Photometry and coherence: wave aspects of the theory of radiation transport

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The problem is discussed of the statistical and wave content of photometric concepts. The photometric and statistical definitions of the concept of radiance of illumination are treated. We stress that photometry describes the limiting case of a statistically quasihomogeneous and quasistationary wave field. The relationship of the phenomenological and statistical approaches to the theory of radiation transport is traced with the example of the problem of diffraction of the radiation from plane sources. The concept is discussed of the generalized radiance, which enables one partially to take into account within the framework of the radiation transport theory diffraction effects by a transition from the phenomenological radiance to the local spectrum of a quasihomogeneous field. We present also the fundamental results on the statistical and wave substantiation of the theory of transport in scattering media.

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1. INTRODUCTION

Photometry, i.e., the discipline of the measurement of light magnitudes, had been formed as independent field of physics long before Maxwell formulated the fundamental laws of electrodynamics. For a long time it had been used successfully in various applications of optics, while remaining invariably a phenomenological theory operating with simple, and more importantly, pictorial geometric-optical concepts, such as ray tubes, independent beams of radiation, etc.

At the end of the 19th and beginning of the 20th centuries, owing to the studies of Khvol'son and Schuster,[1,2] photometry developed further in the form of the theory of radiation transport, which allowed one to describe radiation in scattering media with the aid of phenomenological photometric concepts (see the reviews[3–7] for the history of the problem and the later advances of the phenomenological theory).

For a long time phenomenological photometry[1)] existed as a closed, self-consistent theory that had been constructed practically without any connection with wave optics and which completely ignored the wave and statistical properties of light (not without reason was the transport equation transferred practically unchanged into neutron physics). All that photometry borrowed from the rigorous wave theory reduced in essence to the statement of the need for going to the geometric-optical limit $\lambda \to 0$, as well as the requirement of noncoherence of natural light sources. The "self-evidence" of the geometrical-optical pictures on which photometry is based led to a clearly insufficient interest in its physical bases and to the tendency to treat photometry in isolation and independently of the electrodynamic theory of light (see, e.g., the textbooks on photometry[8,9]).

The situation has changed substantially in the past 10–20 years. The development of statistical optics has opened up a deeper, statistical-wave content of photometry, which has permitted deriving the photometric relationships from first principles, i.e., from Maxwell's equations for random electromagnetic radiation. The nontriviality of these results becomes clear as soon as one even tries to ascribe a more rigorous meaning to the usual photometric concepts. Within the framework of the traditional phenomenological approach, such attempts ultimately prove fruitless and only destroy the

[1)]We shall understand by this name here photometry proper together with the theory of radiation transport.

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initial illusion of the "self-evidence" of the photometric concepts. This remark holds especially true as applied to the theory of radiation transport, in which one deals with the description of radiation in scattering media. While the use of assumptions of going to the geometric-optical limit and of mutual noncoherence of beams travelling in different directions basically suffices for the applicability of the photometry of free radiation (as will be shown below, these assumptions are not necessary!), in the case of a scattering medium the traditional photometric approach faces almost insurmountable difficulties. Many questions of principle actually lie outside the limits of the phenomenological theory. Thus, for example, within the framework of transport theory, it remains unclear how to define the radiance of radiation near an individual scattering inhomogeneity, which may not at all be smooth on the scale of a wavelength, so that the geometric-optics approximation loses its validity. One can give a satisfying answer to this and analogous questions only with the more rigorous statistical approach, in which the radiation and the medium are treated as two interacting random fields (naturally, in the linear theory, to which we shall restrict the treatment here, all the properties of the medium are considered fixed and independent of the properties of the field).

The principal aim of this review is to describe the status of the problem of the statistical and wave content of photometry and the theory of radiation transport, while directing it primarily toward readers accustomed to the classical phenomenological approach. Some of the problems treated below have been dealt with in the reviews of Refs. 10, 11 and the monographs of Refs. 12, 13 which, however, do not present a clear enough picture as a whole.

The wave foundation of transport theory is constructed in this review on the basis of the concept of statistical quasihomogeneity of the radiation (the need for quasihomogeneity was first stressed in Ref. 21 and later in Refs. 14, and 15, which treated the general case of partially polarized radiation in a scattering multimode medium, although this condition had been used in implicit form even earlier in practically all studies on substantiation of photometry; see below).

This review presents a rather detailed list of the literature on the problem of substantiation of the theory of radiation transport. The number of articles on this topic continues to grow unceasingly, exceeding the number of really new results.

The authors dedicate this study to the memory of its initiator, G. V. Rozenberg, whose unexpected demise prevented him from participating directly in writing this review.

2. PHOTOMETRY AND THE STATISTICAL DEFINITION OF THE CONCEPT OF RADIANCE FOR FREE RADIATION

a) The photometric description of free radiation

First of all, let us recall the fundamental results of photometry and the wave theory for radiation in free space. The photometric description of free radiation is given by the spectral radiance $I_\omega = I_\omega(\mathbf{r}, \hat{\mathbf{n}})$, which is defined at each point $\mathbf{r}$ of space for all directions $\hat{\mathbf{n}}$ and frequencies ω (for simplicity we shall consider the radiation pattern to be stationary, i.e., not varying with the time t). As is known,[8,9,16] the wave field in photometry is treated as a set of noncoherent ray beams (i.e., giving independent energy contributions). Thus, if we place at the point $\mathbf{r}$ an arbitrarily oriented putative area having the normal $\hat{\mathbf{z}}$, then it will be traversed by beams in all possible directions (Fig. 1). In agreement with this picture, the magnitude of the radiance I_ω enables one to express the mean energy flux density $\overline{\mathbf{S}}$ and the mean energy density $\overline{W}$ by the usual relationships

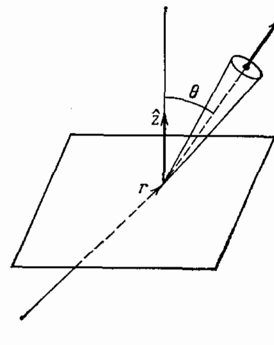

FIG. 1. On the definition of the concept of radiance of radiation.

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}(\mathbf{r}) = \int \hat{\mathbf{n}} I_\omega(\mathbf{r}, \hat{\mathbf{n}})\, d\Omega_n\, d\omega,$$

$$\overline{W} = \overline{W}(\mathbf{r}) = \frac{1}{c}\int I_\omega(\mathbf{r}, \hat{\mathbf{n}})\, d\Omega_n\, d\omega.$$

Here c is the velocity of light. Hence we see that we can define I_ω as

$$I_\omega = \frac{d\overline{S}_z}{d\Omega_n\, d\omega \cos\theta}.$$

Here $\overline{S}_z$, is the component of the flux density $\overline{S}$ in the direction of the unit vector $\hat{\mathbf{z}}$, while $\cos\theta = n_z$.

Further, we shall restrict the treatment for simplicity to the case of monochromatic radiation and examine the corresponding case of the value of the spectral densities

$$\mathbf{S}_\omega = \frac{d\overline{\mathbf{S}}}{d\omega} = \int \hat{\mathbf{n}} I_\omega\, d\Omega_n, \quad W_\omega = \frac{d\overline{W}}{d\omega} = \frac{1}{c}\int I_\omega\, d\Omega_n. \tag{1}$$

For brevity, we omit the arguments or subscripts ω: $\mathbf{S}_\omega = \mathbf{S}, W_\omega = W$.

Elementary considerations involving conservation of energy in the ray tube lead to the fundamental equation for photometry of radiation transport for the radiance $I = I_\omega$ (see, e.g., Ref. 16). In the case of a homogeneous medium as the radiation becomes more remote from the source, the solid angle in which the source is visible declines simultaneously with the broadening of the ray tube. Consequently the radiance proves constant along the ray. We can write this condition for a stationary field in the form of the transport equation

$$\frac{dI}{ds} = 0. \tag{2}$$

Here $d/ds = \hat{\mathbf{n}}\nabla$ is the derivative along the ray.

Thus, in classical photometry we have on the one hand, the relationship (1), which relates the mean flux and energy density with the radiance, and on the other hand, the transport equation (2), which determines the behavior of the radiance of illumination. Here the radiance I is considered to be a nonnegative (energy) quantity, $I \geq 0$, while otherwise it can be arbitrary. The bar over $\bar{\mathbf{S}}$ and $\bar{W}$ in (1) denotes averaging over space on certain scales $L \gg \lambda$ and over a time interval $T \gg \tau$, where λ and τ are the characteristic wavelength and period of the radiation (the latter statement is somewhat vague, as is generally characteristic of the phenomenological theory, although it is "rather self-evident").

Let us compare the photometric relationships (1) and (2) with the results of the rigorous wave theory, where the fundamental quantity is not the radiance, but the random complex amplitude of the wave field. Here we shall not treat quantum effects, restricting the treatment to the framework of the classical theory, which suffices for understanding the essence of the matter.

b) Model of the wave field

We shall use as the initial "wave" model that of a random field φ (scalar for simplicity), since taking into account the polarization of the radiation is not fundamental for what follows, and essentially involves only technical difficulties. We shall assume that the instantaneous flux and energy density of the field are expressed in terms of the real scalar potential as

$$\mathbf{S} = -\dot{\varphi}\nabla\varphi, \quad W = \frac{1}{2}\left[(\nabla\varphi)^2 + \left(\frac{\dot{\varphi}}{c}\right)^2\right]. \tag{3}$$

Here we have $\dot{\varphi} \equiv \partial\varphi/\partial t$. Then the equation of continuity

$$\nabla\mathbf{S} + \dot{W} = \dot{\varphi}\left(-\Delta\varphi + \frac{\ddot{\varphi}}{c^2}\right) = 0,$$

which expresses the law of conservation of energy, reduces for $\dot{\varphi} \neq 0$ to the wave equation

$$\Delta\varphi - \frac{\ddot{\varphi}}{c^2} = 0. \tag{4}$$

For monochromatic radiation we can introduce the complex field u by assuming, as usual, that $\varphi = \mathrm{Re}\, u e^{-i\omega t}$. Then Eq. (4) takes on the form

$$(\Delta + k_0^2)\, u = 0. \tag{5}$$

Here $k_0 = \omega/c$. Also, the average values of $\mathbf{S}$ and W over the period $\tau = 2\pi/\omega$ can be written as

$$\tilde{\mathbf{S}} \equiv \frac{1}{\tau}\int_0^\tau \mathbf{S}\, dt = \frac{i\omega}{4}(u\nabla u^* - u^*\nabla u),$$
$$\tilde{W} = \frac{1}{4}(|\nabla u|^2 + k_0^2 |u|^2). \tag{6}$$

We can naturally treat the field $u = u(\mathbf{r})$ in (5) as random. That is, $u(\mathbf{r})$ is a random function of the space coordinates (for simplicity we shall consider the mean over the statistical ensemble to be zero, $\langle u \rangle = 0$, although this requirement is not necessary; see Ref. 78). Evidently the quantities $\tilde{\mathbf{S}}$ and $\tilde{W}$ also will be random functions. Further, if we adopt the natural assumptions of statistical homogeneity and of spatial ergodicity of u (to fulfill the latter condition in the case of a Gaussian field, it suffices to require a decline in correlation; see, e.g., Refs. 12 and 17), then the means $\langle\tilde{\mathbf{S}}\rangle$ and $\langle\tilde{W}\rangle$ over the statistical ensemble will coincide with the means over space, i.e., with the mean values $\bar{S}$ and $\bar{W}$ that figure in the phenomenological theory:

$$\langle\tilde{\mathbf{S}}\rangle = \lim_{V\to\infty}\frac{1}{V}\int_V \tilde{\mathbf{S}}(\mathbf{r})\, d^3r \equiv \bar{\mathbf{S}}, \quad \langle\tilde{W}\rangle = \bar{W}. \tag{7}$$

For a statistically homogeneous field we can express the means in (7) in terms of the coherence function:

$$\Gamma = \Gamma(\boldsymbol{\rho}) = \langle u(\mathbf{r}_1)\, u^*(\mathbf{r}_2)\rangle.$$

Here we have $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$. Consequently we have

$$\langle\tilde{\mathbf{S}}\rangle = \frac{i\omega}{4}\langle u\nabla u^* - u^*\nabla u\rangle = -\frac{i\omega}{2}\nabla_\rho\Gamma\Big|_{\rho=0},$$
$$\langle\tilde{W}\rangle = \frac{1}{4}(k_0^2 - \Delta_\rho)\,\Gamma\Big|_{\rho=0} = \frac{k_0^2}{2}\Gamma\Big|_{\rho=0}. \tag{8}$$

In the latter relationship we have taken into account the fact that Γ satisfies the wave equation

$$(\Delta_\rho + k_0^2)\,\Gamma = 0, \tag{9}$$

which stems directly from the wave equation (5) for u.

c) Interrelation of photometry and wave theory for free radiation

Let us elucidate the interrelation between the phenomenological expressions (1) and the results of the statistical theory from Sec. 2b. We see from (1) that the radiance $I = I(\mathbf{r}, \hat{\mathbf{n}})$ has the meaning of the angular spectrum of the radiation, and hence it must be somehow related to the spatial spectrum of the random field, i.e., to the Fourier transform of the coherence function

$$\mathscr{Y}_\mathbf{k} \equiv \int \Gamma(\boldsymbol{\rho})\, e^{-i\mathbf{k}\boldsymbol{\rho}}\, d^3\rho\cdot(2\pi)^{-3}. \tag{10}$$

Let us substitute into (9) the inverse expression to (10) that gives the coherence function in terms of the spectrum

$$\Gamma(\boldsymbol{\rho}) = \int \mathscr{Y}_\mathbf{k} e^{i\mathbf{k}\boldsymbol{\rho}}\, d^3\mathbf{k}. \tag{11}$$

Thus we arrive at the equation

$$(-\mathbf{k}^2 + \mathbf{k}_0^2)\,\mathscr{Y}_\mathbf{k} = 0. \tag{12}$$

Hence we see that the spectral density $\mathscr{Y}_\mathbf{k}$ can differ from zero only when $k = k_0$. The physical meaning of this condition amounts to taking into account the wave nature of the radiation, and is associated with the idea that the free field u is formed of running plane waves whose wave vectors $\mathbf{k}$ satisfy the dispersion equation $k = k_0$. Upon taking this into account, we can write $\mathscr{Y}_\mathbf{k}$ in the form of the product of a delta function $\delta(k - k_0)$ with some function $I'_{\hat{n}}$:

$$\mathscr{Y}_\mathbf{k} = I'_{\hat{n}}\,\delta(k - k_0). \tag{13}$$

Here we have $\hat{\mathbf{n}} = \mathbf{k}/k$. Upon substituting (13) and (11) into (8) and comparing the obtained expressions with (1), we find

$$I = \frac{k_0^3\omega}{2} I'_{\hat{n}}. \tag{14}$$

Thus the spectrum of the coherence function of free, statistically homogeneous radiation is expressed in terms of the phenomenological radiance I as

$$\mathscr{W}_{\mathbf{k}} = aI \frac{\delta(k-k_0)}{k_0^2}. \tag{15}$$

Here the proportionality coefficient a in the scalar-field case being treated is equal to $2/k_0\omega$, while in the general case it depends on the nature of the field (for an electromagnetic field $a = c/8\pi$; see Ref. 12, p. 116). The existence of the delta function in (15) corresponds to localization of the spectrum at the dispersion surface $k = k_0$.

Equations (15) and (11) imply that one can express in terms of the radiance I not only the mean flux and energy density of the field in (2), but also the coherence function

$$\Gamma(\boldsymbol{\rho}) = a \int I e^{ik_0 \hat{\mathbf{n}} \boldsymbol{\rho}} \, d\Omega_{\mathbf{n}}. \tag{16}$$

In classical photometry this relationship has not been employed in any way, since the correlation characteristics of radiation are not included in the traditional photometric quantities.

It is interesting to note that, although the relation of the radiance to the correlation functions (16) of the field had already been known for a long time for equilibrium thermal radiation (see, e.g., Refs. 18–20, where relationships of the form of (16) have been used to calculate the correlation functions of thermal radiation), this relation became clear in photometry only after the studies of Dolin,[21,22] who treated the special case of radiation in the small-angle approximation. This is explained by the fact that thermal radiation in free space is an example of a statistically homogeneous field, whereas in transport theory the fundamental interest is directed to describing changes in the radiance of illumination, i.e., taking statistical inhomogeneity into account. Therefore, in order to use (16) in transport theory, one must first generalize it to the case of statistically inhomogeneous fields, which in essence has been done in Refs. 21 and 22.

d) Local spectrum of a quasihomogeneous field

According to (15), the spectrum $\mathscr{W}_{\mathbf{k}}$ of the wave field is proportional to the radiance of the radiation. At the same time, just as the concept of the spectrum $\mathscr{W}_{\mathbf{k}}$ itself, the relationship (15) is applicable only to a statistically homogeneous field u, for which the radiance I does not depend on the space coordinates. Real physical fields are never strictly statistically homogeneous, if only because of their spatial boundedness. Here the need of taking into account the dependence of I on $\mathbf{r}$ for finite physical systems is evident. Fortunately we can considerably weaken the condition of statistical homogeneity: it suffices for a "wave" substantiation of photometry to require that the field be statistically quasihomogeneous. The latter means that the field differs not too strongly from a statistically homogeneous field—in the sense that its statistical characteristics can vary smoothly (on the scale of the range of coherence) from point to point.

In order to describe a quasihomogeneous field, one must generalize the usual concept of the spectrum in (10) so as to take into account the possibility of weak inhomogeneity. We shall call this generalization the *local spectrum* of the fluctuations with the wave vector $\mathbf{k}$ near the point $\mathbf{r}$, and denote it by $\mathscr{W}_{\mathbf{k}}(\mathbf{r})$. Evidently, the local spectrum must be expressed by a linear transformation of the coherence function: $\mathscr{W}_{\mathbf{k}}(\mathbf{r}) = \hat{Q}\Gamma$, where $\hat{Q}$ is a certain linear operator. It proves impossible to introduce the spectrum $\mathscr{W}_{\mathbf{k}}(\mathbf{r})$ in such a way that it strictly satisfies all the intuitive properties of a spectrum, if only because of the approximateness of the very concept of the "wave vector $\mathbf{k}$ near the point $\mathbf{r}$." One can treat various definitions of local spectra (in this regard see Refs. 23–28, where analogous concepts of spectra have been discussed mainly for nonstationary processes). The most widespread definition takes the local spectrum to be the so-called *Wigner function*

$$\mathscr{W}_{\mathbf{k}}(\mathbf{R}) = \int e^{-i\mathbf{k}\boldsymbol{\rho}} \left\langle u\left(\mathbf{R}+\frac{\boldsymbol{\rho}}{2}\right) u^*\left(\mathbf{R}-\frac{\boldsymbol{\rho}}{2}\right)\right\rangle d^3\rho \cdot (2\pi)^{-3}. \tag{17}$$

This was introduced by Wigner into quantum mechanics for completely different purposes (as the first example of quasiprobability).[29,95] We can easily see that the function in (17) is real, and coincides with the ordinary spectrum $\mathscr{W}_{\mathbf{k}}$ of (10) for a statistically homogeneous field u. However, in the presence of statistical inhomogeneity it can take on negative values. Therefore, in the general case one must not attribute an energy interpretation to it. Only in the case of a quasihomogeneous field differing not too strongly from statistical homogeneity does it become admissible to treat the Wigner function as a measure of the intensity fluctuations having the wave vector $\mathbf{k}$ near the point $\mathbf{r}$. The need for this requirement is not a defect of the Wigner function, and is inherent in any other reasonable definition of the local spectrum.

Qualitatively one can write the condition of quasihomogeneity in the form of the inequality

$$\left|\frac{\partial\Gamma}{\partial\mathbf{R}}\right| \ll \left|\frac{\partial\Gamma}{\partial\rho}\right|, \qquad \Gamma = \Gamma(\mathbf{R}, \boldsymbol{\rho}) = \left\langle u\left(\mathbf{R}+\frac{\boldsymbol{\rho}}{2}\right) u^*\left(\mathbf{R}-\frac{\boldsymbol{\rho}}{2}\right)\right\rangle. \tag{18}$$

This implies the smallness of the variations of the coherence function with respect to the argument of the "center of gravity" $\mathbf{R}$ as compared with its variations with respect to the difference variable $\boldsymbol{\rho}$ (for a statistically homogeneous field we have $\partial\Gamma/\partial\mathbf{R} = 0$, and (18) is satisfied to any degree of accuracy). When the inequality (18) is satisfied, we can show by using the wave equations for the coherence function Γ arising from (5) that Eq. (15) will be satisfied approximately for the local spectrum. That is, we have

$$\mathscr{W}_{\mathbf{k}}(\mathbf{R}) = aI(\mathbf{R}, \hat{\mathbf{n}}) \frac{\delta(k-k_0)}{k_0^2}. \tag{19}$$

Here the radiance I can now depend smoothly on $\mathbf{R}$. Upon taking into account the definition of $\mathscr{W}_{\mathbf{k}}(\mathbf{R})$ in (17), this implies an expression analogous to (16):

$$\Gamma(\mathbf{R}, \boldsymbol{\rho}) = a \int I(\mathbf{R}, \hat{\mathbf{n}}) e^{ik_0 \hat{\mathbf{n}} \boldsymbol{\rho}} \, d\Omega_{\mathbf{n}}. \tag{20}$$

Upon substituting this relationship into (8) and thereby neglecting terms of the order of $\nabla_R \Gamma$ in the expression for $\langle \tilde{W} \rangle$, we again obtain photometric relationships of the form of (1), where the mean flux and energy density can now depend smoothly on the spatial coordinates $\mathbf{R}$. Further, the transport equation (2) arises from (20) and from the wave equation for Γ with the condition (18) of quasihomogeneity for $I(\mathbf{R}, \hat{\mathbf{n}})$ taken into account. Thus, we have obtained in the "quasihomogeneous limit" (18) a wave substantiation of the fundamental photometric relationships (1) and (2).

We note that, although the radiance I enables one according to (20) to obtain an expression for the coherence function of the wave field, the inverse expression of the radiance in terms of the coherence function is not unique. This involves the fact that the relationship (20) holds only for a quasihomogeneous field obeying the inequality (18). In other words, not every coherence function can be expanded in an angular spectrum of the type (20). Thus, for free radiation the expansion of the random amplitude u into a spatial spectrum generally contains a contribution from inhomogeneous, damped waves. This contribution corresponds to a statistically inhomogeneous field, and is not taken into account in the expression (19), in which the field is considered statistically homogeneous and formed only of running waves. One can easily derive the simplest of the possible inversions of (20) from (19) by simple integration with respect to k, which gives

$$I = I(\mathbf{R}, \hat{\mathbf{n}}) = \frac{1}{a} \int_0^\infty k^2 \, dk \int \Gamma(\mathbf{R}, \boldsymbol{\rho}) \, e^{-i\mathbf{k}\boldsymbol{\rho}} \, d^3\rho \, (2\pi)^{-3}. \qquad (21)$$

This relationship was first discussed[30] for the problem of scalar radiation from plane sources (see also Ref. 31, where this problem has been treated for vector electromagnetic radiation). One can derive other forms of the expression of I in terms of Γ by first multiplying both sides of (19) by an arbitrary function $\varphi(\mathbf{k})$.

We stress that, when the quasihomogeneity condition is satisfied, one could employ for substantiation of Eq. (1) other definitions of the local spectra differing from (17) instead of the Wigner function, and ultimately arrive at equivalent results. Actually, the essential point for substantiating photometry is not so much the choice of the definition of the local spectrum as the necessary requirement of quasihomogeneity (18), which expresses the restriction on the statistics of the wave field that allows a photometric description.

Up to now we have been speaking of monochromatic radiation, for which the time-dependence of the field is described by the factor $e^{-i\omega t}$. Fully analogous arguments continue to hold as applied to a nonmonochromatic field. Here, naturally, the requirement of quasihomogeneity must be supplemented with the condition of quasistationarity of the field (one can combine these two conditions together under the name of space-time quasihomogeneity).

3. RADIANCE OF PLANE SOURCES AND GENERALIZED PHOTOMETRY

a) The photometric description of the radiation from plane sources

Next in complexity after the description of free radiation is the problem of the radiation from plane sources in free space. A rigorous solution is known for this problem that enables one to compare directly the photometric and wave approaches. First let us examine the photometric description of this problem.

In photometry one can assign to plane radiation sources a radiance distribution $I^0(\mathbf{r}_\perp, \hat{\mathbf{n}})$ in the plane $z = 0$ (hereinafter $\mathbf{a}_\perp$ denotes the component of the vector $\mathbf{a}$ transverse to z). Then the radiance I of radiation for $z > 0$ is expressed in terms of the radiance of the sources I^0 as the solution of the transport equation (2):

$$I = I(\mathbf{r}_\perp, z; \hat{\mathbf{n}}) = I^0\left(\mathbf{r}_\perp - \frac{z\hat{\mathbf{n}}_\perp}{n_z}, \hat{\mathbf{n}}\right). \qquad (22)$$

This corresponds to conservation of radiance along the rays.

By using the radiance of (22), one can express the coherence function of the field in line with (20). This gives

$$\Gamma(\mathbf{R}, \boldsymbol{\rho}) = a \int I^0\left(\mathbf{R}_\perp - R_z \frac{\hat{\mathbf{n}}_\perp}{n_z}, \hat{\mathbf{n}}\right) e^{ik_0\hat{\mathbf{n}}\boldsymbol{\rho}} \, d\Omega_{\mathbf{n}}. \qquad (23)$$

Let us compare this relationship with the results of the wave theory.

b) The wave description and the generalized radiance of plane sources

In the wave theory, instead of the radiance I^0 of the sources in the plane $z = 0$, one fixes the boundary values of the complex amplitude of the field $u^0(\mathbf{r}_\perp)$. This allows one to express the amplitude of the diffracted field u for $z > 0$ by using the Rayleigh method (expansion in plane waves) or by using the known Green's function for this problem (see, e.g., Ref. 12). The result has the form

$$u(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}'_\perp) \, u^0(\mathbf{r}'_\perp) \, d^2r'_\perp. \qquad (24)$$

Here

$$G(\mathbf{r}) = -ik_0 \frac{z}{r} \frac{e^{ik_0 r}}{2\pi r} \qquad (25)$$

is the Green's function, and we have $\mathbf{r} = (\mathbf{r}_\perp, z)$.

Upon termwise multiplication of the expression (24) taken at two points $\mathbf{r}_1$ and $\mathbf{r}_2$ and averaging of the results, we find that the coherence function of the field

$$\Gamma = \langle u(\mathbf{r}_1) \, u^*(\mathbf{r}_2) \rangle = \left\langle u\left(\mathbf{R} + \frac{\boldsymbol{\rho}}{2}\right) u^*\left(\mathbf{R} - \frac{\boldsymbol{\rho}}{2}\right) \right\rangle,$$

$$\mathbf{r}_{1,2} = \mathbf{R} \pm \frac{\boldsymbol{\rho}}{2},$$

for $z_{1,2} > 0$ is expressed in terms of the coherence function of the field at the screen

$$\Gamma^0 = \Gamma^0(\mathbf{R}_\perp, \boldsymbol{\rho}_\perp) = \left\langle u^0\left(\mathbf{R}_\perp + \frac{\boldsymbol{\rho}_\perp}{2}\right) u^{0*}\left(\mathbf{R}_\perp - \frac{\boldsymbol{\rho}_\perp}{2}\right) \right\rangle \qquad (26)$$

as

$$\Gamma(\mathbf{R}, \boldsymbol{\rho}) = \int G\left(\mathbf{R} - \mathbf{R}'_\perp + \frac{\boldsymbol{\rho} - \boldsymbol{\rho}'}{2}\right) G^*\left(\mathbf{R} - \mathbf{R}'_\perp - \frac{\boldsymbol{\rho} - \boldsymbol{\rho}'_\perp}{2}\right) \times \Gamma^0(\mathbf{R}'_\perp, \boldsymbol{\rho}'_\perp) \, d^2\mathbf{R}'_\perp \, d^2\rho'. \qquad (27)$$

Now let us employ the following approximation in (27):

$$G\left(\mathbf{R} - \mathbf{R}'_\perp + \frac{\boldsymbol{\rho} - \boldsymbol{\rho}'_\perp}{2}\right) G^*\left(\mathbf{R} - \mathbf{R}'_\perp - \frac{\boldsymbol{\rho} - \boldsymbol{\rho}'_\perp}{2}\right) \approx |G(\mathbf{R} - \mathbf{R}'_\perp)|^2 e^{ik_0\hat{\mathbf{n}}(\boldsymbol{\rho} - \boldsymbol{\rho}'_\perp)}. \qquad (28)$$

Here the dependences on the difference variables ρ, and $\rho_{\perp'}$ and on the coordinates $\mathbf{R}$ and $\mathbf{R}_{\perp'}$ of the center of gravity are separated (here $\hat{\mathbf{n}} = (\mathbf{R} - \mathbf{R}_{\perp'})/|\mathbf{R} - \mathbf{R}_{\perp'}|$). Then Eq. (27) turns out to acquire the photometric structure of (23). The quantity that plays the role of the radiance I^0 here is

$$I^0(\mathbf{R}_\perp, \hat{\mathbf{n}}) = \frac{1}{a}\left(\frac{k_0}{2\pi}\right)^2 \cos\theta \int \Gamma^0(\mathbf{R}_\perp, \boldsymbol{\rho}_\perp)\, e^{ik_0\hat{\mathbf{n}}\boldsymbol{\rho}_\perp}\, d^2\boldsymbol{\rho}_\perp. \quad (29)$$

In order to derive Eq. (29), it suffices to substitute (28) into (27) and go over from integrating over $\mathbf{R}_{\perp'}$ to integrating over a solid angle near the direction $\hat{\mathbf{n}}$.

In order that the approximation (28) be applicable, it suffices that the following inequalities be satisfied in the region essential for integration:

$$|\mathbf{R}-\mathbf{R}'_\perp| \equiv \frac{R_z}{n_z} \gg \frac{1}{2}|\boldsymbol{\rho}-\boldsymbol{\rho}'_\perp|, \quad \mathbf{R}=(\mathbf{R}_\perp, R_z), \quad (30)$$

$$\frac{R_z}{n_z} \gg \sqrt{k_0\hat{\mathbf{n}}\left(\frac{\boldsymbol{\rho}-\boldsymbol{\rho}'}{2}\right)}\left|\frac{\boldsymbol{\rho}-\boldsymbol{\rho}'_\perp}{2}\right|. \quad (31)$$

The first of these allows one to assume that $(\rho - \rho_{\perp'})/2 \approx 0$ in the amplitude coefficients of the Green's functions of (25), while the second one allows us to keep only the terms linear in $\rho - \rho_{\perp'}$ in the phase coefficients (the quadratic terms in the phases drop out, so that (31) expresses the condition of smallness of the third-order terms).

The importance of the relationship (29) consists of the fact that it relates the characteristics of plane sources in the phenomenological and wave theories. A formula of this type was first derived by Walther.[32] Subsequently it has been treated from various standpoints in many studies (see, e.g., Refs. 33–52, and also the reviews of Refs. 53,54). The very simple problem of the radiation from plane sources possesses an exact wave solution, which has been studied for a long time in detail in the radiophysical literature (see, e.g., Refs. 12, 55, 56). The existence of such great interest in it can be explained by the relative novelty of the "correlation" approach to photometry, the essence of which is traced most distinctly in this problem.

It is important to note that in the general case one can treat (29) only as a certain *generalized radiance* of sources, since it does not possess all the properties of phenomenological radiance. The point is that the conditions of quasihomogeneity of the field generally do not require quasihomogeneity of the sources themselves. In the case of substantially inhomogeneous sources, the field near them will also be substantially inhomogeneous. Hence, in this region it does not allow a photometric description. At the same time the conditions of quasihomogeneity can be fulfilled far from the sources, where the photometric approach becomes valid. It is precisely in this case that we must understand (29) as a generalized radiance not possessing all the properties of the photometric radiance, but nevertheless giving the correct description of the radiation in a region of quasihomogeneity by allowing one to take diffraction effects into account by transport theory. The latter cannot be done if one stays fully within the framework of classical photometry.

We note that, if one employs some other definition of the local spectrum instead of the Wigner function to substantiate photometry, then one can derive other expressions different from (29) for the radiance I^0 of sources in terms of the coherence function (in this regard, see Refs. 32 and 33). These differences involve the above-noted ambiguity of the expression of the radiance in terms of the coherence function of the field. However, all these differences drop out for a quasihomogeneous field, and all the more so for a statistically homogeneous field.

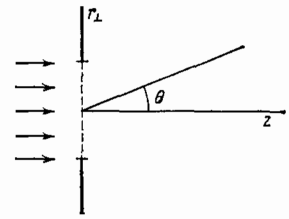

FIG. 2. Diffraction by an aperture.

Let us illustrate what we have said with some examples.

c) Examples

1) Coherent source. Let us study the diffraction of a coherent (i.e., nonfluctuating) plane wave $u^0 = e^{i\mathbf{k}\cdot\mathbf{r}}$ normally incident on an aperture in a plane screen having a characteristic dimension $L \gg \lambda$ (Fig. 2). Although fluctuations are absent in this problem, under certain conditions the coherence function of the field $\Gamma = \langle u(\mathbf{r}_1)u^*(\mathbf{r}_2)\rangle = u(\mathbf{r}_1)u^*(\mathbf{r}_2)$ proves to be quasihomogeneous and hence allows a photometric description.

Actually, upon assuming in the physical-optics approximation that $u^0(\mathbf{r}_\perp) = \theta_\Sigma(\mathbf{r}_\perp)$, where $\theta_\Sigma(\mathbf{r}_\perp)$ is the transmission function of the aperture, which equals unity directly at the aperture and zero outside it, we obtain from (29) the following expression for the generalized radiance of the source:

$$I^0(\mathbf{R}_\perp, \hat{\mathbf{n}}) = \frac{1}{a}\left(\frac{k_0}{2\pi}\right)^2 n_z \int \theta_\Sigma\left(\mathbf{R}_\perp + \frac{\boldsymbol{\rho}'_\perp}{2}\right)\theta_\Sigma\left(\mathbf{R}_\perp - \frac{\boldsymbol{\rho}'_\perp}{2}\right) e^{-ik_0\hat{\mathbf{n}}\boldsymbol{\rho}'_\perp}\, d^2\boldsymbol{\rho}'_\perp. \quad (32)$$

Substituting this expression into (23), we shall estimate the value of the coherence function Γ for observation points lying in the Fraunhofer zone with respect to the aperture where $|R_z \pm (\rho_z/2)| \gg k_0 L^2$. For simplicity we restrict the treatment to the case of small-angle diffraction, so that $n_z \sim 1$. Consequently we obtain

$$\Gamma(\mathbf{R}, \boldsymbol{\rho}) = a\int I(\mathbf{R}, \hat{\mathbf{n}})\, e^{ik_0\hat{\mathbf{n}}\boldsymbol{\rho}}\, d\Omega_n$$
$$\approx \left(\frac{k_0}{2\pi R_z}\right)^2 \int \theta_\Sigma\left(\mathbf{R}'_\perp + \frac{\boldsymbol{\rho}'_\perp}{2}\right)\theta_\Sigma\left(\mathbf{R}'_\perp - \frac{\boldsymbol{\rho}'_\perp}{2}\right)$$
$$\times \exp\left[i\frac{k_0}{R_z}(\mathbf{R}_\perp - \mathbf{R}'_\perp)(\boldsymbol{\rho}_\perp - \boldsymbol{\rho}'_\perp)\right] d^2\boldsymbol{\rho}'_\perp\, d^2\mathbf{R}'_\perp. \quad (33)$$

Here we have $\mathbf{R} = (\mathbf{R}_\perp, R_z)$. One can find exactly the same expression from wave theory by using the small-angle Green's function to solve the diffraction problem.[12]

We note that, although the distribution of the field over the aperture is homogeneous in the given case, the generalized radiance I^0 is no longer homogeneous, but varies from point to point. Thus, for example, if we treat as Σ a slit of width $2L$ unbounded along the y axis, then the calculation of the integral on the right-hand side of (32) yields

$$I^0(\mathbf{R}_\perp, \hat{\mathbf{n}}) = \frac{n_z}{\pi a n_x}\,\delta(n_y)\sin[2k_0 n_x(L - |R_x|)]\,\theta(L - |R_x|).$$

Here $\theta(x)$ is the Heaviside step function. This quantity varies over the surface of the slit, and can acquire negative values.

However, if we substitute this expression into (23), then with the conditions (30) and (31) taken into account for the intensity of radiation $\Gamma|_{\rho=0} = \langle |u|^2\rangle$ the integration over $d\Omega_{\mathbf{n}}$ now yields a nonnegative quantity.

Let us turn to the conditions (30) and (31), which allow one to employ a photometric description for the given problem in terms of the generalized radiance I^0. The vector $\boldsymbol{\rho}$ that enters into these inequalities is given by the spacing of the observation points, and the vector $\boldsymbol{\rho}_{\perp'}$ by the spacing of the points on the aperture ($|\rho_{\perp'}| \lesssim L$), while the vector $\hat{\mathbf{n}}$ estimates the characteristic direction of diffraction ($n_z \sim \cos\vartheta$, where ϑ is the characteristic angle of diffraction). In order to satisfy the condition (30), it suffices to require simultaneous satisfaction of the inequalities $R_z \gg \rho\cos\vartheta \sim \rho$ (i.e., the spacing of the observation points must be small in comparison with their distance from the source) and $R_z \gg L\cos\vartheta$ (the dimension of the source $L\cos\vartheta$ visible in the direction of ϑ is small in comparison with R_z; see Fig. 2). As regards the condition (31), for the case in which the points of observation coincide ($\rho = 0$), upon omitting the numerical coefficients, we have $R_z \gg L\cos\vartheta\sqrt{k_0 L\sin\vartheta}$. This inequality is far weaker than the condition for the Fraunhofer zone $R_z \gg k_0 L^2$, so that the applicability of the generalized radiance of (32) does not require going into the Fraunhofer zone of the aperture.

Thus in this case the generalized radiance of (32) allows one to describe diffraction of radiation by an aperture. The expression (32) remains applicable also in the case of two apertures if we assume that the transmission function θ_Σ is equal to unity at each of them (i.e., $\theta_\Sigma = \theta_\Sigma^{(1)} + \theta_\Sigma^{(2)}$, where $\theta_\Sigma^{(1,2)}$ is the transmission function of each of the apertures). This implies that (32) also describes the interference of apertures, in particular, the two-slit Young interferometer. We note that in this case the generalized radiance of (32) will contain interference terms proportional to the product $\theta_\Sigma^{(1)}\theta_\Sigma^{(2)}$, which prove to be different from zero for values of $\mathbf{R}_\perp$ lying *between* the apertures. This does not correspond at all to physical intuition. The explanation of this paradox is that the usual photometric description in this situation becomes applicable only far from the apertures, in a region for which the apertures act as a single effective source.

2) Quasihomogeneous source. For a quasihomogeneous source, the coherence function $\Gamma^0(\mathbf{R}_\perp, \boldsymbol{\rho}_\perp)$ varies smoothly with the argument $\mathbf{R}_\perp$ as compared with the rapid variation with $\boldsymbol{\rho}_\perp$. For simplicity, let us assume that Γ^0 can be factored, i.e.,

$$\Gamma^0 = \sigma^2(\mathbf{R}_\perp)\, K(\boldsymbol{\rho}_\perp). \tag{34}$$

This condition is not necessary, although in the literature analogous factorization conditions are sometimes included in the definition of quasihomogeneity.[41] Here the dependence of the dispersion $\sigma^2(\mathbf{R}_\perp) \equiv \langle |u^0(\mathbf{R}_\perp)|^2\rangle$ on $\mathbf{R}_\perp$ describes a smooth statistical inhomogeneity with the characteristic scale L_R, while the correlation coefficient $K(\boldsymbol{\rho}_\perp)$ approaches zero when $|\boldsymbol{\rho}_\perp| \gtrsim l_k$, with $l_k \ll L_R$. The expression for the radiance (32) corresponding to (34) has the form

$$I^0(\mathbf{R}_\perp, \hat{\mathbf{n}}) = \frac{1}{a} k_0^2 n_z \sigma^2(\mathbf{R}_\perp)\, \widetilde{K}(k_0 \hat{\mathbf{n}}_\perp). \tag{35}$$

Here we have

$$\widetilde{K}(\varkappa) = \int K(\boldsymbol{\rho}_\perp)\, e^{-i\varkappa_\perp \boldsymbol{\rho}_\perp}\, d^2\boldsymbol{\rho}_\perp \cdot (2\pi)^{-2}. \tag{36}$$

According to (23), the coherence function here is

$$\Gamma = \int k_0^2 n_z \sigma^2 \left(\mathbf{R}_\perp - \frac{\hat{\mathbf{n}}_\perp R_z}{n_z}\right) \widetilde{K}(k_0 \hat{\mathbf{n}}_\perp)\, e^{ik_0 \hat{\mathbf{n}}\boldsymbol{\rho}}\, d\Omega_{\mathbf{n}}. \tag{37}$$

If the field at the screen $z = 0$ is statistically homogeneous, so that $\sigma^2 = \text{const}$, then one can easily reduce this expression to the form

$$\Gamma = \sigma^2 \int\limits_{|\varkappa_\perp| \leqslant k_0} \widetilde{K}(\varkappa_\perp) \exp\left[i\left(\varkappa_\perp \boldsymbol{\rho}_\perp + \rho_z\sqrt{k_0^2 - \varkappa_\perp^2}\right)\right] d^2\varkappa_\perp. \tag{38}$$

This agrees with the well-known result of wave theory for diffraction by a random screen (see Ref. 12, p. 74). In contrast to the exact solution, Eq. (38) neglects the contribution from inhomogeneous waves, which decay exponentially with distance from the plane $z = 0$.[2)]

3) The radiant intensity and the inverse problem—measurement of the coherence function of sources. Let us examine the expression for the radiant intensity corresponding to a source localized in the plane $z = 0$ with dimensions of the order of L. In relation to observation points far from the source (with $z \gg L$), such a source acts as a point source, while the radiant intensity $J(\hat{\mathbf{n}})$ corresponding to it is obtained by integrating the radiance of (32), multiplied by $n_z = \cos\theta$, over the plane of the source $z = 0$:

$$\begin{aligned} J(\hat{\mathbf{n}}) &= n_z \int I^0(\mathbf{R}_\perp, \hat{\mathbf{n}})\, d^2 R_\perp \\ &= \frac{1}{a}\left(\frac{k_0 n_z}{2\pi}\right)^2 \int \Gamma^0(\mathbf{R}_\perp, \boldsymbol{\rho}_\perp)\, d^2\mathbf{R}_\perp\, d^2\boldsymbol{\rho}_\perp \\ &= \frac{1}{a}\left(\frac{k_0 n_z}{2\pi}\right)^2 \langle |\widetilde{u}^0(k_0 \hat{\mathbf{n}}_\perp)|^2\rangle. \end{aligned} \tag{39}$$

Here

$$\widetilde{u}^0(\varkappa_\perp) = \int u^0(\mathbf{R}_\perp)\, e^{i\varkappa_\perp \mathbf{R}_\perp}\, d^2 R_\perp \tag{40}$$

is the spatial spectrum of the field of the source. We see from (39) that the radiant intensity $J(\hat{\mathbf{n}})$ corresponding to the generalized radiance of (32) is non-negative and depends only on the low-frequency (i. e., large-scale) part of the spectrum $\tilde{u}^0(\varkappa_\perp)$, which corresponds to rather small wave numbers $|\varkappa_\perp| = k_0|\hat{\mathbf{n}}_\perp| \leqslant k_0$. The large values $|\varkappa_\perp| > k_0$, which describe exponentially decaying waves, are not taken into account in (32).

This implies that, for a known angular distribution of radiant intensity, one can express the contribution of the low-frequency part of the spectrum u^0 to the coherence function of the field at the aperture. Actually, we shall assume that the field u^0 can be considered statistically homogeneous. For this to be so, the dimensions L of the source must be sufficiently large in comparison with the correlation radius l_c of the field u^0, so that we can neglect edge effects involving the boundedness of the source. Then we obtain from (39) and (40)

[2)]In most cases one can neglect this contribution, although there are exceptions.[57,58]

$$J(\hat{\mathbf{n}}) \approx \frac{1}{a}\left(\frac{k_0 n_z}{2\pi}\right)^2 \Sigma \int \Gamma^0(\rho_\perp)\, e^{-ik_0\hat{\mathbf{n}}_\perp \rho_\perp}\, d^2\rho_\perp. \quad (41)$$

Here Σ is the area of the aperture.

If now we denote the contributions from the low-frequency ($|\varkappa_\perp| < k_0$) and high-frequency ($|\varkappa_\perp| \geqslant k_0$) parts of the spectrum u^0 to the coherence function Γ_0 respectively as $\Gamma^0_{\rm lf}$ and $\Gamma^0_{\rm hf}$, then we can invert the Fourier transform entering into (41) to obtain easily[59]

$$\Gamma^0_{\rm lf}(\rho_\perp) = \frac{a}{\Sigma} \int\limits_{|\hat{\mathbf{n}}_\perp| \leqslant 1} \frac{J(\hat{\mathbf{n}})\, e^{ik_0\hat{\mathbf{n}}_\perp \rho_\perp}}{1 - n_\perp^2}\, d^2 n_\perp. \quad (42)$$

We note that in some cases one can express the function $\Gamma^0_{\rm hf}$ in terms of $\Gamma^0_{\rm lf}$ by using analytic continuation. Here the measurement of the angular dependence of the radiant intensity enables one to reconstruct fully the coherence function Γ_0 of the sources (see Refs. 60–64).

Thus we see that classical photometry and the theory of radiation transport are from the "wave" standpoint a phenomenological theory of the statistically homogeneous random field.

d) Relation of the radiance to the coherence function in the small-angle approximation

In optical applications one often encounters propagation of narrow beams of radiation. For a wave description of such beams, one can use the small-angle approximation, also called the quasioptical approximation or the parabolic-equation approximation. In this approximation, Eq. (21), which expresses the relation of the radiance to the coherence function of the field, can be simplified somewhat. For this purpose let us write, as is usually done, the complex amplitude of a beam propagating along the z axis in the form

$$u = u' e^{ik_0 z}.$$

Here the random amplitude u' varies slowly with z as compared with the rapidly varying phase coefficient. Then the coherence function is represented in the form

$$\Gamma = \langle u_1 u_2^* \rangle = \langle u_1' u_2'^* \rangle\, e^{ik_0\rho_z} \equiv \Gamma' e^{ik_0\rho_z}, \quad (43)$$

where $u_{1,2} = u(\mathbf{r}_{1,2})$, while Γ' is the coherence function for the amplitude u', which varies slowly with varying ρ_z as compared with the "fast" factor $\exp(ik_0\rho_z)$. Upon neglecting this variation, we can write

$$\Gamma' = \Gamma|_{\rho_z = 0} = \Gamma(\mathbf{R}, \rho_\perp). \quad (44)$$

Now, upon substituting (43) and (44) into the general expression (21), we find

$$I = \frac{1}{a}\int\limits_0^\infty k^2\, dk \int \Gamma(\mathbf{R}, \rho_\perp)\, e^{ik_0\rho_z - i\mathbf{k}\rho}\, d^3\rho \cdot (2\pi)^{-3}$$

$$= \frac{1}{a}\int\limits_0^\infty k^2\, dk \int \Gamma(\mathbf{R}, \rho_\perp)\, e^{-ik\hat{\mathbf{n}}_\perp \rho_\perp}\, d^2\rho_\perp\, \delta\left(k_0 - \sqrt{k^2 - k^2 n_\perp^2}\right)$$

$$\times\ (2\pi)^{-2} \approx \frac{1}{a}\left(\frac{k_0}{2\pi}\right)^2 \int \Gamma(\mathbf{R}, \rho_\perp)\, e^{-ik_0\hat{\mathbf{n}}_\perp \rho_\perp}\, d^2\rho_\perp. \quad (45)$$

In the argument of the delta-function, we have assumed approximately that $|\hat{\mathbf{n}}_\perp| \approx 0$, which is allowable in the case of sharply directional beams with a sufficiently small angular width $\delta\vartheta$: $|\hat{\mathbf{n}}_\perp| \lesssim \delta\vartheta \ll 1$.

Thus, in the small-angle approximation the radiance is expressed as the Fourier transform of the coherence function of the field in "difference" coordinates transverse to the direction of propagation. This connection between the radiance and the coherence function was established in the above-cited studies of Dolin.[21,22] We note that (45) agrees in form with the expression (29) for the radiance of plane sources if we set $n_z = 1$ there.

The inverse expression of the coherence function Γ in terms of I can be obtained by going over to the small-angle approximation directly in the general relation (20), which yields

$$\Gamma(\mathbf{R}, \rho) \approx a \int I \exp\left\{ik_0\left[\hat{\mathbf{n}}_\perp \rho_\perp + \left(1 - \frac{\hat{\mathbf{n}}_\perp^2}{2}/2\right)\rho_z\right]\right\} d^2 n_\perp.$$

Here, in contrast to (45), we have retained the terms quadratic in $\hat{\mathbf{n}}_\perp$ in the exponential. Now if we assume $\rho_z = 0$, taking into account the known properties of the Fourier transform for the range of the transverse correlation of the beam $l_\perp$, we easily obtain the estimate

$$l_\perp k_0 \mid n_\perp \mid \sim l_\perp k_0 \delta\vartheta \gtrsim 1$$

or, if we omit the numerical coefficients, $l_\perp \approx \lambda/\delta\vartheta$. Analogously, upon assuming $\rho_\perp = 0$ we find for the range of the longitudinal correlation $l_\parallel$

$$\frac{k_0 n_\perp^2 l_\parallel}{2} \sim \frac{k_0 (\delta\vartheta)^2 l_\parallel}{2} \gtrsim 1,$$

so that $l_\parallel \gtrsim \lambda/(\delta\vartheta)^2$.

e) Limiting resolving power in photometric measurements

The wave nature of radiation imposes fundamental restrictions on the accuracy of photometric measurements that can be viewed as a consequence of the wave uncertainty relationship. The photometric radiance $I(\mathbf{R}, \hat{\mathbf{n}})$ depends on two arguments-the direction of propagation $\hat{\mathbf{n}}$ and the position vector $\mathbf{R}$. Hence it is expedient to find out how compatible an increase in angular resolution is with an increase in coordinate resolution. We can obtain an answer to this question with the example of a very simple measuring device—a lens of diameter D, in whose focal plane the angular distribution of the radiation is being determined (Fig. 3).

The angular resolution $\Delta\theta$ of the lens, which equals λ/D, improves with increasing diameter D of the lens. At the same time, the localization of the radiance is impaired: one

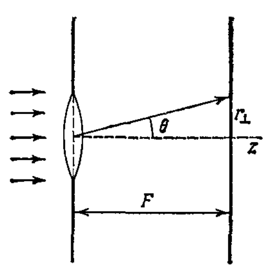

FIG. 3. Recording of radiation in the focal plane of a lens.

should not ascribe the measured radiance $I(\mathbf{R}, \hat{\mathbf{n}})$ to the point $\mathbf{R}$, but to a spatial region of diameter D. That is, the uncertainty ΔR of the coordinate amounts to approximately D. Consequently the product of the uncertainties $\Delta\theta$ and ΔR equals the wavelength:

$$\Delta\theta\cdot\Delta R \approx \frac{\lambda}{D} D = \lambda. \tag{46}$$

Thus an increase in the angular resolution unavoidably entails a loss in resolution in the spatial variables, and vice versa.

If the field in the plane of observation is statistically homogeneous, then one can make the angular resolution arbitrarily high by increasing the diameter D of the lens. At the same time, in a quasihomogeneous field an increase in D exceeding the scale of the quasihomogeneity L_R becomes unreasonable, since when $D \gtrsim L_R$ the distribution of the radiation in the focal plane of the lens is unable to characterize the angular distribution of the radiation. Consequently, the limiting angular resolution $\Delta\theta_{\min}$ amounts to

$$\Delta\theta_{\min} = \frac{\lambda}{L_R}. \tag{47}$$

This requires a lens of diameter $D \lesssim L_R$.

Upon determining the angular distribution of the radiation in the focal plane of the lens, we can determine the coherence function of the field $\Gamma(\mathbf{R}, \boldsymbol{\rho})$ by a Fourier transformation of the form (20). The accuracy with which one knows the coordinate $\mathbf{R}$ of the center of gravity, i.e., the accuracy of localization, approximately equals the diameter D of the lens. With increase in the difference variable $\boldsymbol{\rho}$, the reconstruction of the coherence function qualitatively worsens. The limiting value $\rho_{\max}$ is estimated as $\rho_{\max} \sim \lambda/\Delta\theta = D$ i.e., also as the diameter of the lens.

Similar restrictions also apply to the reconstruction of the time coherence function from the measured value of the spectral density.

f) On the relation between photometric and correlation measurements

Let us examine the conditions under which the readings of a photodetector can be interpreted in photometric terms. The fundamental photometric device is the quadratic photodetector, whose output current i is proportional to the intensity E^2 of the light field averaged over some time T:

$$i \propto \frac{1}{T}\int_0^T E^2(\mathbf{r}, t+\tau)\,d\tau \equiv \overline{E^2(\mathbf{r}, t)}^T \tag{48}$$

(for simplicity we neglect the polarization, treating one of the components of the vector $\mathbf{E}$ as E). Here the averaging time T is determined by the lag time of the detector and in optical measurements usually proves large in comparison with the characteristic period of the light oscillations $\tau = 2\pi/\omega$. If we assume the field to be quasimonochromatic (one can achieve this, for example by fitting the photodetector with a monochromator with a transmission band $\Delta\omega_s \ll \omega$) and write $E = \mathrm{Re}\,E' e^{-i\omega t}$, where E' is the slow complex amplitude of the field, then (48) acquires the form

$$i \propto \frac{1}{2}\mathrm{Re}\,\overline{(|E'|^2 + E'^2 e^{-2i\omega t'})}^T \approx \frac{1}{2}\overline{|E'|^2}^T. \tag{49}$$

Hence the measured current is proportional to the square of the complex amplitude of the field $|E'(\mathbf{r}, t)|^2$ averaged over the time T.

If the input aperture of the photodetector is small enough, while the device itself has a small lag time, so that the slow amplitude E' cannot vary appreciably in the time T, then $\overline{|E'|^2}^T \approx |E'|^2$, and the measured quantity is the *local instantaneous intensity* $|E'(\mathbf{r},t)|^2$. In this case, evidently, the photodetector readings do not admit an interpretation in terms of averages over the ensemble of quantities, since the photocurrent proves to fluctuate with time and from point to point.

On the other hand, if the field is stationary and ergodic in time (see Ref. 12), while the averaging time T is large in comparison with the coherence time τ_{coh} of the field (which equals in order of magnitude the variation time of E', and in the case of a monochromator can be of the order of the reciprocal width of the transmission band $\tau_{\mathrm{coh}} \sim 1/\Delta\omega_s$[3)]), the time T in (49) can be made to approach infinity. Here the time average will equal the average over the ensemble: $\overline{|E'|^2}^T \approx \langle |E'|^2\rangle$. In this case the photocurrent practically does not fluctuate, and will describe the mean intensity of the field at the point $\mathbf{r}$ at the frequency ω. That is, it is proportional to the integral of the photometric radiance I_ω over the directions, $i \sim \int I_\omega d\Omega_n$. If we can consider the radiation to be isotropic, so that $\int I_\omega d\Omega_n = 4\pi I_\omega$, then one measures the spectral density I_ω, which enables one to find the time correlation function $\Gamma(\tau)$ of the radiation. There is also the reverse possibility—to find the spectral density I_ω from measurements of $\Gamma(\tau)$ (Fourier spectroscopy).

Another case, in which the averaging action of the device reduces to averaging over the ensemble, involves spatial ergodicity. Let us assume that the field is strictly monochromatic, thus ruling out time fluctuations of the amplitude E', i.e., considering it to be independent of time (in practice this means that the observation time T is small in comparison with the coherence time of the field τ_{coh}). Then the intensity $\overline{|E'|^2} \approx |E'|^2$ does not depend on the averaging time T and is a random function of the spatial coordinates. Let us examine the typical scheme of photometric measurements shown in Fig. 3. The photodetector lies in the focal plane of the lens, which can be oriented in different directions in order to determine the angular distribution of the radiation, i.e., the radiance $I_\omega(\hat{\mathbf{n}})$. The amplitude of the field in the focal plane E^{F} is proportional to the Fourier transform of the amplitude in a plane immediately in front of the lens:

$$E^{\mathrm{F}}(\mathbf{r}_\perp) \sim \int_\Sigma E'(\mathbf{r}'_\perp)\, e^{ik_0\hat{\mathbf{n}}_\perp \mathbf{r}'_\perp}\, d^2 r'_\perp. \tag{50}$$

Here we restrict our treatment to the small-angle approximation, considering points near the z axis, so that $\hat{\mathbf{n}}_\perp = \mathbf{r}_\perp/F$. Here the photocurrent of the detector is

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3) See also Ref. 65, where the treatment of the coherence time as the characteristic duration of the light trains constituting the field is described in detail.

$$i \sim |E^{F}|^2 \propto \int\limits_{\Sigma}\int\limits_{\Sigma} E'(\mathbf{r}'_{\perp}) E'^{*}(\mathbf{r}''_{\perp}) e^{ik_0\hat{\mathbf{n}}_{\perp}(\mathbf{r}'_{\perp}-\mathbf{r}''_{\perp})} d^2r'_{\perp} d^2r''_{\perp}$$

$$= \iint\limits_{\mathbf{R}'_{\perp}\pm(\rho'_{\perp}/2)\in\Sigma} E'\left(\mathbf{R}'_{\perp}+\frac{\rho'_{\perp}}{2}\right) E'^{*} \times\left(\mathbf{R}'_{\perp}-\frac{\rho'_{\perp}}{2}\right) e^{ik_0\hat{\mathbf{n}}_{\perp}\rho'_{\perp}} d^2R'_{\perp} d^2\rho'_{\perp}. \quad (51)$$

Here the integration is performed over the distribution of the incident light over the lens aperture Σ. The integration over $\mathbf{R}'_{\perp}$ that enters into (51) can be viewed as a spatial averaging over the aperture.

Now let us assume that the random incident field in the plane of the lens is statistically homogeneous and spatially ergodic, and has some finite range of coherence l_{coh}. Then, if the dimension D of the lens is large in comparison with l_{coh}, then the integration over $\mathbf{R}'_{\perp}$ in (51) reduces to statistical averaging and multiplication by the area Σ of the lens:

$$i \propto \Sigma \int \left\langle E'\left(\mathbf{R}'_{\perp}+\frac{\rho'_{\perp}}{2}\right) E'^{*}\left(\mathbf{R}'-\frac{\rho'_{\perp}}{2}\right)\right\rangle e^{ik_0\hat{\mathbf{n}}_{\perp}\rho'_{\perp}} d^2\rho'_{\perp}$$

$$= \Sigma \int \Gamma(\rho'_{\perp}) e^{ik_0\hat{\mathbf{n}}_{\perp}\rho_{\perp}} d^2\rho_{\perp} \propto \Sigma I_{\omega}(\hat{\mathbf{n}}). \quad (52)$$

Here Γ is the coherence function of the field E', and $I_{\omega}(\hat{\mathbf{n}})$ is the radiance of the radiation, which is related to Γ by Eq. (45). We see that in this case the photocurrent proves to be nonfluctuating and proportional to the photometric radiance I_{ω}, owing to the spatial averaging action of the lens.

Yet if the dimension of the lens is small in comparison with the scale of coherence l_{coh}, the averaging action of the lens will be insignificant, so that the photocurrent will be a fluctuating quantity. To average it we must collect the statistics of repeated measurements for different positions of the lens in the plane $z = 0$. This case is analogous to that treated above of averaging over time with a small averaging interval T.

One can find the coherence function $\Gamma(\rho)$ from the measured values of the radiance $I_{\omega}(\hat{\mathbf{n}})$ by using the fundamental relationship (45), which establishes the correspondence between the photometric and field characteristics of the electromagnetic field. In principle the inverse sequence of determining I_{ω} and Γ is possible: first one measures the coherence function $\Gamma(\rho)$, and then calculates the radiance $I_{\omega}(\hat{\mathbf{n}})$. This type of measurement might be called spatial Fourier spectroscopy. In essence, this is precisely what one does in using the Michelson stellar interferometer; see Ref. 16.

In closing this subsection, let us point out some studies treating problems involving the substantiation of the photometry of free radiation. In Ref. 66 the properties of the coherence function of volume (nonplane) sources in free space are discussed (as in the case studied above of plane sources, the exact wave solution of this problem is well known). References 67 and 68 have treated the passage through a lens of a generalized radiance defined in a somewhat different way from that in this review (the possibility of different definitions of generalized radiance was noted above). References 69 and 70 have studied the generalized radiance for the vector electromagnetic problem. Reference 71 has attempted to construct a more detailed theory of transport theory for free electromagnetic radiation; in Ref. 72 the same problem has been treated on the basis of a quantum approach.

4. STATISTICAL THEORY OF RADIATION TRANSPORT IN A SCATTERING MEDIUM

Now let us proceed to the very complex case of radiation in a scattering medium, restricting the treatment only to a brief review of the fundamental results obtained up to now. One can find a more detailed presentation of these problems in Refs. 10–12.

a) The equation of radiation transport

In the phenomenological theory the behavior of the radiance of radiation in a scattering medium is described by the well known transport equation (see, e.g., Refs. 73, 74). In the simplest case of a scalar monochromatic field it has the form

$$\frac{dI}{ds} + \alpha I = \int \sigma(\hat{\mathbf{n}} \leftarrow \hat{\mathbf{n}}') I(\mathbf{r}, \hat{\mathbf{n}}') d\Omega_{\mathbf{n}'}. \quad (53)$$

The left-hand side of (53) describes the attenuation of the radiance I along the ray involving scattering and adsorption and linked under the name "extinction" (α is the extinction coefficient and s is the length of the ray). The right-hand side characterizes the contribution of scattering, with the parameter σ as the scattering cross-section per unit volume. For a weakly scattering medium with continuous fluctuations, σ is usually calculated in the Born approximation, while in the case of widely separated discrete scatterers it is calculated in the approximation of independent particles with a possible correction for a weak correlation of the scatterers. Thus, for example, in the very widespread scalar model of scattering of radiation by continuous fluctuations of dielectric permittivity $\varepsilon = 1 + \tilde{\varepsilon}$, $\langle\varepsilon\rangle = 1$, the Born approximation for the scattering cross-section yields

$$\sigma = \sigma_{\mathrm{B}} = \frac{\pi k_0^4}{2} \int \langle\tilde{\varepsilon}(0)\tilde{\varepsilon}(\rho)\rangle e^{-ik_0(\hat{\mathbf{n}}-\hat{\mathbf{n}}')\rho} d^3\rho (2\pi)^{-3}. \quad (54)$$

For scattering by point dipoles with polarizability β in the single-scattering approximation we have

$$\sigma = \sigma_1 [N + N(N-1)\chi(k_0(\hat{\mathbf{n}} - \hat{\mathbf{n}}'))]. \quad (55)$$

Here N is the number of dipoles per unit volume, $\sigma_1 = k_0^4\beta^2$ is the scattering cross-section for a single dipole (without allowance for the polarization factor), and $\chi(\mathbf{q}) = \langle\exp(i\mathbf{q}\cdot(\mathbf{r}_1 - \mathbf{r}_2))\rangle$ is the characteristic distribution function of the relative coordinates of the dipoles $\mathbf{r}_1$ and $\mathbf{r}_2$. Here the first term in (55) corresponds to scattering by independent particles, while the second takes into account correlation of the positions of the particles (see, e.g., Ref. 12, Secs. 26 and 31).

In the absence of absorption the extinction coefficient α proves to be equal to the total (over all directions) scattering cross-section

$$\alpha = \int \sigma(\hat{\mathbf{n}}' \leftarrow \hat{\mathbf{n}}) d\Omega_{\mathbf{n}'}. \quad (56)$$

(This relationship is called the optical theorem.)

To derive the transport equation (53) in the phenomenological theory, one uses simple arguments of energy balance in a physically infinitely small volume of the scattering medium.

b. Statistical derivation of the equation of radiation transport

In proceeding to the statistical substantiation of the transport equation (53), we first note that the statistical meaning of the radiance as the angular spectrum of the coherence function of the wave field evidently must be kept for radiation in a scattering medium. Of course, here the condition of quasihomogeneity (18), which permits introducing the concept of a localized spectrum, must be satisfied. As before, we shall use as the latter the Wigner function. However, the condition of quasihomogeneity no longer suffices, since the scattering must be in some sense "small" in order that the transport equation may hold. This leads to certain restrictions on the parameters of the scattering medium.

For a "wave" derivation of the transport equation in the linear theory, we can naturally start with the stochastic wave equation for a random field u, considering the statistical characteristics of the scattering medium to be fixed. Such a medium can be either continuous and described by some random field (usually the dielectric permittivity of the medium serves as such a field), or discrete, i.e., consisting of separate scattering particles whose properties and positions are random. Here the course of the argument is practically independent of the model of the medium: for any model of the medium, the stochastic wave equations are used to derive the equations for the statistical moments of the field, with the equation for the second moment (i.e., for the coherence function Γ) serving for deriving the transport equation. The equation for the coherence function is known as the Bethe-Salpeter equation. For radiation in the absence of sources, it can be written in the form (see, e.g., Ref. 12):

$$D_1 D_2^* \Gamma_{12} = K_{12} \Gamma_{12}, \tag{57}$$

Here $\Gamma = \langle u(\mathbf{r}_1) u^*(\mathbf{r}_{12}) \rangle = \langle u_1 u_2^* \rangle$ is the coherence function of the field in the scattering medium, the Dyson operators D_1 and D_2 act on the arguments with subscripts 1 and 2, respectively, and K_{12} is called the intensity operator.

In the general case the operators $D_{1,2}$ and K_{12} are known only in the form of expansions in infinite series. Therefore, in order to employ Eq. (57), one must practically always restrict the treatment to certain approximate values of the operators $D_{1,2}$ and K_{12} involving the assumption of "sufficiently weak" scattering.

One can derive the radiation transport equation from the Bethe-Salpeter equation by methods differing in form but coinciding in physical content: the diagram method,[75,76] the operator method,[77] by asymptotic expansions,[14,78] etc. (see also Refs. 79–93, where the transport equations were derived under the most varied initial assumptions). Actually the essence of all these methods consists of solving the Bethe-Salpeter equation in the geometric-optics approximation under the assumption that the coherence function Γ of the field is quasihomogeneous. Ultimately this enables one to introduce the pithy concepts of the local spectrum (17) and the radiance of radiation (21); here the radiance I will satisfy the transport equation (53). The essential point is that this approach does not apply the method of geometric optics to individual realizations of the field, but only to the equation for a quantity averaged over the ensemble—the coherence function. This immediately removes a number of methodological problems involving the treatment of the concept of the radiance in a scattering medium. In particular, the question that was mentioned in the Introduction loses meaning—what to take as the radiance near sharp scattering inhomogeneities? Actually, as we have seen, radiance is essentially a statistical concept characterizing the ensemble of realizations, or in other words, the quadratic characteristics of the radiation averaged over large regions of space. Therefore it is generally pointless to speak of the value of radiance at a point near an individual inhomogeneity.

The approach that we have just described enables one to go from the rigorous Bethe-Salpeter equation (57) to the ordinary equation of radiation transport (53) by expressing simultaneously the extinction coefficient α and the scattering cross-section per unit volume σ in terms of statistical parameters appearing in the wave theory—the explicit form of these parameters depends on the model of the medium that one employs.

As we have already noted above, when using the Bethe-Salpeter equation one must restrict the treatment to approximate values of the operators $D_{1,2}$ and K_{12}. The most important approximation for the theory of radiation transport, known as the "one-group" approximation, has been proposed in Ref. 94. For a medium having continuous fluctuations, this approximation is a natural generalization of the simplest approximations in which one keeps the first nonvanishing terms of a power-series expansion of the fluctuations—the Bourret approximation for $D_{1,2}$ and the ladder approximation for K_{12}. For a medium with discrete inclusions, the one-group approximation generalizes the approximation of independent particles.

The importance of the one-group approximation for K_{12} involves the fact that this approximation keeps only the terms of the expansion of K_{12} that lead to a "local" scattering cross-section depending on the properties of the medium in a volume of the order of several correlation ranges of the fluctuations. It is precisely this cross-section, which characterizes the local properties of the medium, that is introduced in the phenomenological theory. The one-group approximation for $D_{1,2}$ in the sense that it leads to the law of conservation of energy being satisfied approximately (valid to the accuracy of the omitted many-group terms).

Let us formulate some important results that stem from the "wave" derivation of the transport equation (53).

1. Instead of the heuristic picture of a noncoherent superposition of ray beams on which photometry rests, the wave theory employs the picture of quasihomogeneous random fields. The quasihomogeneous coherence function in a scattering medium is connected with the radiance by the relationship (20) as in free space. Here, however, the wave number k_0 for free space must be replaced by the quantity Re k^{eff}, where k^{eff} is the effective wave number in the scattering medium. It satisfies the dispersion equation for the mean field, which corresponds to the homogeneous Dyson equation $D\langle u \rangle = 0$.

2. In deriving the transport equation, the method of geometric optics is applied, not to find the individual realizations of the field, but to solve the equations for the moments. This eliminates many difficulties in the substantiation of photometry. The "effective medium" in which the radiation is propagating is described by the Dyson equation for the mean field. This enables one to take into account many effects lying outside classical photometry, in particular, the influence of inhomogeneity of fluctuations on the refraction of rays.

3. The transport equation (53) can be derived only under the assumption of weak attenuation of the mean field, i.e., small extinction. In turn, this requires smallness of both true absorption and scattering, so that the field may have the character of running waves within the confines of individual inhomogeneities. If we define the extinction length l_e as the reciprocal of the extinction coefficient α, then the inequality $l_e \gg \rho_\varepsilon$ must be satisfied, where ρ_ε is the range of correlation of the inhomogeneities of the medium.

4. In the wave derivation of the transport equation, the previous phenomenological characteristics—the scattering cross-section σ and the extinction coefficient α—acquire a microscopic meaning. In the wave theory these quantities prove to be expressed as series, the first terms of which differ little from a Born approximation like (54), or (for a discrete medium) from a single-scattering approximation like (55). The subsequent terms of the series, which are calculated by the "one-group" approximation, allow for a possible non-Gaussian character of continuous fluctuations or particle-correlation effects. However, here multipoint distribution functions appear in the theory whose experimental determination at present is difficult. Moreover, theoretical models are lacking that give a sufficiently full statistical description of any real scattering media. All this impedes evaluating the conditions of applicability of one-group approximations, as well as comparing the results of theory and experiment.

5. The diffraction content of the transport equation has become evident in the wave substantiation. In particular, this equation, when supplemented with the Dyson equation for the mean field, has made possible a description of the transformation of coherent radiation into a scattered component[78]; here it turned out that the single-scattering approximation in the theory of radiation transport yields results equivalent to the diffraction theory of single scattering. Moreover, as Dolin[22] has shown, the transport equation in the small-angle approximation is identical with the diffraction parabolic equation.

6. If we start not with the wave equation, but with Maxwell's equations, we can derive a transport equation for the radiance matrix whose evolution describes various polarization effects: birefringence, depolarization of waves in the process of propagation, Rytov rotation of the plane of polarization, etc. (on the question of taking polarization effects into account in transport theory, see Refs. 3, 7, 67, 69, 71, 31, 14, 80, 82–84, 87–88).

7. Up to now we have mainly been treating the case of the three-dimensional problem of scattering theory. There is a large number of studies evaluating the conditions of applicability of transport equation in the one-dimensional problem (see Refs. 96–99 and the literature cited there). The region of applicability of the transport equation in the one-dimensional problem has proven to be substantially narrower than in the three-dimensional case. When backscattering is taken into account in the absence of absorption, this region differs little from the region of applicability of the Born approximation. Within the framework of transport theory, this does not allow one to allow effectively for multiple backscattering. The result concerning the restricted applicability of the transport equation in the one-dimensional problem has been obtained by direct comparison of the wave solutions with the conclusions of transport theory. In all likelihood, it is related to the phenomenon known in solid-state physics of localization of eigenstates in one-dimensional systems.[100–101]

8. In speaking of the theory of radiation transport, one must also mention nonlinear problems. The transport equations in these problems are widely employed and are known under the name of equations for quantum numbers or—in turbulence theory—as kinetic equations for waves.[102,103] The discussion of the nonlinear theory lies outside the scope of this review. We note only that in the nonlinear problems the substantiation of transport theory is closely associated with the problem of the onset of random behavior in dynamic systems.[104–106]

5. CONCLUSION

The principal result of the research on the statistical and wave content of the concepts of classical photometry consists of the fact that these concepts have found their natural interpretation in the terms of statistical optics as the spectral characteristics of the coherence function of a quasihomogeneous and quasistationary random wave field. The most fully developed are the physical bases of the photometry of free radiation, although even here there are some unsolved questions. In particular, the case of nonplanar sources must be studied in greater detail, as well as studying the inverse problems of the type of the problem mentioned in Sec. 3 of reconstruction of the coherence function of sources from the distribution of light intensity. The problems that are associated with the description of individual experiments, also require more detailed study, including the problems, hardly touched upon in this review, of the conditions for space and time ergodicity of radiation, of the choice of the necessary averaging intervals, of the possibility of "self-averaging" of certain parameters of radiation in the process of propagation, etc.

Considerably more points remain unclear in the theory of radiation transport in scattering media. First of all, one should note here the need for quantitative specification of the conditions of applicability of the fundamental equation of radiation transport (53), as well as the need for constructing detailed models for calculating the statistical characteristics of scattering media. Further, since one can substantiate the transport equation only under conditions of sufficiently weak absorption and scattering, the problem, which is important in principle, of describing scattering in highly turbid media, in which near-field effects are essential, i.e., effects of strong mutual irradiation of particles with participation of inhomogeneous waves, still remains open. Finally, the field of activity is open for performing experiments interpretable on the basis of the correlation content of photometric radiance.

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