Radiation damping forces and radiation from charged particles

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A review of the literature on the radiation reaction force on a charged particle shows that the expression given for this force obtained by Lorentz, Abraham, and Dirac is in physically reasonable agreement with the radiation of energy, momentum, and angular momentum, and is successfully used in investigating the motion of particles in a field. A selection of physical solutions by the methods presented herein guarantees that the conservation laws are satisfied. In the first approximation, which is the only one utilized in the majority of physical situations, radiation damping does not depend on assumptions concerning the structure of the charge of the particle. A theory is presented of the losses of energy, momentum and angular momentum by a system of charged particles in the course of their moving together taking into account the external field, the radiation damping forces, and the retarded Lienard-Wiechert forces. Formulas are given for the spectral and angular distribution of the radiation from a system of particles. The concept of a center of a system of events with relativistic particles is utilized in constructing a system of equations for finding the angular momenta of the electromagnetic waves radiated by particles of the system. The angular distribution and the total intensity of the radiation from a system of particles at an arbitrary instant of time is obtained. Using the example of the joint synchrotron radiation from two particles the consistency of all three approaches to the radiation from a system of particles is demonstrated.

CONTENTS

1. RADIATION DAMPING FOR A SINGLE CHARGE

The equations of motion for charged particles include, as is well known, forces of radiation friction, arising as a result of the reaction on the particles of electromagnetic fields produced by their motion. Although under laboratory conditions these forces are usually small compared with the other forces acting on the particles, their presence puts the equations of motion in the form which differs essentially from the form of the equations when such forces are ignored. Therefore the correct definition and the conditions for utilizing the radiation damping forces in the equations of motions of charged particles moving with acceleration have become the subject of numerous investigations in which quite a few contradictions have arisen. Although radiation reaction forces have been applied successfully in calculating the motion of particles in a number of physical situations and repeatedly exhortations have been made to exclude the problem of radiation damping from the "eternal" problems of the mechanics of charged particles and classical electrodynamics, it is still being discussed in the literature, and many fundamental monographs and textbooks on the theory of the electromagnetic field present it evasively. As regards the connection between the radiation damping of a system of relativistic particles and their joint radiation nothing at all has been known until very recent times.

Already in 1871 Stewart[1] opined that a moving body interacting with other bodies within a finite volume by

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means of thermal radiation must experience damping as a result of this radiation. In 1892 Lorentz[2] showed that the force doing work on a particle with charge e which is equal to its loss of energy by radiation in its accelerated quasiperiodic motion contains a third derivative of the position vector of the particle and (in the frame of reference where the speed of the particle is small) is given by

$$\mathbf{f}=\frac{2e^2}{3c^3}\dddot{\mathbf{r}}, \tag{1.1}$$

in virtue of which the equation of motion of the particle in an external electromagnetic field with field intensities $\mathbf{E}$ and $\mathbf{H}$ takes on the form

$$m\ddot{\mathbf{r}}=e\mathbf{E}+\frac{e}{c}[\mathbf{vH}]+\frac{2e^2}{3c^3}\dddot{\mathbf{r}}. \tag{1.2}$$

The force (1.1) has been given the name of radiation friction or radiation damping force since it was possible to show that at least in some easily analyzable cases the effect of the force (1.1) reduces the speed of the particle.

In 1904 Abraham[3] starting from the balance of energy and momentum in the process of radiation found that in an arbitrary frame of reference the force (1.1) has the form

$$\mathbf{f}=\frac{2e^2}{3c^3}\left[\frac{\ddot{\mathbf{v}}}{1-\beta^2}+\frac{\mathbf{v}\,(\mathbf{v}\ddot{\mathbf{v}})}{c^2(1-\beta^2)^2}+\frac{3\dot{\mathbf{v}}\,(\mathbf{v}\dot{\mathbf{v}})}{c^2(1-\beta^2)^2}+\frac{3\mathbf{v}\,(\mathbf{v}\dot{\mathbf{v}})^2}{c^4(1-\beta^2)^3}\right],$$

$$\beta=\frac{v}{c}, \tag{1.3}$$

if one exludes both the term which is infinite for a point particle, and also all the terms which vanish for such a particle. Von Laue[4] showed that (1.3) is obtained from (1.1) by a Lorentz transformation. Planck[5] was one of the first to apply equation (1.2) to an investigation of the damped motion of an oscillator.

The force (1.3) was obtained by eliminating the electromagnetic field from the Maxwell equations and the equation of motion of a particle in a given field, but the role of the neglected terms remained unclear. In this connection a further development of the theory of radiation damping proceeded along two different paths. In the first case (cf., Secs. 2–6 of the present review) the starting points were the concept of an elementary particle, and also the requirements of relativistic covariance of the equations and the exact validity of the laws of conservation of energy, momentum and angular momentum. As is well known, the elementary nature of a particle and relativistic covariance of its description are compatible only if the particle is regarded as a point particle. In the second case (cf., Sec. 7) the influence of a possible structure of the particle on the radiation damping force was emphasized. A consistent treatment of such an effect in the case of a finite number of degrees of freedom of the particle is possible in classical theory only in the nonrelativistic limit. It turns out, however, that all the problems encountered in reality, including radiation damping and at the same time not requiring the application of quantum theory, admit an approximate description which synthesizes the principal features of both points of view (cf., Sec. 8).

2. RADIATION DAMPING FORCE FOR A POINT PARTICLE AND ITS PROPERTIES

In 1938 Dirac[6] considered the problem of the radiation damping force on a point particle in four-dimensional form and showed that the force (1.3) is obtained on the assumption that the particle experiences the limit of half the difference of retarded and advanced fields produced as a result of its motion, if one takes into account the balance of energy and momentum on a three-dimensional surface of an infinitely thin tube surrounding the world-line of the particle. The limit is taken as all the components tend to zero of the four-dimensional vector drawn from the particle to the point in the field, whose three-dimensional part is perpendicular to the velocity of the particle. The term, neglected in deriving (1.3) as a result of its becoming infinite for a point particle, in the process of this limiting transition cancels, and (1.3) turns out to be an exact expression for the three-dimensional force $\mathbf{f}$, which determines the four-dimensional force[1)] $g_i=\{\mathbf{f}/c\sqrt{1-\beta^2},\ i\mathbf{fv}/(c^2\sqrt{1-\beta^2})\}$, which appears in the equation of motion for a point particle:

$$mc\frac{\mathrm{d}u_i}{\mathrm{d}s}=\frac{e}{c}F_{ik}^{\mathrm{ext}}(x)\,u_k+g_i, \tag{2.1}$$

where $\mathrm{d}s=c\,\mathrm{d}t\sqrt{1-\beta^2}$, $u_i=\mathrm{d}x_i/\mathrm{d}s$ is a four-dimensional velocity, F_{ik}^{ext} is the tensor of the external field and

$$g_i=\frac{2e^2}{3c}\left(\frac{\mathrm{d}^2u_i}{\mathrm{d}s^2}+u_iu_k\frac{\mathrm{d}^2u_k}{\mathrm{d}s^2}\right)=\frac{2e^2}{3c}\left[\frac{\mathrm{d}^2u_i}{\mathrm{d}s^2}-u_i\left(\frac{\mathrm{d}u_j}{\mathrm{d}s}\right)^2\right]. \tag{2.2}$$

In transforming one expression (2.2) into another one, and also below, the identities $\mathrm{d}^l/\mathrm{d}s^l\,(u_iu_i+1)=0,\ l=0,1,\ldots$ are taken into account. The facts $u_i=\{\mathbf{v}/(c\sqrt{1-\beta^2}),\ i/\sqrt{1-\beta^2}\}$, $\omega_i=\mathrm{d}u_i/\mathrm{d}s=\{[\mathbf{w}/c^2(1-\beta^2)]+[\mathbf{v}(\mathbf{vw})/c^4(1-\beta^2)^2],\ i\mathbf{vw}/c^3(1-\beta^2)^2\}$ are also utilized. Dirac also showed in the same paper that the force (2.2) can be represented in the form

$$g_i=\frac{e}{c}F_{ik}^{\mathrm{int}}u_k,\quad F_{ik}^{\mathrm{int}}=\frac{2e}{3}(u_i\ddot{u}_k-u_k\ddot{u}_i), \tag{2.3}$$

where the dots indicate differentiation with respect to $\mathrm{d}s$.

Expression (2.2) satisfies the identity

$$g_iu_i=0 \tag{2.4}$$

and leads to the following expressions for the change in the energy, momentum and angular momentum of a particle when it traverses a certain region of space where an external field is acting:

$$p_i(s_2)-p_i(s_1)=\frac{e}{c}\int_{s_1}^{s_2}F_{ik}^{\mathrm{ext}}u_k\,\mathrm{d}s+\int_{s_1}^{s_2}g_i\,\mathrm{d}s$$
$$=\frac{e}{c}\int_{s_1}^{s_2}F_{ik}^{\mathrm{ext}}u_k\,\mathrm{d}s-\frac{2e^2}{3c}\int_{s_1}^{s_2}w_k^2u_i\,\mathrm{d}s+\frac{2e^2}{3c}[w_i(s_2)-w_i(s_1)], \tag{2.5}$$

$$M_{ij}(s_2)-M_{ij}(s_1)=\frac{e}{c}\int_{s_1}^{s_2}(x_iF_{jk}^{\mathrm{ext}}-x_jF_{ik}^{\mathrm{ext}})\,u_k\,\mathrm{d}s+\int_{s_1}^{s_2}(x_ig_j-x_jg_i)\,\mathrm{d}s$$
$$=\frac{e}{c}\int_{s_1}^{s_2}(x_iF_{jk}^{\mathrm{ext}}-x_jF_{ik}^{\mathrm{ext}})\,u_k\,\mathrm{d}s-\frac{2e^2}{3c}\int_{s_1}^{s_2}(u_iw_j-u_jw_i)\,\mathrm{d}s-\frac{2e^2}{3c}\int_{s_1}^{s_2}(x_iu_j-x_ju_i)\,w_k^2\,\mathrm{d}s$$
$$+\frac{2e^2}{3c}\{[x_i(s_2)\,w_j(s_2)-x_j(s_2)\,w_i(s_2)]-[x_i(s_1)\,w_j(s_1)-x_j(s_1)\,w_i(s_1)]\}. \tag{2.6}$$

The first term in each of the right-hand sides of (2.5) and (2.6) is the change in the energy, momentum and angular momentum of a particle under the influence of an external field. The second term of the last expression in (2.5) and the sum of the second and third terms in the last expression in (2.6) are equal, up to their sign as we shall verify below, respectively to the energy, momentum and angular momentum leaving the region, where the particle is moving, together with the electromagnetic field radiated by it. The last terms in (2.5) and (2.6) represent the discrepancies in the balance of energy, momentum and angular momentum associated with the possible accumulation (or expenditure) of these quantities in the field of the charge. An exact balance of energy, momentum and angular momentum without any discrepancy is observed only in two cases, when either $w_i(s_2) = w_i(s_1)$ and $x_j(s_2) = x_j(s_1)$, or $w_i(s_2) = w_i(s_1) = 0$. The former case refers approximately to the quasiperiodic motion of a particle when $s_2 - s_1$ amounts to one or several periods (the fewer, the better is the approximation). The latter case is realized either in the case of arbitrary s_1 and s_2 in the case of motion of a particle in space, where the external field is *entirely* absent, or in the passage of a particle through the *entire* region where an external field acts, with $s_1 = -\infty, s_2 = +\infty$ and it is assumed not only that $\lim\limits_{s\to\pm\infty} w_i(s) = 0$, but also $\lim\limits_{s\to\pm\infty} x_i(s)w_j(s) = 0$. The conditions indicated above limit in an essential manner the selection of physically acceptable solutions of equation (2.1), and we shall refer to this below.

3. DERIVATION OF THE EXPRESSION FOR THE RADIATION DAMPING FORCE

We now pose the question in the reverse order, considering the expressions (2.4), (2.5) and (2.6) (in the absence of a discrepancy) as equations defining the force g_i. First of all the question arises concerning the lack of ambiguity in expression (2.2) as a solution of these equations. Bhabha[7] showed that for a point particle, in contrast to an extended one, the balance of angular momentum does not yet follow from the balance of energy and momentum. He found, that taking into account the flux of angular momentum through the surface of an infinitely narrow tube surrounding the world line of the particle makes (2.2) the unique solution of the above system of equations, if one excludes terms with higher derivatives of the velocity. He also found[8] that one can add to (2.2) an infinite number of terms each of which is a certain polynomial in the components of the vector u_i and its derivatives of increasing orders, but these terms take into account the fact that the particle has not only a charge but also a magnetic moment, while complex systems also have higher electric and magnetic moments. The required number of such terms can be easily taken into account in (2.2), but, firstly, when a particle has charge they in the majority of physical situations give a negligibly small contribution to the radiation damping force since they contain higher derivatives of the velocity, and secondly all the corresponding moments are of quantum origin and taking them into account in a classical theory is not justified. We then return to expression (2.2).

Infeld and Wallace[9] obtained equation (2.1) with the force (2.2) starting from the equations of motion of the general theory of relativity. A number of other authors[10–18] arrived at the same equation but by different paths. A similar equation was obtained by Wheeler and Feynman[19] starting with the concept of total absorption of radiation by other charged particles. On the other hand, in Refs. 20–22 it was shown that the same result can be obtained using only retarded fields in the neighborhood of the world line of a particle and a renormalization of the mass of the particle, since only the use of the limit of half the difference of retarded and advanced fields guarantees automatically mass renormalization. This is not surprising since in the limit of zero distances from the particle the advanced field taken with the opposite sign differs from the retarded field only by the sign of the term corresponding to mass renormalization.

In papers by other authors[23–25] doubts were expressed whether the derivation of expression (2.2) was convincing or correct, or alternatively other expressions for g_i were proposed in its place.[26–37] These proposals did not gain support. In particular, in Refs. 38 and 23 it was shown that the equation proposed in Ref. 30, leads to contradictions. In Ref. 39 it was shown that the equations of Ref. 34 cannot be correct, since in the one-dimensional case they reduce to the equation of motion of a particle without radiation damping. Stöckel,[37] who has enumerated in detail all, in his opinion (with arguments ranging from those entirely justified, but harmless for the theory, to absurd ones) essential defects of equation (2.1) with the force (2.2), proposed a different equation that contradicts relations (2.5) and (2.6). One of the subsequent papers by Dirac,[40] which has not been elaborated in the literature, can be regarded as a negation of radiation reaction on the particle.

4. BALANCE OF ENERGY, MOMENTUM AND ANGULAR MOMENTUM

Expression (2.2) for the radiation damping force which satisfies relations (2.4), (2.5) and (2.6) cannot be represented in the form for which the balance between the change in the energy, momentum and angular momentum transmitted to the particle by the external field, the energy, momentum and angular momentum acquired by the particle, and the energy, momentum and angular momentum carried away by the radiation into the wave zone, holds simultaneously for each instant of time. This problem, in particular the seeming contradiction between the disappearance of the radiation damping force in the case of uniformly accelerated motion (hyperbolic motion for relativistic particles) and the presence of radiation was discussed in Refs. 41–50, 15, 51–60. This balance holds only for the entire period that the particle is in the field. Indeed, in order that (2.4) should hold it is necessary that the coefficients of the two terms in expression (2.2) be equal (in particular, it follows from this that the coefficient of the first term is not equal to zero), and then in order that (2.2) and (2.5) should agree it is necessary that the integral of the first term in (2.2) for the indicated period should vanish. This is possible if the acceleration of the particle w_i is either the same at the limits of integration, or vanishes sufficiently rapidly at each of them. The presence in (2.2) of the aforementioned first term, the integral of the fourth component of which over a finite time interval is sometimes referred to as the Schott energy, reflects the fact that the energy and momentum are accumulated by the field of the particle moving with an acceleration in an external field and are then expended on radiation or, conversely, at first radiation occurs partially or completely at the expense of the energy of the field of the particle, and later the deficit in the energy of the field in replenished. Similarly the integrals of corresponding terms additional to (2.2) and associated with higher moments must vanish.

At one time in Refs. 42 and 45 the opinion was expressed that in case of hyperbolic motion no radiation takes place, since the radiation damping force vanishes. In Ref. 61 a method of altering the boundary conditions for the field was sought for which this radiation is absent. But this radiation occurs[52,53] at the expense of the energy of the field, since the Schott energy is exactly equal to the radiated energy. An objection to this explanation based on the absence of radiation observed by a comoving observer moving with an acceleration, is rejected in Refs. 62, 60, since such an observer has a horizon of events, and all the radiation, as it turns out, occurs beyond this horizon.

We note that the first term in the force (2.2) which prevents a simultaneous balance of energy and momentum is needed to have agreement between (2.6) and the radiated angular momentum.

5. EQUATIONS OF MOTION AND SELECTION OF SOLUTIONS

Equations (1.2) or (2.1) represent systems of three ordinary differential equations in the three coordinates of the particle in an external field specified as a function of the coordinates and velocities of the particle and also of the time. The fourth equation of the system (2.1) is a consequence of the first three in virtue of (2.4). Its most important difference from the equations of mechanics without radiation friction is the presence of the third derivatives of the coordinates of the particles with respect to time [and derivatives of higher orders if additional terms are taken into account in (2.2)]. According to the general theory of systems of ordinary differential equations they have a unique solution (at any rate over a certain interval of time; the solution can have an analytic continuation if the field along the path of the particle is sufficiently smooth) if 9 arbitrary constants are specified, for example, the coordinates, the velocities and the accelerations at the initial instant of time. However, it turns out that arbitrary specification of an initial acceleration leads, as a rule, to physically unacceptable solutions, i.e., solutions that violate the conservation laws. In particular, as we have already seen above, for a particle traversing the region of space where an external field exists the laws of conversation of energy, momentum and angular momentum for a system of particles and the field can be satisfied taking into account radiation damping forces if the accelerations of the particles vanish in regions outside the field. But according to other solutions third order equations a particle can be accelerated indefinitely expending the energy of its own electromagnetic field.

The situation here is reminiscent of that which exists in a number of other divisions of theoretical physics where from all solutions of the basic equations of the theory only those are selected which satisfy certain subsidiary conditions. Thus, in quantum mechanics for finite motion of particles only normalizable solutions of the Schrödinger equation are physically acceptable. In other cases those solutions of the Schrödinger equation are selected which possess a definite symmetry. Of the wave solutions of Maxwell's equations most frequently only the retarded solutions are utilized. In this respect the equations of motion with radiation friction are not exceptional.

A well-known example of a not always acceptable solution of the equations of motion is the solution of equation (1.2) in the absence of an external field that has the form $\mathbf{r} = \mathbf{a} + \mathbf{b}t + \mathbf{c}\exp(3mc^3t/2e^2)$. Only the solution with $\mathbf{c} = 0$ is physically acceptable, while the remaining give "runaway", "self-accelerating" or "nonphysical" solutions. Other examples are solutions of equation (1.2) for a particle in a constant and homogeneous magnetic field and of equation (2.1) for a particle in a constant and homogeneous electric field parallel to the initial velocity and the initial acceleration.[63] For the last problem the expression for the velocity in terms of speed reduces equation (2.1) to the form of equation (1.2). Other examples of exact solutions of equations (1.2) and (2.1) were found by Plass,[50] who showed that the authors of Refs. 30 and 64 who asserted that in certain cases the equations of motion with radiation friction have no physical solutions at all missed just the physically reasonable solutions. In all these examples it turns out that the space of the initial data leading to physically acceptable solutions is the 6-dimensional subspace of the initial 9-dimensional space, with the acceleration at that point being uniquely expressed in terms of the velocity and the field at the initial point. It is just the necessity of selection from among the initial data of such combinations of them which yield physically acceptable solutions that gave rise to the extensive discussion of the validity of the expression for the force of radiation friction. In it, in addition to the authors mentioned above, the authors of Refs. 65–69 also participated.

The necessity of selecting initial data leading to reasonable solutions follows already from matching losses and radiation. Indeed, for a physical solution the integral of the first term of (2.2) proportional to the difference in the accelerations at the limits of integration must vanish not only for any arbitrary motion of particles through the entire region of space occupied by the field, but also for any region without a field. In the opposite case mentioned in the preceding paragraph the solution for the motion of a particle outside the field would have to be regarded as being physical even in the case when $\mathbf{c}$ is different from zero. Consequently, outside the field the particle can have only zero acceleration. Then from the theorem of the uniqueness of the solution of a system of equations with given initial conditions it follows that in the region occupied by the field there is only one solution which corresponds to the position, velocity and zero acceleration at the initial instant specified outside the field, and after emergence from the field the acceleration vanishes. The self-acceleration of a particle after it traverses any arbitrary field turns out to be a physical myth belonging to unacceptable solutions. Therefore Newtonian determinism—determination of accelerations by the positions and velocities of bodies at the same instant of time (the nonrelativistic case) is preserved also taking into account the radiation damping force, the dimensionality of phase space is not altered as the order of the equations is raised, while the force of radiation friction loses any kind of mystical overtones and can be utilized in equations of motion even when it is not small compared to the force due to an external field. The same applies to the equations of higher order arising when terms are taken into account that are additional to (2.2), but in this case asymptotic limitations are imposed also on higher order derivatives of the coordinates.

The fundamental question remains—how to choose the initial acceleration when initial data are specified at the point where there is already an external field, so that also in this case one should not deviate from the physically sensible solution of the equations of motion? Dirac[6] proposed to choose only those solutions for which the acceleration vanishes when the particle traverses the field or in a magnetic field the particle finally comes to rest. Bhabha[8] assumed that one should select solutions of equations (1.2) or (2.1) which depend continuously on the charge of the particle e as $e \to 0$. It is true that Arley[80] noted that in order to be able to expand the solution in powers of the charge continuity is insufficient, but one needs analyticity in the neighborhood of $e = 0$, but it is not known whether the physical solutions are analytical. Caldirola[81] proposed to regard only such a solution physical which for any t reduces to the solution for a free particle if one lets the charge tend to zero. The above recipes that guarantee the choice of the correct solution are useful only in those exceptional cases when one succeeds in obtaining an exact solution of the system of equations and in finding all its solutions that depend on 9 arbitrary constants. In the more general case one has to find the physically acceptable solution by using the method of successive approximations to equations (1.2) or (2.1) assuming that the force g_i is small compared to the force due to the external field, and this we shall discuss in greater detail below.

6. INTEGRAL EQUATIONS. EXISTENCE THEOREM AND SUCCESSIVE APPROXIMATIONS

An automatic selection of physically acceptable solutions of the equations of motions taking radiation damping forces into account is possible if they are formulated in the form of integral equations equivalent both to differential equations and to initial and final conditions. For the nonrelativistic case these equations have been obtained in the papers by Haag[14] and Rohrlich[23] and for equation (2.1) by Rohrlich.[23,71,15] When the final condition $w_i = 0$ is specified then at the expiration of infinite time they have the form

$$w_i(\tau) = \frac{e^{\tau/\tau_0}}{\tau_0} \int_{\tau}^{\infty} [F_i(\tau') - \tau_0 w_j(\tau') w_j(\tau') u_i(\tau')] e^{-\tau'/\tau_0} \, d\tau',$$
$$F_i = \frac{e}{mc^2} F_{ik} u_k, \quad \tau_0 = \frac{2e^2}{3mc^3} \tag{6.1}$$

(τ is the proper time of the particle) or, in the nonrelativistic limit,

$$\mathbf{w}(t) = \frac{e^{t/\tau_0}}{\tau_0} \int_{t}^{\infty} \mathbf{F}(t') e^{-t'/\tau_0} \, dt',$$
$$\mathbf{F} = \frac{e}{m} \left(\mathbf{E} + \frac{1}{c} [\mathbf{vH}] \right), \quad \tau_0 = \frac{2e^2}{3mc^3}. \tag{6.2}$$

The obtained equations can be easily formally integrated further in order to obtain on the left hand side velocities and coordinates respectively, but we omit these obvious relations.

If the initial and finite conditions are specified in a region without a field, relations analogous to (6.1) and (6.2) reduce to the equation $w_i = 0$.

In considering the integral equations (6.1) and (6.2) it strikes us first of all that the acceleration of a particle at each instant of time is expressed not in terms of the past but in terms of the future motion of the particle, so that the acceleration of the particle is determined by fields which the particle has not yet reached, over a time interval of the order of τ_0. This phenomenon is referred to as "preacceleration" and is sometimes regarded as a violation of causality.

Grünbaum and Janis[82] have shown that the apparent violation of causality in physical solutions for the motion of charged particles taking into account radiation damping is a consequence of the Newtonian point of view on the connection between force as the cause of motion and acceleration as its consequence, while one can only speak of the fact that force and acceleration mutually determine each other over the entire time interval between the instant of specifying the initial position and the initial velocity of the particle and the infinitely removed position when the final condition of zero acceleration is imposed. Here it is essential that one of the conditions is specified in the future. One can invoke the analogy with the case when it is necessary to attain a definite final velocity, for example, in launching space vehicles when the velocity at each instant of time is determined taking into account the future action of a force (the difference in the velocities is expressed in terms of the integral over time of future forces). Such a description does not arouse fundamental objections from the point of view of Newtonion mechanics, while the adherents of the Aristotelian point of view, according to which the velocity of a body is determined by the force acting on it at that instant, would have said that there is an unnatural "prevelocity".

Reference 23 sets forth also other arguments in favor of the proposition that the integral equations (6.1) or (6.2) are not at all evidence of the violation of causality, and that they should be regarded as a mathematical instrument for finding the correct classical picture of motion in the course of the *entire* time that the particle is in a field.

The action of the right hand sides of (6.1) and (6.2) can be picturesquely described in the following manner: acceleration at each instant of time obtained with the aid of the integral equations is determined by the requirement that in the immediate future the solution should not deviate from a physically sensible one. For this purpose in (6.1) and (6.2) a sort of reconnaisance into the future takes place.

We note that for an electron $\tau_0 = 0.62 \cdot 10^{-23}s$, while for other charged particles and charged microscopic bodies this time is significantly still smaller. Consequently, all the corresponding effects are far beyond the limits of applicability of classical theory and cannot be directly investigated with the aid of "test" particles. However, they manifest themselves and are entirely observable and measurable in the case of a sufficiently lengthy action, for example, in the case of spiral motion of a particle in a magnetic field.

To find a solution in quadratures of equations (6.1) or (6.2) is not any simpler, but rather more difficult than to solve in quadratures the differential equations (2.1) or (1.2). Therefore (6.1) and (6.2) serve most frequently to construct a method of successive approximations for finding the accelerations at each instant of time. Writing (6.1) and (6.2) in the form

$$w_i(\tau) = \int_0^\infty [F_i(\tau + \alpha\tau_0) - \tau_0\Phi_i(\tau + \alpha\tau_0)]\, e^{-\alpha}\, d\alpha,$$

$$\Phi_i = \tau u_i w_j w_j \tag{6.3}$$

$$\mathbf{w}(t) = \int_0^\infty \mathbf{F}(t + \alpha\tau_0)\, e^{-\alpha}\, d\alpha \tag{6.4}$$

and, expanding the integrands in series in powers of τ_0, we obtain

$$w_{i,n}(\tau) = w_{i,n-1}(\tau) + \tau_0^n (F_i^{(n)}(\tau) - \Phi_i^{n-1}(\tau)) \tag{6.5}$$

$$\mathbf{w}_n(t) = \mathbf{w}_{n-1}(t) + \tau_0^n \mathbf{F}^{(n)}(t). \tag{6.6}$$

In order to evaluate the terms appearing in subsequent approximations (6.5), one needs the values of the derivatives of the acceleration evaluated by means of differentiating (6.3) and substituting a lower approximation than the one into which they are being substituted.

We note that the expansions (6.5) and (6.6) coincide with those which can be obtained directly from the differential equations (1.2) and (2.1). This shows that the solution of equations (1.2) or (2.1) by means of successive approximations, in which at each stage the desired part of the radiation reaction force is regarded as being small compared to the external force and to the sum of the already obtained terms of the radiation reaction force, leads automatically to a physically sensible solution. It is true that the number of terms which have to be retained in such a series depends on the actual smallness of the change in the external force during the time τ_0 in the reference system under consideration.

The outline for calculating the motion of a particle is visualized as follows. At an arbitrary initial point, in the field or outside the field, the initial position and the initial velocity are specified. Then with the aid of (6.5) or (6.6) the acceleration corresponding to the physical solution is calculated at that point. Then a finite step $\Delta\mathbf{r} = \mathbf{v}\,\Delta t$, $\Delta\mathbf{v} = \mathbf{w}\,\Delta t$ is made corresponding to the time step Δt. The same calculations are repeated at the new point. Naturally a strong or rapidly varying field necessitates making the steps Δt small so as not to depart too far from the physical solution due to the finite accuracy of calculations. The programs and parameters of such a calculation, and also the criteria for the agreement of the obtained solution with the integral equations require a detailed investigation.

A number of papers, in particular those of Ezeilo[83] and Plass[84] have been devoted to the problems of the proof of the existence of physical solutions of the equations of motion in fields of a general type. The process of convergence of successive approximations in the solution of the integral equations (6.1) or (6.2) has been investigated in greatest detail by Hale and Stokes.[85] They showed that in order for a physically acceptable solution to exist it is sufficient that the force acting on the particle due to the external field should fall off faster than the first inverse power of the distance from the region of the field at its periphery (if the particle leaves the region of the field) and should increase not faster than the first power of the velocity of the particle. By this they proved that the successive approximations (6.5) or (6.6) obtained with the aid of the integral equations (6.3) or (6.4) are a convergent (under the aforementioned conditions) method for seeking solutions of these integral equations.

These results are confirmed by examples of a numerical solution (by successive approximations) of a number of model problems dealing with one-dimensional motion of repelling particles.[86] In Ref. 87 it was found that in the one-dimensional problem with attractive particles there is no physical solution, which does not contradict the aforementioned theorem since in this problem the condition concerning the asymptotic vanishing of the force in the distant future is not satisfied since there is no outgoing trajectory in general. The last problem becomes soluble if we consider particles as moving in three-dimensional space and do not attempt to apply classical theory at too small impact parameters.

Cases may exist which are not embraced by the conditions of this theorem, for example, motion of a particle in a magnetic field right up to its coming to rest.

Rohrlich[15] noted that the results of Ref. 85 are insufficient to prove the uniqueness of the obtained physical solution. The conditions under which it is unique have not yet been investigated. On the other hand, no problems have been found in which the physical solution for the motion of a particle is not unique.

In Ref. 88 an example is given in which the authors assert that they have found two physical solutions for the problem of a charge initially at rest near a plane layer within which a force is acting to accelerate the charge. However, one of these solutions cannot be regarded as being physically acceptable, since although the acceleration after the charge traverses the layer vanishes, the acceleration differs from zero in the region of space where there is no external field of any kind. We have already mentioned this condition in connection with the discussion of relations (2.5) and (2.6). We note that in practically realizable physical situations such a limitation cannot become needed (therefore it is frequently forgotten), but it cannot be avoided in discussing model problems in which the spatial derivatives of the field are discontinuous.

Naturally the theorem concerning the existence of a physical solution of the equations of motion of a particle does not indicate the numerical procedure which combines accuracy of results with economy of calculations, particularly if the interval of time over which it is necessary to trace the motion of the particle is great. It appears that here the "push-through" method[2)] might be applicable, but the application of this method to the problems under consideration has not yet been investigated in detail.

7. RADIATION DAMPING OF EXTENDED PARTICLES

A number of authors[32,89–100] investigated the possibility of removing the undesirable, from their point of view, properties of the equation of motion of a point particle by introducing a certain finite distribution of the charge of the particle. Such a description with a finite number of degrees of freedom turns out to be possible only for nonrelativistic particles, since otherwise one has to consider the motion of an infinite number of elements of a particle the motion of which is described by integral equations,[100] the properties of which and the possibility of describing stable particles are completely uninvestigated.

On the other hand, at least for electrons, the ideas concerning the existence of a finite structure of the charge of a particle have no independent experimental confirmation. And even if such a structure is manifested, its dimensions are considerably smaller than the distances to which classical electrodynamics is applicable. Therefore within the limits of applicability of classical theory taking into account the structure of a charge appears to exceed the allowable accuracy. Since the present review shows that even for a point particle expression (2.2) for radiation damping is sensible if the necessary selection of solutions is made, taking into account a possible structure of some particles is the subject of subsequent investigations by means of quantum theory.

8. RADIATION DAMPING IN THE FIRST APPROXIMATION

For laboratory and some cosmological situations that have been studied the force of radiation damping is in fact so small, compared to the force due to an external field in the laboratory reference system, that already the first approximation when the acceleration, whose derivative appears in the radiation force, is expressed in terms of the field neglecting radiation damping, turns out to be sufficient to obtain the equations of motion with the required accuracy (cf., Heitler's book[101]). But in other cosmological problems the force of radiation damping can be great compared to the external force in the frame of reference associated with the cosmic object giving rise to the external field, but is small in the frame of reference associated at a certain instant of time with the particle.

However, this does not provide a basis for asserting, as has been done in Refs. 33 and 36, that the equation of motion with the expression for the radiation damping force in terms of the external field in the first approximation is the exact equation of motion. Indeed, such an equation satisfies the balance of energy, momentum and angular momentum only approximately, and not exactly as does (2.1). Similarly, the replacement of the exact equation by an equation obtained after a certain finite number of iterations breaks down the exact balance between energy, momentum and angular momentum lost by the particle and radiated with the field.

Pomeranchuk[67], and also Landau and Lifshitz (Ref. 10 and all the subsequent editions) assumed that equations (1.2) or (2.1) are applicable only when the radiation damping force is small compared with the force due to the field (at least in the reference system in which the velocity of the particle is small), although Ref. 67 is one of the few applications of equation (2.1), in which in the system of reference with respect to which the particle is moving rapidly, the radiation damping force is great compared with the Lorentz force. It is necessary to specify the reference system in which the forces are compared, since the condition of smallness of the force under consideration in an arbitrary reference system is not relativistically invariant,[102] although the condition of smallness in the instantaneous rest system of the particle is invariant.

It is easy to see that the condition formulated in Refs. 67 and 10 for the applicability of the radiation force is not in fact some kind of a limitation on the set of physical situations subject to investigation,[50] since the same authors have shown that the indicated smallness of the radiation damping force in the instantaneous rest system of the particle always holds under the conditions of applicability of classical electrodynamics. On the other hand, although practically (with the exception of some model problems) only the first approximation is used, in the case of a theoretical analysis it is preferable to use the complete expression (2.2) for the radiation damping force as satisfying the conservation laws exactly and therefore not signifying that the accuracy has been inadmissibly exceeded.

In the first approximation the equations of motion of extended particles mentioned in Sec. 7 coincide[98] with the first approximation for the equations of motion for point particles. Consequently, in the first approximation the radiation damping force does not depend on assumptions concerning the structure of the charge of a particle. Thus, the first approximation for the radiation damping force realizes an approximate synthesis of two existing points of view or a compromise between them.

9. APPLICATION OF THE EQUATIONS OF MOTION OF PARTICLES WITH RADIATION FRICTION

A number of papers is devoted to the investigation of the motion of charged particles which do not interact with other particles in a homogeneous and constant magnetic field taking radiation damping into account: Refs. 89, 50, 103, 75, 55, 34, 35, 104, 113. In Refs. 106, 108, 114–116, 35 it was shown that in the case of motion of high energy particles in a strong magnetic field the spectrum of synchrotron radiation and, particularly, its polarization are changed in an essential manner compared with the results of calculations which assume that particles move exactly along a circle. In a real synchrotron particles necessarily move along trajectories close to circles, and are supplied with energy, and this suppresses the effect of radiation damping on the radiation, but the calculation itself of these correcting interactions requires taking radiation friction into account at least in the form of a first iteration. On the other hand, the effect of radiation damping must necessarily be taken into account[103] in analyzing the synchrotron radiation of fast particles in strong magnetic fields of extraterrestrial origin. For synchrotron radiation Shen[108] indicated ranges of values of the particle energy and of the magnetic field within which one should mainly take into account the radiation reaction and quantum effects. In Ref. 106 it is shown that the radiative corrections to the radiation emitted by particles in a magnetic field are more important than the quantum corrections, if the Lorentz factor exceeds 137. Shen and White[112] have shown that the radiation deflection of a beam of particles is greater than its broadening as a result of quantum fluctuations, since the radiation shift is proportional to LH, where L is the path length within the field H, and the fluctuational spread is proportional to $\sqrt{LH}$.

In order to give the reader an idea of the order of magnitude of the radiation effects arising when particles move in a magnetic field we note that the relative contraction of the radius of the orbit during one revolution is for nonrelativistic particles equal to $4\pi H(3H_0)$, where $H_0 = m^2c^4/e^3 = 6.03\cdot 10^{11}T$, while for fast particles this is multiplied[55] by the Lorentz factor. If $H = 2$ T and $\mathscr{E}/mc^2 = 5\cdot 10^4$, then $\Delta r/r = -7\cdot 10^{-7}$. In the case when the particles move in a tube with the ratio of radii equal to 10^{-4}, then already after a hundred revolutions they could collide with the wall of the tube, if necessary corrections and focusing were not introduced. At low energies the radiation changes of the motion can be less important than scattering by the molecules of the medium.

Other cases of motion of particles taking radiation damping into account were investigated in Refs. 97, 50, 117–125. Sections 1–9 of the present article were also the subject of the review of Ref. 126.

10. LOSSES OF ENERGY, MOMENTUM AND ANGULAR MOMENTUM BY A SYSTEM OF PARTICLES

Until now we have taken into account the action on charged particles by an external field and by the radiation field of each particle. But in sufficiently dense beams the particles interact also with one another, and this in certain cases significantly affects both their motion, and the nature of the losses by the particles of energy, momentum and angular momentum and the resultant radiation. In the nonrelativistic case some aspects of this effect were considered in Refs. 127, 128. The coherence of the radiation by a system of particles at sufficiently low frequencies was studied in Refs. 129–131.

Each particle with a charge e_a at the four-dimensional point x_a gives rise at the field point x (also sometimes referred to as the observation point) to the Lienard-Wiechert potential

$$A_i^{(a)}(x) = -e_a u_{ia}((x-x_a)u_a)^{-1}, \tag{10.1}$$

where

$$(x-x_a)^2 = 0. \tag{10.2}$$

Differentiating the relations (10.2) with respect to x and taking account that $\partial x_{ka}/\partial x_j = u_{ka}\partial s_a/\partial x_j$ we find $\partial s_a/\partial x_j = (x-x_a)_j/(x-x_a)u_a$. Then from (10.1) we obtain the field tensor

$$F_{ij}^{(a)}(x) = \frac{e_a}{[(x-x_a)u_a]^3} \times\{[(x-x_a)_i u_{ja} - (x-x_a)_j u_{ia}][1+(x-x_a)w_a] - [(x-x_a)_i w_{ja} - (x-x_a)_j w_{ia}](x-x_a)u_a\}, \tag{10.3}$$

where again conditions (10.2) are taken into account and only the retarded fields are selected.

The equations of motion of a system of N charged particles situated in an external electromagnetic field and interacting by means of an electromagnetic field have the form

$$dp_{ia} = \frac{e_a}{c}F_{ij}(x_a)\,dx_{ja},$$
$$dx_{ia} = (c^2p_{ia}/\mathscr{E}_a)\,dt_a, \quad a = 1, \ldots, N, \tag{10.4}$$

where

$$F_{ij}(x_a) = F_{ij}^{\text{ext}}(x_a) + \frac{2e_a}{3}(u_{ia}\ddot{u}_{ja} - u_{ja}\ddot{u}_{ia}) + \sum_{b\neq a} F_{ij}^{(b)}(x_a). \tag{10.5}$$

The first term in (10.5) is the external field, the second has already been written down in (2.3) (the dots indicate differentiation with respect to ds_a) and the third is the sum of the fields (10.3), acting on each particle due to all the others. On substituting (10.3) into (10.5) x is replaced by x_a, x_a by the coordinates x'_b, a, determined for each pair of particles with indices a and b from the equations $(x_a - x'_{b,a})^2 = 0$ (Fig. 1), and u_a, w_a are replaced by $u'_{b,a}$, $w'_{b,a}$, which refer to the point x'_b, a (the primed instant of time preceeds the unprimed one). A closely similar form of the equations of motion of a system of particles is given, for example, in Ref. 15.

The system of equations of motion for the particles contains not only terms with derivatives of higher order, which we have discussed above, but also terms with retarded arguments. Although there exists mathematical literature concerning differential equations with retarded arguments, there is not much known concerning the properties of the systems (10.4) and concerning the effective methods of their

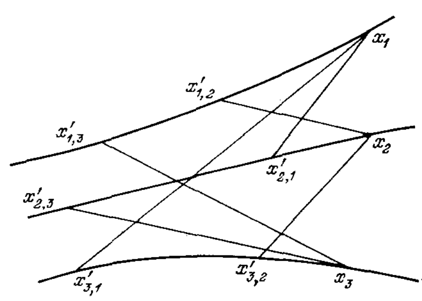

FIG. 1. Coordinates in the case of a retarded interaction between particles.

numerical solution. It is clear only that specification of a finite number of initial data is insufficient for their unique solution, and a specification of the asymptotic properties on the periphery of the field is required.[132,134] However, a case is frequently encountered when at sufficiently great distances between the particles the fields of interaction are considerably weaker than the external field and this makes it possible to specify at such distances the usual initial conditions which determine the asymptotic behavior corresponding to the external field.

The total intensity of the change in the energy, momentum and angular momentum of a system of particles is obtained directly from (10.4) when either the equations of motion have been integrated, or the external field is so strong compared with the other terms of the field (10.5) that the motion of particles can be regarded as given:

$$d\mathcal{P}_i = \frac{1}{c}\sum_a e_a F_{ij}(x_a)\, dx_{ja}, \tag{10.6}$$

$$dM_{ij} = \frac{1}{c}\sum_a e_a \left(x_{ia}F_{jk}(x_a) - x_{ja}F_{ik}(x_a)\right) dx_{ka}. \tag{10.7}$$

The complete field (10.5) appears in (10.6) and (10.7). If the motion of all the particles is known in the same (laboratory) reference system, the time in which we denote by t_0 then we can set $t_a = t_0$ for all a.

Expressions (10.6) and (10.7) coincide up to their sign with the intensities of radiation of corresponding quantities determined (taking retardation into account) by the fluxes in the wave zone, only for systems in which the corresponding fluxes in general do not depend on the time, and in the remaining cases give total losses of energy, momentum and angular momentum which coincide with integrals with respect to time of these fluxes through surfaces infinitely removed from these systems, after integration of (10.6) and (10.7) with respect to time over the whole period of the accelerated motion of the particles or averaging in the case of quasiperiodic motion.

11. SPECTRAL AND ANGULAR DISTRIBUTION OF THE RADIATION FROM A SYSTEM

The concept of spectral distribution of radiation is applicable only to the intensity of radiation averaged over the entire period of motion of the particles or at least over a part of it (if sufficiently high frequencies are being considered) with the angles determining the direction of the radiation being defined in a reference system which is not comoving with the particles. In order to find the spectral and angular distribution of the radiation the potentials and the fields of the particles are expanded into a Fourier integral or a Fourier series at the field point $\mathbf{r}$, t with the aid of harmonics including the field time t which can be assumed to be the same for all the particles of the system and all the directions of the radiation. Therefore the expansion of the retarded potentials of the system of N particles almost literally follows the presentation of the corresponding paragraphs of the book of Ref. 10. In particular, for finite motion of particles with frequency ω_0 the spectrum of the radiation is discrete and the harmonics of the vector potential at large spatial

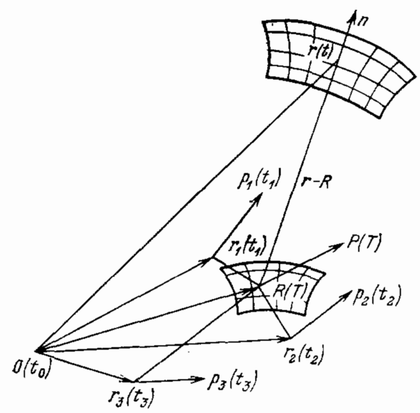

FIG. 2. Radiation events, surface of radiation sources and a distant surface.

distances $\mathcal{R}$ from the system to the field points (Fig. 2) have the asymptotic form

$$\mathbf{A}_n = \frac{\omega_0}{\pi\mathcal{R}} e^{ik\mathcal{R}} \sum_{a=1}^{N} e_a \oint e^{i\omega t - i\mathbf{k}\mathbf{r}_i}\, d\mathbf{r}_a,$$

$$\omega = n\omega_0, \quad \mathbf{k} = n\omega_0\mathbf{n}, \tag{11.1}$$

where n is the number of the harmonic, while $\mathbf{n}$ is the unit vector in the direction of the radiation. We have written the potential (11.1) in the form which is sufficiently accurate when $\mathcal{R} \to \infty$ for finding the fluxes of the energy and the momentum of the field; in finding the flux of the angular momentum it is necessary also to take into account terms of order $\mathcal{R}^{-2}$.

In fact for sufficiently high frequencies in integrals appearing in (11.1) an essential contribution is made only by quite small parts of the trajectories of the particles[132,52] (regions where radiation is being formed).

The angular distribution of the intensity of the nth harmonic of the radiation of energy with respect to axes fixed in the laboratory is obtained in the form

$$dI_n = \lim_{\mathcal{R}\to\infty} \frac{c}{8\pi} \left| [\mathbf{k}\mathbf{A}_n] \right|^2 \mathcal{R}^2\, d\Omega. \tag{11.2}$$

12. ANGULAR DISTRIBUTION OF THE INSTANTANEOUS INTENSITY AND THE TOTAL INTENSITY OF RADIATION BY A SYSTEM OF PARTICLES

In investigating the angular distribution of radiation emitted by a single charge field points are considered with coordinates $\mathbf{r}$, t lying on a sphere of radius $\mathcal{R}$ which is large compared with the length of the waves being studied, and whose center is at the point where the charge is situated at the instant t_0, with the direction towards the element of the sphere being indicated by the unit vector $\mathbf{n}$, and a limiting transition is then made as $\mathcal{R} \to \infty$. By this method, in particular, it has been verified in Ref. 136 that the energy and the momentum of the radiation from a particle constitute a four-dimensional vector and agree with (2.5).

For a system of radiating particles one has to characterize in a relativistically covariant manner the position of the system as a whole at the instant t_0 of laboratory time. The center of events of radiation with coordinates $\mathbf{r}_a$, t_a, which are not only at different points, but also as a rule not simultaneous in the reference system under consideration, and giving rise to a field at the point $\mathbf{r}$, t which lies in the direction $\mathbf{n}$, is[137] the four-dimensional vector X defined in Refs. 138, 139 and having the coordinates

$$\mathbf{R}=\frac{1}{Mc^2}\sum_{a=1}^{N}\Big\{\mathbf{r}_a\left(\mathscr{E}_a-\frac{c^2\mathbf{P}\mathbf{p}_a}{\mathscr{E}+Mc^2}\right)+c\mathbf{P}\mathbf{r}_a\left[\frac{1}{\mathscr{E}+Mc^2}\left(c\mathbf{p}_a+\mathbf{P}\frac{\mathscr{E}_a}{Mc}\right)-\frac{\mathbf{P}}{NMc}\right]-ct_a\left[c\mathbf{p}_a+\frac{\mathbf{P}}{Mc}\left(\frac{c^2\mathbf{P}\mathbf{p}_a}{\mathscr{E}+Mc^2}-\frac{\mathscr{E}}{N}\right)\right]\Big\}, \quad (12.1)$$

$$T=\frac{1}{M^2c^4}\sum_{a=1}^{N}\left[\mathbf{P}\mathbf{r}_a\left(\mathscr{E}_a-\frac{\mathscr{E}}{N}\right)+t_a\left(\frac{\mathscr{E}^2}{N}-c^2\mathbf{P}\mathbf{p}_a\right)\right], \quad (12.2)$$

where $\mathbf{P}=\Sigma_{a=1}^{N}\mathbf{p}_a$, $\mathscr{E}=\Sigma_{a=1}^{N}\mathscr{E}_a$, $Mc^2=\sqrt{\mathscr{E}^2-c^2\mathbf{P}^2}$, $\mathscr{E}_a$, $\mathbf{p}_a$ are the energies and momenta of the particles at the instants of radiation t_a. In Ref. 138 it is shown that the coordinates of the center, just as the coordinates of events, transform according to the representation of the Poincare group as the inertial reference system is changed. The vector X_i is canonically (by means of $8N$-dimensional Poisson brackets) conjugate to the momentum P_i of the system of events. In the nonrelativistic limit if the times of events coincide, $\mathbf{R}$ determines the center of inertia the system of particles. In Ref. 139 it is shown that the determination of the center X for N particles contains N indeterminate constants α_a which are subject only to the condition that their sum should be equal to unity. But in applying to the system of radiating particles the symmetry condition which we shall touch upon below yields uniquely that $\alpha_a = 1/N$, and this has been utilized in (12.1) and (12.2). One can show that the requirements that X and P should be canonically conjugate, that X should be canonically independent of the relative momenta in the system of particles, that a correct nonrelativistic transition is made to the symplectic structure of the usual coordinate-momentum phase space and the symmetry determine the vector X uniquely.[3)] The time of the system as a whole in (12.2) we set equal to the laboratory time t_0.

Let the field point $\mathbf{r}$ at the instant t be separated from $\mathbf{R}$ by a distance $\mathscr{R}$ in the direction $\mathbf{n}$ (Fig. 2), and let the instant t be the same for all the field points. In virtue of (10.2) we have

$$t_a=t-\frac{1}{c}\sqrt{(r-r_a)^2}. \quad (12.3)$$

The spatial coordinates $\mathbf{R}$ of the centers of radiation events in different directions $\mathbf{n}$ need not coincide. In the general case they form a two-dimensional closed shell (the surface of radiation sources) which has at each instant of time t_0 its own shape and lies within the system of particles. In certain cases, for example, for particles moving in one plane this shell shrinks into a double layered area, and in even more particular cases—into a one-dimensional segment of a curve, or more accurately into an infinitely narrow sleeve. In the nonrelativistic limit the shell of the radiation sources shrinks into a point coincident with the position of the center of inertia of the system at the instant t_0. The field points separated by $\mathscr{R}$ from the corresponding points of the surface of radiation sources form a surface which goes over as $\mathscr{R}\to\infty$ into an infinitely removed sphere (portions of the surface of radiation sources and of the infinitely removed surface are shown in Fig. 2).

We consider the sum of the fields (10.3) produced by the charges of the system on the distant surface at the instant t of field time. In order to do this we add and subtract in all the differences $x_i - x_{ia}$ in (10.3) the coordinates X_i, we add the fields at $\mathbf{r}$, t and go to the limit as $\mathscr{R}\to\infty$. As a result of this construction we have $x_i - X_i = \mathscr{R}\mathfrak{n}_i$, $\mathfrak{n}_i = (\mathbf{n}, in_0)$, $\mathbf{n}^2 = 1$, $n_0 = (t-T)/\mathscr{R}$. With the aid of (12.2) we find that

$$\lim_{\mathscr{R}\to\infty} n_0=1, \quad \lim_{\mathscr{R}\to\infty}\mathfrak{n}^2=0. \quad (12.4)$$

The sum of the fields (10.3) contains both terms which decrease as $\mathscr{R}\to\infty$ like $\mathscr{R}^{-1}$ and also terms which fall off more rapidly. In investigating the fluxes of energy and momentum carried off by radiation the last terms are not needed. We then find the principal part $T_{ik}^{(2)}$ of the energy-momentum tensor at the distant surface:

$$T_{ik}^{(2)}=\frac{\mathfrak{n}_i\mathfrak{n}_k}{4\pi\mathscr{R}^2}\sum_{a=1}^{N}\sum_{b=1}^{N}\frac{e_ae_bf_af_b}{(\mathfrak{n}u_a\cdot\mathfrak{n}u_b)^3}, \quad f_{ja}=u_{ja}\mathfrak{n}w_a-w_{ja}\mathfrak{n}u_a; \quad (12.5)$$

here and below $\mathfrak{n}$ denotes the quartet $(\mathbf{n}, i)$.

The flux of energy and momentum across an element of an infinitely distant sphere referred to an element of the solid angle and an element of field time is equal to

$$d\mathscr{F}_i=\lim_{\mathscr{R}\to\infty} c\sum_{\alpha=1}^{3}T_{i\alpha}\mathfrak{n}_\alpha\mathscr{R}^2\,d\Omega\,dt. \quad (12.6)$$

The element dt can be expressed with the aid of (12.2) and the substitution $t_a = t - [\mathbf{n}(\mathbf{r}-\mathbf{r}_a)/c]$ which follows from (10.3) for $\mathscr{R}\to\infty$ in terms of the element of collective time, if we take into account that $dt_a = dt/[1-(\mathbf{n}\mathbf{v}_a/c)]$ follows from (12.3):

$$\begin{aligned} dt=dt_0\Big\{1+\sum_{a=1}^{N}\frac{1}{1-\frac{1}{c}\mathbf{n}\mathbf{v}_a} &\times\Big[\frac{\mathbf{n}\mathbf{v}_a}{cN}+\frac{1}{M^2c^4}\Big(\mathbf{P}\mathbf{v}_a\left(\mathscr{E}_a-\frac{\mathscr{E}}{N}\right)+\mathbf{P}\mathbf{r}_a\dot{\mathscr{E}}_a\\ &+c\mathbf{n}\mathbf{v}_a\left(\frac{\mathbf{P}^2}{N}-\mathbf{P}\mathbf{p}_a\right)-\mathbf{n}\mathbf{r}_ac\mathbf{P}\dot{\mathbf{p}}_a\\ &+\sum_{b=1}^{N}\Big(\dot{\mathbf{p}}_a\mathbf{r}_b\left(\mathscr{E}_b-\frac{\mathscr{E}}{N}\right)+c\mathbf{n}\mathbf{r}_b\left(2\frac{\mathbf{P}\dot{\mathbf{p}}_a}{N}-\mathbf{p}_b\dot{\mathbf{p}}_a\right)\\ &-\frac{1}{N}\dot{\mathscr{E}}_a\mathbf{P}\mathbf{r}_b-2\frac{\mathscr{E}\dot{\mathscr{E}}_a-c^2\mathbf{P}\dot{\mathbf{p}}_a}{M^2c^4}\\ &\times\left(\mathbf{P}\mathbf{r}_b\left(\mathscr{E}_b-\frac{\mathscr{E}}{N}\right)+c\mathbf{n}\mathbf{r}_b\left(\frac{\mathbf{P}^2}{N}-\mathbf{P}\mathbf{p}_b\right)\right)\Big)\Big)\Big]\Big\}^{-1}, \end{aligned} \quad (12.7)$$

where $\dot{\mathscr{E}}_a$ and $\dot{\mathbf{p}}_a$ denote the derivatives of $\dot{\mathscr{E}}_a$ and $\dot{\mathbf{p}}_a$ respectively with respect to the times of radiation events t_a taken at instants t_a. For a single particle we find as usual $dt = dt_0(1-\mathbf{n}\mathbf{v}/c)$.

The total intensity of radiation of energy and momentum per unit laboratory time which corresponds to the state of the system of particles at the instant t_0 can be found by integrating the right hand side of (12.6) over the solid angle:

$$\frac{d\mathscr{P}_i}{dt_0} = \frac{c}{4\pi} \int d\Omega \mathfrak{n}_i \frac{dt}{dt_0} \sum_{a=1}^{N} \sum_{b=1}^{N} \frac{e_a e_b f_a f_b}{(\mathfrak{n}u_a \cdot \mathfrak{n}u_b)^3}. \qquad (12.8)$$

Two of the most important methods of choosing angles specifying the vector **n** consist of the fact that the angles are tied either to the arrangement within which the particles are moving (laboratory angles), or to the motion of the group of particles (angles with respect to the comoving system of axes).

For $N > 1$ the coordinates, velocities, accelerations, momenta and energies of the particles and their derivatives in evaluating the integral (12.8) must be taken at the instants of radiation t_a, with this distinction depending on the direction of the radiation. In virtue of (12.3) as $\mathscr{R} \to \infty$ we obtain a system of $N(N-1)/2$ equations

$$c(t_a - t_b) = \mathbf{n}(\mathbf{r}_a - \mathbf{r}_b), \qquad (12.9)$$

of which only $N-1$ are independent. These $N-1$ equations and the equation $T = t_0$ are the complete system of N equations for finding numerically the $N \times t_a$ for given t_0 and **n**. Substituting the obtained times for each point of the sphere in (12.7) and (12.8) we obtain by numerical integration the total radiation of energy and momentum per unit laboratory time corresponding to the instant t_0.

The radiation from a system of particles found by this method should not depend on the method of numbering them. In order to guarantee this it is necessary that the system of equations for determining the times of radiation events should transform into itself upon a permutation of the numbers of particles. Equations (12.9) indeed satisfy this condition, while the equation $T = t_0$ satisfies this condition only when T is taken in a form symmetric with respect to all the particles and this has already been utilized above.

For a single particle we find from (12.2) $t_1 = t_0$, the shell of radiation sources coincides with the point at which the particle is situated at the instant t_0, the integral over the sphere is easily evaluated in the case of arbitrary motion of the particle and from (12.8) the well known expression

$$d\mathscr{P}_i = \frac{2e^2}{3c} w^2 u_i \, ds, \qquad (12.10)$$

is obtained which agrees with (2.5).

For the flux of angular momentum, taking into account in the energy-momentum tensor terms of order $\mathscr{R}^{-3}$, we obtain

$$\frac{dM_{ik}}{dt_0} = \int d\Omega \frac{dt}{dt_0} \{[(X+\mathfrak{m})_i T^{(2)}_{k\alpha} - (X+\mathfrak{m})_k T^{(2)}_{i\alpha}] \mathscr{R}^2 + (\mathfrak{n}_i T^{(3)}_{k\alpha} - \mathfrak{n}_k T^{(3)}_{i\alpha}) \mathscr{R}^3\} \mathfrak{n}_\alpha, \qquad (12.11)$$

where

$$\mathfrak{n}_i T^{(3)}_{kl} - \mathfrak{n}_k T^{(3)}_{il} = \frac{1}{4\pi\mathscr{R}^3} \sum_{a=1}^{N} \sum_{b=1}^{N} \frac{e_a e_b}{(\mathfrak{n}u_a \cdot \mathfrak{n}u_b)^3} \Big[(\mathfrak{n}_i q_{ka} - \mathfrak{n}_k q_{ia}) f_a f_b \mathfrak{n}_l - (\mathfrak{n}_i f_{ka} - \mathfrak{n}_k f_{ia}) f_b q_a \mathfrak{n}_l - (\mathfrak{n}_i f_{ka} - \mathfrak{n}_k f_{ia}) \mathfrak{n} g_b \mathfrak{n}_l + (\mathfrak{n}_i f_{ka} - \mathfrak{n}_k f_{ia}) f_{lb} \mathfrak{n}(q_a + q_b) - \frac{1}{2} (\mathfrak{n}_i \delta_{kl} - \mathfrak{n}_k \delta_{il}) f_a f_b \cdot \mathfrak{n}(q_a + q_b) \Big],$$

$$g_{ia} = u_{ia} \left(1 + q_a w_a - 3 \frac{\mathfrak{n}w_a \cdot q_a u_a}{\mathfrak{n}u_a}\right) + w_{ia}(3 q_a u_a - q_a w_a),$$

$$q_{ia} = X_i + \mathfrak{m}_i - x_{ia},$$

$$\mathbf{m} = 0, \quad \mathfrak{m}_4 = i \lim_{\mathscr{R} \to \infty} \mathscr{R}(n_0 - 1) = i \left\{ \mathbf{nR} - \frac{1}{M^2 c^4} \sum_{a=1}^{N} \left[c\mathbf{P}\mathbf{r}_a \left(\mathscr{E}_a - \frac{\mathscr{E}}{N}\right) + \mathbf{n}\mathbf{r}_a \left(\frac{\mathscr{E}^2}{N} - c^2 \mathbf{P}\mathbf{p}_a\right)\right]\right\}. \qquad (12.12)$$

For a single particle integration over the sphere yields

$$\frac{dM_{ik}}{ds} = \frac{2e^2}{3c} (u_i w_k - u_k w_i + (x_i u_k - x_k u_i) w^2), \qquad (12.13)$$

which agrees with (2.6).

The coincidence of results obtained from formulas from Sections 10, 11 and 12 (in the case of a complete traversal of the region of the field or averaging in the case of periodic motion) is guaranteed by the laws of conservation of energy, momentum and angular momentum for a system composed of particles and field. We have already seen this agreement in the case of radiation from a single particle, and for a system of particles it is demonstrated on the example given below. In order to clarify the difference in the sense of times appearing in the formulas given above we consider a simplified case when the particles of the radiating system move in an accelerated manner only during a finite time interval Δt_0 according to the laboratory clock, and it is observed with the aid of a device which is situated at a sufficiently great distance $\mathscr{R}$ from the system and encompassing the solid angle $\Delta\Omega$. We assume that during the time Δt_0 the center of the system undergoes only a small displacement. Then the observer will note the presence of radiation during a time interval Δt which sometimes can be considerably shorter than Δt_0. For this case the integral over time over the interval Δt_0 of the fourth component of (10.6) indicates the energy lost by the system, the integral of the fourth component of (12.8) over Δt_0 and $\Delta\Omega$ yields the energy that has passed through the detector (after the lapse of the retardation time $\mathscr{R}/c$) during laboratory time interval Δt_0, while a similar integral of the fourth component of (12.6) indicates an energy that has passed through the detector during the field time interval Δt. The last two energies are close to each other for sufficiently small Δt_0 and $\Delta\Omega$.

13. JOINT SYNCHROTRON RADIATION FROM TWO PARTICLES

We consider the simplest case when two identical particles are moving with the same velocity v along circles of the same radius a, lying in planes parallel to the xy plane and separated from each other along the z axis by an amount h. Assume that one of the circles is shifted with respect to the other along the y axis by an amount b, that the origin of coordinates is taken half way between the centers of the circles, that the particles move in the same direction with one particle being ahead of the other by the angle φ, and the laboratory time is measured from the instant when the middle of the segment joining the positions of the particles crosses the x axis. Radiation from these two particles can be in phase for sufficiently small φ, h/a and b/a. The circle towards which both trajectories of the particles tend as these quantities vanish we shall refer to as the central one.

The radiation from the system at each instant of time t_0 is concentrated around the straight line tangent to the central circle at the point with the angle coordinate $\omega_0 t_0$, $\omega_0 = v/a$. We shall take this straight line as the polar axis of the comoving coordinate system. The system of equations for determining the instants of radiation t_1 and t_2 takes on the form

$$\frac{1}{2}(t_1+t_2)=t_0, \quad \gamma = d+\beta\sin\gamma\cos\theta, \tag{13.1}$$

where

$$\gamma=\beta\frac{c(t_1-t_2)}{2a}+\frac{\varphi}{2},$$

$$d=\frac{1}{2}\left[\varphi+\beta\frac{b}{a}(\sin\omega_0 t_0\sin\theta\cos\psi+\cos\omega_0 t_0\cos\theta)-\beta\frac{h}{a}\sin\theta\sin\psi\right], \tag{13.2}$$

while θ and ψ are the angles specifying the direction $\mathbf{n}$ in the comoving system of axes.

For each t_0 and $\mathbf{n}$ the system (13.1) is solved by convergent successive approximations, starting with $t_1 = t_2 = t_0$. The kinematic factor (12.7) is simplified:

$$\frac{dt}{dt_0}=\frac{[1-(1/c)\,\mathbf{n}\mathbf{v}_1]\,[1-(1/c)\,\mathbf{n}\mathbf{v}_2]}{1-[\mathbf{n}(\mathbf{v}_1+\mathbf{v}_2)/2c]}. \tag{13.3}$$

Then with the aid of (12.8) we find the intensity of radiation

$$\frac{d\mathscr{E}}{dt_0}=\frac{ce^2\beta^4}{4\pi a^2}\int\limits_0^{\pi}\sin\theta\,d\theta\int\limits_0^{2\pi}\frac{d\psi}{E}\left[\frac{A_+}{A_-^3}-(1-\beta^2)\frac{F_+^2A_+}{A_-^5}+\frac{A_-}{A_+^3}-(1-\beta^2)\frac{F_-^2A_-}{A_+^5}\right.$$
$$-2(1-\beta^2\cos2\gamma)\frac{F_+F_-}{A_+^2A_-^2}$$
$$\left.-\frac{2\beta\sin2\gamma}{A_+A_-}\left(\frac{F_+}{A_-}-\frac{F_-}{A_+}\right)+\frac{2\cos2\gamma}{A_+A_-}\right], \tag{13.4}$$

where

$$A_\pm = 1-\beta(\cos\gamma\cos\theta\pm\sin\gamma\sin\theta\cos\psi),$$
$$E = 1-\beta\cos\gamma\cos\theta,$$
$$F_\pm = \cos\gamma\sin\theta\cos\psi\pm\sin\gamma\cos\theta, \tag{13.5}$$

and γ is substituted from (13.1). The surface of radiation sources for this problem turns out to be the segment of the straight line perpendicular to the central circumference at the point with the angular coordinate $\omega_0 t_0$ and lying in its plane. The length of this segment vanishes quadratically for small φ, h/a and b/a.

The difference in the intensity of the joint emission of energy by two particles from the sum of the intensities of two independent particles can be conveniently characterized by the ratio R of the total intensity according to (13.4) to the doubled intensity of radiation from a single particle under the same conditions. In the case of sufficiently small shifts φ, h/a and b/a we obtain with the aid of (13.1), (13.3) and (13.4) the approximate expression

$$R(\varphi,h,b)=2-\frac{5+21\beta^2+9\beta^4}{10}\frac{\varphi^2}{(1-\beta^2)^3}-\frac{\beta^2(2+3\beta^2)}{10}\frac{(h/a)^2}{(1-\beta^2)^2}$$
$$-\frac{\beta^2(7-16\beta^2)}{70}\left(\frac{b}{a}\right)^2\frac{\sin^2\omega_0t_0}{(1-\beta^2)^2}$$
$$-\frac{\beta^2(14+163\beta^2+68\beta^4)}{70}\left(\frac{b}{a}\right)^2\frac{\cos^2\omega_0t_0}{(1-\beta^2)^3}$$
$$-\frac{\beta^2(189+275\beta^2+26\beta^4)}{70}\frac{b\varphi}{a}\frac{\cos\omega_0t_0}{(1-\beta^2)^3}+\dots. \tag{13.6}$$

Numerical calculation using (13.4) agrees with (13.6) and with terms of higher orders not exhibited here which can be obtained, and shows that the ratio $R(\varphi, h, b)$ falls off rapidly from $R = 2$ as φ, h/a and $b\cos\omega_0t_0/a$ increase, crosses the level $R = 1$ and approaches it from below. The point of the most rapid falling off of the curve $R(\varphi, 0, 0)$ lies at $\varphi \sim 0.5\,(1-\beta^2)^{3/2}$, of the curve $R(0, h, 0)$—at $h \sim 0.8 \times (1-\beta^2)$, and of the curve $R(0,0,b)$ at $(t_0 = 0)$—at $b \sim 0.2a(1-\beta^2)^{3/2}$. The wavelength at the maximum of the radiation spectrum is close to twice the distance between the particles in the direction of their motion at which the sharp falling off of the curve $R(\varphi, 0, 0)$ occurs. The size of the region in which the radiation is in phase in the direction h exceeds this wavelength by a factor of $\sim 5/\sqrt{1-\beta^2}$.

Of greatest interest is the case when $\beta^2 > 7/16$ and ω_0t_0 is close to $\pi/2$, or more accurately, $\cos^2\omega_0t_0 < 9(1-\beta^2)/245$. In this case $R(0, 0, b)$ does not fall off but increases from $R = 2$ as b increases from $b = 0$, goes through a maximum, and its smooth falling off occurs (if $\omega_0t_0 = \pi/2$) for $b \sim 25a(1-\beta^2)$. This size of the region where the radiation is in phase exceeds by a factor of $\sim 100/\sqrt{1-\beta^2}$ the wavelength at the maximum of the radiation spectrum. Thus, the region where the radiation is in phase has quite different axes, the greatest of which greatly exceeds the intuitive estimate of the distances between particles for which interference of the emitted waves is possible. The intuitive estimate is of the order of the wavelength of the emitted wave. The curve $R(0, 0, b)$ for $\omega_0t_0 = \pi/2$ is shown in the case $1/\sqrt{1-\beta^2} = 3$ in Fig. 3. It has a maximum $R = 2.44$ for $b = 0.28a$.

In the case $\varphi = h = 0$ it is of interest to examine the time dependence of the ratio R and to compare it with the time

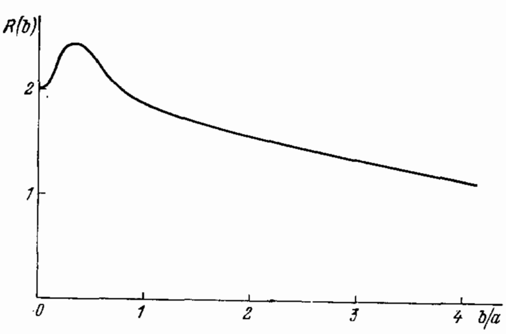

FIG. 3. Total intensity of radiation from a pair of particles as a function of the shift b for $\beta = 2\sqrt{2}/3$.

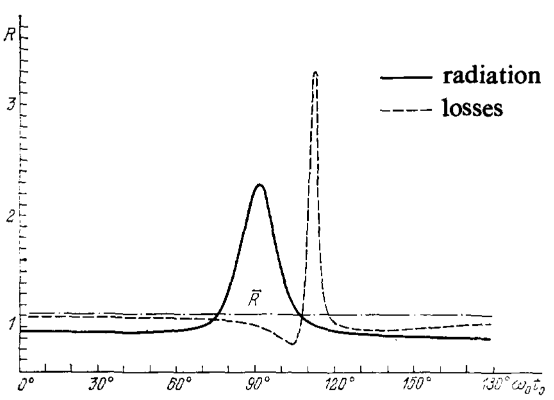

FIG. 4. Intensity of energy losses by a pair of particles (dashed curve) and the intensity of radiation from them (solid curve) as a function of $\omega_0 t_0$ for $b = 0.28$ and $\beta = \sqrt{2}/3$.

dependence of the energy losses, obtained from (10.6) by changing the sign. This comparison is given in Fig. 4 for the same value of β and for $b = 0.28a$. Integration of the curves shown there shows that their averages coincide and are equal to $\overline{R} = 1.041$. If $b = 0$, calculations with the aid of (10.6) give exactly the expansion (13.6) and the curves $R(\varphi, h, 0)$.

Calculation with the aid of (13.4) shows that in the general case the angular distribution of the joint radiation of two particles is broader than the angular distribution of the radiation of a single particle, which has the width of the order of the reciprocal Lorentz factor. But if $\omega_0 t_0$ is close to $\pi/2$ and the shift b is close to the value corresponding to the maximum of the total intensity, the radiation is compressed into a beam which is considerably narrower (in the region of sufficiently small angles), than the radiation from a single particle.

Utilizing the formulas for the spectral decomposition (11.1) and (11.2) we obtain for the problem under consideration the spectral and angular distribution

$$dI_n^{(2)} = 4\cos^2\left[\frac{n}{2}\left(\varphi + \beta\frac{b}{a}\sin\theta_L \sin\psi_L + \beta\frac{h}{a}\cos\theta_L\right)\right] \times dI_n^{(1)}(\theta_L), \qquad (13.7)$$

where n is the number of the harmonic, $dI_n^{(1)}(\theta_L)$ is the spectral and angular distribution of the synchrotron radiation of a single particle which was already found by Schott[46], and the laboratory angles θ_L and ψ_L have been used. If $\varphi \ll \pi/n$, in the sector $\cos\theta_L < \pi a/(n\beta h)$, $\sin\psi_L < \pi a/n\beta b \sin\theta_L$) the radiation spectrum coincides with the radiation spectrum of independent particles, but is multiplied by 4 in place of 2 (by N^2 instead of N for N particles). For N particles such an increase in radiation is possible if all the particles are shifted in one plane from the central circumference along a single straight line, with the maximum of the intensity of the radiation being perpendicular to the indicated straight line, while in the other directions and for other shifts of the trajectories of the particles the high frequency part of the spectrum turns out to be suppressed to the extent that in place of a factor of N^2 once again the factor N appears.

In the case of small φ, h/a and b/a the expression (13.7) can be expanded in powers of these quantities, integrated over the angles and summed over the spectrum. Then we again obtain expression (13.6) in which averaging over time has been made.

On the example discussed above we have demonstrated the agreement between all three possible approaches to the study of radiation (formulas (10.6), (11.2) and (13.4) respectively) and, in particular, the justification for using expression (2.3) for the radiation damping force and for using the concept of the center of a system of events with relativistic particles. This example also shows that the theory of radiation from a system of particles can provide the possibility of foreseeing new phenomena that are manifested in the case of joint radiation by particles and which could be observed and useful.

Investigation of other examples shows that (12.8) enables one to trace in the nonrelativistic limit the transition to multipole radiation from a system of particles.

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[1)]Here and below four-dimensional vectors are taken in the form having imaginary fourth components, and the summation indices in four-dimensional scalar products have been omitted in a number of formulas.

[2)]Consultation with a number of mathematicians and computer scientists produced the consensus that the Russian term "metod progonki" translated here as the "push-through" method has not as yet acquired a generally accepted English equivalent. A brief description of the method is available in Ref. 140, which states that it may be used to solve boundary value problems for ordinary linear differential equations when the "shooting method"[141] or "metod pristrelki"[142] is not effective.

[3)]An attempt to introduce different expressions for **R** and T which are symmetrical with respect to particles, but associated with internal momenta which depend not only on the individual momenta and energies, but also on the masses of the particles, leads in the problem discussed in §14 to a contradiction between the different approaches to the radiation from a system of particles.

[1]B. Stewart, In: Br. Assoc. Adv. Sci. Rep. 41st Meet. Notices and Abstracts, Edinburgh, 1871, p. 45.
[2]H. A. Lorentz, Arch. Neer. Sci. Ex. Nat. **25**, 363 (1892). The Theory of Electrons, Teubner, Leipzig, 1909, p. 49, 253.
[3]M. Abraham, Z. Phys. **14**, 236 (1904); Theorie der Strahlung, Teubner, Leipzig, 1905, Vol. 2, §§13–15.
[4]M. von Laue, Ann. Phys. (Leipzig) **28**, 436 (1909).
[5]M. Planck, Vorlesungen über die Theorie der Wärmestrahlung, J. Barth, Leipzig, 1906, §§104-111.
[6]P. A. M. Dirac, Proc. R. Soc. London **167**, 148 (1938).
[7]H. J. Bhabha, Proc. Indian Acad. Sci. **A10**, 324 (1939).
[8]H. J. Bhabha, Phys. Rev. **70**, 759 (1946).
[9]L. Infeld and P. R. Wallace, Phys. Rev. **57**, 797 (1940).
[10]L. D. Landau and E. M. Lifshitz, Teoriya polya, Gostekhizdat, M., 1941, §§75, 76 [Engl. Transl. of later ed. The Classical Theory of Fields, Addison-Wesley, Reading, Mass., 1951].
[11]C. J. Eliezer and A. W. Mailvaganam, Proc. Cambridge Philos. Soc. **41**, 184 (1945).
[12]A. A. Sokolov, Vestn-Mosk. Univ. No. 2, p. 33 (1947).
[13]D. D. Ivanenko and A. A. Sokolov, Klassicheskaya teoriya polya (Classical Theory of Fields), Gostekhizdat, M., 1948, §§34, 36 [Germ. Transl., Klassische Feldtheorie, Akademie-Verlag, Berlin, 1953].
[14]R. Haag, Z. Naturforsch. **10a**, 752 (1955).
[15]F. Rohrlich, Classical Charged Particles, Addison-Wesley, N. Y., 1965, §§6-6, 6-8.
[16]R. Tabensky, Phys. Rev. **D13**, 267 (1976).

[17]E. Tirapequi, Am. J. Phys. **46**, 634 (1978).
[18]E. G. P. Rowe, Nuovo Cimento **B73**, 226 (1983).
[19]J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).
[20]P. Havas, Phys. Rev. **74**, 456 (1948).
[21]C. Teitelboim, Phys. Rev. **D1**, 1572 (1970), **D3**, 297 (1971), **D4**, 345 (1971).
[22]A. O. Barut, Phys. Rev. **D10**, 3335 (1974).
[23]F. Rohrlich, Ann. Phys. (N.Y.) **13**, 93 (1961).
[24]J. Kalckar and O. Ulfbeck, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. **39**(6), 1(1974).
[25]M. Sorg, Z. Naturforsch. **31a**, 1500 (1976).
[26]A. Belousov, Zh. Eksp. Teor. Fiz. **9**, 658 (1939).
[27]F. Bopp, Ann. Phys. (Leipzig) **42**, 573 (1942/43).
[28]M. Schönberg, Phys. Rev. **67**, 122 (1945).
[29]J. L. Lopes and M. Schönberg, Phys. Rev. **67**, 122 (1945).
[30]C. J. Eliezer, Rev. Mod. Phys. **19**, 147 (1947); Proc. R. Soc. London Ser. A **194**, 543 (1948).
[31]M. A. Markov, Zh. Eksp. Teor. Fiz. **16**, 800 (1946).
[32]K. Wildermuth, Z. Naturforsch. **10a**, 450 (1955).
[33]T. Erber, Fortschr. Phys. **9**, 343 (1961).
[34]T. C. Mo and C. H. Papas, Phys. Rev. **D4**, 3566 (1971).
[35]W. B. Bonnor, Proc. R. Soc. London Ser. A **337**, 591 (1974).
[36]J. C. Herrera, Phys. Rev. **D15**, 453 (1977).
[37]H. Stöckel, Fortschr. Phys. **24**, 417 (1976).
[38]C. Zin, Nuovo Cimento **6**, 1 (1949).
[39]J. Huschilt and W. E. Baylis, Phys. Rev. **D9**, 2479 (1974).
[40]P. A. M. Dirac, Proc. R. Soc. London Ser. A **209**, 291 (1951).
[41]M. Born, Ann. Phys. (Leipzig) **30**, 1 (1909).
[42]M. von Laue, Die Relativitätstheorie, Viewig, Braunschweig, 1919, vol. 1, §20 (1st ed: 1911).
[43]G. A. Schott, Electromagnetic Radiation, Cambridge University Press, 1912, Ch. XI.
[44]G. A. Schott, Philos. Mag. **29**, 49 (1915).
[45]W. Pauli, Relativitätstheorie, In: Encyklopedie der mathematischen Wissenschaften, 1921, Bd. 5_2, Nr. 4, Leipzig, 1920 [Russ. Transl. Teoriya otnositel'nosti, Nauka, M., 1983, p. 136] [Engl. Transl., Theory of Relativity, Pergamon Press, N. Y. 1958].
[46]D. L. Druckey, Phys. Rev. **76**, 543 (1949).
[47]H. Bondi and T. Gold, Proc. R. Soc. London Ser. A **229**, 416 (1953).
[48]T. Fulton and F. Rohrlich, Ann. Phys. (N.Y.) **9**, 499 (1960).
[49]C. Leibovitz and A. Peres, Ann. Phys. (N.Y.) **25**, 400 (1963).
[50]G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).
[51]A. Kovetz and G. E. Tauber, Am. J. Phys. **37**, 382 (1969).
[52]A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **56**, 2035 (1969) [Sov. Phys. JETP **29**, 1093 (1969)].
[53]V. L. Ginzburg, Usp. Fiz. Nauk **98**, 569 (1969) [Sov. Phys. Usp. **12**, 565 (1970)].
[54]W. T. Grandy, Jr., Nuovo Cimento **A65**, 738 (1970).
[55]J. C. Herrera, Nuovo Cimento **B70**, 12 (1970).
[56]A. A. Sokolov and M. M. Kolesnikova, Vestn. Mosk. Univ. Fiz. Astronomiya **12**, 198 (1971).
[57]T. P. Mitchell, J. Chirivella, and J. E. Lingerfelt, Plasma Phys. **13**, 387 (1971).
[58]J. Cohn, Am. J. Phys. **46**, 225 (1978).
[59]P. Candelas and D. W. Sciama, Phys. Rev. **D27**, 1715 (1983).
[60]L. Herrera, Nuovo Cimento **B70**, 12 (1970) (same as Ref. 55).
[61]S. R. Milner, Philos. Mag. **41**, 405 (1921).
[62]D. G. Boulware, Ann. Phys. (N.Y.) **124**, 169 (1980).
[63]W. Wessel, Z. Phys. **92**, 407 (1934).
[64]H. Steinwedel, Fortschr. Phys. **1**, 7 (1953).
[65]La Théorie du rayonnement et les quanta: Solvay Conf. Rep.- Bruxelles, 1911 (Ed.) P. Langevin and M. de Broglie, Gauthier-Villars, Paris, 1912, pp. 42-46.
[66]L. Page, Phys. Rev. **11**, 376 (1918); **12**, 371 (XXXX).
[67]I. Pomeranchuk, J. Phys. USSR **2**, 65 (1940).
[68]A. Loinger, Nuovo Cimento **6**, 360 (1949).
[69]E. Gora, Phys. Rev. **84**, 1119 (1951).
[70]H. Steinwedel, Z. Naturforsch. **7a**, 292 (1952).
[71]F. Rohrlich, Nuovo Cimento **21**, 811 (1961).
[72]T. C. Bradbury, Ann. Phys. (N. Y.) **19**, 323 (1962).
[73]A. A. Sokolov and B. A. Lysov, Phys. Rev. **128**, 2422 (1962).
[74]H. Arzeliès Rayonnement et dynamique du corpuscule chargé fortement accéléré, Gautier-Villars, Paris, 1966, Ch. XII-XIV.
[75]A. A. Sokolov, V. Ch. Zhukovskiĭ, M. M. Kolesnikova, N.S. Nikitina and O. E. Shishanin, Izv. Vyssh. Uchebn. Zaved. Fiz. **12**, No. 2, 108 (1969) [Sov. Phys. J. **12**, No. 2, 218 (1969)].
[76]V. L. Ginzburg, V. N. Sazonov and S. I. Syrovatskiĭ, Usp. Fiz. Nauk **94**, 63 (1969) [Sov. Phys. Usp. **11**, 34 (1968)].
[77]J. M. Cloetens, B. Castillo, E. de Escobar and R. Merlos, Nuovo Cimento **A62**, 247 (1969).
[78]K. S. Cheng, J. Math. Phys. (N.Y.) **19**, 1656 (1978).
[79]M. Sorg, Z. Naturforsch. **33a**, 619 (1978).
[80]N. Arley, Phys. Rev. **71**, 272 (1947).
[81]P. Caldirola, Nuovo Cimento **5**, 99 (1948).
[82]A. Grünbaum and A. I. Janis, Am. J. Phys. **46**, 337 (1978).
[83]J. O. C. Ezeilo, Quart. J. Math. **11**, 64 (1960).
[84]G. N. Plass, Phys. Rev. Lett. **4**, 248 (1960).
[85]J. K. Hale and A. P. Stokes, J. Math. Phys. (N.Y.) **3**, 70 (1962).
[86]J. Huschilt and W. E. Baylis, Phys. Rev. D **13**, 3256 (1976).
[87]W. E. Baylis and J. Huschilt, Phys. Rev. D **13**, 3262 (1976).
[88]W. E. Baylis and J. Huschilt, Phys. Rev. D **13**, 3237 (1976).
[89]P. Caldirola, Nuovo Cimento **3**, Suppl. No. 2, 297 (1956).
[90]O. Bergmann, Acta Phys. Austriaca **13**, 33 (1960).
[91]D. J. Kaup, Phys. Rev. **152**, 1130 (1966).
[92]E. J. Moniz and D. H. Sharp, Phys. Rev. **D10**, 1133 (1974).
[93]H. Levine, E. J. Moniz and D. H. Sharp, Am. J. Phys. **45**, 75 (1977).
[94]W. Maass and J. Petzold, J. Phys. **All**, 1211 (1978).
[95]H. M. Franca, G. C. Margues and A. J. da Silva, Nuovo Cimento **A48**, 65 (1978).
[96]P. Caldirola, Riv. Nuovo Cimento 2(13), 1 (1979).
[97]J. C. Herrera, Phys. Rev. **D21**, 384 (1980).
[98]V. L. Ginzburg, Teoreticheskaya fizika i astrofizika, Nauka, M., 1981 [Engl. Transl. Theoretical Physics and Astrophysics, Pergamon Press, Oxford, 1979].
[99]L. de la Peña, J. L. Jimenéz and R. Montemayor, Nuovo Cimento **B69**, 71 (1982).
[100]P. Petzold, Ann. Phys. (Leipzig) **37**, 425 (1980); **38**, 451 (1981).
[101]W. Heitler, The Quantum Theory of Radiation, Clarendon Press, Oxford, 1954, p. 25-38 (1st ed. 1936) [Russ. Transl. IL, M., 1956, p. 39].
[102]C. Teitelboim, D. Villaroel and Ch. G. van Weert, Riv. Nuovo Cimento 3(9), 1 (1980).
[103]V. L. Ginzburg and S. I. Syrovatskiĭ, Usp. Fiz. Nauk **87**, 65 (1965) [Sov. Phys. Usp. **8**, 674 (1966)].
[104]G. Gibson and E. J. Lauer, Phys. Rev. **117**, 1188 (1960).
[105]G. E. Gernet, Dokl. Akad. Nauk SSSR **168**, 63 (1966) [Sov. Phys. Dokl. **11**, 405 (1966)].
[106]C. S. Shen, Phys. Rev. Lett.. **24**, 410 (1970).
[107]N. D. Sen Gupta, Phys. Lett. **A32**, 103 (1970).
[108]C. S. Shen, Phys. Rev. **D6**, 2736 (1972).
[109]N. D. Lubart, Phys. Rev. **D9**, 2717 (1974).
[110]J. C. Herrera, Phys. Rev. **D10**, 4297 (1974).
[111]A. V. Borisov and Yu. V. Grats, Izv. Vyssh. Uchebn. Zaved. Fiz. No. 3, 87 (1972) [Sov. Phys. J. **15**, 377 (1972)].
[112]C. S. Shen and D. White, Phys. Rev. Lett. **28**, 455 (1972).
[113]V. G. Bagrov, I. M. Ternov and N. I. Fedosov, Dokl. Akad. Nauk SSSR **263**, 1339 (1982) [Sov. Phys. Dokl. **27**, 333 (1982)].
[114]D. V. Gal'tsov and V. A. Losev, Yad. Fiz. **17**, 109 (1973) [Sov. J. Nucl. Phys. **17**, 57 (1973)].
[115]D. V. Gal'tsov and V. A. Losev, Vestn. Mosk. Univ. Fiz. Astronomiya No. 1, 37 (1975).
[116]P. Pascual and L. Mas, Phys. Rev. **D9**, 2482 (1974).
[117]A. A. Sokolov, Zh. Eksp. Teor. Fiz. **18**, 280 (1948).
[118]M. Schwartz, Phys. Rev. **123**, 1903 (1961).
[119]J. L. Anderson and J. W. Ryon, Phys. Rev. **181**, 1765 (1969).
[120]J. E. Gunn and J. P. Ostriker, Astrophys. J. **165**, 523 (1971).
[121]M. Grewing and H. Heintzmann, Phys. Rev. Lett. **28**, 381 (1972).
[122]J. Jaffe, Phys. Rev. **D5**, 2909 (1972).
[123]A. O. Barut and D. Villaroel, J. Phys. **A8**, 156 (1975).
[124]O. A. Barut and D. Villaroel, J. Phys. **A8**, 1537 (1975).
[125]E. A. Jackson, J. Math. Phys. (N.Y.) **25**, 1584 (1984).
[126]N. P. Klepikov, O sile tormozheniya izlucheniem Obzor Radiation Damping Forge: A Review), VINITI, M., No. 4923-82, 1982.
[127]R. E. Kates, Phys. Rev. **D25**, 2487 (1982).
[128]R. E. Kates and A. Rosenblum, Phys. Rev. **D25**, 2494 (1982).
[129]L. Arzimovich and I. Pomeranchuk, J. Phys. USSR **9**, 267 (1945).
[130]V. L. Ginzburg and L. M. Ozernoĭ, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **9**, 221 (1966).
[131]V. V. Zheleznyakov, Astron. Zh. **44**, 42 (1967) [Sov. Astron. **11**, 33 (1967)].
[132]R. D. Driver, Ann. Phys. (N.Y.) **21**, 122 (1963).
[133]R. D. Driver and M. J. Norris, Ann. Phys. (N.Y.) **42**, 347 (1967).
[134]P. Havas, In: Proc. of Intern. School of Physics "Enrico Fermi", Course LXVII (Ed.) J. Ehlers, Bologna, 1979, p. 74.
[135]J. Schwinger, Phys. Rev. **75**, 1912 (1949).

[136]A. Schild, J. Math. Anal. Appl. **1**, 127 (1960).
[137]N. P. Klepikov and S. N. Sokolov, Izluchenie sistemy relyativistskikh chastits (Radiation by a System of Relativistic Particles), Preprint IFVÉ (Institute of High Energy Physics) 84-57, Serpukhov, 1984.
[138]N. P. Klepikov and A. N. Shatniĭ, Yad. Fiz. **31**, 841 (1980) [Sov. J. Nucl. Phys. **31**, 434 (1980)]; Teor. Mat. Fiz. **46**, 50 (1981) [Teor. Math. Phys. (USSR) **46**, 32 (1981)].
[139]M. Datta and N. P. Klepikov, Teor. Mat. Fiz. **48**, 210 (1981) [Teor. Math. Phys. (USSR) **48**, 704 (1981)].
[140]Matematicheskaya éntsiklopediya (Mathematical Encyclopedia), (Ed.) I. M. Vinogradov *et al.*, Sovetskaya Entsiklopediya, M., 1984, Vol. 4, pp. 642-643.
[141]D. M. Young and R. T. Gregory, A Survey of Numerical Mathematics, Addison-Wesley, Reading, Mass., 1973, Vol. II, p. 576.
[142]Ref. 140, p. 640.

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