a random walk on a fractal.<sup>14</sup>

We shall now consider the backbone of an infinite cluster. Since the resistance of a cluster is equal to the resistance of its backbone (if the electrodes are sufficiently far apart), the exponents for the whole cluster and its backbone are identical. Then, in discussing a random walk on the backbone, we find [in full analogy with Eq. (5.10)] that

$$\zeta = (\theta_b + 2) - \left( d - \frac{\theta_b}{\nu} \right), \quad (5.12)$$

where  $\theta_b$  is the exponent of anomalous diffusion on the backbone of a cluster. Equating Eqs. (5.10) and (5.12), we obtain<sup>60</sup>

$$D'_{w,b} - D_b = D'_w - D. \quad (5.13)$$

The internal dimensionality of a random walk on a backbone is less than the internal dimensionality of a walk on the

whole cluster: it is closer to 2, which is the dimensionality of an ordinary random walk. This is due to a change in the nature of recurrence: wandering along dead ends results in repeated passage through sites close to the point of attachment of the dead ends to the backbone and when the backbone is considered on its own, this contribution is excluded.

Other types of random walks have also been considered. For example, diffusion on lattice animals is considered in Ref. 95, whereas in Refs. 92, 96, and 97 a study has been made of self-avoiding random walks on fractal lattices.

### 5.3. Equation for diffusion on a fractal

In the preceding section we have dealt with the asymptotic behavior of various quantities. We can now pose the question: what should be the equation for diffusion on a fractal and what is the solution of this equation? In this section we shall follow the treatment in Refs. 98 and 99 which deal specifically with this topic.

We shall consider the density of the probability  $p(\mathbf{r}, t)$  of finding a particle located at the point  $0$  at the beginning of the process  $t = 0$  and at a point  $\mathbf{r}$  at a time  $t$ . Calculations for a regular fractal (in the form of a triangular Sierpiński curve) show that the function  $p(\mathbf{r}, t)$  is nonanalytic and has singularities on small and large scales. We can consider a smooth envelope of this function  $P(\mathbf{r}, t)$ . The function  $P(\mathbf{r}, t)$  is described by the equation

$$\frac{\partial P(\mathbf{r}, t)}{\partial t} = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left[ \mathcal{K} r^{D-1-\theta} \frac{\partial P(\mathbf{r}, t)}{\partial r} \right] \quad (5.14)$$

(this equation is derived in spherical coordinates). The quantity  $\mathcal{K}$  is the generalized diffusion coefficient. The solution of the above equation is

$$P(\mathbf{r}, t) = \frac{2+\theta}{\Gamma(D/(2+\theta))} \left[ \frac{1}{\mathcal{K}^{2+\theta} t} \right]^{D/(2+\theta)} \exp \left[ -\frac{r^{2+\theta}}{\mathcal{K}^{2+\theta} t} \right] \quad (5.15)$$

and it yields

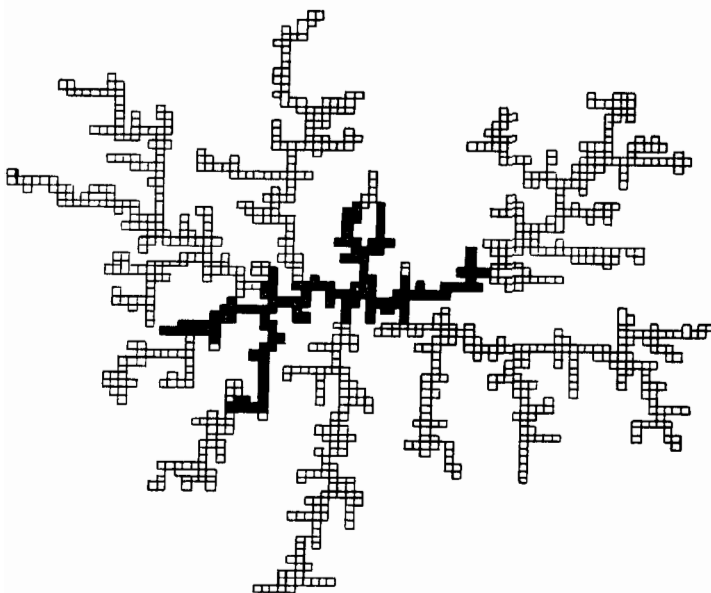


FIG. 13. Random walk of 2500 steps on a fractal of 1000 sites (dendrite).<sup>14</sup>



$$\langle r^2(t) \rangle = \frac{[\mathcal{K}'(2+\theta)2t]^{2/(2+\theta)} \Gamma((D+2)/(\theta+2))}{\Gamma(D/(2+\theta))} \quad (5.16)$$

and

$$P(0, t) = \frac{2+\theta}{\Gamma(D/(2+\theta))} [\mathcal{K}'(2+\theta)2t]^{-D/(2+\theta)}. \quad (5.17)$$

The expression (5.15) agrees excellently with the results of rigorous calculations for a triangular Sierpiński curve. Another approach which gives the same results and is based on the formalism of the path integrals is presented in Ref. 100. Equation (5.14) is an analog of the diffusion equation in Euclidean space written down using spherical coordinates:

$$\frac{\partial P(r, t)}{\partial t} = \frac{\mathcal{K}'}{r^{d-1}} \frac{\partial}{\partial r} \left[ r^{d-1} \frac{\partial P(r, t)}{\partial r} \right]. \quad (5.18)$$

The derivation of Eq. (5.14) is based on the relationship between diffusion and conductivity, and the procedure used to derive it is standard. The probability of finding a particle in a spherical layer between  $r$  and  $r + dr$  is  $m(r, t)dr$  and it obeys the relationship

$$\frac{\partial m(r, t)}{\partial t} = \frac{\partial}{\partial r} I(r, t), \quad (5.19)$$

where  $I(r, t)$  is the probability flux. The probability  $m(r, t)$  is related to  $P(r, t)$  and to the number of sites  $N(r)$  inside the layer:

$$m(r, t) = P(r, t) N(r). \quad (5.20)$$

We can close the system by writing down the equation that relates linearly  $I(r, t)$  and  $\nabla P(r, t)$ :

$$I(r, t) = \sigma(r) \frac{\partial P(r, t)}{\partial r}, \quad (5.21)$$

where the coefficient  $\sigma(r)$  is proportional to the total (integrated over the angles) conductivity at a distance  $r$ . Using Eqs. (5.19)–(5.21), we obtain

$$\frac{\partial P(r, t)}{\partial t} = N^{-1}(r) \frac{\partial}{\partial r} \left[ \sigma(r) \frac{\partial P(r, t)}{\partial r} \right], \quad (5.22)$$

The quantity  $N(r) = \partial n(r)/\partial r$ , where  $n(r)$  is the number of sites within a sphere of radius  $r$ , has the value  $N(r) = Dr^{D-1}$  if  $a_0 = 1$ . Instead of  $\sigma$  we shall introduce the “conductivity” per site proportional to the diffusion coefficient:  $\sigma(r) = \mathcal{K}'(r)N(r)$ . We can describe  $\mathcal{K}'(r)$  by the expression mentioned earlier:  $\mathcal{K}'(r) = \mathcal{K}'r^{-\theta}$ . Substituting all this in Eq. (5.22), we obtain Eq. (5.14).

We shall make one other comment. In Euclidean space the case of an isotropic diffusion coefficient is not common. On the other hand, in the case of fractals the diffusion coefficient is typically isotropic.

We shall begin with an example. We shall consider a triangular Sierpiński curve. We shall assume that on the smallest scale the resistance between the vertices  $A$  and  $B$  of a unit cell is  $r_x$ , whereas that between  $A$  and  $C$  or  $B$  and  $C$  is  $r_y$ . Replacing, as usual, the triangle with a star, we find that the renormalized resistance (i.e., resistances on a  $2a_0$  scale) are

$$\begin{aligned} r_x' &= r_{AB'} = r_x + \frac{2r_x r_y}{r_x + 2r_y}, \\ r_y' &= r_{AC'} = r_y + \frac{r_y(r_x + r_y)}{r_x + 2r_y}. \end{aligned} \quad (5.23)$$

The ratio of these resistances, representing the anisotropy of the resistance  $X$ , is renormalized in accordance with the law

$$X' = \frac{r_x'}{r_y'} = g(X) = X \frac{X+4}{2X+3}. \quad (5.24)$$

The  $X' = g(X)$  transformation has one stable fixed point  $X^* = 1$  and, consequently, on a large scale the resistances between all the vertices of a fractal are the same, irrespective of how the resistances of the bonds in a unit cell of a fractal are selected. On a large scale ( $X \approx 1$ ) the quantity  $\Delta = X - 1$  is renormalized in accordance with the law  $\Delta' = g'(1)\Delta = 4\Delta/5$ . The corresponding geometric exponent is  $\ln(4/5)/\ln 2 = -0.322$ .

This behavior, representing isotropization of the conductivity, is generally typical of fractals. Let us consider some isotropic (in respect of structure) self-similar system with different bond resistances. Let us assume that the conductivities of certain bonds, such as those parallel to the  $OX$  axis, are equal to  $\sigma_x$  and the remainder to  $\sigma_y$ . The scaling transformation (i.e., the crossover to a fractal which is  $n$  times greater) renormalizes these conductivities as follows:

$$\begin{aligned} \sigma_x' &= f_x(\sigma_x, \sigma_y), \\ \sigma_y' &= f_y(\sigma_y, \sigma_x). \end{aligned} \quad (5.25)$$

We shall now assume that all the bonds in our fractal are current-carrying. Then, the functions  $f_x$  and  $f_y$  are monotonic in each of the arguments: it follows from  $a > b$  that  $f_{x,y}(a, c) > f_{x,y}(b, c)$  and  $f_{x,y}(c, a) > f_{x,y}(c, b)$ . An increase in the conductivity of any specific bonds cannot reduce the conductivity of the system as a whole.

In the case of identical arguments, we have

$$f_{x,y}(\sigma, \sigma) = n\sigma,$$

where  $n$  is the scaling transformation coefficient. We shall consider the specific case when  $\sigma_x > \sigma_y$ . We then have

$$\frac{\sigma_x'}{\sigma_y'} = X' = \frac{f_x(\sigma_x, \sigma_y)}{f_y(\sigma_y, \sigma_x)} < \frac{f_x(\sigma_x, \sigma_x)}{f_y(\sigma_y, \sigma_y)} = X. \quad (5.26)$$

Therefore, if the initial value is  $X > 1$ , then the values of  $X$  decrease during the subsequent iterations and they approach unity. For  $X < 1$ , the value of  $X$  increases as a result of iteration until it reaches unity. The exception to this rule is the case when the rigorous inequality is not obeyed, i.e., when the functions  $f_x$  and  $f_y$  are independent of the second of their arguments. The dependence of  $X'$  on  $X$  in the limit  $X \rightarrow 1$  is described by a certain function  $g(X)$  and the derivative of this function at  $X = 1$  determines the corresponding geometric exponent

$$\omega = \frac{\ln g'(1)}{\ln n}. \quad (5.27)$$

This disappearance of the anisotropy on approach to the critical point has long been known to be a feature of percolation problems: in the effective-medium approximation such isotropization was studied in Ref. 101 and the critical behavior of the anisotropy  $X$  was first considered in Ref. 102. The anisotropy of the conductivity near the percolation threshold has been studied both numerically<sup>102–105</sup> and in real experiments.<sup>106</sup> The corresponding thermodynamic exponent is  $\Lambda = -\omega\nu$ :  $X \propto \tau^\Lambda$ . In the two-dimensional case

this exponent is approximately 0.86, whereas in the three-dimensional case it is close to 1/3.

## 6. SPECTRAL AND RELATED DIMENSIONALITIES

### 6.1. Spectral dimensionality, dimensionality of a recurrence set, and dimensionality of random walks

The characteristics of a random walk on a cluster include also another very important exponent which is known as the spectral or fracton dimensionality.<sup>86,107</sup> We shall now consider the transport equation describing a random walk of a particle:

$$\frac{d}{dt} p_i(t) = \sum_j V_{ij} [p_j(t) - p_i(t)], \quad (6.1)$$

where  $p_i(t)$  is the probability of finding a particle at a site  $i$  at a moment  $t$ ;  $V_{ij}$  is the probability of a transition from a site  $i$  to a site  $j$  per unit time. We shall assume that  $V_{ij} = 1$  for neighboring sites connected by a whole bond and that  $V_{ij} = 0$  in all the other cases. This equation differs only in respect of the order of magnitude from the equation describing vibrations of a random net:

$$\frac{d^2}{dt^2} x_i(t) = \sum_j \omega_{ij} [x_j(t) - x_i(t)]; \quad (6.2)$$

here,  $\omega_{ij} = 1$  if a given bond is whole. The elasticity forces are then regarded as isotropic and a displacement is a scalar quantity. An equation of the (6.1) type describes also the electronic properties of such a random crystal in the tight-binding approximation with a correlated diagonal and off-diagonal disorder, described by the Hamiltonian

$$\hat{H} = \sum_j |j\rangle e_j \langle j| + \sum_{i,j} |i\rangle V_{ij} \langle j|, \quad (6.3)$$

where  $e_i = -\sum_j V_{ij}$ ;  $V_{ij} = 1$  for a whole bond and  $V_{ij} = 0$  for a broken bond. As usual, we can introduce the Green's function of the relevant equations in the time or energy representation. In the time representation  $G_{ik}(t)$  is a solution of, for example, Eq. (6.1) subject to the initial condition  $G_{ik}(t=0) = \delta_{ik}$ . A diagonal matrix element  $G_{kk}(t)$  simply denotes the probability that a particle is at the original site  $k$  at a moment  $t$ . The appropriate Green's functions for regular fractals can be calculated exactly.<sup>108</sup> If we consider the Fourier transform of such a function, we find that the density of the eigenstates of the relevant equation is

$$N(\xi) = -\frac{1}{\pi} \text{Im} \langle G_{00}(\xi + i0) \rangle. \quad (6.4)$$

In the problem of lattice vibrations, which differs from the problems of diffusion and of an electron spectrum by the replacement of the first derivative with the second, i.e., by the replacement of  $\xi$  with  $\omega^2$ , the density of states is

$$\mathcal{N}(\omega) = \omega N(\omega^2). \quad (6.5)$$

We can establish the relationship between this expansion and the earlier approaches by considering the asymptotic

behavior of  $G_{00}(t)$  in the case of a random walk on a percolation cluster or some other fractal lattice. We shall assume that at the moment  $t = 0$  a particle executing such a random walk is at the site 0. After  $n \gg 1$  steps, its average distance from the starting point is  $l \propto n^{1/(2+\theta)}$ . If we consider a sphere with the radius equal to this distance, we find that the number of lattice sites inside the sphere is proportional to  $l^D$ . If, for the sake of simplicity, we assume that the probability of finding the final point of a random walk within this sphere is the same for every site, we can see that the probability that after a time  $t$  a particle is at the starting site amounts to

$$P_0(t) \propto t^{-D/(2+\theta)}. \quad (6.6)$$

Knowing the asymptotic behavior of the probability of remaining at the starting site, we can find the asymptotic form of the density of states. In the problem of diffusion and in the solution of the Schrödinger equation of a random crystal with correlated diagonal and off-diagonal disorder the asymptotic behavior of  $N(\xi)$  in the limit  $\xi \rightarrow 0$  is

$$N(\xi) \propto \xi^{D/(2+\theta)-1}, \quad (6.7)$$

whereas in the problem of lattice vibrations, we have

$$\mathcal{N}(\omega) \propto \omega^{[2D/(2+\theta)]-1}. \quad (6.8)$$

We shall now consider a regular lattice in a Euclidean space of any dimensionality  $d$ . The density of the phonon states for a regular lattice (in the case of a homogeneous lattice there is only an acoustic mode) is described by the following asymptotic expression in the limit  $\omega \rightarrow 0$ :

$$\mathcal{N}(\omega) \propto \omega^{d-1}. \quad (6.9)$$

For a fractal structure (percolation cluster, gel at the formation threshold) the quantity

$$D_f = 1 + \lim_{\omega \rightarrow 0} \ln \mathcal{N}(\omega) = \frac{2D}{2+\theta} = 2 \frac{D}{D_w} \quad (6.10)$$

is an important characteristic of the vibrational spectrum of an object, which transforms into a Euclidean dimensionality in the case of a regular lattice. This quantity is known as the spectral or fracton dimensionality of the system. The word "dimensionality" does not mean in this case that there is any set which has these dimensions (in the sense used in Sec. 3.2). The spectral dimensionality can be found also for a more general case of both nonisotropic elastic forces<sup>109</sup> and long-range interactions.<sup>110</sup> The above case is, however, the most important. In this case the fracton dimensionality  $D_f$  is related to the recurrence dimensionality  $D_{\text{rec}}$  of a set of moments  $t_i$  at which a random walker returns to the starting point  $O$  (Refs. 111 and 112) and to the fractal (Hausdorff) dimensionality of a random walk. The first relationship is given by

$$D_{\text{rec}} = 1 - \frac{D_f}{2}, \quad (6.11)$$

which is valid if  $D_f < 2$  (which is always true of percolation clusters; see Sec. 6.3). If we consider a random walk on any set with  $D_f > 2$ , we find that  $D_{\text{rec}}$  vanishes.

In the case of regular lattices in the space of  $d$  dimen-

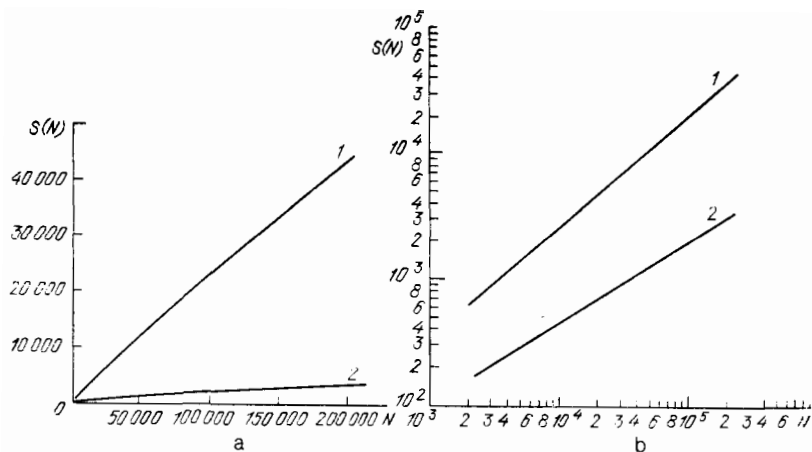


FIG. 14. Dependences of the numbers of different sites visited in the course of a random walk on the number of steps  $N$  (Ref. 113). Linear (a) and double logarithmic (b) scales: 1) Euclidean lattice; 2) percolation cluster.

sions  $D_{\text{rec}}(d)$  the recurrence dimensionality behaves as follows:  $D_{\text{rec}}(1) = 1/2$ ,  $D_{\text{rec}}(d \geq 2) = 0$ ; in the space of two dimensions the number of returns to the original point is proportional to  $\ln t$ , whereas in the space with  $d \geq 3$  a random walk is self-avoiding, i.e., there is no return to the starting point.<sup>113</sup>

The quantity  $D_w = D_s/2$  also has a simple meaning. The number of different sites visited in the course of a random walk of  $n$  steps is  $S_n \propto n^{D_w/2}$  (see Fig. 14 and Ref. 114), so that  $D_w$  is the fractal dimensionality of a random walk, in contrast to the internal dimensionality  $D'_w = 2D/D_f = D/D_w$ .

The exponent  $D_{\text{rec}}$  is related directly to the kinetics of fractal reactions and can be determined experimentally. By way of example, we shall consider Ref. 115. The investigated material was a substitutional solid solution of naphthalene  $C_{10}H_8$  in  $C_{10}D_8$ . A study was made of the transfer of energy by triplet excitons T, the radiative decay of which was spin-forbidden.<sup>116</sup> The mechanism of noncoherent energy transfer was found to involve random jumps of excitons from one  $C_{10}H_8$  molecule to another not too far from the former. The critical percolation concentration of  $C_{10}H_8$  was about 8%.

When two triplet excitons reach the same site, they fuse to form one singlet exciton:



which decays radiatively. The reaction rate proportional to the probability of an encounter of two excitons obeys the following equation in the homogeneous regime:

$$\frac{dn_S}{dt} = n_T^2, \quad (6.13)$$

where  $n_T$  and  $n_S$  are the concentrations of triplet and singlet excitons, respectively. In the fractal regime the above equation becomes

$$\frac{dn_S}{dt} = n_T^2 t^{-D_{\text{rec}}}, \quad (6.14)$$

so that an explicit time dependence appears in the equation. For  $D_{\text{rec}} = 0$ , i.e., in the case of a multidimensional regular lattice, the above equation reduces to the preceding one. An experimenter can study all the quantities which occur in Eq. (6.14):  $dn_S/dt$  is proportional to the fluorescence intensity

$I_n(t)$ , whereas  $n_T$  is proportional to the phosphorescence intensity  $I_{\text{ph}}(t)$  (at a different frequency) which is associated with the absorption of triplet excitons by impurities (luminescence centers).

The experiments mentioned above indicated that when the  $C_{10}H_8$  concentration was 8–12%, the recurrence dimensionality was  $D_{\text{rec}} \approx 0.35$ , in good agreement with the theoretical value.

We analyzed this investigation in detail because the problem of the luminescence of mixed molecular crystals and dyes in porous matrices is currently highly topical from the point of view of investigating the fractal dynamics experimentally. These problems are discussed in Refs. 117–126, where both binary chemical reactions and the capture of wandering particles by fixed traps are considered.

## 6.2. Eigenstates on fractals. Quantum percolation model

We shall now make some comments about the nature of the eigenfunctions of Eqs. (6.1)–(6.3) for fractal lattices. These functions correspond to localized states known as fractons. The problem of localization and crossover from the fracton to the phonon behavior is considered in Ref. 107. Localization of vibrational and electron states has been demonstrated for some models of regular fractals.<sup>127,128</sup> The conclusion of localization<sup>7,91</sup> is supported by the following arguments. Let us consider a  $\beta$  function, similar to that used in the scaling theory of localization<sup>129</sup>:

$$\beta(g) = \frac{d \ln g(L)}{d \ln L}, \quad (6.15)$$

where  $g$  is a dimensionless conductivity, i.e., a quantity of the order of  $\sigma L^{d-2}$ . In our case we have  $\beta(g) = -\zeta = \text{const}$ . Then, the inequality  $\beta(g) < 0$  is used, by analogy with Ref. 129, to draw the conclusion of localization. This conclusion is based on the relationship between  $g(L)$  and the sensitivity of the wave functions to changes in the boundary conditions, i.e., it is based on one of the definitions of localization used in Refs. 33 and 130.

However, the  $\beta$  function is the average and, therefore, a far too coarse characteristic of the system. It follows that the finer and the most interesting aspects of the problem are then ignored.

It should be pointed out that the problem of localization of eigenstates on a fractal lattice (percolation cluster) has been investigated for some time (at least beginning from 1972) in connection with the problem of quantum percolation.<sup>131-139</sup> This is the problem of a model binary alloy of very different components in which it is assumed that type A atoms form a substitutional alloy with a substance B and that the energy levels of these atoms differ so much that in a study of an energy band formed by levels of atoms of one type we can ignore the existence of atoms of the other type. This model is described by the Hamiltonian

$$\hat{H} = \sum_{i,j} V_{ij}|i\rangle\langle j|, \quad (6.16)$$

which differs from Eq. (6.3) by the absence of diagonal matrix elements. In the case of this problem it is known that near  $x = x_c$ , when an infinite cluster of atoms of type A has already been formed, the states are localized at small aggregates of atoms belonging to this cluster and there is no cluster conductivity.

Such localization of eigenstates in the model in question is not surprising: it describes the limiting case of the usual Anderson localization in an alloy.<sup>33</sup> The localization of eigenstates at the regular fractals, which at first sight seem to be very ordered structures, is somewhat more surprising. The reason is as usual the absence of translational invariance. In spite of the very close analogy of the problems, the present author is not aware of any cross-fertilization from the work on quantum percolation to that on fracton characteristics.

We shall consider in greater detail the nature of localized states in the case of fractal lattices, their behavior on transition to the homogeneous regime, and the characteristic differences between random and regular fractals, which in this case are important.

We shall begin by presenting some results of a quantum percolation model. The most striking distinguishing feature of this problem is the existence of special localized states in the case of an infinite cluster, known as the Kirkpatrick-Eggarter states.<sup>131</sup> Figure 15 shows the simplest configurations that give rise to such states. The black dots are atoms of type B and the positions of type A are labeled by the amplitude of the wave functions of the relevant state.<sup>132</sup> The existence of such states and their energy are completely independent of the configuration of the remaining sites in the system:

they are completely insensitive to the boundary conditions and may coexist with the exponentially (according to Anderson) localized and delocalized states. In the density-of-states spectrum these levels form a fine structure representing  $\delta$ -like peaks.

The localization effect and the spectrum of eigenstates of regular fractals were discussed in Refs. 127 and 128. A triangular Sierpiński curve was considered in Ref. 128 and an additional regular model was discussed in Ref. 127. The main conclusions of these two investigations agree. In both cases it is reported that regular fractals have states similar to the Kirkpatrick-Eggarter states (they are called molecular states in Ref. 128). An example of such a state on the lowest scale of a triangular Sierpiński curve is shown in Fig. 15. Similar states exist also for larger scales. In the case of the Sierpiński curve and its analogs in spaces with larger numbers of dimensions the fraction of such states is  $d/(d+1)$ . The remaining fraction  $1/(d+1)$  represents hierarchical states with the amplitudes different from zero on small and large scales, but with a definite average localization length. The average localization length of all the states, except those corresponding to  $E = 0$ , is finite and the state with  $E = 0$  is the only one which is delocalized. The hierarchical states are fully analogous to those which are Anderson-localized in the quantum percolation model.

The transition to the homogeneous regime was discussed in Ref. 127. A system of Sierpiński curves of the  $n$ th generation (i.e., of the  $a_0 2^n$  scale) forming a regular triangular lattice was considered. As before, the majority of the states was found to be insensitive to the boundary conditions, but there should also be delocalized states representing  $\sim (2/3)^n$  of the total number. It is natural to assume that the narrow bands of these delocalized states form from hierarchical states and very extended Kirkpatrick-Eggarter states.

The fine structure of the density of states of regular fractals is also of interest (Fig. 16). Since all the states are localized, there is no continuous component in the spectrum. Discrete levels are grouped in a sequence of "nests" inside which the structure of the levels is exactly as before: the spectrum has a fractal structure resembling the structure of the density of states in one-dimensional systems (see, for example, Ref. 140). In each of these "nests" the scaling behavior is described by its own exponent. The fracton dimensionality exponent is the quantity associated with the average density of states in a "nest" of levels grouped around  $E = 0$ .

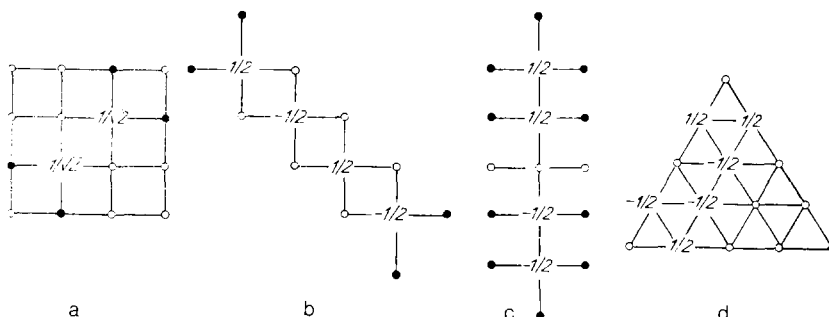


FIG. 15. Simple Kirkpatrick-Eggarter states for a square lattice (a-c) and for the lowest scale of a triangular Sierpiński curve (d).

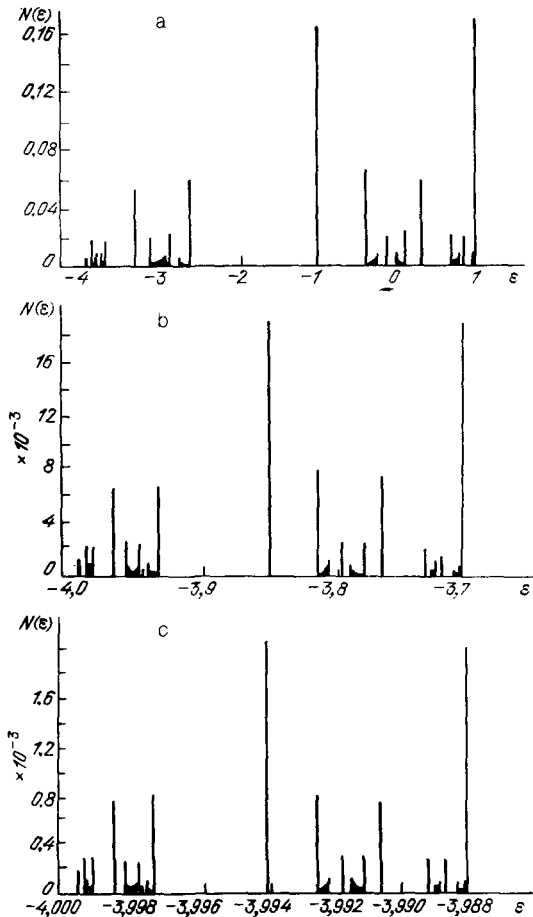


FIG. 16. Spectrum of the density of eigenstates of the Schrödinger equation for a triangular Sierpiński curve<sup>127</sup>: a) density of states; b) fine structure of one of the peaks in Fig. 16a; c) fine structure of one of the peaks in Fig. 16b. Scaling-invariant structure of the peaks can be seen.

There is a significant difference between the models of regular fractals and the quantum percolation model: in the latter case the coordination number (number of sites with which a given lattice site  $Z$  is associated) is a random quantity. The consequences are as follows:

1) in the case of a regular fractal, Eqs. (6.3) and (6.16) differ only in respect of the shift of the energy origin by an amount  $ZV$ , whereas in the disordered case the electron and phonon problems differ by the absence or presence of diagonal disorder and may belong to different classes of universality;

2) there are no dead ends in regular fractal lattices.<sup>127,128</sup> A lattice of triangular Sierpiński curves is essentially a regular triangular lattice with a complex basis, so that it is not surprising that for any value of  $n$  (i.e., no matter how large is the crossover length  $\xi$ ) the system includes delocalized states. However, in the case of a percolation cluster the first delocalization states appear when  $x$  exceeds considerably  $x_c$ ; the correlation length  $\xi$  is then not so large. The reason for this is the existence of dead ends.<sup>136,141</sup> The existence in a regular model of a single delocalized state with

$E = 0$  at  $Z = \text{const}$  and the absence of such a state in the quantum percolation model is clearly also a consequence of the disorder of  $Z$ .

### 6.3. "Superuniversality" of the spectral dimensionality

The fracton dimensionality calculated from the known values of  $t$ ,  $\beta$ , and  $\nu$  is very close to  $4/3$  for any spatial dimensionality  $2 \leq d \leq 6$ . In the case of a Bethe lattice, i.e., in the classical region, the equality  $D_f = 4/3$  is exact. It has been suggested that this equality is exact for any dimensions of space (Alexander-Orbach hypothesis<sup>86,142</sup>), i.e., the fracton dimensionality is superuniversal.

The Alexander-Orbach hypothesis has immediately attracted much attention. Indeed, this hypothesis postulates a universal relationship between the dynamic exponents  $t$  and the static components  $\beta$  and  $\nu$ : it follows from this hypothesis and from Eqs. (4.3), (5.9), and (6.10) that the hyperscaling relationship applies:

$$t = \frac{1}{2} [(3d - 4)\nu - \beta]. \quad (6.17)$$

Substitution into it of the values from Table I shows that it is satisfied quite reasonably and it provides a satisfactory approximate rule.

The question is now: is the relationship between  $t$  [or, better,  $\zeta = (t/\nu) - d + 2$ ] and  $\nu$  and  $\beta$  a general property of the fractal geometry or of only special classes of fractals? If  $d = 2$ , it follows from Eq. (6.17) that  $\zeta = D/2$ . We shall now turn back to Fig. 12. The relevant dependence is represented by the dashed line. We can see that even the slopes of this line and of a curve near which all the points are grouped have different signs. The relevant calculations for infinitely ramified models of regular fractals can be found in Refs. 55 and 143. In the case of a randomly selected model of a regular fractal the Alexander-Orbach relationship is invalid.

The Alexander-Orbach hypothesis immediately met with serious criticism.<sup>144,145</sup> The problem of the classes of fractals to which the superuniversality can be applied at all and whether percolation clusters belong to this class was considered in Refs. 84, 146, and 147. In particular, it was concluded in Ref. 84 that the superlinearity applies to percolation clusters with  $d \geq 3$  and for  $d = 2$  we have  $\zeta = 1$  and not  $D/2$ . All these treatments are based on certain likelihood considerations. However, we shall now consider more rigorous calculations and numerical experiments.

The behavior of the exponent  $\zeta$  was investigated in Ref. 148 using the  $\varepsilon$  expansion. For  $\zeta' = \zeta/\nu$  the expansion was  $\zeta' = 1 + \varepsilon/42 + O(\varepsilon^2)$ , where  $\varepsilon = 6 - d$ , whereas the Alexander-Orbach relationship gives  $\zeta' = 1 + O(\varepsilon^2)$ . Therefore, this relationship is disobeyed even in the first order in respect of  $\varepsilon$ . This conflict (like the results obtained for any finite order of  $\varepsilon$ ) tell us nothing about the extent to which Eq. (6.17) is valid as an approximate relationship.

Several numerical experiments involving modeling of the percolation conductivity with the aim of determining more accurately the ratio  $t/\nu$  and, consequently,  $\zeta$  have been made for the case when  $d = 2$ . Calculations of this kind have to be very accurate and a computer has to be powerful: it is necessary to determine the difference in the third significant

place of the power exponent. A numerical modeling of the conductivity of a random net in the form of a strip of size  $350 \times 10^6$  sites was made in Ref. 149 by the method of finite scaling and it gave the value  $t/\nu = 0.973 \pm 0.005$ , corresponding to  $D_f = 1.332 \pm 0.003$  (this differs by less than 1% from  $4/3$ ). Calculations were carried out using a CDC-205 vector computer. In Ref. 150 it was reported that a direct calculation for a random walk gave  $t/\nu = 0.970 \pm 0.009$ , whereas in Ref. 151 a model with superconducting bonds gave  $S/\nu = 0.977 \pm 0.010$ . In the last investigation the three-dimensional case was also simulated. Thirty hours of the machine time on a CRAY-1 computer were needed to obtain the relevant curve. More detailed discussions of these numerical experiments can be found in Ref. 107.

## 7. CONCLUSIONS

We have discussed the main dimensional properties of percolation clusters. Since the phase transition in percolation models is of purely geometric nature, the critical behavior exponents of various quantities are naturally governed by the geometric characteristics of a system. The properties associated with the self-similarity of clusters are governed by their fractal dimensionality, the maximum velocity of propagation of excitations is governed by the chemical dimensionality, the behavior of the diffusion coefficient, of the conductivity, and of the asymptotic density-of-states spectrum are all determined by the spectral (fracton) dimensionality. The relationships between many of the critical exponents have been established.

The main dimensional characteristics of clusters and the relationships between them are listed in Table III. All the relationships between the dimensionalities with one another and with other exponents (apart from that identified by an asterisk, postulated in Ref. 80 and known to be invalid in the  $d \neq 2$  case) apply to any dimensions in space that do not exceed 6. The numerical values are given for the most thoroughly investigated two-dimensional case. The references to

the relevant work can be found in the present review. Most of these values have been obtained in experiments involving direct numerical modeling. A considerable scatter is encountered only in the relatively poorly investigated quantity known as the hull dimensionality  $D_h$ . The value identified by an asterisk was obtained by modeling the hull of a cluster by a random walk, whereas that identified by a double asterisk was calculated using Eq. (4.6).

A new concept of dimensionality introduced in the late seventies has stimulated studies of the critical behavior of percolation models. These studies have shed much light on the behavior of such systems, have made it possible to formulate new models, and to achieve a better (although not yet final) understanding of many long-investigated properties of the critical behavior of some quantities, such as the conductivity. However, not all the problems have yet been solved.

At present we do not know what is the relationship between the chemical dimensionality  $D_c$  and the ordinary thermodynamic exponents and, in particular, it is not known whether this exponent can be expressed in terms of  $\beta$  and  $\nu$  using two-exponent scaling. The meaning of the hull dimensionality  $D_h$  has not yet been finally established. The problem of the behavior of dimensionalities in the classical range  $d \geq 6$  (which is interesting from the practical point of view in connection with, for example, modeling of complex communication networks) has not yet been finally solved. In the case of the fracton dimensionality the exponent is most probably independent and cannot be expressed in terms of  $\beta$  and  $\nu$ . The question is whether this is indeed so. These questions remain to be answered. Naturally, introduction of new quantities or the use of new models will bring new problems.

As already pointed out, percolation theory provides a satisfactory model of many experimental situations. The number of experimental studies in which any dimensionality apart from fractal has been determined is as yet small. Apparently, their number will increase. From the experimental point of view the very existence of self-similarity is not very

TABLE III. Dimensionality characteristics of percolation clusters (two-dimensional case).

Name of quantity	Designation	Relationship with other exponents	Numerical value
Dimensionality of cluster	$D$	$D = d - \frac{\beta}{\nu} = \frac{\beta + \gamma}{\nu}$	$\frac{91}{48} = 1.896$
Dimensionality of backbone	$D_b$	$D_b = d - \frac{\beta_b}{\nu}$	$1.67 \pm 0.17$
Chemical dimensionality	$D_c$	?	$1.72 \pm 0.02$
Dimensionality of "geodesic"	$D_R$	$D_R = \frac{D}{D_c}$	$1.10 \pm 0.02$
Dimensionality of outer boundary (hull)	$D_h$	$D_h = D(p + \kappa)$	$1.72 - 1.76$
		$D_h = 1 + \frac{1}{\nu}$ *	$1.751 \pm 0.002$ *
			$1.72 \dots **$
Dimensionality of random walk	$D_w$	$D_w = \frac{D_t}{2} = \frac{D}{2 + \theta}$	$0.67$
Internal dimensionality of random walk	$D'_w$	$D'_w = D + \zeta = 2 + \theta$	$2.846 \pm 0.016$
Fracton (spectral) dimensionality	$D_f$	$D_f = \frac{2D}{2 + \theta} = \frac{2D}{D'_w}$	$1.33$



interesting compared with those features of the behavior of the electrical conductivity, diffusion coefficient, elasticity, and their anisotropy, and also of the specific heat which can be exhibited by fractal systems. Real systems with long-range forces and different correlations may prove to be much more complex than the simplest models discussed above. In such cases the dimensional characteristics are not statistical but depend on the controlling parameters and this will make the problems even more interesting. All that we have said about percolation models should apply also to other systems with fractal geometry mentioned in the Introduction.

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