Kinematic dynamo in random flow

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The growth of a magnetic field in a given random flow of a well-conducting liquid is considered. The known Lagrange solution for the transport of a frozen-in magnetic field is utilized. Magnetic diffusion is taken into account by averaging the result of this transport over a set of random trajectories. This permits a derivation of the equations for the mean magnetic field and its moments, as well as an investigation of the true (random) magnetic field. The field and its moments increase exponentially in the limit of large magnetic Reynolds numbers, and the field distribution becomes intermittent. The analysis is devoted mainly to streams that are restored after a definite time interval, but stationary flows with stochastic properties are also discussed.

CONTENTS

1. INTRODUCTION

The idea of a hydromagnetic dynamo, advanced by Larmor,[1] stemmed from attempts to understand the nature of the earth's and sun's magnetism. Its gist is that motions of a conducting liquid enhance an initially weak magnetic field in the absence of external electromotive forces.

The dynamo idea met first with skepticism. First, consideration of symmetric situations (the usual simplification device) led to negative results. It turned out that neither spherically symmetric nor planar flows produce growing magnetic field.[2,3] Second, it was found that in a well-conducting medium, when the role of motion is particularly significant, the magnetic field is frozen into the medium and no increase in the number of magnetic lines is possible.[4]

Later, however, individual counter examples were devised, which demonstrated that neither of the two objections are general in character. Herzenberg[5] was the first to obtain an undamped solution for flows concentrated on two spheres. Dynamos for axisymmetric flow on cylindrical surfaces were constructed in Refs. 6 and 7. (The known Cowling theorem,[4] that a dynamo for axisymmetric solution is impossible, means only that a growing magnetic field has no axial symmetry.) There are well known examples that generalize these solutions to the case of several rotors[8,9] and toroidal vortices.[10,11] Braginskiĭ[12] constructed a theory for a dynamo based on small deviations from cylindrical symmetry. For a development of this theory see Refs. 13 and 14.

The answer to the second objection to the dynamo concept, concerning the possibility of multiplication of magnetic force lines in a medium with high conductivity, was first given by Alfven.[15] Figure 1, taken from this paper, shows clearly how this can happen. The initial magnetic-field loop is stretched to double its size. Two oppositely directed sections of the field are then pinched together and the initial loop is split into two by the action of the molecular magnetic diffusivity ν_m. After superimposing the two loops obtained, say by a simple shift, we obtain a doubled loop with a diameter equal to the initial one and with double the initial flux through its cross section. Of course, magnetic diffusivity plays the principal role in this mechanism. The characteris-

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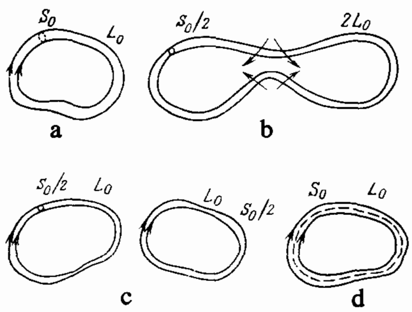

FIG. 1. Magnetic-line multiplication method proposed by Alfven. The division of the loop into two (the transition from b to c) is due to the action of magnetic diffusivity.

tic loop separation time is of the order of $(\Delta L)^2/\nu_m$, where ΔL is the range of approach of the oppositely directed field sections. Clearly, in the limit as $\nu_m \to 0$ (at ΔL determined only by the flow and thus independent of ν_m) the effectiveness of the process tends to zero.

At the 1971 symposium on liquid dynamics in Krakow, Ya. B. Zel'dovich disclosed another perspicuous mechanism (first published in Ref. 16) of amplifying a magnetic field (Fig. 2). While somewhat similar to Alfven's mechanism, it is qualitatively different in character. The similarity lies in the first step — doubling the length of the loop. The twisting into a figure-of-eight and the subsequent combination constitutes qualitatively a three-dimensional operation that leads to doubling of the magnetic flux — this is fundamentally a new aspect. The rate of field increase in no longer dependent on magnetic diffusivity: $H = H_0 \cdot 2^n$, where n is the number of doubling-up operations.

In the limit of small ν_m, the first mechanism (Fig. 1) can be naturally called a slow dynamo, and the second (Fig. 2) a fast one. We note that in the second mechanism the magnetic diffusivity should play a role in the formation of the spatial form (in the topology-changing loop breaking) but not in the field growth rate. The concepts of slow and fast dynamo were formulated in Ref. 17, although the effectiveness of a dynamo such as in Fig. 2 was noted back in Ref. 16. An exponentially growing solution in a specified conducting flow is called a fast dynamo if the field growth rate γ does not decrease as $\nu_m \to 0$,[1)] and slow if $\gamma \to 0$ as $\nu_m \to 0$. It became subsequently clear that solutions of one more intermediate type are possible, whereby the growth rate of an individual mode of the field is positive only in a finite range of variation of ν_m and becomes negative as $\nu_m \to 0$.

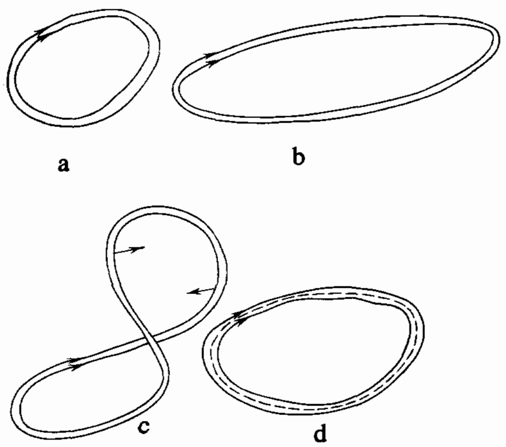

FIG. 2. Ya. B. Zel'dovich's figure-of-eight — the fast-dynamo mechanism. The magnetic-flux growth rate is independent of magnetic diffusivity.

A concrete solution of this type was constructed for a periodic three-dimensional flow.[18] The distinguishing feature of this flow is that besides the "laminar" sections it contains regions, each of which is filled everywhere densely (stochastically) by an individual stream line.

The solutions noted above, of the type given in Refs. 5–11, belong to the slow-dynamo class. Their common feature is they represent stationary $\mathbf{v} = \mathbf{v}(\mathbf{x})$ flows over surfaces (of spheres, a cylinder, a torus, etc.). Solutions of this type are possible also for periodic velocity fields (v_x, v_y, v_z) that depend on two coordinates.[19] Incidentally, such flows are topologically equivalent to flows over surfaces.

A rapid dynamo is realized by a nonstationary velocity field $\mathbf{v}(t, \mathbf{x})$. This can be intuitively perceived with the figure-of-eight (Fig. 2) as the example, although the question of the feasibility of a fast dynamo in a stationary flow is still moot. We emphasize that a nonstationary flow can produce also a slow dynamo. This is Backus's known example[21] with the velocity field turned off, inasmuch as the duration of intervals with $\mathbf{v} = 0$ is determined in it by the magnetic diffusivity ν_m (all but the first harmonic should attenuate).

A fast dynamo is naturally realized by random turbulent velocity fields that are usually observed in nature. It is customary to consider velocity fields with stationary statistical characteristics. In such flows it is possible to simplify the dynamo problem considerably. This was first illustrated by Parker[22] as well as in the well-known papers of Steinbeck, Krause, Rädler and their followers (see, e.g., Ref. 23 and 24) with the dynamo problem for the average magnetic field in a reflectionally-asymmetric turbulent flow, and by Kazantsev[25] with the problem of generation of the second moment (correlation function) in a reflectionally-symmetric short-correlated flow. According to contemporary notions both are examples of a fast dynamo. In a fast dynamo the field growth rate does not depend on the magnetic diffusivity in the limit as $\nu_m \to 0$, so that one can start out with the explicit Lagrangian solution of the induction equation. The Lagrangian approach to the description of large-scale magnetic fields was developed by Kraichnan[26] and applied in Refs. 27 and 28 to the problem of a turbulent dynamo in a reflectionally-invariant medium. The main problem in this approach is the next step — correct allowance for the small magnetic diffusivity. A naive approach consisting of adding a customary diffusivity term [such as $\nu_m \Delta(\mathbf{H})$] to the equations for the average field or of the second moment, is not justified *a priori* and is generally speaking incorrect if for no other reason than that in the general case these equations are integral, despite the fact that the initial induction equation is differential (cf. Ref. 28). A consistent account for the magnetic induction within the framework of the Lagrangian approach can be given by considering not one Lagrangian trajectory but a bundle of random Wiener trajectories, followed by averaging over them, as is done in the theory of Brownian motion.[38,46] This approach to the problem of a kinematic dynamo in random flow was developed by us recently and is the main content of the present review. It includes naturally, with some generalization, results concerning the average-field and the mean-square dynamo. The important fact is that conclusions can be drawn concerning the behavior of all the moments and of the true magnetic field in random flow. It became clear that in the limit of low magnetic diffusivity the growth rate increases with increasing number of the moment. Therefore a study of the second moment (see, e.g., Refs. 27 and 28) was not enough to solve the problem posed by Batchelor,[29] that of generation of small-scale magnetic fields.

The structure of the magnetic field generated in a random stream is intermittent in the limit of small ν_m, i.e., the field distribution has high and widely spaced peaks. This is proved formally by the aforementioned behavior of the growth rates of the field moments. Intermittency was first observed in numerical experiments for two-dimensional MHD turbulence[30,31] and in direct integration of the three-dimensional simultaneous induction and Navier-Stokes equations for an incompressible conducting fluid driven by a random short-correlated Gaussian force.[32] Altogether, a computer experiment is of great importance for the theory of the hydromagnetic dynamo. It touches upon practically all the applications of this theory to cosmic magnetic fields. We shall not dwell in the present review on applications, which are the subject of a recent monograph by Zel'dovich *et al.*[20] (see also the earlier monographs by Moffatt,[23] Parker,[33] and Krause and Rädler[24]). We note only new work on magnetic-field generation, performed mainly via successful computer experiments.

Ivanova and Ruzmaĭkin[34] studied the excitation conditions and found the oscillation periods of a three-dimensional mean magnetic field as applied to large-scale solar fields, the three-dimensionality of which manifests itself in a number of observable phenomena (sector structure, coronal holes, etc.). Numerical investigations of the behavior of the correlation function of a field in an isotropic mirror-symmetric random stream are reported in Refs. 35 and 36. We note also computer experiments on the behavior of magnetic fields in stationary periodic (nonrandom) flows,[18,27] using the maximum capabilities of modern computers.

Another useful approach, besides computer methods, in applications of dynamo theory is the use of asymptotic methods of solving moment equations (see Sec. 5).

Speaking of applications, it must be emphasized that the hydrodynamic dynamo is not the only mechanism of magnetic-field generation under cosmic (and all the more so under laboratory) conditions. Under certain conditions, for example in peculiar A-stars and neutron stars, the principal role may be played by thermal e.m.f.'s.[39–42] On the role of thermal e.m.f.'s and other mechanisms in the generation of magnetic fields under laboratory conditions see, e.g., Ref. 43. Under consideration are also magnetic-field-generation mechanisms that combine the actions of hydrodynamic motions and external electromotive forces.[44]

2. GENERAL SOLUTION OF THE CAUCHY PROBLEM FOR THE INDUCTION EQUATION

a) Freezing-in. The most popular concept in magnetohydrodynamics of a highly conducting fluid is freezing-in. In one sense it means, as in Kelvin's known hydrodynamic theorem, conservation of the magnetic flux through any contour that moves together with an ideally conducting fluid. In another sense it is said that the magnetic field is deformed in the same way as a linear element that joins two infinitesimally close particles of the fluid.

One other aspect of the freezing-in concept is of importance to us. The induction equation in an ideally conducting incompressible fluid

$$\frac{dH_i}{dt}=H_k\nabla_k v_i,\qquad \frac{d}{dt}=\frac{\partial}{\partial t}+v_k\nabla_k\qquad (i,\ k=1,\ 2,\ 3)\quad (2.1)$$

is equivalent to field transport along the Lagrangian trajectory of a liquid particle. In fact, we shall solve this equation by successive approximations.[33] Let $\xi=\mathbf{x}(t=0)$ be the initial position of the fluid particle. After a time Δt we have then

$$H_i(\xi+\Delta x,\ \Delta t)=H_i(\xi)+\frac{dH_i}{dt}\Delta t$$
$$=\frac{\partial(\xi_i+v_i(\xi)\Delta t)}{\partial\xi_j}H_j(\xi)=\frac{\partial x_i(\xi,\ \Delta t)}{\partial\xi_j}H_j(\xi).$$

Repetition of this procedure leads to the equation

$$H_i(\mathrm{x},\ t)=H_i(\mathrm{x}(\xi,\ n\Delta t),\ n\Delta t)$$
$$=\frac{\partial x_i(\xi,\ n\Delta t)}{\partial x_k(\xi,\ (n-1)\Delta t)}\cdots\frac{\partial x_l(\xi,\ 2\Delta t)}{\partial x_m(\xi,\ \Delta t)}\frac{\partial x_m(\xi,\ \Delta t)}{\partial\xi_j}H_j(\xi)$$
$$=\frac{\partial x_i(\xi,\ t)}{\partial\xi_j}H_j(\xi).\qquad (2.2)$$

Thus, to determine the magnetic field under freezing-in conditions it suffices to know only the trajectory

$$x_i(\xi,\ t)=\xi_i+\int_0^t v_i(x,\ s)\,ds,\qquad (2.3)$$

which is determined completely by specifying the velocity field. We have written the trajectory in the form $\mathbf{x}=\mathbf{x}(\xi,t)$. It can be written in inverted form:

$$\xi_i=x_i-\int_0^t v_i(\xi,\ t-s)\,ds.\qquad (2.4)$$

To obtain this equation it suffices to carry out the procedure of deriving (2.2) backward in time.

b) Multiplicative integral. From the second line of Eq. (2.2) we see that the transport of the magnetic field along the trajectory in a finite time interval can be represented as the product of transports over infinitesimally small segments of the trajectory. In the limit as $\Delta t\to 0$ we obtain an infinite product or, in mathematical language, a multiplicative integral (see, e.g., Ref. 45 concerning this concept)

$$G_{ij}(t,\xi,\mathbf{x})=\prod_{s=0}^{t}\left(\delta_{ij}-\frac{\partial v_i(\xi,\,t-s)}{\partial x_j}\,ds\right)$$

$$=\prod_{s=0}^{t}\left(\delta_{ij}+\frac{\partial v_i(x,\,s)}{\partial x_j}\,ds\right)=\frac{\partial x_i}{\partial \xi_j}, \tag{2.5}$$

which shows explicitly how the transport (Green's) function is connected with the velocity field; here

$$H_i\ (\mathbf{x},\ t)\ =G_{ij}H_j\ (\xi). \tag{2.6}$$

For greater clarity, we write down the differential equation obtained for G_{ij} by substituting (2.6) into (2.1):

$$\frac{dG_{ij}}{ds}=-G_{ej}\,(t,\,s,\,\mathbf{x},\,\xi_s)\,\frac{\partial v_i\,(t-s,\,\xi_s)}{\partial x_l}. \tag{2.7}$$

The solution of this equation can be formally written in the form of a matrix exponential equivalent to the multiplicative integral (2.5).

c) Wiener trajectories. The Lagrangian solution obtained in subsection (a) is remarkable in that it can be generalized to include the case when magnetic diffusivity takes part in the transport of the magnetic field, i.e., when the field satisfies the complete induction equation

$$\frac{dH_i}{dt}=H_k\nabla_k v_i+\nu_m\Delta H_i,\quad \nabla_i H_i=0. \tag{2.8}$$

It is known from the theory of Brownian motion that the diffusion process can be described as the average of the motion over a bundle of random trajectories (see, e.g., Refs. 38 and 46). The coordinates $\mathbf{w}_t$ of a random trajectory are at each instant of time Gaussian random quantities with a mean square deviation proportional to $t^{1/2}$ (under the condition $\mathbf{w}_{t-0}=0$) and with independent increments in time. Therefore the random trajectories that generalize (2.4) can be defined by the equation

$$\xi_t=\mathbf{x}-\int_0^t v\,(t-s,\,\xi_s)\,ds+\sqrt{2\nu_m}\,\mathbf{w}_t, \tag{2.9}$$

where $\mathbf{w}_t$ is a random (Wiener) process having the properties

$$M\mathbf{w}_t=0,\quad M\,(w_{ti}w_{tj})=\delta_{ij}t. \tag{2.10}$$

The symbol M denotes averaging over the $\mathbf{w}_t$ distribution. Equation (2.9) describes an aggregate of random trajectories that arrive at the point $\mathbf{x}$ under consideration by the instant of time t (Fig. 3). We emphasize that the initial coordinate ξ_t, in contrast to $\mathbf{x}$, is a random quantity. Therefore (2.9) does not admit of an inversion $\mathbf{x}=\mathbf{x}(\xi,t)$ similar to (2.3). Moreover, the trajectory (2.9) is not differentiable, inasmuch as $\Delta\xi$ as $t\to 0$ is proportional not to Δt but to $\sqrt{\Delta t}$. It is said that the random trajectory has a derivative of order $\frac{1}{2}$.

The transport of the magnetic field along one random trajectory is equivalent to the solution (2.6) of Eq. (2.1) without the Laplacian. The true solution is obtained by averaging over all the random trajectories that arrive at a given point x at the instant t:

$$H_i\,(\mathbf{x},\ t)=M_{\mathbf{x}}G_{ij}\,(t,\,\xi_t,\,\mathbf{x})\,H_{0j}'(\xi_t). \tag{2.11}$$

In the mathematical sense the symbol $M^{\mathbf{x}}$ means integrating

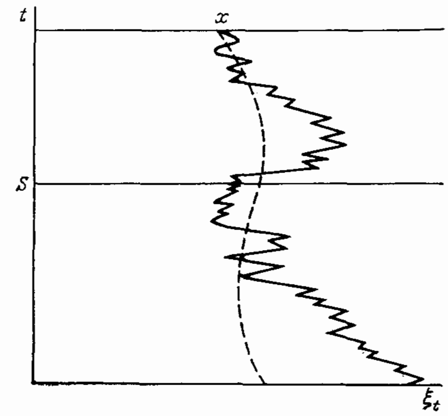

FIG. 3. In an ideally conducting medium, the field is transported along a Lagrangian trajectory (dashed line). Allowance for magnetic diffusivity is equivalent to consideration of a set of random trajectories that reach the point x at the instant t (one of them is shown by the sinuous line) followed by averaging the result of the transport over all the trajectories.

over a Wiener measure, in which the integration is with respect to $\mathbf{w}$, and the trajectories converge to the point $\mathbf{x}$. This general solution of the Cauchy problem for the induction equation (2.8) is verified by substitution in the equation and by using (2.7) and (2.10).

To demonstrate more clearly that such a pseudo-Lagrangian solution satisfies an equation with a Laplacian, it is convenient to digress from the vector properties of the field and consider the heat-conduction equation in a moving medium:

$$\frac{\partial T}{\partial t}+(\mathbf{v}\nabla)\,T=\varkappa\Delta T. \tag{2.12}$$

The general solution of this scalar equation has the simple form:

$$T\,(\mathbf{x},\ t)=M_{\mathbf{x}}T_0\,(\xi_t). \tag{2.13}$$

We shall show that (2.13) is indeed the solution of (2.12). The temperature at the instant $t+\Delta t$, by virtue of the independence of the increments of the Wiener process, is expressed in terms of the temperature at the instant t by the same equation:

$$T\,(\mathbf{x},\ t+\Delta t)=M_{\mathbf{x}}T\,(\xi_{\Delta t},\ t)$$

$$=M_{\mathbf{x}}\left[T\,(\mathbf{x},\ t)+\frac{\partial T}{\partial x_i}\,(\xi_{\Delta t}-x)_i+\frac{1}{2}\,\frac{\partial^2 T}{\partial x_i\,\partial x_j}\,(\xi_{\Delta t}-x)_i\,(\xi_{\Delta t}-x)_j\right]. \tag{2.14}$$

To retain the terms of order Δt we had, by virtue of (2.10) to take into account the terms with the second derivatives in the Taylor expansion of the right-hand side of the solution. It is here that account is taken of the properties of the random trajectory which is not differentiable, but has only a derivative of degree $\frac{1}{2}$. It suffices now to use the fact that $(\xi_\Delta - x)_i = v_i\Delta t+\sqrt{2\varkappa w_{\Delta ti}}$, and the properties of (2.10), and to take the limit as $t\to 0$. A contribution in first order in t is made to the second term of the expansion (2.14) only by $v_i\Delta t$, and to the third term only by the terms

$M(w_{\Delta ti} w_{\Delta tj}) = \delta_{ij} \Delta t.$

Thus, the considered method of describing magnetic diffusivity is mathematically corroborated; for details see Refs. 38 and 46. It is similar to the known Feynman path-integral method in quantum mechanics. Its advantage and convenience when applied to the problem of the hydromagnetic dynamo lie in the use of an explicit Lagrangian solution of the induction equation. It remains to average this solution over an aggregate of random trajectories with a known statistical weight. Of course, in the general case path integration is also a complicated task. However, in the case of small ν_m, which is of greatest interest and closest to the Lagrangian approach, and when deriving the moment equations, this method yields results that are difficult to obtain (at least technically) by other methods.

It is appropriate to note that a random trajectory (unlike a Lagrangian one) and transport of a field along it are a fiction. Of physical meaning is the field averaged over the entire aggregate of trajectories. The averaging makes the situation non-invariant to time reversal, as should be the case when diffusion is taken into account.

3. PRELUDE TO DYNAMO. LINEAR VELOCITY FIELD

A conducting-fluid stream moving at constant velocity cannot alter a magnetic field, by virtue of the Galilean invariance of magnetohydrodynamics. The simplest velocity field in which the kinetic energy can be converted into magnetic is a linear field $v_i = c_{ik} x_k$, where $i, k = 1,2,3$ and $c_{ik}(t)$ is a coordinate-independent tensor. Such a velocity distribution is well known in cosmology as Hubble's law, which conserves homogeneity in the sense that an observer moving with the matter sees at any point an identical picture of the departure (or approach) of surrounding particles.[47]

A linear velocity field is usually understood as a local approximation to a smooth general-type velocity field. It is used in hydrodynamics in this sense to describe small-scale turbulence.[48-50] This approximation seems even more attractive in the kinematic-dynamo problem,[51-53] since a magnetic field, unlike a vortex, is not connected by a definite relation with the velocity. Consideration of magnetic-field transport in a linear velocity field is useful in many respects. First, this problem admits of an exact solution for a matrix $c_{ik}(t)$ of general form, particularly a random one.[54] The results are obtained for the true (and not averaged) magnetic field. It is easy to track the application of the concepts of the multiplicative integral and of the product of a large number of independent random matrices. Second, the result turns out to be instructive from the physical viewpoint. Any magnetic field that decreases rapidly enough at infinity attenuates exponentially, and its magnetic energy increases exponentially because of the very rapid growth of the size of the field-occupied region. (A similar effect was first indicated in hydrodynamics by Pearson[55].) This is evidence that the local approximation cannot give the final answer to the question of enhancement of a magnetic field as $t \to \infty$. It is necessary in principle to know the form of the velocity field at large scales.

We present below a qualitative analysis of the behavior of a magnetic field in a linear velocity field. See Ref. 54 for a more general and rigorous treatment.

The infinitesimally small vector δx_i that connects two close liquid particles obeys in a linear field the equation

$$\frac{d\delta x_i}{dt} = c_{ik} \delta x_k. \tag{3.1}$$

Assume for simplicity that the matrix c_{ik} is constant and diagonal, with elements $c_1 > c_2 > c_3 \neq 0$, and the liquid is incompressible, div $\mathbf{v} = 0$, i.e.,

$$c_1 + c_2 + c_3 = 0. \tag{3.2}$$

Clearly, at least one of these three constants is positive, so that the corresponding component of the vector δx_i grows exponentially. Under freezing-in conditions the corresponding magnetic-field component initially oriented along the vector δx_i will grow similarly. It turns out, however, that allowance for even arbitrarily small but finite diffusion leads to the opposite result. A field that decreases rapidly enough at infinity (but more slowly than $|\mathbf{x}|^{-3}$) attenuates as $t \to \infty$.[2])

It is convenient to carry out the analysis in Fourier space. A distinction must then be made between two c_{ik} matrices:

a) $c_1 > 0$, $0 > c_2 > c_3$ (*stretching into a filament*). In this case the wave vectors k_2 and k_3 increase like $k_{02}\exp(|c_2|t_3)$ and $k_{03}\exp(|c|t)$, and the vector k_1 decreases exponentially. It follows hence that almost all the Fourier harmonics of the magnetic field attenuate sharply like

$$\exp\left(-\nu_m \int_0^t k^2(s)\, ds\right),$$

i.e., like an exponential in the exponent. Exceptions are harmonics with $k_0 = 0$ and harmonics from an exponentially narrow cone having an axis k_{01} (Fig. 4a), for which

$$\nu_m \int_0^t k^2\, ds = O(1).$$

The cone has an elliptic cross section with semi-axes proportional to $\exp(-|c_2|t)$ and $\exp(-|c_3|t)$. Since the field is solenoidal ($H_k \cdot \mathbf{k} = 0$), the directions of the harmonics with the wave vectors of the cone are almost perpendicular to the cone axis or, more accurately, form an "orthogonal

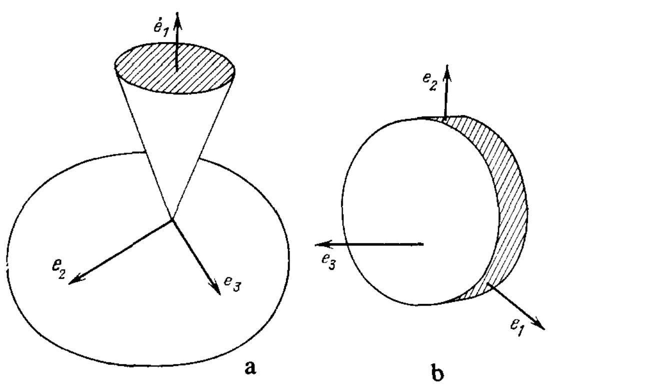

FIG. 4. Cone in Fourier space, within which the magnetic-field harmonics increase exponentially, and outside of which they attenuate like an exponential in the exponent. a) $c_2 < 0$; b) $c_2 > 0$.

cone." The projection of the harmonics of the orthogonal cone on the k_{01} axis is of the order of $k_{01}/k_{02} \sim \exp(-|c_2|t)$, so that the harmonics increase like $\exp(c_1 t)\cdot\exp(-|c_2|t)$. The first factor is due to the stretching of the field along the k_{01} axis.

The magnetic field in x-space can be estimated as the product of the amplitude of the growing harmonic and the volume of the cone, the latter being proportional to $\exp[-(|c_2| + |c_3|)t]$. As a result, the field attenuates as

$$\mathbf{H}(t, \mathbf{x}) \sim \int \mathbf{H}_\mathbf{k}\, d^3k_0 \sim \exp(-|c_2|t). \qquad (3.3)$$

The region occupied by the magnetic field, however, increases exponentially because of the stretching along the first axis. The total magnetic energy increases therefore:

$$\int \mathbf{H}^2\, d^3x \sim \exp[(c_1 - 2|c_2|)t] = \exp[(|c_3| - |c_2|)t], \qquad (3.4)$$

since $|c_3| > |c_2|$.

It is curious that the magnetic diffusivity does not enter in the answer. It played, however, a principal role, since it eliminated practically all the field harmonics. The finite diffusivity stabilizes the scale of the field along the axes 2 and 3 ($\propto \nu_m^{1/2}$) and does not prevent stretching along the first axis, making the field filamentlike. The reader can easily verify that if we put initially $\nu_m = 0$ we obtain an increasing field $H_1 \propto \exp(c_1 t)$, i.e., that it is not allowed to reverse the sequence of the transitions $t \to \infty$ and $\nu_m \to 0$.

b) $c_1 < c_2 > 0, 0 > c_3$ (*flattening into a pancake*). In this case the cone is made up of wave vectors close to the (k_{01}, k_{02}) plane, see Fig. 4b. The cone is defined by the condition

$$\nu_m \int_0^t k^2\, ds \sim \nu_m |c_3|^{-1} k_{03} \exp(|c_3|t) = O(1).$$

Its aperture decreases therefore like $\exp(-|c_3|t)$. The harmonic having the maximum growth rate is directed along the first axis ($\sim \exp(c_1 t)$), and its wave vector is close to the third axis. The field decreases consequently like

$$\mathbf{H}(t, \mathbf{x}) \sim \exp(c_1 t) \exp(-|c_3|t) = \exp(-c_2 t). \qquad (3.5)$$

Its distribution flattens into a pancake of thickness $\sim \nu_m$. To estimate the total magnetic energy it is necessary to multiply $H^2(t,\mathbf{x})$ by the volume occupied by the field. Since the stretching is now in the (k_{01}, k_{02}) plane the volume increases like $\exp[(c_1 + c_2)t]$, and the total energy like $\exp[(|c_3| - 2c_2)t] = \exp[(c_1 - c_2)t]$.

We note the degenerate case $c_2 = 0$, which corresponds to planar flow $(v_x, v_z), v_y = 0$. In this case the magnetic field becomes stabilized at an exponential growth of its total energy. This result does not contradict the theorem that forbids planar flow,[3,17,23] since it is implied in this theorem that the velocity field does not increase at infinity. In the case considered by us the component H_y in view of the condition $v_y = 0$, is damped exponentially independently of the remaining components. This damped field, however, serves as an undamped source for the two-dimensional field (H_x, H_z) in view of the exponential growth of the region occupied by H_y.

We consider for simplicity a diagonal matrix c_{ik}. Let us assess the role of the off-diagonal terms. The usual reasoning is the following. A matrix of general form can be resolved into antisymmetric and symmetric parts. The former is eliminated by transforming to a reference frame that rotates with constant angular velocity $e_{ikl}c_{kl}/2$(e_{ikl} is a unit pseudotensor). The remaining symmetric matrix is reduced to diagonal form with time-independent eigenvalues. The principal axes of this matrix, however, have a complicated, albeit periodic, time dependence. The resultant problem therefore appears to be very difficult.

Actually, the off-diagonal terms are not so dangerous. We illustrate this using the simple example of Couette flow, where only c_{12} differs from zero. Diagonalization of this matrix by transforming to a rotating coordinate frame leads to a matrix with two equal and opposite eigenvalues. Owing to the periodic transformation of the axes into each other, the exponential stretchings of the field will be periodically accompanied by exponential flattenings. In the absence of magnetic diffusivity the field will increase with time linearly. Therefore allowance for finite ν_m leads to an exponential damping of the field. For a rigorous treatment see Ref. 54.

4. THE KINEMATIC DYNAMO PROBLEM

a) Statement of problem. We shall attempt to define the kinematic dynamo and classify the possible dynamo solutions. Without striving for formalized statements, we shall describe the principal aspects and some fine points of this problem.

In a liquid at rest with a magnetic diffusivity ν_m an initial magnetic field $\mathbf{H}_0(\mathbf{x})$ having a characteristic scale L is dissipated after a time of the order of L^2/ν_m. The motion of the liquid leads to an inductive effect $\mathbf{v} \times \mathbf{H}$ capable of amplifying the magnetic field by converting kinetic energy into magnetic. Roughly speaking, the kinematic-dynamo problem consists of answering the following question: for what specified velocity fields does the inductive action prevail over dissipation, or at least offsets its action? Formally this involves obtaining for the induction equation (2.8), with initial condition $\mathbf{H}(0,\mathbf{x}) = \mathbf{H}_0(\mathbf{x})$, solutions that are not damped as $t \to \infty$. We shall consider for simplicity the flow of an incompressible fluid and regard the magnetic diffusivity as a constant scalar. The consequences of disregarding these assumptions will be discussed in Sec. (c).

This formulation of the problem, however is incomplete. It is necessary first of all to formulate the boundary conditions. When the stream is specified in a bounded region, say in a star surrounded by a medium having another diffusivity coefficient (in particular, by a vacuum), it is natural that in the absence of permeability discontinuities the magnetic field must be continuous on the boundary of the region. For flow in an unbounded region, the boundary conditions are imposed on the function $\mathbf{H}_0(\mathbf{x})$. This field is usually specified by the distribution of currents concentrated in a bounded region. Then

$$\mathbf{H}_0(\mathbf{x}) = O(|\mathbf{x}|^{-3}), \quad |\mathbf{x}| \to \infty. \qquad (4.1)$$

Moreover, the distribution of the currents that produce the initial field must have a finite magnetic moment proportional to the integral $\int \mathbf{H}_0\, d^3 x$, taken in the sense of the principal value.

In the preceding section, with a linear velocity field as the example, we pointed out the importance of this boundary condition: if the magnetic field decreases more slowly than in (4.1), the conclusions can be qualitatively different even if the magnetic energy is finite. Consideration of the linear velocity field has also taught us that it is not enough to formulate the dynamo problem merely in the form of the question whether magnetic energy can increase without limit (cf. Ref. 23, §6.1). When the magnetic energy increases the magnetic field can also decrease if the region occupied by the field increases.

Inasmuch as in a liquid at rest the magnetic diffusivity acts on the field "exponentially," interest attaches in the dynamo problem to solutions that grow with time exponentially[3)] (or more rapidly).

In a stationary velocity field $\mathbf{v}(\mathbf{x})$ it is natural to speak of an eigenvalue problem. When the velocity field is concentrated in a finite region surrounded by vacuum, the spectrum is discrete. For an unbounded region, the spectrum of a damped field is obviously continuous when $\mathbf{v} = 0$. The presence of the flow, however, can make the spectrum discrete (in analogy with the action of the potential in the Schrödinger equation).

In a nonstationary velocity field, one of course cannot speak of an eigenvalue problem. If the field is periodic in time, however, the quasienergy concept[56] can be used, i.e., we can consider solutions whose shape repeats in every velocity period, and whose spectrum is discrete. In another case of practical importance, that of a stochastic flow with stationary characteristics, one can pose an eigenvalue problem for the average magnetic moment and its higher moments.

One can consider also random flows whose statistical characteristics are not stationary, but repeat after a finite time. For the average field characteristics it is natural here, too, to pose a quasi-eigenvalue problem. But one can raise the question of the growth of the true field rather than the averaged one, by considering the Cauchy problem with an initial field. This formulation of the problem is meaningful, inasmuch as for long times the magnetic field will vary, with unity probability, at a rate determined by the leading Lyapunov exponent.

It can thus be stated that in a large set of flows, the magnetic field varies asymptotically as $\exp(\gamma t)$, where the growth rate γ can be, depending on the type of flow, an eigenvalue, a quasi-eigenvalue, or a Lyapunov exponent. Generally speaking, the growth rate is a complex number. An illustrative example of a periodic magnetic field corresponding to imaginary γ is the solar cycle.[22–24,33,20]

b) Classification of dynamos. The growth rate is a function of the magnetic diffusivity, or more accurately, when expressed in units of v/l, it depends on the dimensionless magnetic Reynolds number $\mathrm{Re}_m = lv/\nu_m$, where l and v are the characteristic scale and amplitude of the velocity field. For simplicity we shall hereafter regard ν_m as the reciprocal of the magnetic Reynolds number. Obviously, at large ν_m, when magnetic diffusion prevails over the action of the velocity field, we have $\mathrm{Re}\,\gamma < 0$. If a dynamo is possible, then $\mathrm{Re}\,\gamma$ vanishes at a certain $(\nu_m)_{\text{crit}}$ and becomes positive with further decrease of the magnetic diffusivity. The behavior of the magnetic-field growth rate in the vicinity $(\nu_m)_{\text{crit}}$ depends substantially on the actual form of the velocity field.[23] As $\nu_m \to 0$, however, one can speak of certain general regularities.[17,20] Furthermore, the case of small ν_m is a feature of most applications of the hydromagnetic-dynamo theory (e.g., $\nu_m \sim 10^{-8}$ in the convective shells of the sun and of stars similar to it, $\sim 6\cdot 10^{-3}$ in the liquid core of the earth, and $1.2\cdot 10^{-2}$ in the "Superphoenix" power plant under construction in a number of European countries).

Depending on the form of velocity field, $\mathrm{Re}\,\gamma$ can tend as $\nu_m \to 0$ to a positive value independent of ν_m (fast dynamo); after going through a positive maximum, it can become again negative (intermediate dynamo), and finally tend to zero while remaining positive (slow dynamo). For clarity, these three types of solution are qualitatively shown in Fig. 5.

This classification stems from the premises concerning the possible character of the flow of the conducting fluid. A fast dynamo is possible in a general nonstationary three-dimensional velocity field. A clear example is the Zel'dovich figure-of-eight cited in the Introduction (Fig. 2). Included in this type are the known turbulent flows with medium or zero vorticity.[23–28,20,36] It will be shown in Sec. 6 that a fast dynamo is always realized at sufficiently low magnetic diffusivity in a uniform random stream that becomes replenished within a finite time. Of course, in a spatially bounded stream that occupies too small a volume, $\mathrm{Re}\,\gamma$ may possibly not become positive. We have marked this case (fast dissipation) by the dashed curves in Fig. 5. A flow that produces a fast dynamo features a combination of topological complexity (stochasticity of instantaneous trajectories, i.e., the absence of integral surfaces of the velocity field) with a nonstationary character (e.g., in the case of a dynamo with medium vorticity we have $\mathrm{Re}\,\gamma \propto \alpha^2$, where $\alpha \sim \tau(\mathbf{v}\cdot\text{curl}\,\mathbf{v})$, and τ is the velocity field correlation time). Our review is in fact devoted mainly to the fast dynamo.

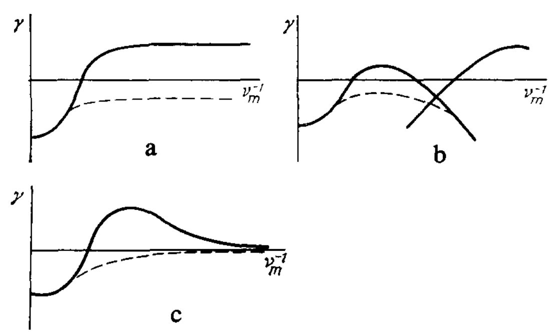

FIG. 5. Typical dependences of the field growth rate on the magnetic Reynolds number for the cases of fast (a), intermediate (b) and slow (c) dynamo. The dashed curves mark the situation when the moving fluid occupies too small a volume for the field to become self-excited.

An intermediate dynamo is possible in a stationary three-dimensional stream in which exponentially running off trajectories are present. An example of such a dynamo is given in Ref. 38. A solution with the decrement shown in Fig. 5b was obtained in Ref. 57 (superfast dissipation); see Sec. 7. Actually the behavior of $\gamma(\nu_m)$ for an intermediate dynamo is more complicated. With increasing magnetic Reynolds number the first mode attenuates rapidly, but is replaced by the second, with smaller scale but with larger amplitude and with a wider section in which $\mathrm{Re}\,\gamma > 0$. A hypothesis can be advanced that alternating modes with $\mathrm{Re}\,\gamma > 0$ appear and approach as $\nu_m \to 0$ the growth rate typical of a fast dynamo.

A slow dynamo can be produced by two-dimensional flows. Alfven's illustrative example was shown in the Introduction (see Fig. 1), although it is not so simple, in view of the fact that it is nonstationary (the flows are shut off in one plane and turned on in another). A large class of laminar flows over stationary surfaces, which are of the slow-dynamo type, is mentioned in the Introduction. The trajectories of these flows, apart from special cases such as the Hubble flow with v increasing towards infinity, can run off only in power-law fashion, since they coincide with the level lines of the stream function. The reason why the magnetic field can grow exponentially is that the magnetic diffusivity of the vector field is capable of interlinking different field components. This is not the case for surfaces with isotropic normal curvature, i.e., for a plane or a sphere.[17] A dynamo is therefore impossible for planar and spherical flows (the exclusion theorems); the corresponding $\mathrm{Re}\,\gamma$ is shown in Fig. 5c.

The question of the character of the dynamo in nonstationary flows on two-dimensional surfaces other than a plane and a sphere remains uninvestigated. A concrete example of such a dynamo may be nonstationary (say, "turbulent") flow of a conducting fluid over a cylindrical surface.

So far we have spoken of solutions that increase exponentially. Are there any solutions that grow more rapidly, particularly such in which the magnetic field or the energy becomes infinite after a finite time? Solutions of explosive type $H \sim (t - t_0)^{-p}$, $p > 0$, are usually associated with nonlinear effects, since they satisfy an equation of the type $\partial H / \partial t \sim H^{(1+p)/p}$. Under certain conditions, however, such solutions are possible also for an induction equation linear in the magnetic field. They stem from singularities of the velocity field, for example from an unlimited decrease of its spatial scale. When these singularities are eliminated the explosive solutions no longer occur. In fact, multiplying the induction equation by H_i and integrating over the entire volume we obtain the inequality

$$\frac{\partial \mathscr{E}}{\partial t} \leqslant \mathscr{E} \max_{\mathbf{x}} \left(\frac{\partial v_k}{\partial x_i} \right),$$

where $\mathscr{E}$ is the magnetic energy, and the maximum is taken over all the points of space and over all the components of the velocity deformation tensor. It is clear therefore that a faster than exponential growth of the magnetic energy is possible only in the case when the tensor $\partial v_k / \partial x_i$ is unbounded, i.e., when the velocity is finite but its characteristic scale decreases without limit. The last situation seems possible in principle when the kinematic viscosity tends to zero. A sequence of skin-effect layers with thicknesses that decrease without limit are produced then in the velocity field. The construction of such a solution of the induction equation is procedurally of interest, but is apparently technically complicated. An explosive-type solution was constructed in Ref. 58 for a moderate-field equation with locally homogeneous and isotropic, but specularly asymmetric turbulence. Such a solution is obtained when the average helicity of the turbulence increases at infinity, but the initial magnetic field is present in the entire space and decreases at infinity like exp $(-|\mathbf{x}|^2)$. Analogous solutions are known in quantum mechanics for some potentials, say those that lead to falling to the center.

c) Remarks on inhomogeneous and anisotropic magnetic diffusivity. Compressible-fluid flows. In media with inhomogeneous and (or) anisotropic magnetic diffusivity, the conditions for generating a magnetic field may change in favor of possible self-excitation of the field. In particular, some of the exclusion theorems can be lifted. This is clear even from the fact that inhomogeneous and anisotropic electric conductivities can simulate the windings of dynamo machines.

A characteristic and frequently encountered example of inhomogeneous magnetic diffusivity is a bounded volume of a conducting fluid (e.g., a star or the core of a planet) surrounded by vacuum or by a medium having a different conductivity. Allowance for the inhomogeneity of ν_m is known to reduce in this case to formulation of boundary conditions on the interface between the media. The question of the extent to which allowance for the inhomogeneities of the magnetic diffusivity weakens the exclusion theorems for plane and spherical flows or allows generation of fields that are independent of one of the coordinates, has not yet been resolved in general form for motion over surfaces. It is known that a dependence of ν_m on only one coordinate is insufficient, but examples of a dynamo for ν_m that depends on two coordinates have been proposed. An exclusion theorem was proved in Ref. 60 for a stationary velocity field that satisfies the condition $\nabla \times [\nu_m^{-1} \mathbf{v}(\mathbf{x})] = 0$, i.e., for a special Beltrami-type flow, $\nabla \times \mathbf{v} = \mathbf{v} \nabla \nu_m / \nu_m$.

The generation conditions are substantially altered by the assumption of magnetic diffusivity.[61] Ohm's law in an isotropic medium at rest takes the form[62]

$$E_j = \nu_{jl}\, \mathrm{rot}_l\, \mathbf{H}, \tag{4.2}$$

where ν_{jl} is the magnetic diffusivity tensor (the reciprocal of the conductivity tensor), which we shall assume to be constant and independent of the magnetic field. If the anisotropy of the medium is determined by one polar vector q_j, this tensor is symmetric and can be represented in the form

$$\nu_{jl} = \nu_0 \delta_{jl} + \nu_1 q_j q_l. \tag{4.3}$$

Allowance for the regular motion reduces to addition of a term $-\mathbf{v} \times \mathbf{H}/c$ to the right-hand side of Ohm's law. In the equation for the mean field there appear in a random stream additional contributions proportional to the symmetric part of the tensor of the first derivatives,[23,24] in the form

$e_{jkm} q_m q_l\, (\partial H_l/\partial x_k + \partial H_k/\partial x_l)$.

In the case of molecular diffusion such a term is excluded because it does not guarantee a positive growth rate of the entropy. In the hydrodynamic situation (kinematic dynamo) these arguments are untenable because of the presence of e.m.f.'s due to the interaction of the velocity and the magnetic field and causing conversion of kinetic energy into magnetic.

It is shown in Ref. 61 that the induction equation with an anisotropic magnetic-diffusivity tensor (4.2) has at $\nu_1 \ll \nu_0$ growing solutions that depend on one of two coordinates, $v_y = \Omega z$ in the simplest planar Couette flow. For example, a magnetic field initially localized as $\exp(-x^2)$ when $t \to \infty$, spreading in accordance with the diffusion law with a diffusivity coefficient ν_0, increases at a rate $\gamma = (\nu_1/4\nu_0)\,\Omega\, q_y q_z$. The condition that γ be positive reduces to the requirement that the vector q_j not be parallel or perpendicular to the flow direction, the latter coinciding with the symmetry axis y, as well as to the condition that the signs of ν_1 and Ω be the same. Such a solution can be compared[61] with the known unipolar dynamo,[23] although the presence of the winding and of the edge of the disk makes the latter inhomogeneous.

The greater part of the present review is devoted to the dynamo produced by flows of an incompressible conducting fluid. For stationary flows in the absence of an external force field, the incompressibility condition is formulated as smallness of the hydrodynamic velocities v compared with the speed of sound c_s. For a fluid in nonstationary flow to be incompressible it suffices that the time l/c_s needed by an acoustic signal to travel a distance over which the velocity changes noticeably be short compared with the characteristic velocity-change time τ. Under astronomical conditions the influence of gravitational fields is important. For incompressibility to obtain in the stationary situation it suffices to have the characteristic velocity-change scale much smaller than the scale of density change $\rho/\nabla\rho$ (altitude scale). This condition is usually not satisfied in convective shells of stars. Compressibility must also be taken into account when nonstationary astrophysical conditions are considered, such as the compression of interstellar clouds in the course of star creation.

The induction equation in compressible flow

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{v}\nabla)\mathbf{H} = (\mathbf{H}\nabla)\mathbf{v} - \mathbf{H}\nabla\mathbf{v} + \nu_m \nabla^2 \mathbf{H} \tag{4.4}$$

contains an additional term $-\mathbf{H}\nabla\mathbf{v}$. Clearly, this term influences the dynamo substantially when it is of constant sign, i.e., under nonstationary compression and explosion conditions, or in the presence of sources and sinks. There have been few studies of dynamos under these conditions. We note that in the case of practical interest, star creation, the compression lasts a finite time and leads to enhancement of the magnetic field (according to $H \propto \rho^{2/3}$ in the trivial spherically symmetric case; a more detailed analysis and references to work in this field can be found in Chap. 14 of Ref. 20), which can be regarded as an independent process that supplements the dynamo action.

Equation (4.4) can also be studied by methods under development in this paper. For application, interest attaches to the problem of the magnetic field in the so-called acoustic turbulence,[63,64,73] when terms of order of $(v/c_s)^2$ cannot be neglected.

5. SHORT-CORRELATED RANDOM FLOW

The construction of a fast dynamo in a nonstationary three-dimensional laminar flow is a problem incredibly difficult to investigate even numerically. Fortunately, consideration of nonstationary random flows of practical interest greatly simplifies the problem, since even velocity fields with simple statistical characteristics can operate as fast dynamos. We consider in this section the simplest random flow with short (δ-like) time correlations, first applied to the dynamo problem by A. P. Kazantsev.[25] It is convenient to conceive of such a velocity field as the limit of velocities $v^\Delta(t,\mathbf{x})$ that are constant in t over intervals of lengths Δt:$(0, \Delta t)$, $(\Delta t, 2\Delta t)$, . . . and are independent in each different interval. In the limit as $\Delta t \to 0$ we have

$$\langle v_i(t, \mathbf{x})\, v_j(t', \mathbf{y})\rangle = 2\frac{l}{v}\,\delta(t-t')\, V_{ij}(\mathbf{x}, \mathbf{y}),$$

where l and v are the characteristic scale and velocity, while the angle brackets denote averaging over the velocity-field distribution. At small Δt, the velocity v^Δ is thus of the order of $v(l/v\,\Delta t)^{1/2} \sim (\Delta t)^{-1/2}$. For the sake of simplicity we shall assume everywhere in this Section, except in Subsec. (b), that there is no average velocity, $\langle v_i(t,\mathbf{x})\rangle = 0$, and omit the dimensional factor $2l/v$.

Let us study the behavior of the average characteristics of the magnetic field.

a) Equation for the average magnetic field. An equation for the average magnetic field in a turbulized stream was first proposed by Steinbeck, Krause, and Rädler and then corroborated by a number of workers who used, in the case of large magnetic Reynolds numbers of interest to us, a number of restrictive assumptions, e.g., that the velocity field is Gaussian or two-scaled (see, e.g., Refs. 23, 24, 65 and the references therein). The existence of growing solutions of this equation is due to the fact that the flow is on the average vortical.

The authors of Ref. 66 proposed a simple and rigorous derivation of the equations for the average fields and for other moments under the sole assumption that the velocity field is δ-correlated. Of particular importance is the dropping of the assumption of a two-scale velocity field. Although a differential equation for $\langle \mathbf{H}\rangle$ is indeed obtained in a δ-correlated flow, allowance for the finite correlation time in the case when the velocity field is not two-scale leads to an integral equation; see Subsec. (d).

The idea underlying the derivation of an equation for the mean field is the following. Specifying the magnetic field at the instant t, we obtain the field after a short time interval $t + \Delta t$ by using Eq. (2.11)

$$H_i(t+\Delta t, \mathbf{x}) = M_{\mathbf{x}}\,[G_{ij}(\Delta t, \xi_{\Delta t}, \mathbf{x})\, H_j(t, \xi_{\Delta t})]. \tag{5.1}$$

We average it over the velocity field, and change the order of the averaging in the right-hand side of (5.1), $\langle M_{\mathbf{x}}\rangle \to M_{\mathbf{x}}\langle\ \rangle$. The $\langle G_{ij} H_j\rangle$ correlation can be calculated by breaking up the operation of averaging over the velocity field into two stages. We first average over the interval from 0 to t, when only the field H_j that depends on the prior history is averaged. The averaged field is smooth and can, in analogy with (2.14), be expanded in a Taylor series in which only terms of order not higher than Δt are retained, thereby retaining also the term

$$\frac{1}{2}\frac{\partial^2 \langle H_j\rangle}{\partial x_k \partial x_l}(\xi_{\Delta t}-x)_k(\xi_{\Delta t}-x)_l.$$

With the same accuracy we have from (2.9) and (2.5), recognizing that $v^{\Delta}\propto(\Delta t)^{-1/2}$,

$$(\xi_{\Delta t}-x)_k=\sqrt{2\nu_m}\,w_k+v_k(t,\mathbf{x})\,\Delta t+\sqrt{2\nu_m}\frac{\partial v_k}{\partial x_l}\int\limits_0^{\Delta t}w_l\,dt+\frac{1}{2}v_l\frac{\partial v_k}{\partial x_l}(\Delta t)^2,$$

$$G_{ij}(\Delta t,\xi_{\Delta t})=\delta_{ij}-\frac{\partial v_i(t,\mathbf{x})}{\partial x_j}\Delta t-\sqrt{2\nu_m}\frac{\partial^2 v_i}{\partial x_j\partial x_l}\int\limits_0^{\Delta t}w_l\,dt+\frac{(\Delta t)^2}{2}\frac{\partial}{\partial x_k}\left(v_i\frac{\partial v_k}{\partial x_j}-v_k\frac{\partial v_i}{\partial x_j}\right). \quad (5.2)$$

We now average $G_{ij}\langle H_j\rangle$ over the velocity field in the interval from t to $t+\Delta t$. Owing to the short memory time, correlations of the type

$$\left\langle v_k\frac{\partial v_i}{\partial x_j}\cdot\frac{\partial\langle H_j\rangle}{\partial x_k}\right\rangle=\left\langle v_k\frac{\partial v_i}{\partial x_j}\right\rangle\frac{\partial\langle H_j\rangle}{\partial x_k}.$$

are split off. Averaging finally over the Wiener trajectories with allowance for Eqs. (2.10), denoting the total average by $B_i(T,\mathbf{x})$, and taking the limit as $\Delta t\to 0$, we obtain the sought equation

$$\frac{\partial B_i(t,\mathbf{x})}{\partial t}=-\left\langle v_k\frac{\partial v_i}{\partial x_j}\right\rangle\frac{\partial B_j}{\partial x_k}+\frac{1}{2}\frac{\partial}{\partial x_k}\left\langle v_l\frac{\partial v_k}{\partial x_j}-v_k\frac{\partial v_i}{\partial x_j}\right\rangle B_j+\frac{\partial}{\partial x_j}\left[\left(\nu_m\delta_{kj}+\frac{1}{2}\langle v_k v_j\rangle\right)\frac{\partial B_i}{\partial x_k}\right], \quad (5.3)$$

where $\langle\ \rangle$ denotes the spatial part of the velocity-field correlator in coinciding points of space. When converting to dimensional units it must be multiplied by $2l/v$.

It is interesting that in the approximation considered there is no linkage of the different field components. Such a linkage appears, however, when account is taken of the finite correlation time.

Equation (5.3) is valid in the general inhomogeneous anisotropic case, the only assumption of importance being that the velocity-field correlations be instantaneous. In homogeneous turbulence we have $\partial\langle v_l v_k\rangle/\partial x_l=0$ and the equation takes the simpler form

$$\frac{\partial B_i}{\partial t}=\left\langle v_l\frac{\partial v_k}{\partial x_l}\right\rangle\frac{\partial B_l}{\partial x_k}+\left(\nu_m\delta_{kl}+\frac{1}{2}\langle v_k v_l\rangle\right)\frac{\partial^2 B_i}{\partial x_k\partial x_l}.$$

The constant three-index tensor preceding the first derivative is antisymmetric in the indices i and k. It can therefore be represented in the form

$$\left\langle v_i\frac{\partial v_k}{\partial x_l}\right\rangle=e_{ikm}\alpha_{lm}-\delta_{il}q_k+\delta_{kl}q_i. \quad (5.4)$$

Multiplying this equation by e_{ikm} and using the incompressibility and homogeneity of the flow we can verify that $q_i=0$ and α_{lm} is a symmetric tensor

$$\alpha_{ik}=\frac{1}{2}\langle v_i\omega_k+v_k\omega_i-\delta_{ik}v_l\omega_l\rangle,\quad \omega=\text{curl }\mathbf{v},$$

whose trace is equal to $\pm\alpha_{ii}=-\frac{1}{2}\langle\mathbf{v}\cdot\text{curl }\mathbf{v}\rangle$. By virtue of the symmetry, α_{ik} can always be reduced to diagonal form. It can also be seen that generation of the mean field is possible also at zero average vorticity, when $\langle v_i\omega_k\rangle\neq 0$.[24,66]

The polar vector q_j does not vanish in the weakly inhomogeneous case, when one can still speak of the representation (5.4). It enters then in the mean-field equation in the form curl $[\mathbf{q}\times\mathbf{B}]$, i.e., it describes transport of the field with effective velocity q_j (the diamagnetic effect[3,16,24]).

The most frequently used in astrophysical applications is an isotropic weakly inhomogeneous approximation wherein, when account is taken of the average velocity $\mathbf{V}$, the equation for the mean field takes the form

$$\frac{\partial B}{\partial t}=\text{curl }[\mathbf{V}\times\mathbf{B}]+\text{curl }\alpha\mathbf{B}-\text{curl}\left(\beta\,\text{curl}\frac{\mathbf{B}}{\mu}\right), \quad (5.5)$$

where α in dimensional form is equal to $-(\tau/3)\langle\mathbf{v}\cdot\text{curl }\mathbf{v}\rangle$, β includes the molecular and turbulent magnetic diffusivity, while μ is the turbulent magnetic permeability (whose gradient is equal to q_j). Of course, the assumption of a δ-correlated random velocity-field component (or the assumption of a two-scale velocity field, which leads to the same equation), is not satisfied under real conditions. It will be shown in Subsection (d), however, that the solutions of Eq. (5.5) are close to the solutions of the more rigorous integral equations that take into account flow with a finite correlation time.

b) Asymptotic solutions. Assuming for simplicity that $\mu=1$ and $\beta=$ const, we can rewrite (5.5) in the form

$$\frac{\partial B}{\partial t}=R_\omega\text{curl }[\mathbf{V}\times\mathbf{B}]+R_\alpha\text{ curl }\alpha\mathbf{B}+\Delta\mathbf{B}, \quad (5.6)$$

where

$$R_\omega=\frac{\Omega_0 L^2}{\beta},\quad R_\alpha=\frac{\alpha_0 L}{\beta}$$

are dimensionless numbers that characterize the field-generation sources. Equation (5.6) is much simpler than the initial induction equation, since it admits of growing solutions with high symmetry. In the simplest case when $R_\omega=0$ and $\alpha=$ const the solution takes the form

$$\mathbf{B}=\mathbf{B}_0\exp(\bar{\gamma}t+i\mathbf{k}\mathbf{x}),\quad \bar{\gamma}=\pm R_\alpha k-k^2. \quad (5.7)$$

The spatial part of the growing solution is represented in a Cartesian coordinate frame by a certain combination of six linearly independent helical vectors of the type (sin kz, cos kz, 0), (cos kz, $-$ sin kz, 0) etc., with cyclic permutation of x, y, and z. The generated field is then of the form of a right-hand ($R_\alpha>0$) or a left-hand ($R_\alpha<0$) standing helical wave. The generation in unbounded space has no threshold, i.e., it occurs at all ν_m, and with increasing ν_m only the scale of the generated field increases.[4)] The fastest to grow is the mode having the wave vector $k_0=R_\alpha/2$ and $\bar{\gamma}_{max}=R_\alpha^2/4$.

It should be noted that in the approximation $\alpha=$ const the quantity R_α has a Pickwickian sense, since there is no characteristic scale L in the problem (it is better to speak simply of α and β). Physically meaningful is the problem with a weakly inhomogeneous function $\alpha(\mathbf{x})$, where L is the characteristic scale of variation of the average vorticity. We have then, in order of magnitude, $R_\alpha \sim \varepsilon L/l$. Since $\alpha \sim \varepsilon v$, $\varepsilon \lesssim 1$; $\beta \sim lv$ (l is the characteristic energy-carrying scale of the velocity field). The scale $Lk_0^{-1} = 2L/R_\alpha \sim l\varepsilon^{-1}$ of the average magnetic field lies between l and L. The case of maximum vorticity $\varepsilon \sim 1$ is not dangerous, see Sec. (a) (cf. Moffatt,[23] §9.2).

At large R_α one can construct, using V. P. Maslov's elaboration of the WKB method, an asymptotic solution similar to the solution with constant average vorticity[67] and having the form

$$\mathbf{B}(\mathbf{x}, t) = [\varphi_0(\mathbf{x}) + R_\alpha^{-1}\varphi_1(\mathbf{x}) + \ldots] \exp[\bar{\gamma}t + iR_\alpha S(\mathbf{x})],$$
$$\bar{\gamma} = \gamma_2 R_\alpha^2 + \gamma_1 R_\alpha + \gamma_0 + \ldots . \quad (5.8)$$

The field configuration in the leading order is determined by the functions φ_0 and S. The maximum growth rate is determined by γ_2. To estimate the excitation threshold $(R_\alpha)_{\rm crit}$ at which $\mathrm{Re}\,\gamma = 0$, we must know also γ_1. To determine these quantities (5.8) must be substituted into (5.6).

The maximum growth rate for the function $\alpha(\mathbf{x})$ with a maximum at an isolated point is equal to

$$\bar{\gamma}_{\max} = \frac{1}{4} R_\alpha^2 - \frac{5}{2} R_\alpha + O(1), \quad (5.9)$$

where $(R_\alpha)_{\rm crit} = 10$. With changing character of the extremum (concentration of α on lines and planes) the excitation threshold is lowered.[67] An excitation threshold of the same order of magnitude was obtained earlier in a number of numerical computations with various $\alpha(\mathbf{x})$.[68,24]

At $R_\omega \neq 0$ the operator of the right-hand part of the initial equation (6.6) is not self-conjugate (for $R_\omega = 0$ it was self-conjugate in the leading orders of R_α^2 and R_α). The eigenfunctions will therefore be complex. In the approximation $|R_\omega| \gg R_\alpha^2$ ($\alpha\omega$ dynamo) it is customary to characterize the problem by the dimensionless dynamo number $D \equiv R_\alpha \cdot R_\omega$.

This approximation is usually valid in convective shells of stars, particularly in the solar convective zone. In a thin shell the solution takes the form of a dynamo wave

$$\mathbf{B} \sim \exp\left(\bar{\gamma}t + i\omega t - ikx - iqz - \frac{pz^2}{2}\right)$$

(the z axis is perpendicular to the shell). The mean-field excitation threshold can be estimated from the condition $k_0 R \gtrsim 1$. Where $k_0^{-1} \sim D^{-1/3}$ is the characteristic scale of the generated field and R is the radius of the star (such that the field can fit into the star). We obtain[69]

$$D \gg \left(\frac{4\pi}{R}\right)^3 \sqrt{\left|\alpha \frac{\partial \Omega}{\partial z}\right|_{\max}},$$

where Ω is the angular velocity of the star rotation. The period of the dynamo wave is $\approx 10 D^{-2/3} (\alpha \partial\Omega/\partial z)_{\max}^{-2/3}$. These asymptotic relations are valid for sufficiently uniform distributions of the sources α and $\partial\Omega/\partial z$ over the thickness of the convective shell. For concentrated and nonoverlapping sources, the period is proportional to $\ln^{-1} D$.[70]

The foregoing estimates demonstrate the capabilities of the asymptotic methods. A detailed model-dependent and numerical investigation of dynamo waves and their use to explain solar and stellar activity can be found in a number of papers and monographs.[71,72,23,24,33]

c) Study of second moment. The equation for the equal-time correlator tensor $\mathscr{H}_{ij}(t,\mathbf{x},\mathbf{y}) \equiv \langle H_i(t,\mathbf{x}) H_j(t,\mathbf{y})\rangle$ is not as easy to derive in the general inhomogeneous anisotropic case as the equation for the mean magnetic field.[66] The non-equal-time correlation can be obtained in this case if $\mathscr{H}_{ij}$ and the Green's function of the mean-field problem are known.

This tensor equation has been analyzed so far only in a homogeneous isotropic mirror invariant velocity field whose correlation tensor is of the form[50]

$$\langle v_i(\mathbf{x}) v_j(\mathbf{y})\rangle = \frac{lv}{3}\left[F(r)\delta_{ij} + \frac{r}{2}\frac{dF}{dr}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right],$$
$$r_i = x_i - y_i, \quad i, j = 1, 2, 3, \quad (5.10)$$

where we have separated the dimensional factor

$$l = \int_0^\infty F(r)\,dr$$

which is the correlation length. The longitudinal correlation function $F(r)$ is dimensionless, is equal to unity at zero, has a positive Fourier transform, and decreases at infinity.

It is natural to assume that the fastest growing solution has the same symmetry

$$\mathscr{H}_{ij}(r, t) = \frac{1}{3}\left[W(r, t)\delta_{ij} + \frac{r}{2}\frac{dW}{dr}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right], \quad (5.11)$$

although the question of growing solutions of lower symmetry has remained uninvestigated.

The determination of the longitudinal correlation function $W(r,t)$ reduces, as shown back in 1967 by A. P. Kazantsev,[25] to solution of an equation similar to the Schrödinger equation with variable mass, but without the imaginary unity in front of the derivative with respect to time

$$\frac{1}{2}\frac{\partial \psi}{\partial t} = \frac{1}{2m(r)}\frac{\partial^2 \psi}{\partial r^2} - U(r)\psi, \quad (5.12)$$

where

$$\psi = \frac{r^2}{3\sqrt{2m}} W, \quad \frac{1}{2m} = \nu_m + \frac{1 - F(r)}{3},$$
$$U(r) = \frac{1}{6r}\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr^3 F}{dr}\right) + \frac{1}{mr^2} - \frac{1}{8m^3}\left(\frac{dm}{dr}\right)^2.$$

The solution is sought in the form $\psi(r,t) = \psi(r)\exp(2\gamma^{(2)}t)$, after which the problem reduces to finding the eigenvalues and the eigenfunctions $\psi(r)$ in a potential $U(r)$, with zero boundary conditions at zero and at infinity. The existence of growing solutions is indicated in Refs. 25, 27, and 28. The problem was investigated in detail in Ref. 36 for a number of typical forms of $F(r)$.

The potential for the longitudinal velocity correlation that depends exponentially on r^2 is of the form shown in Fig. 6a. At large ν_m and as $U(r) \to 2\nu_m/r^2$ there are also no growing solutions (bound states). With decreasing ν_m a well appears and is followed, after some critical $(\nu_m)_{\rm crit} \sim 1/50$, by a level, i.e., self-excitation of the field takes place. With further decrease of ν_m the potential tends to a limiting shape of the well, which is separated by barriers from zero and infinity.

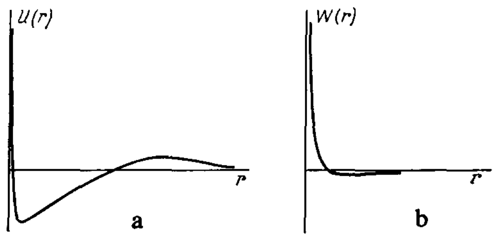

FIG. 6. Typical forms of the potential and of the correlation function in the problem of the evolution of the second moment of the field.

The first barrier becomes the molecular diffusion, and the second the turbulent one.

In contrast to the problem of the mean field in a stream that has no mirror symmetry and is unbounded in space, in this case a threshold appears and is due to the requirement that a potential well exist. From the qualitative viewpoint this can be understood as follows. Field generation calls for the nearby particles to run off in the turbulent stream up to a certain scale, beyond which it gives way to turbulent diffusion of the individual particles. Obviously, this scale should exceed the scale in which the magnetic-field diffusivity due to the molecular ν_m is substantial. A spatially bounded stream must naturally satisfy another requirement, that the dimensions of the region greatly exceed the characteristic velocity-field scale. To our knowledge, however, no such problem has been considered so far.

Since the Kazantsev equation[27] (5.12) is self-conjugate, the growth rate of $\gamma^{(2)}$ is real and in the limit $\nu_m \to 0$ is independent[36] of ν_m, i.e., the flow in question is a fast dynamo. The approach to the limit turns out to be very slow $\gamma^{(2)} - \gamma_0^{(2)} \sim (\ln \nu_m)^{-2}$ (this formula was obtained by O. V. Artamonova). The spatial dependence of the correlation function $\langle H_i H_i \rangle = (2r^2)^{-1} \partial r^3 W/\partial r$ is shown in Fig. 6b. With decreasing ν_m it "presses" against the coordinate axis, and always has a tail with a negative value of $\langle H_i H_i \rangle$, beginning with $r \sim \nu_m{}^{1/2}$ and going through a minimum at $r \sim \nu_m{}^{1/4}$. The last characteristic scale corresponds to the position of the bottom of the potential well.

We have assumed so far that the magnetic-field distribution is uniform. Let us discuss the evolution of an initially nonuniform field distribution.[36] Clearly, such a field will grow via the described dynamo process and will be transported by turbulent (and molecular) diffusion. Ya. B. Zel'dovich called attention to the similarity of this problem to the known problem of front propagation, first studied in biology by Kolmogorov, Petrovskiĭ and Piskunov, and widely used in combustion theory.[74]

For a magnetic-energy distribution localized at the initial instant, the solution is obviously of the form

$$\mathscr{E}(x, t) = \frac{1}{(4\pi D t)^{3/2}} \exp\left(2\gamma^{(2)} t - \frac{x^2}{4Dt}\right),$$

where $D = 1/2m(\infty) = \nu_m + \nu_T$ is the total diffusion coefficient, and $\gamma^{(2)}$ is the growth rate obtained above. The argument of the exponential can be represented in the form

$$2\gamma^{(2)} t - \frac{x^2}{4Dt} = -\frac{1}{4Dt}\left(|x| - 2\sqrt{2\gamma^{(2)} D}\, t\right) \times \left(|x| + 2\sqrt{2\gamma^{(2)} D}\, t\right),$$

from which it can be seen that the surface $\mathscr{E}$ = const propagates with a velocity $2\sqrt{2\gamma^{(2)} D}$. We note that this result remains in force also when account is taken of the nonlinearity that limits the growth of the energy.

d) Allowance for finite correlation time. In real turbulent flows, the correlation time is known to be of the order of l/v. Nonetheless, the δ-correlated random flow considered in the preceding section is in a certain sense a fair approximation, since the characteristic field growth time exceeds l/v. Indeed, the reciprocal growth rates $\bar{\gamma}^{-1}$ of the mean field are usually much larger than l/v [see (5.9)], since $\alpha \ll v$; in a mirror-symmetry situation, $(\gamma^{(2)})^{-1}$ is somewhat larger than l/v even in the limit as $\nu_m \to 0$.[36]

On the other hand, allowance for a finite correlation time is undoubtedly important. First, it leads to quantitative changes in the growth rate and in the form of the generated field. Second, new physical effects appear. A. P. Kazantsev pointed out that the δ-like approximation is patently unsuitable for the description, say, of acoustic turbulence, whose frequency spectrum has a strong maximum at the sound frequency and a weak tail at $\omega = 0$ [see Subsec. (d)].

To generalize the mean-field equation (5.3) we can use the same random velocity field with which we approximated δ-correlated flow (see the start of the section), but we shall not let the characteristic time Δt of the velocity-field restoration tend to zero, but regard it as finite independent parameter, which we shall designate by τ.

Assume thus that $\mathbf{v}$,$(\mathbf{x}, t)$ is a random solenoidal velocity field distributed and independent over the time intervals $(0,\tau)$, $(\tau, 2\tau)$.... For simplicity we stipulate also homogeneity of the velocity field, i.e., that its statistical characteristics in each restoration interval be invariant to a shift of the spatial coordinates.

Such a velocity field is nonstationary in time. In particular, its temporal correlation function depends on two times, but not on their difference. It is therefore possible to reconcile the vanishing of the correlations after a finite time with positiveness of the flow energy, something that might be impossible for flow whose statistical characteristics are constant in time, by virtue of the known Bochner-Khinchin theorem that the Fourier transform of a correlation function is positive. The flow considered by us is stationary only at discrete times $n\tau$, $n = 0, 1, 2, 3 \ldots$. Therefore, if we are interested in processes that occur over times considerably longer than τ, such a flow can be regarded as stationary. The restoration time has the meaning of memory time, and in this sense it is similar to the correlation time. From the physical viewpoint the restoration can be visualized as a periodic shaking up of the fluid, say when energy is supplied. All the points are shaken up here simultaneously, so that no spatial discontinuities of $\mathbf{v}(\mathbf{x})$ occur. Time discontinuities are apparently inessential, since the induction equation does not contain $\partial \mathbf{v}/\partial t$, although these discontinuities can, of course be smoothed out without changing the result. To this end it is necessary to shake up at random instants of time or ensure a continuous energy supply. This gives rise to a temporal correlation function that depends on the time difference and, say, decreases exponentially as a function of this difference.

An equation for the mean magnetic field in the model with restoration was obtained in Ref. 75 by averaging over the random trajectories. This equation relates the value of the field at the instant $(n+1)\tau$ with the field at the instant $n\tau$. In the general case it turns out to be integral, inasmuch as after a finite time τ the bundle of random trajectories diverges over a finite distance. This equation takes in Fourier space the form

$$B_i\left((n+1)\tau, \mathbf{k}\right) = \Pi_{ij}(\tau, \mathbf{k})\, B_j(n\tau, \mathbf{k}). \tag{5.13}$$

The transfer function Π_{ij} is expressed in terms of the Lagrangian coordinate as follows:

$$\Pi_{ij}(\tau, \mathbf{k}) = M_{\mathbf{x}}\{\langle G_{ij}(\tau, \xi)\exp(i\xi_\tau \mathbf{k})\rangle\}. \tag{5.14}$$

It can be shown[75] that at small $\mathbf{k}$ (the two-scale approach) or small τ (the δ-correlated approximation) the integral equation (5.13) is equivalent to the differential equation (5.6) in the sense that on going from $t = n\tau$ to $t = (n+1)\tau$ the latter has the very same transfer function.

Let us calculate the growth rate of a mean magnetic field that satisfies Eq. (5.13), using a concrete example.

Let the coordinate ξ_τ and the matrix $G_{ij}(\tau,\xi)$ have a joint Gaussian distribution with parameters σ and $\alpha > 0$, so that

$$M\langle\xi_\tau\rangle = 0, \quad M\langle G_{ij}\rangle = \delta_{ij},$$
$$M\langle\xi_{\tau i}\xi_{\tau j}\rangle = 2\tau\sigma^2\delta_{ij}, \quad M\langle\xi_{\tau i}G_{jl}\rangle = e_{ijl}\alpha\tau.$$

This is equivalent to specifying on a time segment of length τ all the correlators of a certain (generally speaking, not Gaussian) velocity field.

In this example the operator $M\langle\ \rangle$ in (5.14) denotes simple averaging over a Gaussian distribution, so that

$$\Pi_{ij}(\tau, \mathbf{k}) = (\delta_{ij} + ie_{ijl}k_l\alpha\tau)\exp(-\tau\sigma^2k^2).$$

The mean-field growth rate, as is clear from (5.13), is determined by the leading eigenvalue of the matrix $\tau^{-1}\ln\Pi_{ij}$:

$$\bar{\gamma} = \frac{1}{\tau}\ln(1+\alpha\tau k) - \sigma^2k^2. \tag{5.15}$$

Let us compare it with the growth rate $\bar{\gamma}_j = \alpha k - \beta k^2$ obtained from the differential equation (5.6). We see that $\bar{\gamma}_d$ with $\beta = \sigma^2 + (\alpha^2\tau/2)$ is a good approximation in the region of small $k < (\alpha\tau)^{-1}$ (recall that $k_{\max} \sim \alpha/\beta$). In the region of larger k, but smaller than the dissipation scale, we have $\gamma > \gamma_d$. Interestingly, the average vorticity contributes to the turbulent diffusion. Therefore the customarily employed representation of turbulent diffusion as a parameter independent of vorticity leads at small k to some undervaluation of $\beta (= \sigma^2)$, and hence to an over-valued growth rate $\bar{\gamma}_d = \alpha k - \beta k^2$. We note that we are dealing here with the contribution of a homogeneous vorticity. If we regard $\alpha = \alpha(\mathbf{x},t)$ as a random function then, in Kraichnan's opinion (see Refs. 23 and 33) the fluctuations of α make a negative contribution to the coefficient of turbulent diffusion.

The use of a differential equation such as (5.6) is thus perfectly valid for the description of the evolution of large-scale magnetic fields. It is required here that the magnetic-field scale k^{-1} exceed the characteristic scale l of the velocity field. We have $kl < 1$ in the case $\alpha \sim v$ of maximum vorticity, and $kl < \varepsilon^{-1}$ at $\alpha \sim \varepsilon v$. Going outside the framework of the two-scale approach in the case of small average vorticity adds therefore only small corrections to the growth rates. This justifies the application of Eq. (5.6) of the mean-field dynamo theory to the solar cycle, even though the two-scale approach is regarded as unsatisfactory in this case (see, e.g., Refs. 71 and 76).

The equations for the second and higher moments of the magnetic field can also be generalized to include the case of finite τ (see Sec. 6). For an explicit calculation of the kernel of this equation one must specify the form of the correlator $\langle\xi_t\eta'_{t'}\rangle$, where ξ_t and $\eta_{t'}$ are two Wiener trajectories. We note that equations for the second moment were derived earlier in the direct-interaction approximation[77] and in the scheme used by U. Frisch to close a chain of moment equations (see, e.g., Ref. 35). The investigations of these equations in the cited papers lead to results close to those of Subsec. (c).

e) Role of temporal spectrum of velocity field. The main shortcoming of the short-correlated approximation is not the shortness of the correlation time but the fact that it corresponds to a continuous spectrum and consequently does not reflect at all the singularities of the temporal spectrum of the flow. This fact was first pointed out by A. P. Kazantsev. For example, the so-called acoustic turbulence[63,64] has a strong peak at the sound frequency and a weak tail at zero frequency (acoustic flow; Fig. 7), which can be described only in the short-correlated approximation. In the simplest quadratic approximation it can be shown, with acoustic turbulence as the example,[86] that allowance for the character of the temporal spectrum of the flow leads to new physical effects, although it does not influence the coefficient of turbulent diffusion (which is determined as before by the zeroth point of the spectrum) and does not change the conclusion that self-excitation of the magnetic field is possible.

f) Transformation properties of the diffusivity coefficients. The question can be raised whether the initial mean magnetic field can be determined from the field known at a later instant. If the coefficients α and β have definite signs such a problem is ungrounded, since the evolution of the field is accompanied by dissipation and by a change of the smoothness of the solution. A similar situation, as is well known, obtains already for the simplest scalar diffusion or heat-conduction equation when considered going backward in time.

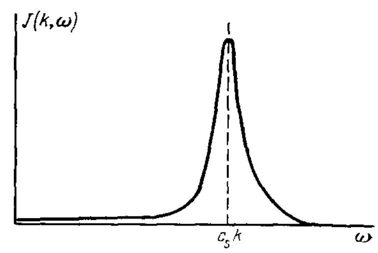

FIG. 7. Temporal spectrum of acoustic turbulence. The spectrum has a peak at acoustic frequency and differs from zero at $\omega = 0$.

Ya. B. Zel'dovich[85] points out that D is a scalar relative to transformation of three-dimensional space, but reverses sign when t is replaced by $-t$. He develops in this connection a new approach to finding the turbulent-diffusion coefficient. An exact solution of the scalar diffusion problem is obtained for a simple velocity field that has only one Fourier component. The expression for the turbulent diffusion coefficient contains as a factor the molecular-diffusion coefficient D, and this ensures a correct behavior of the result with respect to time reversal. The paper cited considers, in the spirit of cascade-renormalization ideas, the diffusion coefficient of a scalar in a velocity field with a wide spectrum that presumably describes the turbulence. For isotropic turbulence, a differential equation was derived for the turbulent-diffusion coefficient. Its solution is obtained in the form

$$D_T = D\sqrt{\text{const}\cdot\frac{v^2l^2}{D^2} + O(1)},$$

and not directly in the form $D_T = \tau\sqrt{\overline{v^2}}$ in which the necessary transformation properties with respect to time reversal are lost.

One can write also in similar form an expression for the average helicity tensor:

$$\alpha_{ij} = i\nu_m e_{inl}\int\frac{k^2k_j\Phi_{nl}(\mathbf{k},\omega)}{\omega^2+\nu_m^2k^4}\,d^3k\,d\omega,$$

see §7.8 of Ref. 23, where Φ_{nl} is the velocity-field spectral tensor. We note that when Φ_{nl} does not vanish at zero frequency, the tensor α_{ij} does not depend on ν_m in the limit of small ν_m, since

$$\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2+\nu_m^2k^4} = \frac{\pi}{\nu_m k^2}.$$

Thus, α is a pseudoscalar under coordinate transformation and reverses sign following the substitution $t\to -t$.

6. THE DYNAMO THEOREM

The mean magnetic field and its second moment are insufficient for a complete description of the behavior and distribution of the field in a random flow. In particular, they cannot be used to describe so important a phenomenon as recurrence.

Actually the direct way of studying the true (random) magnetic field in a random field with restoration turns out to be simpler. The reason is that over long time-periods the evolution of a random magnetic field as governed by the induction equation is subject to the central limit theorem. This theorem is known to ensure the existence of two determined quantities, mean value and variance. The first of these quantities turns out in this case to be the growth rate, positive in the limit of low magnetic diffusivity, while the second is responsible for the recurrence. The statement that a magnetic field grows exponentially in a three-dimensional random stream with restoration, and that the field has a recurrent structure, comprises the dynamo theorem presented in the present Section.[82]

a) Magnetic-field transport function in a restorable flow. We consider the unbounded random velocity field considered in Secs. 5 and 6, which is restored after a finite time τ. In the case of total freezing-in ($\nu_m = 0$) the magnetic-field transport along a Lagrangian trajectory of any fluid particle on a segment $[(l-1)\tau, l\tau]$ is fully described by the matrix function $G_l(t,\xi,\mathbf{x})$ (see Chap. 2), where ξ defines the trajectory in question. By virtue of the recurrence, each of the matrices G_l is unimodular, i.e., has a unit determinant. The complete transport function at the instant $t = n\tau$ is equal to the product

$$G_t = \prod_{l=1}^{n} G_l. \qquad (6.1)$$

Thus, over long times ($n\to\infty$) the problem reduces to calculation of a product of a large number of independent random matrices. Such an algebraic problem was first solved by Furstenberg.[83] Unfortunately, the mathematical formalism used in the original paper is hardly comprehensible to a physicist or even a mathematician who does not specialize in algebra. We shall therefore attempt, in the context of the problem of the behavior of a frozen-in field, to find a more lucid and simpler form of the solution, as undertaken by V. N. Tutbalin,[84] who likewise used an extension of Furstenberg's theory. When magnetic diffusivity is taken into account the problem is no longer reducible to algebraic, and products of random operators must be calculated. At small ν_m, however, this problem can be solved by the technique of averaging over random trajectories. It is important that the growth rates of the field and of its moments coincide in the limit as $\nu_m\to 0$ with growth rates for the case of complete freeze-in.[82]

To understand qualitatively the result of the action of a large number of random matrices on a given vector $\mathbf{H}_0$, we imagine a two-dimensional sphere with radius equal to the length of this vector. Under the influence of one matrix G_l with unity determinant the sphere is transformed into an ellipsoid having the same volume. Let the coefficient of elongation along the x axis be $\lambda > 1$. By virtue of area conservation, contraction along the y axis takes place, with a coefficient λ^{-1}. The length of the vector after the transformation becomes $(\lambda^2H_{0x}^2 + \lambda^{-2}H_{0y}^2)^{1/2} = H_0(\lambda^2\cos^2\varphi + \lambda^{-2}\sin^2\varphi)^{1/2}$, where φ is the polar angle. It exceeds the length $H_0 = (H_{0y}^2 + H_{0y}^2)^{1/2}$ of the initial vector when $\cos\varphi > (1+\lambda^2)^{-1/2}$. We see hence that the set of directions φ for which $\mathbf{H}_0$ is stretched will exceed one-half. In other words, an arbitrarily directed vector will be stretched with probability greater than one-half. Therefore the action of a large number of matrices G with independent elongation directions and with different elongation coefficients will result in an increase of the length of the vector.

We proceed now to a rigorous treatment, from which it follows that in the general situation the growth will be exponential.

b) Lyapunov's exponents and random basis. The results of the action of a large number of independent random matrices G_l on a given nonzero initial vector $\mathbf{H}_0$ is such that, apart from certain degenerate cases, for each velocity-field realization there exists, at long times compared with the replenishment time, a basis (e_1, e_2, e_3) in which the magnetic field varies exponentially with definite exponents $\gamma_1 > \gamma_2 > \gamma_3$ that are independent, in contrast to the basis, of the velocity-field realization and are connected, by virtue of incompressibility, by the relation

$$\gamma_1 + \gamma_2 + \gamma_3 = 0. \tag{6.2}$$

It is clear even from (6.2) that if the γ_i are not identically zero then at least the leading exponent γ_1 is always positive.

Let us determine the values of γ_i (called in mathematics the Lyapunov exponents) and the basis e_i ($i = 1, 2, 3$), and prove their existence and that γ_1 is always positive.[54]

The action of a sequence of matrices $G_1, G_2, \ldots, G_n$ on a vector $\mathbf{H}_0$ changes its magnitude and rotates it during each step. It is natural to separate the amplitudes and the phase factors of the resultant vectors $R_n \equiv |\mathbf{H}_n|$, $\varkappa_n = \mathbf{H}_n/|\mathbf{H}_n|$.

A sequence of random points formed on a sphere by unit phase vectors

$$\varkappa_0 = \frac{\mathbf{H}_0}{|\mathbf{H}_0|}, \quad \varkappa_1 = \frac{G_1\mathbf{H}_0}{|G_1\mathbf{H}_0|} = \frac{G_1\varkappa_0}{|G_1\varkappa_0|}, \ldots,$$
$$\varkappa_n = \frac{\mathbf{H}_n}{|\mathbf{H}_n|} = \frac{G_n\varkappa_{n-1}\ldots G_1\mathbf{H}_0}{|G_n\varkappa_{n-1}|\ldots|G_1\mathbf{H}_0|} \tag{6.3}$$

is a Markov chain, since each successive step $\varkappa_{n-1}\to\varkappa_n$ is connected only with the value $\varkappa_{n-1}$ and with a matrix G_n that is independent of the prior history.

Taking the norm of the last vector of (6.3) and recognizing that $|\varkappa_n| = 1$, we find for the field amplitude

$$R_n = |G_nG_{n-1}\ldots G_1\mathbf{H}_0|$$
$$= |G_n\varkappa_{n-1}|\,|G_{n-1}\varkappa_{n-2}|\ldots|G_1\varkappa_0|\,|\mathbf{H}_0|.$$

At large n the logarithm of the amplitude is thus a sum of a large number of independent quantities. Consequently, by virtue of the law of large numbers, there exists a limit

$$\frac{1}{n\tau}\ln\frac{R_n}{|\mathbf{H}_0|} \underset{n\to\infty}{=} \frac{1}{n\tau}\sum_{l=1}^{n}\ln|G_l\varkappa_{l-1}| \equiv \gamma_1 \tag{6.4}$$

(the leading Lyapunov exponent). This exponent determines the growth rate of the frozen-in magnetic field, so that the question of its sign is of fundamental importance. From Eq. (6.2), as already noted, it is obvious that if all the Lyapunov exponents are not identically zero, the leading exponent $\gamma_1 > 0$. Since the exponents γ_i are determined by the eigenvalues λ_i of the matrix G_t, it is clear that it is necessary to have $|\lambda_1| > 1$ at least at some individual points of space and to stipulate that a fluid particle land at these points frequently enough, i.e., good miscibility of the flow is necessary. It is required formally that the velocity field distribution not be concentrated on a subset of matrices with $|\lambda_i| = 1$,[5)] i.e., that it have a good (not δ-like) probability distribution density on a set of unimodular matrices. The restoration ensures furthermore that the particles will not "squat" on points with $|\lambda_1| = 1$. It must be emphasized that generally speaking the velocity field need not be three-dimensional in order to prove that γ_1 is positive and hence that the transport function G_t is exponential. The condition that the magnetic field have zero divergence, however, excludes a growth of the field in two-dimensional flows.[82] We shall prove below that γ_1 is positive in the simple case of a constant distribution density of the matrices G_l on a group of unimodular matrices (isotropic distribution). For a proof that γ_1 is positive under general assumptions concerning the form of the velocity-field distribution over the restoration interval see Refs. 54 and 82.

The limit (6.4) should be understood as a relation that is satisfied with unity probability, i.e., it is valid for an overwhelming majority of H, but not for all. In fact, let us subject $\mathbf{H}$ to the inverse transformation $G_1^{-1}G_2^{-1}\ldots G_n^{-1}$, where G_n^{-1} is the inverse of the matrix G_n. By the same reasoning as above, the resultant vector varies like $\exp(|\gamma_3|t)$, where γ_3 is the "leading exponent backwards in time," i.e., the lowest-order exponent, $\gamma_3 < 0$. There exists thus a direction

$$\mathbf{e}_3^{(n)} = \frac{G_1^{-1}G_2^{-1}\ldots G_n^{-1}\mathbf{H}_0}{|G_1^{-1}G_2^{-1}\ldots G_n^{-1}\mathbf{H}_0|}, \tag{6.5}$$

along which the field decreases like $\exp(-|\gamma_3|t)$. From the relation $|G_{n+1}G_n\ldots G_1e_3^{(n+1)}| \sim \exp(-|\gamma_3|t)$ it follows that the vector $\mathbf{e}_3^{(n+1)}$, just as $\mathbf{e}_3^{(n)}$, is a decreasing one, i.e., on going from n to $n+1$ the vector $\mathbf{e}_3^{(n)}$ varies weakly. This means that there exists a limit $\mathbf{e}_3^{(n)}\to\mathbf{e}_3$, $n\to\infty$, i.e., a basis vector corresponding to γ_3: $G_t\mathbf{e}_3 \sim \mathbf{e}_3\exp(\gamma_3 t)$.

To introduce the second Lyapunov exponent and the corresponding vector[6)] we examine the action of the matrix product $G_n\ldots G_1$ on a plane made up of a pair of orthogonal unit vectors $\varkappa_n$ and $\boldsymbol{\theta}_n$. This action generates a Markov chain on a three-dimensional manifold of bivectors, which is singled out by the relations $|\varkappa_n| = |\boldsymbol{\theta}_n| = 1$ and $(\varkappa_n\cdot\boldsymbol{\theta}_n) = 0$ (it is called the Stiefel manifold). Indeed, let us form the vectors $G_{n+1}\varkappa_n$ and $G_{n+1}\boldsymbol{\theta}_n$, and let us then orthogonalize and normalize them. As a result we have

$$(\varkappa_{n+1}, \theta_{n+1}) = \left(\frac{G_{n+1}\varkappa_n}{|G_{n+1}\varkappa_n|}, \frac{G_{n+1}\theta_n - G_{n+1}\varkappa_n\dfrac{(G_{n+1}\varkappa_n, G_{n+1}\theta_n)}{(G_{n+1}\varkappa_n)^2}}{\left|G_{n+1}\theta_n - G_{n+1}\varkappa_n\dfrac{(G_{n+1}\varkappa_n, G_{n+1}\theta_n)}{(G_{n+1}\varkappa_n)^2}\right|}\right).$$

Let there now be two arbitrary vectors $\mathbf{H}_0$ and $\tilde{\mathbf{H}}_0$ stretched by a product of independent random matrices G_t. We transform from $\tilde{\mathbf{H}}_n$ to the vector

$$\mathbf{H}_n' = \tilde{\mathbf{H}}_n - \mathbf{H}_n\frac{(\mathbf{H}_n\tilde{\mathbf{H}}_n)}{(\mathbf{H}_n\mathbf{H}_n)},$$

which is orthogonal to $\mathbf{H}_n$ and introduce the amplitude and phase factors of the vectors $\mathbf{H}_n$ and $\mathbf{H}_n'$: $\varkappa_n = \mathbf{H}_n/|\mathbf{H}_n|$, $R_n = |\mathbf{H}_n|$ and $\boldsymbol{\theta}_n = \mathbf{H}_n'/|\mathbf{H}_n'|$, $\rho_n = |\mathbf{H}_n'|$. The area of the parallelogram stretched on the considered pair of vectors is $S_n = |\mathbf{H}_n\times\mathbf{H}_n'| = |\mathbf{H}_n\times\tilde{\mathbf{H}}_n|$. This quantity, just as R_n, depends multiplicatively on the Markov chain $(\varkappa_n, \boldsymbol{\theta}_n)$. Indeed,

$$S_{n+1} = \rho_{n+1}R_{n+1} = |G_{n+1}\mathbf{H}_n \times G_{n+1}\tilde{\mathbf{H}}_n|$$
$$= \rho_nR_n\,|G_{n+1}\varkappa_n \times G_{n+1}\theta_n|$$
$$= S_n\,|G_{n+1}\varkappa_n \times G_{n+1}\theta_n|.$$

As $n\to\infty$ there exist therefore the limit $n^{-1}\ln(\rho_n R_n)$ and the limit

$$\frac{1}{n}\ln\rho_n \underset{n\to\infty}{=} \frac{1}{n}\ln\left|\tilde{\mathbf{H}}_n - \mathbf{H}_n\frac{(\mathbf{H}_n\tilde{\mathbf{H}}_n)}{(\mathbf{H}_n\mathbf{H}_n)}\right| \equiv \gamma_2, \tag{6.6}$$

by which we mean the second Lyapunov exponent. We shall show that if the matrices G_n have an isotropic distribution the exponent γ_2 is rigorously smaller than γ_1. Indeed,

$$\gamma_2+\gamma_1=\frac{\ln S_n}{n}=\frac{1}{n}\sum_{l=0}^{n-1}\ln|G_{l+1}\varkappa_l\times G_{l+1}\theta_l|$$

$$=\frac{1}{n}\sum\ln|G_{l+1}\varkappa_l|+\frac{1}{n}\sum\ln|G_{l+1}\theta_l|$$

$$+\frac{1}{n}\sum\ln\sin(\widehat{G\varkappa_l G\theta_l}).$$

By virtue of the isotropy, the first sums are equal and yield $2\gamma_1$, while the third is equal to the average over the distribution of the G matrices: $\langle\ln\sin(G\varkappa\, G\,\theta)\rangle$. Although the vectors $\varkappa$ and θ are orthogonal, $G\varkappa$ and $G\,\theta$ have unity probability of not being orthogonal, since the matrix distribution is not degenerate. Therefore $\langle\ln\sin(G\varkappa\, G\,\theta)\rangle<0$ and hence $\gamma_2<\gamma_1$. It follows therefore by virtue of (6.2) that the leading exponent γ_1 is strictly positive and the least exponent γ_3 is strictly negative.

The exponent γ_2 can be of either sign. When the distribution of the matrices G_n is symmetric with respect to substitution $G_t\to G_t^{-1}$, i.e., when the flow is symmetric under time reversal, we have $\gamma_1=-\gamma_3$ and $\gamma_2=0$.

To construct the second basis vector $\mathbf{e}_2$ we must consider the action of the inverse matrices G_n^{-1} on the plane containing the vector $\mathbf{e}_3$. This yields the maximum compression plane. For $\mathbf{e}_2$ one can choose a vector lying in this plane and orthogonal to $\mathbf{e}_3$.

Similarly, by considering the action of a matrix product on the parallelepiped formed by the three vectors $\mathbf{H}_0$, $\tilde{\mathbf{H}}_0$, $\tilde{\tilde{\mathbf{H}}}_0$, we can introduce also a third Lyapunov exponent

$$\gamma_3=\lim_{n\to\infty}\frac{1}{n}\ln|\tilde{\tilde{\mathbf{H}}}_n|, \tag{6.7}$$

the projection of $\tilde{\tilde{\mathbf{H}}}_n$ on the $(\mathbf{H}_n,\tilde{\mathbf{H}}_n)$ plane, which we have introduced by considering the action of inverse matrices on a vector.

The third basis vector $\mathbf{e}_1$ is constructed orthogonal to $\mathbf{e}_2$ and $\mathbf{e}_3$.

c) Evolution of a transport matrix along a random trajectory. The existence of Lyapunov exponents and of a basis means that as the matrix G_t evolves along a random Wiener trajectory it assumes asymptotically, as $t\to\infty$, a definite form that contains $\exp(\gamma_i t)$ as factors. Knowledge of this form makes it possible to express the Lyapunov exponents in terms of elements of the matrix G_t, i.e., in terms of the characteristics of the velocity field.

To find the asymptotic behavior of the transport matrix we expand it in terms of simpler components. Namely, we represent G_t as a product of an orthogonal matrix (rotation matrix) U by the upper-triangular matrix K:

$$G_{ij}(t,\xi_t,\mathbf{x},\mathbf{w})=U_{il}(t,\xi_t,\mathbf{x},\mathbf{w})\,K_{lj}(t,\xi_t,\mathbf{x},\mathbf{w}). \tag{6.8}$$

The expansion is technically implemented as follows (see, e.g., Ref. 84). The rows g_1, g_2, and g_3 of the matrix G are orthonormalized, starting with the upper one, and used to form a new basis $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$. The transition to this from the initial E_1, E_2, E_3 is via the matrix U. The diagonal elements of the matrix K have the following meaning: K_{33} is the length of the lower row g_3, K_{22} is the length of the component of the middle row g_2, which is orthogonal to the lower row, while K_{11} is the length of the first-row component which is orthogonal to g_1 and g_2.

We derive now equations for the evolution of the matrices U and K along a Wiener trajectory. At the instant $t+\Delta t$ we have

$$G_{ij}(t+\Delta t,\xi_{t+\Delta t},\mathbf{x},\mathbf{w})$$

$$=\left(\delta_{ie}-\Delta t\frac{\partial v_e(\Delta t,\xi_t)}{\partial x_i}\right)G_{ej}(t,\xi_t,\mathbf{x},\mathbf{w}).$$

Substituting here the expansion (6.8), using the fact that an arbitrary matrix can be represented as a sum of an antisymmetric and an upper-triangular matrix

$$-U_{ik}^{-1}\frac{\partial v_e}{\partial x_k}U_{ej}=A_{ij}+B_{ij}$$

and letting $\Delta t\to 0$, we obtain the differential equations

$$\frac{dU}{dt}=UA,\quad \frac{dK}{dt}=BK. \tag{6.9}$$

The first is a closed nonlinear equation for the orthogonal matrix that determines the orientation of the matrix G_t. After solving this equation, we obtain directly from the second equation of (6.9) the elements of the upper-triangular matrix:

$$K_{ii}(t)=K_{ii}(0)\exp\left(\int_0^t B_{ii}(s)\,ds\right)$$

$$=K_{ii}(0)\exp[n\tau\gamma_i+\sqrt{n\tau}\,\zeta_i(n\tau)], \tag{6.10}$$

where

$$\gamma_i=\lim_{n\to\infty}\frac{1}{n\tau}\int_0^{n\tau}B_{ii}(s)\,ds=\langle B_{ii}\rangle,\qquad i=1,2,3;$$

there is no summation over i, while

$$\zeta_i=(n\tau)^{-1/2}\int_0^{n\tau}(B_{ii}-\gamma_i)\,ds,$$

has according to the central limit theorem a normal distribution as $n\to\infty$.

So far we have considered the behavior of a frozen-in field. To take into account a finite magnetic-diffusivity it is necessary (by virtue of the independence of the initial distributions of the magnetic field and of the velocity) to average the function $G_t=G_n\ldots G_1$ over all the random trajectories that arrive by the instant t at the considered point $\mathbf{x}$. If the natural condition $v^2\tau\gg\nu_m$ [see (2.9)], which is equivalent to $R_m\gg 1$, is satisfied each function G_t deviates slightly from the corresponding Lagrangian matrix that describes the evolution during one restoration step. Since, however, it is necessary to follow the correct order of the transitions to the limits ($n\to\infty$ and then $\nu_m\to 0$), we must prove that the subset of random trajectories on which the growth rate deviates considerably from γ_i has a small statistical weight. The corresponding proof is given in our earlier article,[82] from which it follows that $\gamma_i(\nu_m) \to \gamma_i$ as $\nu_m \to 0$, i.e., the Lagrangian approach is valid for flow with restoration in the limit of small magnetic diffusivity.

d) Recurrence. In the absence of magnetic diffusivity the Lyapunov exponents are not random and by virtue of homogeneity are the same at all spatial points. The corrections $t^{-1/2}\xi_i(t)$ to them, however, are random, the trajectories also depend on the time (although the variances ξ_i do not increase with time). This means that the field in the liquid particle does not increase uniformly, and the deviations of the growth rates are $\propto t^{-1/2}$. Examining the field at a given point, we also observe this recurrence, inasmuch as different trajectories arrive at this point at different instants.

Recurrence of the generated magnetic field follows also from the behavior of the moments of the field. In fact, consider the function

$$f(p) = \lim_{n\to\infty} \frac{1}{n\tau} \ln \langle |\mathbf{H}_n|^p \rangle, \tag{6.11}$$

where the angle brackets denote averaging over the distribution of the matrices G_n. Obviously, $\gamma(=\gamma_1)$ coincides with the derivative $\mathrm{d}f/\mathrm{d}p$ at zero, and $f(p)/p \equiv \gamma^{(p)}$ determines the growth rate of the p-th moment of the modulus of the field. The function (6.11) for the case of random flows with restoration takes the form shown in Fig. 8 (Refs. 54 and 82). It is seen from this that the growth rate is faster the higher is the number of the moment

$$\gamma < \gamma^{(2)} < \gamma^{(4)} < \ldots. \tag{6.12}$$

Let us prove that these inequalities remain in force also when magnetic diffusivity is taken into account in the limit as $\nu_m \to 0$.

The equations that describe the evolution of the tensor of the equal-time p-moment of the field, $m_p(t, x_1, \ldots, x_p) \equiv \langle H_{i_1}(t,\mathbf{x}_1) H_{i_2}(t,\mathbf{x}_2) \ldots H_{ip}(t,\mathbf{x}_p)\rangle$, are obtained by expressing the field at the instant $t=(n+1)\tau$ in terms of the field at the instant $n\tau$, in accordance with Eq. (2.11):

$$m_p[(n+1)\tau, \mathbf{x}_1, \ldots, \mathbf{x}_p]$$

$$= \langle M_{\mathbf{x}} m_p [n\tau, \mathbf{x}_1 + \xi_\tau(\mathbf{x}_1, \nu_m), \ldots, \mathbf{x}_p + \xi_\tau(\mathbf{x}_p, \nu_m)]$$

$$\times G_n[\mathbf{x}_1 + \xi_\tau(\mathbf{x}_1, \nu_m)] \ldots G_n[\mathbf{x}_p + \xi_\tau(\mathbf{x}_p, \nu_m)]\rangle, \tag{6.13}$$

where $\xi_\tau(\mathbf{x}_i,\nu_m) \simeq \xi_\tau(\mathbf{x}_i,0) + (2\nu_m)^{1/2} w_\tau^{(i)}$ by virtue of the condition $v^2\tau \gg \nu_m$, while $w_\tau^{(i)}$ are p independent versions of a Wiener process. For even p, this integral equation has a symmetric kernel that is expressed in terms of the joint probability density of the Lagrangian coordinates $\xi_\tau(x_i,0)$ and the matrices $G_n(x_i + y_i)$, as well as in terms of the Gaussian density $\exp(-\sum_{i=1}^{p} y_i^2/4\nu_m) \times (2\pi\tau\nu_m)^{3/2}$. In view of the translational invariance of the flow, the kernel depends only on the differences $\mathbf{x}_2 - \mathbf{x}_1, \ldots, \mathbf{x}_p - \mathbf{x}_1$. As $\nu_m \to 0$, owing to the degeneracy of the joint densities, the kernel acquires δ-like singularities on planes of the type $\mathbf{x}_{i_1} = \mathbf{x}_{i_2} = \ldots = \mathbf{x}_{i_l}$, $i_l \leqslant p$ (because l end points of random trajectories coincide). The strongest singularity is obtained when all the points coincide, $\mathbf{x}_1 = \ldots = \mathbf{x}_p$. Corresponding to this singularity is an "eigenfunction" $\prod_{i=2}^{p} \delta(\mathbf{x}_i - \mathbf{x}_1)$ and an

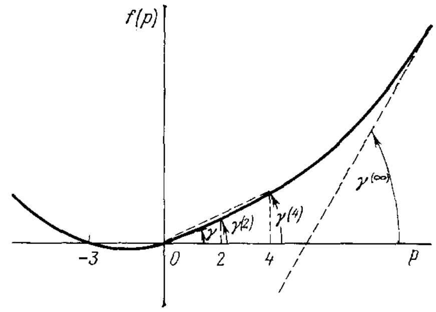

FIG. 8. Concave function whose derivative at zero is equal to the field growth rate, while $f(p)/p$ determines the growth rate of the pth moment of the modulus of the field.

"eigenvalue" $\tau^{(p)}$ (called the resolvent's singular point).

From the convergence of the integral-operator kernels as $\nu_m \to 0$ follows also convergence of the eigenvalues $\gamma_{\nu_m}^{(p)} \to \gamma^{(p)}$.

The exponents $\gamma^{(p)}$ increase [see (6.12)] and accumulate near $\gamma^{(\infty)}$, which is finite by virtue of the velocity limitation. It appears that the inequalities $\gamma^{(l)}(\nu_m) < \gamma^{(p)}(\nu_m)$ and $l < p$ hold for all $\nu_m < (\nu_m^{(p)})_{\rm crit}$. We note that the excitation thresholds $(\nu_m)_{\rm crit}^{-1}$ for all the successive moments decrease with increasing number of the moment (Fig. 9).

Let us explain why the inequalities (6.12) mean recurrence in the distribution of $\mathbf{H}(t,\mathbf{x})$. For the magnetic-energy density as $t\to\infty$ we have $\ln\langle \mathbf{H}^2(t,0)\rangle / t \to 2\gamma^{(2)}$. The probability of finding at the point $\mathbf{x}$ a strong deviation from this background is, according to the Chebyshev inequality

$$P\{|\mathbf{H}|(t, \mathbf{x})| > \exp \eta t\} \leqslant \exp[-2(\eta - \gamma^{(2)})t],$$

whence it is seen that the fraction of the points in space at which $\eta > \gamma^{(2)}$ ("peaks") is exponentially small and decreases exponentially with increasing t. Let at the same time ν_m be such that $\gamma^{(4)} > \gamma^{(2)}$ and $\eta < \gamma^{(4)}$. The growth of the fourth moment is then obviously determined precisely by these widely spaced "peaks." The energy contained in them is very high, but in view of the large spaces between the peaks they make no noticeable contribution to the average energy density. At still smaller ν_m there appears an entire hierarchy of peaks, with the more widely spaced and taller ones responsible for the growth of the higher moments.

We expect this hierarchy to be infinite for each $\nu_m \neq 0$.

e) Remarks on the case of planar motion. It was explicit-

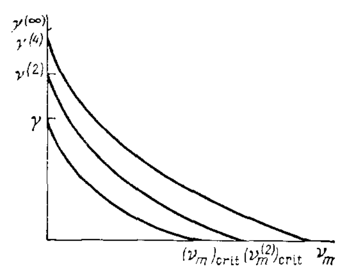

FIG. 9. Expected dependences of the growth rates of the field and of its moments with decreasing magnetic diffusivity.

ly assumed above that the motion is three-dimensional. The statement, however, that the leading Lyapunov exponent is positive at $\nu_m = 0$, i.e., that the magnetic field increases when frozen-in, does not depend on the dimensionality. In particular, it is valid also in the case of planar flow. At first glance this contradicts Zel'dovich's known[3] antidynamo theorem, which can be generalized to include the case when the velocity depends on three coordinates[17] and is valid even if the initial requirement that the magnetic field have no divergence is dispensed with (remark by U. Frisch).

The paradox is resolved by the fact that the Zel'dovich theorem requires $\nu_m \neq 0$, while the limit $\gamma(\nu_m) \to \gamma(0)$ as $\nu_m \to 0$ can be taken only in the three-dimensional case. The qualitative cause is a circumstance known from probability theory, that two-dimensional trajectories are reflexive and three-dimensional ones are not. As a result, at $\nu_m = 0$, two close trajectories coalesce on a plane and diverge in a 3 D space. Let us consider the situation in more detail.

Let $\boldsymbol{\eta}_t = \xi_t^{(\mathbf{x})} - \xi_t^{(\mathbf{y})}$ be the difference between two trajectories (2.9) that emerge from the points $\mathbf{x}$ and $\mathbf{y}$ and are constructed in accordance with two independent Brownian processes $\mathbf{w}_t^{(1)}$ and $\mathbf{w}_t^{(2)}$. We confine ourselves for simplicity to the velocity field (5.10) which is δ-correlated in time. For the mean value $u(t, \mathbf{x} - \mathbf{y}) = \langle M g(\boldsymbol{\eta}_t) \rangle$ we then obtain by the method indicated in Sec. 5 an equation of the diffusion type with a variable diffusivity coefficient

$$\frac{\partial u}{\partial t} = [\nu_m + F(0) - F(r)] \Delta u, \quad u_{t=0} = g(r), \quad r = |\mathbf{x} - \mathbf{y}|.$$

The behavior of this process in the two-dimensional case is qualitatively different at $\nu_m = 0$ and at $\nu_m \neq 0$. No such difference occurs in three dimensions. This difference is easiest to note in the form of the invariant measure of the process $\boldsymbol{\eta}_t$. It is obtained from the equation $A^* \pi = 0$, where $A = [\nu_m + F(0) - F(r)]\Delta$ is the generating operator of the diffusion $\boldsymbol{\eta}_t$. We have $\pi(r) = (\nu_m = F(0) - F(r))^{-1}$. The invariant measure is bounded from above and from below when $\nu_m > 0$. When $\nu_m = 0$ the function $\pi(r) \sim r^{-2}$ is integrable as $r \to 0$ in the 3 D case but not in the 2D one. The nonintegrability of $\pi(r)$ in the vicinity of zero in 2 D space means that the process $\boldsymbol{\eta}_t$ lasts an anomalously long time in the vicinity of zero, i.e., when $\nu_m = 0$ two 2 D trajectories $\xi_t^{(x)}$ and $\xi_t^{(y)}$ come infinitely close as $t \to \infty$. For $\nu_m > 0$, on the other hand, as $t \to \infty$ these trajectories have an overwhelming probability of being separated by a distance of the order of $t^{-1/2}$ in both 2 D and 3 D space.

The difference between the two- and three-dimensional cases can be explained in a different and a simpler manner. Let us find the time required for the diffusion process $\boldsymbol{\eta}_t$ to emerge from a finite region, a spherical layer $\delta \lesssim r < 1$. The quantity $T(\mathbf{x}) = M\tau(\mathbf{x})$, where $\tau(\mathbf{x})$ is the time required to leave this region, obeys the equation $AT(\mathbf{x}) = -1$ with the condition $T = 0$ on the boundary.[46] Since $F(0) - F(r) \approx c^{-1} r^2, c = \text{const}$, we have in the 2 D case

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -\frac{c}{r^2},$$

and in the 3 D case

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = -\frac{c}{r^2}$$

with identical boundary condition $T(\delta) = T(1) = 0$. The solution of these equations is elementary:
in the two-dimensional case

$$\frac{1}{c} T_\delta(r) = \frac{1}{2}(\ln\delta - \ln r)\ln r \xrightarrow[\delta \to 0]{} \infty,$$

i.e., the process lasts an infinitely long time in the vicinity of zero;
in the three-dimensional case

$$\frac{1}{c} T_\delta(r) = \ln\delta \cdot \left(\frac{1}{r} - 1\right)\left(\frac{1}{\delta} - 1\right)^{-1} - \ln r \xrightarrow[\delta \to 0]{} \ln\frac{1}{r} < \infty.$$

Owing to the presence of a qualitative discontinuity $\nu_m = 0$, $\nu_m \neq 0$ for 2 D motion, neither the growth rates of the moment equations nor the Lyapunov exponent γ is a continuous function at the point $\nu_m = 0$. For the second moment this can be easily proved also directly.

For 3 D motion, all the probability characteristics of a single, a pair, etc., of Lagrangian trajectories vary continuously as $\nu_m \to 0$.

7. STATIONARY VELOCITY FIELDS WITH STOCHASTIC PROPERTIES

The proof of the dynamo theorem was based essentially on the nonstationarity (restoration) of the random flow. The case of a stationary random three-dimensional flow as applied to the dynamo problem is practically uninvestigated. The distinguishing feature of such flows is the presence of stationary regions (traps) from which a fluid particle can escape usually only via molecular diffusion.[78,79] On the one hand, in three-dimensional stochastic flows the nearby particles run off exponentially, i.e., an effect exists that causes exponential growth of the magnetic field. On the other hand, the presence of trapping regions that are everywhere densely filled with individual stream lines decreases greatly the size of the generated field and enhances the role of magnetic diffusivity. The competition between these two effects can produce under certain conditions, as already noted in Sec. 4, a dynamo of intermediate type.

Actually, the noted singularities of stochastic flows are possessed by trajectories of velocity fields describable by nonrandom smooth functions. A simple example of such a flow was indicated by V. I. Arnol'd and was numerically investigated in Ref. 80:

$$\mathbf{v} = (A\cos y + B\sin z,\ B\cos z + C\sin x,\ C\cos x + A\sin y). \quad (7.1)$$

When none of the constants A, B, and C is zero, for example if $A = B = C = 1$, the field streamline equations (7.1) have no integrals (i.e., the velocity field is essentially three-dimensional) and finite regions exist, each of which is filled densely everywhere with an individual stream line. We note that div $\mathbf{v} = 0$ and that curl $\mathbf{v} = \mathbf{v}$, i.e., this field has maximum vorticity. This, however, is still not enough for a fast dynamo.

Before we discuss the direct solution of the dynamo problem in the velocity field (7.1), we consider a simplified modification of this flow[57] which, albeit formally exotic, reveals clearly the main features of the evolution of the magnetic field.

a) Role of shift and of exponential stretching of fluid particles. Let the flow region be a three-dimensional compact manifold which is constructed in Cartesian coordinates as a product of a two-dimensional torus $(x,y,z)\to(x+1,y,z)$, $(x,y+1,z)$ by a segment $0\leqslant z\leqslant 1$, with the end-point tori identified in accordance with the rule

$$(x,\ y,\ z)\to(2x+y,\ x+y,\ z+1), \tag{7.2}$$

i.e., with some twist. We consider thus a flow that is periodic in x and y, and subjected when shifted along z to the transformation (7.2) that conserves area and has eigenvalues

$$\lambda_1=\frac{3+\sqrt{5}}{2}\approx 2.11,\quad \lambda_2=\frac{3-\sqrt{5}}{2}\approx 0.34,\quad \lambda_1\lambda_2=1.$$

If x and y are replaced by coordinates directed respectively along the eigenvectors corresponding to λ_2 and λ_1, one can define in the manifold under consideration a Riemannian metric that is invariant to the transformations of x, y, and z written out above:

$$ds^2=e^{-2\mu z}\ dp^2+e^{2\mu z}\ dq^2+dz^2,\quad \mu=\ln\lambda_1\approx 0.75. \tag{7.3}$$

In this Riemannian space the flow takes the very simple form $\mathbf{v}=(0,0,v)$, where $v=\text{const}$, i.e., $\operatorname{div}\mathbf{v}=\operatorname{curl}\mathbf{v}=0$. It is important, however, that owing to the non-Euclidean character of the metric each liquid particle is stretched in the q direction and compressed in the p direction. The shift of $\mathbf{v}$ is specified explicitly, and the tension is produced by the metric properties of the space.

The tension, obviously, does not act on the z-component of the magnetic field, so that this component is damped. An equation for the p-component of the field is obtained from the equation for the q-component by letting $\mu\to-\mu$. It suffices therefore to consider the evolution of $H_q(p,q,z,t)\equiv H$ in the Riemannian metric (7.3):

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)H=\mu v\,H+\nu_m(\nabla^2-\mu^2)\,H. \tag{7.4}$$

The solution, naturally, is periodic in x and y or in p and q, wherein all but the zeroth harmonic should decrease with increasing $|z|$. The last requirement follows from the condition that H be analytic and from the property (7.2) according to which a shift along z is equivalent to raising the numbers of the harmonics.

Equation (7.4) has growing solutions that are periodic in z only if the initial field is independent of x and y, i.e., it corresponds to a zeroth harmonic unbounded in z.

From the viewpoint of the dynamo problem, however, greater interest attaches to solutions that decrease with $|z|$. It might seem at first glance that it is always possible to choose ν_m or the number of the harmonic such as to obtain an exponential growth because of the term μvH that describes the action of the exponential stretching of the particles on the field. The shift along z (the term $v\partial/\partial z$), however, is equivalent to raising the numbers of the harmonics of the expansion in x and y at fixed z. Therefore any harmonic (except the zeroth) moves in the course of time into the region of ever increasing wave numbers, where the dissipation $\nu_m(\nabla^2-\mu^2)H$ becomes substantial.

For a perspicuous illustration (see Ref. 57 for a more rigorous treatment) of the pernicious action of a constant shift we transform from H to the variable $T=H\exp(-\mu vt+\mu^2\nu_m t)$ and replace the dissipative term in (7.4) by a one-dimensional one, i.e., we consider an equation of the heat-conduction type

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)T=\nu_m\frac{\partial^2 T}{\partial z^2},$$

and stipulate here $T(0)=T(1)=1$, which is equivalent to a decrease with $|z|$. The solution of this problem, say by separating the variables, is trivial. As a result we have

$$T(z,\ t)=\exp\left(-\frac{v^2}{4\nu_m}t-\pi^2n^2\nu_m t+\frac{v}{2\nu_m}z\right)\sin\pi nz,$$

$$n=1,\ 2,\ \ldots$$

We have thus for the variable $H\propto\exp(\gamma t)$

$$\gamma=\mu v-\frac{v^2}{4\nu_m}-\pi^2n^2\nu_m-\mu^2\nu_m.$$

The function $\gamma(\nu_m)$ has the characteristic form shown in Fig. 5b of Sec. 4. As $\nu_m\to 0$ and $\gamma\to-v^2/4\nu_m$, the maximum is reached at $\nu_m=v/2\sqrt{\pi^2+\mu^2}$ and is equal to $\gamma_{max}=[\mu-(\pi^2+\mu^2)^{1/2}]\,v$, i.e., a dynamo is impossible here. A more detailed analysis of the problem (7.4) shows that by virtue of the exponential decrease of the wave numbers, $\mathbf{k}=\mathbf{k}_0\exp(\mu vt)$, most harmonics of the magnetic field attenuate very sharply, like $\exp(-\nu_m k^2(t)t)$, i.e., like an exponential in an exponent. The resultant exponential damping is due to harmonics that have near-zero wave numbers. This is similar to the situation in the problem of the linear velocity field (see Sec. 3), except that there ($v_i=c_{ik}x_k$) the shift was not constant but tended to zero as $x\to 0$ and to infinity as $x\to\infty$.

b) Intermediate dynamo in periodic flow. Numerical integration of the induction equation in a velocity field (9.1) with $A=B=C=1$ was carried out in Ref. 37. The eigenvalue with the largest real part was calculated for the operator $(\mathbf{v}\nabla)\mathbf{H}-(\mathbf{H}\nabla)\mathbf{v}+\nu_m\nabla^2$. Disregarding for a while the zero eigenvalue,[81] the critical value of the Reynolds number at which Re γ becomes positive is approximately $(\mathrm{Re}_m)_{crit}\approx 27$. We consider in this case precisely $\mathrm{Re}_m=lv/\nu_m$ rather than ν_m^{-1}, because the velocity field (7.1) has a special symmetry that makes the characteristic scale equal not to the lattice period 2π but one-third as large. In addition, we note that $v\sqrt{3}$. The maximum growth rate is reached when Re_m is of the order of 40, and at $\mathrm{Re}_m\sim 60$ Re γ again becomes negative. We note that $(\mathrm{Re}\,\gamma)_{max}$ turned out to be very small, of the order of 10^{-2}. In addition, Im γ exceeds Re γ by approximately two decades, i.e., the solution oscillates very rapidly in time.

It is instructive to compare this result with the behavior of the growth rate in the model example of the preceding section, where γ [see (7.5)] turned out always to be negative. This is due to the fact that the exponent μ, which characterizes the stretching of the stream, is real. For the velocity field (7.1), the corresponding quantity is complex. It is easy to verify with the aid of (7.5), by choosing μ complex, that Re γ

can have a positive maximum and in this case Im γ must without fail be large.

Simple dimensional estimates show that the same behavior of $\gamma(\nu_m)$ as $\nu_m \to 0$ as in (7.5) should also be expected for the field (7.1). Indeed, during the first stage of the evolution of a smooth initial field it becomes enhanced because of the exponential run-off of the fluid particles and the simultaneous reduction of its scale. This reduction of scale in the presence of magnetic diffusion cannot continue infinitely and must be stopped by the diffusion. For that eigenfunction having the largest eigenvalue which describes the field behavior over long times, the contribution of the term $\nu_m \nabla^2 H$ should be one of the principal ones. It is easily seen, however, that it can compete with the principal term $(\mathbf{v}\cdot\nabla)\mathbf{H}$ only if the characteristic scale of the field is of the order of ν_m (and not of $\nu_m^{1/2}$ as usual) and if $\gamma \sim \nu_m^{-1}$. Since the generator term is small in this case $((\mathbf{H}\nabla)\mathbf{v}/\nu_m \Delta \mathbf{H} \sim \nu_m)$, it drops out of the induction equation and γ turns out to be negative.

A hypothesis was advanced in Sec. 4 (Fig. 6b) that this behavior of the growth rate for an individual mode can be combined with a regime of change of modes. As a result, even though γ decreases for an individual mode, the values of $(\mathrm{Re}\,\gamma)_{\max}$ for different types of modes approach asymptotically as $\nu_m \to 0$ a growth rate that is independent of ν_m and is typical of the fast dynamo. We present some arguments in favor of this hypothesis.

As shown in the model example of Subsec. (a), Re γ becomes negative $(-v^2/4\nu_m)$ because of the presence of the shift term. In the real flow (7.1) this term cannot vanish at nonzero A, B, and C because there are no velocity-field integrals. It is natural to assume, however, that each succeeding mode with a smaller characteristic scale decreases the contribution of this term, so that $|(\mathbf{v}\nabla)\,\mathbf{H}| \sim \delta_n v k H$, when δ_n are successively decreasing numbers. In the model example given above $\delta_n \equiv 1$. We have then in place of (7.5)

$$\gamma_n = \mu v - \delta_n \frac{v^2}{4\nu_m} - (\pi^2 n^2 + \mu^2)\,\nu_m.$$

Recognizing also that the exponent for the velocity field expansion is complex, $\mu = \mu_1 + i\mu_2$. We find that the maximum growth rate of the nth mode is reached at $\mathrm{Re}_m = 2a\delta_n^{-1}$, $a \equiv \sqrt{\mu_1^2 - \mu_2^2 + \pi^2 n^2}$. In this case $(\mathrm{Re}\,\gamma)_{\max} = \mu_1 v - \delta_n v a \to \mu_1 v$, while $(\mathrm{Im}\,\gamma)_{\max} = -\mu_2 v(\delta_n \mu_1 a^{-1} - 1)$ is large at small n and tends to $\mu_2 v$ as $n \to \infty$.

8. CONCLUSION

Let us formulate once more the main results reported in this paper.

A magnetic field contained in a random statistically homogeneous stream with a finite correlation time increases exponentially with time in the limit of large magnetic Reynolds numbers. The distribution of the generated field is recurrent, i.e., inhomogeneous in space and time. This is manifest in the fact that the higher moments of the field have higher growth rates. In particular, the fraction of the volume which contains practically all (say, 90%) of the generated energy decreases exponentially with time.

The results were obtained not by solving directly the induction equation in a specified velocity field, but by using modern ergodic-theory methods to investigate the asymptotic behavior of the product of a random operator. We note that similar methods were successfully used in quantum theory of disordered systems (the theory of Anderson and Mott). Similar effects can be expected in a large class of problems in which equations with random operators are used (problems of heat conduction, separation of matter by gravity, and others).

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1) More accurately, one should speak of an increase of the dimensionless magnetic Reynolds number $\mathrm{Re}_m = lv/\nu_m$, by measuring γ in units of v/l, where l and v are the characteristic scale and velocity of the flow. Here and below γ and ν_m are expressed in these units.

2) Such an initial field is generated by current distribution in a finite volume. A slower decrease of the field may turn out to be insufficient. In the trivial case $(H_1)_0 = H_0$, $H_2 = H_3 = 0$ there always exists a solution $H_1 = H_0 \exp(c,t)$.

3) Naturally, solutions of the type $t^\mu \exp(\gamma t)$ are possible, which in particular can be of a power type for $\gamma = 0$. However, this latter case is degenerate since usually γ can vanish only for a certain "critical" value of ν_m.

4) Generation by a stream concentrated in a bounded region requires a sufficiently large region and a sufficiently small ν_m, and this imposes restrictions on the two numbers Re_m and R_α.

5) Examples of such matrices are the rotation matrices U or matrices of the form $G_n = A\,U_n A^{-1}$, where A is a constant matrix. For such matrices we have

$$G_n G_{n-1} \ldots G_1 = AU_n \ldots U_2 U_1 A^{-1} = A\tilde{U}A^{-1}.$$

6) If only the exponent γ_2 is of interest, it can be derived from Eq. (6.2) alone.

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