A. A. Migdal. *Lattice models in quantum chromodynamics and results of computer calculations.* Until quite recently, the theoretical physics of elementary particles was restricted to a very modest range of equipment. While experimental physics demanded enormous accelerators, and applied physics ultrapowerful computers, theoretical physics has had to make do with pen and paper. The reason for this was that the ruling theory at the time was quantum electrodynamics (QED) which was amenable to a remarkably simple and effective theoretical procedure, namely, perturbation theory. The same applies to the theory of electroweak interactions that has recently unified the theory of weak and electromagnetic forces.

At the same time, the strongest known forces in nature—the strong interactions—have for a long time remained a puzzle. On the other hand, once quantum chromodynamics (QCD) was formulated for the description of strong interactions, it turned out that perturbation theory had a very restricted range of applicability to QCD. In particular, perturbation theory is valid for the description of virtual processes at ultra-short distances. For example, a virtual photon may be produced in a collision between a fast electron and positron, which, in turn, creates a quark-antiquark pair. Perturbation theory is valid so long as the two members of the pair do not depart from one another to a distance of the order of $r_c = 10^{-13}$ cm, i.e., the interaction between the quarks can be described as being the result of gluon exchange. This gives Coulomb forces, as in QED, but with a logarithmically growing constant instead of e^2:

$$g^2_{\text{eff}}(r) = \frac{C}{\ln(r_c/r)}, \qquad (1)$$

where C is a universal numerical constant. Perturbation theory ceases to be valid when $r \gtrsim r_c$. We expect that, in this region, the forces continue to grow, reaching a constant value for $r \to \infty$, so that the potential $V(r)$ increases linearly. We think that this surprising phenomenon is due to strong and highly fluctuating gluon fields. Such fields are very different from the usual set of quanta and require new methods of description.

This problem arose some ten years ago and although only moderate progress has been achieved during the last ten years, we are at least on firm ground now. Although methods comparable in simplicity and efficacy with the Feynman perturbation theory have not been discovered, numerical methods are now available that are capable of a precision of the order of 10–20%. The precision can be improved by using specialized computers and by combining analytical with numerical methods.

Before we describe these methods, it will be useful to say a few words about why we consider the problem to be important and why we are not satisfied with a precision of only 20%. The point is that, according to modern ideas, all forces of nature are based on non-Abelian gauge fields, as is QCD. Large-scale fluctuations develop in these fields above a certain scale. We expect that a hierarchy of such scales will be encountered at ultrashort distances in the interior of quarks. It follows that QCD methods of solution will also be valid in future unified theories. This means that we shall have to have reliable and precise methods in order to be able to separate weaker interactions from the background of stronger ones. As we move into the interior of quarks, physical experiments will become more and more difficult, and numerical experiments will necessarily become more valuable. They will yield more accurate predictions at low energies, at which experiments can also be made more precise. By verifying these predictions, it will be possible to choose between

different hypotheses on the interaction at ultrashort distances.

This question has a further aspect. Quantitative QCD solutions present us with an exceedingly difficult problem—a kind of Everest for theoretical physics. It demands combined efforts by representatives of different areas of science, i.e., theoretical physics, computational mathematics, and cybernetics. Usually, all adjacent areas gain as a result of the solution of a problem. In our case, it may be expected that the problem will stimulate advances in computer science.

The first specialized computers for QCD have already been developed in the USA. They take the form of a network of computers working in parallel, and their memory and speed are better by an order of magnitude than Cray-1. They are also much simpler and cheaper. This has been achieved as a result of narrow specialization, but the authors of these projects consider that they will be useful not only in QCD, but also in a broad range of theoretical problems in physics and technology. One way or another, this range of investigations is attracting considerable attention in the West.

Let us now proceed to a description of numerical methods for QCD. So far, there are three such methods, namely, the strong-coupling expansion, the recursion relations, and the Monte Carlo method. They all make use of the technical device of four-dimensional Euclidean space and approximate it by a hypercubic lattice.

Euclidean space arises as follows. Instead of the S-matrix ($S = \exp(-iHt)$), commonly used in quantum mechanics, one considers the temperature density matrix $\rho = \exp(-H/T)$ that corresponds to the analytic continuation from real to imaginary time. The advantage of the density matrix is that sums over intermediate states converge absolutely. By taking the limit as $T \to 0$, we can extract from the density matrix information of the spectrum of low-lying excitations and, by introducing the corresponding external fields, we obtain information on other low-energy parameters, e.g., magnetic moments, beta-decay constants, and so on.

We can return to the S-matrix by analytic continuation to imaginary T but, at the present level of precision of numerical experiments, it is premature to consider this. The most convenient quantity for numerical calculation is the trace of the density matrix for which the famous Feynman-Katz formula

$$Z = \operatorname{tr}\rho = \int DA \exp\left(\int_0^{T^{-1}} dx_4 \int d^3x \mathscr{L}_E\right), \quad A\left(\frac{1}{T}\right) = A(0); \qquad (2)$$

was developed more than forty years ago, where $\mathscr{L}_E$ is the Euclidean Lagrangian, in which the sign of the kinetic energy is reversed after the substitution $t = -ix_4$: if $\mathscr{L} = E^2 - H^2$, then $\mathscr{L}_E = -E^2 - H^2$. For the moment, we do not consider quarks and investigate field fluctuations. These fluctuations are expressed in that the vector potential A_μ can assume any value with probability proportional to the exponential in (2). Since this exponential is bounded and decreases for high field strengths, the integral differs from the Feynman integral in Minkowski space by the fact that it can be given a rigorous mathematical meaning.

This demands a procedure for approximating the functional integral by a multidimensional integral in a way similar to that by which an ordinary integral is approximated by a Riemann sum. As was first recognized by Wilson, this approximation must conserve gauge invariance if the phenomenon of quark confinement is to be retained. On the lattice, gauge invariance is formulated as follows. The vector potential is associated with the midpoint of the edge (xy) and the transfer matrix $u_{xy} = \exp[ig(y_\mu - x_\mu)A_\mu((x+y)/2)]$ is introduced. In QCD, the vector potential is a 3×3 Hermitian matrix and the transfer matrix is a unitary one. Gauge transformations multiply the transfer matrix from the left by $s^{-1}(x)$ and from the right by $s(y)$. It is readily verified that, when $x \to y$, these generalized gauge transformations become identical with the usual transformations: $A_\mu(x) \to s^{-1}(x)(A_\mu(x) + (ig)^{-1}\partial/\partial x_\mu)s(x)$.

The lattice analog of field strength is the transfer matrix over the elementary closed contour drawn around the boundary of the lattice $u\square = u_{xy}u_{yz}u_{zt}u_{tx}$. The lattice analog of action, i.e., the integral of the Lagrange function over space, is the following sum over all the boundaries of the lattice:

$$S = \sum_{\square} g_0^{-2} \operatorname{tr}(u_{\square} - 1). \qquad (3)$$

It is readily verified that the new definition becomes identical with the old one in the limit of zero lattice spacing. All that remains is to determine the measure of integration over the transfer matrices which now appear as the dynamic variables. The requirement of gauge invariance means that this measure must coincide with the so-called Haar invariant measure, which is well known to mathematicians.

Lattice gauge theories were first examined by Wegner in 1971 in statistical mechanics. They were rediscovered by Wilson in 1974, who generalized them to the Yang-Mills theory and applied them to the confinement problem. Wilson's outstanding paper determined the development of the theory for many years. An important quantity introduced by Wilson was the potential between heavy quarks

$$V(R) = -\lim \frac{1}{L} \ln W(R, L), \quad L \to \infty; \qquad (4)$$

where

$$W(R, L) = \left\langle \operatorname{tr} \prod_{\square} u_{xy} \right\rangle \qquad (5)$$

is the trace of the transfer matix over a rectangular contour, averaged over fluctuations, where one side L of the contour tends to infinity. Confinement corresponds to an increase in this potential with distance R.

Wilson used an expansion in $1/g_0^2$ to show that W decreases as exp(-area within the contour) for sufficiently large g_0^2. The potential thus increases as σR, where σ is the coefficient in front of the area in the exponential in W. In other words, this coefficient can be interpreted as the force between distant heavy quarks.

The $1/g_0^2$ expansion was subsequently investigated in greater detail by Balian, Drouffe and Itzykson and by Kogut and Susskind in 1974. It was found that it had limited convergence, and a phase transition liberating the quarks could be expected for sufficiently small g_0^2.

On the other hand, in the real world, we are interested precisely in small $g_0^2 = g^2(a) = C/\ln(r_c/a)$ because g_0^2 has the meaning of the effective charge at the lattice separation. Thus, the central question of the theory was the behavior of the force $\sigma(g_0^2)$ with decreasing seed charge g_0^2. If σ remains constant for small g_0^2, the quarks are confined, and if $\sigma = 0$ beginning with certain $g_0^2 < g_{0C}^2$, the quarks become liberated.

To solve this problem, the present author developed in 1975 an approximate functional integration procedure that did not make use of an expansion into a series in either $1/g_0^2$ or g_0^2. The basic idea of this method can be described as follows. Consider a closed volume in 4-space, for example, a 4-cube, and integrate over all fields, i.e., over all transfer matrices within this volume. This results in the so-called z-functional which depends on the fields on the boundary of the volume. If we now take two neighboring volumes and integrate the product of the Z-functionals for both volumes over the partition between them, the result is the Z-functional for the combined volume.

This gives a nonlinear functional equation for the Z-functionals—the so-called recursion relation. In this form, the equation is exact but, because of the infinite number of degrees of freedom, it cannot be treated by numerical methods. A procedure was then developed for isolating the principal degrees of freedom which, in the gauge theory, correspond to transfer matrices over "typical" contours on the boundary of the volume. This results in a closed equation of sufficiently simple structure that can be solved numerically. For small spatial scales, this reproduces existing results of perturbation theory, including the logarithmic law for the effective charge. For distances $r \gtrsim r_C$, new and disturbing results are obtained.

It is found that the forces between the quarks do, in fact, reach a constant limit, and this transition from Coulomb forces to constants occurs very rapidly in a narrow interval of distances. The force σ is an exponential function of g_0^{-2}:

$$\sigma \sim a^{-2} \exp\left(-\frac{2C}{g_0^2}\right) = a^{-2} \exp\left(\ln \frac{a^2}{r_c^2}\right) = r_c^{-2}, \qquad (6)$$

and remains finite at $a = 0$, i.e, in local theory. A phase transition releasing the quarks does not, therefore, occur. The nonanalyticity noted in the expansions if g_0^{-2} is due to the rapid transition from Coulomb forces to constants, and not to the liberation of quarks. This was the first theoretical indication of quark confinement in the local theory, and the first calculation outside the framework of perturbation theory.

However, the recursion relation was not exact but approximate. Its precision was of the order of 30%. To improve it, Kadanoff, Martinelli, Parisi and others have constructed a special perturbation theory for which this relation is the zero-order approximation. Additional degrees of freedom are turned on gradually, so that the number of degrees of freedom remains finite in each order. From the technical point of view, this perturbation theory is not simple and, so far, the first two orders have been obtained only for two-dimensional models, but it has been verified that the precision improves.

Quite recently, Tomboulis has proved an inequality according to which the recursion relation gives the lower bound for the force $\sigma(g_0^2)$. The recursion relation thus provides us with the possibility of proving quark confinement.

However, direct evaluation of the integral by the Monte Carlo method[1] has turned out to be simpler for practical QCD calculations. The basis for the application of the Monte Carlo method is the crossover phenomenon found in the recursion relation. In particular, since the force between the quarks reaches a constant limit very rapidly, and without an intermediate law, the infinite-volume limit is reached on a lattice that is only a few times larger than r_C. The lattice constant a can then be only a few times smaller than r_C, and the effective constant on the lattice is small enough for the local limit to appear.

Such ideas seemed at first to lie in the realm of science fiction. In fact, the 1976 paper in which the Monte Carlo method for lattice gauge theories was developed and proposed as a means of calculating the hadron mass spectrum was rejected by the editorial board of JETP on the grounds that the calculations should have been made first. This could not be done at the time, and the paper was not published.

However,the idea has survived. It was born again in 1979, when Creutz in the USA carried out the first successful calculation of $\sigma(g_0^2)$. This confirmed the qualitative crossover picture, and his data are satisfactorily described by the exponential law. The calculations were performed on a lattice consisting of 10^4 points for the SU(2) group.

An explosion of activity began at that point. Several hundred papers have now been published, and new groups of researchers continue to appear. This is so because the Monte Carlo method can be simply programed: the problem is transferred from the human to the electronic brain. The difficulty is the slow convergence: the statistical error decreases only as the square root of the time. There is also a systematic error due to the finite size of the lattice. Since the size of the required memory increases as the fourth power of the size of the lattice, this is the weakest point of the situation.

As long as quarks are not included the demands placed on the computer are not too excessive. One can work with lattices of 8^4 or every 4^4 and obtain reasonable results. This can be achieved with relatively small machines, such as VAX-11/780 and our ES-10-60. Practically all the low-energy quantities have been calculated, including the mass spectrum of glueballs, the critical temperature of vacuum at which quarks are liberated, and the energy density and topological charge of vacuum. In all cases, reasonable results that agree between the different groups to within 10–20% have been obtained.

Real difficulties emerge with the appearance of quarks. It has not been possible to devise a Monte Carlo algorithm for Fermi fields, so that one has to start with computations of the Green's function for quarks in lattice gauge fields, and then average over the field configurations. Most of the machine time is then expended in inverting the enormous matrices corresponding to the Green's functions. Moreover, when quarks are present, the lattice must be much greater in order not to "feel" the presence of the boundary.

Thus, although the work of Parisi *et al.* has resulted in

reasonable values for the hadron mass spectrum, these results cannot be regarded as final because of the finite-lattice effects. It would appear that the $10^3 \times 20$ lattice used in one of the latest calculations by the Italian group is still too small. We note that this calculation took about 30 hours of Cray-1 at the French Ministry of Electricity.

It is thus clear that the Monte Carlo method in QCD with quarks has reached the limit of the possibilities of modern computers and does not as yet yield reliable numbers for the mass spectrum. This could be reached with a computer performing 10 billion operations per second and a storage of a few million 8-byte words. A supercomputer of this kind is planned for 1987 by CDC, according to its Vice-President Schmidt.

The alternative is to develop specialized machines in which several hundred computers work in parallel, each being responsible for its own portion of the lattice. The machine that has just been built by Christ *et al.* at Columbia University utilizes, under the control of a VAX-11/780 host computer, a two-dimensional lattice consisting of 256 nodes (memory + processor). Each node is responsible for its own plane of a 4-dimensional lattice. It has a memory of 128 kb and costs in the neighborhood of two or three thousand dollars. The machine can perform 4 billion operations per second. It is designed for calculations of the hadron spectrum on lattices of 16^4–32^4.

In our own country, lattice gauge theories are being studied by about 20 people. We are urgently in need of machine time, which we have to find in roundabout ways. Unfortunately, despite the fact that the idea of these calculations was put forward by us 3 years before the Americans, various technical difficulties have prevented us from exploiting this advantage.

To eliminate the lag, the QCD theoreticians should have access to at least one modern fast computer. What is required is collaboration between theoreticians and engineers, so that specialized computers, such as the Christ mesh, can be developed. Finally, a search should be made for a more efficient numerical method that would be an advance on the Monte Carlo procedure and would combine numerical methods with analytical techniques. There is much hope for methods using the equations of motion for the traces of transfer matrices. It may well be that this will be the way forward in developing a successful quantitative theory employing numerical methods only at some of the stages.

Yu. M. Makeenko, Usp. Fiz. Nauk **143**, 161 (1984) [Sov. Phys. Usp. **27**, No. 6 (1984)].