Superfluid properties of ^{3}He-A

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The unique superfluid properties which distinguish the A phase of ^{3}He from other superfluids (He II, the B phase of ^{3}He, and electrons in superconductors) are described. These properties result from the specific breaking of the invariance of the state under gauge transformations and rotations in orbital space, so that all the superfluid properties are related in a fundamental way to the dynamics of textures of the liquid-crystal-anisotropy vector. The flow of the superfluid component of the liquid, for example, is not a potential flow; its vortical part is related to textures. The relationship between the structure of the superfluid velocity and the nature of the symmetry breaking is analyzed in detail. Simple phenomenological considerations are used to derive an expression for the current and to explain the strange term in the current which is not present in the model of a Bose gas of molecules having a Cooper-pair structure in the A phase. The existence of a normal density at $T = 0$ and the nonlocal properties of the density of the intrinsic orbital angular momentum of the liquid are discussed. The coexistence of superfluid properties with a nonpotential superfluid flow and also with the vanishing of the critical Landau velocity for the creation of excitations is discussed. Conditions for the global and local stability of the superfluid flow are described. The changes in stability upon changes caused in the topological structure of the order parameter by external conditions are discussed. The relaxation dynamics of a superfluid flow is analyzed. An instanton mechanism for phase slippage is analyzed; this mechanism leads to observable oscillations analogous to the ac Josephson effect in superconductors. Principles for constructing a generalized two-fluid Landau hydrodynamics taking an orbital variable into account are outlined. Continuous and singular vortex structures which arise upon rotation are discussed, as are their topology, phase transitions between them, and some other topologically nontrivial structures: a vortex with a free end, a boojum, and an instanton. The basic experiments in which the unique superfluid properties are observed are discussed. Some other existing or possible systems in which a nonstandard breaking of gauge invariance leads to unusual properties are cited; examples are a neutron star and a superconductor with the A-phase structure. In the latter medium there is a structure similar to a Dirac magnetic monopole. This structure is discussed.

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INTRODUCTION

The discovery of the A phase of ^{3}He renewed interest in the problem of superfluidity. Because of the particular structural features of the order parameter in this substance, its superfluid properties are fundamentally different from those of all other known superfluids: He II, the B phase of ^{3}He, and the electrons in a superconductor. Suffice it to say that the velocity of the superfluid component of the A phase is not a potential velocity, and the Landau critical velocity is zero. These circumstances contradict the conventional wisdom that the flow of the superfluid component must be a potential flow and that the emission of excitations must be forbidden if superfluidity is to exist.

For ^{3}He-A we need to reexamine all the questions raised by superfluidity: 1) Does friction accompany the motion of the superfluid component? 2) Are there stable superflows of the nature of the persistent flow of He II in a closed channel or of an electric current in a closed superconductor? 3) What are the relaxation mechanisms for the superfluid current? 4) Is there an analog of the ac Josephson effect? That is, do oscillations of the flow arise under steady external conditions? 5) What is the structure of the A phase in a rotating vessel? 6) Is there a two-velocity hydrodynamics for the A phase, as is characteristic of He II, and do all the consequences of such a hydrodynamics exist (second sound, fourth sound, the mechanocaloric effect, etc.)? Among the many other questions that might be asked is how do the orbital variables and an anisotropy affect the superfluid properties.

As we will see, most of these questions can be answered by analyzing the structure of the order parameter of the A phase. In other words, most of the phenomena associated with superfluidity are consequences of the symmetry of the order parameter and depend only slightly on the particular medium which is the carrier of this structure. The distinction between one medium and another of the same symmetry is usually expressed in the numerical coefficients, which do not alter the qualitative picture. For example, all the properties of superfluid ^{4}He—a real liquid with a strong interaction—can be found by analyzing the model of a slightly nonideal Bose gas. This model retains the most important point: the structure of the order parameter. Correspondingly, the properties of the A phase can be found by analyzing a model of a slightly nonideal Fermi gas with an interaction potential of the type which gives rise to the *p*-pairing with the A-phase structure (the so-called weak-coupling approximation). Another model which gives a qualitative description of the superfluid properties of the A phase is the model of a slightly nonideal Bose gas of diatomic molecules for which the wave function of the relative motion of the atoms in a molecule has the symmetry of the wave function of a Cooper pair of the A phase.

In the discussion below we will be using both these models, which lead to qualitatively identical results in most cases.

1. STRUCTURE OF THE ORDER PARAMETER AND THE SUPERFLUID VELOCITY

a) Structure of the order parameter

In ^{3}He, as in superconductors, we have the Cooper phenomenon: Spin-1/2 quasiparticles form bound states or "Cooper pairs" on the Fermi surface. Cooper pairs with an integer spin are bosons, whose condensation into a Bose condensate gives rise to superfluidity. In contrast with the electron Cooper pairs in a superconductor, the pairs in ^{3}He have a nonzero spin, $S = 1$, and an angular momentum $L = 1$ of the relative motion of the quasiparticles in the pair (a so-called orbital angular momentum). Cooper pairs are thus anisotropic in both shape and magnetic structure. By virtue of the macroscopic coherence, Cooper pairs are oriented identically, so that the superfluid phases of ^{3}He have, on the whole, a coherent magnetic structure and are therefore ordered magnetic materials (or antiferromagnetic materials), and they have a spatial anisotropy and are therefore liquid crystals. Mineev[1] has reviewed the magnetic and liquid-crystal properties of the superfluid phases of ^{3}He.

The A phase has two anisotropy axes: the magnetic axis **d** and the orbital or liquid-crystal axis **l**. The meaning of these vectors can be seen easily in the model of a Bose gas of molecules having the structure of Cooper pairs in an A phase. The unit vectors **d** and **l** determine the quantization axes of the spin and orbital angular momenta, respectively, of the molecule. The projection of the spin (S) of the molecule onto the **d** axis is zero ($S_{\mathbf{d}} = 0$); i.e., the spins of the molecule are oriented equiprobably in the plane perpendicular to **d**, so that the molecule has no average spin, and the A phase therefore has no average nuclear magnetic moment: The A phase is a uniaxial antiferromagnet. The projection of the angular momentum L onto the **l** axis is 1 ($L_{\mathbf{l}} = 1$); i.e., the orbital angular momenta of the pairs are directed along **l**. The A phase thus has an internal orbital motion around **l**. In Subsection 1g we will see how this motion is manifested.

The wave function of a molecule in a state with these projections breaks up into the product of the spin wave function $\chi_{\alpha\beta}$ and the orbital wave function $f(R)Y_{11}(\mathbf{n})$:

$$\psi_{\alpha\beta}(\mathbf{R}) = \chi_{\alpha\beta} f(R)\, Y_{11}(\mathbf{n}), \tag{1}$$

where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$ is the coordinate of the relative motion of the atoms in the molecule, and $\mathbf{n} = \mathbf{R}/R$. The spinor corresponding to $S = 1, S_d = 0$ is (see §57 of Ref. 2)

$$\chi_{\alpha\beta} = i d_i (\sigma_i \sigma_y)_{\alpha\beta}, \tag{2}$$

where the σ_i are the Pauli matrices. The spherical harmonic $Y_{11}(\mathbf{n})$, which describes the orbital motion with $L = 1$, $L_l = 1$, is (§57 of Ref. 2)

$$Y_{11}(\mathbf{n}) = \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} = \sqrt{\frac{3}{8\pi}} (n_x + i n_y), \tag{3}$$

where θ and φ are the polar and azimuthal angles of the vector $\mathbf{n}$ in the coordinate system with polar axis along $\mathbf{l}$ (Fig. 1). The unit vectors $\mathbf{\Delta}'$ and $\mathbf{\Delta}''$ in this figure are the directions of the coordinate axes in the plane perpendicular to $\mathbf{l}$, so that we can write

$$n_x + i n_y = (\mathbf{\Delta}' + i\mathbf{\Delta}'', \mathbf{n}) \equiv (\boldsymbol{\psi}\mathbf{n}). \tag{4}$$

By virtue of the macroscopic coherence of the molecules in a Bose condensate, their wave functions must be identical. This assertion means that all molecules have the same complex vector $\boldsymbol{\psi} = \mathbf{\Delta}' + i\mathbf{\Delta}''$ and the same real vector $\mathbf{d}$. These vectors are degeneracy parameters of the A phase, since the energy does not depend on their orientation, if we ignore the weak spin-orbit or so-called dipole interaction, which tends to orient $\mathbf{d}$ and $\mathbf{l} = (i/2)[\boldsymbol{\psi}\boldsymbol{\psi}^*]$ parallel to each other.[1] States of the A phase thus have five degrees of freedom: three Euler angles specifying the orientation of the triad of unit vectors $\mathbf{\Delta}'$, $\mathbf{\Delta}''$, $\mathbf{l}$, and two angles specifying the direction of the vector $\mathbf{d}$. In the discussion below we will consider only the orbital part of the degeneracy parameter ψ, since it is this part which, by virtue of its complex nature, is pertinent to the superfluid properties of the A phase.

The complex vector degeneracy parameter $\boldsymbol{\psi}$ in the A phase is analogous to the complex scalar degeneracy parameter $e^{i\Phi}$ in He II, where Φ—the phase of the order parameter—is identical for all atoms of the Bose condensate of wave function $\psi = |\psi| e^{i\Phi}$. The quantity serving as the order parameter in ^{3}He, which is analogous to the complex function ψ in He II and which vanishes above the superfluid transition temperature T_c, is the complex matrix A_{ik}. This is the matrix of coefficients in the expansion of the wave function

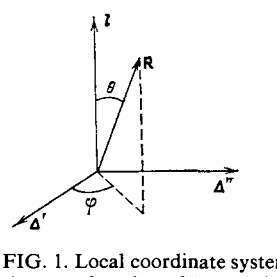

FIG. 1. Local coordinate system with the unit vectors $\mathbf{\Delta}'$, $\mathbf{\Delta}''$, $\mathbf{l}$, in which the wave function of the pair has the form in (1).

$\psi_{\alpha\beta}$ of a molecule with $S = 1$, $L = 1$ in nine eigenfunctions corresponding to states with different projections S and L:

$$\psi_{\alpha\beta}(\mathbf{R}) = i (\sigma_i \sigma_y)_{\alpha\beta} f(R) A_{ik} n_k.$$

At equilibrium in the A phase, the order parameter takes on the values [see (1)–(4)].

$$A_{ik} = C(T)\, d_i \psi_k,$$

where the constant C vanishes at $T = T_c$.

b) Current and superfluid velocity in the model of a Bose gas of molecules

In He II, in a slightly inhomogeneous state, the degeneracy parameter Φ depends on the coordinate of the atom, $\mathbf{r}$, giving rise to a coherent superfluid current of atoms of the Bose condensate. At $T = 0$, this current is

$$\mathbf{j} = \frac{\hbar}{2im_4} (\psi^* \nabla\psi - \psi\nabla\psi^*) = \rho \frac{\hbar}{m_4} \nabla\Phi = \rho \mathbf{v}^s,$$

where ρ is the density of ^{4}He atoms. In the A phase, in a slightly inhomogeneous state, the degeneracy parameter $\boldsymbol{\psi}$ is a function of the coordinate of the center of mass of the molecule, $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$. There may also be a coherent current of Cooper pairs in this case. In the model of a Bose gas of molecules, the current at $T = 0$ is found by multiplying the current of a single molecule,

$$\int d^3R \left[f^* Y_{11}^* \left(\frac{\hbar}{2im_3} \frac{\partial}{\partial \mathbf{r}_1} + \frac{\hbar}{2im_3} \frac{\partial}{\partial \mathbf{r}_2} \right) f Y_{11} + \text{c.c.} \right]$$
$$= \frac{\hbar}{4im_3} (\psi_l^* \nabla \psi_l - \psi_l \nabla \psi_l^*) = \frac{\hbar}{m_3} \Delta_l' \nabla \Delta_l''$$

by the density of molecules, $\rho/2$, where ρ is the density of ^{3}He atoms and m_3 is the ^{3}He mass:

$$\mathbf{j} = \rho \frac{\hbar}{2m_3} \Delta_l' \nabla \Delta_l''. \tag{5}$$

To this expression we must add the current which flows because a molecule has an angular momentum $\hbar\mathbf{l}$ equal to (§115 of Ref. 2)

$$\frac{1}{2} \operatorname{rot} \left(\frac{\rho}{2m_3} \hbar\mathbf{l} \right).$$

The coefficient of proportionality between the current and the density ρ in (5) is naturally called the "superfluid velocity,"

$$\mathbf{v}^s = \frac{\hbar}{2m_3} \Delta_l' \nabla \Delta_l'', \tag{6}$$

so that the total current density in a Bose liquid of molecules with the A-phase structure is, at $T = 0$,

$$\mathbf{j} = \rho \mathbf{v}^s + \frac{1}{2} \operatorname{rot} \left(\frac{\rho}{2m_3} \hbar\mathbf{l} \right). \tag{7}$$

We will see in the following subsection that in a real A phase the current also has another term with the curl (rot) of $\mathbf{l}$; this term does not change the superfluid velocity.

Let us mention some of the properties of the superfluid velocity $\mathbf{v}^s$. If the field of the vector $\mathbf{l}$ is fixed in space, the parameter $\boldsymbol{\psi}$ is left with only a single degree of freedom: rotation through an arbitrary angle around the $\mathbf{l}$ axis. We assume that $\mathbf{l}$ is uniform; then all values of the parameter $\boldsymbol{\psi}$ can be found by specifying some fixed value $\boldsymbol{\psi}_0$ and acting on it with the rotation matrix $\overleftrightarrow{\mathbf{R}}(-\mathbf{l}\Phi)$, which performs a rotation through an angle $-\Phi$ around the $\mathbf{l}$ axis:

$$\boldsymbol{\psi} = \overleftrightarrow{\mathbf{R}}(-\mathbf{l}\Phi)\, \boldsymbol{\psi}_0 = e^{i\Phi} \boldsymbol{\psi}_0. \tag{8}$$

We see that when the $\mathbf{l}$ field is uniform the angle Φ plays the role of the phase of the Bose condensate. According to (6), the superfluid velocity

$$\mathbf{v}^s = \frac{\hbar}{2m_3} \nabla\Phi \quad (9)$$

has the same form as for an ordinary Bose condensate in He II. In the general case in which $\mathbf{l}$ depends on the coordinates, however, the superfluid velocity can no longer be represented as the gradient of a phase. Taking the curl of both sides of (6), we find that curl $\mathbf{v}^s$ is not zero but is related to the gradients of the vector $\mathbf{l}$ by the relation

$$\text{rot}\,\mathbf{v}^s = \frac{\hbar}{4m_3} e_{ikl} l_i [\nabla l_k, \nabla l_l], \quad (10)$$

which is called the Mermin-Ho relation.[3] The superfluid flow is thus a potential flow only for a certain class of vector fields $\mathbf{l}$: those for which the right side of (10) vanishes. Included in this class of fields is, for example, a planar distribution of $\mathbf{l}$.

c) Strange term in the current in the model of a Fermi gas

A calculation of the current in the weak-coupling model[4] leads to a term in the current which is not found in (7). The appearance of this new term, which contradicts the simple model of a Bose condensate of molecules [this model leads to (7)], seemed so strange that many investigators continued to doubt its existence until it became clear that this term results from a nonremovable singularity in the state density at two points on the Fermi surface where the gap in the excitation spectrum vanishes. We will not go through the calculations[4] here, but we will show how to derive the final result from some simple arguments.[5,6]

In contrast with a Bose liquid, whose molecules are produced by a pairing of atoms in real space, the atoms in a Fermi liquid combine into Cooper pairs on the Femi surface in momentum space. The atoms which pair up have oppositely directed momenta $\mathbf{k}$ and $-\mathbf{k}$, so that the resultant momentum of the pair is zero at equilibrium. The angular part of the wave function of a pair with a given $\mathbf{k}$ is found by replacing $\mathbf{R}$ by $\mathbf{k}$ in (3):

$$\psi_{\mathbf{k}} = \Delta_0 (\mathbf{\Delta}' + i\mathbf{\Delta}'', \mathbf{n}), \quad \mathbf{n} = \mathbf{k}/k. \quad (11)$$

The wave function $\psi_{\mathbf{k}}$ is normalized in such a manner that its modulus

$$|\psi_{\mathbf{k}}| = \Delta_0 |[\mathbf{n}, \mathbf{l}]| \quad (12)$$

coincides with the gap in the excitation spectrum $E_{\mathbf{k}}$:

$$E_{\mathbf{k}} = \sqrt{\left(\frac{k^2}{2m_3} - \varepsilon_F\right)^2 + |\psi_{\mathbf{k}}|^2}. \quad (13)$$

In contrast with ordinary s pairing in superconductors, the gap in the spectrum depends on $\mathbf{k}$: It vanishes if $\mathbf{k}\|\mathbf{l}$ and reaches a maximum value Δ_0 at $\mathbf{k}\perp\mathbf{l}$. The phase of the wave function of the pair,

$$\Phi_{\mathbf{k}} = \text{arctg}\, \frac{\mathbf{\Delta}''\mathbf{k}}{\mathbf{\Delta}'\mathbf{k}} \quad (14)$$

also depends on $\mathbf{k}$ and has a singularity at $\mathbf{k}\|\mathbf{l}$, where it is not defined.

The current density is determined by $\Sigma_{\mathbf{k}} \mathbf{k} n(\mathbf{k}, \mathbf{r})$, where $n(\mathbf{k}, \mathbf{r})$ is the distribution function of the Fermi particles. The coherent change in the distribution function at $T=0$ due to a slight inhomogeneity can be found by noting that the phase of the wave function is, in the classical limit, the mechanical action $s = (\hbar/2)\Phi$ (§17 in Ref. 2). The momenta in the distribution function are thus shifted by an amount ∇S, and the coordinates are correspondingly shifted by $-(\partial S/\partial \mathbf{k})$. As a result we have

$$\delta n(\mathbf{k}, \mathbf{r}) = (1/2)\left(\nabla n \frac{\partial \Phi_{\mathbf{k}}}{\partial \mathbf{k}} - \frac{\partial n}{\partial \mathbf{k}} \nabla \Phi_{\mathbf{k}}\right), \quad (15)$$

and an integration by parts gives us the following expression for the current density:

$$\begin{aligned} \mathbf{j} &= \sum_{\mathbf{k}} \mathbf{k}\,\delta n(\mathbf{k}, \mathbf{r}) \\ &= \frac{1}{2}\sum_{\mathbf{k}} n_{\mathbf{k}} \nabla\Phi_{\mathbf{k}} + \frac{1}{2}\nabla_i\left(\sum_{\mathbf{k}} \mathbf{k} n \frac{\partial \Phi_{\mathbf{k}}}{\partial k_i}\right) \\ &\quad - \frac{1}{2}\sum_{\mathbf{k}} \mathbf{k} n_{\mathbf{k}} \left(\nabla \frac{\partial}{\partial \mathbf{k}} - \frac{\partial}{\partial \mathbf{k}} \nabla\right)\Phi_{\mathbf{k}}. \end{aligned} \quad (16)$$

Substituting in $\Phi_{\mathbf{k}}$ from (14), we find $\rho\mathbf{v}^s$ and $(1/2)\,\text{rot}(\rho\hbar\mathbf{l}/2m_3)$ for the first and second terms in (17), where $\rho = \Sigma_{\mathbf{k}} n(\mathbf{k}, \mathbf{r})$, and $\mathbf{v}^s$ is given by (6). Accordingly, the difference between the current in a Fermi liquid and that in a Bose liquid, (7), lies in the third term in (16), which would vanish if $\Phi_{\mathbf{k}}$ had no singularity at $\mathbf{k}\|\mathbf{l}$, i.e., where the gap in the excitation spectrum is zero. Because of this singularity, the mixed derivatives of $\Phi_{\mathbf{k}}$ are not equal and instead differ by a δ-function of $\mathbf{k}_\perp$, where $\mathbf{k}_\perp = \mathbf{k} - \mathbf{l}(\mathbf{kl})$:

$$\left(\nabla \frac{\partial}{\partial \mathbf{k}} - \frac{\partial}{\partial \mathbf{k}} \nabla\right)\Phi_{\mathbf{k}} = 2\pi\,(\mathbf{l}\,\text{rot}\,\mathbf{l})\,(\mathbf{kl})\,\delta(\mathbf{k}_\perp). \quad (17)$$

We then find the following expression for the current in the weak-coupling model at $T=0$:

$$\mathbf{j} = \rho\mathbf{v}^s + \frac{1}{2}\,\text{rot}\left(\frac{\rho}{2m_3}\hbar\mathbf{l}\right) - \frac{\hbar}{2m_3} C_0 \mathbf{l}\,(\mathbf{l}\,\text{rot}\,\mathbf{l}), \quad (18)$$

where

$$C_0 = \frac{m_3}{4\pi^2}\int_{-\infty}^{\infty} dk_\parallel k_\parallel^2 n(0, 0, k_\parallel), \quad k_\parallel = (\mathbf{kl}). \quad (19)$$

For a stepped distribution function $n = 2\Theta[\varepsilon_F - (k^2/2m_3)]$, as in a Fermi gas, we would have

$$C_0 = m_3 k_F^3/3\pi^2,$$

which is the same as the density of a Fermi gas, ρ. This agreement is of course only approximate, since the pairing alters the system, and C_0 differs from ρ by an amount of the order of $\rho(\Delta_0/\varepsilon_F)^2$. In a real ^{3}He–A liquid, C_0 may be quite different from ρ.

Without going into detail we would like to point out that that singularity in the wave function of the pair which gives rise to the unusual term with C_0 in the current is stable with respect to small changes in the order parameter,[7] although the system may leave the A phase. The reason is that the A phase is one of a family of states whose wave function has a vortical singularity in momentum space: The phase $\Phi_{\mathbf{k}}$ changes by 2π when the $\mathbf{l}$ axis in momentum space is circumvented. A vortex which is stable with respect to small perturbations leads to a singularity in the state density at points at which the vortex intersects the Fermi surface, $\mathbf{k} = \pm k_F \mathbf{l}$, and it is this singularity which produces the strange term in the current. The gap in the excitation spectrum vanishes at these points.

The model of a nonideal Fermi gas thus changes the term associated with curl $\mathbf{l}$, i.e., that part of the current which stems from the angular momentum of the pair, but it does not alter the determination of the superfluid velocity $\mathbf{v}^s$. The identical expressions for $\mathbf{v}^s$ in the two models suggest that the definition of $\mathbf{v}^s$ and therefore its nonpotential nature are general structural properties of the order parameter, i.e., are associated with a symmetry breaking which is characteristic of the given structure. Let us examine this question in more detail.

d) Superfluid velocity and joint deviations from rotational and gauge invariance

In the A phase, as in ^{4}He, the invariance of the state under a gauge transformation is disrupted specifically, under such a transformation a state with an order parameter ψ goes into another state with an order parameter $e^{i\alpha}\psi \neq \psi$, where α is a parameter of the transformation. Simultaneously, the invariance of the state under a three-dimensional rotation in coordinate space is disrupted: Any three-dimensional rotation which can be specified by an orthogonal matrix $\overleftrightarrow{\mathbf{R}}$ sends the state with ψ into a state with another order parameter $\overleftrightarrow{\mathbf{R}}\psi \neq \psi$. The A phase is analogous in this respect to a liquid crystal, whose states also change upon spatial rotations. The A phase is peculiar, however, in that the combined invariance is preserved: A state does not change upon a simultaneous gauge transformation $\psi \rightarrow e^{i\alpha}\psi$ and rotation through an angle α around the $\mathbf{l}$ axis. According to (8), ψ does not change upon this combined transformation:

$$\psi \rightarrow e^{i\alpha}\overleftrightarrow{\mathbf{R}}(\mathbf{l}\alpha)\,\psi = \psi. \tag{20}$$

It is because of this circumstance that the superfluid velocity $\mathbf{v}^s$ must be related to the rotational degrees of freedom. To show this, we write the order parameter ψ as

$$\psi = (\Delta' + i\Delta'')\,e^{i\Phi}. \tag{21}$$

We have replaced three variables (the three angles specifying the orientation of the triad of unit vectors Δ', Δ'', $\mathbf{l}$) by four variables, by artificially distinguishing a phase variable Φ. We assume that Φ changes only under a gauge transformation, while the unit vectors Δ', Δ'', and $\mathbf{l}$ change only under rotations. Since the state of the system is not changed by the composite transformation

$$\Phi \rightarrow \Phi + \alpha, \quad \Delta' + i\Delta'' \rightarrow \overleftrightarrow{\mathbf{R}}(\mathbf{l}\alpha)\,(\Delta' + i\Delta'') \tag{22}$$

all physical quantities must depend on only certain combinations of the variables Δ', Δ'', and Φ, namely, combinations which do not change under transformation (22). These combinations are $\mathbf{l} = [\Delta', \Delta'']$ and

$$\mathbf{v}^s = \frac{\hbar}{2m_3}\,(\nabla\Phi + \Delta_i'\nabla\Delta_i''). \tag{23}$$

The latter combination is an invariant definition of the superfluid velocity which is invariant under (22) and which differs from that in the case of He II, where we have $\mathbf{v}^s = (\hbar/m_4)\nabla\Phi$, in that this velocity is necessarily affected by the orbital variables. Definition (13) does not differ from definition (6), since the phase variable Φ is incorporated in Δ' and Δ'' in (6). All that we need to note is that the variables Δ' and Δ'' in (6) and (23) differ in transformation properties: In (23) they are invariant under a gauge transformation, since they are purely liquid-crystal variables, while in (6) they transform in accordance with $\Delta' + i\Delta'' \rightarrow e^{i\alpha}(\Delta' + i\Delta'')$ under a gauge transformation.

The unusual symmetry breaking in the A phase, which leaves the combined gauge-rotational symmetry unchanged, leads to a unique relationship between the superfluid and liquid-crystal properties of the A phase. Consequences of this relationship are the nonpotential nature of the superfluid flow and other phenomena discussed in this review.

e) Current and energy of the A phase at $T \neq 0$

In the model of a Fermi liquid, the normal motion is associated with Fermi excitations of the system which arise against the background of the superfluid ground state. At $T \neq 0$ these excitations make a contribution to the current in addition to the background coherent current (18):

$$\mathbf{P} = \sum_{\mathbf{k}} \mathbf{k}\nu_{\mathbf{k}}, \tag{24}$$

where $\nu_{\mathbf{k}}$ is the distribution function of the excitations. At a local equilibrium, $\nu_{\mathbf{k}}$ is characterized by a local temperature and a local velocity $\mathbf{v}^n$ of the normal motion:

$$\nu_{\mathbf{k}} = \left(\exp\frac{E_{\mathbf{k}} - \mathbf{k}\mathbf{v}^n}{T} - 1\right)^{-1}, \tag{25}$$

where $E_{\mathbf{k}}$ is the local energy of the excitations. In a slightly inhomogeneous liquid, $E_{\mathbf{k}}$ would differ from (13) in that it would also depend on the phase ($\Phi_{\mathbf{k}}$) of the wave function of the pair. This dependence can be easily found by generalizing the known expression for the excitation energy in the presence of a superfluid flow: $E_{\mathbf{k}} \rightarrow E_{\mathbf{k}} + \mathbf{k}\mathbf{v}^s$ in He II (Ref. 8), where $\mathbf{v}^s = (\hbar/m_4)\nabla\Phi$, or in a Fermi liquid with s pairing,[9] where $\mathbf{v}^s = (\hbar/2m)\nabla\Phi$. Here it should simply be recalled that the phase of the wave function of a Cooper pair depends on $\mathbf{k}$. As a result we find[5]

$$E_{\mathbf{k}} = \sqrt{\left(\frac{k^2}{2m_3} - \varepsilon_F\right)^2 + |\psi_{\mathbf{k}}|^2} + \frac{1}{2m_3}\,\mathbf{k}\nabla\Phi_{\mathbf{k}}. \tag{26}$$

Expressing $\nabla\Phi_{\mathbf{k}}$ in (26) in terms of $\mathbf{v}^s$ and curl $\mathbf{l}$ with the help of (14), and substituting (26) and (25) into (24), we find the following expression for the current in the approximation linear in $\mathbf{v}^s$ and curl $\mathbf{l}$ (we are ignoring terms $\sim \nabla\rho$):

$$\mathbf{j} = \overleftrightarrow{\rho}^s\mathbf{v}^s + \overleftrightarrow{\rho}^n\mathbf{v}^n + \frac{\hbar}{2m_3}\overleftrightarrow{C}\,\mathrm{curl}\,\mathbf{l} \tag{27}$$

with the tensor coefficients

$$\rho^s_{ij} = \rho^s\delta_{ij} - \rho_0 l_i l_j,$$

$$\rho^n_{ij} = (\rho - \rho^s)\,\delta_{ij} + \rho_0 l_i l_j, \quad C_{ij} = C\delta_{ij} - C_0 l_i l_j. \tag{28}$$

The coefficients ρ^s, ρ_0, C and C_0 have the temperature dependence

$$C = \frac{1}{2}(\rho^s - \rho_0) = \frac{1}{2}C_0,$$

$$\rho^s_{ij} = 3\rho\int\frac{d\Omega}{4\pi}\,n_i n_j\,(1 - Y(\mathbf{n}, T)), \tag{29}$$

where $Y(\mathbf{n}, T)$ is the Yosida function

$$Y(\mathbf{n}, T) = \frac{1}{2m_3 T}\int_0^\infty dk k\,\mathrm{sech}^2\frac{E_{\mathbf{k}}}{2T}, \tag{30}$$

which vanishes at $T = 0$ and is equal to 1 at $T = T_c$.

In a real Fermi liquid, the temperature dependence of the coefficients ρ^s, ρ_0, C, and C_0 would be more complicated,[4,10] but the structure of expression (27) would remain the same.

Let us write a general expression for the energy of a slightly inhomogeneous state. Like the current, the energy depends only on combinations which are invariant under transformation (22), i.e., on $\mathbf{l}$ and $\mathbf{v}^s$:

$$F=\int dV\left\{\frac{1}{2}\rho^s(\mathbf{v}^s-\mathbf{v}^n)^2-\frac{1}{2}\rho_0(\mathbf{l},\ \mathbf{v}^s-\mathbf{v}^n)^2+\frac{\hbar}{2m_3}C(\mathbf{v}^s-\mathbf{v}^n,\mathrm{curl}\,\mathbf{l})-\frac{\hbar}{2m_3}C_0(\mathbf{l}\,\mathrm{curl}\,\mathbf{l})(\mathbf{v}^s-\mathbf{v}^n,\mathbf{l})+\frac{1}{2}\left(\frac{\hbar}{2m_3}\right)^2K_1(\nabla\mathbf{l})^2+\frac{1}{2}\left(\frac{\hbar}{2m_3}\right)^2K_2(\mathbf{l}\,\mathrm{curl}\,\mathbf{l})^2+\frac{1}{2}\left(\frac{\hbar}{2m_3}\right)^2K_3[\mathbf{l},\mathrm{curl}\,\mathbf{l}]^2+\mathbf{j}\mathbf{v}^n-\frac{1}{2}\rho(\mathbf{v}^n)^2-L_0\mathbf{l}\frac{1}{2}\mathrm{curl}\,\mathbf{v}^n\right\}. \quad (31)$$

The dependence on $\mathbf{v}^n$ is found through a Galilean transformation from the coordinate system in which $\mathbf{v}^n=0$. The term curl $\mathbf{v}_n$ stems from the internal angular momentum of the Cooper pairs, $L_0\mathbf{l}$. The particular form of expression (31) for the energy is characteristic of this type of disruption of the symmetry and does not depend on the particular medium which is the carrier of the structure. The particular medium determines only the relationships among the coefficients ρ^s, ρ_0, C, C_0, K_2, K_r, K_3, and L_0. In the model of a Bose gas, for example, we would have $\rho_0=C_0=0$, $L_0=(\hbar/2)\rho$.

f) The normal component at $T=0$

The vanishing of the gap in the excitation spectrum has a curious consequence. Considering the terms in the current which are nonlinear in $\mathbf{v}^s$ and curl $\mathbf{l}$, we find that the density of the normal component is nonzero, $\rho^n_{ij}\neq 0$, even at[11] $T=0$. To prove this assertion, we note that at $T=0$ the quasiparticle distribution function (27) degenerates into a step function:

$$\nu_{\mathbf{k}}=\theta(-E_{\mathbf{k}}+\mathbf{k}\mathbf{v}^n). \quad (32)$$

For ordinary s pairing (in superconductors), the gap in the excitation spectrum does not vanish anywhere at $T=0$. Consequently, if the difference $\mathbf{v}^s$-$\mathbf{v}^n$ is sufficiently small, states with a negative effective energy $E-\mathbf{k}\mathbf{v}^n<0$ will not exist, and there will be no excitations at $T=0$. In the A phase, the gap in the excitation spectrum vanishes at $\mathbf{k}_\perp=0$, so that for nonzero $\mathbf{v}^s-\mathbf{v}^n$ and curl $\mathbf{l}$ there are states with a negative effective energy $E_{\mathbf{k}}-\mathbf{k}\mathbf{v}^n<0$. Although the density of these states is small, of the order of the small quantities $\mathbf{v}^s-\mathbf{v}^n$ and curl $\mathbf{l}$, these states make a substantial nonlinear contribution to the excitation current (24) at $T=0$. Substituting (32) and (26) into (24), we find

$$\rho^n_{ij}(T=0)\sim\rho\frac{k_F}{m_3\Delta_0}l_il_j|[\mathbf{l},\ \mathrm{curl}\,\mathbf{l}]|+\rho\frac{k_F^2}{\Delta_0^2}l_il_j(\mathbf{v}^s-\mathbf{v}^n,\ \mathbf{l})^2; \quad (33)$$

where k_F is the momentum on the Fermi surface. We note that the dependence of $\hat{\rho}^n$ on curl $\mathbf{l}$ is nonanalytic. The coefficients in expression (33) were calculated in a more rigorous theory in Refs. 12 and 13.

g) Orbital angular momentum in the A phase

As follows from Subsection 1a, each Cooper pair has an internal orbital angular momentum $\hbar\mathbf{l}$. We would thus expect that the entire liquid would have an orbital angular momentum in a homogeneous state. The magnitude of this angular momentum, however, depends strongly on the model. In a Bose gas of isolated molecules, the rotational motion of the molecules is easily separated from the translational motion, so that in this model we can introduce the concept of a density of the internal orbital angular momentum, which is equal at $T=0$ to the product of the density of molecules, $\rho/2$, and their internal angular momentum $\hbar\mathbf{l}$: $\mathbf{L}_0=(\hbar/2)\rho\mathbf{l}$.

In a Fermi liquid these molecules overlap markedly, since the dimension of a Cooper pair (of the order of the coherence length $\xi\sim10^{-16}$–10^{-15} cm; see the review by Mineev[1]) is much larger than the interatomic distance a. In this situation it is difficult to distinguish the translational motion of the Cooper pairs from their rotational motion. It thus becomes difficult to determine the density of the internal angular momentum, if only because a local characteristic of the liquid of this sort may not exist at all. In singling out a part of the volume in which we wish to calculate the internal angular momentum, we must necessarily include those Cooper pairs which intersect the boundary of the volume. Their contribution, like that of the surface currents to the orbital angular momentum, is comparable to the resultant orbital angular momentum of the internal pairs. The orbital angular momentum is thus not of a local nature in the A phase. To demonstrate this point, we consider the change in the angular momentum upon a local change in the density ρ and the vector $\mathbf{l}$ against the background of a homogeneous state, working from an integral relationship between the angular momentum and the current, $\int dV\mathbf{L}=\int dV[\mathbf{r}\mathbf{j}]$. Expressing the current variation in terms of $\delta\rho$ and $\delta\mathbf{l}$ with the help of (18), we find the variation of the orbital angular momentum to be

$$\delta\mathbf{L}=\frac{\hbar}{2}\mathbf{l}\delta\rho+\frac{\hbar}{2}(\rho-C_0)\delta\mathbf{l}. \quad (34)$$

Equation (34) shows that the density of the orbital angular momentum is not a total differential in a Fermi liquid, where $C_0\neq0$, demonstrating the nonlocal nature of the angular momentum. Nevertheless, the quantity $L_0=(\hbar/2)(\rho-C_0)$ found from a variation with $\delta\mathbf{l}$, has a definite physical meaning: It is that part of the orbital angular momentum which is related to the rotational, rather than translational, motion of the Cooper pairs. For a Bose gas this is obvious, since we would have $L_0=(\hbar/2)\rho$. In the A phase, this conclusion is indicated by the following circumstances: First, it is this quantity which appears in the energy (31), forming along with the local angular velocity of the rotation of the liquid, $\boldsymbol{\omega}=(1/2)\mathrm{curl}\,\mathbf{v}^n$, a term $-L_0\mathbf{l}\boldsymbol{\omega}$ which is characteristic of a liquid with an internal angular momentum. Second, L_0 appears in the dynamic equation for the vector $\mathbf{l}$, which is analogous at low temperatures to the equation for the magnetic moment in a magnetic material with an angular momentum L_0 (Ref. 11).

The quantity L_0 is very small, $L_0\sim\hbar\rho(\Delta_0/\varepsilon_F)^2\ \ln(\varepsilon_F/$

Δ_0), specifically because of the strong overlap of Cooper pairs, since $\Delta_0/\varepsilon_F \sim a/\xi \ll 1$. We can thus assert that the strong overlap of Cooper pairs causes their orbital motion to transform basically into a translational motion at the surface of the vessel. Only a small fraction of the angular momentum, $\sim \hbar\rho(a/\xi)^2$, corresponds to a local rotational motion of pairs. It is easy to see the origin also of the first term in (34). Even when there is a strong overlap of pairs, an excess (δN) of pairs in a given volume element gives rise to an orbital angular momentum $\hbar \mathbf{l}\delta N$ in this volume element. This angular momentum is related to the currents which flow over the surface of this volume. For the same reason, the total angular momentum of an A phase with a homogeneous distribution of $\mathbf{l}$ in a vessel containing N ^{3}He atoms is $(\hbar/2)N\mathbf{l}$. For more details on the orbital angular momentum we refer the reader to Ref. 11 and the bibliography there.

2. SUPERFLUIDITY OF THE A PHASE

a) Why the superfluid component superflows

As can be seen from (27), the current of the A phase consists of three parts. The first, characterized by the velocity $\mathbf{v}^n$, describes the normal motion. The density of the normal component, $\overleftrightarrow{\rho}^n$, is a tensor, in contrast with the scalar density in He II and ^{3}He–B. In the A phase we can thus clearly see the arbitrariness of this separation of the liquid into two components. Specifically, we cannot represent the liquid as consisting of two types of atoms—superfluid and nonsuperfluid—because in this case both densities would be scalars. It is thus more appropriate to say that in the liquid there are several types of motions, one of which occurs at the normal velocity $\mathbf{v}^n$. At low T, the Fermi excitations of the system are entrained in this motion, while the entire liquid is entrained above T_c. This motion clearly experiences a friction, both an internal friction and a friction exerted by the walls. Do the other types of motion with $\mathbf{v}^s$ and curl $\mathbf{l}$, experience a similar friction? The nonpotential nature of the velocity $\mathbf{v}^s$ and the vanishing of the Landau critical velocity[8] $v_L = \min(E_{\mathbf{k}}/|\mathbf{k}|)$, at which excitations begin to be created, cast doubt on the superfluidity of these motions. To answer the question of whether the superfluid component in the A phase superflows, we first consider He II. We analyze the reason for the superfluidity of its superfluid component. We consider the flow of He II through a closed channel. The velocity of the normal component of He II falls continuously to zero because of friction exerted by the wall and by viscosity, which causes the slowing to penetrate into the interior of the liquid. The superfluid velocity of the liquid, on the other hand, cannot fall continuously to zero: Stopping of the superfluid flow requires a certain type of disruption of the coherent superfluid state. If the wave function of the Bose condensate, $\psi = \sqrt{\rho_s}\, e^{i\Phi}$, is continuous everywhere, and if its phase Φ is defined everywhere, i.e., if ρ_s vanishes nowhere, then the circulation of the superfluid velocity $\mathbf{v}^s = (\hbar/m_4)\nabla\Phi$ over any contour around the channel takes on a quantized value

$$\oint \mathbf{v}^s\, d\mathbf{r} = \frac{2\pi\hbar}{m_4} N; \tag{35}$$

here the integer N specifies that Φ changes by $2\pi N$, and ψ returns to its original value. Because N is discrete, the velocity $\mathbf{v}^s$ cannot decrease continuously. A substantial expenditure of energy is usually required to produce discontinuities in a superfluid liquid; if these processes are ignored, then a flow with minimum energy in a set of flows with a given $N \neq 0$ is stable. This flow is nondissipative, since it does not require an external source of energy. Since there is no overall dissipation, there *is* no local dissipation in any volume element of the liquid. In other words, despite the fact that there is a velocity difference $\mathbf{v}^s - \mathbf{v}^n$ in each volume element, there is no consequent dissipative decrease in this difference. This conclusion means that the superfluid component can move without resistance in a vessel of arbitrary geometry, including an unclosed channel. This is the fundamental property of superfluidity. This property of course does not mean that there is absolutely no mechanism for relaxation of the superfluid velocity. The superfluid motion may prove unstable with respect to the formation of various structures which are inhomogeneous in space and time, with discontinuities in the phase of the order parameter which give rise to an ac Josephson effect, which we will be discussing below. The property of superfluidity means that there is no homogeneous relaxation of the superfluid velocity $\mathbf{v}^s$ toward $\mathbf{v}^n$.

b) Superfluidity and the Landau criterion

How are these arguments affected by the possibility of the creation of excitations (phonons, rotons, or Fermi quasiparticles)? Excitations are created when the velocity of the superfluid motion (more precisely, $\mathbf{v}^s - \mathbf{v}^n$) exceeds the Landau critical velocity $v_L = \min(E_{\mathbf{k}}/k)$. The resulting quasiparticles do not alter the continuity of the phase of the condensate wave function $\psi = \sqrt{\rho_s}\, e^{i\Phi}$, so that the velocity $\mathbf{v}^s$ in a closed channel does not change. A dissipation of the superfluid flow $\rho^s \mathbf{v}^s$ can occur only through a decrease in ρ^s, and this decrease does occur, since the normal density $\rho^n = \rho - \rho^s$ increases because of the creation of additional quasiparticles. Whether the superfluid flow falls to zero in the process depends on the particular conditions. In a Bose liquid, both cases are possible: For example, the creation of a large number of excitations can change the excitation spectrum, with the result that the increase in the density of the normal component is brought to a halt, and the system acquires an equilibrium value ρ^s which depends on $v^s - v_L$. In a Fermi liquid, the increase in the number of excitations is brought to a halt as soon as the Fermi excitations fill the states with a negative effective energy $E_k^0 + (\mathbf{k}, \mathbf{v}^s - \mathbf{v}^n) < 0$, and the liquid again goes into an equilibrium state with a flow. This is precisely the situation in the A phase, in which we have $v_L = 0$ because there is no gap in the excitation spectrum. The excitations are formed at an arbitrarily low velocity $\mathbf{v}^s$, but the number of states with negative energy is proportional to $(\mathbf{v}^s - v^n)^2$. The Fermi excitations thus rapidly fill negative energy levels, the dissipation stops, and the superfluid current circulates with a slightly altered density ρ^s. At $T = 0$ we thus have [see (33)]

$$\rho - \rho^s \sim \frac{k_F^2}{\Delta_0^2} (\mathbf{v}^s - \mathbf{v}^n)^2. \tag{36}$$

The Landau velocity thus does not generally constitute a critical velocity above which a dissipationless flow of the superfluid component exists.

c) The A phase is moving even in the ground state

We turn now to the condition for a potential flow. We will show that this is not a necessary condition for superfluidity. To establish the property of superfluidity in the case of He II we made use of the circumstance that dissipationless current states were possible in principle in the liquid. Such states also exist in the A phase, although its superfluid velocity is not a potential velocity. We will now prove this assertion.

We consider the ground state of a liquid in a spherical vessel. Because of the boundary conditions (l is directed normal to the boundary of the vessel), the liquid cannot be in a uniform state. Two possible distributions of the field l are shown in Fig. 2. In the first case, the singularity in the field l are shown in Fig. 2. In the first case, the singularity in the field l is at the center, while in the second case it is at the surface of the vessel. In general, it is easy to see that if there were not singularities it would not be possible to construct a configuration which satisfied the boundary conditions. We note that in both configurations shown here the velocity $\mathbf{v}^s$ is nonzero. That this is true can also be seen from the Mermin-Ho relation (10). If we are interested in axisymmetric distributions of the fields l and $\mathbf{v}^s$, we can express $\mathbf{v}^s$ directly in terms of l by working from (10) (Ref. 3):

$$\mathbf{v}^s = \frac{\hbar}{2m_3}\hat{\varphi}\frac{1-(\mathbf{l}\hat{\mathbf{z}})}{\rho}, \tag{37}$$

where $\hat{\mathbf{z}}$, $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\varphi}}$ are the unit vectors of a cylindrical coordinate system. In a spherical vessel with ^{3}He–A there is thus always a nonzero flow with a velocity $\mathbf{v}^s$, even in the ground state.[14] There is accordingly no uniform relaxation of $\mathbf{v}^s - \mathbf{v}^n$. The superfluid component of the A phase is thus a superfluid not only in name. Analogously, it can be shown that a flow with curl l also experiences no resistance.

The potential nature of the flow is thus not a necessary condition for superfluidity. Nor is it a sufficient condition. To demonstrate this point, we consider a two-component liquid, e.g., a solution of two normal liquids. Although the flow of one of the components with respect to the other may be a potential flow, e.g., uniform over space, there is a local friction between the components, and the velocity difference undergoes a uniform relaxation. For superfluidity to occur, there must exist an order parameter in terms of whose spatial derivatives the superfluid velocity can be expressed. In other words, we need a disruption of gauge invariance. The A phase is a superfluid because there is a disruption of the gauge invariance. The fact that this disruption is coupled in a nontrivial manner with a breaking of rotational symmetry does not eliminate the superfluidity, but it does lead to some unusual superfluid properties.

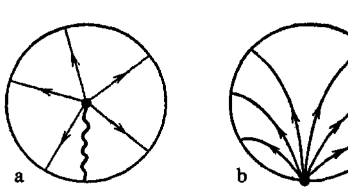

FIG. 2. Two states of the field l in a spherical vessel. a—There is a point singularity in the field l at the center of the vessel, and from this point emerges a singularity in the field $\mathbf{v}^s$: a vortex (wavy line); b—there is a point singularity at the surface of the vessel: a boojum.

d) Vortex with a free end and the boojum

Which of the states in Fig. 2 is the ground state? From (37) we see that for the configuration in Fig. 2a the superfluid velocity is singular on a radius descending from a singular point. Near this radius the velocity $\mathbf{v}^s$ is

$$\mathbf{v}^s = \frac{\hbar}{m_3\rho}\hat{\varphi}, \tag{38}$$

the same as the velocity field near a quantized vortex in He II. There are no other singularities anywhere in the liquid. We thus observe a surprising phenomenon: A quantized vortex may terminate in the volume of the liquid.[15,16] This would be absolutely impossible in either He II or ^{3}He-B and also in a superconductor. This fact is a simple consequence of the topological properties of the order parameter in the A phase: A singularity of the vector field l having a nonzero topological charge is a source of a vortex line.[15] Because of the energy of the vortex, states of the system with a singularity in the volume are not favored, and a ground state exists only if the singularity moves off to the surface (Fig. 2b). In this case the vortex contracts to a point, so that the singularity at the surface of the vessel has the configuration of a point vortex. The circulation of $\mathbf{v}^s$ along a contour along the surface and circumventing the singularity is $2\pi\hbar/m_3$. This point vortex was first studied by Mermin,[17] who named it a "boojum" (a mysterious object from Lewis Carroll's poem *The Hunting of the Snark*; how this term came to be adopted in the physics literature has been related in a delightfully humorous way by Mermin[18]). The reader is referred to Ref. 19 for a topological classification of surface defects in ordered media which are similar to the boojum in the A phase. Boojums play an important although not governing role in the relaxation of a superfluid flow (Section 3).

e) Topological stability of a macroscopic flow

We saw in Section 1 that the superfluid velocity $\mathbf{v}^s$ could not undergo a uniform relaxation; a superfluid flow can relax only as a result of processes which are nonuniform over space and time. We know that in He II the decay of the superfluid current results from the creation of, or the motion of existing, structural defects in the channel: quantized vortex lines at whose axes the superfluid state is disrupted. The phase Φ, which changes by $2\pi N$ as the vortex axis is circumvented, is not defined on the axis itself. It follows from the continuity of the order parameter $\psi = \sqrt{\rho_s}e^{i\Phi}$ that the quantity ρ^s, a measure of the superfluid state, vanishes on the vortex axis.

Figure 3 illustrates the decrease in the circulation of the superfluid velocity in He II in a closed channel by a single quantum through the creation and motion of a vortex with a single quantum of circulation. This process would ordinarily require a large expenditure of energy, in order to increase the

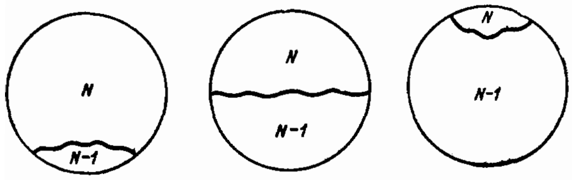

FIG. 3. Passage of a vortex (wavy line) across the cross section of a channel at successive times during a decrease in the superfluid flow in He II by one quantum of circulation ($N \to N-1$).

size of the vortex as it sweeps over the cross section of the channel. A state with $N \neq 0$ is thus extremely stable, a consequence of the topology of the system. The current may be macroscopically large if the topological charge of the state N is large.

f) Topological characteristic of flows in the A phase

We wish to determine whether the A phase has an integer invariant of motion which ensures the stability of macroscopic flows. It might seem at first glance that we could resolve this question by starting in the same way as in He II. As in He II, we could introduce a condensate phase Φ, which would have to change by $2\pi N$ upon a circuit of the channel, and this change would seemingly give us stability. That this line of argument is erroneous can be seen simply from Eq. (21). Without changing the degeneracy parameter ψ, we can use combined transformation (22) to distill the dependence $\Phi(\mathbf{r})$ into an $\mathbf{r}$ dependence of the unit vectors $\mathbf{\Delta}'$ and $\mathbf{\Delta}''$. In other words, without changing the state, we can convert the shift of the phase Φ by $2\pi N$ into a shift by $2\pi N$ of the angle through which the vectors $\mathbf{\Delta}'$ and Δ'' are rotated around $\mathbf{l}$. In this case we are dealing with three-dimensional rotations, and we have no guarantee that a change in the angle of the rotation around $\mathbf{l}$ will not be undone in some fashion through a rotation of the unit vectors around other directions. The problem thus reduces to one of determining whether there are distinct configurations of the field of unit vectors, which do not continuously convert into each other, and how these configurations are related to the magnitude of the superfluid flow. Questions of this type are resolved by topological methods (see Ref. 20 and the reviews in Refs. 21–23).

The solution procedure is very simple. We examine the changes in the order parameter as we go along some arbitrary closed contour γ around the channel, noting the path Γ traced out by the values of the degeneracy parameter over its range R. We assume that Γ is closed, so that in this circumvention we return to a value of the degeneracy parameter $\psi(\mathbf{r}_0)$ at the same point $(\mathbf{r}_0)$ in the channel from which we started (Fig. 4). In the space R there may be different sets of closed contours which cannot be continuously deformed into each other. It is intuitively clear, and confirmed by a rigorous analysis, that the number of different configurations—not convertible into each other—of the field of the degeneracy parameter in a channel is determined by the number of different contours in R space (which cannot be converted into each other).

Let us check this picture for He II. The space of the degeneracy parameter, R (the range of the phase Φ), is a circle. On this circle we can easily construct various types of

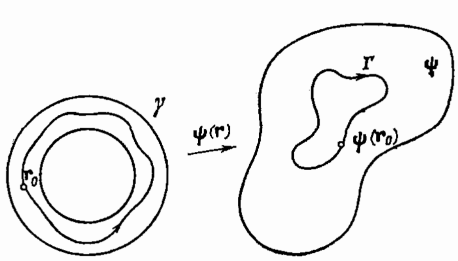

FIG. 4. The contour γ circumventing the channel is mapped by the field of the degeneracy parameter $\psi(\mathbf{r})$ into a continuous contour Γ in the space R.

closed contours which cannot be converted into each other. These are the contours, $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$, which go around the circle $1, 2, \ldots, N$ times in the positive direction (N could be negative). These contours also correspond to configurations with 1, 2, ... quanta of the circulation of the superfluid velocity.

In the A phase, in order to describe the sets of closed contours in the unit-vector space R we need a convenient way to represent this space. All possible orientations of the triad of unit vectors $\mathbf{\Delta}', \mathbf{\Delta}'', \mathbf{l}$ can be specified conveniently by means of the orthogonal matrix of three-dimensional rotations $\overleftrightarrow{\mathbf{R}}$ (Subsection 1d), which transforms this triad from the fixed position ψ_0 into all other positions. In other words, the range of ψ is the range of the matrix $\overleftrightarrow{\mathbf{R}}$, which is related to ψ by

$$\psi = \overleftrightarrow{\mathbf{R}}\psi_0. \tag{39}$$

Rotations are conveniently specified by the vector $\boldsymbol{\alpha}$, whose direction specifies the rotation axis and whose modulus specifies the rotation angle. The range of all nonequivalent values of the vector $\boldsymbol{\alpha}$ is a sphere of diameter π, since a rotation through an angle greater than π can always be represented as a rotation through an angle less than π. A rotation through an angle $2\pi > |\boldsymbol{\alpha}| > \pi$, for example, is equivalent to a rotation through an angle $2\pi - |\boldsymbol{\alpha}| < \pi$ around the opposite axis. It follows that rotations through an angle π around oppositely directed axes are equivalent; i.e., diametrically opposite points on the surface of a sphere are equivalent. Figure 5 shows all possible types of closed contours in the space R. A closed contour of the type Γ_0 can be deformed continuously to a point, so that the corresponding inhomogeneous state can be continuously transformed into a state with a constant order parameter, i.e., without a flow. The contour Γ_1 is also closed, since it begins and ends at points with the same ψ. It cannot, however, be deformed into a contour of type Γ_0. There are no other closed contours. For example, the contour Γ_2, which passes twice in succession

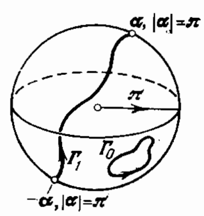

FIG. 5. Two sets of closed contours in the space of three-dimensional rotations.

along Γ_1 (Fig. 6), can be continuously contracted to a point; i.e., it belongs to the set of contours Γ_0. This circumstance can be described algebraically by

$$\Gamma_1\Gamma_1 = \Gamma_0, \tag{40}$$

i.e., the product of two contours of set Γ_1 is a contour of set Γ_0. This differs from the corresponding law in the case of a circle, where we would have $\Gamma_n \cdot \Gamma_m = \Gamma_{n+m}$.

The A phase in a closed channel can thus be in only two topologically different states. If a state corresponds to the set of contours Γ_0, it can be continuously transformed into a rest state. If, on the other hand, it corresponds to set Γ_1, then it can be continuously transformed into a state with the least energy, (31), in this set, i.e., into the state with the smallest flow, which corresponds in order of magnitude to a flow with a single quantum of velocity circulation in He II, i.e., with $v^s \sim \hbar/m_3L$, where L is the length of the channel.

g) How can the flow in the A phase be continuously reduced?

Let us consider in more detail how a macroscopic flow relaxes into one of these states. We assume that the field **l** is directed along the flow (the z axis). This orientation of **l** is favored from the energy standpoint, because of the term with $\rho_0 > 0$ in the energy in (31). The order parameter in this current state depends on the coordinates in the following way:

$$\psi = \overleftrightarrow{R}(\alpha(z)\hat{z})(\hat{x}+i\hat{y}). \tag{41}$$

The angle (α) specifying the rotation of the unit vectors Δ' and Δ'' around the $\hat{z}$ axis runs over the values from 0 to $2\pi N$ as the channel is circumvented. Figure 7 shows the path (Γ_N) traced out by the order parameter in the space R; this path represents N sequential paths Γ_1. The contours Γ_1 are directed vertically along z, showing that rotations are being performed around the $\hat{z}$ axis. Pairs of neighboring contours Γ_1 can annihilate continuously, as shown in Fig. 6. As a result, we pass in succession through states of a homogeneous flow with $\mathbf{l}\|\hat{z}$ but with a flow which decreases by two quanta in each instance (Fig. 7, a and c). In the intermediate states (Fig. 7b), **l** varies continuously in space (a curvature of the contour Γ in R means that rotations are performed not only around $\hat{z}$ but also around other axes, and these other rotations involve the vector **l**).

We thus see that, in contrast with He II, the topology does not guarantee stability of a macroscopic superfluid flow. In other words, the flow is globally unstable. The flow can undergo a continuous relaxation, and in the course of this relaxation we would find not a creation of discontinuous singular formations (quantized vortices), as in He II but a

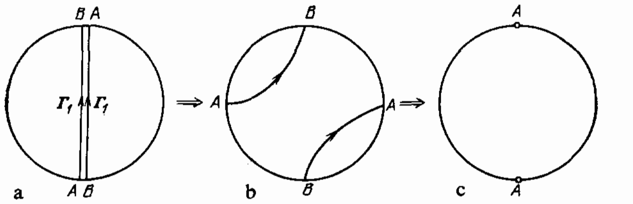

FIG. 6. Contraction to a point of the closed contour Γ_2 (ABA), which is initially (a) the product of two closed contours of set Γ_1 (AB and BA; the points A and B in part (a) are equivalent).

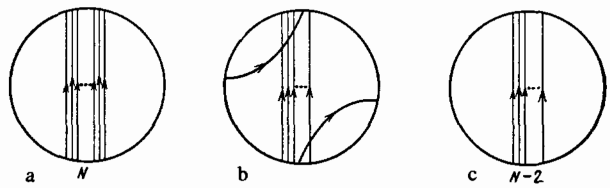

FIG. 7. Representation in the space R of the continuous decrease in the flow by two quanta of circulation in the A phase.

continuous and inhomogeneous change in the vector field **l**, i.e., the formation of continuous textures.

h) Stability of the flow near a wall and in a magnetic field

In contrast with the topological instability of the flow in a channel, a flow near a surface in an A phase is topologically stable.[17] The field **l** is fixed at the boundary of a channel, so it cannot lead to a damping of the flow. As a result, we find a unique situation in which there is no flow in the channel (or it is very small), but a flow is circulating in a narrow layer along the surface. The flow at the surface can relax only through the creation and motion of point vortices on the surface, i.e., boojums.

Second, in the A phase there is the unique possibility of regulating the topology by applying a magnetic field.[24] As a result, as we will see in Section 5, a macroscopic flow in an A phase in a closed channel in a magnetic field becomes globally stable.

i) Local stability of a superflow in the A phase

Since there are no formidable topological obstacles to prevent a continuous relaxation of a macroscopic superfluid flow in the A phase, such a flow is far simpler to disrupt than in He II. The relaxation process may occur without an activation energy, if there is a monotonic decrease in the energy in (31) in the course of the transformation from the state in Fig. 7a to the state in Fig. 7c; alternatively, an energy barrier may be surmounted if the elastic energy of the field **l** in (31) in the intermediate state (Fig. 7b) exceeds the increase in the kinetic energy of the liquid due to the decrease in $\mathbf{v}^s$. However, even if there is a barrier of this type, it would be much simpler to surmount than a barrier dictated by the topology, whose surmounting would require the creation of discontinuities in the ordered state. The existence of a barrier depends on the relation between the coefficients in the energy in (31). Analysis of a flow with a uniform **l** directed along the flow shows that it is locally stable if[25]

$$\left[C_0+\frac{1}{2}(\rho^s-\rho_0)\right]^2<\rho_0K_3. \tag{42}$$

In the model of a Bose gas of molecules we would have $\rho_0 = C_0 = 0$, and condition (42) would never hold; i.e., the flow would always be stable. In a real A phase, which can be described well by the weak-coupling model near T_c, inequality (42) may or may not hold at a given temperature, depending on the width of the channel. This circumstance is a consequence of the spin-orbit (dipole) interaction between the orbital vector **l** and the spin vector **d** (see the review by Mineev[1]). In narrow channels, with a diameter smaller than the dipole length $\xi_D \sim 10^{-3}$ cm, **d** does not depend on the orbital vector **l** and is unrelated to the superfluidity. In wide channels, with a diameter greater than ξ_D, the dipole interaction orients **d** along **l**, and the elastic energy of the field **d** is added to the elastic energy of the field **l**, effectively increasing the coefficient K_3 in (42). Let us consider the two cases separately.

In narrow channels we would have

$$\rho_0 = C_0 = \frac{1}{2}\rho^s = \frac{2}{3}K_3 = 2K_1 = 2K_2, \tag{43}$$

and condition (42) would not hold. In other words, such a flow would be unstable. It can be shown that at these temperatures this flow can be transformed into a state without a flow with a monotonic decrease in energy.[25]

j) The helix—a spiral current structure

In wide channels the coefficients in the energy in (31) are

$$\rho_0 = C_0 = \frac{1}{2}\rho^s = \frac{2}{5}K_3, \quad K_1 = K_2 = K_3, \tag{44}$$

and condition (42) holds; i.e., the superfluid flow is locally stable. As the temperature is lowered, condition (42) may also cease to hold for wide channels at some $T = T^*$. Analysis of a flow at T just below T^* shows that, although the local energy minimum corresponding to the homogeneous flow disappears, a new local minimum arises which corresponds to a flow with a spiral structure in the field of the vector **l** (a helix)[26–28]:

$$\begin{gathered}\boldsymbol{\psi} = \exp\,[i\,(u - v)\,z]\,\overleftrightarrow{\mathbf{R}}(-v\hat{\mathbf{z}})\,\overleftrightarrow{\mathbf{R}}(\beta\hat{\mathbf{y}})\,(\hat{\mathbf{x}} + i\hat{\mathbf{y}}),\\ \mathbf{l} = \hat{\mathbf{z}}\cos\beta + (\hat{\mathbf{x}}\cos vz - \hat{\mathbf{y}}\sin vz)\sin\beta;\end{gathered} \tag{45}$$

where β is the angle made by **l** with the axis of the spiral, $\hat{\mathbf{z}}$, and $(2\pi/v)$ is the pitch of the spiral. The velocity $\mathbf{v}^s$ in the helix is directed along the flow,

$$\mathbf{v}^s = \frac{\hbar}{2m_3}\hat{\mathbf{z}}\,(u - v\,(1 - \cos\beta)), \tag{46}$$

but the current also has the structure of a helix, by virtue of the tensor nature of the superfluid density ρ^s. An interesting feature of the current in a helicoidal structure is that the terms with curl **l** contribute to the constant component of the current, i..e., part of the macroscopic superfluid flow is created by the texture.

We again note that the absence of topological restrictions makes the stability of locally stable macroscopic flows extremely tenuous, and experimentally a relaxation of the flow can be observed directly through space-time variations of the vector **l** (Ref. 29). The kinetic energy of the flow is transferred by the normal component due to the rotational friction which arises upon a local precession of the vector **l**. To find quantitative characteristics of the flow relaxation we must analyze the system of hydrodynamic equations.

3. HYDRODYNAMIC EQUATIONS OF THE A PHASE

a) Principle for constructing these equations

In this section we discuss the derivation of a closed system of nonlinear hydrodynamic equations for the A phase and certain consequences of this system. This system must describe the dynamics of both those variables which are characteristic of an ordinary liquid (the mass density ρ, the current density **j**, and the entropy S) and those variables which are associated with the degeneracy parameters $\boldsymbol{\psi}$ (**l** and $\mathbf{v}^s$). The phenomenological hydrodynamic equations are a generalization of the equations of the Landau-Khalatnikov two-velocity hydrodynamics.[8] Some new points which must be taken into account stem from the new variable **l**, not present in the case of He II. In the first place, a corresponding equation must be constructed for this variable. Second, we need to take into account the circumstance that the condition curl $\mathbf{v}^s = 0$ in He II is replaced by the Mermin-Ho condition (10), which expresses curl $\mathbf{v}^s$ in terms of **l**. Third, because of the liquid-crystal anisotropy of the A phase, with an anisotropy axis along **l**, the coefficients in the equation are of a tensor nature. Otherwise, the principles for constructing the equations are the same: 1) All the conservation laws must hold (mass, energy, momentum, and angular momentum). 2) The dissipative function describing the conversion of dynamic energy into heat and the increase in entropy must be positive definite and must vanish at local equilibrium. 3) The symmetry conditions must be satisfied; i.e., the equations must not change upon a Galilean transformation, a rotation, or a displacement of the coordinate system. 4) The equations must have solutions which correspond to an equilibrium state in a vessel in uniform rotation, and this equilibrium state must not depend on the time in the coordinate system of the vessel.[30,31] All these requirements, especially if we are not interested in the terms of high order in the nonlinearities, leave us with little latitude, specifically, some uncertainty regarding the phenomenological coefficients in the equations: the dynamic coefficients (of the type of ρ^s, ρ_0, and C_0) and the kinetic coefficients (the thermal conductivity and the various viscosity coefficients). This last bit of latitude is eliminated by comparing with the linear dynamic equations derived in the microscopic theory.[32]

b) Dissipative function and equilibrium conditions

Before we write equations which meet these requirements, we wish to discuss the dissipative function R, whose form follows from the conditions for a local equilibrium, under which it vanishes. At local equilibrium, the entropy of a liquid in a closed vessel must have a local maximum, provided that the total energy, the mass, the momentum, and the angular momentum remain constant:

$$\begin{gathered}\int dV\,\Big\{\delta S - a\delta\varepsilon - b\delta\rho - \mathbf{c}\delta\mathbf{j} - \mathbf{d}\,[\mathbf{r},\,\delta\mathbf{j}]\\ - \Big(\mathbf{e},\ \mathrm{rot}\,\mathbf{v}^s - \frac{\hbar}{4m_3}e_{imn}l_i\,[\nabla l_m,\ \nabla l_n]\Big)\Big\} = 0.\end{gathered} \tag{47}$$

Here, a, b, **c**, and **d** are constant Lagrange multipliers; **e** is a coordinate-dependent Lagrange multiplier, which ensures that the Mermin-Ho relation (10) is satisfied; and $\delta\varepsilon$ is an energy variation, which is related thermodynamically to the other variables by

$$\begin{gathered}\delta E = \int dV\delta\varepsilon = \int dV\,[T\delta S + \mu\delta\rho + \mathbf{v}^n\delta\mathbf{j}\\ + \mathbf{h}\delta\mathbf{l} + (\mathbf{j} - \rho\mathbf{v}^n)\,\delta\mathbf{v}^s],\end{gathered} \tag{48}$$

where

$$\mu=\frac{\delta E}{\delta\rho}\Big|_{S,\,\mathbf{l},\,\mathbf{v}^s,\,\mathbf{j}}$$

is the chemical potential, and the quantity

$$\mathbf{h}=\frac{\delta E}{\delta\mathbf{l}}\Big|_{\rho,\,S,\,\mathbf{v}^s,\,\mathbf{j}}$$

is analogous to the molecular field in a nematic liquid crystal.[33] Here we have also used the following relations, which follow from (31) and (27):

$$\frac{\delta E}{\delta\mathbf{v}^s}\Big|_{S,\,\rho,\,\mathbf{j},\,\mathbf{l}}=\mathbf{j}-\rho\mathbf{v}^n,\quad \frac{\delta E}{\delta\mathbf{j}}\Big|_{S,\,\rho,\,\mathbf{v}^s,\,\mathbf{l}}=\mathbf{v}^n. \tag{49}$$

Relations (49) are actually quite general, independent of the particular form of the energy, since they are a consequence of Galilean and gauge invariance.

Substituting (48) into (47), and equating the coefficients of $\delta\rho$, δS, $\delta\mathbf{l}$, $\delta\mathbf{v}^s$, and $\delta\mathbf{j}$ to zero, we find the following conditions, which determine the local equilibrium:

$$T=\text{const},\quad \mu=\text{const},\quad \mathbf{v}^n=[\omega,\,\mathbf{r}]+\mathbf{v},$$

$$\omega=\text{const},\quad \mathbf{v}=\text{const},\quad (\nabla,\,\mathbf{j}-\rho\mathbf{v}^n)=0,\quad [\mathbf{l},\,\tilde{\mathbf{h}}]=0,$$

$$\tilde{\mathbf{h}}=\frac{\delta E}{\delta\mathbf{l}}\Big|_{\mathbf{v}^s}+\frac{\delta E}{\delta v_i^s}\Big|_{\mathbf{l}}\frac{\delta v_i^s}{\delta\mathbf{l}}\equiv\mathbf{h}-\frac{\hbar}{2m_3}[\mathbf{l},\,(\mathbf{j}-\rho\mathbf{v}^n,\,\nabla)\,\mathbf{l}]. \tag{50}$$

The dissipation function must vanish at local equilibrium, so that it could depend on the hydrodynamic variables only in certain combinations, which vanish under conditions (50), namely, the following quantities:

$$\nabla T,\ \nabla\mu,\ B_{ik}=\frac{1}{2}(\nabla_k v_i^n+\nabla_i v_k^n),$$

$$\text{curl curl}\,\mathbf{v}^n,\ [\mathbf{l},\,\tilde{\mathbf{h}}],\ (\nabla,\,\mathbf{j}-\rho\mathbf{v}^n). \tag{51}$$

In the quadratic approximation in these expressions, if we ignore the higher gradients (including the term curl curl $\mathbf{v}^n$), the dissipative function is

$$R=\int dV\Big\{\frac{1}{2}\nu_{iklm}B_{ik}B_{lm}+\frac{1}{2\gamma}[\mathbf{l},\,\tilde{\mathbf{h}}]^2+\frac{1}{2}\varkappa_{ij}\nabla_iT\nabla_jT+\frac{1}{2}\zeta(\nabla,\,\mathbf{j}-\rho\mathbf{v}^n)^2+\xi_{ij}B_{ij}(\nabla,\,\mathbf{j}-\rho\mathbf{v}^n)+\xi l_m+l_je_{ikm}B_{ij}\tilde{h}_k\Big\}. \tag{52}$$

In a uniform state with a nonzero difference $\mathbf{v}^s-\mathbf{v}^n$, there is no dissipation, since the dissipation function vanishes. In other words, there is no uniform relaxation of the superfluid velocity, a characteristic of superfluidity (Subsection 2a).

All the kinetic coefficients except ξ appear in the hydrodynamic equations of He II and in the hydrodynamics of a uniaxial nematic liquid crystal. The second viscosity coefficients ζ and ξ_{ij} are found in the case of He II (Ref. 8), but in contrast with the He II, where ξ_{ij} is isotropic and equal to $\zeta_1\delta_{ij}^s$, in ^{3}He-A ξ_{ij} is anisotropic: $\xi_{ij}=\xi_0\delta_{ij}+\xi_1l_il_j$. The thermal conductivity tensor $\varkappa_{ij}=\varkappa_0\delta_{ij}+\varkappa_1l_il_j$ is characteristic of a liquid crystal with a uniaxial anisotropy. The tensor ν_{iklm} contains five viscosity coefficients, also characteristic of a uniaxial anisotropy.[33] In nematic liquid crystals, the coefficient γ describes a rotational friction which arises upon rotation of the anisotropy axis with respect to the medium.[33] In ^{3}He-A this coefficient is called the Cross-Anderson orbital viscosity coefficient, after the investigators who first calculated it.[34] Only the coefficient ξ lacks an analog in liquid crystals. The reason is that the anisotropy vector $\mathbf{l}$ in the A phase and the director $\mathbf{n}$, which specifies the anisotropy axis in a nematic liquid crystal, have different symmetry properties. Time reversal in the A phase reverses the direction of the vector $\mathbf{l}$: $\mathbf{l}(t)=-\mathbf{l}(-t)$, since the wave function (1) [or (2)] is converted into its complex conjugate upon time reversal. The term with ξ in (52) contains an odd power of $\mathbf{l}$ and is linear in $\mathbf{v}^n$, so it does not change upon time reversal. In a nematic liquid crystal, states with $\mathbf{n}$ and $-\mathbf{n}$ are indistinguishable, so that the physical quantities can contain only even powers of $\mathbf{n}$.

c) Hydrodynamic equations

Here are hydrodynamic equations for the A phase which satisfy the requirements listed in Section 1 (these equations are derived in Refs. 35, 36, and 31, among other places):

$$\frac{\partial\rho}{\partial t}+\nabla\mathbf{j}=0, \tag{53}$$

$$\frac{\partial S}{\partial t}+\nabla\mathbf{Q}=\frac{2R}{T}+\nabla\frac{\partial R}{\partial\nabla T}, \tag{54}$$

$$\frac{\partial\mathbf{v}^s}{\partial t}+\nabla\Big(\mu+\mathbf{v}^n\mathbf{v}^s+\frac{\hbar}{4m_3}\mathbf{l}\ \text{curl}\ \mathbf{v}^n\Big)+\frac{\hbar}{2m_3}e_{imn}l_i\nabla l_m\frac{\partial l_n}{\partial t}=\nabla\frac{\partial R}{\partial(\nabla,\,\mathbf{j}-\rho\mathbf{v}^n)}, \tag{55}$$

$$\frac{\partial j_i}{\partial t}+\nabla_k\pi_{ik}=\nabla_k\frac{\partial R}{\partial B_{ik}}, \tag{56}$$

$$\frac{\partial l_i}{\partial t}+(\mathbf{v}^n\nabla)\,l_i-\alpha(\delta_{ij}-l_il_j)\,l_nB_{jn}+\frac{1}{2}[\mathbf{l},\,\text{curl}\ \mathbf{v}^n]+\eta[\mathbf{l},\,\tilde{\mathbf{h}}]_i=-\frac{\partial R}{\partial\tilde{h}_i}. \tag{57}$$

In (54) and (56), $\mathbf{Q}$ and π_{ik} are reversible heat and momentum fluxes:

$$\mathbf{Q}=S\mathbf{v}^n+\frac{1}{T}\beta[\mathbf{l},\,\nabla T], \tag{58}$$

$$\pi_{ij}=P\delta_{ij}+j_iv_j^n+v_i^s(\mathbf{j}-\rho\mathbf{v}^n)_j+\frac{\partial\varepsilon}{\partial\nabla_j\mathbf{l}}\nabla_i\mathbf{l}+\frac{\hbar}{4m_3}e_{ijk}l_k(\nabla,\,\mathbf{j}-\rho\mathbf{v}^n)-\frac{1}{2}\alpha(l_i\tilde{h}_{\perp j}+l_j\tilde{h}_{\perp i})+\frac{1}{2}e_{ijk}[\mathbf{l},\,\tilde{\mathbf{h}}]_k+(\gamma_{il}^{(1)}e_{jpq}+\gamma_{jq}^{(2)}e_{ipl}+\gamma_{jl}^{(3)}e_{ipq}+\gamma_{iq}^{(3)}e_{jpl})\,l_p\nabla_qv_l^n; \tag{59}$$

where P is the pressure

$$P=-\varepsilon+TS+\mu\rho+\mathbf{v}^n\mathbf{j}. \tag{60}$$

The reactive coefficients α, β, and η in Eqs. (53)–(57) are arbitrary; the tensors γ in (59) are symmetric, satisfy the conditions

$$\gamma_{\|}^{(1)} = \gamma_{\|}^{(2)} = \gamma_{\|}^{(3)}, \quad \gamma_{\perp}^{(1)} = \gamma_{\perp}^{(2)}, \tag{61}$$

and are otherwise arbitrary. These coefficients are determined along with the kinetic coefficients from the microscopic theory.[32]

That Eqs. (53)–(57) are consistent with the Mermin-Ho relation (10) can be seen by taking the curl of each side of Eq. (55). We find

$$\frac{\partial}{\partial t}\left(\mathrm{rot}\ \mathbf{v}^s - \frac{\hbar}{4m_3} e_{imn} l_i [\nabla l_m, \nabla l_n]\right) = 0, \tag{62}$$

which means that if (10) holds initially then it will continue to hold at all subsequent times.

Equations (53)–(57) describe all the superfluid effects. At a constant value of **l**, which is the situation near boundaries or in narrow channels, for example, Eqs. (53)–(56) differ from the two-velocity hydrodynamic equations of He II essentially only by the tensor nature of the coefficients, if we ignore several other, relatively small terms (the term with **l**curl $\mathbf{v}^n$ in the equation for $\mathbf{v}^s$ and the terms with β and γ_{ik}). Therefore in the case $\mathbf{l} = \mathrm{const}$, all the superfluid effects characteristic of He II are reproduced, e.g., the mechanocaloric effect and fourth sound (oscillations of $\mathbf{v}^s$ and of the density ρ with the normal component halted in narrow channels). Experimental observation of this effect would be rigorous proof of the superfluidity of new phases of ^{3}He. The other terms in the equations may give rise to new effects. The term **l**curl $\mathbf{v}^n$ in the equation for $\mathbf{v}^s$, for example, gives rise to an interesting "gauge wheel,"[37] which has not yet been observed. This effect may be summarized by saying that the rotation of the normal component with a nonzero gradient $\nabla(\mathbf{l}\,\mathrm{curl}\,\mathbf{v}^n)$ generates a translational motion of the superfluid component.

d) Goldstone modes

In the A phase, in addition to the ordinary Goldstone mode, which is characteristic of a superfluid liquid and which is either second sound or fourth sound, depending on the experimental conditions, there are also some Goldstone modes, associated with four other degrees of freedom in the degeneracy parameters of the A phase. Two of them—the degrees of freedom of the vector **d**—give rise to the spin waves[1] which are observed in NMR experiments. The other two are related to the dynamics of the vector **l**.

1) *Fourth sound*. The propagation velocity of fourth sound depends on the propagation direction.[38] To demonstrate this point, we write the equations of motion in a narrow gap between two plane surfaces with $\mathbf{v}^n = 0$ and $\mathbf{l} = \mathrm{const}$ in the linear approximation, ignoring dissipation and temperature changes:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho^s \mathbf{v}^s - \rho_0 \mathbf{l}(\mathbf{l}\mathbf{v}^s)) = 0,$$
$$\frac{\partial \mathbf{v}^s}{\partial t} + \frac{\partial \mu}{\partial \rho}\nabla\rho = 0. \tag{63}$$

From (63) we have a wave equation for the density,

$$\frac{\partial^2 \rho}{\partial t^2} = \rho^s \frac{\partial \mu}{\partial \rho}\left[\Delta\rho - \frac{\rho_0}{\rho^s}(\mathbf{l}\nabla)^2\rho\right], \tag{64}$$

from which it follows that the propagation velocity of fourth sound, c_4, along the direction of **n** depends in the following way on the orientation of **l**:

$$c_4^2(\mathbf{n}) = c_1^2 \frac{\rho^s}{\rho}\left[1 - \frac{\rho_0}{\rho^s}(\mathbf{nl})^2\right], \tag{65}$$

where c_1 is the propagation velocity of ordinary sound,

$$c_1^2 = \rho \frac{\partial \mu}{\partial \rho}. \tag{66}$$

Equation (65) can be used to determine experimentally the anisotropy of the superfluid density (see Ref. 39, for example).

2) *Second sound*. The hydrodynamic equations have solutions which describe second sound (oscillations of the temperature, $\mathbf{v}^s$ and $\mathbf{v}^n$ at an essentially constant current **j**). The velocity of the second sound,

$$c_2^2(\mathbf{n}) = \frac{S^2}{\rho}\frac{\partial T}{\partial S}\left(\frac{\rho_\perp^s}{\rho_\perp^n}\mathbf{n}_\perp^2 + \frac{\rho_\|^s}{\rho_\|^n}\mathbf{n}_\|^2\right), \tag{67}$$

turns out to be extremely small (of the order of 1 cm/s), however, because the entropy of the liquid is low at such low temperatures. Consequently, the incorporation of a dissipation causes the imaginary part of the frequency to become much larger than the real part at the wavelengths attainable experimentally. As a result, the second sound is unobservable at the existing dimensions of measurement cells, in either the A phase or the B phase. Second sound has been observed experimentally in the A_1 phase, where, because of a specific combined invariance, the velocity of second sound increases sharply due to a coupling with spin waves (Section 6).

3. *Dynamics of the vector* **l**. Equation (57) for **l** is superficially reminiscent of the equation for the director **n** in a nematic liquid crystal.[33] We write this equation in the linear approximation, in the absence of flows, and in broad channels, where **d**‖**l**, assuming that the coefficient η, which stems from the internal orbital angular momentum L_0, is essentially always small in comparison with γ^{-1}:

$$\gamma \frac{\partial \mathbf{l}}{\partial t} = \left(\frac{\hbar}{2m_3}\right)^2 K_3[\Delta\mathbf{l} - \mathbf{l}(\mathbf{l}, \Delta\mathbf{l})]. \tag{68}$$

This is of the nature of a diffusion equation.

The variable **l** is a very slow variable because γ is large:

$$\gamma \sim 6{,}6\cdot 10^{-6}(1 - T/T_c)^{3/2}\ \mathrm{g/(cm^{-1}\cdot s^{-1})}. \tag{69}$$

Consequently, the vector **l** can be assumed to be constant at all times in the sound process (as long as we remain within the range of applicability of the hydrodynamic equations). The dynamics of the vector **l** becomes important only in slow hydrodynamic flows, where it plays a governing role in the decay of the superfluid flows.

4. RELAXATION OF SUPERFLUID FLOWS IN THE A PHASE

a) Effective friction between the superfluid and normal components

A Manchester group[29] carried out the first experiment in which a decay of a superfluid flow due to the dynamics of the vector **l** was observed directly. This experiment involved the vibrations of a rotational pendulum consisting of a stack of 25 toroidal channels $4.9\cdot 10^{-3}$ cm in diameter with a major-circle radius of $7.5\cdot 10^{-2}$ cm and a resonant vibration frequency ~ 60 hz. This experiment revealed that upon the transition to the superfluid state in both the A phase and the B phase there is a shift of the vibration frequency, which occurs because the superfluid component does not participate in the vibrations; the result is a change in the inertial mass of the pendulum. The frequency shift depends in very different ways on the vibration amplitude in the A and B phases. In the A phase, the shift decreases rapidly at large vibration amplitudes, and this behavior is not found in the B phase. Consequently, in the A phase there is a strong effective friction between $\mathbf{v}^s$ and $\mathbf{v}^n$, which forces the superfluid component to follow the normal component, with the result that the inertial mass of the pendulum becomes nearly the same as in a normal liquid, and the shift of the resonant frequency nearly disappears. This mechanism for the relaxation of a superfluid flow, which does not occur in the B phase, is related to the dynamics of the vector $\mathbf{l}$. The relaxation of the flow also affects the damping of the pendulum. This experiment showed that if a transition to the B phase occurs then the damping is essentially unchanged, but it does change considerably in the A phase.

Let us make a quantitative comparison of theory and experiment. For this purpose we consider those equations from the system (53)–(57), and those terms in these equations, which are pertinent to the experiment. The normal component in the narrow channels which were used moves as a whole along with the pendulum, because of the high viscosity of the normal component (the viscous penetration depth $\sqrt{\nu/\omega}$, where ν is the kinematic viscosity, exceeds the diameter of the channel). For slow motions we can ignore the change in the density. Of all the dissipative coefficients the most important is the orbital viscosity coefficient γ, so this is the only coefficient which we will retain. The temperature has only a minor effect on the dynamics of $\mathbf{l}$ when the normal component is retarded. We write the necessary equations in a coordinate system moving with the pendulum [$\mathbf{a}(t)$ is the instantaneous acceleration of the pendulum]:

$$\nabla \mathbf{j} = 0, \quad \mathbf{j} = \overleftrightarrow{\rho}^s \mathbf{v}^s + \frac{\hbar}{2m_3} \overleftrightarrow{C} \operatorname{curl} \mathbf{l}, \tag{70}$$

$$\frac{\partial \mathbf{v}^s}{\partial t} + \nabla \mu + \frac{\hbar}{2m_3} e_{imn} l_i \nabla l_m \frac{\partial l_n}{\partial t} = -\mathbf{a}, \tag{71}$$

$$\gamma \frac{\partial \mathbf{l}}{\partial t} = -\mathbf{h}_\perp + \frac{\hbar}{2m_3} [\mathbf{l}, (\mathbf{j}\nabla) \mathbf{l}]. \tag{72}$$

Let us examine Eq. (71) in more detail. Since the channel is closed, we have $\nabla_\mu = 0$. If $\mathbf{l}$ remains constant over time, $\mathbf{v}^s$ follows the acceleration and remains fixed in the laboratory coordinate system. As we saw in Section 2, however, the topological instability of the flow causes the superfluid velocity to relax to $\mathbf{v}^n$ by virtue of the dynamics of the vector $\mathbf{l}$. This circumstance is seen in Eq. (71) in the appearance of an effective friction force

$$\mathbf{F}_{fr} = \frac{\hbar}{2m_3} e_{imn} l_i \nabla l_m \frac{\partial l_n}{\partial t}, \tag{73}$$

which is exerted on the superfluid component by the normal component. This force can be estimated from dimensional considerations. The scale distance (z_0) and the scale time (t_0) are

$$z_0 \sim \frac{\hbar}{m_3 (v^s - v^n)},$$
$$t_0 \sim \frac{\rho^s}{\gamma} (\mathbf{v}^s - \mathbf{v}^n)^2. \tag{74}$$

The value of t_0 is found by substituting z_0 into Eq. (68) for $\mathbf{l}$. The effective friction force btween the superfluid and normal components is thus of the form[41–43]

$$\mathbf{F}_{fr} = -\frac{1}{\beta} \frac{\rho^s}{\gamma} (\mathbf{v}^s - \mathbf{v}^n)^3, \tag{75}$$

where β is a coefficient of the order of unity. The effective equation for the superfluid velocity is thus

$$\frac{\partial \mathbf{v}^s}{\partial t} + \nabla (\mu + \mathbf{v}^s \mathbf{v}^n) + \frac{1}{\beta} \frac{\rho^s}{\gamma} (\mathbf{v}^s - \mathbf{v}^n)^3 = -\mathbf{a}. \tag{76}$$

Analysis of the experimental data with Eq. (75) yields $\beta \sim 4$.

A more obvious manifestation of the effect of the orbital dynamics on the properties of the superfluid flow was observed in the experiments of Ref. 44, where stable, long-lived oscillations of the vector $\mathbf{l}$ were observed. In this case these oscillations can be interpreted as an analog of the ac Josephson effect for the A phase.

b) The ac Josephson effect in the A phase

The original purpose of the experiments of Ref. 44 was to use ultrasound to measure the orientation of the vector $\mathbf{l}$ in a heat flow in the presence of walls and a magnetic field. [In a heat flow $\mathbf{Q} = S\mathbf{v}^n$, in the absence of a mass flux, $j = 0$, a velocity difference

$$\mathbf{v}^s - \mathbf{v}^n \sim \frac{\rho}{\rho^s} \frac{\mathbf{Q}}{S} \tag{77}$$

arises and orients $\mathbf{l}$ along $\mathbf{Q}$ by virtue of the second term in the energy in (31).] For this purpose, magnetic fields of various strengths and directions were applied to change the orientation of the field $\mathbf{l}$ by virtue of the spin-orbit interaction,[1] and the ultrasonic damping, which is sensitive to the orientation of $\mathbf{l}$, was measured. Unexpectedly, a certain procedure of applying and then completely removing a field gave rise to large, periodic oscillations in the intensity of the sound signal transmitted through the sample. These oscillations, which imply the onset of oscillations in the vector field $\mathbf{l}$, proved very stable, lasting for hours in several cases.

Some of the mystery can be stripped from these oscillations by recalling that an analogous effect is observed in superconductors, where, at a given potential difference, i.e., under steady-state external conditions, an oscillatory regime arises and can be observed, for example, by measuring the electromagnetic radiation which it induces. This regime results from the action of two opposing factors: On the one hand, the current should increase because of the potential difference across the Josephson contact; on the other hand, the current decreases because of some phase-slippage mechanism, e.g., a motion of vortices in the cross section perpendicular to the current direction. The effective friction force which results tends to cancel the effect of the electric field. As a result, an oscillatory equilibrium current with periodic phase slippage is established. If one specified the current across the contact rather than the potential difference, there would be no resistance at a low current, while at a high current an oscillatory regime would arise with a potential difference across the contact.

The oscillations in the A phase evidently arise in a similar manner.[42,45] At a given heat flux density, i.e., at a given $\mathbf{v}^s-\mathbf{v}^n$, the flow can occur in two regimes: either a steady-state regime without friction and with a homogeneous field $\mathbf{l}$ or a dissipative regime, with a difference in chemical potentials arising between the edges of the channel. In this regime there is a relaxation of the superflow because of friction force (73), which cancels the resulting difference in chemical potential, so that on the average we have $\langle \partial \mathbf{v}^s/\partial t \rangle = 0$ [see (76), where $\mathbf{a}$ must be set equal to zero]. In this regime the vector $\mathbf{l}$ evolves continuously, causing a continuous conversion of kinetic energy into heat because of the orbital viscosity. A motion of this type, which frequently occurs in phase-slippage processes, is periodic. Its period is determined from considerations based on the dimensionality of Eq. (74), which gives a good description of the experimental situation.

c) Instanton phase slippage

The field $\mathbf{l}$ can evolve in different ways. Phase slippage can occur, for example, through a motion in the flow cross section of so-called nonsingular vortices, which have a fountain-shaped distribution of the field $\mathbf{l}$ (Ref. 46; we will discuss these vortices in the next section). The experimental data instead imply a vortex-free oscillatory regime. In this regime the distribution of the field $\mathbf{l}$ depends on only a single coordinate, the coordinate z, along the flow.[41,42] On the z, t plane (t is the time) the field $\mathbf{l}$ forms a two-dimensional periodic structure with a cell area

$$\Delta S \sim z_0 t_0 \tag{78}$$

[see (74)].

Two possible versions of the structure are shown in Fig. 8. Each cell of the structure contains a definite integer topological invariant[42] ν, which is equal to the power of the mapping of the two-dimensional torus (a cell with periodic boundary conditions) onto a sphere (the range of the vector $\mathbf{l}$). This invariant is described analytically by

$$\nu = \frac{1}{4\pi} \int\limits_{\Delta S} dt\, dz \left(\mathbf{l}, \left[\frac{\partial \mathbf{l}}{\partial z}, \frac{\partial \mathbf{l}}{\partial t}\right]\right) \tag{79}$$

and tells us the number of times the vector $\mathbf{l}$ sweeps out a unit sphere in the course of its motion, while the coordinates z and t run around the cell ΔS. For the configurations in Figs. 8a and b, we have $\nu = 2$ and $\nu = 1$, respectively.

Taking an average over the spatial and temporal periods of the friction force (73),

$$\langle F^z_{\mathrm{fr}} \rangle = \frac{\hbar}{2m_3} \int \frac{dt\, dz}{\Delta S} \left(\mathbf{l}, \left[\frac{\partial \mathbf{l}}{\partial z}, \frac{\partial \mathbf{l}}{\partial t}\right]\right) = \frac{2\pi\hbar}{m_3} \frac{\nu}{\Delta S}, \tag{80}$$

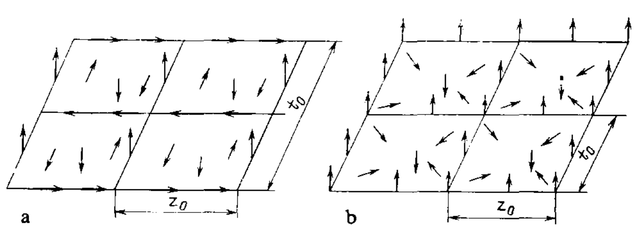

FIG. 8. Space-time periodic structure of the field $\mathbf{l}(z, t)$ in the course of the ac Josephson effect in the A phase. The topological charge (79) of the cell of the structure is 2 in part a) and 1 in part b).

we see that ν being different from zero is a necessary condition for the existence of a dissipative oscillatory regime.

The cell of the periodic structure is directly analogous to instantons in field theory[47]: dynamic structures with a topological charge. In field theory an instanton is a process of a transition between two vacuums with different topological charges. In the A phase an instanton also has a topological charge ν and performs as transition between "vacuum" states of the liquid (states with a uniform distribution of $\mathbf{l}$) which differ by an integral number of quanta of circulation of the superfluid velocity, $2\pi\hbar\nu/m_3$. The ac Josephson effect in the A phase is therefore a periodic array of instantons in a space-time continuum.

In contrast with a superconductor, in which only one of two regimes can prevail (either a dissipative regime or an ac Josephson effect) at a given electric current, in the A phase both regimes can be stable at a given counterflow $\mathbf{v}^s-\mathbf{v}^n$, as can be seen by working from a crude model which can be used to analyze the trajectory of the sytem in phase space.[48] A transition between regimes can be induced by a strong perturbation of the system, as is verified experimentally, where it is necessary to turn a rather strong field on and off in a rather complicated manner.

5. THE ROTATING A PHASE

a) Free energy of a liquid in a rotating vessel

The theoretical conclusions regarding the possibility of a continuous vortex motion of the superfluid component of the A phase have been confirmed in experiments with a rotating A phase,[49–51] in which continuous vortex structures arise.

Let us consider equilibrium dissipationless structures which arise in a rotating vessel. The conditions for the equilibrium of the A phase are given by Eqs. 50. In order to make these equations compatible with the boundary conditions at the surface of the vessel for the normal velocity $\mathbf{v}^n$, we must set $\boldsymbol{\omega}$ in (50) equal to the angular rotation velocity of the vessel. Conditions (50) are obviously satisfied for many different local-equilibrium states. Among these states, a state with a maximum entropy for a given $\mathbf{d}/a = -\boldsymbol{\omega}$ is absolutely stable [see (47)]. This situation corresponds to a state with a minimum value of the functional

$$\tilde{F} = F - \boldsymbol{\omega} \int dV\, [\mathbf{r}, \mathbf{j}], \tag{81}$$

where $F = E - TS$ is the free energy. Since we have $\mathbf{v}^n = [\boldsymbol{\omega}, \mathbf{r}]$ at equilibrium, we find from (31) the following expression for $\tilde{F}$:

$$\begin{aligned}\tilde{F} = \int dV \Big\{ &\frac{1}{2}\rho^s (\mathbf{v}^s - [\boldsymbol{\omega}, \mathbf{r}])^2 - \frac{1}{2}\rho_0 (\mathbf{l}, \mathbf{v}^s - [\boldsymbol{\omega}, \mathbf{r}])^2 \\ &+ \frac{\hbar}{2m_3} C (\mathbf{v}^s - [\boldsymbol{\omega}, \mathbf{r}], \operatorname{curl} \mathbf{l}) \\ &- \frac{\hbar}{2m_3} C_0 (\mathbf{l}, \mathbf{v}^s - [\boldsymbol{\omega}, \mathbf{r}]) (\mathbf{l} \operatorname{curl} \mathbf{l}) \\ &+ \frac{1}{2}\left(\frac{\hbar}{2m_3}\right)^2 K_1 (\nabla \mathbf{l})^2 + \frac{1}{2}\left(\frac{\hbar}{2m_3}\right)^2 K_2 (\mathbf{l} \operatorname{curl} \mathbf{l})^2 \\ &+ \frac{1}{2}\left(\frac{\hbar}{2m_3}\right)^2 K_3 [\mathbf{l}, \operatorname{curl} \mathbf{l}]^2 \Big\} + \frac{1}{2} \int dV \rho\, [\boldsymbol{\omega}, \mathbf{r}]^2. \end{aligned} \tag{82}$$

We consider motions with low values of ω, so that the change in the density ρ can be ignored and the last term in (82) discarded, under the assumption that it is a constant and does not affect the minimization.

The corresponding functional for He II is

$$\tilde{F} = \int dV \frac{1}{2} \rho^s (\mathbf{v}^s - [\boldsymbol{\omega}, \mathbf{r}])^2, \qquad (83)$$

where $\mathbf{v}^s = \hbar \nabla \Phi / m_4$. A straightforward minimization of (83) with respect to $\mathbf{v}^s$ yields $\mathbf{v}^s = [\boldsymbol{\omega}, \mathbf{r}]$ for the superfluid component of the rigid-body rotation; this result of course contradicts the potential nature of the superfluid flow. The condition of rigid-body rotation in He II can be satisfied only on the average, and for this purpose a system of quantized singular vortices must form. Since each vortex has a single quantum of circulation, equal to $2\pi\hbar/m_4$, the density of vortices must be $m_4\omega/\pi\hbar$, so that the average vorticity of the corresponding rigid-body rotation is

$$\langle \operatorname{curl} \mathbf{v}^s \rangle = \hat{\mathbf{z}} \frac{2\pi\hbar}{m_4} \frac{m_4 \omega}{\pi\hbar} = 2\boldsymbol{\omega}. \qquad (84)$$

The energy per unit length of a vortex is

$$\frac{E}{L} = \pi \rho^s \left(\frac{\hbar}{m_4} \right)^2 \ln \frac{r_\omega}{a},$$

where $r_\omega \sim (\pi\hbar/m_4\omega)^{1/2}$ is the distance between vortices, and a is the radius of the vortex core. Hence the energy density $\tilde{F}$ for He II is (per unit volume of the vessel)

$$\frac{\tilde{F}}{V} = \frac{\hbar\omega}{m_4} \rho^s \ln \frac{r_\omega}{a}. \qquad (85)$$

b) Topology of continuous vortex structures in the A phase

In the A phase, the superfluid velocity can in principle be of a rigid-body nature, not on the average but exactly, since the velocity $\mathbf{v}^s$ is not necessarily of a potential nature. There is accordingly no need for the formation of discontinuities. It follows from the Mermin-Ho relation (10) that with $\mathbf{v}^s = [\boldsymbol{\omega}, \mathbf{r}]$ gradients of $\mathbf{l}$ should arise along the coordinates x and y, which are transverse with respect to the rotation axis. These gradients should be, in order of magnitude,

$$\left(\frac{\partial \mathbf{l}}{\partial x} \right)^2 \sim \left(\frac{\partial \mathbf{l}}{\partial y} \right)^2 \sim \frac{m_3 \omega}{\hbar}. \qquad (86)$$

Substituting (86) into $\tilde{F}$ in (82), and using $K_{1,2,3} \sim \rho^s$, we find the following estimate of the energy F for a nonsingular periodic structure:

$$\frac{\tilde{F}}{V} = \lambda \frac{\hbar\omega}{m_3} \rho^s, \qquad (87)$$

where λ is of the order of unity. Expression (87) differs from the corresponding expression for He II, (85), in the absence of the large logarithm. This difference is a consequence of the difference between the continuous distribution of the degeneracy parameter in the A phase and the discontinuous distribution in He II. The coefficient λ in (87) has not yet been calculated exactly, and the exact equilibrium configuration of the fields $\mathbf{l}$ and v^s has not yet been found. It is possible, however, to draw some qualitative conclusions.[52] A configuration of the field $\mathbf{l}$ with gradients along x and y which are constant on the average should naturally be a two-dimensional periodic structure, like a system of singular vortices in He II. The characteristic period of this structure must be of the order of the distance between the vortices in He II, $r_\omega \sim |\nabla_i \mathbf{l}|^{-1} \sim (\hbar/m_3\omega)^{1/2}$. The cell area can be found exactly from topological considerations. The field of the vector $\mathbf{l}(x, y)$ in a cell describes a mapping of the cell (a two-dimensional torus, by virtue of the periodic boundary conditions) onto the sphere over which the vector $\mathbf{l}$ varies. The integral topological invariant—the degree of the mapping—is given by analytic expression (79), in which t and z must be replaced by the two-dimensional coordinates x and y:

$$\nu = \frac{1}{4\pi} \int_{\Delta S} dx\, dy \left(\mathbf{l} \left[\frac{\partial \mathbf{l}}{\partial x}, \frac{\partial \mathbf{l}}{\partial y} \right] \right). \qquad (88)$$

The average vorticity $\langle \operatorname{curl} \mathbf{v}^s \rangle = 2\boldsymbol{\omega}$ can be expressed in terms of ν and the cell area ΔS with the help of the Mermin-Ho relation (10):

$$2\boldsymbol{\omega} = \langle \operatorname{rot} \mathbf{v}^s \rangle = \frac{1}{\Delta S} \int dx\, dy \operatorname{rot} \mathbf{v}^s$$

$$= \hat{\mathbf{z}} \frac{\hbar}{2m_3} \int \frac{dx\, dy}{\Delta S} \left(\mathbf{l}, \left[\frac{\partial \mathbf{l}}{\partial x}, \frac{\partial \mathbf{l}}{\partial y} \right] \right) = \frac{2\pi\hbar}{m_3} \hat{\mathbf{z}} \frac{\nu}{\Delta S}. \qquad (89)$$

The area of the cell of the continuous periodic structure is thus quantized:

$$\Delta S = \frac{\pi\hbar}{m_3 \omega} \nu. \qquad (90)$$

As can be seen from (89), the topological invariant ν must be nonzero. It is easy to see from energy considerations that states with large values of ν in the cell are not favored from the energy standpoint. We would thus expect that ν would take on the value 1 or 2. We have already seen some illustrative field configurations with $\nu = 2$ and $\nu = 1$, in parts a and b, respectively, of Fig. 8. The numerical analysis carried out in Ref. 53 by means of trial functions of a definite type indicates that the structure with $\nu = 2$ is preferred. That analysis, however, did not take into account all possible realizations, so that the structure question remains open.

The distribution of the field $\mathbf{v}^s$ in the cell is interesting. The circulation along the cell boundary C,

$$\oint_C d\mathbf{r}\mathbf{v}^s = \hat{\mathbf{z}} \int_{\Delta S} dx\, dy \operatorname{rot} \mathbf{v}^s = \frac{2\pi\hbar}{m_3} \nu \qquad (91)$$

is also quantized; i.e., the cell is a quantized vortex, which has no singularities anywhere, in contrast with the quantized vortices in He II. A nonsingular quantized vortex with $\nu = 1$ was first analyzed in Refs. 54 and 46. In addition to nonsingular vortices, the A phase may contain singular vortices, an array of which may be preferred from the energy standpoint to a nonsingular structure under certain conditions. We will therefore examine in more detail the properties of vortices in the A phase.

c) Topology of vortices in the A phase

In He II, the vortices are topologically stable singular structural defects in the form of lines. We need to determine which of the various possible, topologically distinguishable configurations of the field of the unit vectors in the A phase have singularities on a line. The sets of linear defects are found in the same way as the sets of flows in an annular channel (Subsection 2f). We have to run a closed contour γ around the line of interest and determine what this contour is mapped into in the space of the order parameter. We already know that only two sets of contours, Γ_0 and Γ_1, are

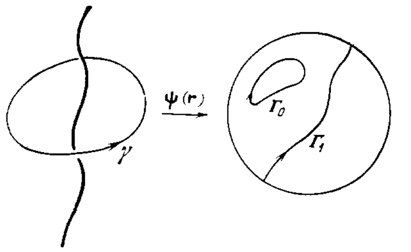

FIG. 9. The contour γ, which circumvents a singular line in the A phase, is mapped by the field of the degeneracy parameter into a closed contour of one of two possible sets, Γ_0 and Γ_1, in the space R.

possible in this space (Fig. 9). Consequently, there are only two possible sets of singular lines corresponding to the contours Γ_0 and Γ_1. A singular line with the index $N = 1$, corresponding to contour Γ_1, is topologically stable, since the contour Γ_1 does not leave its set upon an arbitrary deformation of the field ψ, the contour γ, and the line itself. A singular line with $N = 0$ can continuously disappear. Upon the coalescence of two singular lines with $N = 1$, the lines mutually annihilate by virtue of Eq. (40), which can be written

$$1 + 1 = 0. \qquad (92)$$

Figure 10 shows two examples of linear singularities with $N = 1$. Figure 10a shows a vortex with a single quantum of the circulation of the superfluid velocity, $2\pi\hbar/2m_3$ (in ^{3}He, the quantum of circulation includes the doubled mass of the atom, $2m_3$: the mass of a pair):

$$\mathbf{l} = \hat{\mathbf{z}}, \quad \Delta' + i\Delta'' = e^{i\varphi}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}),$$
$$\mathbf{v}^s = \frac{\hbar}{2m_3}\nabla\varphi = \frac{\hbar}{2m_3\rho}\hat{\varphi}, \quad \oint \mathbf{v}^s\, d\mathbf{r} = \frac{2\pi\hbar}{2m_3}, \qquad (93)$$

where ρ, z, φ are the coordinates of a cylindrical coordinate system. Figure 10b shows an entity from the same set but without any sort of superfluid flow:

$$\mathbf{l} = \hat{\rho}, \ \Delta' + i\Delta'' = \hat{\varphi} + i\hat{\mathbf{z}}, \quad \mathbf{v}^s = 0. \qquad (94)$$

The entity has a singularity in the field $\mathbf{l}$ and is called a "radial disgyration." Since both line defects belong to the same set, they can transform continuously into each other. Upon such a transformation, this circulation of the superfluid velocity around a line is not conserved. Consequently, in contrast with He II, the topological invariant which characterizes line singularities in the A phase is not related to quantization of the circulation. The quantization of circulation which occurs in a cell of a periodic structure of a rotating A phase, (91), is related to a completely different topological invariant, ν, which characterizes continuous periodic distributions of the field $\mathbf{l}$.

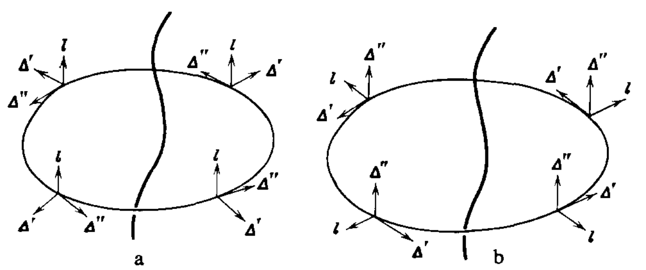

FIG. 10. Line singularities in the A phase corresponding to the set Γ_1 of closed contours. (a)—Singular vortex; (b)—radial disgyration.

d) Singular vortex and structural phase transition

A line defect of the set $N = 1$, which has the lowest energy and thus actually occurs, is—in contrast with other defects of the set, which may continuously relax into it—a vortex with a single quantum of circulation but with a far more complicated structure than the vortex described by Eq. (93) (Refs. 55–57)

We wish to point out that a singular vortex of the field of the superfluid velocity also has no singularity, since only the field $\mathbf{l}$ has a singularity. The vorticity curl $\mathbf{v}^s$ is concentrated within a region $\sim\xi_D$ in size around the vortex axis, in a so-called soft core of the vortex [in contrast with the "hard core" of a vortex, $\sim\xi$, which arises because of a singularity in the field $\mathbf{l}$ (Fig. 11)].

At low angular rotation velocities, at which $\ln(r_\omega/\xi_D)$ is large, a nonsingular periodic structure is preferred from the energy standpoint to an array of singular vortices. In the following section we will see that the situation changes in a magnetic field, and an array of singular vortices becomes preferred. We may thus observe a structural phase transition upon a change in the field. Figure 1 shows how a transition of this type from a singular structure to a nonsingular structure may actually occur. For simplicity we are assuming that the field $\mathbf{l}$ in the singular vortex has the configuration shown in Fig. 10a. In the course of the transition, pairs of vortices with $N = 1$ combine to form vortices with two quanta of circulation (Fig. 12b), which belong to the set $N = 0$ by virtue of (92) and which can therefore be dissipated through a deformation of the field $\mathbf{l}$. Vortices with free ends—monoples—form in the course of the dissipation; these monopoles move upward and downward, consuming the singular vortices (Fig. 12c). As a result, a nonsingular state forms (Fig. 12d). In this state, the superfluid velocity, which is given in the axisymmetric case by (37), changes from zero at the center to $\hbar/m_3\rho$ at the periphery. In other words, we find a nonsingular vortex with two quanta of circulation, which forms a cell of a nonsingular periodic structure.

e) Rotation in a magnetic field

We have seen that the unusual superfluid properties of the A phase result from a certain structure of the order parameter. This structure can be changed by external means, e.g., a magnetic field, and the behavior of the A phase can thus also be changed. The magnetic field $\mathbf{H}$ acting on the

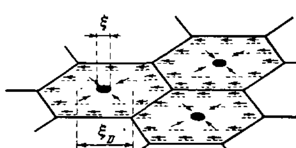

FIG. 11. Array of singular vortices in a rotating vessel (line defects with a minimum energy in the class $N = 1$). The distributions of the fields $\mathbf{l}$ and $\mathbf{d}$ are shown by arrows and dashed lines, respectively.

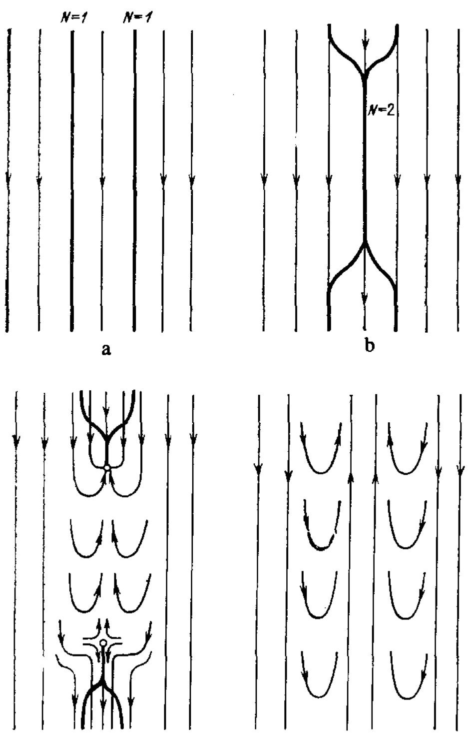

FIG. 12. The coalescence of two singular vortices (a), which gives rise to the formation of a nonsingular vortex (d). In the intermediate state, an unstable singular vortex with two quanta of circulation (b) and then breaks up, forming monopoles—vortices with free ends (c)—which move off to the surface of the vessel.

spin vector **d** also influences **l** through the dipole interaction, orienting **l** perpendicular to **H** (Ref. 1). In this case the degeneracy parameter ψ is

$$\psi = e^{i\Phi}(\mathbf{h} + i[\mathbf{l}, \mathbf{h}]),$$
$$\mathbf{h} = \frac{\mathbf{H}}{|\mathbf{H}|}. \tag{95}$$

The degeneracy parameter has two degrees of freedom: the condensate phase Φ and the angle through which the vector **l** rotates in the plane perpendicular to **H**. These variables are independent; there is no rotation of **l** in the plane which can eliminate the shift of the phase Φ. The superfluid velocity $\mathbf{v}^s = (\hbar/2m_3)\nabla\Phi$ is unrelated to the orbital variable **l**, so that for a planar distribution of **l** the right side of the Mermin-Ho relation (10) vanishes, and $\mathbf{v}^s$ assumes a potential nature. Since the phase Φ is specified on a circle, the basic superfluid properties of the A phase in a magnetic field must be analogous to the properties of He II and ^{3}He-B. In particular, the vortices must have a quantized circulation with a quantum of circulation equal to $2\pi\hbar/2m_3$. Vortices with even and odd values of the circulation have different core structures, however. The size of the core of a vortex with an even number of quanta is the length λ, which is a characteristic of the interaction between **H** and **l**:

$$\lambda = \begin{cases} \xi_D \dfrac{25}{H}, & H < 25\ \text{G}, \\ \xi_D, & H > 25\ \text{G}. \end{cases} \tag{96}$$

Near and inside the core, the magnetic field is no longer capable of constricting **l** to the plane perpendicular to **H**, and the order parameter varies over the entire space of three-dimensional rotations. Inside a region of size λ, an even vortex has no singularities anywhere, since it belongs to the set $N = 0$ in the space of three-dimensional rotations. The number of circulation quanta $2\pi\hbar/2m_3$ of an even vortex is equal to two times the topological invariant ν in (88), where the integral is carried out along a surface intersecting the vortex. An odd vortex, in contrast, belongs to the set $N = 1$, so that within the soft core of dimension λ there is a hard core with a dimension of the order of the coherence length ξ, where the parameter ψ is not defined. As a result, even and odd vortices have different types of energy. For example, the energy of a vortex with $N = 2$ is

$$\frac{E_2}{L} = 1{,}4\,\frac{\pi}{2}\,\frac{\hbar^2}{m_3^2}\,\rho^s \ln\frac{R}{\lambda}, \quad N = 2. \tag{97}$$

The energy of a vortex with $N = 1$, on the other hand, essentially does not change in a magnetic field.

What occurs in a rotating vessel as the magnetic field is increased from zero?[55] In weak fields, with λ larger than the cell size r_ω, a nonsingular periodic structure (Subsection 5b) is not perturbed. As the field is increased, and λ becomes smaller than r_ω, nonsingular quantum vortices in the cell become well localized with a core dimension $\sim\lambda$ [see Fig. 13, which illustrates the case in which the cell of the structure is a nonsingular vortex with two quanta of circulation; this situation corresponds to Eq. (91) to a unit value of the topological invariant ν in (88)].

f) Experiments with vortices in the A and B phases

Vortices in the A phase were recently detected experimentally[49–50] at Helsinki Technological University, where a rotating minilaboratory has been constructed for studying superfluid ^{3}He during rotation. The vortices were detected from the appearance of a new, "vortex" absorption peak in NMR, due to the excitation of spin waves localized at the soft core of vortices. Theoretically, both singular and nonsingular vortices in the magnetic field $H \sim 300$ G used in the NMR experiments should give rise to a vortex peak, since vortices of both types have a soft core. Comparison of the theoretical results of Ref. 56 for the position of the vortex peak with the experimental data implies that what we are actually seeing are nonsingular vortices, despite the fact that they are less favored than singular vortices in such fields. It is simpler for a nonsingular vortex to be produced than it is for a singular vortex, which requires the formation of a hard core. Consequently, as the nonsingular vortices are rapidly produced they occupy all the positions in the periodic struc-

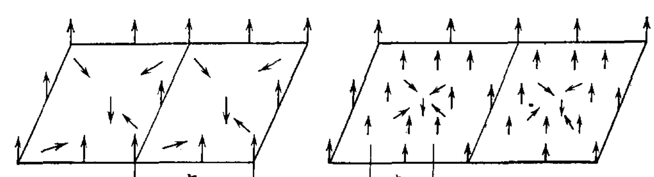

FIG. 13. (a)—Continuous change in a nonsingular periodic structure in rotating ^{3}He-A as the magnetic field is increased; (b)—the nonsingular vortices become well localized.

ture, leaving no room for the singular vortices. Transitions from one structure to another are hindered because (Fig. 12) the splitting of a nonsingular vortex into two singular vortices—the process inverse to that shown in this figure—requires a large expenditure of energy. The reader is referred to Refs. 51, 58, and 59 for more details on the vortices in the A phase and on their effect on NMR.

Vortices have also been discovered in the B phase of ^{3}He (Refs. 60 and 61). Although these vortices were expected to be similar to the vortices in He II, there is an important difference between the two: in the core structure. While superfluidity is disrupted ($\rho^s = 0$) in the core of a vortex in He II, the core of a vortex in the B phase may contain other superfluid phases with a continuous distribution of the vorticity curl $\mathbf{v}^s$, without a disappearance of superfluidity. A core may contain both the A phase and another phase with a nontrivial disruption of gauge invariance: a β phase with ferromagnetically ordered spins.[62] Experiments reveal a first-order phase transition from one core structure to another at $T = 0.6\ T_c$ ($P = 29.4$ atm). In addition, a magnetic moment of a vortex has been detected; it is concentrated in the core.[63] The existence of a spontaneous magnetic moment in the core of a vortex is a result of a specific disruption of the symmetry in the B phase, which preserves invariance under a combined rotation of the spin and orbital spaces. Consequently, the orbital angular momentum of the liquid associated with the motion around the vortex gives rise to a spin angular momentum. The reader is referred to Refs. 61–65 for more details on vortices in the B phase.

g) Vortex formation in a magnetic field

In concluding this section we note that a magnetic field also affects the stability of a flow. A macroscopic flow in a channel becomes stable because of a quantization of the flow along the channel, as in He II. The current can decay at low velocities only as a result of a formation of nonsingular quantized vortices with a core size $\sim\lambda$ [see (96); the formation of such vortices involves an activation energy]. The decay of a current due to instanton dynamics of the vector $\mathbf{l}$ or to the formation of nonsingular vortices (not involving an activation energy) can become important only at velocities exceeding[46] $\hbar/m_3\lambda$. By varying the magnetic field we have a unique opportunity to study the dependence of the vortex formation on the radius of the vortex core; corresponding experiments may cast light on the problem of vortex formation in He II, which remains unresolved. The critical velocity v_c at which the flow of the A phase in a channel becomes dissipative has been observed experimentally.[66] The velocity v_c has turned out to be independent of the temperature and approximately equal to $\hbar/m\xi_D$ ($v_c \sim 0.5$ mm/s) in strong fields ($\mathbf{H} \gtrsim 40$ G). With decreasing field, v_c decreases, reflecting the increase in the core radius λ [see (96)] with decreasing field. The broadening of the NMR absorption line observed when the critical velocity is exceeded is evidence of intense formation of vortices. The same value has been observed for the critical velocity in experiments with rotation,[49–51] with the implication that nonsingular vortices are being produced.

6. SYSTEMS SIMILAR TO THE A PHASE

We have seen that the unique nature of the superfluid properties of the A phase is a consequence of a peculiar symmetry breaking, in which the order parameter is not invariant under gauge transformations but is invariant under a combination of a gauge transformation and a rotation in orbital space. Can other substances with a similar symmetry breaking exist?

a) The A_1 phase of ^{3}He

In strong magnetic fields there is yet another superfluid phase of ^{3}He—the A_1 phase[1]—in a narrow temperature range between normal ^{3}He and ^{3}He-A. In this liquid, the only atoms which become paired are those whose spins are oriented along the field. The spins of the Cooper pairs are thus oriented identically: along the field. The orbital structure of a Cooper pair is the same as in the A phase, i.e., as described in (1). As a result, many of the superfluid properties of the A and A_1 phases are similar. Nonsingular vortices can exist in both phases (the core radius of such vortices is of the order of ξ_D in strong fields in each phase), and surface vortices or boojums can also exist in both phases. The nonsingular vortices and the boojums cause a relaxation of the flow respectively in the interior of the channel and at its surface. There are, on the other hand, some important distinctions, associated with the circumstance that the general structure of the order parameter, including its spin part, is different for these two phases. The total order parameter, with spin taken into account, is specified by a matrix $A_{\mu k}$ which can be written as follows for the A_1 phase:

$$A_{\mu k} = \mathrm{const}\cdot(d'_\mu + i d''_\mu)(\Delta'_k + i\Delta''_k)\, e^{i\Phi}; \tag{98}$$

here $\mathbf{d}'$ and $\mathbf{d}''$ are orthogonal unit vectors whose vector product $\mathbf{s} = [\mathbf{d}', \mathbf{d}'']$ specifies the direction of the spin angular momentum of the pair. The unit vectors $\mathbf{d}'$, $\mathbf{d}''$, and $\mathbf{s}$ transform as vectors under a rotation of the coordinate system in spin space. The symmetry breaking in the A_1 phase is even more interesting than that in the A phase. The order parameter does not change if the gauge transformation is accompanied by a corresponding rotation of either the orbital or spin space (or of both spaces simultaneously). In other words, two combined transformations with the parameters α and α',

$$\left.\begin{aligned} &\Phi \rightarrow \Phi + \alpha, \\ &\mathbf{d}' + i\mathbf{d}'' \rightarrow \overleftrightarrow{\mathbf{R}}(-\alpha'\mathbf{s})(\mathbf{d}' + i\mathbf{d}''), \\ &\Delta' + i\Delta'' \rightarrow \overleftrightarrow{\mathbf{R}}((\alpha'+\alpha)\mathbf{l})(\Delta' + i\Delta''), \end{aligned}\right\} \tag{99}$$

do not change $A_{\mu k}$. The superfluid velocity $\mathbf{v}^s$, defined in a manner invariant under these transformations, is

$$\mathbf{v}^s = \frac{\hbar}{2m_3}(\nabla\Phi + \Delta'_k\nabla\Delta''_k + d'_\mu\nabla d''_\mu) \tag{100}$$

and satisfies a modified Mermin-Ho relation [see (10)]:

$$\mathrm{rot}\ \mathbf{v}^s = \frac{\hbar}{4m_3}(e_{imn} l_i [\nabla l_m,\ \nabla l_n] + e_{\alpha\mu\nu} s_\alpha [\nabla s_\mu,\ \nabla s_\nu]). \tag{101}$$

It follows in particular from the form of the superfluid velocity that a relaxation of the superfluid current could in principle occur through oscillations of the vector $\mathbf{s}$. In reality, however, this does not occur, because $\mathbf{s}$ is tied quite rigidly to the magnetic field.

There are some consequences of this symmetry breaking which can actually be observed. The mass flow with respect to the normal component, $\partial\varepsilon/\partial \mathbf{v}^s = \mathbf{j} - \rho \mathbf{v}^n$ [see (49)], is simultaneously a superfluid flow of spin, since the spins of the pairs are oriented identically. Consequently, second sound in the A_1 phase consists of coupled oscillations of the temperature and the magnetization.[67,68] Because of the pronounced spin rigidity, the propagation velocity of the second sound is not low, as it is in the A phase (Subsection 3d), and is instead approximately equal to the spin-wave velocity $\sim 10^3$ cm/s, so that these waves could be observed experimentally.[69]

b) The A phase in neutron stars

A system similar to an A phase can arise in a neutron star.[14] It is believed that the same type of Cooper pairing occurs in a neutron star as in superfluid ^{3}He, i.e., with an orbital angular momentum, $L = 1$ and a spin $S = 1$. Because of the strong spin-orbit interaction, a state with a total angular momentum $J = 2$ is established. The order parameter, as in ^{3}He, is a complex matrix A_{ik}, which must be symmetric and traceless because of restrictions on the total angular momentum J:

$$A_{ik} = A_{ki}, \quad A_{ii} = 0. \tag{f}$$

The order parameter in a phase with a nontrivial symmetry breaking is

$$A_{ik} = \text{const}\cdot e^{i\Phi}(\Delta_i' + i\Delta_i'')(\Delta_k' + i\Delta_k''); \tag{102}$$

here, as in the A phase, $\boldsymbol{\Delta}'$ and $\boldsymbol{\Delta}''$ are orthogonal unit vectors whose vector product $\mathbf{l} = [\boldsymbol{\Delta}', \boldsymbol{\Delta}'']$ specifies simultaneously the directions of the orbital angular momentum and the spin of the pair. The order parameter does not change upon a simultaneous gauge transformation and rotation of the spin-orbit space around the axis

$$\Phi \to \Phi + \alpha,$$
$$\boldsymbol{\Delta}' + i\boldsymbol{\Delta}'' \to \mathbf{R}\left(\frac{\alpha}{2}\mathbf{l}\right)(\boldsymbol{\Delta}' + i\boldsymbol{\Delta}''). \tag{103}$$

The invariantly defined superfluid velocity

$$\mathbf{v}^s = \frac{\hbar}{2m_n}(\nabla\Phi + 2\Delta_i'\nabla\Delta_i''), \tag{104}$$

where m_n is the mass of the neutron, satisfies the modified Mermin-Ho relation

$$\text{rot}\,\mathbf{v}^s = \frac{\hbar}{2m_n} e_{imn} l_i [\nabla l_m, \nabla l_n]. \tag{105}$$

In a rotating neutron star with the structure in (102) there should be a nonsingular periodic structure in the field of the vector $\mathbf{l}$, in contrast with the singular array of vortices which should exist if another phase exists. This difference may have experimentally observable consequences.

c) Superconducting A phase and the magnetic monopole

Some interesting effects should also be expected in those—as yet undiscovered—superconductors in which Cooper pairing occurs in a p state with the structure of the A phase. Here are just two of the most surprising properties of such superconductors: There will be no Meissner effect, because the superfluid velocity is not a potential velocity, and there can be monopole-like defects, by which we mean formations with a field of the vector potential $\mathbf{A}$ which is the same as that of the $\mathbf{A}$ field near a Dirac monopole. Let us consider the latter effect. For simplicity we write the energy of the superconductor in the London limit in a simplified form, retaining only the two constants ρ^s and K of the seven possible constants in (31):

$$F = \int dV \left\{ \frac{1}{2}\rho^s \left(\mathbf{v}^s - \frac{e}{m_e c}\mathbf{A}\right)^2 + \frac{1}{8\pi}(\text{rot}\,\mathbf{A})^2 + \frac{1}{2}K(\nabla_i \mathbf{l})^2 \right\}. \tag{106}$$

The equations found by minimizing F under the Mermin-Ho condition (10) are

$$\left.\begin{aligned} &\text{rot rot}\,\mathbf{A} = 4\pi\mathbf{j}, \\ &K(\Delta\mathbf{l})_\perp + \frac{\hbar}{2m_e}[\mathbf{l}, (\mathbf{j}\nabla)\mathbf{l}] = 0, \\ &\mathbf{j} = \rho^s\left(\mathbf{v}^s - \frac{e}{m_e c}\mathbf{A}\right). \end{aligned}\right\} \tag{107}$$

We seek a solution of these equations at distances from the defect exceeding the characteristic depth to which the magnetic field penetrates into ordinary superconductors:

$$\lambda = \frac{m_e c}{e}(4\pi\rho^s)^{-1/2}. \tag{g}$$

In this case the vector potential $\mathbf{A}$ completely cancels the superfluid velocity in the flow, $\mathbf{A} = (m_e c/e)\mathbf{v}^s$, and the solution describing the monopole-like defect is ($\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ are the unit vectors of a spherical coordinate system)

$$\mathbf{l} = \hat{\mathbf{r}}, \quad \mathbf{v}^s = \frac{\hbar}{2m_e}\hat{\boldsymbol{\varphi}}\frac{1-\cos\theta}{r\sin\theta},$$
$$\mathbf{A} = \frac{m_e c}{e}\mathbf{v}^s = \frac{\hbar c}{2e}\hat{\boldsymbol{\varphi}}\frac{1-\cos\theta}{r\sin\theta}. \tag{108}$$

This defect is a point singularity in the field of the vector $\mathbf{l}$, from which a line singularity of the fields $\mathbf{v}_s$ and $\mathbf{A}$ emerges: a vortex with two quanta of circulation (Fig. 14). While the magnetic field in an ordinary superconductor falls off exponentially with distance from the vortex by virtue of the Meissner effect, in this case the magnetic field

$$\mathbf{H} = \text{rot}\,\mathbf{A} = \frac{m_e c}{e}\text{rot}\,\mathbf{v}^s = \frac{\hbar c}{2e}\frac{\hat{\mathbf{r}}}{r^2} \tag{109}$$

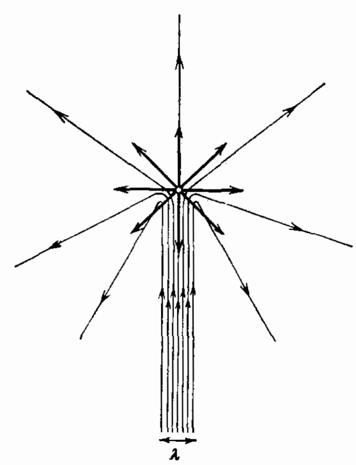

FIG. 14. Monopole-like defect in a hypothetical superconducting A phase. Heavy lines—the field of the vector $\mathbf{l}$; light lines—the magnetic field $\mathbf{H}$.

falls off in a power-law fashion with distance from the center of the point singularity. The magnetic charge of this monopole-lie defect is $\hbar c/2e$. The magnetic flux, $2\pi\hbar c/e$, which is directed outward from the point defect is cancelled by the magnetic flux which flows toward the defect along the vortex. The field **A** has exactly the same form as the vector potential of a Dirac monopole. If a genuine magnetic monopole[70,71] happened to alight on this monopole-like defect, the line singularity in the field $\mathbf{v}^s$ would disappear completely. We would be left with only a point singularity in the field **l**, at whose center there would be a magnetic monopole.

CONCLUSION

In summary, the unique properties of the A phase and of similar systems stem from the peculiar symmetry breaking, which preserves invariance under a composite transformation including a gauge transformation.

The internal symmetry also determines other properties of the superfluid phases of ^{3}He: magnetic, liquid-crystal, electrical, etc. The symmetry approach has proved particularly effective in studying some vortex structures recently detected in superfluid ^{3}He-B. A study of the unusual breaking of the so-called relative spin-orbit symmetry in the B phase led to the experimental discovery of a magnetic moment concentrated in the hard cores of the quantum vortices which arise in a rotating vessel.[63] The symmetry approach has been taken to describe the phase transition observed in the rotating B phase, which occurs in the cores of vortices.[60,61] As a result it has been learned that there are five types of vortices, all having the same number of circulation quanta but differing in the symmetry structure of the core.[62] The superfluidity is disrupted in the cores of some vortices, as it is in the vortices of He II, while the cores of other vortices are superfluid, as they are in the A phase. There are vortices with a spontaneously broken parity, which exhibit a spontaneous electric polarization along the vortex axis. For other vortices with a spontaneously broken composite parity, there is an undamped super-fluid flow in the core along the vortex axis; etc.

In terms of the wealth of broken symmetries and the variety of topologically nontrivial entities, the unique superfluid properties of the ^{3}He phase are unmatched and can compete with vacuums in modern field theories. It is thus not surprising that study of the symmetry in superfluid ^{3}He has proved useful in many other fields of physics.

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