Precursor and lateral waves during pulse reflection from the separation boundary of two media

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CONTENTS

1. INTRODUCTION

Possibly it is somewhat strange that the current literature (both original papers and monographs) contains inaccurate and sometimes simply erroneous slips concerning features generated during reflection of variously shaped pulses from the separation boundary of two media (see, for example, Refs. 1–4). The most popular problem encountered here is that of the "precursor", or the seeming violation of the causality principle in solving the problem of reflection of a narrow pulse of plane sound waves from the separation boundary of two media at an angle larger than that of total reflection. This problem is related to the fact that in treating the problem within the plane wave approximation the reflected wave appears at the observation point somewhat earlier than the arrival time of the reflected planar pulse. The reason for precursor occurrence is the contact existing at all times of the incident pulse with the separation boundary. A correct treatment of the causally stated problem,[5] in which the incident pulse contacts the boundary at a finite moment of time, makes it possible to clarify that the precursor represents the trailing edge of the lateral wave, naturally reaching the observation point after a finite time. The second aspect often mentioned in studies devoted to reflection of sound pulses from the separation boundary of two media is the interpretation of the lateral wave. In most of these studies it is assumed that the lateral wave is related to the quasistatic portion of the reflected disturbance. Incidentally, it is more reasonable to associate this not with the quasistatic, but with the wave disturbance excited by a source moving along the boundary with a speed higher than the speed of sound in the medium. This interpretation of reflected and transmitted waves was expressed by Heaviside and Frank.[6,7] The wave nature of this disturbance is implied, in particular, also by the fact that there exists a region in which the falling-off of the disturbance with distance in the lateral wave is the same as for the reflected wave. We note that this circumstance is not sufficiently clearly discussed in the original studies.[1–3] The next aspect to which we wish to turn attention is the problem of reflection from the boundary of a spherical concentrated disturbance. As follows from the studies devoted to this problem, here also something remains unclear (see, for example, Ref. 8). This is related to the fact that during incidence of a spherical delta-shaped disturbance on the boundary, i.e., a disturbance described by a delta-function of the argument $t-(R/c_1)$ (R is the distance from the source of the initial disturbance to the observation point, and c_1 is the speed of sound), a disturbance distribution can be realized in the reflected field, described by the equation $p\sim[(t-(R/c_1))]^{-1}$. In this case, for spherical disturbances of arbitrary shape the reflected field is described by a corresponding integral in the principal value sense near the singularity $t\to R/c_1$. In this connection it is obvious that to obtain a finite magnitude of a reflected signal the shape of the initial pulse must be described by a continuous function, while for increasing front steepness of this pulse the magnitude of the reflected pulse also increases (for a finite jump of the function describing the shape of the incident pulse the magnitude of the reflected field becomes infinite).[8–10] A similar situation also occurs in the case of incidence of a plane concentrated pulse on the boundary.

In the present study we propose to clarify the problem under consideration specifically taking into account the features mentioned above. In Section 2 we briefly consider incidence of a monochromatic plane wave on a boundary. The necessity of doing that in the present review is dictated by the fact that essentially all features generated during pulse reflection are already manifested in the simple case of reflection of monochromatic waves. The point is that the reflec-

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tion coefficient of a monochromatic wave for incidence angles exceeding the angle of total internal reflection depends on angular frequency even in the absence of dispersion in both media, leading to a shift of the phases of the Fourier harmonics forming the reflected pulse. This is what explains the generation of a reflected distributed disturbance in the case of a concentrated incident pulse.

In Section 3 we consider on the basis of Ref. 5 the problem of reflection of a semi-infinite pulse, and investigate effects due to the possibility of existence of a precursor. The problem of the reflection of a cylindrical disturbance is considered in section 4. This problem was solved in Refs. 11, 12 (see also Refs. 5, 13). To interpret the features of this system it became necessary to determine in a special case the transmitted wave field as well.

Finally, in Section 5 we consider the reflection problem of a spherical pulse by the same method as in the preceding sections. In this case the determination of the reflected field involves more awkward calculations than in the preceding sections. Possibly, the inaccurate and erroneous conclusions in a number of studies are related to a misinterpretation of the studies of Savage and Town,[9,14] where reflection of a spherical pulse from the separation boundary between two liquids was first treated. In this connection we discuss in the present review in detail the case of a spherical pulse by another method, which is simpler and clearer. Another advantage of the treatment utilized is that it is a natural consequence of the method discussed in the preceding sections of the present review for simpler configurations.

2. INCIDENCE OF A MONOCHROMATIC PLANE WAVE ON A BOUNDARY. THE FRESNEL EQUATIONS

In the present review the problem of pulse reflection from the separation boundary between two media is treated using the example of reflection of sound waves from the separation boundary between two liquids, located above and below the plane $z = 0$ and characterized by densities ρ_1, ρ_2 and speeds of sound c_1 and c_2, respectively. We assume that the sound signal is incident from the side of the medium with the lower speed of sound ($c_1 < c_2$), since the features related to total reflection are generated precisely in this situation. Similar effects also occur in the case of reflection of electromagnetic waves from the separation boundary between two dielectrics,[12,15–17] as well as reflection of sound pulses from a liquid-solid and solid-solid separation boundaries.[11,13,18]

As is well-known, reflection of monochromatic plane sound waves from the separation boundary between two liquids is described by the Fresnel equations for the reflection (V) and transmission (W) coefficients. If the pressure p in the incident wave is determined by the real part of the expression $p_0 \exp\{-i\omega[t-(y/c_1)\sin\mu + (z/c_1)\cos\mu]\}$, then

$$-\left(\frac{y}{c_1}\right)\sin\mu + \left(\frac{z}{c_1}\right)\cos\mu\Big]\Big\}, \text{ TO}$$

$$V = \frac{m\cos\mu - \sqrt{n^2-\sin^2\mu}}{m\cos\mu + \sqrt{n^2-\sin^2\mu}},$$

$$W = \frac{2m\cos\mu}{m\cos\mu + \sqrt{n^2-\sin^2\mu}}, \tag{2.1}$$

where $m = \rho_2/\rho_1$, $n = c_1/c_2$, μ is the angle of incidence of the wave at the boundary, and $(n^2 - \sin^2\mu)^{1/2} = i \times (\sin^2\mu - n^2)^{1/2}$ for $\sin\mu > n$. For $\sin\mu > n = \sin\mu_0$ (the incidence angle is larger than the angle of total internal reflection) the reflection coefficient becomes complex, while, obviously, $|V| = 1$. For this case the reflection coefficient is conveniently represented in the form

$$V(\omega) = \exp(i\varphi \operatorname{sign}\omega),$$
$$\cos\varphi = \frac{m^2\cos^2\mu + n^2 - \sin^2\mu}{m^2\cos^2\mu - n^2 + \sin^2\mu}. \tag{2.2}$$

The dependence on frequency ω indicated in Eq. (2.2) follows from the requirement of reality of the expression for the pressure in the reflected wave or from the condition that the pressure tends to zero in the transient wave as the observation point is removed from the boundary toward infinity. The reflection coefficient (2.1), (2.2) has the hermitian property: $V(\omega) = V^*(-\omega)$. Upon wave reflection from the separation boundary between two media there occurs a phase change of the reflected wave in the case $\mu > \mu_0$. If a wave is incident, given by the expression

$$p = p_0\cos\omega[t + \left(\frac{z}{c_1}\right)\cos\mu - \left(\frac{y}{c_1}\right)\sin\mu]$$
$$= p_0\cos\omega\xi, \quad \xi = t + \left(\frac{z}{c_1}\right)\cos\mu - \left(\frac{y}{c_1}\right)\sin\mu,$$

then the pressure in the reflected wave is given by the equation

$$p_{\text{ref}} = p_0\cos(\omega\xi_1 + \varphi), \quad \xi_1 = t - \frac{z}{c_1}\cos\mu - \frac{y}{c_1}\sin\mu.$$

Important for the subsequent description of incidence at the boundary of concentrated disturbances is, as already mentioned, the nonanalytic dependence of V on ω for $\omega \to 0$, following from Eq. (2.2). The latter makes the steepest descent method inapplicable for determining the reflected field in the case of incidence on the boundary of a spherical concentrated pulse (compare with Ref. 2).

3. INCIDENCE ON A BOUNDARY OF A SEMI-INFINITE DELTA-SHAPED PULSE OF SOUND WAVES. THE PRECURSOR

Using Eq. (2.2), one can easily obtain an expression for the reflected field of a narrow pulse of plane waves, which also will contain an expression for the precursor, i.e., the signal reaching the observation point somewhat earlier than the moment of arrival there of the reflected concentrated pulse. The precursor concept appeared in the literature[1,3,19] essentially in this manner, though in the causally stated problem, when contact of the incident pulse with the boundary occurs at a certain finite moment of time $t = t_0$, there exist, naturally, no precursors.

Consider the reflection of a delta-shaped planar pulse ($p = p_0\delta(\xi)$, where $\delta(\xi)$ is the delta-function), incident under an angle larger than the angle of total internal reflection. It is assumed that the incident pulse has constant contact with the boundary during the whole time interval $-\infty < t < \infty$ (the noncausal statement of the problem).[10,20–24] The incident signal is then described by the relation

$$p=\frac{p_0}{2\pi}\int\limits_{-\infty}^{\infty}e^{i\omega\xi}\,d\omega=\frac{p_0}{\pi}\int\limits_{0}^{\infty}\cos\omega\xi\,d\omega, \qquad (3.1)$$

while the reflected field is, according to Eq. (2.2),

$$p_{\mathrm{ref}}=\frac{p_0}{2\pi}\int\limits_{-\infty}^{\infty}e^{i\omega\xi_1}V(\omega)\,d\omega=\frac{p_0}{\pi}\int\limits_{0}^{\infty}\cos(\omega\xi_1+\varphi)\,d\omega$$

$$=p_0\left[\cos\varphi\delta(\xi_1)-\frac{\sin\varphi}{\pi}\frac{P}{\xi_1}\right], \qquad (3.2)$$

where P/ξ_1 defines the principal value of the corresponding integral for $\xi_1\to 0$, i.e.,

$$\int\frac{P}{\xi_1}f(\xi_1)\,d\xi_1=\oint\frac{f(\xi_1)}{\xi_1}\,d\xi_1,$$

and $f(\xi_1)$ is a continuous function (for $\xi_1\to 0$). We note that by multiplying Eq. (3.2) by $\exp(-i\omega_0 t)$ ($\omega_0>0$) and replacing t by $t-t_1$, and then integrating the expression obtained over t_1 within the limits $-\infty<t<\infty$, then, naturally, an equation is obtained for the reflected field, corresponding to the case of incidence of a monochromatic plane wave on the boundary, i.e.,

$$P_{\mathrm{ref}}=P_0\exp\left[-i\omega\left(t-\frac{y}{c_1}\sin\mu-\frac{z}{c_1}\cos\mu\right)+i\varphi\right].$$

The first term in Eq. (3.2), described by the delta-function, refers to the concentrated reflected pulse, having the same shape as the incident pulse. Though there is no dispersion in both media, for $\mu>\mu_0$ the boundary "reprocesses" the incident signal in such a manner that a distributed disturbance appears in the reflected field—a precursor, corresponding to the second term in Eq. (3.2): $Y=-(p_0P/\pi\xi_1)\sin\varphi$, existing in the whole time interval $-\infty<t<\infty$, i.e., "arriving" at the observation point faster than the reflected concentrated signal. Moreover, the magnitude of the disturbance in the precursor can reach higher values for small ξ_1, i.e., for $\mu>\mu_0$ the boundary, in some sense, exerts a focusing action on the incident concentrated pulse.

Of course, if the quantity incident on the boundary is not a delta-shaped pulse, but is distributed in space (and in time), with the shape of this pulse described by a continuous function $f(\xi)$, the reflected field has no singularities. In this case the reflected disturbance is described by some integral expression, in which an expression of type (3.2) appears as a Green's function, and the presence of a singularity for $\xi_1\to 0$ under the integral sign leads to the consequence that the latter must be understood in the principal value sense.[10,20,21] These comments also apply to the case of incidence of a spherical pulse on a boundary, this case being discussed in detail in section 5. It must not be thought that expression (3.2) is deprived of any meaning. On the contrary, as will be clear from the following, it is a consequence of the causal statement of the problem, when the moment of contact of the incident planar pulse with the boundary is $t_0\to-\infty$ (see also Ref. 5). Attempts have been made in a number of studies to correct Eq. (3.2) without invoking a causal statement of the problem. Thus, it is stated in a well-known monograph[4] that an expression of type (3.2) can be utilized in the description of a reflected wave only when $t>(y\sin\mu+z\cos\mu)/c_1$. For an incident planar pulse having contact with the boundary in the whole infinite time interval, a solution is constructed in Ref. 1, differing from that described by Eq. (3.2) and not containing an expression corresponding to the precursor. This solution, however, is obviously of little interest since, as is easily shown, it corresponds to postulating a certain given disturbance (unrelated to the incident pulse propagating in the first medium) in the second medium as well. It must be mentioned that in Refs. 1, 2 it is necessary to specify more precisely the path of integration L_ω in the complex ω-plane for the expression describing the reflected signal, by extending L_ω symmetrically with respect to the $\mathrm{Re}\,\omega=0$ axis into the region $\mathrm{Re}\,\omega<0$ (for more detail see Ref. 5).

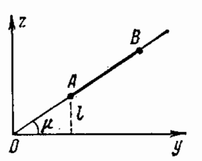

FIG. 1

To explain the meaning of occurrence of signals of the precursor type it is necessary to consider a system in which a semi-infinite pulse of sound waves, propagating in the first medium toward the boundary, contacts the boundary by its edge at some moment of time $t=t_0$ (see Ref. 5).

As in Section 2, let c_1, c_2, ρ_1, ρ_2 be the speeds of sound and densities, respectively, in the upper ($z>0$) and lower ($z<0$) media, with the speed of sound in the lower medium higher than in the upper medium, which is precisely the situation in which the features are generated related to refraction of waves incident at an angle larger than that of total internal reflection. It is assumed that up to the moment of time $t=0$ there is no sound disturbance in the whole space. Further, we stipulate that at $t=0$ the pressure in the upper medium is described by the equation:

$$p=\frac{A}{\pi}\frac{\lambda}{(y-z\,\mathrm{ctg}\,\mu)^2+\lambda^2}\Pi(z-l)\,e^{-\varepsilon z},$$
$$\frac{\partial p}{\partial t}=0,\quad \varepsilon>0,\quad \varepsilon\to 0, \qquad (3.3)$$

where

$$\Pi(z-l)=\begin{cases}1, & z>l,\\ 0, & z<l,\end{cases}$$

y,z are the coordinates of the observation point, μ is the angle between the plane of location of the largest initial disturbances and the boundary (the line AB in Fig. 1), and λ characterizes the width of the initial disturbance (see Eq. (3.14) and the comments to this equation). Below, whenever possible, we put $\varepsilon=0$. It is required to find the pressure field in both media, i.e., to find the solution of the wave equations

$$\frac{\partial^2 p_1}{\partial t^2}-c_1^2\left(\frac{\partial^2 p_1}{\partial y^2}+\frac{\partial^2 p_1}{\partial z^2}\right)=\frac{A}{\pi}\frac{\lambda\delta'(t)\,\Pi(z-l)}{(y-z\,\mathrm{ctg}\,\mu)^2+\lambda^2}e^{-\varepsilon z},$$
$$z>0, \qquad (3.4)$$

$$\frac{\partial^2 p_2}{\partial t^2}-c_2^2\left(\frac{\partial^2 p_2}{\partial y^2}+\frac{\partial^2 p_2}{\partial z^2}\right)=0,$$
$$z<0, \qquad (3.5)$$

satisfying the boundary conditions

$$p_1 = p_2, \tag{3.6}$$

$$m\frac{\partial p_1}{\partial z} = \frac{\partial p_2}{\partial z}; \tag{3.7}$$

where $\delta'(t) = d\delta/dt$ is the derivative of the delta-function with respect to time, and p_1 and p_2 are the wave pressures in the first and second medium. In this case most of the attention is devoted to determining $p_1 = p + p_{\rm ref}$, where p and $p_{\rm ref}$, respectively, are the pressures in the incident and reflected disturbances. We represent the expression for the field of the incident wave in the form of a Fourier integral:

$$p = \frac{A}{(2\pi)^3}\int\int\int \frac{\omega \exp\left[-i\omega t + ik_z(z-l) + ik_y(y - l\,\mathrm{ctg}\,\mu) - \lambda |k_y|\right] d\omega\, dk_y\, dk_z}{(k_z + k_y\,\mathrm{ctg}\,\mu - i\varepsilon)\left[\omega^2 - c_1^2(k_y^2 + k_z^2)\right]}, \tag{3.8}$$

where, as usual, it follows from the condition $p(t<0) = 0$ that in integrating over ω the singular points are bypassed from above ($\mathrm{Im}\,\omega > 0$). In the integration over k_z the pole $k_z = -k_y\,\mathrm{ctg}\,\mu$ is bypassed from below ($\varepsilon \to +0$), therefore at $z<1$ only the pole $k_z = -((\omega^2/c_1^2) - k_y^2)^{1/2}$, $\mathrm{Im}\,k_z < 0$ ($\mathrm{Im}\,\omega > 0$) gives a contribution to (3.8). Taking into account what was said, we obtain

$$p = -\frac{iA}{8\pi^2 c_1^2}\int\int \frac{\omega \exp\left[-i\omega t + i\sqrt{(\omega^2/c_1^2) - k_y^2}\,(l-z) + ik_y(y - l\,\mathrm{ctg}\,\mu) - \lambda|k_y|\right]}{\left[k_y\,\mathrm{ctg}\,\mu - \sqrt{(\omega^2/c_1^2) - k_y^2}\right]\sqrt{(\omega^2/c_1^2) - k_y^2}} \times d\omega\, dk_y. \tag{3.9}$$

We seek a reflected wave field in the form

$$p_{\rm ref} = \int\int F(\omega, k_y) \exp\left(-i\omega t + i\sqrt{\frac{\omega^2}{c_1^2} - k_y^2}\,z + ik_y y\right) d\omega\, dk_y,\quad z > 0, \tag{3.10}$$

and a transmitted one—in the form

$$p_{\rm tr} = \int\int G(\omega, k_y) \exp\left(-i\omega t - i\sqrt{\frac{\omega^2}{c_2^2} - k_y^2}\,z + ik_y y\right) d\omega\, dk_y,\quad z < 0. \tag{3.11}$$

Substituting then these expressions into the boundary conditions (3.6), (3.7), we obtain the following reflected and transmitted waves

$$p_{\rm ref} = -\frac{iA}{8\pi^2 c_1^2}\int\int \left(m\sqrt{\frac{\omega^2}{c_1^2} - k_y^2} - \sqrt{\frac{\omega^2}{c_2^2} - k_y^2}\right)\omega \exp\left(-i\omega t + i\sqrt{\frac{\omega^2}{c_1^2} - k_y^2}\,z' + ik_y y' - \lambda|k_y|\right)$$

$$\times\left[\left(m\sqrt{\frac{\omega^2}{c_1^2} - k_y^2} + \sqrt{\frac{\omega^2}{c_2^2} - k_y^2}\right)\sqrt{\frac{\omega^2}{c_1^2} - k_y^2}\left(k_y\,\mathrm{ctg}\,\mu - \sqrt{\frac{\omega^2}{c_1^2} - k_y^2}\right)\right]^{-1} d\omega\, dk_y, \tag{3.12}$$

$$p_{\rm tr} = -\frac{iAm}{4\pi^2 c_1^2}\int\int \frac{\omega \exp\left[-i\omega t + i\sqrt{(\omega^2/c_1^2) - k_y^2}\,l - i\sqrt{(\omega^2/c_2^2) - k_y^2}\,z + ik_y y' - \lambda|k_y|\right]}{\left(m\sqrt{(\omega^2/c_1^2) - k_y^2} + \sqrt{(\omega^2/c_2^2) - k_y^2}\right)\left(k_y\,\mathrm{ctg}\,\mu - \sqrt{(\omega^2/c_1^2) - k_y^2}\right)}\, d\omega\, dk_y,$$

$$z' = z + l,\quad y' = y - l\,\mathrm{ctg}\,\mu.$$

We carry out the following variable replacement in Eq. (3.12):

$$\omega = c_1 k_y' x,\quad k_y = k_y'. \tag{3.13}$$

It is seen from (3.13 that $\mathrm{Im}\,x > 0$ for $k_y > 0$, and $\mathrm{Im}\,x < 0$ for $k_y < 0$ (we recall that $\mathrm{Im}\,\omega > 0$). We further note that for convergence of the integrals (3.12) over k_y it is necessary to impose the following condition on the value of the imaginary part of x for $c_1 t > z + l$ ($x = x_1 + ix_2$):

$$\begin{array}{ll} 0 < x_2 < \frac{\lambda}{\beta}, & k_y > 0, \\ -\frac{\lambda}{\beta} < x_2 < 0, & k_y < 0, \end{array}\quad \beta = c_1 t - \frac{z' x_1}{\sqrt{x_1^2 - 1}} > 0. \tag{3.14}$$

For $\beta < 0$ x_2 can be selected arbitrarily. Conditions (3.14) indicate that, unlike the L_ω integration path, which can be selected arbitrarily in the ω-plane, with the only restriction of bypassing the singular points of the integrand of expression (3.12) from above, the integration path L_x is squeezed (for $\lambda \to 0$) near the real x_1 axis. It is important to note that it is specifically the introduction of the finite source width λ into the initial disturbance (3.3) which renders conditions (3.14) compatible, and consequently also the variable replacement (3.13). If one puts $\lambda = 0$ at the outset, then following the replacement (3.13) the integrals over k_y will diverge.

Carrying out the integration over k_y, we obtain the relation

$$p_{\rm ref} = -\frac{A}{8\pi^2}\int\limits_{L_x} \frac{xV(x)\,dx}{\sqrt{x^2-1}\,(\mathrm{ctg}\,\mu - \sqrt{x^2-1})\,(c_1 t x - \sqrt{x^2-1}\,z' - y' - i\lambda)} + \text{c.c.}; \tag{3.15}$$

where

$$V(x)=\frac{m\sqrt{x^2-1}-\sqrt{n^2x^2-1}}{m\sqrt{x^2-1}+\sqrt{n^2x^2-1}},$$

and the integration contour L_x lies above the real axis

$$[-\infty+ix_2, \infty+ix_2],\quad 0<x_2<\lambda/\beta,\quad n=c_1/c_2,$$

$$\lim_{x\to\infty}\sqrt{x^2-1}=x,\quad \lim_{x\to\infty}\sqrt{n^2x^2-1}$$

In the integral (3.15) it is convenient to make one more replacement:

$$\theta=\frac{\sqrt{x^2-1}}{x},\quad x=\frac{1}{\sqrt{1-\theta^2}},\quad \sqrt{1-\theta^2}\underset{\theta\to\infty}{=}-i\theta. \tag{3.16}$$

The pressure in the reflected wave is then determined by the expression

$$p_{\text{ref}}=-\frac{A}{8\pi^2}\oint_{L_\theta}\frac{V(\theta)\,d\theta}{\sqrt{1-\theta^2}\,(\sqrt{1-\theta^2}\operatorname{ctg}\mu-\theta)\,(c_1t-\theta z'-(y'+i\lambda)\sqrt{1-\theta^2})}+\text{c.c.}; \tag{3.17}$$

where

$$V(\theta)=\frac{m\theta-\sqrt{\theta^2-\alpha^2}}{m\theta+\sqrt{\theta^2-\alpha^2}},$$

$$0<\alpha^2\equiv 1-n^2<1,\quad \lim_{\theta\to\infty}\sqrt{\theta^2-\alpha^2}=\theta. \tag{3.18}$$

The L_θ contour consists of a loop encompassing the point $\theta=1$, segments along the imaginary axis, and a semicircle of infinitely large radius (see Fig. 2). Relationship (3.17) can be interpreted as a plane wave expansion of the reflected signal. The variable of integration θ is the cosine of the angle of incidence of the plane wave, while real θ values correspond to homogeneous plane waves, and complex values—to inhomogeneous ones.

In Eq. (3.17) $\theta_{1,2,3,4}=\pm 1, \pm\alpha$ are branching points. We make a cut from $\theta=-1$ to $\theta=1$ below the real axis; $\theta_5=\cos\mu$ is then a pole. But if we make a cut joining the points $\theta=\pm 1$ above the real axis (see Eq. (3.17)), then $\theta_6=\cos\mu$ will be a pole. The contribution to (3.17) is provided only by the poles determined by the equation

$$c_1t-\theta z=(y+i\lambda)\sqrt{1-\theta^2}, \tag{3.19}$$

since only they are located inside the L_θ contour.

As shown by detailed analysis, the roots of Eq. (3.19), located on the selected sheet of the Riemann surface (see Eq. (3.18)), are

$$\theta_7=\frac{c_1tz'+(y'+i\lambda)\sqrt{R^2-c_1^2t^2}}{R^2},\quad y'>0,\quad t>\frac{z'}{c'}, \tag{3.20}$$

$$\theta_7=\frac{c_1tz'-(y'+i\lambda)\sqrt{R^2-c_1^2t^2}}{R^2},\quad y'<0,\quad t>\frac{z'}{c_1}; \tag{3.21}$$

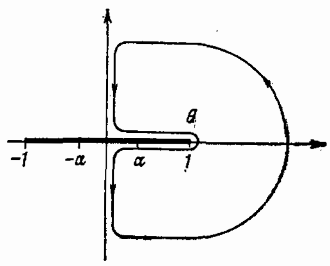

FIG. 2. ($\theta=\theta_7$)

where $R^2=(y+i\lambda)^2+z'^2$, and the roots must be understood in the following sense: when $R_0^2\equiv y'^2+z'^2>c_1^2t^2$ the real parts of the roots in (3.20), (3.21) are positive, while when $R_0^2<c_1^2t^2$, then, both when $y'>0$ and $y'<0$, the solution of Eqs. (3.19) is represented in the form

$$\theta_7=\frac{c_1tz'+i\sqrt{c_1^2t^2-R^2}\,(y'+i\lambda)}{R^2},\quad \operatorname{Re}\sqrt{c_1^2t^2-R^2}>0. \tag{3.22}$$

It also follows from the analysis of Eq. (3.19) that for $c_1t<z'$ the poles lie on another sheet of the Riemann surface, while there are no singular points of the integrand expression in Eq. (3.17) inside the L_θ contour, and (3.17) vanishes. We further mention that the poles determined by (3.20), (3.21) are located inside the L_θ contour on different sides of the real Re θ axis (for $y'>0$ Im $\theta_7>0$). Further calculating the integral (3.17), we obtain the final answer

$$p_{\text{ref}}=-\frac{iA}{4\pi}\,\frac{V(\theta_7)}{(\sqrt{1-\theta_7^2}\operatorname{ctg}\mu-\theta_7)\,\Omega}+\text{c.c.},\quad z>0, \tag{3.23}$$

where

$$\Omega=\begin{cases}\sqrt{R^2-c_1^2t^2}\,\operatorname{sign} y, & R_0>c_1t,\\ i\sqrt{c_1^2t^2-R^2}, & R_0<c_1t,\end{cases}$$

and θ_7 is determined from expressions (3.20)–(3.22).

We also present here the equation for the pressure in the incident wave, obtained similarly to Eq. (3.23)

$$p=-\frac{iA}{4\pi}\,\frac{1}{(\sqrt{1-\theta_7^2}\operatorname{ctg}\mu-\theta_7)\,\Omega}+\text{c.c.},\quad z<l, \tag{3.24}$$

where in comparison with (3.23) one must make the following replacement in the expressions for θ_7 and Ω: z' is replaced by $z_1=l-z$. Thus, an exact expression for the pressure in the reflected wave has been obtained in terms of elementary functions.

We analyze Eq. (3.23) in the simple limiting case $l=0$, i.e., $t_0=0$. Let $y>0$, and let the observation point be near the boundary, i.e., $z\to 0$; the expression for the reflected wave field is then simplified:

$$p_{\text{ref}}=-\frac{iA}{4\pi}\,\frac{V(\theta_7)}{(\sqrt{1-\theta_7^2}\operatorname{ctg}\mu-\theta_7)\sqrt{(y+i\lambda)^2-c_1^2t^2}}+\text{c.c.},$$

$$\theta_7=\sqrt{1-\frac{c_1^2t^2}{(y+i\lambda)^2}}. \tag{3.25}$$

For $t \to y/c_1$ Eq. (3.25) contains a singularity of type $(y - c_1^2t^2)^{-1/2}$, corresponding to reflection of a cylindrical wave excited by the edge of the initial disturbance. Until the moment of time $\tau = \min\{t_1, t_2\}$, $t_1 = y/c_1 \sin\mu$, $t_2 = y/c_2$ the field of the reflected wave is $\sim\lambda$, and for $\lambda \to 0$ it is vanishingly small (in this case Eq. (3.25) is approximately represented for $\lambda \to 0$ by the sum of two complex conjugate imaginary quantities). The reflected wave field appears at $t > \tau$. Here it is necessary to distinguish two cases: a) $t_1 < t_2$, $\sin\mu < c_1/c_2$ (the angle of incidence is smaller than the angle of total internal reflection), and b) $\sin\mu > c_1/c_2$ (the angle of incidence is larger than the angle of total internal reflection).

Consider initially the case in which $\sin\mu < c_1/c_2$. Expression (3.25) has a singularity for $t \to t_1$, $\lambda \to 0$. To isolate this singularity it is necessary to put $\lambda = 0$, $\theta_7 = \cos\mu$, in the expressions in (3.25) not having singularities for $t \to t_1$, $\lambda \to 0$, and the relation $c_1 t \operatorname{ctg}\mu - ((y + i\lambda)^2 - c_1^2t^2)^{1/2}$ must be expanded in powers of λ. Omitting the simple computations, by means of the equation

$$\delta(x) = \frac{1}{\pi} \lim_{\lambda \to 0} \frac{\lambda}{\lambda^2 + x^2},$$

we obtain

$$p_{\text{ref}} = \frac{1}{2} AV(\cos\mu)\, \delta\left(y - \frac{c_1 t}{\sin\mu}\right). \tag{3.26}$$

We turn now to describe the features of the case in which the incidence angle is larger than the angle of total internal reflection, i.e., $\sin\mu > c_1/c_2 = n$ $(t_1 > t_2)$. For $t \to t_2$ $\operatorname{Im} V(\theta) \sim \lambda$, and, since in this case the field is $\sim iV(\theta)$ + c.c., the disturbance in this time interval is small ($p_{\text{ref}} \sim \lambda$). For $t > t_2$ the reflection coefficient is complex, and the reflected wave field becomes nonvanishing, though the concentrated disturbance, located on the line $y = c_1 t/\sin\mu$, moving along the boundary with velocity $c_1/\sin\mu$, does not approach the point under consideration. This disturbance is partially due to the fact that a cylindrical wave propagates from the edge of the planar initial disturbance, with a lateral wave excited at its contact with the boundary, with the moment of its arrival at the given boundary point characterized by the equality $t = t_2$. Besides the mechanism mentioned, a contribution to the lateral wave is provided by the distributed disturbance, excited in the second medium ($z < 0$), concentrated on the line $y = c_1 t/\sin\mu$, $t > t_0 = 0$, $z = 0$. The expression for the lateral wave field is easily found from Eq. (3.25):

$$p_l = -\frac{A}{\pi y} \frac{m\sqrt{\sin^2\chi - n^2}}{(m^2\cos^2\chi + \sin^2\chi - n^2)(\sin\chi \operatorname{ctg}\mu - \cos\chi)}, \tag{3.27}$$
$$\sin\chi = \frac{c_1 t}{y}, \qquad t_2 < t < t_1.$$

For $|t - t_1| \ll t_1$, $y \gg \lambda \to 0$ we restrict ourselves, wherever necessary, to quantities of order $(t - t_1)/t_1$, λ/y only, and find after several transformations

$$p_{\text{ref}} = \frac{A}{2\pi} \frac{1}{m^2\cos\mu + \sin^2\mu - n^2} \left[\frac{(m^2\cos^2\mu - \sin^2\mu + n^2)\lambda}{\lambda^2 + a^2} - \frac{2m\cos^2\mu\sqrt{\sin^2\mu - n^2}\, a}{\lambda^2 + a^2}\right], \quad \lambda \to 0, \quad t > t_1, \tag{3.28}$$

where $a = (t - t_1)c_1/\sin\mu$, $\sin\mu > n$.

For $t \to \infty$, $y \to \infty$ the region in which Eq. (3.28) is valid becomes arbitrarily large. Comparing (3.28) for the indicated t, y $(t \to \infty, y \to \infty)$ with expression (3.2), obtained by using the noncausal statement of the problem, we see that these equations coincide. This noted fact justifies the statement that the precursor obtained in the noncausal statement of the problem, when contact of the planar concentrated disturbance with the boundary is constant during the whole time interval $-\infty < t < \infty$, is fully understood and corresponds to the asymptotic solution (for $t \to \infty$, $y \to \infty$) of the causal problem, when contact of the incident disturbance with the boundary occurs at the finite moment of time $t = t_0$. Thus, the precursor represents a trailing front of the lateral wave (3.28), excited during contact of the incident radiation with the separation boundary.

It follows from Eq. (3.28) that the reflected wave field is represented in the form of two parts: a concentrated disturbance, described by a delta-function, and a distributed disturbance, described by the second term in (3.28). A similar situation also occurs in reflection of a spherical pulse.[8,9,14] Using Eq. (3.28) as a Green's function for an initial pressure distribution of arbitrary shape, it is obvious that this distribution must be described by continuous functions of coordinates (since integrals of expressions of type (3.28) must be understood as integrals in the principal value sense for $a \to 0$). This implies that for initial fields for which the front steepness increases the reflected wave will increase logarithmically in the corresponding region.[8-10,20,21]

We turn to a brief description of the fields in the region $y < 0$. In this region a lateral wave, existing at $c_1 t < |y| < c_2 t$, is the first to arrive at the observation point, followed by a cylindrical one ($c_1 t > |y|$), and there is no plane wave in this case. This lateral wave is due to the same factors as the lateral wave existing in the region $y > 0$. We present the expression for the lateral wave

$$p_l = \frac{A}{\pi} \frac{m\cos\chi\sqrt{\sin^2\chi - n^2}}{(\cos\chi \operatorname{ctg}\mu + \sin\chi)(m^2\cos^2\chi + \sin^2\chi - n^2)\sqrt{y^2 - c_1^2t^2}},$$
$$y < 0, \quad \sin\chi = \frac{c_1 t}{|y|}, \quad \frac{|y|}{c_2} < t < \frac{|y|}{c_1}. \tag{3.29}$$

The location of the disturbances in space is illustrated by Fig. 3 ($\lambda \to 0$). The straight lines 1, 2 determine, respectively, the positions of incident and reflected concentrated planar disturbances. The semicircle 5 determines the boundary of the cylindrical disturbance due to the edge of the planar initial distribution. The straight lines 3 and 4 represent the boundary of the region of existence of the lateral wave.

In connection with Fig. 3 we note the following. The disturbances corresponding to the reflected and transmitted

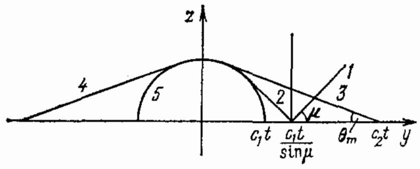

FIG. 3

fields can, naturally, be obtained by considering the radiation from sources induced by the incident wave at the boundary. The expressions for these sources are easily provided, noting that the incident wave is given only in the region $z>0$, i.e., $p_{\rm inc} = \Pi(z)\,p(t,y,z)$, where $p(t,z,y)$ is determined by (3.24). In that case $p_{\rm ref}$ and $p_{\rm tr}$ satisfy the equations

$$\frac{\partial^2 p_{\rm ref}}{\partial t^2} - c_1^2 \Delta p_{\rm ref} = c_1^2\left[-\delta'(z)\,p_0(t,\,y) - 2\delta(z)\,p'_{0z}(t,\,y)\right] = J, \qquad z>0, \tag{3.30}$$

$$\frac{\partial^2 p_{\rm tr}}{\partial t^2} - c_2^2 \Delta p_{\rm tr} = J, \qquad z<0, \tag{3.31}$$

$$p_0 \equiv p(t,\,y,\,0), \quad p'_{0z} = \frac{\partial p}{\partial z}(t,\,z,\,y)\,|_{z=0},$$

with boundary conditions (3.6), (3.7), while, naturally,

$$p_{\rm ref}(t<t_0,\,y,\,z) = p_{\rm tr}(t<t_0,\,y,\,z) = 0, \quad t_0 = \frac{l}{c_1}.$$

As shown by analysis similar to the preceding, J consists of two different parts. The first part corresponds to a source concentrated on the line $y = c_1 t/\sin\mu$, whose value in the comoving coordinate system is constant. Such a source is generated by a plane incident disturbance crossing the separation boundary (see Fig. 3). The second source is distributed at the boundary, and is localized within the strip $|y| < c_1 t$ (it is assumed here that $l=0$ and that the moment of contact of the cylindrical disturbance with the boundary is $t_0 = 0$). This source corresponds to a distributed cylindrical pulse, propagating from the edge of the plane disturbance, crossing the separation boundary (as is well known, a cylindrical signal is distributed in space also in the case when the initial disturbance is localized on a straight line). Using the sources introduced above, one easily explains several features of the system under consideration. Firstly, if $c_1/\sin\mu < c_2$ (the angle of incidence is larger than the angle of total internal reflection), then a source concentrated on the line $y = c_1 t/\sin\mu$ provides Mach radiation only in the first medium (a reflected concentrated pulse with an angle of reflection equal to the angle of incidence), while there is no transmitted concentrated pulse in this case (the condition of supersonic source motion is not satisfied in this case). The distributed source due to a cylindrical divergence is time-dependent, and therefore provides radiation in both media, with this radiation in the second medium near the boundary being localized in the strip $|y| < c_2 t\,(l=0)$. Thus, the disturbance moves along the boundary of the first medium with a velocity c_2 larger than c_1, which is what leads to the appearance of a reflected signal propagating with a speed higher than the sound speed in the first medium. As could be expected, the boundaries of the localization region of this disturbance are inclined to the $z=0$ plane by the Mach angle $\sin\theta_m = c_1/c_2$ (see Fig. 3). In view of what was said the nature also becomes obvious of appearance of a signal of a precursor type in the system considered, a signal which does not decrease for $t\to\infty$, $y\to\infty$. Since the source concentrated at $y = c_1 t/\sin\mu$ moves along the boundary with a velocity smaller than the speed of sound in the second medium, a quasistatic disturbance from this source is formed in it for $t\to\infty$, $y\to\infty$, moving along with the source with velocity $c_1/\sin\mu$.[1)] As already noted, this signal is independent of time in the comoving coordinate system ($y' = y - c_1 t'\sin\mu$) also near the boundary $p \sim 1/y'$. In turn, the indicated disturbance, moving along the boundary with a velocity $c_1/\sin\mu$ larger than the speed of sound in the first medium provides to it a distributed Mach radiation (due to the distribution property of $p\sim 1/y'$) at an angle equal to the reflection angle. It must be noted that for a planar source of shape $J = J(y - (c_1 t/\sin\mu))$, moving along this plane in a direction perpendicular to the y-axis, Mach radiations from separate elementary bands Δy, composing the source, do not interfere with themselves. In this case the value of the Mach radiation is directly derived from that of the source (of course, an essential aspect here is the fact that the source depends only on the combination $y - c_1 t/\sin\mu$). It also remains to clarify specifically with what is the precursor shape mentioned above for a concentrated incident disturbance (see Eqs. (3.2), (3.28)) connected. For this we find the induced solution $p_{\rm i.tr}$ of Eq. (3.31), corresponding to the field excited by the source formed by the intersection of the incident pulse with the boundary. For the precursor it is sufficient to take into account only an incident planar pulse, i.e.,

$$p_0 = A\,\frac{\lambda_0}{\lambda_0^2+\xi_0^2}, \quad \xi_0 = t - \frac{y}{v}, \quad v = \frac{c_1}{\sin\mu},$$

$$p'_{0z} = -\frac{2A}{v_1}\,\frac{\lambda_0\xi_0}{(\lambda_0^2+\xi_0^2)^2}, \quad v_1 = \frac{c_1}{\cos\mu}, \quad \lambda_0 \to 0.$$

One can then seek a solution of Eq. (3.31) in the form $p_{\rm i.tr}(z,\xi_0)$, i.e., this solution in a coordinate system moving with velocity v is a steady-state one (independent of time), and

$$p_{\rm i.tr} = \int J_{k_z,\,k_\xi} \exp(ik_z z + ik_\xi \xi) \times\left[k_z^2 c_2^2 + k_\xi^2\left(\frac{c_2^2}{v^2} - 1\right)\right]^{-1} dk_z\, dk_\xi,$$

where

$$J_{k_z,\,k_\xi} = \frac{iAc_1^2}{4\pi}\begin{cases}\left(k_z + \dfrac{k_\xi}{v_1}\right)e^{-k_\xi\lambda_0}, & k_\xi>0,\\[2mm] \left(k_z + \dfrac{k_\xi}{v_1}\right)e^{k_\xi\lambda_0}, & k_\xi<0.\end{cases}$$

After simple calculations we find

$$p_{\rm i.tr} = \frac{Ac_1^2}{2\pi c_2^2}\,\frac{(-\lambda_0+\sqrt{\gamma}\,z)+(2\xi_0/\sqrt{\gamma}\,v_1)}{(-\lambda_0+\sqrt{\gamma}\,z)^2+\xi_0^2}, \tag{3.32}$$

$$\gamma \equiv v^{-2} - c_2^{-2}, \quad z<0.$$

A disturbance of this shape, moving as a whole along the boundary with velocity $c_1/\sin\mu > c_1$, provides Mach radiation in to the first medium, with the shape of this radiation, described by Eq. (3.28), reproducing the boundary value $p_{\rm i.tr}$ without distortion. Equation (3.32) must be supplemented by the solution of the homogeneous equation (3.31), of the same shape as (3.32). Furthermore, using boundary conditions (3.6) and (3.9), one can find $p_{\rm tr}$.

A detailed analysis of the transmitted pulse for various shapes of the incident signal was performed in Ref. 25. In this study was noted the fruitfulness of representing the transmitted waves in the form of radiation fields of a source moving along the separation boundary. Indeed, the asymptotic expression for the transmitted wave field for angles larger than the angle of total internal reflection is proportional to $(A_1/r) + (B/r_2)$, where r is the distance from the source to the observation point,

$$A_1 \sim \int_{-\infty}^{\infty} p_{\rm inc}\,dt, \quad B \sim \int_{-\infty}^{\infty} t p_{\rm inc}\,dt.$$

Such expansions correspond to a representation of source fields in the form of multipole series, with the role of the charge being played by the total momentum of the incident wave. The expansion written above corresponds to quasistatic fields of a filamentary source, which, without altering its shape, moves with a velocity $c_1/\sin\mu$, smaller than the speed of sound in the second medium.

For a bell-shaped form of an incident planar disturbance the problem was solved in Refs. 3, 28. As follows from Eq. (3.32), at $z=0$ the pressure field consists of two parts: a concentrated disturbance, described by $\delta(\xi_0)$, and a distributed disturbance, related to the expression P/ξ_0. By comparing Eqs. (3.2) and (3.28) with (3.32) it is easily seen that the precursor shape totally repeats the shape of the steady-state distributed disturbance, moving along the boundary with constant velocity $c_1/\sin\mu$.

We turn now to describe the localization regions of lateral waves in the system under consideration. As already noted, in the case of initial conditions (3.3) there exist two incident waves (planar and cylindrical from the edge of the planar disturbance). Therefore there exist also two types of lateral waves. In the region restricted by the planes $z = -(y - c_2t)\,\mathrm{tg}\,\theta_m$, $z = -[y - (c_1t/\sin\mu)]\,\mathrm{tg}\,\mu$ (lines 2, 3 in Fig. 3) there exist lateral waves from both sources: a disturbance not decreasing for $t\to\infty$, $y\to\infty$, $y - (c_1t/\sin\mu) = \mathrm{const}$, related to a source localized at $y = c_1t/\sin\mu, z=0$ (see Eqs. (3.2), (3.28)), and a cylindrical disturbance decreasing as y^{-1} for $t\to\infty$, $y\to\infty$, formed by the cylindrical incident wave (the source is localized at $|y|<c_1t$, $z=0$). In the regions bounded by the planes $z = -[y - (c_1t/\sin\mu)]\,\mathrm{tg}\,\mu$, $z = (y + c_2t)\,\mathrm{tg}\,\theta_m$ and by the cylindrical surface $y^2 + z^2 = c_1{}^2t^2$ there exists a cylindrical lateral wave. In the regions bounded by the planes 2, 4 and the cylindrical surface 5 (see Fig. 3), along with the lateral cylindrical wave there exists a disturbance from a concentrated source localized at $y = c_1t/\sin\mu$.

We turn now briefly to the case in which the angle of incidence is smaller than the angle of total internal reflection. If $c_2 < c_1/\sin\mu$, then the velocity of the concentrated source $v = c_1/\sin\mu$, moving along the boundary, is larger than the speed of sound in the second medium, and a quasistatic disturbance which would move along with the source cannot be formed. At the same time the possibility appears to Mach radiation propagated at the refraction angle (first noted by Heaviside[6]). In view of what was said, the reflected field retains the nature of the concentrated pulse, repeating the shape of the incident signal. Similarly to (3.28), for $t\to\infty, y\to\infty$ we have

$$p = A\delta\left(t - \frac{y\sin\mu - z\cos\mu}{c_1}\right),$$

$$p_{\rm ref} = AV\delta\left(t - \frac{y\sin\mu + z\cos\mu}{c_1}\right), \tag{3.33}$$

and then we find from the boundary condition (3.6)

$$p_{\rm tr} = AW\delta\left(t - \frac{y\sin\mu_1 - z\cos\mu_1}{c_2}\right), \tag{3.34}$$

where $\sin\mu_1 = c_2\sin\mu/c_1$, and V and W are defined by Eq. (2.1). Naturally, Eqs. (3.33), (3.34) are easily found directly from Eqs. (3.4), (3.5) without the right hand sides and the boundary conditions (3.6), (3.7). For $c_2 < c_1/\sin\mu$ (see Fig. 3) there exists a lateral wave related to the cylindrical disturbance in the region bounded by the plane $z = (y - c_2t)\mathrm{tg}\,\theta_m$ and the cylindrical surface $x^2 + y^2 = c_1{}^2t^2$ $(c_1 < c_2)$, while, unlike the preceding case $(c_2 > c_1/\sin\mu)$, in this region there exist no disturbances unrelated to the lateral wave. It is interesting to note that the lateral wave reaches the observation point later than the plane wave (the point c_2t is located to the left of the point $c_1t/\sin\mu$). It is precisely due to this, therefore, that no seeming violation of the causality principle is generated within the plane wave approximation.

4. INCIDENCE OF A CYLINDRICAL DISTURBANCE ON A BOUNDARY[5,11,13]

As is well known, in considering the problem of incidence of a spherical pulse on the separation boundary between two nondispersive media there are certain difficulties in interpreting the results obtained, related, apparently, to the relative complexity of the expressions for the reflected field.[9,14] At the same time, in the case of a cylindrical disturbance the expression describing the pressure in the reflected field contains only elementary functions, and all features of the process (similar in many cases to a spherical pulse) can be easily explained. Naturally, the reflection problem of a cylindrical disturbance is also of intrinsic interest, and contains some features not present in the case of a spherical pulse.

We assume that the source, being an infinitely long filament, is parallel to the boundary, located at a distance l from it in the first medium $(z>0)$ (see preceding Section). To find the sound field one must start from the wave equations (3.4), (3.5) and the boundary conditions (3.6), (3.7) with $p_i(t<0) = 0, i = 1, 2$, replacing the right hand side of (3.4) by a source corresponding to a cylindrical disturbance of the shape

$$J = \frac{A}{\pi}\frac{\lambda}{\lambda^2 + y^2}\delta(z - l)\,\delta(t). \tag{4.1}$$

Comparing Eq. (4.1) with the right hand side of (3.4), we see that the solution of Eqs. (3.4), (3.5) with a source (4.1) can be obtained directly from the results of the preceding section, if in the corresponding equations $\mu = \pi/2$, $\mathrm{ctg}\,\mu = 0$ and multiplies the integrand expressions for $p, p_{\rm ref}$, and $p_{\rm tr}$ by $-k_z/\omega = -\theta/c_1$. Thus, for example, from (3.20) – (3.24) we have for the pressure in the reflected wave

$$p_{\text{ref}} = -\frac{iA}{4\pi c_1}\frac{V(\theta_7)}{\sqrt{R^2-c_1^2t^2}} + \text{c.c.}, \quad y>0, \quad t>0, \tag{4.2}$$

where θ_7 is determined primarily from Eqs. (3.20)–(3.22) with the replacement of y' by y, and $R^2 = (y+i\lambda)^2 + z'^2$. Putting in (4.2) $\lambda = 0$, a simple expression is obtained

$$p_{\text{ref}} = -\frac{iA}{4\pi c_1}\frac{V(\theta_0)}{\sqrt{R_0^2-c_1^2t^2}} + \text{c.c.}, \quad y>0, \tag{4.3}$$

$$\theta_0 = \frac{c_1tz' + y\sqrt{R_0^2-c_1^2t^2}}{R_0^2}, \quad R_0^2 = z'^2 + y^2, \quad t>0.$$

It is seen from Eq. (4.3) that the reflected signal exists at some point of space only when the expression for it, $V(\theta_0)/(R_0{}^2 - c_1^2t^2)^{1/2}$, is complex. This is possible for two reasons: either $\theta_0 < \alpha$, i.e., the angle of incidence is larger than the angle of total internal reflection and the coefficient $V(\theta_0)$ is complex (the region of existence of the lateral wave for $R_0 > c_1t$), or for $t > R_0/c_1$ (the region of existence of a reflected cylindrical wave). The moment of arrival of the lateral wave at a given point of space t_0 is determined by the condition $\theta_0 = \alpha$, giving

$$t_l = \frac{z' + y\,\mathrm{tg}\,\theta_m}{c_1}\cos\theta_m, \quad y>0, \tag{4.4}$$

$$\sin\theta_m = \frac{c_1}{c_2} = n.$$

Equation (4.4) corresponds to the Mach condition for a source concentrated in the plane $z = 0$, $|y| < c_1t$, whose boundaries move with velocity $c_2 > c_1$. Since the reflected cylindrical wave reaches the given point of space at the moment of time $t = R_0/c_1$, then, as seen from Eq. (4.4), the lateral wave reaches the observation point earlier than the reflected cylindrical wave. The expression for the pressure in the lateral wave is

$$p_{\text{ref}.l} = -\frac{Am}{\pi c_1}\frac{\theta_0\sqrt{\alpha^2-\theta_0^2}}{\sqrt{R_0^2-c_1^2t^2}\,[\theta_0^2(m^2-1)+\alpha^2]}, \tag{4.5}$$

$$\theta_0 < \alpha, \quad y>0, \quad R_0/c_1 > t > t_6.$$

It is interesting to note that for $R_0 \gg c_1t$ $p_{\text{ref}.l}$ is independent of time. Thus, for $y \gg z'$

$$p_{\text{ref}.l} \approx -\frac{Am}{\pi c_1R_0^2}\frac{y\sqrt{\alpha^2-y^2/R_0^2}}{\alpha^2+[y^2(m^2-1)/R_0^2]}, \tag{4.6}$$

$$y>0, \quad R_0 \gg c_1t, \quad t>t_6, \quad \theta_0<\alpha.$$

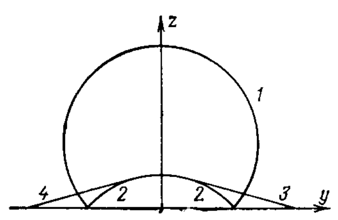

FIG. 4

For $R_0 \to c_0t$ ($\lambda = 0$) the expressionfor $p_{\text{ref}.l} \to \infty$ behaves as $(R_0^2 - c_1^2t^2)^{-1/2}$. The same singularity also exists in the expression for the reflected cylindrical wave, described by the equation (see Eq. (4.3))

$$p_{\text{ref}} = -\frac{A}{2\pi c_1}\frac{\theta_0^2(m^2+1)-\alpha^2}{\sqrt{c_1^2t^2-R_0^2}\,[\theta_0^2(m^2-1)+\alpha^2]}, \tag{4.7}$$

$$\theta_0 < \alpha, \quad c_1t > R_0 \ (c_1t \to R_0).$$

The noted circumstance is related to the fact that, unlike the planar and spherical cases, for which concentrated disturbances of the following respective types can propagate in the medium: $\delta(\xi)$, $\xi = t - (y/c_1)\sin\mu + (z/c_1)\cos\mu$ and $\delta(t - R/c_1)/R$, for cylindrical symmetry the existence of concentrated disturbances of the type $\delta(t - R/c_1)\,f(R)$ ($R = (y^2+z^2)^{1/2}$, and $f(R)$ is an arbitrary function) is impossible. Thus, in the case under consideration the pressure in the incident wave is

$$p = -\frac{iA}{4\pi c_1}\frac{1}{\sqrt{y^2+(z-l)^2-c_1^2t^2}} + \text{c.c.}$$

In Fig. 4 part of circle 1, whose equation is $y^2 + (z-l)^2 = c_1^2t^2$, determines the boundary of the incident wave, the part of circle $2y^2 + (z+l)^2 = c_1^2t^2$ shows the boundary of the reflected cylindrical wave, and the straight lines 3, 4, whose equations are $(z + l \pm y\,\mathrm{tg}\,\theta_m)\cos\theta_m = c_1t$, determine the boundary for the existence of the lateral wave.

In the case of the symmetry considered here it is easy to find also the transmitted wave.[11–13] We dwell briefly on this problem. Similarly to Eq. (3.17), we have from (3.12), putting $\mu = \pi/2$ and multiplying the integrand expression by $-k_z/\omega$,

$$p_{\text{tr}} = -\frac{A}{8\pi^2c_1}\oint_{L_\theta}\frac{W(\theta)\,d\theta}{\sqrt{1-\theta^2}\,[c_1t+\sqrt{\theta^2-\alpha^2}\,z-\theta l-(y+i\lambda)\sqrt{1-\theta^2}]}, \quad z<0, \tag{4.8}$$

where the integration path is the same as in (3.17), and $W(\theta) = 2m\theta/m\theta + \sqrt{\theta^2-\alpha^2}$. In what follows we put $l = 0$; a contribution to (4.8) is then provided by the pole of the integrand expression, determined from the equation

$$c_1t + \sqrt{\theta^2-\alpha^2}\,z = (y+i\lambda)\sqrt{1-\theta^2}, \tag{4.9}$$

whence, as can be shown, only one value turns out to be significant

$$\theta_n = \sqrt{\frac{1}{R^4}\left[-c_1tz + (y+i\lambda)\,n\sqrt{R^2-c_2^2t^2}\right]^2+\alpha^2}, \tag{4.10}$$

$$y>0, \quad |z|<c_2t, \quad z<0, \quad \mathrm{Re}\,\theta_n>0,$$
$$\mathrm{Im}\,\theta_n>0, \quad R^2 = z^2+(y+i\lambda)^2.$$

Using Eq. (4.10), and letting λ tend to zero, we find

$$p_{\text{tr}} = -\frac{iA}{4\pi c_1}\frac{W(\theta_n)\sqrt{\theta_n^2-\alpha^2}}{n\theta_n\sqrt{R_0^2-c_2^2t^2}} + \text{c.c.}, \quad R_0^2 = z^2+y^2. \tag{4.11}$$

As follows from Eq. (4.11), $p_{\text{tr}} = 0$ for $R_0 > c_2t$, since in this case $\theta_n > \alpha$ (see Eq. (4.10)). Near the boundary we have for $z\to 0$

$$p_{\mathrm{tr}} = -\frac{iA}{2\pi c_1}\,\frac{m}{m\sqrt{y^2-c_1^2t^2}+n\sqrt{y^2-c_2^2t^2}} + \text{c.c.} \quad (4.12)$$

Part of the transmitted wave, existing in the region $c_1t < y < c_2t$, forms the lateral wave.

We turn attention to the fact that, unlike the incident and reflected fields, the pressure in the transmitted wave is finite for $R = c_1t$, and is determined by the equation

$$p_{\mathrm{tr}} = -\frac{Am}{\pi c_1 a y}. \qquad (4.13)$$

Naturally, this conclusion also follows from the fact that for $z\to +0$, $y\neq 0$ the value of the total pressure $p + p_{\mathrm{ref}}$ in the first medium does not contain singularities for $y\to c_1t$, and equals $-Am/\pi c_1 a y$.

5. INCIDENCE OF A DELTA-SHAPED SPHERICAL PULSE ON A BOUNDARY[8,9,14]

In the present Section we consider the case in which a concentrated spherical pulse is incident on the separation boundary between two media. Obviously, this case is the most complex to study among all those considered in the present review. As already mentioned, the reflection problem of a spherical pulse from the separation boundary between two media was first solved by Savage,[9] and then, in more detail, by Town.[14] In the present review we provide the solution of this problem by a different method, namely, we use a method of direct calculation of the integral for p_{rel}, used in the preceding Sections for other shapes of initial disturbances. This approach is simpler and easier to visualize than those used earlier (see Ref. 14).

In the light of the problems considered in the present study, the case of spherical symmetry is interesting in that by the limiting transition $l\to\infty$ (l is the distance from the boundary to the localization point of the initial disturbance) one can obtain the solution of the problem of reflection of a planar disturbance incident on the boundary. Therefore it is of interest to clarify the features related to occurrence of a precursor.[19]

According to what was said above, we assume that initially a concentrated sound disturbance is created in the upper medium ($z>0$) at a distance l from the boundary.

To find the sound field it is then necessary to start from Eqs. (3.4), (3.5) and boundary conditions (3.6), (3.7) with $p_i(t<0) = 0$, $i = 1, 2$, while the right hand side of Eq. (3.4) must be replaced by a concentrated source near the point $\rho = 0$, $z = 1$ of the initial disturbance:

$$J = \frac{-Ac_1^2}{2\pi}\,\delta(t)\,\delta(z-l)\,\frac{\lambda}{(\lambda^2+\rho^2)^{3/2}},\quad \lambda\to 0,$$

$$\rho^2 = x^2+y^2. \qquad (5.1)$$

Here we used the well-known representation of the two-dimensional delta-function

$$\delta(\rho)/2\pi\rho = (2\pi)^{-1}\lim \lambda(\lambda^2+\rho^2)^{-3/2}.$$

The necessity of representing the source in this form is dictated by the following method of calculation (see Eq. (5.9) and the comments to Eq. (3.14)). Thus, for the Fourier component of the incident wave

$$p(\omega, k) = \int p(t, \mathbf{r})\,e^{i(\omega t-\mathbf{kr})}\,dt\,d\mathbf{r},\quad \mathbf{r} = \{x, y, z\},$$

we have:

$$p(\omega, k) = \frac{Ae^{-ik_zl}}{\omega^2/c^2-\varkappa^2-k_z^2}\int_0^\infty \frac{\rho\lambda J_0(\varkappa\rho)\,d\rho}{(\lambda^2+\rho^2)^{3/2}}, \qquad (5.2)$$

where $\varkappa^2 = k_x^2 + k_y^2$, and $J_0(\varkappa\rho)$ is a Bessel function.

Using the well-known equation

$$\int_0^\infty \frac{\rho\lambda J_0(\varkappa\rho)\,d\rho}{(\lambda^2+\rho^2)^{3/2}} = e^{-\lambda\varkappa}, \qquad (5.3)$$

by means of Eq. (5.2) we obtain the following expression for the pressure in the incident wave

$$p(t, \mathbf{r}) = \frac{A}{(2\pi)^3}\int_{L_\omega} d\omega \int_{-\infty}^{\infty} dk_z \int_0^\infty \frac{\exp[i(-\omega t-k_zz_1)-\lambda\varkappa]\,\varkappa J_0(\varkappa\rho)}{\omega^2/c_1^2-\varkappa^2-k_z^2}\,d\varkappa, \qquad (5.4)$$

$$z_1 = -z+l.$$

From the condition $p(t<0) = 0$ it follows that on the integration path L_ω $\operatorname{Im}\omega > 0$. In the region $z<l$ one easily carries out the integration over k_z in Eq. (5.4). As a result we find

$$p(t, \mathbf{r}) = -\frac{iA}{8\pi^2}\int_{L_\omega} d\omega \int_0^\infty \frac{\varkappa\exp[-i\omega t+i\sqrt{(\omega^2/c_1^2)-\varkappa^2}\,z_1-\lambda\varkappa]\,J_0(\varkappa\rho)\,d\varkappa}{\sqrt{(\omega^2/c_1^2)-\varkappa^2}}, \qquad (5.5)$$

where $\sqrt{(\omega^2/c_1^2-\varkappa^2}\to\omega/c_1$ for $\omega\to\infty$, $\omega_2>0$, and the cuts in the ω plane are made as follows: $|\omega_1| > \varkappa c_1$, $\omega_2 = 0$.

As in the preceding sections, we carry out in (5.5) a replacement of variables similar to (3.13), i.e., $\omega = c_1\varkappa' x$, $\varkappa = \varkappa'$. Then

$$p(t, \mathbf{r}) = \frac{A}{8\pi^2}\frac{\partial}{\partial t}\int_{L_x}\frac{dx}{x\sqrt{x^2-1}}\int_0^\infty J_0(\varkappa\rho) \times\exp[+i\varkappa(\sqrt{x^2-1}\,z_1-c_1tx)-\lambda\varkappa]\,d\varkappa$$

$$= \frac{A}{8\pi^2}\frac{\partial}{\partial t}\int_{L_x} f_1(x)\,dx, \qquad (5.6)$$

where the cuts in the x plane are characterized as follows: $|x_1| > 1$, $x_2 = 0$, $x = x_1 + ix_2$. It follows from Eq. (5.6) that the integration contour lies above the real axis, but, unlike the integration over ω, the path of integration over x cannot be continued arbitrarily in the region $\operatorname{Im} x > 0$. To refine this circumstance we consider in more detail the internal integral over $\varkappa$ in Eq. (5.6). Using the equation

$$J_0(\varkappa\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(i\varkappa\rho \sin\theta)\, d\theta,$$

we have

$$I(b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\zeta \int_0^{\infty} d\varkappa \exp(i\varkappa\rho \sin\zeta - b\varkappa),$$

$$b = i\left(c_1 t x - \sqrt{x^2 - 1}\, z_1\right) + \lambda. \tag{5.7}$$

The convergence condition of integral (5.7)

$$\operatorname{Re} b > 0 \tag{5.8}$$

imposes a restriction on the value of x, i.e., on the choice of the integration path L_x; more specifically, assuming on L_x $x_2 \to 0$, $x_2 > 0$ ($x = x_1 + ix_2$), we obtain from (5.8) that the following condition must be satisfied

$$0 < x_2 < \frac{\lambda}{c_1 t - (x_1 z_1 / \sqrt{x_1^2 - 1})}$$

$$\text{for} \quad |x_1| > \frac{c_1 t}{\sqrt{s}}, \quad s = c_1^2 t^2 - z_1^2. \tag{5.9}$$

If $|x_1| < c_1 t/\sqrt{s}$, $|x_1| \ll x_2$, the inequality $\operatorname{Re} b > 0$ is satisfied automatically. Condition (5.9) is consistent only for $\lambda \neq 0$. It is just in connection with this that in the source representation used by us in the form (5.1) one cannot put $\lambda = 0$ at the outset.

Assuming now that Eq. (5.8) is satisfied, and introducing the new variable of integration $Y = \exp(i\zeta)$, we find

$$I(b) = \frac{i}{\pi\rho} \int_{|Y|=1} \frac{dY}{Y^2 - (2bY/\rho) - 1} = -\frac{1}{\rho\sqrt{a^2+1}}, \quad a = \frac{b}{\rho}. \tag{5.10}$$

The contribution to $I(b)$ is provided by that pole of the integrand expression in Eq. (5.10)

$$Y_0 = a + \sqrt{a^2 + 1}, \tag{5.11}$$

at which $|Y_0| < 1$, and this determines the sign of $(a^2 + 1)^{1/2}$.

It must be mentioned that condition (5.9) determines the position of the singular points in the complex x-plane relative to the integration contour. Indeed, the branching points $x^{1,2} = \pm 1$ lie below the L_x contour. The branch points, determined from the relations $a = \pm i$ or

$$c_1 t x \pm \rho - i\lambda = \sqrt{x^2 - 1}\, z_1, \tag{5.12}$$

have small imaginary parts for $R_0 = (\rho^2 + z_1^2)^{1/2} > c_1 t$:

$$x_2 = \frac{\lambda}{c_1 t - (x_1 z_1 / \sqrt{x_1^2 - 1})}. \tag{5.13}$$

Taking into account the position of the cuts in (5.6) in the x-plane, we find that Eq. (5.12) has only two roots, localized in the regions $x_2 > 0$, $|x_1| > c_1 t/\sqrt{s}$. Therefore, it follows from condition (5.9) that both branching points, determined by (5.12), lie above the integration path L_x. When the condition $c_1 t > R_0$ is satisfied the imaginary parts of the branching points, determined by (5.12), have a finite imaginary part for $\lambda \to 0$:

$$|x_2| = \frac{z_1}{s} \sqrt{c_1^2 t^2 - R_0^2},$$
$$R_0 < c_1 t, \quad \lambda \ll \rho, \quad \lambda \ll c_1 t\, |x|, \quad |x|\, z_1 \gg \lambda, \tag{5.14}$$

and therefore their position relative to L_x is also determined. (The L_x contour is strongly squeezed toward the $x_2 = 0$ axis.)

Using (5.6), (5.7), and (5.10), we find

$$p(t, r) = \frac{A}{8\pi^2} \frac{\partial}{\partial t} \int_{L_x} \frac{dx}{x\sqrt{x^2-1}\sqrt{\rho^2 - (\sqrt{x^2-1}\, z_1 - c_1 t x + i\lambda)^2}}, \tag{5.15}$$

$\sqrt{x^2 - 1} \to x$

for $x_2 > 0$, $x \to \infty$, $\sqrt{\rho^2 - (\sqrt{x^2-1}\, z_1 - c_1 t x + i\lambda)^2} = R_0$

for $x = 0$, $\lambda = 0$.

Performing the variable replacement (3.16), we have

$$p(t, r) = \frac{iA}{8\pi^2 R_0} \frac{\partial}{\partial t} \int_{L_\theta} \frac{d\theta}{e(\theta)} = \frac{A}{8\pi^2 R_0} \frac{\partial I(t, r)}{\partial t}, \tag{5.16}$$

$$e(\theta) = \sqrt{(\theta - \psi_1)(\theta - \psi_2) + 2i(\lambda/R_0)(\theta\cos\delta - \cos\delta_1)\sqrt{1-\theta^2}}, \tag{5.17}$$

$$\psi_{1,2} = \frac{1}{R_0^2}\left(c_1 t z_1 \pm \rho\sqrt{R_0^2 - c_1^2 t^2}\right) = \cos(\delta \mp \delta_1),$$

$$\cos\delta = \frac{z_1}{R_0}, \quad \cos\delta_1 = \frac{c_1 t}{R_0}.$$

Singular points of the integrand expression are the branching points, which for $c_1 t > z_1$, $R_0 > c_1 t$ are determined by the relations

$$\theta_{1,2} = \cos(\delta - \delta_1) \pm i \frac{\lambda}{R_0} \frac{\sin^2(\delta \mp \delta_1)}{\sin\delta_1} \tag{5.18}$$

(the cut connecting the points $\theta = \pm 1$ lies along the real axis). If also $c_1 t < z_1$, then there are no singular points in the integrand expression (5.16) (they are located on another sheet of the Riemann surface, and $p(c_1 t < z_1) = 0$). For $R_0 < c_1 t$, $|\sqrt{R_0^2 - c_1^2 t_2}| \gg \lambda$, $z_1 \gg \lambda$, $\rho \gg \lambda$ the branching points coincide with $\psi_{1,2}$ (see Eq. (5.17)).

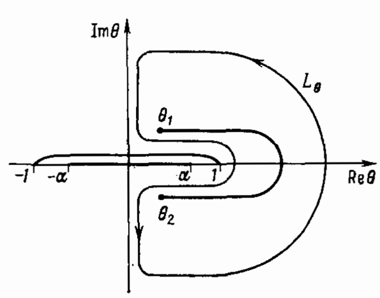

FIG. 5

As shown by straightforward analysis, in (5.16) the integration contour L_θ can be selected to be the same as in (3.17) (see Fig. 2). We recall that the variable of integration θ denotes the cosine of the angle of incidence of homogeneous and inhomogeneous planar pulses. Figure 5 shows the positions of the branching points of the integrand expression in (5.16) relative to L_θ for $R_0 > c_1 t$.

For $R_0 > c_1 t$ the integral (5.16) reduces to integration over the edges of the cut, connecting the points θ_1 and θ_2 (see Fig. 5). As a result we find

$$I(t,r) \approx 2i \int\limits_{\substack{1 \\ 0<\operatorname{Im}\theta\to 0}}^{\theta_1} \frac{d\theta}{e(\theta)} + 2i \int\limits_{\substack{\theta_2 \\ 0>\operatorname{Im}\theta\to 0}}^{1} \frac{d\theta}{e(\theta)} = 2i \int\limits_{1}^{\theta_1} \frac{d\theta}{e(\theta)} + \text{ c.c.} \approx O(\lambda,\, t), \tag{5.19}$$

$$\theta_1 \approx \theta_2, \quad O(\lambda,\, t) \sim \lambda, \quad \left| 1 - \frac{c_1^2 t^2}{R_0^2} \right| \gg \frac{i\lambda}{R_0}, \quad \frac{\lambda}{R_0} \ll 1,$$

since the integration paths in both integrals can be selected over different sides of the real axis arbitrarily close to the latter ($\lambda \to 0$). For $R_0 < c_1 t$ the branching points $\theta_{1,2}$ also lie inside L_θ. In this case, more accurately for $\sqrt{c_1^2 t^2 - R_0^2} \gg \lambda$, $\rho \gg \lambda$, $z_1 \gg \lambda$, as already noted, one can omit in (5.16) expressions of order λ, and then

$$I(t,\, r) = -2\pi, \quad R_0 < c_1 t. \tag{5.20}$$

Taking into account (5.19), (5.20), we find that the pressure in the incident disturbance is described by the following expression:

$$p(t,\, R) = -\frac{A}{4\pi R_0} \delta\left(t - \frac{R_0}{c_1} \right). \tag{5.21}$$

Of course, the Green's function of the wave equation in unbounded space can also be found by other methods. The advantage of the method provided is that it is very convenient for finding the field of the reflected wave. To find the expression for the reflected wave it is necessary to multiply the integrand expression in (5.16) by the Fresnel coefficient $V(\theta)$ (see Eq. (3.18)), and replace z_1 by $z' = z + l$. Thus, we obtain

$$p(t,\, r) = \frac{iA}{8\pi^2 R_1} \frac{\partial}{\partial t} \oint \frac{V(\theta)\, d\theta}{e_1(\theta)} = \frac{iA}{8\pi^2 R_1} \frac{\partial}{\partial t} J_1(t,\, r), \tag{5.22}$$

where $R_1 = \sqrt{z'^2 + \rho^2}$, $\cos\delta^{(1)} = z'/R_1$, $\cos\delta_1^{(1)} = c_1 t/R_1$, $\psi_{1,2}^{(1)} = \cos(\delta^{(1)} \mp \delta_1^{(1)})$, and

$$e_1(\theta) = \sqrt{(\theta - \psi_1^{(1)})(\theta - \psi_2^{(1)}) + 2i(\lambda/R_1)(\theta\cos\delta^{(1)} - \cos\delta_1^{(1)})\sqrt{1-\theta^2}},$$

the cut connecting the branching points $\theta = \pm\alpha$ is over the real axis (see Fig. 5). For $z' > c_1 t$ $p_{\text{ref}} = 0$, since in this case there are no singular points inside L_θ.

Consider the region located ahead of the spherical front $R_1 = c_1 t$. As easily seen, for the region $R_1 > c_1 t$ the expression for p_{ref} is similar to (5.19), i.e.,

$$p_{\text{ref}} = \frac{iA}{4\pi^2 R_1} \frac{\partial}{\partial t} \left(\int\limits_{\substack{1 \\ 0<\operatorname{Im}\theta\to 0}}^{\theta_1(z')} \frac{V(\theta)\, d\theta}{e_1(\theta)} + \int\limits_{\substack{\theta_2(z') \\ 0>\operatorname{Im}\theta\to 0}}^{1} \frac{V(\theta)\, d\theta}{e_1(\theta)} \right), \tag{5.23}$$

where $|1 - (c_1^2 t^2/R_1^2)| \gg \lambda/R_1$, $\lambda/R_1 \ll 1$, $\theta_1(z')$, $\theta_2(z')$ are determined by (5.18) with the replacement of z_1 by z'. As in (5.18), for $\alpha < \operatorname{Re}\theta_1(z')$, p_{ref} $(R_1 > c_1 t) \simeq O(\lambda)$, while for $\alpha > \operatorname{Re}\theta_1$ both integrals in (5.23) cancel each other in the interval $\alpha < \theta < 1$, while cancellation does not take place in the interval $\operatorname{Re}\theta_1(z') < \theta < \alpha$. This is related to the fact that the root $(\theta^2 - \alpha^2)^{1/2}$, appearing in the reflection coefficient $V(\theta)$ (see (3.18)) on the segment $\operatorname{Re}\theta_1(z') < \theta < \alpha$ (in the first integral of (5.23)), is defined as $i|\sqrt{\alpha^2 - \theta^2}|$ (the upper side of the cut $-\alpha < \theta < \alpha$), while in the second integral of (5.23) it is defined as $-i|\sqrt{\alpha^2 - \theta^2}|$ (the lower side of the cut $-\alpha < \theta < \alpha$). Taking into account what was said above and omitting all quantities of order λ in (5.23), we obtain an expression for the reflected wave ($\lambda/R_1 \ll |1 - c_1 t/R_1|$)

$$p_{\text{ref}} \approx -\frac{Am\Pi(\alpha - \Psi_1^{(1)})}{\pi^2 R_1} \frac{\partial}{\partial t} \times \int\limits_{\Psi_1^{(1)}}^{\alpha} \frac{\theta\sqrt{\alpha^2 - \theta^2}\, d\theta}{[\theta^2(m^2-1) + \alpha^2]\sqrt{(\theta - \Psi_1^{(1)})(\theta - \Psi_2^{(1)})}}, \tag{5.24}$$

$$\alpha > \Psi_1^{(1)}, \quad \Pi(\alpha - \Psi_1^{(1)}) = \begin{cases} 1, & \alpha > \Psi_1^{(1)}, \\ 0, & \alpha < \Psi_1^{(1)}. \end{cases}$$

The disturbance described by (5.24) reaches the observation point earlier than the reflected spherical front, characterized by $R_1 = c_1 t$. Therefore, expression (5.24) corresponds to a lateral wave. Its moment of arrival t_l at the observation point is determined from the condition $\theta_1(z') = \alpha$, whence it follows that

$$t_l = \cos(\delta_1^{(1)} - \theta_m) \frac{R_1}{c_1}, \quad \frac{z'}{R_1} < \cos\theta_m, \tag{5.25}$$

where $\sin\theta_m = n$. Thus, a lateral wave exists in the time interval $t_l < t < R_1/c_1$.

Using Eq. (5.24), one can easily obtain an expression for the lateral wave near its leading ($t \to t_l$) and trailing ($t \to R_1/c_1$) edges. For $t \to t_l$ we have from Eq. (5.24)

$$p_{\text{ref}} = -\frac{Ac_1 n}{2\pi R_1^2 m \alpha^{1/2} \sin^{1/2}\delta^{(1)} \sin^{3/2}(\delta^{(1)} - \theta_m)}, \quad 0 < t - t_l \ll t_l. \tag{5.26}$$

It follows from Eq. (5.26) that near its leading front the lateral wave decreases, just as in the case of a point harmonic source,[3,27–29] i.e., $p_{\text{ref}} \sim R_1^{-2}$.

It was noted earlier that for a planar pulse incident on the boundary the lateral wave near the reflected planar pulse decreases inversely proportionally to the distance from this pulse (it is precisely this part of the lateral wave which determines the precursor). A similar situation also occurs for a spherical pulse. Indeed, for $0 < (R_1/c_1) - t \ll R_1/c_1$ one easily finds from (5.24), retaining only terms of order $[(R_1/c_1) - t]^{-1}$,

$$p_{\text{ref}} \approx \frac{Am\gamma_0 \sqrt{\alpha^2-\gamma_0^2}}{2\pi^2 R_1 [\gamma_0^2(m^2-1)+\alpha^2][t-(R_1/c_1)]}, \qquad (5.27)$$

where $\gamma_0 = c_1 t z'/R_1^2$, $\alpha > \gamma_0$, $\lambda/R_1 \ll |1-(c_1 t/R_1)|$.

We turn now to determining the disturbance field in the region behind the spherical reflecting front, i.e., in the region $R_1 < c_1 t$. Here one must distinguish two cases: $\gamma_0 > \alpha$ (the angle of incidence is smaller than the angle of total internal reflection), and $\gamma_0 < \alpha$ (the angle of incidence is larger than the angle of total internal reflection). For

$$\frac{z'}{R_1} > \alpha, \quad \frac{\rho^2}{R_1^2}\left(\frac{c_1^2 t^2}{R_1^2} - 1\right) \gg \frac{\lambda}{R_1}$$

one can neglect in (5.22) terms of order λ. In that case (5.22) reduces to integrating around the cut connecting the points $\Psi_1^{(1)}(z')$, $\Psi_2^{(1)}(z')$.

It is of interest to determine the behavior of p_{ref} directly behind the reflected front, i.e., for $\sqrt{(c_1^2 t^2/R_1^2) - 1} \ll z'/R_1$. In this case the points $\Psi_1^{(1)}(z')$, $\Psi_2^{(1)}(z')$ are arbitrarily close to each other, and Eq. (5.22) is easily computed. Since for $z'/R_1 > \alpha$, $R_1 > c_1 t$, $\sqrt{(R_1^2/c_1^2) - t^2} \to 0$, $\lambda \to 0$ the integral appearing in (5.22) (see Eq. (5.23)) vanishes, while for $z'/R_1 < \alpha, R_1 < c_1 t$ the integral J_1 in (5.22) equals $2\pi i V(\theta = z'/R_1)$, so that

$$p_{\text{ref}} = -\frac{AV(z'/R_1)}{4\pi R_1}\,\delta\left(t - \frac{R_1}{c_1}\right). \qquad (5.28)$$

Thus, for $z'/R_1 < \alpha$, near the front $R_1 = c_1 t$ the shape of the reflected pulse repeats the shape of the incident pulse.

The situation is more complex when the angle of incidence is larger than the angle of total internal reflection. For

$$\gamma_0 < \alpha, \quad R_1 < c_1 t, \quad \frac{\rho^2}{R_1^2}\left(\frac{c_1^2 t^2}{R_1^2} - 1\right) \gg \frac{\lambda}{R_1}$$

one can also omit in (5.22) terms of order λ under the radical, and (5.22) reduces to calculating J_1, encompassing the cut connecting the points $\Psi_{1,2}(z')$:

$$\begin{aligned}
&V = V_1 + iV_2, \quad V_1 = \frac{\theta^2(m^2+1)-\alpha^2}{\theta^2(m^2-1)+\alpha^2},\\
&V_2 = -\frac{2m\theta\sqrt{\alpha^2-\theta^2}}{\theta^2(m^2-1)+\alpha^2},\\
&J_1 = \oint \frac{(V_1+iV_2)\,d\theta}{\sqrt{(\theta-\gamma_1)^2+\nu^2}} = J_{11} + J_{12},\\
&J_{11} = \oint \frac{V_1\,d\theta}{\sqrt{(\theta-\gamma_1)^2+\nu^2}},\\
&\nu^2 = \frac{\rho^2}{R_1^2}\left(\frac{c_1^2 t^2}{R_1^2} - 1\right) \to 0, \quad \gamma_1 = \frac{z'}{R_1}.
\end{aligned} \qquad (5.29)$$

For arbitrary t, R values this integral, as well as in the case $R_1 > c_1 t$, cannot be expressed in terms of elementary functions, except for the cases $z' = 0$ and $\rho = 0$. At the same time it is easily computed for the range of values $R_1 \to c_1 t$. In this case the θ values on L_θ can be assumed to be real; then, in the integral J_1 the L_θ contour encompasses the branching points $\Psi_{1,2}(z')$ located arbitrarily closely ($\nu \to 0$). Since there are no other singular points inside L_θ, then

$$J_{11} = 2\pi i V_1\left(\frac{z'}{R_1}\right), \quad R_1 < c_1 t. \qquad (5.30)$$

For $\nu \to 0$ Eq. (5.29) contains a logarithmically diverging expression ($\theta \to \gamma_1$), which is what determines the most slowly decreasing part of the disturbance upon moving away from the sphere $R_1 = c_1 t$. As a result we have

$$\begin{aligned}
&J_{12} = 4V_2\left(\frac{z'}{R_1}\right)\ln\frac{\rho\sqrt{R_1^2 - c_1^2 t^2}}{R_1^2},\\
&\nu \to 0, \quad R_1 < c_1 t.
\end{aligned} \qquad (5.31)$$

By means of (5.24), (5.27), (5.30), and (5.31), we find as $R_1 \to c_1 t$ for the regions adjacent on both sides to the sphere $R_1 = c_1 t$:

$$\begin{aligned}
&p_{\text{ref}} = -\frac{A}{4\pi R_1}\left[V_1\left(\frac{z'}{R_1}\right)\delta\left(t - \frac{R_1}{c_1}\right) + \frac{1}{\pi}\frac{V_2(z'/R_1)}{t - R_1/c_1}\right],\\
&\frac{|t - R_1/c_1|}{t} \ll \frac{z'}{R_1}, \quad \frac{\rho^2}{R_1^2}\left|\frac{c_1^2 t^2}{R_1^2} - 1\right| \gg \frac{\lambda}{R_1}.
\end{aligned} \qquad (5.32)$$

Besides the concentrated disturbance, repeating the shape of the incident pulse, Eq. (5.32) contains a part related to the distributed disturbance. In the sense mentioned above the situation here is analogous to the case of incidence of a planar pulse on the boundary (see Eqs. (3.2) and (3.28)). It must be noted, however, that (5.32) does not apply to very small values of $t - (R_1/c_1)$. A more accurate treatment instead of the term $[t - (R_1/c_1)]^{-1}$ in (5.32) leads to the expression

$$\frac{t-(R_1/c_1)}{[t-(R_1/c_1)]^2 + (\lambda^2\rho^2/c_1^2 t^2 R_1^2)} = \frac{P}{t-(R_1/c_1)},$$

i.e., there is full consistency between (5.32) and (3.2), (3.28). Moreover, for $l \to \infty, A \to \infty, A/l = \text{const}$ the distributed part of the signal, described by (5.32), can be interpreted as a disturbance related to the precursor, since the "arrival" of this signal at the observation point leads the moment of arrival of the concentrated pulse at this point by an arbitrarily long time interval.

Indeed, for $l \to \infty$, $A/l = \text{const}$ Eq. (5.32) corresponds to (3.2), (3.28), since for $l \gg |y|$, $l \gg z$

$$\begin{aligned}
&R = \sqrt{x^2 + (y + l\sin\mu)^2 + (z - l\cos\mu)^2}\\
&\quad \approx l + y\sin\mu - z\cos\mu,\\
&R_1 \approx l + y\sin\mu + z\cos\mu, \quad t = \frac{l}{c_1} + \tau,
\end{aligned}$$

where l_1 is the distance from the localization point of the initial disturbance to the origin of coordinates, and τ is time, measured from the moment of arrival of the sound signal at the origin of coordinates.

It must be stressed that the asymptotic expansion of (5.32) is valid for $|t - R_1/c_1|/t \ll z'/R_1$ and can not be used when the radiator and detector are located near the boundary ($z' \to 0$). The reflected wave field at $z' \to 0$ is expressed in terms of elementary functions.[19] For a lateral wave we obtain from (5.24)

$$p_l = \begin{cases} 0, & t < \dfrac{\rho}{c_2},\\[2mm] -\dfrac{Am^2\alpha c_1^2 t}{2\pi\rho^3[\alpha^2 + (m^2-1)(1 - c_1^2 t^2/\rho^2)]^{3/2}}, & \dfrac{\rho}{c_2} < t < \dfrac{\rho}{c_1}, \end{cases} \qquad (5.33)$$

and the reflected spherical wave repeats the shape of the incident one with a reflection coefficient $V = -1$:

$$p_{\rm ref} = \frac{A}{4\pi\rho}\,\delta\left(t - \frac{\rho}{c_1}\right), \quad t \geqslant \frac{\rho}{c_1}. \tag{5.34}$$

In the plane wave approximation ($R \to \infty$) the total field in the upper medium is for grazing incidence $p = p_{\rm inc} + p_{\rm ref} = 0$. This fact is the content of the so-called Lloyd paradox: "For grazing incidence there is no total field in the upper and lower media". The resolution of the Lloyd paradox consists of taking into account the sphericity of the incident pulse. In that case a lateral wave is present in the upper medium (see Eq. (5.33)).

It is interesting to note that the asymptotic equations given for the reflected spherical pulse (5.28), (5.32), corresponding to reflection of plane waves, though describing correctly the shape of the fundamental part of the signal, nevertheless do not determine its total momentum

$$P = \int_{-\infty}^{\infty} p\,\mathrm{d}t.$$

To explain this fact we calculate the total momentum of the reflected signal. Integrating (5.22) over time, we obtain

$$P = -\frac{m-1}{4\pi R_1 (m+1)}.$$

The same result can also be reached easily by integrating the original wave equations (3.4), (3.5) over time and then solving the Laplace equations with boundary conditions (3.6), (3.7).[14] At the same time integration of relations (5.28), (5.32) provides results differing from those given above. The resolution of the problem of the "vanished pulse" is that the parts of the reflected field dropping off faster than $1/R_1$ (lateral waves and signals arriving after the fundamental pulse), though small in amplitude ($\sim 1/R_1^2$), have a long duration ($\sim R_1$), and provide a contribution to the total momentum of the reflected field which is comparable in order of magnitude to the fundamental signal. This result, naturally, follows from direct computation of the momentum of the lateral wave (for more detail see Town's paper[19]).

CONCLUSION

Above we considered problems of reflection of variously shaped pulses from the separation boundary between two liquids. In studying the reflection problem of a semi-infinite pulse from the separation boundary it has been determined that the precursor generated during reflection of a planar pulse is related to excitation of that part of the lateral wave, whose field in the reference system $y' = y - (c_1 t/\sin\mu)$ is independent of the time. In considering the causal formulation of the problem the moment of signal arrival at the observation point is determined by the leading edge of the lateral wave. We considered reflection of a cylindrical shape pulse from the separation boundary between two liquids.[5] Similar results for reflection of electromagnetic pulses from the separation boundary between two dielectrics were obtained in Ref. 12. Reflection of cylindrical pulses from a liquid-solid separation boundary was investigated in Refs. 11, 13. Exact expressions were obtained in the studies mentioned[5,11–13] in terms of elementary functions for the reflected fields, which makes it possible to study lateral waves in detail, as well as different kinds of surface waves, generated during reflection from a solid.[11,13]

The problem of reflection of a spherical pulse from the separation boundary between two liquids, first solved in Ref. 9, and later treated in more detail in Ref. 14, has been analyzed in the third Section. The results of these studies are also contained in Ref. 8. The case of reflection of a spherical pulse from the boundary of a solid was investigated in Ref. 30; see also Refs. 31, 32. One or another variant of the Fourier or Laplace method, with subsequent computations of multiple integrals, has been used in the studies quoted to obtain original expressions for the reflected field. In foreign studies (particularly in the U.S.A.) this method of obtaining the expression for the reflected field is called the Cagniard or the Cagniard-Hoop method.[33–35] The unified approach adopted in the present study for reflection problems of variously shaped pulses is also based on the Fourier method.

The most substantial aspect of the method used in the present review, as well as in Ref. 5, is that for initial conditions we use a disturbance having a finite, though small, localization width, characterized by the parameter λ (only in the final expressions λ tends to zero). This approach makes it possible to avoid a number of complications in the computation of Fourier integrals describing the reflected field for singular initial conditions (delta-shaped initial disturbances). This fact is well illustrated by the circumstance that following integration over k_y (see Eq. (3.15)) or $\varkappa$ (see Eq. (5.15)), in the single contour integrals for the reflected field the integration paths L_x are compressed near the $\operatorname{Im} x = 0$ axis by conditions (3.14), (5.9). Naturally, L_x cannot pass along the real axis, since it contains singular points of the integrand expressions, and, as is very important, the branching points $x = \pm n^{-1}$.

It must be noted that the method developed here, as well as the Cagniard-Hoop method, is as a rule effective only for nondispersive media, when the reflection and refraction coefficients $V(\omega, k)$ and $W(\omega, k)$ are homogeneous functions of their variables.

In conclusion we point out that the computational procedure employed in this article can also be successfully used also in problems of the reflection of pulses from anisotropic media.[36,37]

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[1)] The situation here is essentially similar to the electrodynamic problem of formation of a quasistatic field by a charge, appearing at some moment of time $t = 0$ and then starting to move with a velocity smaller than that of light.

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