Energy spectrum of a metal and its singularities

M. I. Kaganov

S. I. Vavilov Institute of Physics Problems, Academy of Sciences of the USSR
Usp. Fiz. Nauk **145**, 507–527 (March 1985)

The Fermi surface and the **p**-space region adjacent to it shape the spectrum of the elementary excitations of a metal, the fermions (electrons and holes), as well as the bosons (phonons). Electron-phonon interaction renormalizes the dispersion laws of the elementary excitations. Their lifetime therefore becomes finite and the dependence of the energy on the quasimomentum has singularities. The features of these singularities are intimately related to the local geometry of the Fermi surface (to its shape, curvature, presence or absence of lines of parabolic points); this distinguishes them from other singularities (e.g., those due to phonon-phonon interaction). A unique role is played by the singularities, due to parabolic points on the Fermi surface, of the sound velocity as a function of the propagation direction, since these singularities are produced by electrons that have an infinite lifetime in a perfect crystal. The results cited formulate the general premises concerning the elementary-excitation spectrum of a metal and continue in this sense the semi-phenomenological approach developed by I. M. Lifshitz and his school.

TABLE OF CONTENTS

1. SIMPLEST MODEL

The simplest but quite productive model of a metal combines the Drude-Lorentz-Sommerfeld approach with the Debye approach. The metal electrons are characterized in this model by their density n, i.e., the number of electrons per unit volume (for monatomic metals it is usually assumed equal to $Z/a^3 = Zn_i$, where Z is the valence of the metallic atom, a^3 is the volume per atom of the crystal cell, and n_i is the ion density). The phonons are characterized by the Debye temperature θ (or frequency ω_D, with $\theta = \hbar\omega_D$, or by the speed of sound s). Assuming the metal-electron mass to be equal to the mass m of the free electron, we can calculate the Fermi energy ε_F of the conduction electrons and their velocity v_F, as well as the Fermi momentum p_F (it is determined by the density n):

$$p_F = (3\pi^2 n)^{1/3}\hbar = (3\pi^2 Z)^{1/3}\hbar/a, \qquad v_F = \frac{p_F}{m}, \quad \varepsilon_F = \frac{p_F^2}{2m}. \tag{1}$$

The maximum wave vector in the Debye model is $k_m = (6\pi^2)^{1/3}/a$. Therefore

$$\theta = \hbar\omega_D = (6\pi^2)^{1/3}\hbar s a^{-1}. \tag{2}$$

Owing to the substantial differences between the electron and ion masses ($M \gg m$) we have[1)]

$$\frac{s}{v_F} \sim \left(\frac{m}{M}\right)^{1/2} \ll 1, \quad \frac{\theta}{\varepsilon_F} \sim \left(\frac{m}{M}\right)^{1/2} \ll 1. \tag{3}$$

These estimates remain valid also in a more realistic study of electrons and phonons, and corroborate the adiabatic approximation that permits the use of perturbation theory when the electron-phonon interaction is taken into account. Figure 1 shows the described single-particle spectrum of the model "metal."

2. CONDUCTION ELECTRONS—QUASIPARTICLES

According to the contemporary representation of a metal[2)] (see Ref. 1 § 61), the conduction electrons form a Fermi liquid, while the individual carriers are quasiparticles that obey Fermi statistics (fermions). Their states are described in terms of the band theory by the quasimomentum **p** and by the band number α, and are frequently designated by $|\alpha, \mathbf{p}\rangle$; we denote the number of electrons in a state $|\alpha, \mathbf{p}\rangle$ by $n_{\mathbf{p}_\alpha}$, with $n_{\mathbf{p}_\alpha} = 0$ or 1. The ground state of the electron subsystem of a metal is similar to the ground state of a Fermi gas: the Fermi surface

$$\varepsilon_\alpha(\mathbf{p}) = \varepsilon_F \tag{4}$$

separates the occupied states from the free ones ($n_{\mathbf{p}_\alpha} = 1$ at

 0038-5670/85/030257-12$01.80

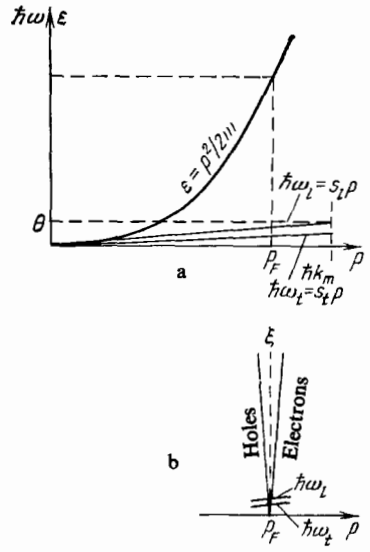

FIG. 1. Electron and phonon spectra of a model in the Drude-Lorentz-Sommerfeld-Debye model (schematic). a) Dependence of electron (ε) and phonon ($\hbar\omega$) energies on the momentum at $\hbar k_m > p_F$; b) dependence of electron (hole) and phonon energy under the same conditions. The letters l and t mark the phonon polarization.

$\varepsilon_\alpha(\mathbf{p}) < \varepsilon_F$ and $n_{\mathbf{p}_\alpha} = 0$ at $\varepsilon_\alpha(\mathbf{p}) > \varepsilon_F$). If the energy bands overlap, the Fermi surface can have cavities in certain bands. The local minima of the dispersion law $\varepsilon_\alpha = \varepsilon_\alpha(\mathbf{p})$ can cause several cavities to the present in one band (with the same number α). The total volume τ_F of the Fermi surface obeys the Luttinger theorem[2]:

$$\frac{2\tau_F}{(2\pi\hbar)^3} = n_e, \tag{5}$$

where n_e is the number of electrons in the partially filled bands ($n_e = \Sigma_\alpha n_\alpha$, n_α is the number of electrons in band α, and the summation is over only the partially filled bands).

It must be emphasized that *the existence of a Fermi surface does not mean neglect of the interaction between the electrons*. The quasiparticle energy $\varepsilon_\alpha(\mathbf{p})$ takes into account not only electron-ion interaction but also interaction of electrons with one another. The most important feature of the ground state of a metal can be the fact of *partial* filling of its quasimomentum space (of the periodic $\mathbf{p}$ space). Excitation of the electron system of a metal means "creation" of electrons and holes,[3)] and their energies of $|\mathbf{p} - \mathbf{p}_F^\alpha| \ll \hbar/a$ are

$$\xi(\mathbf{p}) = \begin{cases} \varepsilon_\alpha(\mathbf{p}) - \varepsilon_F \approx (\mathbf{p} - \mathbf{p}_F^\alpha)\,\mathbf{v}_F^\alpha & \text{(for electrons)}, \\ \varepsilon_F - \varepsilon_\alpha(\mathbf{p}) \approx (\mathbf{p}_F^\alpha - \mathbf{p})\,\mathbf{v}_F^\alpha & \text{(for holes)} \end{cases} \tag{6}$$

and are linear functions of the quasimomentum; $\mathbf{p}_F^\alpha$ is a point on the αth cavity of the Fermi surface, and

$$\mathbf{v}_F^\alpha = \frac{\partial\varepsilon_\alpha(\mathbf{p})}{\partial\mathbf{p}}\bigg|_{\mathbf{p}=\mathbf{p}_F^\alpha} \tag{7}$$

is the electron velocity and coincides with the normal to the Fermi surface (we note that $\xi_\alpha(\mathbf{p}) > 0$).

In the case of degeneracy, the Fermi surface has self-intersection points that are as a rule conical (see Ref. 3, § 2). Near these points, the electron and hole dispersion law is quadratic (an example is graphite, whose Fermi surface has conical points; see Ref. 3, § 11, as well as Ref. 4). The characteristic interval over which the electron or hole energy $\xi_\alpha(\mathbf{p})$ varies is determined by the band structure. In good metals the width of each band is of the order of the atom energy $\varepsilon_\alpha \sim h^2/ma^2 \sim e^2/a$. The interaction between the electrons complicates not only the actual calculations but also the principles underlying the description of the electron system (in the sense of transforming from particles to antiparticles), but does not change the order-of-magnitude estimates; the average electron-electron interaction energy involved in the Fermi-liquid description is approximately equal to the average energy of the interaction of the electrons with the ion core. The spectrum structure is made much more complicated by the periodicity of the space in which the electrons move. The Fermi surface is a periodic structure with a period specified by the crystal geometry:

$$\varepsilon_\alpha(\mathbf{p} + 2\pi\hbar\mathbf{b}) = \varepsilon_\alpha(\mathbf{p}), \tag{8}$$

where $\mathbf{b}$ is an arbitrary reciprocal-lattice vector. The minimum length of the reciprocal-lattice vector is $\sim 1/a$. Therefore $\hbar/a$ is the characteristic dimension of the Fermi surface. For open surfaces it is the period, and for closed ones it is the size of each cavity that repeats periodically in each unit cell of $\mathbf{p}$ space. (Depending on their geometric properties,[3] we can have open or closed Fermi surfaces and electron or hole Fermi surfaces.[4)] Metals whose hole and electron parts of the Fermi surface are equal are called compensated.) Crossing or near-crossing of the bands can change these estimates somewhat: most metals have besides "large" cavities also some whose sizes are considerably smaller than those of the reciprocal-space cell.[5] Investigation of magnetic breakdown[6] has shown that Fermi-surface cavities of different bands ($\alpha \neq \alpha'$) very frequently come close to one another and the barriers between them are small. As a rule, however, the smallness referred to here does not alter greatly the estimates based on the gas model (see above); viz., even on a small Fermi-surface cavity the electron velocity exceeds greatly that of sound, and the electron effective mass m^* is of the other of the free-electron mass m[5)] and is considerably less than the ion mass M. Naturally, the condition $(m^*/M)^{1/2} \ll 1$ under which the adiabatic approximation holds remains valid.

The remark concerning small cavities of Fermi surface does not apply to metals of the first group of the periodic table (1A: Li, Na, K, Rb, Cs and 1B: Cu, Ag, Au)—their Fermi surfaces simply have no cavities. This well established fact is explained by band calculations based on the electron structure of metallic atoms.[7]

3. PHONONS

The oscillations of metal ions can of course not be described quantitatively by the Debye theory. One must start from the real periodic arrangement of the ions in the crystal lattice. The vibrational spectrum of a crystal is determined by the dependence of the oscillation frequencies on the quasiwave vector[6)] $\mathbf{k}$ ($\hbar\mathbf{k}$ is the quasimomentum):

$$\omega_\nu = \omega_\nu(\mathbf{k}), \qquad \nu = 1, 2, \ldots, 3r, \tag{9}$$

where r is the number of ions in the unit cell. For three of these branches (the acoustic ones) the frequencies vanish at $\mathbf{k} = 0$ and are linear in $\mathbf{k}$ at $ak \ll 1$:

$$\omega_a = s_a(\mathbf{n})\, k, \quad a = 1, 2, 3; \quad \mathbf{n} = \frac{\mathbf{k}}{k}; \tag{10}$$

$s_a(\mathbf{n})$ is the sound velocity of the ath branch. The choice of the origin in periodic $\mathbf{k}$ space is arbitrary (the periods, naturally, are equal to $2\pi\mathbf{b}$). It would therefore be more accurate to say that the $\mathbf{k}$-space cell contains a point at which the frequencies of three out of the $3r$ vibration branches vanish in accord with Eq. (9) (if the quasiwave vector is reckoned from this point). This point coincides with the center of the first Brillouin zone of the crystal. With the origin so chosen in $\mathbf{k}$ space, the quasiwave vector $\mathbf{k}$ is fully equivalent to the wave vector of a macroscopic sound wave if $ka \ll 1$. This correspondence allows us to derive the elasticity-theory equations from the dynamic equations for crystal-lattice vibrations (see, e.g., Ref. 9).

Quantization of the ion-displacement waves introduces phonons, which are particles that obey Bose-Einstein statistics (bosons), and whose occupation numbers $N_{\mathbf{k},\nu}$ determine the degree of excitation of the crystal lattice ($N_{\mathbf{k},\nu} = 0, 1, 2, \ldots$ are integers). In the ground state there are no phonons at all: $N_{\mathbf{k},\nu} \equiv 0$—the space of the quasiwave vectors is empty. Of course, *this does not mean that the ions do not execute zero point oscillations.*

The conduction electrons must be taken into account in the calculation of the dynamic matrix of a crystal, if for no other reason than that the electrons compensate for the Coulomb repulsion of the ions and a stable system of positively charged ions could not exist without electrons. The electron and phonon spectra must be calculated in a self-consistent manner. The basis of such calculations is the adiabatic approximation, which makes possible the use of some variant of perturbation theory (see, e.g., Ref. 9). In the simplest but physically quite lucid approach, the role of the electrons (in the calculation of the phonon spectrum) reduces to screening of the field produced by a vibrating ion.[10]

4. ELECTRONS, PHONONS, AND ELECTRON-PHONON INTERACTION

An established tradition completes the self-consistent calculation of the energy spectrum of a metal by formulating the Hamiltonian of the quasiparticles [of the fermions (electrons and holes) and of the bosons (phonons)]

$$\mathscr{H} = \sum_{\alpha,\,\mathbf{p}} \xi_\alpha(\mathbf{p})\, \hat{a}^+_{\alpha\mathbf{p}} \hat{a}_{\alpha\mathbf{p}} + \sum_{\nu,\,\mathbf{p}} \hbar\omega_\nu(\mathbf{p})\, \hat{b}^+_{\nu\mathbf{p}} \hat{b}_{\nu\mathbf{p}} + \mathscr{H}_{\text{int}}; \tag{11}$$

where $\hat{a}^+_{\alpha\mathbf{p}}$ and $\hat{a}_{\alpha\mathbf{p}}$ are the fermion operators for creation and annihilation of an electron in a state $|\alpha, \mathbf{p}\rangle$ [or of a hole, depending on the value of $\mathbf{p}$; see Eq. (6)], the spin index being included in α; $\hat{b}^+_{\nu\mathbf{p}}$, $\hat{b}_{\nu\mathbf{p}}$ are the boson operators for creation and annihilation of a phonon in a state $|\nu, \mathbf{p}\rangle$. When comparing (9) with (10) it must be recognized that $\mathbf{p} = \hbar\mathbf{k}$ for phonons. The Hamiltonian $\mathscr{H}_{\text{int}}$ describes the interaction between the quasiparticles. Expression (11) is somewhat arbitrary. To make it more specific it is possibly necessary to use different expressions for $\mathscr{H}_{\text{int}}$ in the electron and phonon Hamiltonians (cf. Refs. 44 and 45). $\mathscr{H}_{\text{int}}$ has terms that contain more than two creation or annihilation operators (anharmonicities). It is natural to distinguish between the phonon-phonon interaction

$$\mathscr{H}_{\text{ph, ph}} = \sum_{1,\,2,\,3} \Phi_{1;\,2,\,3} \hat{b}^\dagger_1 \hat{b}_2 \hat{b}_3 + \text{H.c.}, \tag{12}$$

the electron-electron interaction

$$\mathscr{H}_{\text{e, e}} = \sum_{1,\,2;\,3,\,4} \Psi_{1,\,2;\,3,\,4} \hat{a}^\dagger_1 \hat{a}^\dagger_2 \hat{a}_3 \hat{a}_4 + \text{H.c.} \tag{13}$$

and the electron-phonon interaction, the one in which we are mainly interested:

$$\mathscr{H}_{\text{e, ph}} = \sum_{1,\,2,\,3} \lambda^{\text{I}}_{1,\,2,\,3} \hat{a}^\dagger_1 \hat{a}_2 \hat{b}_3 + \lambda^{\text{II}}_{1,\,2,\,3} \hat{a}^\dagger_1 \hat{a}_2 \hat{b}^\dagger_3 + \text{H.c.} \tag{14}$$

The terms with the smallest number of creation and annihilation operators have been written out. The subscripts 1, 2, 3, ... designate α_1, $\mathbf{p}_1$ etc. or ν_1 $\mathbf{p}_1$ etc. The summation over the $\mathbf{p}_i$ is within the limits of one unit cell. The amplitudes $\Phi \ldots$, $\Psi \ldots$, $\lambda \ldots$ must each contain a factor that ensures the quasimomentum conservation; for example, for phonon-phonon interaction

$$\mathbf{p}_2 + \mathbf{p}_3 = \mathbf{p}_1 + 2\pi\hbar\mathbf{b},$$

where $\mathbf{b}$, as always, is the reciprocal-lattice vector, such that all the vectors of the initial and final states are in one unit cell of $\mathbf{p}$ space.

We have already noted that a detailed treatment of the electron and phonon spectra of the metal does not change the results based on the simplest estimates [see Ftn. 1)]. Therefore

$$v_{\text{F}} \sim \frac{\hbar}{am}, \quad s \sim \frac{\hbar}{a\sqrt{Mm}}, \quad \omega_{\text{D}} \sim \frac{s}{a}. \tag{15}$$

Since $a \sim \hbar^2/e^2 m$, the last estimates can be recast in the quite simple form

$$v_{\text{F}} \sim \frac{1}{137} c, \quad s \sim \frac{c}{137}\sqrt{\frac{m}{M}},$$
$$\theta \sim \frac{mc^2}{(137)^2}\sqrt{\frac{m}{M}}, \quad \frac{1}{137} \approx \frac{e^2}{\hbar c}. \tag{15'}$$

For order-of-magnitude estimates of the lattice anharmonicities [of the coefficients in (12) and (14)] we note that the operator of the $(\nu, \mathbf{k})$th Fourier component of the displacement vector is proportional to $(\hat{b}^+_{\nu\mathbf{k}} + \hat{b}_{\nu\mathbf{k}})/2$, and the proportionality coefficient is of the order of

$$\sqrt{\frac{1}{MN}\frac{\hbar}{\omega_\nu(\mathbf{k})}};$$

N is the number of crystal cells. Using this, we obtain[11]

$$\Phi \ldots \sim \theta\sqrt{\frac{\theta}{Ms^2}}, \quad \lambda \ldots \sim \sqrt{\theta\varepsilon_{\text{F}}}. \tag{16}$$

We have left out the factor $N^{-1/2}$, which is cancelled out in the calculation of an actual process on account of the summation over the quasimomenta. In addition, the estimate (16) does not contain any dependence on the quasimomentum; such a dependence can decrease substantially the amplitudes (for example, when dealing with emission of a long-wave acoustic phonon, the amplitude $\lambda \ldots$ contains an additional factor ak; see Ref. 12, § 79).

The electron-electron interaction amplitude $\Psi \ldots$ is determined by the Coulomb interaction between two elec-

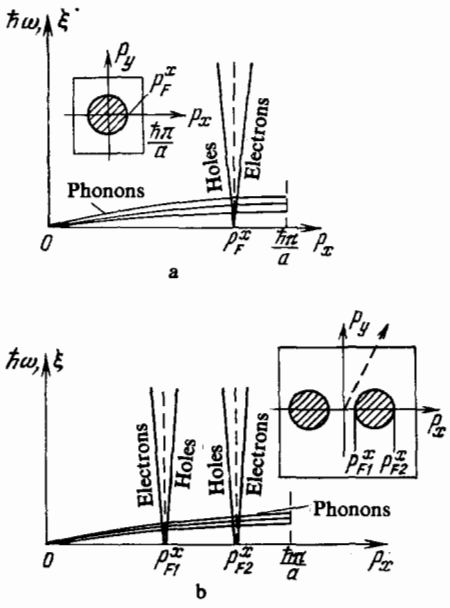

FIG. 2. Schematic representation of electron and phonon spectra of metals. a) The electron Fermi surface encloses the origin; b) origin outside the electron Fermi surface (the electron and hole branches change places in the case of a hole Fermi surface). Insets—Fermi surfaces. The dashed lines show the direction in which there are no holes and the electron energy differs substantially from the Fermi energy. The region of states occupied by electrons ($\varepsilon < \varepsilon_F$) is shown shaded in the insets.

trons; to be sure, some part of the interaction "entered" in the dispersion law of the quasiparticles (electrons and holes). In addition, at large electron-electron distances the interaction is certain to be neutralized (screened) by redistribution of other electrons[7] (all the remaining ones, except the two considered[13]).

It might seem that a characteristic feature of the Hamiltonian (11) with the interaction specified by Eqs. (12)–(14) is that the electron-phonon and electron-electron interactions are large,[8] thereby casting doubts on the possibility of using the concept of almost-independent quasiparticles (electrons and phonons). Actually, however, in all but exceptional cases (on some of which we shall dwell below) the substantial difference between the electron and ion masses makes the interaction effectively small, and in the treatment of electrons the small parameter is not Ψ, but the degree of excitation of the electron subsystem, T/ε_F and ξ/ε_F respectively when macroscopic or microscopic problems are solved (T is the temperature). In other words, an important role is played by the *degeneracy* of the metal electrons, which makes the number of quasiparticles (electrons, holes) small and their interaction therefore not too large.

In nature there are no ideal defect-free metals, infinite crystals, absolute zero temperature and other objects simplified at the will of the theoretical physicist. In the study of the spectrum of elementary excitations of a macroscopic body, however, it is natural to start out with just a scheme that is simplified to the limit. We shall assume therefore that the metal is at zero temperature and has no defects whatever. This simplification enables us to investigate just those intrinsic dissipation mechanisms that are possessed by even an ideal body and limit the lifetime τ of the quasiparticles.[9] Comparison of the time governed by the intrinsic dissipation mechanism with that governed by defects makes it possible to ascribe a quantitative meaning to the concept of a "defect-free crystal." The assumption $T = 0$ is not only convenient but also natural for the investigation of the properties of individual quasiparticles.

5. LOW-ENERGY STATES

Figure 2a shows the dependence of the metal-quasiparticle energy on the quasimomenta, neglecting the anharmonicities, in a "good" crystallographic direction. Although the figure is reminiscent of Fig. 1b, there are substantial differences, principal among which is the anisotropy: at another direction the dependence can differ not only quantitatively (in the values of all the parameters) but also qualitatively. For example, if the cavity of the Fermi surface does not contain the origin (we recall that the latter is set by the phonon branches), there may be no electrons (or holes) at all in some direction, while in another direction the number of electron-hole branches is doubled (Fig. 2b).

We confine ourselves to a discussion of only the metal states with the lowest energies ($\varepsilon \lesssim 0$). If the metal has several atoms per unit cell, it is necessary to add the optical branches to the acoustic branches of the phonon spectrum (see Figs. 1 and 2).

Phonons are quanta of matched oscillation of the ions and electrons that maintain the local neutrality of the metal (see below). An electron liquid can contain also oscillations of another type, relative to practically immobile ions. These oscillations, which are typical of any two-component plasma, have as a result of violation of the local neutrality high frequencies $\omega \sim \omega_L = (4\pi ne^2/m)^{1/2}$. The energy of the plasmons (of the quasiparticles corresponding to these oscillations) is of the order of the Fermi energy:

$$\hbar\omega_L \sim \sqrt{\frac{4\pi\hbar^2 e^2 n}{m}} \sim \sqrt{\frac{\hbar^2}{ma^2}\frac{e^2}{a}} \sim \varepsilon_F,$$

since $$\frac{e^2}{a} \sim \frac{\hbar^2}{ma^2} \sim \varepsilon_F$$

(see above). The plasmons are therefore not shown in Figs. 1 and 2.

Under certain conditions,[14] quasiacoustic electron-hole oscillations with a linear dispersion law might apparently be observed in compensated metals (the amplitude of the charge-density oscillations in these modes is zero). To our knowledge, however, these oscillation branches have not yet been observed.

By placing the metal in a magnetic field we change the character of the electron motion in it. The electron rotation at the cyclotron frequency $\omega_c = eH/m^*c$ makes possible the existence of a variety of undamped waves in the conductors, and quantization gives rise to unusual boson quasiparticles (helicons, dopplerons, and others) with energies considerably lower than ε_F and $\hbar\omega_L$. Were we to investigage the influence of the magnetic field, these quasiparticles would occupy "important" positions in the figures and in the exposition. We note only, however, that both the plasmons and

the quasiparticles brought about by the magnetic field are modified photons—electromagnetic-field quanta[10] (an exception is the longitudinal plasmon, whose motion is not accompanied by oscillations of the magnetic field).

The Bose modes are not confined to phonons, even in the absence of a magnetic field. If the metal has a spin structure, the oscillations of the average magnetic moments (spin waves) generate distinctive quasiparticles, magnons, which obey Bose-Einstein statistics. The characteristic magnon energy (the analog of the Debye energy) is determined by the Curie temperature θ_C (for ferromagnets) or the Néel temperature θ_N (for antiferromagnets. As a rule $\theta_C, \theta_N \ll \varepsilon_F$, and in this sense magnons are similar to phonons. Assuming that the investigated metals are paramagnets, we shall not deal with the magnon branches of the spectrum.[11]

A unique place among the low-energy oscillations of the electron subsystem of a metal is occupied by oscillations of the conduction-electron distribution function; these oscillations can be called "quasiwaves." Let $f_{\mathbf{k}}(\mathbf{p}, t)$ be the kth component of the nonequilibrium part of the electron distribution function. In the collisionless limit it can oscillate at a frequency that depends both on the wave vector $\mathbf{k}$ and on the quasimomentum $\mathbf{p}$ (see Ref. 19):

$$\omega = \mathbf{kv}(\mathbf{p}). \tag{17}$$

This relation can be treated as the dispersion law of the "quasiwaves." If the electron system is taken out of the equilibrium position and left "to its own resources," it oscillates, after the magnetic fields have decayed (in a time determined by the Landau damping[19]), at frequencies given by Eq. (17).

If the initial perturbation is weak enough and the oscillations are linear, the degeneracy of the electron liquid singles out the Fermi velocities from among all the others, i.e.,

$$\omega = \mathbf{kv}(\mathbf{p}_F) \tag{17'}$$

($\mathbf{p}_F$ belongs to the Fermi surface). Although electron density does not fluctuate when "quasiwaves" propagate, the "quasiwaves" are quite real. Thus, second absorption by metal electrons is in the collisionless limit (see below) is in essence a resonance between the sound wave and the "quasiwave." We shall return below to the role of "quasiwaves" in the high-frequency properties of metals.

6. INTERACTION: RENORMALIZATIONS AND SINGULARITIES

Strictly speaking, $\xi_\alpha(\mathbf{p})$ and $\hbar\omega_\nu(\mathbf{p})$ are not the energies of the quasiparticles (of the electrons, holes, and phonons, respectively). The interaction described by $\mathscr{H}_{\text{int}}$ renormalizes the quantities $\xi_\alpha(\mathbf{p})$ and $\hbar\omega_\nu(\mathbf{p})$, and is in addition the cause of the quasiparticle finite lifetime. The most striking example of renormalization is the appearance of the gap Δ in the hole or electron spectrum, due to production of Cooper pairs, and hence to the transition to the superconducting state (see, e.g., Ref. 1, Chap. V). Although this phenomenon is the consequence of electron-electron interaction via exchange of virtual phonons, its description is outside the scope of the present article. We confine ourselves to describing the electron and hole spectrum in the superconducting

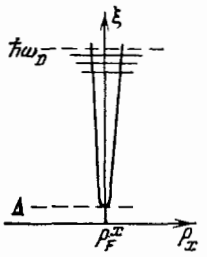

FIG. 3. The superconducting gas Δ is many times smaller than the Debye energy.

state (Fig. 3) and emphasize that we are mainly interested in electrons and holes having energies substantially higher than the gap Δ, the latter being of the order of one or several degrees. From among the characteristic energies encountered in this context, the lowest is $ms^2 \sim 10$ K, and in the overwhelming majority of cases larger than Δ. Failure to take the Cooper pairing into account calls for no justification at all if we confine ourselves to metals that do not become superconducting even at absolute zero—there are quite a few of them.

Before we report the results of a number of specific investigations of electron and phonon interactions, we make two essential remarks.

First. Special attention will be paid to long-wave phonons. They can be treated classically, for at $ak \ll 1$ the quasiwave vector $\mathbf{k} = \mathbf{p}/\hbar$ coincides with the true wave vector (see below), and a second wave with definite values of $\mathbf{k}$ and ω can regarded as a coherent state of a large number of phonons ($Nk \gg 1$).

The *second* remark pertains to the quasimomentum dependences of the quantities $\xi_\alpha(\mathbf{p})$ and $\hbar\omega_\nu(\mathbf{p})$. To analyze the analytic properties of the excitation spectrum it can be assumed that the periodic functions $\xi_\alpha = \xi_\alpha(\mathbf{p})$ and $\hbar\omega_\nu = \hbar\omega_\nu(\mathbf{p})$ have no singularities,[12] or if the latter are unavoidable, they are weaker than those resulting from the renormalization due to $\mathscr{H}_{\text{int}}$ (see below).

The functions $\xi_\alpha(\mathbf{p})$ and $\hbar\omega_\nu(\mathbf{p})$ can apparently acquire singularities either because of the approximate calculation method (these will be simply disregarded, assuming that the calculation is consistent enough[13]), or because the momentum space of the electrons in the ground state is *not uniformly filled* (a dynamic calculation of the metal characteristics, with account taken of the interaction between all the crystal particles, can include integration over the $\mathbf{p}$ space whose inhomogeneous filling—the existence of a discontinuous Fermi step—can in principle be the cause of the singularities.[14] We shall verify, however, that the singularities referred to below are connected as a rule with integration over the Fermi surface; their character and position in the $\mathbf{p}$ space of the phonons and electrons are determined by the local structure of the Fermi surface and by its dimensions. It is assumed that the functions $\xi_\alpha(\mathbf{p})$ and $\hbar\omega_\nu(\mathbf{p})$ do not have *these* singularities.

We repeat: the quasiparticle interaction described by the Hamiltonian $\mathscr{H}_{\text{int}}$ renormalizes the dispersion laws of the electrons (holes) and of the phonons, and also limits the quasiparticle lifetime. We shall write down the standard equations for the imaginary parts of the electron and phonon energies, for owing to the presence of the δ functions corre-

sponding to the energy conservation law their analysis is more lucid than the analysis of the expressions for the renormalization of the energies themselves (of the real parts), and the singularities of the energies can be investigated by using the integral relation between the real and imaginary parts that generalize the Kramers-Kronig relations.[15] We shall use perturbation theory, since an additional complication does not change the qualitative picture, and in most cases there is simply no need to go beyond perturbation theory.

Thus

$$\delta\varepsilon(\mathbf{p}) = -\pi \sum_{\nu} \int d^3p' \, |M^{\nu}_{\mathbf{p},\mathbf{p}'}|^2 \times \delta(\varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p}') - \hbar\omega_{\nu}(\mathbf{p}-\mathbf{p}')) (1 - n_{\mathbf{p}'}), \tag{18}$$

$$\delta\omega_{\nu}(\mathbf{q}) = -\pi \int d^3p \, |M^{\nu}_{\mathbf{p},\mathbf{p}+\mathbf{q}}|^2 (n_{\mathbf{p}} - n_{\mathbf{p}+\mathbf{q}}) \times \delta(\varepsilon(\mathbf{p}) + \hbar\omega_{\nu}(\mathbf{q}) - \varepsilon(\mathbf{p}+\mathbf{q})),$$

where

$$M^{\nu}_{\mathbf{p},\mathbf{p}'} = \sqrt{2}\,\Lambda^{\nu}_{\mathbf{p}\mathbf{p}'} \, |\mathbf{p}-\mathbf{p}'| \, (2\pi\hbar)^{-3/2} [\rho\hbar\omega_{\nu}(\mathbf{p}-\mathbf{p}')]^{-1/2}, \tag{19}$$

and $\Lambda^{\nu}_{\mathbf{p}\mathbf{p}'}$ tends to the deformation potential as $\mathbf{p}' \to \mathbf{p}$ and is of the order of ε_F (see Refs. 20 and 21), and ρ is the density of the metal [by comparing (18) and (19) with (14) we easily establish the connection between the amplitude $\lambda^{I} \ldots$, $\lambda^{II} \ldots$, and $M \ldots$; we have confined ourselves to the one-band approximation only to simplify matters, and have therefore left out the band-number index α]; just as before, $n_{\mathbf{p}}$ is the Fermi step; the phonon quasimomentum is designated by $\mathbf{q}$. Expressions (18) are written not in terms of electrons and holes. The notation used here is more convenient for geometric interpretation. It is clear that by adding and subtracting the energy ε_F in the argument of the δ function we can easily transform from $\varepsilon(\mathbf{p})$ to $\xi(\mathbf{p})$ [cf. Eq. (6)].

Of course, the electron-electron and electron-phonon interactions (12) and (13) also contribute to Im $\varepsilon(\mathbf{p})$ and to Im $\hbar\omega_{\nu}(\mathbf{q})$. They have been left out, since we wish to focus on the electron-phonon interaction. To be sure, it must be borne in mind here that the electron-phonon damping mechanism is the basic one only at $|\varepsilon(\mathbf{p}) - \varepsilon_F| > ms^2$, and that at lower energies the principal role is assumed by the electron-electron interactions, particularly via phonons,[22,21]—this interaction leads to a quadratic relation for Im $\varepsilon(\mathbf{p})$:

$$\mathrm{Im}\,\varepsilon(\mathbf{p}) \sim \varepsilon_F \left(\frac{\varepsilon(\mathbf{p})-\varepsilon_F}{\varepsilon_F}\right)^2. \tag{20}$$

7. ELECTRON-SPECTRUM RENORMALIZATION DUE TO THE ELECTRON-PHONON INTERACTION

The influence of the electron-phonon interaction on the electron spectrum was investigated by Migdal,[22] who assumed a quadratic and isotropic electron spectrum and Debye phonons. The renormalization was reduced, naturally, to renormalization of the effective mass. The equations obtained by A. B. Migdal were generalized in Ref. 21 to include the case of arbitrary electron and phonon dispersion.

At first glance, the concepts "renormalization" and "arbitrary dispersion law" are incompatible. What sense is there in renormalization of the function $\varepsilon = \varepsilon(\mathbf{p})$ if it is only *assumed* known, even though in fact very little is known of it? (We refer not so much to the approximate numerical value of $\varepsilon(\mathbf{p})$ as to the functional dependence of ε on $\mathbf{p}$.)

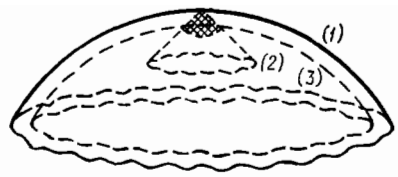

FIG. 4. Illustrating expression (18): integration region in p′ space as $|\varepsilon(\mathbf{p}) - \varepsilon_F| \to 0$ (shaded). 1) $\varepsilon(\mathbf{p}') = \varepsilon(\mathbf{p})$ surface; 2) surface (23); 3) Fermi surface.

The point is that the renormalization covers a small $\mathbf{p}$-space region directly adjacent to the Fermi surface $\Delta p \sim \hbar\omega_D/v_F$, and in this interval the renormalization $\delta\varepsilon(\mathbf{p})$ changes by an amount $\sim\hbar\omega_D$. This leads to a finite (in the m/M scale) change of the electron velocity: only if $\Delta p \gg ms$ does the velocity "take on" its value $\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}$ which is not perturbed by the phonons (see Refs. 22 and 21). In addition, the smallness of the interval Δp affected by the renormalization allows us to obtain for Re $\delta\varepsilon$ an expression in the form of an integral over the Fermi surface[22]:

$$\mathrm{Re}\,\delta\varepsilon(\mathbf{p}) = \oint \frac{dS_F}{v(\mathbf{p}_F)} \, |M_{\mathbf{p},\mathbf{p}_F}|^2 \ln \frac{\varepsilon(\mathbf{p}) - \varepsilon_F - \hbar\omega(\mathbf{p}-\mathbf{p}_F)}{\varepsilon(\mathbf{p}) - \varepsilon_F + \hbar\omega(\mathbf{p}-\mathbf{p}_F)}, \tag{21}$$

from which it follows that

$$\mathrm{Re}\,\delta\varepsilon(\mathbf{p}) \sim \begin{cases} \varepsilon(\mathbf{p}) - \varepsilon_F & \text{at} \quad |\varepsilon(\mathbf{p}) - \varepsilon_F| \ll \hbar\omega_D, \\ (\varepsilon(\mathbf{p}) - \varepsilon_F)^{-1} & \text{at} \quad |\varepsilon(\mathbf{p}) - \varepsilon_F| \gg \hbar\omega_D. \end{cases} \tag{22}$$

As shown in Ref. 21, the transformation from one relation to the other must be accompanied by a singularity, possibly by more than one (see Sec. 8).

8. ELECTRON LIFETIME AND ITS SINGULARITIES

Let us return to Eq. (18). The presence under the integral sign of a δ function and of a factor $(1 - n_{\mathbf{p}'})$ that is nonzero at $\varepsilon(\mathbf{p}') > \varepsilon_F$ shows that the integration is carried out (in $\mathbf{p}'$ space) over that part of the surface

$$\varepsilon(\mathbf{p}') + \hbar\omega(\mathbf{p} - \mathbf{p}') = \varepsilon(\mathbf{p}), \tag{23}$$

which lies outside the Fermi surface. Since $\hbar\omega_D \ll \varepsilon_F$, the surface (23) is close to the equal-energy surface

$$\varepsilon(\mathbf{p}') = \varepsilon(\mathbf{p}), \tag{24}$$

and if the interaction is with acoustic phonons,[16] the two surfaces are tangent at one point. As $|\varepsilon(\mathbf{p}') - \varepsilon(\mathbf{p})| \to 0$, the surface (23) near the tangency point is the cone

$$\Delta p_n = -\frac{s}{v(\mathbf{p})} \, |\mathbf{p} - \mathbf{p}_\varepsilon|, \quad \Delta p_n = p'_n - p_n; \tag{24′}$$

p'_n and p_n are the projections of the vectors $\mathbf{p}'$ and $\mathbf{p}$ on the normal to the surface (34) at the point $\mathbf{p}$, while $\mathbf{p}_\varepsilon$ is a point on the surface (24). According to (18) we have (Fig. 4)

$$\mathrm{Im}\ \delta\varepsilon(\mathbf{p}) \approx \lambda_e(\mathbf{p}_F)\,|\,\varepsilon(\mathbf{p})-\varepsilon_F\,|^3, \quad |\,\varepsilon(\mathbf{p})-\varepsilon_F\,| \ll \hbar\omega_D,$$

$$\lambda(\mathbf{p}_F) = -\frac{\pi\sum_a |\,\Lambda^a_{\mathbf{p}_F}\,|^2}{3\,(2\pi\hbar)^3\,\rho s_a^4 v_F}, \tag{25}$$

$$\Lambda^a_{\mathbf{p}_F} = \Lambda^a_{\mathbf{p},\mathbf{p}'}\Big|_{\substack{\mathbf{p}=\mathbf{p}_F\\ \mathbf{p}'=\mathbf{p}_F}}, \quad \mathbf{v}_F = \mathbf{v}(\mathbf{p}_F);$$

here $\mathbf{p}_F$ is the quasimomentum corresponding to the Fermi-surface point closest to $\mathbf{p}$ ($\mathbf{p}$ and $\mathbf{p}_F$ lie on the same normal to the Fermi surface; $s_a = s_a/(\mathbf{v}_F/v_F)$ is the sound velocity in the direction of the normal, and the summation is over the three acoustic branches). Equation (25) generalizes Migdal's result (see Ref. 22) and shows that the cubic dependence of $\mathrm{Im}\ \delta\varepsilon(\mathbf{p})$ on $|\varepsilon(\mathbf{p})-\varepsilon_F|$ is not connected with any simplifying assumptions concerning the conduction-electron spectrum. It can be seen that the local properties of the Fermi surface manifest themselves via the factor $\lambda(\mathbf{p}_F)$, which of course change as $\mathbf{p}$ "moves" around the Fermi surface, but if the Fermi surface is smooth there are no grounds for expecting $\lambda(\mathbf{p}_F)$ to have any singularities at all. If, however, the Fermi surface has a conical point $\mathbf{p}_F^c$ at which the velocity vanishes, Eq. (25) no longer holds for quasimomenta $\mathbf{p}$ that are close to $\mathbf{p}_F^c$. It can be shown that in a small $\mathbf{p}$-space region around the conical point the imaginary part of $\delta\varepsilon(\mathbf{p})$, which is due to electron-phonon interaction, is zero and is determined by the interaction of the electrons with one another [see (20)]. The reason why the electron-phonon interaction is "turned off" is that an electron with a velocity lower than that of the sound cannot emit (absorb) a phonon.

At $|\varepsilon(\mathbf{p})-\varepsilon_F| \gg \hbar\omega_D$ the entire surface (23) is outside the Fermi surface, and $\mathrm{Im}\ \delta\varepsilon(\mathbf{p})$ depends little on the energy—it "saturates":

$$\mathrm{Im}\ \delta\varepsilon(\mathbf{p}) \approx -\pi \oint \frac{dS_F}{v(\mathbf{p}_F)}\,|\,M_{\mathbf{p}_F,\mathbf{p}}\,|^2. \tag{26}$$

The word "saturates" is in quotes, since $M_{\mathbf{p}_F,\mathbf{p}}$ depends on $\mathbf{p}$ but varies in the interval $\sim\hbar/a$. The dynamic character of the expressions obtained ($\mathrm{Im}\ \delta\varepsilon(\mathbf{p})\to 0$ as $M\to\infty$) can be seen from the order-of-magnitude estimate

$$|\mathrm{Im}\ \delta\varepsilon(\mathbf{p})| \sim \begin{cases} \theta\left(\dfrac{\varepsilon(\mathbf{p})-\varepsilon_F}{\theta}\right)^3, & |\,\varepsilon(\mathbf{p})-\varepsilon_F\,| \ll \theta, \\ \theta, & |\,\varepsilon(\mathbf{p})-\varepsilon_F\,| \gg \theta, \end{cases} \tag{27}$$

and $\theta \sim M^{-1/2}$. Comparison of (20) and (27) shows that the second of these equations is valid at $|\varepsilon(\mathbf{p})-\varepsilon_F| \gg \theta^2/\varepsilon_F$.

The transition from the cubic dependence to saturation is accompanied by singularities. These include an *obligatory* one, which appears at those values $\mathbf{p}=\mathbf{p}_c$ at which the entire surface (23) is outside the Fermi surface[17] (Fig. 5). At $\mathbf{p}\approx\mathbf{p}_c$

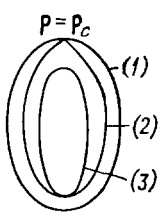

FIG. 5. Locations in $\mathbf{p}'$ space of the surfaces encountered in the text at $\mathbf{p}=\mathbf{p}_c$. 1) $\varepsilon(\mathbf{p}')=\varepsilon(\mathbf{p})$, 2) surface (23), 3) Fermi surface.

TABLE I.

$\mathrm{Im}\ \delta\varepsilon(\mathbf{p})$	Jump to derivative	$\Delta \ln(\varepsilon_F/\|\Delta\|)$
$\mathrm{Re}\ \delta\varepsilon(\mathbf{p})$	$\Delta \ln(\varepsilon_F/\|\Delta\|)$	Jump to derivative

and $\varepsilon \lesssim \varepsilon_c \equiv \varepsilon(\mathbf{p}_c)$ the line of intersection of the Fermi surface with the surface (23) in an ellipse. This enables us to calculate the singular part (SP) of $\mathrm{Im}\ \delta\varepsilon(\mathbf{p})$ (see Ref. 21):

$$\mathrm{SP}\ \mathrm{Im}\ \delta\varepsilon(\mathbf{p}) = \begin{cases} K_c(\varepsilon_c-\varepsilon(\mathbf{p})), & \varepsilon(\mathbf{p})<\varepsilon_c, \\ 0, & \varepsilon(\mathbf{p})>\varepsilon_c, \end{cases} \tag{28}$$

$K_c = \pi|M_c|^2 m_c^*/v_c$, while $m_c^* = \sqrt{\tilde{M}_1\tilde{M}_2} \sim \sqrt{mM}$ is a parameter with dimension of mass, and appears when the argument of the δ function is expanded in the tems of the deviation from the tangency point; the quatities contained in the coefficients are taken at the tangency point.

If the Fermi surface has dents and necks, intermediate singularities are practically always possible (their geometric locus is located between the Fermi surface and the geometric locus of the points $\mathbf{p}_c$). The singularities of particular interest are those due to a change in the topology of the line of intersection of the surface (23) with the Fermi surface. According to Ref. 21, the singular point has the following structure:

$$\mathrm{SP}\ \mathrm{Im}\ \delta\varepsilon(\mathbf{p}) = K_c'\Delta \ln\frac{\varepsilon_F}{|\,\Delta\,|}, \quad \Delta = \varepsilon(\mathbf{p})-\varepsilon(\mathbf{p}_c'); \tag{29}$$

K_c' differs from the factor in (28) in that $(\tilde{M}_1\tilde{M}_2)^{1/2}$ is replaced by $|\tilde{M}_1\tilde{M}_2|^{1/2}$ (in this case $\tilde{M}_1\tilde{M}_2<0$) and in that the factor does not contain π.

As already mentioned, it is easy to establish a correspondence between the singularities of $\mathrm{Im}\ \delta\varepsilon(\mathbf{p})$ and $\mathrm{Re}\ \delta\varepsilon(\mathbf{p})$ (Table I). It can be seen that the interaction of the electrons with the phonons *must* produce a singularity of the type $\Delta\ln|\Delta|$ in the electron dispersion law of a normal metal. But this means (formally!) that the electron velocity $\mathbf{v}=\partial\varepsilon/\partial\mathbf{p}$ becomes infinite at $\mathbf{p}=\mathbf{p}_c$. This circumstance calls for a deeper insight into the "cutoff" mechanism. In this case there is no need to take the retardation into account [see Ftn. 13], since the natural "cutoff" mechanism is the finite lifetime of the electron with energy $\varepsilon(\mathbf{p})>\varepsilon_F$. Therefore the diverging logarithm is replaced by $\ln(\varepsilon_F/\theta)\sim\ln(M/m)$, a characteristic factor most frequently encountered in those cases when the standard perturbation theory (the adiabatic approximation) fails.

The last equations can claim only a qualitative description of the electron spectrum, for in this case perturbation theory is not useful for revealing the singularities. The coefficients in (28) and in (29) are close to unity, and even after the substitution $\ln(\varepsilon_F/|\Delta|)\to\ln M/m$ is made the change of the velocity at $\mathbf{p}\approx\mathbf{p}_c$ exceeds the velocity $\partial\varepsilon/\partial\mathbf{p}$ at $\mathbf{p}=\mathbf{p}_c$. Judging from Migdal's paper,[22] a rigorous treatment does not change the qualitative picture of the spectrum renormalization. Migdal, however, did not investigate the singularities of $\mathrm{Im}\ \delta\varepsilon$ and $\mathrm{Re}\ \delta\varepsilon$. It is furthermore possible that an analysis that goes outside the framework of perturbation theory will necessitate a review of the formulas listed in the table. It happens frequently that the exact formulas differ from the

approximate ones in that the diverging term is transferred from the numerator to the denominator! At any rate, the results presented here (see also Ref. 21) mean that something "interesting" occurs at $\mathbf{p} \approx \mathbf{p}_c$, and they undoubtedly retain the same form, which draws attention to the immediate vicinity of the Fermi surface, in the rigorous theory.

9. PHONON LIFETIME AND ITS SINGULARITIES

Actively interacting with the phonons are the *electrons* from a thin layer (thickness $\sim ms$) of the quasimomentum space around the Fermi surface. But *all* the phonons take part in this interaction. This manifests itself in the fact that phonon-spectrum singularities due to electron-phonon interaction can be produced over a large range of quasimomentum values.[23]

In order not to clutter up the exposition with "minutiae" (see Refs. 23 and 24), we simplify Eq. (18) for $\operatorname{Im} \delta\omega_\nu(\mathbf{q})$, using the smallness of the phonon energy compared with the Fermi energy:

$$\operatorname{Im} \delta\omega_\nu(\mathbf{q}) = -\pi\hbar\omega_\nu(\mathbf{q}) \int |M^\nu_{\mathbf{p}\ \mathbf{p}+\mathbf{q}}|^2 \delta(\varepsilon(\mathbf{p}) - \varepsilon_F) \times \delta(\varepsilon(\mathbf{p}+\mathbf{q}) - \varepsilon_F)\, d^3p \qquad (30)$$

(we made the substitution $-\partial n_p/\partial\varepsilon \to \delta(\varepsilon - \varepsilon_F)$. So long as the Fermi energy and its analog shifted by $-\mathbf{q}$ intersect, with $q < 2p_F$, the integral in (30) differs from zero, and it is well known that[25,26]

$$\frac{\operatorname{Im}\delta\omega}{\omega} \sim \frac{s}{v_F} \sim \sqrt{\frac{m}{M}}. \qquad (31)$$

When q reaches the value $2p_F$, the integral vanishes jumpwise, and a singularity of the type $\Delta\ln|\Delta|$ (the Migdal-Kohn singularity) appears in the spectrum, with $q - 2p_F = \Delta$ in this case; $2p_F$ is here arbitrary—the critical values $\mathbf{q}_c$ of the phonon quasimomentum $\mathbf{q}$ are equal to the diameters (inner and outer) in a specified direction of the unit vector $\mathbf{q}/q$[18].

Of course, the phonon velocity does not become infinite at $\mathbf{q} = \mathbf{q}_c$; allowance for the finite lifetime of the electron leads to replacement of $\ln(q_c/\Delta)$ by $\ln(\varepsilon_F/\hbar\omega_D)$, since the electrons that take part in the formation of the Migdal-Kohn singularity have an energy higher by $\sim\hbar\omega_D$ than the Fermi energy (cf. the preceding section).

The Migdal-Kohn singularities are not the only ones in the phonon spectrum. If the Fermi surface has dents, necks (or, in other words, if the Fermi surface has *lines of parabolic points*), the phonon spectrum has, besides the Migdal-Kohn singularities, the so-called Taylor singularities[30] at those values of $\mathbf{q} = \mathbf{q}_c'$ at which the line of intersection of the $\varepsilon(\mathbf{p}+\mathbf{q}) = \varepsilon_F$ surface with the Fermi surface changes its topology, and the vector $\mathbf{q}_c'$ connects those points on the Fermi surface at which the velocities (normals) are *parallel* (at the end points of the diameters that determine the value of $\mathbf{q}_c$, the velocities are *antiparallel*). The geometric locus of the Taylor singularities and their character were investigated in detail in Ref. 23. We call attention here only to the following: whereas the Migdal-Kohn singularities occur only at large values of the phonon quasimomentum ($\mathbf{q}_c \sim \hbar/a$), the lengths of the critical quasimomenta of the Taylor singularities, if they exist, must vanish at definite directions $\varkappa = \mathbf{q}/q$ (Fig. 6).

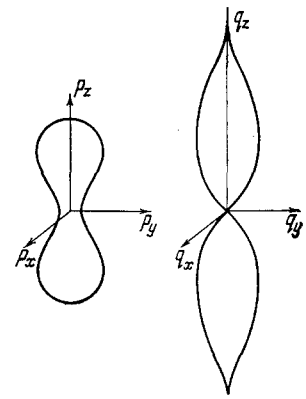

FIG. 6. Geometric locus of the phonon-spectrum Taylor singularities due to a "dumbbell" type Fermi-surface cavity (one of the possibilities, see Ref. 23).

Let us dwell on this question in somewhat greater detail. We can retain $\hbar\omega$ in the argument of the δ function [cf. (30) and (18)] without going outside the framework of the approximation made (the transition from Fermi-step differences to a derivative, and replacement of the latter by a δ function are corroborated by the inequalities $T \ll \varepsilon_F$ and $\hbar\omega \ll \varepsilon_F$). We put

$$W_{\mathbf{q}}(\xi, \eta) = \varepsilon(\mathbf{p}+\mathbf{q}) - \hbar\omega(\mathbf{q}) - \varepsilon_F, \qquad (32)$$

where ξ and η are dimensionless orthogonal coordinates on the Fermi surface over which the integration is carried out in (30); the area element is $dS_F = p^2(\xi, \eta)\, d\xi\, d\eta$ (the coefficient of $d\xi\, d\eta$ is written in this form to emphasize that it is of the order of $p_F^2 \sim \hbar^2/a^2$), and the volume element is $d^3p = [p^2(\xi, \eta)/v(\xi, \eta)]\, d\xi\, d\eta\, d\varepsilon$. Singularities of the integral (30) as functions of the phonon quasimomentum $\mathbf{q}$ appear when the argument of the δ function has a multiple zero, i.e.,

$$W_{\mathbf{q}}(\xi, \eta) = 0, \quad \frac{\partial W_{\mathbf{q}}}{\partial\xi} = \frac{\partial W_{\mathbf{q}}}{\partial\eta} = 0. \qquad (33)$$

Three equations in five variables (the two coordinates ξ and η and the three components of the vector $\mathbf{q}$) have in the general case a two-dimensional set of solutions. Thus, by specifying the direction of the vector $\mathbf{q}$ we can find the values of $\mathbf{q}_c$ and $\mathbf{q}_c'$ (Taylor singularities may be absent if the surface has no parabolic-point lines, see above). Another procedure is also possible. We let the length of the phonon quasimomentum q in (3) and in (32) tend to zero.[19] Equation (32) then takes the form

$$W_\varkappa(\xi, \eta) = \varkappa\mathbf{v}(\xi, \eta) - \frac{s(\varkappa)}{v(\xi, \eta)}, \quad \mathbf{v} = \frac{\mathbf{v}}{v}, \quad s(\varkappa) = \frac{\hbar\omega(\mathbf{q})}{q}. \qquad (34)$$

The function $W_\varkappa(\xi, \eta)$ differs from $W_{\mathbf{q}}(\xi, \eta)$ by a factor qv. As shown in Ref. 31 (see also Ref. 32), each point of any parabolic-points line on the Fermi surface (if such points exist) produces a singularity of the integral (30) as a function of the $\varkappa$ direction. The critical direction $\varkappa_c$ is uniquely determined by the local geometry of the Fermi surface at the point (ξ_c, η_c) that gives rise to the singularity (it is called critical). Singularities can be of two types, a finite jump or a logarithmic divergence. As always, a correspondence exists between the singularities of $\operatorname{Im}\delta\hbar\omega_\nu(\mathbf{q})$ and of $\operatorname{Re}\delta\hbar\omega_\nu(\mathbf{q})$ (Table II). If the critical point is located at the intersection of two lines of parabolic points, the singularity is enhanced and becomes of fractional-power type (as a rule[32] the power is low, 1/6 or 1/4).

TABLE II. Type of singularity.

Im $\delta\varepsilon$	Jump	Logarithmic divergence
Re $\delta\varepsilon$	Logarithmic divergence	Jump

In this case (as $q\to 0$ and $\varkappa\to\varkappa_c$), elimination of the divergence calls for a more thorough analysis than in the cases considered before, since the electrons that produce the singularity have Fermi energy and therefore (in the absence of defects) have infinite lifetime. We shall discuss this question in the next section.

10. LONG-WAVE PHONONS

The propagation and absorption of sound in metals is described by a system consisting of the elasticity-theory equations and the electrodynamics equations[20] (Maxwell equations). The current density in the Maxwell equations and the electron-governed density of the force **f** in the elasticity equations are expressed in terms of the displacement vector **u** and the electrostatic-field strength **E** with the aid of the solution of the Boltzmann equation for the electron distribution function. We shall write down an expression only for **f**, without going beyond the collisionless limit:

$$\mathbf{f}=\frac{iek}{\rho}\langle\Lambda R v_l\rangle E_l+\frac{i\omega k^2}{\rho}\langle\Lambda R\Lambda_l\rangle u_l,\quad R=\frac{1}{i(\mathbf{kv}-\omega)+0}; \tag{35}$$

Here Λ is a vector with components $\Lambda_{il}\varkappa_l$, $\varkappa=\mathbf{k}/k$, $\Lambda_{ik}=\lambda_{ik}-\langle\lambda_{ik}\rangle/\langle 1\rangle$ is the deformation-potential tensor renormalized by the electroneutrality condition;

$$\langle\ldots\rangle=\frac{2}{(2\pi\hbar)^3}\oint\ldots\frac{dS_F}{v(\mathbf{p}_F)} \tag{35'}$$

with the integration carried out over the Fermi surface. The tensor λ_{ik} is of the same origin as the matrices $M\ldots$ and $\lambda\ldots$ in Eqs. (14) and (18), differing only by inessential factors and, of course, by the limiting transition $\mathbf{p}'\to\mathbf{p}$ [in the expression for $\Lambda^{\nu}_{\mathbf{p},\mathbf{p}'}$; see Eq. (19)].

In investigations of the propagation of a sound wave, the electric field intensity should be determined from the Maxwell equations and substituted in the expression for the force; this leads to a *renormalization* of the tensor of the moduli that relate the vectors **f** and **u**. The renormalization is usually[32] carried out in two stages: first to be eliminated is the longitudinal field and

$$\mathbf{f}=\frac{i\omega k^2}{\rho}\left(\langle\Lambda R\Lambda_l\rangle-\frac{\langle\Lambda R(\mathbf{kv})\rangle\langle(\mathbf{kv})R\Lambda_l\rangle}{\langle(\mathbf{kv})R(\mathbf{kv})\rangle}\right)u_l$$
$$+\frac{iek}{\rho}\left(\langle\Lambda R v_\alpha\rangle-\frac{\langle\Lambda R(\mathbf{kv})\rangle\langle(\mathbf{kv})Rv_\alpha\rangle}{\langle(\mathbf{kv})R(\mathbf{kv})\rangle}\right)E_\alpha; \tag{36}$$

the subscript l numbers three projections, and α numbers two (in a plane perpendicular to the vector **k**). Equation (36) is in essence necessary to show that exclusion of the longitudinal field is not essential: the terms (in the curly brackets) that result from the renormalization are smaller by an approximate factor v_F/s than the principal terms[21] ($v_F/s\sim\varepsilon_F/\theta\sim(M/m)^{1/2}$). Elimination of the transverse components E_α is much more important. According to Ref. 32

$$\mathbf{f}=\frac{ik^2\omega}{\rho}\left(\langle\Lambda R\Lambda_l\rangle+\frac{4\pi i\omega e^2}{c^2}\sum_{\alpha=1}^{2}\frac{\langle\Lambda R v_\alpha\rangle\langle v_\alpha R\Lambda_l\rangle}{k^2-\frac{4\pi i\omega e^2}{c^2}\langle v_\alpha^2 R\rangle}\right)u_l. \tag{37}$$

This formula shows that allowance for the renormalization is necessary in actual calculations of the sound velocity in a metal and of the sound absorption coefficient. With increasing wave vector, however, the second and third terms in the curly brackets become continuously smaller, and at $\delta_L k\gg(s/v_F)^{1/2}$, where $\delta_L=c/\omega_L\sim 10^{-5}$ cm, they can be neglected. It must be emphasized that the "exclusion" of those terms which are due to renormalization takes place long before the condition that the sound waves be macroscopic ($ak\ll 1$) is violated. This justifies at the same time the possibility of disregarding the role of the magnetic field when electron-phonon interaction is considered at arbitrary value of the phonon quasimomentum **q**, with exception of the immediate vicinity of the origin ($q/\hbar\ll\delta_L^{-1}(s/v_F)^{1/2}\sim 10^3$ cm^{-1}).

If the Fermi surface has parabolic-point lines, the moduli contained in Eqs. (35)–(37), as functions of the direction of $\varkappa$, have singularities [compare the expression for R in (35) with Eq. (34)]. It can be easily shown, however, that in those cases when the singularity is produced by one point on the Fermi surfaces, the infinites are cancelled out. Thus, strictly speaking, the coefficient of proportionality of the force f to the displacement vector **u** can become infinite only when no transverse electric field is excited at all[22] (for example, from symmetry considerations, when longitudinal sound propagates in a "good" direction).

Equations (35)–(37) do not take into account the Fermi-liquid interaction of the electrons. Since the sound frequencies are relatively low ($\omega/kv_F\sim s/v_F\ll 1$) the Fermi-liquid interaction, which is always important in the investigation of electromagnetic properties of metals,[24] plays no essential role in this case.

To conclude this section we consider a case, albeit rare, when no transverse field is excited and we can therefore not eliminate the divergence. Does a collisionless limit exist in this case and must scattering be taken into account? We note that in an ideal crystal (without impurities or periodicity disturbances) the electrons that participate in the interaction with the sound have infinite mean free paths, since their energy is equal to the Fermi energy (see the end of the preceding section).

It can be shown[33,32] that there exists a *resonant* self-limitation mechanism. To consider it we must forego perturbation theory, meaning the following: in expression (35) for the force at $\mathbf{E}=0\,\omega$ must be taken to mean the *sought* renormalized freuquency, i.e., the root of the dispersion equation obtained by solving the elasticity-theory expressions in which account is taken of the sound-wave interaction with the conduction electrons. The dispersion equation determines the finite value of the complex sound velocity, whose imaginary part describes the damping. Thus, if the damping coefficient Γ has a logarithmic divergence in perturbation theory, allowance for the resonant self-limitation mechanism leads to the finite value

$$\frac{\Gamma_i}{\omega}\sim\frac{s}{v_F}\ln\frac{v_F}{s}. \tag{38}$$

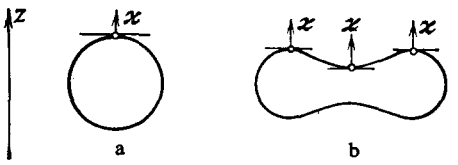

FIG. 7. Positions of critical points on the Fermi surface at a specified direction of $\varkappa$. a) The Fermi surface is a sphere, one critical point; b) dumbbell-shaped Fermi surface, three critical points.

The self-limitation mechanism was named resonant, since a resonant interaction is indeed realized between the sound wave and those conduction electrons on the Fermi surface for which $\varkappa_c \cdot \mathbf{v}_c = s$ [in other words, the sound wave is at resonance with a quasiwave having a dispersion law $\omega = k(\varkappa_c \cdot \mathbf{v}_c)$; see the next section].

11. QUASIWAVES OR NONEXPONENTIALLY DAMPED WAVES IN A METAL

An electromagnetic wave incident on the surface of a metal excites in it electromagnetic oscillations that attenuate over the skin-layer depth. In the collisionless limit the metal can be regarded as an electron plasma whose dielectric function is expressed in the classical approximation ($\hbar\omega \ll \varepsilon_F$, $T = 0$) in the form of an integral over the Fermi surface:

$$\varepsilon_{ik} = \delta_{ik} + \frac{4\pi i e^2}{\omega} \langle v_i R v_k \rangle , \tag{39}$$

[see Eqs. (35) and (35')]. As already stated, at relatively low frequencies there are no undamped oscillations in the spectrum of the electron plasma of a metal. This means that the solution of the dispersion equation

$$D(\omega, \mathbf{k}) = 0 , \tag{40}$$

obtained with the aid of (39) has only complex solutions. The asymptotic form of the field in the interior of a sample is not always determined by the zeros of the dispersion function $D(\omega, \mathbf{k})$. Let $D(\omega, \mathbf{k})$ have at $k_z = k_z^c$ a singularity (as a function of k_z; $k_x = k_y = 0$); the asymptotic form of the field then has the following structure[34,32]:

$$\mathbf{E} \propto e^{-i[(\omega t - k_z^c z)]} (k_z^c z)^{-\alpha} . \tag{41}$$

The exponent α is determined by the structure of the singularity. The formulation of the problem and its solution (41) presuppose normal incidence of the electromagnetic wave on the surface of a metal that occupies the half-space $z > 0$. It can be seen from (39) and (35) that the singularities of the dispersion function $D(\omega, k_z)$ are produced by multiple zeros of the denominator R, i.e.,

$$k_z^c = \frac{\omega}{(\varkappa \mathbf{v}_F)_{extr}} , \quad \varkappa \| Oz, \quad \varkappa \mathbf{v}_F > 0, \tag{42}$$

and the extremum of $(\varkappa \cdot \mathbf{v}_F)$ is taken with respect to the variables ξ and η [see (33) and (34)]. If the Fermi surface is spherical, there is only one value of k_z^c equal to ω/v_F. It corresponds to a limiting point (Fig. 7a). If the Fermi surface is more complicated (the general case), there are several values of k_z^c (Fig. 7b). When the vector $\varkappa$ has a critical direction (meaning that the line $\varkappa \cdot \mathbf{v}_F(\xi, \eta) = 0$ changes at $\varkappa = \varkappa_c$ its topology[14,31]; Fig. 8), $(\varkappa \cdot \mathbf{v}_F)_{extr}$ vanishes. Analysis shows[24] that in this case account must be taken of the quantum corrections [of the terms $\sim \hbar^2 k^2/m^*$; cf. (32) and (34)], and $ak_z^c \sim (\omega/v_F)(\varepsilon_F/\hbar\omega)^{1/2}$—a curious example in which the finite phonon momentum *must* be taken into account when the distribution of a *macroscopic* electromagnetic field in a metal is considered ($ak_z^c \sim (\hbar\omega/\varepsilon_F)^{1/2} \ll 1$).

We named the described asymptotic values of the electromagnetic field, as well as the residual oscillations of the electron distribution function [see (17) ff], *quasiwaves* to emphasize thereby that these are forced, rather than natural, solutions.

When speaking of the dielectric function of the electron plasma of a metal, we cannot disregard the Fermi-liquid interaction of the electrons. Allowance for this interaction alters expression (39) (see Ref. 25):

$$\varepsilon_{ik}^{FL} = \delta_{ik} + \frac{4\pi i e^2}{\omega} \langle v_i R (v_k - \omega J_k) \rangle . \tag{43}$$

The vector $\mathbf{J}$ satisfies an integral equation that contains the Landau matrix[36] $f(\mathbf{p}, \mathbf{p}')$ that describes the Fermi-liquid interaction (at $f(\mathbf{p}, \mathbf{p}') = 0$ and $\mathbf{J} = 0$). Allowance for the Fermi-liquid interaction results in a substantial restructuring of the singularities of the dielectric tensor ε_{ik}^{FL}. According to the integral equation given here,[37,24] the values of the vector $\mathbf{J}$ at those Fermi-surface points (ξ_c, η_c) at which the denomina-

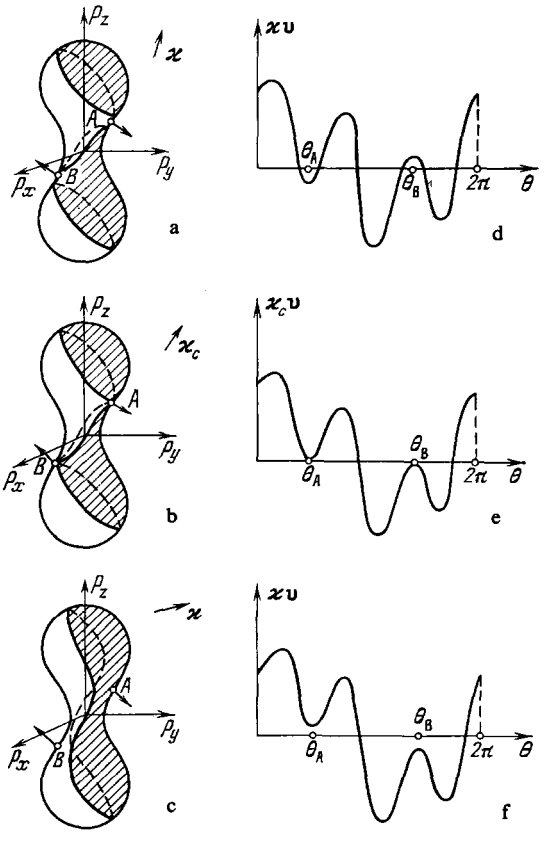

FIG. 8. The strips $\varkappa \cdot \mathbf{v} = 0$ (thick lines on a dumbbell-type Fermi surface (a–c) and dependence of $\varkappa \cdot \mathbf{v}$ on the angle θ (d–f). a, d) $\varkappa \cdot \mathbf{v}_A < 0$, $\varkappa \cdot \mathbf{v}_B > 0$; b, e) $\varkappa_c \cdot \mathbf{v}_{A,B} > 0$, $\varkappa \cdot \mathbf{v}_B < 0$. Points A and B—parabolic points, θ—angle measured from the dumbbell axis ($0 < \theta < 2\pi$).

tor R has a multiple zero are equal to $\mathbf{v}_c/\omega$ ($\mathbf{v}_c \equiv \mathbf{v}(\xi_c, \eta_c)$) so that the integral over the Fermi surface *does not diverge* (nature adhors inifinities !). It must be stated that the change of the analytic properties[23] of ε_{ik}^{FL} (compared with ε_{ik}) does not alter excessively the structure of the field in the quasiwave, and affects as a rule only the cofficient in expressions of the type (41).

12. CONCLUDING REMARKS

The Fermi surface forms in the **p**-space region adjacent to it, as we have attempted to describe, a spectrum of elementary excitations of a metal, and furthermore not only of fermions (electrons and holes) but also of bosons (phonons). Electron-phonon interaction leads to renormalization of the elementary-excitation dispersion laws. The lifetimes of the elementary excitations are therefore finite, and singularities appear in the dependence of the energy on the quasimomentum. The characteristics of these singularities are closely related to the local geometry of the Fermi surfaces and this is what distinguishes them from other singularities (due, e.g., to phonon-phonon interaction[24]).

The anisotropy of the Fermi surfaces, especially the presence of parabolic-point lines on them, expands greatly the spectrum of the phonon singularities in metals. A prominent place is occupied by singularities of the acoustic-phonon velocities (along the propagation direction), since these singularities are formed by electrons located directly on the Fermi surface and have an infinite lifetime. These singularities are therefore not smeared out and the diverging quantities, if perturbation theory is used, turn out to be finite if a consistent account of the resonant interaction of a sound wave with electrons whose velocity projections on the sound wave vector is equal to the sound velocity.

The results reported here formulate *general* ideas concerning the spectrum of the elementary excitations of a metal, and continue in this sense the semiphenomenological approach developed by I. M. Lifshitz and his school (see Refs. 3 and 39; a brief history of this trend in the electron theory of metals can be found in Ref. 39). Our notions concerning metals are at present such that we can describe with relatively high accuracy their Fermi surfaces, and be assured that parabolic-point lines and their intersections (flattened sections) exist on the Fermi surface.

Some experiments have revealed anomalies (singularities) due to flattened sections: a logarithmic increase of the sound absorption coefficient in Cu with increasing kl was observed in Ref. 40 (these results are explained in Ref. 41); it was shown in Ref. 42 that the existence of a flattened section on the Fermi surface of Ga makes the absorption coefficient temperature-dependent at $kl \gg 1$; an anomaly of the phonon dispersion law, due to local geometry of the Fermi surface of Cu, was revealed in Ref. 43 by inelastic scattering of neutrons. All these experimental results agree with the theoretical premises reported here and in the cited papers. It must nevertheless be stated that the number of experimental studies aimed at revealing the qualitative singularities of the phenomena due to the complicated shape of the Fermi surface is very small[25].

I am of the impression that the theory of metals, having accumulated much experimental data on the "geography" of the Fermi surface, is about to experience a new resurgence of interest in the dramatic events that take place between the quasiparticles that live on the Fermi surface. The theory has advanced many predictions, thereby hurling a challenge to the experiments.

I take the opportunity to thank L. P. Pitaevskiĭ and H. Eschrig for several important (to me) conversations that prompted me to reason out and write what turned out to be the present article. In addition, I thank T. Yu Lisovskaya and A. G. Plyavenek with whom many of the results reported here were obtained. The ideas expressed by us all in the preparation of original papers have found reflection in the text.

[1)] The chain of order-of-magnitude relations given below explains the origin of the estimates exhibited here and emphasizes their approximate character:

$$\omega_D^2 \sim \frac{\varkappa}{M}, \quad \varkappa \sim \frac{U}{a^2}, \quad U \sim \varepsilon_F, \quad \frac{1}{a^2} \sim p_F^2/\hbar^2, \quad s \sim \omega_D a,$$

$\varkappa$ is the lattice rigidity and U the energy of interaction between neighboring atoms; the virial theorem can be used to justify the relation $U \sim \varepsilon_F$.

[2)] We refer here, first, to single crystals, and second, to normal (nonsuperconducting) metals (superconductivity is discussed later).

[3)] The term "hole" is used here to denote a free state with energy lower than the Fermi energy.

[4)] The term "hole" does not have here the same meaning as above, but stands for a free state with negative effective mass; the energy corresponding to this state *exceeds* the Fermi energy!

[5)] Even though the effective mass can be differently defined, there is no need to refine the definition of m^*, since we are dealing here with a qualitative picture.

[6)] To describe the properties of a metal it is, of course, not enough to know the quasiparticle dispersion laws. One must know the wave functions of the quasiparticle stationary states (the Bloch waves for electrons and holes, and the vibration-polarization vectors for phonons). We confine ourselves to a description of the energy spectrum.

[7)] Remember that the redistribution takes place against a background of positively charged ions.

[8)] The phonon-phonon interaction is small according to the parameter

$$\left(\frac{\theta}{Ms^2}\right)^{1/2} \sim \left(\frac{\theta}{\varepsilon_F}\right)^{1/2} \sim \left(\frac{m}{M}\right)^{1/4} \ll 1.$$

[9)] The lifetime $\tau = \hbar/\Gamma$, where $\Gamma = \mathrm{Im}E$ and E, the complex "energy" of the quasiparticle, is a pole of the single-particle Green's function (Ref. 11, §§ 8 and 65).

[10)] See Refs. 15, 16, and 17.

[11)] The spin waves predicted by Silin[18] can and do propagate in a magnetic field in normal (nonmagnetic) metals, owing to the Fermi-liquid interaction.

[12)] By "singularities" we mean here and hereafter that the functions and/or (!) their derivatives become infinite.

[13)] For example, if it turns out in the calculation of the dispersion law of some quasiparticle that its velocity at some point of **p** space becomes infinite, then it is obvious that this cannot happen if the retardation has been consistently taken into account).

[14)] In a number of papers[9] a singularity is defined as a substantial difference between the dispersion law of a real phonon from the law obtained on the basis of a model more primitive than the one used in the cited papers (e.g., without allowance for the contribution of the electrons or without allowance for the interaction between the electron and the ion core). We emphasize that we *start out* with *good* wave functions $\xi_\alpha(\mathbf{p})$ and $\hbar\omega_\nu(\mathbf{p})$, which comprise, together with the lattice geometry, the Fermi surface, and the amplitudes $\Phi \ldots$, $\Psi \ldots$, and $\lambda \ldots$ the model of the metal in the normal state.

[15)] Furthermore, calculation of the singularities of the quasiparticle *lifetimes* eliminates the $\mathscr{H}_{int}$ ambiguity due to virtual processes [see the

text following Eq. (11)].

[16)]This is the only case (a monatomic atom) considered here. It is shown in Ref. 21 that interaction with optical phonons, without changing the $\delta\varepsilon(\mathbf{p})$ dependence as $|\varepsilon(\mathbf{p}) - \varepsilon_F| \to 0$, introduces in this relation an additional singularity, viz., a jump of the derivative at that value of $\varepsilon(\mathbf{p}) - \varepsilon_F$ at which the surface (23) is tangent to the Fermi surface.

[17)]The electron and phonon spectra specify the geometric locus of the singular points $\mathbf{p}_c$ around each cavity of the Fermi surface; in the general case the "distance" from the point $\mathbf{p}_c$ and the Fermi surface is of the order of *ms*.

[18)]The singularity described was observed by Migdal[22] and Kohn.[27] Enhancement of the singularity in the case of cylindrical and/or planar Fermi surfaces was predicted by A. M. Afanas'ev and Yu. Kagan,[28] while M. I. Kaganov and A. I. Semenenko[29] associated the character of the singularity with the *local* geometry of the Fermi surface.

[19)]This procedure corresponds to study of the section of the geometric locus at small q (Fig. 6).

[20)]In a review "Dynamic Equations of Elasticity Theory of Metals," recently published in Uspekhi Fizicheskikh Nauk (Sov. Phys. Uspekhi)[20] (and already referred to here), V. M. Kontorovich considered the entire aggregate of questions connected with propagation and absorption of sound in metals (in particular, the history of the question can be learned from the references cited). The availability of this review allows us to restrict ourselves here to a concise exposition.

[21)]The insignificant role of the renormalization due to exclusion of the longitudinal field is a consequence of the low sound velocity. When dealing with electromagnetic high-frequency properties, changing from σ_{ik} to $\sigma^{\perp}_{\alpha\beta} = \sigma_{\alpha\beta} - \sigma_{\alpha x}\sigma_{x\beta}/\sigma_{xx}$ alters substantially the conductivity-tensor components.

[22)]Recall that we are dealing with the case $\varkappa = \varkappa_c$ and when the Fermi surface has *one* critical point (ξ_c, η_c). From the formal point of view a longitudinal electric field also eliminates the divergences. To achieve real cancellation, however, it would be necessary that the mean free path be indeed infinite.[32]

[23)]What becomes infinite is not ε^{FL}_{ik} but $\partial\varepsilon^{FL}_{ik}/\partial\mathbf{k}$.

[24)]As shown by L. P. Pitaevskiĭ,[38] each crystal must without fail contain bound states of phonons (biphonons), owing to the effective attraction between two phonons.

[25)]We purposefully do not dwell on theoretical and experimental studies of metal properties in strong magnetic fields—this is a subject by itself.

[1]E. M. Lifshitz and L. P. Pitaevskii, Statisticheskaya fizika, Ch. 2, Nauka, M., 1978 [Engl. Transl., Statistical Physics, Part 2, Pergamon Press, Oxford, 1980].

[2]J. M. Luttinger, Phys. Rev. **119**, 1153 (1960).

[3]I. M. Lifshitz, M. Ya. Azbel', and M. I. Kaganov, Élektronnaya teoriya metallov. Nauka, M., 1971 [Engl. Transl. Electron Theory of Metals, Consultants Bureau, N.Y. 1973.

[4]T. W. McClure, Phys. Rev. **198**, 612 (1957). Ph. Nozieres, *ibid.* **109**, 1510 (1958).

[5]A. P. Cracknell and K. C. Wong, The Fermi Surface, Clarendon Press, Oxford, 1973 [Russ. Transl. Atomizdat, M., 1978.]

[6]R. W. Stark and L. M. Falicov, Progr. Low Temp. Phys. **5**, 235 (1967). M. I. Kaganov and A. A. Slutskin, Phys. Rept. **98**, 190 (1983).

[7]See, e.g., V. Heine, M. L. Cohen, and D. Weaire, Theory of Pseudopotentials in Solid State Physics, Vol. 24 (ed. H. Ehrenreich, F. Seitz and D. Turnbull) Academic Press, N.Y., 1970. [Russ. Transl., Mir, M., 1973.]

[8]A. M. Kosevich, Osnovy mekhankiki kristallicheskoĭ reshetki (Principles of Crystal-Lattice Mechanics), Nauka, M., 1972.

[9]E. G. Brovman and Yu. M. Kagan, Usp. Fiz. Nauk **112**, 369 (1974) [Sov. Phys. Usp. 17, 369 (1974)]. Yu. Kagan, V. V. Pushkarev and A. Kholas, Zh. Eksp. Teor. Fiz. **84**, 1494 (1983) [Sov. Phys. JETP **57**, 870 (1983)].

[10]J. Ziman, Principles of the Theory of Solids, Cambridge, Univ. Press, 1964, Chap. 6, § 12. [Russ. Transl., Mir, M., 1974].

[11]V. L. Gurevich. Kinetika fononnykh sistem (Kinetics of Phonon Systems), Nauka, 1980, Chap. 1, § 6.

[12]E. M. Lifshitz and L. P. Pitaevskii, Fizicheskaya kinetika, Nauka, M., 1979, [Engl. Transl. Physical Kinetics, Pergamon Press, Oxford, 1981.]

[13]R. E. Peierls, Quantum Theory of Solids, Clarendon Press, Oxford, 1955.

[14]R. N. Gurzhi and V. M. Kontorovich, Zh. Eksp. Teor. Fiz. **55**, 1105 (1968) [Sov. Phys. JETP **28**, 577 (1969)].

[15]E. A. Kaner and V. G. Skobov, Usp. Fiz. Nauk **89**, 367 (1966) [Sov. Phys. Usp. **9**, 480 (1967)]; Adv. Phys. **17**, 69 (1968).

[16]B. Ya. Demikhovskiĭ and A. P. Protogenov, Usp. Fiz. Nauk **118**, 101 (1976) [Sov. Phys. Usp. **19**, 53 (1976)].

[17]P. M. Platzman and P. Wolf, Waves and Interactions in Solid-State Plasma, Suppl. 13 to Solid State Phys., Academic Press, N.Y., 1973. [Russ. Transl., Mir, M., 1975].

[18]V. P. Silin, Zh. Eksp. Teor. Fiz. **35**, 1243 (1958) [Sov. Phys. JETP **8**, 870 (1959)].

[19]L. D. Landau, Zh. Eksp. Teor. Fiz. **16**, 574 (1946).

[20]V. M. Kontorovich, Usp. Fiz. Nauk **142**, 265 (1984) [Sov. Phys. Usp. **27**, 134 (1984)].

[21]M. I. Kaganov and T. Yu. Lisovskaya, Zh. Eksp. Teor. Fiz. **80**, 2445 (1981) [Sov. Phys. JETP **53**, 1280 (1981)].

[22]A. B. Migdal, *ibid.* **34**, 1438 (1958) [*ibid.* 7, 996 (1958)].

[23]M. I. Kaganov, M. I. Plyavenek, and M. Hitschold, *ibid*, 82, 2030 (1982) [**55**, 1167 (1982)].

[24]M. I. Kaganov and A. G. Plyavenek, *ibid.* **88**, 249 (1985) [61, 145 (1985)].

[25]A. B. Pippard, Phil. Mag. **2**, 1147 (1957).

[26]A. I. Akhiezer, M. I. Kaganov, and G. Ya. Lyubarskiĭ, Zh. Eksp. Teor. Fiz. **32**, 837 (1957) [Sov. Phys. JETP **5**, 685 (1957)].

[27]W. Kohn, Phys. Rev. Lett. **3**, 393 (1959).

[28]A. M. Afanas'ev and Yu. Kagan, Zh. Eksp. Teor. Fiz. **43**, 1456 (1962) [Sov. Phys. JETP **16**, 1030 (1963)].

[29]M. I. Kaganov and A. I. Semenenko, *ibid.* **50**, 630 (1966) [**23**, 419 (1966)].

[30]P. L. Taylor, Phys. Rev. **131**, 1995 (1963).

[31]G. T. Avanesyan, M. I. Kaganov, and T. Yu. Lisovskaya, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 381 (1977) [JETP Lett. **25**, 355 (1977)]; Zh. Eksp. Teor. Fiz. **75**, 1786 (1978) [Sov. Phys. JETP **48**, 900 (1978)].

[32]M. I. Kaganov, V. M. Konotorovich, T. Y. Lisovskaya, and N. A. Stepanova, Zh. Eksp. Teor. Fiz. **85**, 1675 (1983) [Sov. Phys. JETP **58**, 975 (1983)].

[33]V. M. Kontorovich and N. A. Stepanova, Pis'ma Zh. Eksp. Teor. Fiz. **18**, 381 (1973) [JETP Lett. **18**, 223 (1973)].

[34]G. E. M. Reuter and E. H. Sondheimer, Proc. R. Soc. London Ser. A., **195**, 336 (1948). T. I. Ivanovski and M. I. Kaganov, Zh. Eksp. Teor. Fiz. **83**, 2320 (1982) [Sov. Phys. JETP **56**, 1345 (1982)]. V. G. Peschanskiĭ, V. S. Likhtsier, and A. Tada, Fiz. Met. Metallov. **56**, 855 (1983).

[35]V. P. Silin, Zh. Eksp. Teor. Fiz. **33**, 495 (1957) [Sov. Phys. JETP **6**, 387 (1958)]; see also Ref. 12.

[36]L. D. Landau, Zh. Eksp. Teor. Fiz. **30**, 1058 (1956) [Sov. Phys. JETP **3**, 920 (1956)].

[37]M. I. Kaganov and A. G. Plyavenek, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 94 (1983) [JETP Lett. **38**, 109 (1983)].

[38]L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **70**, 738 (1976) [Sov. Phys. JETP **43**, 382 (1976).

[39]M. I. Kaganov and I. M. Lifshitz, Usp. Fiz. Nauk **129**, 487 (1979) [Sov. Phys. Usp. **22**, 904 (1979)].

[40]R. E. Macfarlane and J. A. Rayne, Phys. Rev. **162**, 532 (1967).

[41]I. M. Suslov, Fiz. Tverd. Tela (Leningrad) **23**, 1652 (1981) [Sov. Phys. Solid State **23**, 963 (1981)].

[42]V. D. Fil', V. I. Denisenko, and P. A. Bezuglyĭ, Abstracts, 21st All-Union Conf. on Low-Temperature Physics, Khar'kov, 1980, Vol. 3, p. 98.

[43]L. Weiss and H. Eschrig, Phys. Stat. Sol. (b) **114**, 419 (1982).

[44]G. G. Maksimov, Zh. Eksp. Teor. Fiz. **69**, 2236 (1975) [Sov. Phys. JETP **42**, 1138 (1975).

[45]E. G. Brovman and Yu. M. Kagan, in: Dynamical Properties of Solids, G. K. Horton and A. A. Maradudin, eds., North Holland, Amsterdam, 1974, Vol. 1, p. 191.

Translated by J. G. Adashko