Theory of current states in narrow superconducting channels

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The properties of narrow superconducting channels carrying a direct current are reviewed. Among the topics covered are the stability of the normal state of the current-carrying channel and the mechanism for a transition from this normal state to the superconducting state. In a homogeneous channel, the transition occurs through the formation of a critical nucleus and is a first-order phase transition. In a channel with inhomogeneities, the transition is quite different. In this case the normal state can exist only down to a certain value of the current, below which the normal state is absolutely unstable. The review is devoted primarily to the theory of the resistive state of narrow channels, which exists at currents above the critical Ginzburg-Landau current. The description is based on the concept of phase-slippage centers. Phenomenological models are discussed, as is a model of a fluctuational excitation of phase-slippage centers. The results obtained from the microscopic dynamic theory of superconductivity are discussed at length. Among these results are the voltage-current characteristic of the resistive state, the abrupt change in the voltage on this characteristic, and the structure of the phase-slippage centers.

CONTENTS

1. INTRODUCTION

This review deals with the properties of narrow superconducting channels carrying a direct current. "Narrow channels" here are those conductors whose transverse dimensions are small (or, in practice, of the order of) the magnetic-field penetration depth and the coherence length $\xi(T)$. These conditions are satisfied by narrow strips and whiskers. For tin, for example, the transverse dimensions must be of the order of a few tenths of a micron to one micron. We assume that the samples are quite long—longer than the electric-field penetration depth l_E which will be defined below. Experimentally, the properties of narrow superconducting channels have been studied quite thoroughly. The states which occur in such channels are quite unusual, and their explanation frequently requires appealing to new developments in the microscopic theory ("new" in comparison with the classical ideas regarding superconductivity). For the theoretician, on the other hand, these entities are convenient because the problem is effectively one-dimensional, with all quantities depending on only the coordinate along the length of the sample. As a result, the mathematical difficulties are eased, and the physics of the phenomenon emerges more clearly.

Let us briefly summarize what is known about narrow, current-carrying, superconducting channels. The Ginzburg-Landau theory predicts that at temperatures below the superconducting transition temperature such a channel can be in either a homogeneous superconducting state or a normal state, depending on the magnitude of the current flowing through the channel. Specifically, at low currents the channel is in a superconducting state. When the current density is increased above the so-called Ginzburg-Landau critical current j_c the homogeneous superconducting state disappears, and the channel should go into the normal state. This description, however, is a simplification of the actual situation in at least two regards.

First, experiments show that above the Ginzburg-Lan-

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dau critical current the superconducting state does not disappear completely but instead converts into a so-called resistive state in which superconductivity and a static electric field exist simultaneously. In other words, there is a finite potential difference across the sample, while by other measures the sample is a superconductor. This state is evidence that the simple absence of electrical resistance is not a fundamental property of superconductivity. Generally speaking, resistive states are not confined to the case of narrow channels. The best-known example is the flow of current in the mixed state of a type II superconductor,[1,2] where the generation of an electric field is associated with the motion of vortices driven by the current flowing through the sample. In narrow superconducting channels, however, the resistive state is unrelated to the motion of any defects of the superconducting structure and represents a qualitatively new phenomenon. The properties of the resistive state will be the subject of the main part of this review.

Second, the Ginzburg-Landau theory does not take up the question of the particular mechanism which is responsible for the transition from the normal state to the superconducting state as the current is reduced. We know that the normal state of a channel of infinite length is stable "in the small" (i.e., with respect to infinitesimally small fluctuations) at an arbitrarily weak current. This effect is explained by arguing that a Cooper pair which arises in a fluctuational manner is accelerated by the electric field in the sample until it acquires a high velocity and breaks up. This conclusion is not, however, extended to fluctuations of finite size. Since the electric field penetrates a finite depth l_E into the superconducting region, a critical nucleus of the superconducting phase can appear in the normal phase. In this critical nucleus, the electric field is quite weak and cannot prevent the Cooper instability of the normal state. Nuclei exceeding the critical size grow and eventually fill the entire sample, which is thereby put in a superconducting state. The transition from the normal state to the superconducting state in a homogeneous narrow channel of finite length with a current is thus essentially a first-order phase transition. The size of the critical nucleus depends on the current, increasing with increasing current. Above a certain j_2 a critical nucleus cannot exist. This current j_2 is usually considerably higher than the Ginzburg-Landau critical current. We can thus assume that the current interval $j_c < j < j_2$ corresponds to the region in which the resistive state discussed above exists, but this question requires further study.

The events which occur at the ends of the channel and near the various inhomogeneities in the channel play an important role in the transition of a channel from the normal state to the superconducting state. The inhomogeneities and the boundary with the normal metal (for example, at the contacts with the normal conductors which are the leads to the measuring instruments) significantly promote the appearance of a superconducting nucleus. There exists a critical current j_1 such that at $j < j_1$ the normal state near an inhomogeneity or an SN boundary is absolutely unstable with respect to the formation of an infinitesimally small superconducting nucleus, which subsequently grows over time and expands, in such a manner that the entire channel becomes a superconductor after a sufficiently long time.

These phenomena are the subject of the present review. We emphasize that this is a review of the theory of these phenomena; we will be discussing the experimental results only to the extent required to draw a sketch of the events in question. We make no claim for a comprehensive discussion of the experimental results. Furthermore, we will not attempt to cover everything in the literature; we will discuss only the basic ideas and accomplishments of the theory.

2. DYNAMIC EQUATIONS

The behavior of superconductors in the presence of an electric field is far from a steady state and must be described by the dynamic equations of superconductivity. Unfortunately, the system of time-varying equations for superconductors is extremely complicated in its general form; furthermore, the equations for the superconducting properties contain generalized kinetic equations for the distribution function of the excitations. It is exceedingly difficult to follow the phenomena of interest here when forced to work with a general system of equations of this type. The approach which has customarily been taken to obtain specific results from the microscopic theory has been to restrict the parameters of the theory (most commonly the temperature) to certain intervals in which the complete system of dynamic equations can be simplified substantially without giving up the features which are important for the phenomena of interest.

This is the approach which we will also take in the present review. We will write out a comparatively simple system of dynamic equations which can be derived from the microscopic theory of superconductivity in a narrow temperature interval near the critical temperature for the superconducting transition, T_c. This system of equations will be used to derive some quantitative results. The qualitative results obtained without appealing to specific dynamic equations are of course more general in nature.

We denote by Δ and χ the modulus and phase of the superconducting order parameter. We also introduce the gauge-invariant electromagnetic potentials $\mathbf{Q} = \mathbf{A} - (\hbar c/2e)\nabla\chi$ and $\Phi = \varphi + (\hbar/2e)\partial\chi/\partial t$ where $\mathbf{A}$ and φ are the ordinary electromagnetic potentials. In this section of the review we write a system of dynamic equations which contains only the superconducting parameters Δ, $\mathbf{Q}$, and Φ; this system of equations can be obtained from the microscopic theory near the critical temperature if the changes in these parameters over space and time are sufficiently slow. More specifically, we require $Dk^2, \omega \ll \tau_{ph}^{-1}$, where D is the diffusion coefficient, τ_{ph} is the inelastic electron-phonon relaxation time, and k and ω are the characteristic wave vector and characteristic frequency of the problem. These equations are quite general and hold in a temperature range which is quite accessible experimentally. These equations were originally derived in their most general form by Kramer and Watts-Tobin[3] (see also the later paper by Watts-Tobin *et al.*[4] and the papers by Golub[5] and Shön and Ambegaokar[6]). These equations are

$$-\frac{\pi}{8T}\sqrt{4\tau_{ph}^2\Delta^2+\hbar^2}\frac{\partial\Delta}{\partial t}+\frac{\pi\hbar}{8T}D\nabla^2\Delta-\frac{\pi\hbar}{8T}D\left(\frac{2e}{\hbar c}Q\right)^2\Delta+\frac{T_c-T}{T}\Delta-\frac{7\zeta(3)}{8\pi^2}\frac{\Delta^3}{T^2}=0, \quad (2.1)$$

$$\frac{1}{\sqrt{4\tau_{ph}^2\Delta^2+\hbar^2}}\Delta^2\Phi+\frac{D}{\hbar c}\operatorname{div}(\Delta^2 Q)=0, \quad (2.2)$$

$$j=\sigma E-\frac{\sigma\pi\Delta^2}{2\hbar cT}Q, \quad (2.3)$$

where

$$E=-\frac{1}{c}\frac{\partial Q}{\partial t}-\nabla\Phi. \quad (2.4)$$

From the condition for electrical neutrality,

$$\operatorname{div} j=0, \quad (2.5)$$

and (2.2) and (2.3) we find

$$D\nabla^2\Phi+\frac{D}{c}\operatorname{div}\frac{\partial Q}{\partial t}=\frac{\pi}{2T}\frac{\Delta^2\Phi}{\sqrt{4\tau_{ph}^2\Delta^2+\hbar^2}}. \quad (2.6)$$

Equation (2.6) describes the relaxation of the so-called electron-hole disbalance. We will not discuss this phenomenon in detail here; the interested reader is referred to the original papers[7-10] and the reviews (Refs. 11 and 12, for example). We do wish to mention the circumstances which are of greatest importance for the discussion below. The gauge-invariant potential $\Phi=\varphi+(\hbar/2e)\partial\chi/\partial t$ can be written as the difference $\Phi=(1/e)(\mu_p-\mu_e)$, where $\mu_e=-e\varphi$ is the chemical potential of the normal quasiparticles, reckoned from the Fermi level, and $\mu_p=(\hbar/2e)\partial\chi/\partial t$ is the chemical potential of the Cooper pairs per particle. At equilibrium we always have $\mu_p=\mu_e$ and $\Phi=0$. In a nonequilibrium situation, however, the chemical potentials of these two particles species may be different. In a superconductor, this deviation from equilibrium can also be described in terms of a disbalance of the populations of the electron-like and hole-like branches of the energy spectrum. This deviation from equilibrium has a characteristic relaxation time τ_Q, so that electron diffusion causes the difference $\Phi=(1/e)(\mu_p-\mu_e)$ to decay over distances $l_E=\sqrt{D\tau_Q}$. It is not difficult to see that Eq. (2.6) describes specifically this process; the relaxation of Φ results from spatial dispersion (the term containing $\partial\Phi/\partial t$ is small at $\omega\ll\tau_{ph}^{-1}$. Equation (2.6) determines a characteristic length l_E, given in the case $\Delta=\text{const}$ by

$$l_E=\left(\frac{2DT}{\pi\Delta^2}\right)^{1/2}(4\tau_{ph}^2\Delta^2+\hbar^2)^{1/4}. \quad (2.7)$$

This length is the penetration depth of the static electric field ($\partial Q/\partial t=0$, $E=-\nabla\Phi$).

In the gap-free case, $\tau_{ph}\Delta\ll\hbar$, Eqs. (2.1)–(2.3) become the equations of the time-dependent Ginzburg-Landau theory.[13,14] In the opposite limit, $\Delta\tau_{ph}\gg\hbar$, Eq. (2.1) is the same as the dynamic equation for a gap-free superconductor, which was first derived by Gor'kov and Éliashberg[15]; Eq. (2.6) in this case determines the penetration depth of the electric field,

$$l_E=\sqrt{D\tau_{ph}\frac{4T}{\pi\Delta}}, \quad (2.8)$$

in agreement with the result of Refs. 9 and 10.

The relaxation of the potential Φ results from an interaction of the condensate with excitations. In a gap-free superconductor this interaction is strong, so that the electric-field penetration depth is quite small, $l_E=(2DT\hbar/\pi\Delta^2)^{1/2}$, of the order of the coherence length $\xi(T)$, where

$$\xi(T)=\sqrt{\frac{\pi D\hbar}{8(T_c-T)}}. \quad (2.9)$$

In a superconductor with a gap, on the other hand, the interaction of the condensate with excitations can occur only indirectly (through phonons in this model), so that in this case the length l_E in (2.8) contains the long electron-phonon collision time τ_{ph} and is large in comparison with $\xi(T)$.

From Eqs. (2.1)–(2.3) we see that the characteristic frequency is of order

$$\omega_{GL}=\frac{\pi\Delta_{GL}^2}{2T\hbar}, \quad (2.10)$$

where

$$\Delta_{GL}=\sqrt{\frac{8\pi^2T(T_c-T)}{7\zeta(3)}} \quad (2.11)$$

is the equilibrium value of the order parameter. The condition for the applicability of these equations, ω, $Dk^2\ll\tau_{ph}^{-1}$, thus requires that the temperature be quite close to T_c:

$$\frac{T_c-T}{T}\ll\frac{\hbar}{\tau_{ph}T_c}. \quad (2.12)$$

The values of the quantity $\tau_{ph}T/\hbar$ are large for real materials (Table I). From the standpoint of condition (2.12), the best experimental situations are those in Pb, In, and perhaps Sn.

In some cases it will be convenient to use Eqs. (2.1)–(2.3) in dimensionless form. In such cases we adopt as the characteristic length and the characteristic time the quantities $\xi(T)$ and $\tau_{GL}=\omega_{GL}^{-1}$, respectively; the order parameter is divided by its equilibrium value Δ_{GL}; and the current is expressed in units of $\pi\sigma\Delta_{GL}^2/4eT\xi$. In terms of these units the critical Ginzburg-Landau current is $j_c=2/3\sqrt{3}\simeq0.385$. In dimensionless variables, our equations are

$$-u\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{1/2}\frac{\partial\Delta}{\partial t}+\nabla^2\Delta+(1-\Delta^2-Q^2)\Delta=0, \quad (2.13)$$

$$u\Delta^2\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{-1/2}\Phi+\operatorname{div}(\Delta^2Q)=0, \quad (2.14)$$

$$j=-\frac{\partial Q}{\partial t}-\nabla\Phi-\Delta^2Q, \quad (2.15)$$

$$\nabla^2\Phi+\operatorname{div}\frac{\partial Q}{\partial t}=u\Delta^2\left(\frac{\Delta^2}{\Gamma^2}+1\right)^{-1/2}\Phi. \quad (2.16)$$

The gauge-invariant potentials in these units are $\Phi=\varphi+\partial\chi/\partial t$, and $Q=A-\nabla\chi$. We will ignore the magnetic field because of the narrowness of the sample, and we will assume $Q=-\nabla\chi$. In Eqs. (2.13) – (2.16) we have introduced a depairing factor

$$\Gamma=\frac{\hbar}{2\tau_{ph}\Delta_{GL}}=\frac{\pi\hbar}{8\sqrt{u}}\frac{1}{\tau_{ph}T_c}\left(\frac{T_c-T}{T}\right)^{-1/2} \quad (2.17)$$

TABLE I. Values of the parameter $\tau_{ph}k_BT_c/\hbar$ for various superconductors.

Material	T_c, K	τ_{ph}, s	$\tau_{ph}k_BT_c/\hbar$
Pb	7.2	$2\cdot10^{-11}$	$2\cdot10$
In	3.4	10^{-10}	$4\cdot10$
Sn	3.8	$3\cdot10^{-10}$	10^2
Al	1.2	10^{-8}	10^3

and the numerical parameter $u = \pi^4/14\zeta(3) \approx 5.79$.

The gap-free situation corresponds to $\Delta \ll \Gamma$. The depairing factor Γ depends on the temperature. In a narrow neighborhood of T_c, specifically $1-(T/T_c) \ll (\hbar/\tau_{ph} T)^2$, the factor Γ is large, $\Gamma \gg 1$, so that we would always be dealing with a gap-free situation. For those temperatures at which experiments are customarily carried out, however, the factor Γ is usually much less than unity, so that the product $\tau_{ph} T_c/\hbar$ is quite large. In such cases, therefore, when Δ is of the order of its equilibrium value ($\Delta \sim 1$ in our units), the inequality $\Delta \gg \Gamma$ usually holds. This inequality corresponds to the presence of a gap in the energy spectrum. In this situation, the electric-field penetration depth is given in order of magnitude by $l_E \sim (u\Gamma)^{-1/2}$, as can be seen from (2.16), and is much larger than $\xi(T)$ ($\xi = 1$ in our units).

Equations (2.13)–(2.15) can also be written in complex form by introducing the complex order parameter $\psi = \Delta \exp(i\chi)$:

$$-u\left(\frac{|\psi|^2}{\Gamma^2}+1\right)^{-1/2}\left[\frac{\partial\psi}{\partial t}+i\varphi\psi+\frac{1}{2\Gamma^2}\psi\frac{\partial|\psi|^2}{\partial t}\right] + \nabla^2\psi+\psi-|\psi|^2\psi=0, \qquad (2.18)$$

$$\mathbf{j} = -\nabla\varphi + \frac{1}{2i}(\psi^*\nabla\psi-\psi\nabla\psi^*). \qquad (2.19)$$

3. ANALYSIS OF THE STABILITY OF THE NORMAL STATE OF A CHANNEL

a) Infinitely long homogeneous channels

In this section we analyze the stability of the normal state of a current-carrying superconducting channel. We first consider a homogeneous channel which is so long that its boundaries can be ignored.

We begin the stability of the normal state of such a current-carrying channel with respect to infinitesimally small fluctuations of the order parameter. For infinitesimally small values of the order parameter the gap-free situation, $\Delta\tau_{ph} \ll \hbar$, always prevails, so that the superconductor can be described by the time-dependent Ginzburg-Landau equations. For an analysis we use the equation in the form in (2.18), (2.19); we linearize these equations for $|\psi| \ll \Gamma$, 1. In this case we have $\varphi = -jx$ and

$$-u\frac{\partial\psi}{\partial t}+iujx\psi+\frac{\partial^2\psi}{\partial x^2}+\psi=0, \qquad (3.1)$$

where the x axis runs along the length of the channel. A solution of this equation can be written in the form[16]

$$\psi(x,t)=\sqrt{\frac{u}{4\pi t}}\exp\left(\frac{t}{u}-\frac{j^2t^3}{12u}\right)\exp(ijxt) \times \int_{-\infty}^{\infty} dy\,\psi(y,0)\exp\left[-\frac{ijt}{2}(x-y)-\frac{u(x-y)^2}{4t}\right],$$

where $\psi(y, 0)$ describes an initial ($t=0$) fluctuation of ψ. The term t^3 in the exponential function describes the acceleration of Cooper pairs by the electric field and causes all the infinitesimally small fluctuations to decay as $t\to\infty$. Fluctuations with an initial amplitude $|\psi(x,0)| = \Delta_0$, for example, and with characteristic dimensions greater than ξ evolve in accordance with

$$\Delta = \Delta_0 \exp\left(\frac{t}{u}-\frac{j^2t^3}{3u}\right).$$

According to the work by Gor'kov[17] and Kulik,[18] we can thus conclude that the normal state is stable with respect to infinitesimally small fluctuations.

The behavior of a superconducting channel in an electric field depends strongly on just how the electric field is produced. If the superconducting sample and the external field sources form a system in which a static electric field is maintained in the superconducting channel (as, for example, in the case in which the electric field is produced in the channel by induction), then the normal state will always be stable. This stability is clear from the acceleration of Cooper pairs by the electric field which we have just mentioned. The situation is different where there is a state with a given current in the system. In this case, fluctuations of finite amplitude become important. Since the electric field penetrates only to a depth l_E into the superconducting region, there may arise a finite-amplitude superconducting fluctuation of such a nature that the electric field in the superconducting region is substantially suppressed and cannot prevent the Cooper instability of the normal state. A fluctuation of this type serves as the critical nucleus of a superconducting phase in the normal phase. All the nuclei with dimensions greater than the critical expand and propagate to fill the entire channel.

The critical-nucleus problem involves a study of nonlinear equations, so it is an extremely complicated problem from the mathematical standpoint. Numerical methods can be of much assistance here. Watts-Tobin *et al.*[4] have numerically solved the critical-nucleus problem for system of equations (2.18), (2.19). Figure 1, taken from Ref. 4, shows the shape of the critical nucleus (in a steady state, but not at equilibrium) for various values of the parameter Γ for a current $j = 0.25$. With decreasing current, the amplitude of the critical nucleus decreases; i.e., a transition to the superconducting state occurs more easily at lower currents. With increasing current, however, the amplitude of the critical nucleus increases, and above a certain j_2 a critical nucleus cannot exist. Estimates yield

$$j_2 \sim j_0 \frac{l_E^2}{\xi^2}. \qquad (3.2)$$

Figure 2, also taken from Ref. 4, shows j_2 as a function of the parameter Γ. In the limit $\Gamma\to 0$, the asymptotic behavior $j_2 = 0.030\Gamma^{-1}$ is found, in terms of the dimensionless units.

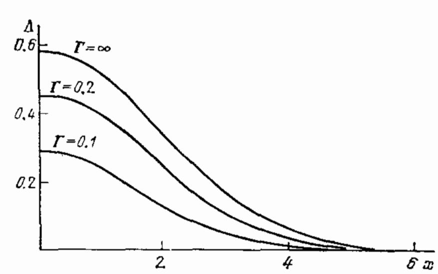

FIG. 1. Shape of the critical nucleus according to a numerical solution of Eqs. (2.18) and (2.19) for the current[4] $j = 0.25$. The nucleus is symmetric with respect to the point $x = 0$; only the right half of the nucleus is shown.

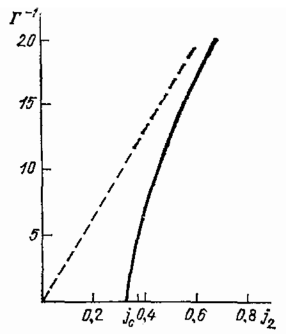

FIG. 2. Dependence of j_2 on Γ^{-1}. At $\Gamma = \infty$ the current j_2 is 0.335, while at $\Gamma = 0.18$ it is $j_2 = j_c = 0.385$. The dashed line is the asymptotic behavior.

In terms of the ordinary units, we would have

$$j_2 = \frac{\sigma\pi\Delta^2_{GL}}{4lT\xi}\cdot 0.030\Gamma^{-1}.$$

The transition from the normal state to the superconducting state in the presence of a current thus results from the formation of a critical nucleus and is therefore a first-order phase transition.

The current j_2 is the boundary above which the normal state of the channel is absolutely stable. As mentioned above, l_E is usually much larger than $\xi(T)$, so that the current j_2 is considerably higher than j_c. If the current (j) flowing through the channel is less than j_c, then the channel will go from the normal state to a homogeneous superconducting state. In this state, the entire current will be carried by the superconducting electrons, $j = j_s$, where the superconducting current $j_s = \Delta^2\nabla\chi$ is related to the modulus of the order parameter by

$$j_s = \Delta^2\sqrt{1-\Delta^2}.$$

We know[19] that a superconducting current state of this type can exist only in the current interval $0 < j < j_c$. If on the other hand, the current j lies in the interval $j_c < j < j_2$, then the transition from the normal state will occur not to a homogeneous superconducting state but to a resistive state, which will be described in Sections 4–7 below.

b) Effect of an SN boundary and of inhomogeneities on the stability of the normal state of a current-carrying channel

A transition accompanied by the formation of a critical nucleus involves overcoming an energy barrier, which is extremely high because of the macroscopic dimensions of the sample (the width and thickness of the barrier are hundreds and thousands of times greater than the atomic dimensions). As a result, the probability for the formation of a critical nucleus must generally be very small.

Up to this point we have been discussing a homogeneous superconducting channel of infinitely great length. By analogy with ordinary first-order phase transitions we would ask what role is played by the ends of the superconducting channel and by inhomogeneities in the channel in the transition from the normal state to the superconducting state in the presence of a current. Since the superconducting sample in an actual experiment is connected to the measuring instruments by means of normal contacts, it is natural to consider first the problem in which a narrow superconducting channel is connected at its ends to normal conductors, so that SN boundaries form in the contact regions (Fig. 3). The effect of an SN boundary on the stability of the normal state of a current-carrying channel was studied in Ref. 20. Let us take a brief look at the results of that paper.

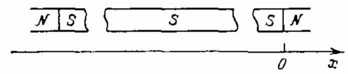

FIG. 3. Narrow superconducting channel (S) whose ends are in contact with normal conductors (N).

We consider a semi-infinite, narrow conducting channel which fills the region $x < 0$ (Fig. 3). We assume the condition $\Delta = 0$ holds at the boundary with the normal metal, as it does if the contact is with a "good" normal metal, i.e., if the order parameter in it decays over distances less than $\xi(T)$ in the superconductor.

The solution for an infinitesimally small nucleus is found from the linearized version of Eq. (3.1). We set

$$\psi = e^{-i\omega t} f(x). \qquad (3.3)$$

For $f(x)$ we find the equation

$$[1 + iu(\omega + jx)] f + \frac{\partial^2 f}{\partial x^2} = 0, \qquad (3.4)$$

whose solution can be expressed in terms of Bessel functions of order 1/3:

$$f(x) = [1 + iu(\omega + jx)]^{1/2} Z_{1/3}(z), \qquad (3.5)$$

where $Z_{1/3}$ is one of the solutions of the Bessel equation, and

$$z = \frac{2i}{3uj}[1 + iu(\omega + jx)]^{3/2}.$$

Figure 4 shows the z interval in the complex plane corresponding to changes in x from $-\infty$ to $+\infty$. A solution which decays in the limit $x \to -\infty$ can be found by choosing $Z = H^{(2)}_{1/3}$, where $H^{(2)}_{1/3}$ is the Hankel function of the second kind. We thus write

$$f(x) = [1 + iu(\omega + jx)]^{1/2} H^{(2)}_{1/3}(z). \qquad (3.6)$$

Figure 4 shows a cut along the negative real z semiaxis corresponding to the determination of the principal branch of the function $H^{(2)}$. As we move away from $-\infty$ along x, the argument of the Hankel function goes out of the region in which the principal branch of $H^{(2)}$ is defined. An analytic

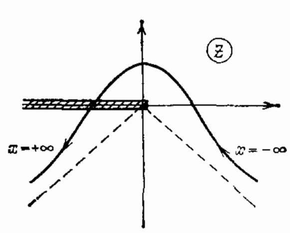

FIG. 4. Range of the argument of the Hankel function, $H^{(2)}_{1/3}(z)$, in the complex plane as x varies from $-\infty$ to $+\infty$. The cut corresponds to the definition of the principal branch of the function.

continuation under the cut yields

$$H^{(2)}_{1/3}(z)=H^{(2)}_{1/3}(z')+e^{i\pi/3}H^{(1)}_{1/3}(z')$$
$$=2e^{i\pi/6}\left[J_{1/3}(z')\cos\frac{\pi}{6}-Y_{1/3}(z')\sin\frac{\pi}{6}\right], \qquad (3.7)$$

where $z=e^{i\pi}z'$, and $J_{1/3}$ and $Y_{1/3}$ are respectively the Bessel and Neumann functions. We see from (3.7) that in the limit $x\to+\infty$ the function $H^{(2)}_{1/3}(z)$ would increase exponentially, meaning that there would be no infinitesimally small steady-state solution of Eq. (3.1) in a channel of finite length, in accordance with the result of the preceding section. In our case, however, the superconducting channel is restricted to the region $x<0$, and we must require $H^{(2)}_{1/3}(z)=0$ at $x=0$. The roots of the Bessel function $J_{1/3}(z')\cos(\pi/6)-Y_{1/3}(z')\sin(\pi/6)$ lie on the positive real z' semiaxis. We need the smallest root, which corresponds to $z'=s_1\approx 2.383$. Equating $z=s_1e^{i\pi}$ at $x=0$, we find the following condition on the frequency:

$$-iu\omega=\left[1-\left(\frac{j}{j_1}\right)^{2/3}\right]-i\sqrt{3}\left(\frac{j}{j_1}\right)^{2/3}, \qquad (3.8)$$

where the critical current j_1 is determined by

$$uj_1=\frac{4\sqrt{2}}{3s_1}\simeq 0.791.$$

Using the numerical value of the parameter u, $u=5.79$, we find

$$j_1\approx 0.137\approx 0.356 j_c.$$

It can be seen from expression (3.8) that at a current $j<j_1$ we have a growth rate $\mathrm{Re}(-i\omega)>0$, and an infinitesimally small solution grows over time. If $j>j_1$, the infinitesimally small solution decays. To determine the subsequent fate of an infinitesimally small nucleus, we consider Eqs. (2.18) and (2.19) in the region $|j_1-j|\ll j_1$, and we consider the nonlinear terms in these equations. Here we have

$$\varphi=-jx-\varphi_s, \quad \varphi_s=\int_x^0 dx\cdot j_s,$$

where

$$j_s=\frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right). \qquad (3.9)$$

Again writing ψ in the form in (3.3), we find, for $|\mathrm{Re}(-i\omega)|\ll 1$,

$$[1+iu(\omega+jx)]f+\frac{\partial^2 f}{\partial x^2}+\left[iu\varphi_s-\frac{iu}{2\Gamma^2}(\omega+jx)|f|^2+|f|^2\right]f=0. \qquad (3.10)$$

We set

$$f=Cf_0(x)+f_1(x),$$

where $C\ll 1$ is a positive constant, and $f_0(x)$ is of the form in (3.6). The small correction $f_1\sim C^3$ also satisfies the conditions $f_1(x)=0$ *at* $x=0$ and $f_1(x)\to 0$ in the limit $x\to-\infty$. Since the solution of linear, homogeneous equation (3.4) is orthogonal with respect to the nonlinear part of Eq. (3.10), we find the following equation for the frequency:

$$-iu\omega=\left(\frac{2}{3}\frac{j_1-j}{j_1}-\alpha'\right)-i\left(\sqrt{3}+\frac{2}{\sqrt{3}}\frac{j-j_1}{j_1}+\alpha''\right), \qquad (3.11)$$

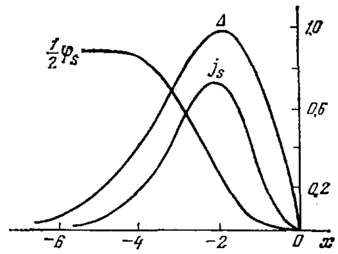

FIG. 5. Result of a numerical solution of Eq. (3.4) with $j=j_1$. The function $f(x)$ is normalized to satisfy the condition $\max|f(x)|=1$. To obtain the actual values of Δ, φ_s, and j_s it is necessary to multiply the result for $|f(x)|$ by C and to multiply the results for j_s and φ_s by C^2, where C is defined by (3.16).

where the quantities α' and α'' are determined by integral expressions which contain the nonlinear part of Eq. (3.10).

To analyze the stability of a small superconducting nucleus we need α'. After a numerical evaluation of the corresponding integrals we find

$$\alpha'=C^2\left(I_1-I_2u-\frac{I_3}{2\Gamma^2}\right), \qquad (3.12)$$

where

$$I_1\approx 0.981, \quad I_2\approx 0.496, \quad I_3\approx 0.372.$$

In the evaluation of the integrals, the function $f_0(x)$ is normalized to satisfy $\max|f_0(x)|=1$. Figure 5 shows the results for $|f_0(x)|$, φ_s, and j_s. It can be seen from (3.11) and (3.12) that a small solution at $j<j_1$ could be stable only if $I_1-I_2u-I_3/2\Gamma^2>0$. Since $u>I_1/I_2\approx 1.98$, however, we have $\alpha'=-bC^2$, where $b=I_2u-I_1+(I_3/2\Gamma^2)>0$, and the growth rate of the solution,

$$\mathrm{Re}(-i\omega)=\frac{1}{u}\left(bC^2+\frac{2}{3}\frac{j_1-j}{j_1}\right), \qquad (3.13)$$

will always be positive for $j<j_1$. Thus, for $j<j_1$, an infinitesimally small solution will grow with time. If $j>j_1$, on the other hand, then the curve described by the expression

$$C(j)=\sqrt{\frac{2}{3b}\left(\frac{j-j_1}{j_1}\right)}, \qquad (3.14)$$

determines the threshold for the stability of the normal state with respect to the formation of a nucleus with an amplitude C near the SN boundary. Figure 6 shows a sketch of the curve $C(j)$.

These results refer to nuclei with small amplitudes $C\ll\Gamma$, 1. Equations (2.18) and (2.19) have been integrated numerically[20] in order to determine the behavior of a nucleus after it has acquired a finite amplitude. Let us examine the results.

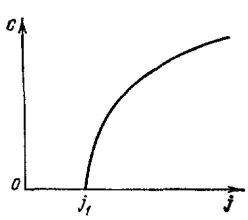

FIG. 6. Sketch of the current dependence of the amplitude of a critical nucleus, $C(j)$.

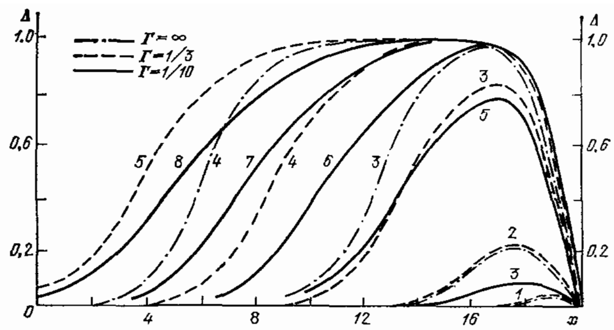

FIG. 7. Results of numerical calculations describing the growth and expansion of a superconducting nucleus at the current $j = 0.1$ and at the parameter values $\Gamma = \infty$, $\Gamma = 1/3$, and $\Gamma = 0.1$. Curves 1, 2, etc., correspond to the successive times $t_1 = 0$, $t_2 = 30$, $t_3 = 60$, etc., $[t_n = 30(n-1)]$.

When $j > j_1$, small perturbations decay with time. On the other hand, there exists a critical amplitude $C(j)$ such that an initial perturbation with a greater amplitude, $C > C(j)$, will grow even under the condition $j > j_1$. This behavior is in agreement with that described above.

Small perturbations grow in the case of currents $j < j_1$; the initial growth of the amplitude of the nucleus continues until it reaches a value near unity. At this point the nucleus begins to expand, and its boundary moves into the interior of the superconductor. This behavior is analogous to the expansion of a superconducting domain as studied in Refs. 21 and 22 on the basis of the time-dependent Ginzburg-Landau equations. The velocity of the boundary of the nucleus and its slope both decrease with decreasing Γ at a given current. With decreasing current, the velocity of the boundary increases, while its slope decreases. Figure 7 shows the results of numerical calculations for the parameter values $\Gamma = \infty$, $\Gamma = 1/3$, and $\Gamma = 1/10$ with the current held at $j = 0.1$.

We thus see that conditions favoring the appearance of a superconducting nucleus against the background of the normal state arise in a superconductor near a boundary with a normal metal. The reason is that near the SN boundary nearly the entire current in the superconductor is carried by normal excitations, so that the current of Cooper pairs and their velocity are small. In other words, the boundary prevents the acceleration of Cooper pairs by the electric field, facilitating the formation of a superconducting nucleus.

The role played by inhomogeneities can be analyzed in a completely analogous way. Following Ref. 23, we consider an inhomogeneity of such a nature that the critical temperature of the channel depends on the coordinate x:

$$T_c = \begin{cases} T_c, & |x| > d, \\ T_{c1}, & |x| < d. \end{cases}$$

If the dimension of the inhomogeneity is small in comparison with $\xi(T)$, then by considering the problem over dimensions of order $\xi(T)$ we can replace the term $(T_c - T)/T_c$ in Eq. (2.1) by $[T_C - T + (T_{c1} - T_c)2d\delta(x)]/T_c$. When we switch to dimensionless units, an additional terms $a\delta(x)$ arises in Eq. (2.18), where the parameter

$$a = \frac{2d}{\xi_0}\frac{T_c - T_{c1}}{T_c}\left(\frac{T_c - T}{T_c}\right)^{-1/2}$$

is a measure of the "intensity" of the inhomogeneity. The parameter a may be either greater or less than unity in absolute value. If the inhomogeneity weakens the superconductivity ($T_{c1} < T_c$), then we have $a > 0$; if the inhomogeneity instead strengthens the superconductivity, we have the opposite result.

Because of the δ-function, the condition at the point $x = 0$ becomes

$$\psi(+0) = \psi(-0), \quad \frac{\partial\psi(+0)}{\partial x} - \frac{\partial\psi(-0)}{\partial x} = a\psi(0). \qquad (3.15)$$

For an infinitesimally small nucleus with $x \neq 0$ we again find Eq. (3.4), and the solution of this equation is again of the form in (3.5), but now $Z_{1/3}(z)$ at $x < 0$ and $x > 0$ must consist of different linear combinations of the Hankel functions $H^{(1)}$ and $H^{(2)}$ satisfying the respective decay conditions $x \to -\infty$ and $x \to +\infty$. Joining these functions with the help of conditions (3.15), we find the critical current j_1 as a function of the inhomogeneity parameter a. We will write expressions for $j_1(a)$ only for certain limiting cases:

$$j_1(a) = \begin{cases} \dfrac{2\sqrt{6}}{us}\, j_c, & a \to +\infty, \\[2mm] \dfrac{2\sqrt{3}}{u\ln(2/|a|)}\, j_c, & a \to \pm 0, \\[2mm] \dfrac{\sqrt{3}}{2u}\, j_c \left(\dfrac{2\pi|a|}{3\Gamma^2(2/3)}\right)^3, & a \to -\infty. \end{cases}$$

In the case $a \to +\infty$, the result is the same as that for the case of the SN boundary, discussed above. In a homogeneous channel, with $a \to 0$, the critical current j_1 tends toward zero, reflecting the fact that the normal state is stable in a homogeneous current-carrying channel.

Analysis of the nonlinear equations reveals that the picture of the stability of the normal state is precisely the same as that in the case of an SN boundary, discussed above. At currents $j < j_1$ the normal state is absolutely unstable. A growing nucleus arises near an inhomogeneity and expands; it eventually fills the entire sample. This process is unrelated to the overcoming of an energy barrier, and it does not require an activation energy. At currents $j > j_1$ the normal state is stable with respect to infinitesimally small fluctuations but unstable with respect to the formation of nuclei with amplitudes above a certain critical value. The physical reason why the growth of the superconducting nucleus is facilitated near an inhomogeneity is that the inhomogeneity limits the velocity of the Cooper pairs, preventing their acceleration by the electric field, as in the case of an SN boundary.

In this discussion of the stability of a normal state with a current we have seen that the transition from the normal state to a superconducting state in the presence of a current occurs through the formation and growth of a supercritical nucleus and is essentially a first-order phase transition. If $j < j_c$, then the growth of a nucleus leads to an ordinary superconducting state, while if $j > j_c$ the final state is a resistive state. As mentioned earlier, at currents above the current j_2 given by (3.2) the normal state of a homogeneous channel is absolutely stable. For this reason, the normal state at currents $j < j_2$ may be called "supercooled," by analogy with metastable states in thermodynamics. The size of the supercritical nucleus drops to zero at $j \leqslant j_1$. The current j_1 is thus a lower boundary on the current of a supercooled normal state. At currents below j_1, the normal state cannot exist in a homogeneous channel.

In the sections of the review which follow we take up the resistive state, which occurs in the current interval $j_c < j < j_2$ (in some cases, the resistive state may also extend partially into the region $j < j_c$).

4. GENERAL IDEAS REGARDING THE RESISTIVE STATE

The easiest way to detect the existence of the resistive state experimentally is to study the voltage-current characteristic of the sample. The voltage-current characteristics of pure and not very long samples were measured in Refs. 24–27. The curves in Fig. 8a are taken from Ref. 25. They clearly show voltage steps. As the length of the sample is increased, and as the electron mean free path is reduced, the steps become less well defined, and the curve becomes smoother. For a while the curve runs nearly parallel to an Ohm's-law line and then departs from it.[28,29] This behavior is shown in Fig. 8b, taken from Ref. 29. Figure 8c shows the initial part of this characteristic on a larger scale. Although the curve has become almost completely smooth, the derivative $\partial V/\partial J$ exhibits traces of the voltage steps.

The experimental voltage-current characteristics can be compared with that derived from the Ginzburg-Landau theory,[19] shown in Fig. 9. At a current exceeding the critical Ginzburg-Landau current the superconductivity should be disrupted according to this theory, and the sample should abruptly switch from a superconducting state to a normal state. The primary distinction between the characteristics in Fig. 8 and that in Fig. 9 is that in practice there are broad ranges of the current and the voltage in which superconductivity exists against the background of a constant electric field. This is the state which we call the "resistive state." We

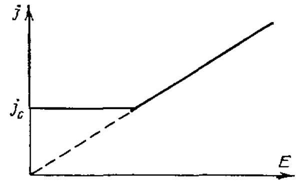

FIG. 9. Voltage-current characteristic of a narrow channel according to the Ginzburg-Landau theory.

wish to point out that it does not arise in the classical Ginzburg-Landau theory.

An important feature of the characteristics of channels of finite length is the presence of voltage steps, which are observed in the cases of both whiskers[24] and samples in the form of narrow strips.[25–27] Samples in the resistive state also have some important time-dependent properties. The foremost of these properties is a time-dependent Josephson effect, which is exhibited by samples exposed to microwave radiation.[25] Generation of lower frequencies is also observed; the radiation is generated at the ends of the sample.[28,30]

We see that the behavior of narrow superconducting channels in the resistive state is greatly different from the classical picture of superconductivity. Furthermore, the small transverse dimensions of the samples rule out an explanation of these phenomena in terms of the ordinary motion of vortices or regions of the normal phase through the sample. It can thus be argued that the resistive state in narrow channels is a new type of superconducting state.

The primary task of the theory is to explain the most unexpected circumstance: how superconductivity can exist in a sample in which there is a constant electric field. It is well known that Cooper pairs have a charge, so that in an electric field they should be accelerated until the superconductivity is disrupted. As we will see, however, this is not what occurs in the resistive state. The motion of Cooper pairs in a superconductor and, in general, the behavior of a superconductor in an electromagnetic field are determined not by the vector potential **A** and the scalar potential φ of the electromagnetic field but by the gauge-invariant potentials

$$\mathbf{Q} = \mathbf{A} - \frac{\hbar c}{2e}\nabla\chi, \quad \Phi = \varphi + \frac{\hbar}{2e}\frac{\partial\chi}{\partial t}, \tag{4.1}$$

which were introduced in Section 2. We can use the potentials (4.1) to write

$$\frac{\partial \mathbf{v}_s}{\partial t} = \frac{e}{m}(\mathbf{E} + \nabla\Phi), \tag{4.2}$$

where the velocity ($\mathbf{v}_s$) of the Cooper pairs is expressed in terms of the gauge-invariant potential **Q** by

$$\mathbf{v}_s = -\frac{e}{mc}\mathbf{Q},$$

so that Eq. (4.2) describes the acceleration of Cooper pairs which we just mentioned. If the velocity of the Cooper pairs is not to increase to infinity the electric field must, on the average, be cancelled out by the term with $\nabla\Phi$. It follows that the resistive state of the superconductor must be a very nonequilibrium state with $\Phi \neq 0$ and $\mu_e \neq \mu_p$, where μ_e and μ_p are the chemical potentials of the quasiparticles and Cooper pairs, respectively (Section 2). In a superconductor, how-

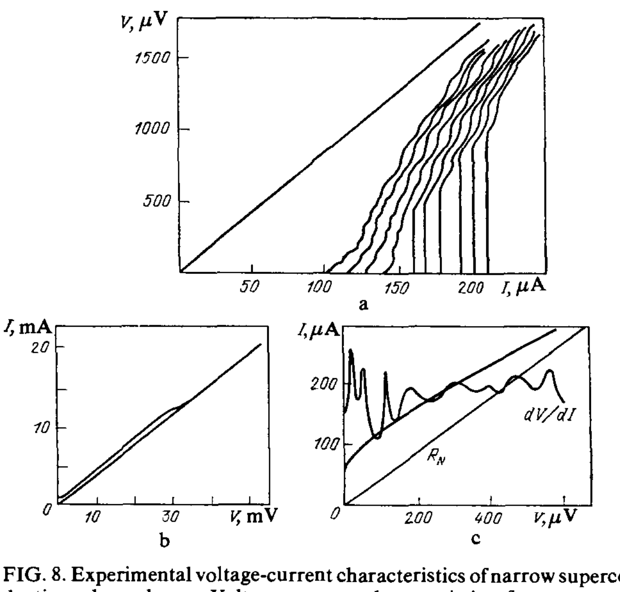

FIG. 8. Experimental voltage-current characteristics of narrow superconducting channels. a—Voltage-current characteristics for narrow tin strips, with voltage steps[25] (the different curves correspond to different temperatures); b—voltage-current characteristic for a long, narrow tin film with a short electron mean free path[29]; c—the first part of the characteristic of part b on a larger scale.

ever, the potential Φ cannot be arbitrarily large, for otherwise the superconductivity would again be disrupted. Let us assume that the sample is infinitely long. In this case the potential difference ($\delta\varphi$) between sufficiently remote points can take on very large values. If Φ is to be kept finite, this potential difference $\delta\varphi$ must be offset by a corresponding difference in the chemical potentials of the pairs, $\delta\mu_p$, or, equivalently, by a corresponding rate of increase in the phase difference between these points.

In principle, there are two ways to describe the picture. The first static model assumes that a structure which is periodic along the length of the sample is established in the sample and that the parameter of the superconducting order vanishes at the points of maximum $|\Phi|$. The adjacent superconducting regions have different values of the chemical potential μ_p; the difference is equal to the potential difference $\delta\varphi$ between these regions. In each superconducting region we thus have $\mathbf{E} = -\nabla\Phi$, $\partial \mathbf{Q}/\partial t = 0$ and $\mu_p = \text{const}$. The potential Φ is kept bounded because the potential difference $\delta\varphi$ offsets the difference between the chemical potentials of the pairs in the adjacent superconducting regions. At points where the order parameter vanishes, the macroscopic phase coherence is disrupted, and μ_p and thus Φ are discontinuous. This picture is sketched in Fig. 10, where Δ is the modulus of the order parameter, and x is the coordinate along the sample. A similar static model has been suggested by Fink and Poulsen[31–34] and Galaĭko *et al.*[27,35–41]

This picture, however, ignores an extremely important circumstance: The adjacent superconducting regions have different pair chemical potentials $\mu_p = (\hbar/2)\partial\chi/\partial t$. Consequently, the phase difference between adjacent regions will increase with time. Since the transition region along x between these regions (i.e., the region in which Δ is approximately zero) has a width of order ξ, the values of the complex order parameter in these regions will interact strongly with each other. To see this, we write the complex order parameter $\psi = \Delta e^{i\chi}$ as

$$\psi = \Delta^{(1)} \exp\left(2i\mu_p^{(1)} t \frac{1}{\hbar}\right) + \Delta^{(2)} \exp\left(2i\mu_p^{(2)} t \frac{1}{\hbar}\right),$$

where $\Delta^{(1,2)} \exp(2i\mu_p^{(1,2)} t (1/\hbar))$ are the values of the order parameter in two adjacent superconducting regions. It is clear that the modulus of the order parameter will oscillate with time in the region in which $\Delta^{(1)}(x)$ and $\Delta^{(2)}(x)$ overlap, and the static picture should be disrupted here. We are thus led to ask to what extent the static model corresponds to reality at all. This topic is discussed in more detail in Section 7; at this point we will simply note that, as we will show below, the oscillations occur only in some extremely narrow neighborhoods of the points at which phase coherence is disrupted, and all quantities oscillate only very slightly over essentially the entire interval between adjacent points of this type.

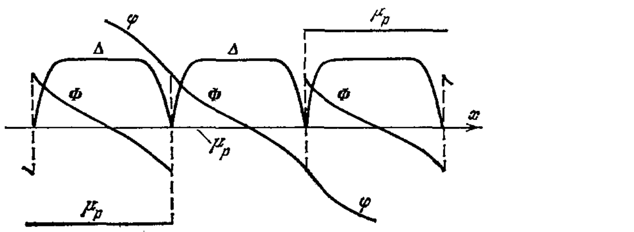

FIG. 10. Sketch of the structure of the order parameter and of the potentials μ_e, μ_p, and Δ in the static model of the resistive state.

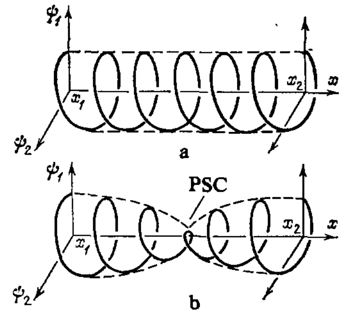

FIG. 11. Representation of the complex order parameter in the space $\{\psi_1, \psi_2, x\}$. The only way in which the number of loops in the spiral can be reduced is for the radius of the turns to vanish at some point.

The second description assumes a time variation from the very outset. This is the description which we will have in mind in the discussion below. As we have already mentioned, if Φ is to be kept finite the potential difference $\delta\varphi$ between remote points x_1 and x_2 in the sample must be offset by the rate of increase of the phase difference between these points:

$$\delta\varphi + \frac{\hbar}{2e}\frac{\partial}{\partial t}\delta\chi \approx 0. \tag{4.3}$$

This interpretation is conveniently illustrated in a space in which the real and imaginary parts of the complex order parameter $\psi = \psi_1 + i\psi_2$ are plotted along two axes, while the coordinate x (along the sample) is plotted along the third axis[42] (Fig. 11). In a homogeneous state the modulus $\Delta = \sqrt{\psi_1^2 + \psi_1^2}$ is constant, and the turns of the spiral in Fig. 11 are of constant radius. As time elapses, the phase difference $\delta\chi$ increases, and the turns of the spiral move closer together. This tendency, however, cannot continue forever. The phase difference $\delta\chi$ determines the condensate velocity

$$v_s \sim \frac{\hbar\delta\chi}{m(x_2 - x_1)},$$

which will thus increase, and should again disrupt the superconductivity. If this is not to happen, some mechanism must operate to cause a reduction in the phase difference, which is increasing with time. It is clear from Fig. 11 that if the spiral is to be able to lose one loop the radius of the spiral, Δ, must vanish at some point x between x_1 and x_2.

The points at which the order parameter vanishes and its phase undergoes jumps equal to a multiple of 2π, are known as "phase slippage centers" (PSCs).

For superconductivity to exist in the sample, the phase slippage must occur repeatedly in time; the relationship between the average time (t_0) between phase jumps and the average voltage between points x_1 and x_2, V, can be determined from (2.3). Since the phase difference slips by 2π upon the elimination of each loop, we find the following equation by averaging (2.3) over the time:

$$2eV = \frac{2\pi\hbar}{t_0}. \tag{4.4}$$

This is the ordinary Josephson relation. We will offer a rigorous derivation of a similar relation below, based on the topological properties of phase slippage centers.

There are two possible mechanisms for the formation of phase slippage centers. First, they may form as a result of thermodynamic fluctuations in the system. The probability for such an event is proportional to $\exp(-\delta F/T)$, where δF is the energy barrier between two homogeneous states before and after the phase slippage. It is clear that this process is more likely to occur in the immediate vicinity of the critical temperature, where the barrier δF is low. This mechanism was proposed by Langer and Ambegaokar,[42] who should also be credited with developing the general picture of phase slippage described above.

As we move away from the critical temperature the probability for the formation of phase slippage centers as a result of fluctuations falls off sharply, and internal and therefore more fundamental factors come into play. One might say that the process by which phase slippage centers are excited in a superconducting channel due to a sufficiently large direct current flowing through the channel is analogous to a self-excited oscillation. From a more formal standpoint, this assertion means that the formation of phase slippage centers, i.e., the oscillations of the order parameter at certain points in a sample, is a consequence of nonlinearities of the limiting-cycle type which are inherent in the system.

If a superconducting channel has structural inhomogeneities then the formation of phase slippage centers will occur more probably at "weak points," where the order parameter is suppressed by extraneous factors. In long and homogeneous samples, however, the phase slippage centers should form periodically along the coordinate and in time because of the spatial and temporal homogeneity.

It is convenient to examine the formation of phase slippage centers in the two-dimensional (2*D*) space-time[43] $\{x; ct\}$ (Fig. 12). The circles in Fig. 12 are phase slippage centers, i.e., space-time points where the modulus of the order parameter vanishes, $\Delta = 0$. We assume that these points form a periodic structure in the space $\{x; ct\}$. It is easy to see that the elimination of n loops as Δ vanishes in Fig. 11 is equivalent to the requirement that the phase of the order parameter change by $2\pi n$, where n is an integer, as we move along a closed contour around a phase slippage center in the $\{x; ct\}$ space-time. The phase slippage centers can thus be thought of as topological singularities of the vortex type in the 2*D* space-time $\{x; ct\}$.

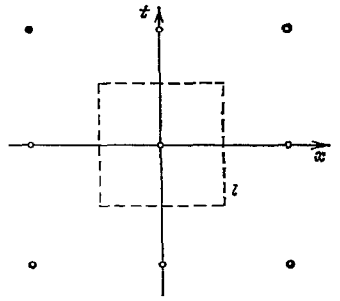

FIG. 12. The space $\{x; ct\}$. The circles are phase slippage centers, i.e., points of such a nature that going around them along a closed contour results in a change in the phase by $2\pi n$. Contour l bounds an elementary cell of the structure of phase slippage centers in the space-time $\{x; ct\}$.

A quantization rule on the electric field "flux" in the space $\{x; ct\}$, analogous to the quantization of magnetic flux in coordinate space, can be derived[43] for topological singularities of this type. In our 2*D* space-time we introduce the 2*D* vectors

$$\boldsymbol{\rho} = \{x,\ ct\},\quad \mathbf{q} = \{Q_x,\ -\Phi\},\quad \mathbf{a} = \{A_x,\ -\varphi\}.$$

These vectors are related by

$$\mathbf{q} = \mathbf{a} - \frac{\hbar c}{2e}\frac{\partial \chi}{\partial \boldsymbol{\rho}}. \tag{4.5}$$

We integrate the vector **q** along a closed contour around a phase slippage center. From (4.5) we find

$$\oint_l \mathbf{q}\, d\boldsymbol{\rho} = \oint \mathbf{a}\, d\boldsymbol{\rho} - \frac{\hbar c}{2e}\oint \frac{\partial \chi}{\partial \boldsymbol{\rho}}\, d\boldsymbol{\rho}. \tag{4.6}$$

Using the Stokes theorem, we can express the first integral on the right side in terms of an integral of rot **a** (rot = curl) over the area enclosed by the contour l. It is not difficult to see that

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \frac{\partial \varphi}{\partial x} = (\text{rot}\ \mathbf{a})_3.$$

The second integral on the right side of (4.6) gives us a phase shift $2\pi n$ as we go around the phase slippage center; we thus have

$$\oint_l \mathbf{q}\, d\boldsymbol{\rho} = \oint_S E\, dx \cdot c\, dt - \frac{2\pi\hbar c}{2e}\, n.$$

If we consider an isolated center, we conclude that the integral along ther infinitely remote contour on the left vanishes. In the case of a periodic system of phase slippage centers, this integral vanishes if we choose as the integration contour the boundary of an elementary cell of the phase slippage center in the space $\{x, ct\}$. We thus find

$$\int_{S_0} E\, ds = \varphi_0 n, \tag{4.7}$$

where $\varphi_0 = \pi\hbar c/e$ is the quantum of "flux," numerically equal to the quantum of magnet flux in coordinate space, and the integration is extended to the elementary cell in the space $\{x, ct\}: ds = dx c dt$. "Quantization rule" (4.7) is a generalization of the Josephson relation to the case in which the potential drop varies and is distributed over the sample. If E remains constant in time, then we immediately find from (4.7) the ordinary Josephson relation, (4.4). Quantization rule (4.7) is extremely useful, since it expresses the electric field averaged over time and space (strictly speaking, this is what is measured experimentally), in terms of the periods of the slippage-center structure over time (t_0) and along the coordinate L:

$$\langle E\rangle = \frac{\varphi_0 n}{c t_0 L}. \tag{4.8}$$

5. PHENOMENOLOGICAL THEORIES OF THE RESISTIVE STATE

a) Fluctuational excitation of phase slippage centers

Let us take a brief look at the basic qualitative theories which have been proposed for describing the resistive state.

When the current flowing through the sample is small (and the exciting force is small) the primary mechanism creating phase slippage centers must be thermodynamic fluctuations in the system. The probability for such fluctuations is proportional to $\exp(-\delta F/T)$, so that they occur only in the immediate vicinity of T_c, where the activation energy is low: $\delta F \sim (F_n - F_s)\xi(T)s_0 \propto (T_c - T)^{3/2}$. We now have experimental proof that thermodynamic fluctuations are indeed responsible for the appearance of a resistive state at temperatures only very slightly different from the critical temperature,[44–46] $1-(T/T_c) \lesssim 10^{-4}$. This effect was first discussed by Little.[47] A more systematic theory was derived by Langer and Ambegaokar[42] and developed further by Halperin *et al.*[48,49] Here we will outline the basic results of Langer and Ambegaokar's theory.[42]

As was shown above, the rate at which the phase slips, i.e., the frequency at which phase slippage centers are formed, is related to the voltage across the part of the superconducting channel where the phase slippage center is formed by relation (4.8) or, equivalently, (4.4). In order to find the resistance of a superconducting channel we must thus relate the frequency of the fluctuational formation of phase slippage centers to the current through the sample. In the theory of Langer and Ambegaokar this problem is solved in the following way.

We consider a channel of length L_0, and we assume for definiteness that the order parameter satisfies cyclic boundary conditions at the ends of this channel; in other words, we are assuming that the spiral in Fig. 11 has an integer number of loops. We assume that the wave function of the superconducting electrons, which is proportional to the order parameter, can be written as follows in the homogeneous current state:

$$\psi_k = \Delta_k \exp(ikx), \tag{5.1}$$

where $k = 2\pi n/L_0$. The corresponding current density

$$j = \frac{e\hbar}{2im}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

is equal to $j_k = (e\hbar/m)k\Delta_k^2$. The free energy $[a = a_0(T_c - T)]$

$$F = \int d^3r \left[\frac{\hbar^2}{2m}|\nabla\psi|^2 - a|\psi|^2 + \frac{b}{2}|\psi|^4\right] \tag{5.2}$$

in current state (5.1) is

$$F\{\psi_k\} = s_0L_0\left[\left(\frac{\hbar^2}{2m}k^2 - a\right)\Delta_k^2 + \frac{b}{2}\Delta_k^4\right], \tag{5.3}$$

where s_0 is the cross-sectional area of the channel. It is easy to calculate the extent to which the free energy of a sample in state (5.1) with n loops differs from that of a sample in a state with $n-1$ loops. This difference is

$$\delta F_1 = \frac{\partial F}{\partial k}\frac{2\pi}{L_0} = \frac{2\pi\hbar s_0}{e}j.$$

The frequency of the transitions $k \rightleftarrows k - 2\pi/L_0$ between states is thus proportional to the quantity

$$\gamma = \Omega(T)\exp\left(-\frac{\delta F_0}{T} \pm \frac{\delta F_1}{2T}\right). \tag{5.4}$$

In the theory of Langer and Ambegaokar the coefficient of the exponential function, $\Omega(T)$, is written as

$$\Omega(T) = \frac{2\pi N_e}{\tau},$$

where $N_e = S_0L_0n_e$ is the number of electrons in the sample (n_e is the number density of electrons), and τ is a characteristic time of the microscopic processes; this time is left undetermined in this theory. The quantity δF_0 is the energy barrier between the states ψ_k and $\psi_{k-(2\pi/L_0)}$. Finding the probability flux of the system from state ψ_k to state $\psi_{k-2\pi/L'}$ and equating it to $2eV/\hbar$ in accordance with (4.4), we find

$$\frac{2eV}{\hbar} = \frac{4\pi n_e s_0 L_0}{\tau}\,\mathrm{sh}\left(\frac{\delta F_1}{2T}\right)\exp\left(-\frac{\delta F_0}{T}\right).$$

Langer and Ambegaokar suggested that the time τ in this expression is the same as the relaxation time which appears in the expression for the conductivity of a normal metal, $\sigma_n = ne^2\tau/m$. If we adopt this suggestion, we can easily derive the resistance of the sample in the limit $j \to 0$. Using (5.4), we find

$$\frac{\rho_s}{\rho_n} = \frac{2\pi^2\hbar^2n^2s_0^2}{mT}\exp\left(-\frac{\delta F_0}{T}\right). \tag{5.5}$$

To complete the calculation of the resistance in (5.5) we must find the energy barrier separating states which differ in that the phase differences at the ends of the channel in these states differ by 2π. The energy barrier is a saddle point of the functional (5.2) in the space of the functions $\psi(x)$. The function $\psi_0(x)$, which corresponds to the "saddle-point solution," must therefore satisfy the condition $\delta F/\delta\psi = 0$; i.e., it must be a solution of the Ginzburg-Landau equation,

$$-\frac{\partial^2\psi}{\partial x^2} - \psi + |\psi|^2\psi = 0, \tag{5.6}$$

written in terms of the dimensionless units of the Ginzburg-Landau theory [x is expressed in units of $\xi(T)$, and ψ is expressed in units of $\psi_{GL} = \sqrt{a/b}$]. In terms of these units the current density is

$$j = \frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right). \tag{5.7}$$

It is convenient to separate the modulus and phase of the order parameter $\psi = \Delta e^{i\chi}$ in these equations; i.e., it is convenient to transform to the gauge-invariant variables $\Delta = |\psi|$ and $Q = -\partial\chi/\partial x$ (here $A = 0$). Equations (5.6) and (5.7) become

$$\frac{\partial^2\Delta}{\partial x^2} + (1 - \Delta^2 - Q^2)\Delta = 0, \qquad j = -\Delta^2 Q. \tag{5.8}$$

By virtue of electrical neutrality, the current density is constant along the sample:

$$\frac{\partial j}{\partial x} = 0. \tag{5.9}$$

Equations (5.8) and (5.9) have the first integral

$$\left(\frac{\partial\Delta}{\partial x}\right)^2 + \Delta^2 - \frac{\Delta^4}{2} + \frac{j^2}{\Delta^2} = \text{const.} \tag{5.10}$$

The structure of the solution $\Delta(x)$ can be visualized quite

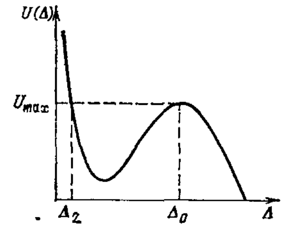

FIG. 13. Sketch of the potential $u(\Delta)$ for Eq. (3.10).

easily by making use of a mechanical analogy: the motion of a particle in the potential

$$u(\Delta)=\Delta^2+\frac{j^2}{\Delta^2}-\frac{\Delta^4}{2}. \quad (5.11)$$

Here Δ is playing the role of the coordinate of the particle, while x is playing the role of the time. The potential u is shown in Fig. 13. Bounded solutions are possible for Δ only if the value of the constant on the right side of (5.10) is less than or equal to $u_{max}(j)$ (Fig. 13). In this case the solution $\Delta(x)$ is a function which is periodic in x and which can be described implicitly by

$$x=\int_{\Delta_2}^{\Delta}\frac{d\Delta}{\sqrt{C-u(\Delta)}}, \quad (5.12)$$

where the origin is put at the point $\Delta=\Delta_2$. We need to construct a solution which asymptotically becomes a homogeneous state $\Delta=\Delta_0$ with the given current j as $x\to\pm\infty$. We obtain such a solution by choosing $C=u_{max}$; in this case the period becomes infinite, and we have

$$x=\int_{\Delta_2}^{\Delta}\frac{d\Delta}{\sqrt{u_{max}-u(\Delta)}}. \quad (5.13)$$

The behavior of this solution is sketched in Fig. 14. The value of Δ_0 for a homogeneous current state is related to j by

$$j=\Delta_0^2\sqrt{1-\Delta_0^2}. \quad (5.14)$$

Equation (5.14) has a solution only if $j\leqslant j_c$; for $j>j_c$, no homogeneous current state is possible.

Solution (5.13), shown in Fig. 14, is a function which corresponds to the maximum free energy $F\{\psi\}$ along that path in the space of the functions ψ which connects two states which lie before and after the formation of a phase slippage center, i.e., before and after the slippage of the loop in Fig. 11; this path is along states with the lowest possible free energy.

We are interested here in small currents, $j\to 0$. In this case, the solution of (5.13) becomes

$$\Delta=\operatorname{th}\frac{|x|}{\sqrt{2}}. \quad (5.15)$$

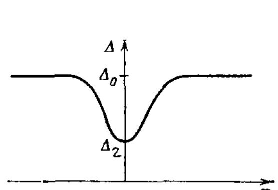

FIG. 14. The Langer-Ambegaokar solution (3.13) for Eq. (3.10).

This solution crosses zero at $x=0$ and corresponds directly to the time at which the phase slippage center is formed.

Substituting (5.15) into (5.2) we can now easily calculate the height of the barrier:

$$\delta F_0=\frac{8\sqrt{2}}{3}\frac{a_0^{3/2}}{2b}s_0(T_c-T)^{3/2}. \quad (5.16)$$

Expressions (5.5) and (5.16) constitute the basic result of the Langer-Ambegaokar theory.

It should be noted, however, that estimates of the coefficient of the exponential function, $\Omega(T)$, in expression (5.4) for the fluctuation probability in the Langer-Ambegaokar theory may raise serious doubt. An exact calculation would require a direct analysis of the dynamics of the transition of the system through the potential barrier during the formation of the slippage center. This problem has been solved by McCumber and Halperin[48] on the basis of the time-dependent Ginzburg-Landau equations supplemented with Langevin random forces to describe the fluctuations. According to the results of Ref. 48, the coefficient $\Omega(T)$ in expression (5.4) is

$$\Omega(T)=\frac{N(T)}{\tau(T)},$$

where $\tau(T)=\pi\hbar/8(T_c-T)$ is the characteristic relaxation time of the modulus of the order parameter in the time-dependent Ginzburg-Landau theory, and $N(\tau)$ is the effective number of statistically independent subsystems over the length of the sample, which is proportional to the ratio L_0/ξ, where L_0 is the length of the channel. For small currents we find

$$\Omega(T)=\frac{\sqrt{3}L_0}{\xi\cdot 2\pi^{3/2}\tau(T)}\sqrt{\frac{\delta F}{T}}.$$

Using this expression, we find that the coefficient of the exponential function in McCumber and Halperin's theory is about ten orders of magnitude smaller than the result derived by Langer and Ambegaokar.

b) Spontaneous formation of phase slippage centers

As was mentioned earlier, the fluctuational mechanism for the formation of phase slippage centers is important only in a very small neighborhood of T_c: $1-(T/T_c)\lesssim 10^{-4}$. Outside this especially narrow interval the probability for fluctuations is exceedingly small. If, in addition, the current through the sample is small, then there will be no mechanism to form phase slippage centers, and the voltage across the superconducting channel will be zero. As the current is raised above a certain value (which depends on the length of the sample), the conditions required for the spontaneous excitation of phase slippage centers become satisfied in the system (the system enters the limiting-cycle attraction region), and initially one phase slippage center forms in the sample. This event is seen on the voltage-current characteristic as a voltage jump (Fig. 8a). With a further increase in the current in the sample, two, three, etc., centers can form, and their formation will be accompaied by corresponding voltage jumps on the voltage-current characteristic. This is the situation in a sample of finite length. In an infinitely long sample, the total number of slippage centers is always large, and the density of these centers increases smoothly with increasing current. The corresponding voltage-current characteristic is a smooth curve (Fig. 8b).

It is now clear that it will be exceedingly difficult, if possible at all, to derive an exact solution for this nonlinear problem. We will thus first consider some qualitative theories which describe the properties of phase slippage centers.

A crude picture of the excitation of a slippage center has the potential difference causing an increase in the velocity of the superconducting condensate. At a certain time, and at a certain place in the sample, this velocity reaches a value such that the superconducting state becomes unstable, so that the order parameter vanishes at this place. When Δ vanishes, the phase coherence is disrupted. Since the sample is below the critical temperature, the formation of a superconducting condensate begins again around this point. The superconductivity is restored with a different phase, however, so that the difference between the phases on the right and left of the centers differs by 2π from the phase difference in the original state. After a certain time, determined by (4.4), the process repeats itself.

The formation of phase slippage centers has been simulated numerically by Rieger *et al.*[50] They took the time in which a center is formed to be the time in which the free energy of a region of the sample, as it is increasing because of the acceleration of Cooper pairs by the electric field, becomes higher than the free energy of the state which would prevail in this region if the phase difference between its ends were 2π smaller. This condition is extremely artificial, of course, and the procedure of Ref. 50 is good only as a first step in solving the problem of the excitation of phase slippage centers.

Let us take a qualitative look at the establishment of the voltage across a phase slippage center. We will be following the model of Skocpol, Beasley, and Tinkham.[26]

We write the total density of the current flowing through the superconducting channel as the sum of a superconducting part j_s and a normal part $j_n = \sigma E$:

$$j=\sigma E+j_s=-\sigma\left(\frac{\partial\Phi}{\partial x}+\frac{1}{c}\frac{\partial Q}{\partial t}\right)+j_s. \qquad (5.17)$$

All quantities on the right side of (5.17) depend on the time, changing periodically with a period $t_0 = 2\pi/\omega_J$, where ω_J is the Josephson frequency. We are interested in the time average of the voltage, which is the quantity which would actually be measured in a real experiment. Taking the average of (5.17) over time, we see that the term with $\partial Q/\partial t$ vanishes because of the periodicity. We find

$$j=-\sigma\frac{\partial\overline{\Phi}}{\partial x}+\overline{j}_s. \qquad (5.18)$$

In the Skocpol-Beasley-Tinkham model the equation for the time average of the potential Φ is written in the simple form [cf. Eqs. (2.6) and (2.7)]

$$\frac{\partial^2\Phi}{\partial x^2}=\frac{1}{l_E^2}\Phi, \qquad (5.19)$$

where l_E is assumed independent of x. In this case, the solution of (5.19) is

$$\overline{\Phi}=c\,\mathrm{sh}\left(\frac{x-x_1}{l_E}\right),$$

where c is a constant, and x_1 is the point at which $\overline{\Phi}=0$. We assume for definiteness that the ends of the superconducting channel are connected to massive superconducting "banks," which are at equilibrium, so that $x = x_1$ corresponds to the end of the channel. Using (5.19) we find

$$j-\overline{j_s(x)}=-\frac{\sigma}{l_E}c\,\mathrm{ch}\left(\frac{x-x_1}{l_E}\right).$$

We assume that the sample has a finite length and that there is only a single phase slippage center in the channel, at the point $x = x_0$. The change in the potential between x_1 and the slippage center is

$$\overline{\Phi}_1=c\,\mathrm{sh}\,\frac{x_0-x_1}{l_E}.$$

Adding the potential change $\overline{\Phi}_2$ between x_0 and the other point (x_2) at which $\Phi = 0$ (the other end of the channel), we find

$$V=\overline{\Phi}_1+\overline{\Phi}_2=\frac{l_E}{\sigma}\left(\mathrm{th}\,\frac{x_0-x_1}{l_E}+\mathrm{th}\,\frac{x_2-x_1}{l_E}\right)(j-\overline{j}_s(x)). \qquad (5.20)$$

If $|x_0 - x_1|, |x_2 - x_0| \gg l_E$, then

$$\overline{V}=\frac{2l_E}{\sigma}(j-\overline{j}_s(x_0)),$$

and the differential resistance for a single slippage center is $\rho_d = 2l_E/\sigma$.

The quantity $\overline{V}$ is the difference between the time averages of the chemical potentials of the Cooper pairs to the right and left of the slippage center. The spatial potential distribution near a phase slippage center predicted by the Skocpol-Beasley-Tinkham model has been confirmed in a brilliant experiment by Dolan and Jackel.[51]

If the superconducting channel has no inhomogeneities, then the first slippage center will form in the middle, so we find from (5.20)

$$\overline{\Phi}=-(j-\overline{j}_s(x_0))\frac{l_E}{\sigma}\frac{\mathrm{sh}\,[(x-x_1)/l_E]}{\mathrm{ch}\,(L_0/2l_E)},$$

where $L_0 = x_2 - x_1$ is the length of the channel. It is now a simple matter to derive an expression for the superconducting current:

$$j_s=j+\sigma\frac{\partial\overline{\Phi}}{\partial x}=j-[j-\overline{j}_s(x_0)]\frac{\mathrm{ch}\,[(x-x_1)/l_E]}{\mathrm{ch}\,(L_0/2l_E)}.$$

The superconducting current j_s reaches a maximum at the ends of the channel (which are connected to the superconducting banks), given by

$$j_{s,\,\max}=j-\frac{j-\overline{j}_s(x_0)}{\mathrm{ch}\,(L/2l_E)}.$$

Clearly, this current cannot exceed the Ginzburg-Landau critical current j_c. On the other hand, as the total current j is increased the condition $j_{s,\max} \leqslant j_c$ will eventually be violated at some current. In this case, a solution with a single slippage center in the channel becomes impossible, and one more slippage center must form. Correspondingly, there will be a voltage jump on the voltage-current characteristic. With a further increase in the current, three, etc., slippage centers form and are seen as corresponding voltage jumps on the voltage-current characteristic. The Skocpol-Beasley-Tinkham model thus also furnishes a qualitative explanation for the voltage jumps on the voltage-current characteristic.[52]

We will return to this topic for a more detailed discussion in Section 7, where we will use equations from the microscopic theory.

The simple physical arguments above give us a key to an understanding of the nature of the resistive state. It is important to note, however, that these arguments should be in agreement with the known dynamic properties of superconductivity, which have been established on the basis of the microscopic theory. This question is the subject of the following sections of this review.

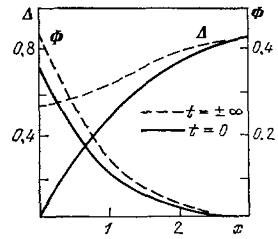

FIG. 15. Profiles of Δ and of the potential μ_e along the coordinate at various times according to the results of numerical calculations[53] for the equations of the time-dependent Ginzburg-Landau theory. $u = 5.79$, $j = j_{\min} = 0.326$.

6. RESULTS OF NUMERICAL SOLUTIONS OF THE DYNAMIC EQUATIONS FOR THE RESISTIVE STATE

As we have already stated, the phase-slippage processes are inherently very nonlinear, so that it is an extremely complicated matter to derive a mathematically exact analytic solution of even the comparatively simple dynamic equations (2.1)–(2.4); no solution has been found so far. Numerical methods can be of considerable assistance in reaching an understanding of the phenomenon.

The first results by this approach were obtained by Kramer and Baratoff.[53] They studied the time-dependent Ginzburg-Landau equations, which apply to gap-free superconductivity, which corresponds to the limit $\Gamma \gg 1$ in terms of Eqs. (2.18)–(2.19). In this limit, the gap-free situation is provided by an electron-phonon interaction:

$$-u\left(\frac{\partial}{\partial t}+i\varphi\right)\psi+\frac{\partial^2\psi}{\partial x^2}+(1-|\psi|^2)\psi=0\,, \qquad (6.1)$$

$$j=-\frac{\partial\varphi}{\partial x}+\frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right). \qquad (6.2)$$

There are corresponding equations for gap-free superconductors with a high concentration of magnetic impurities.[54] In the latter case it is necessary to set $u = 12$.

Kramer and Baratoff studied Eqs. (6.1)–(6.2) for the two values $u = 5.79$ and $u = 12$ and obtained the following results.

1. At a current j below a certain $j_{\min}$ ($j_{\min} = 0.326$ for $u = 5.79$ and $j_{\min} = 0.284$ for $u = 12$), perturbations against the background of the purely superconducting state die out, and the superconductor returns to a homogeneous current state with an order parameter satisfying the condition $j = \Delta^2\sqrt{1-\Delta^2}$. We note that $j_{\min} < j_c$.

2. At a current $j > j_2$ ($j_2 = 0.335$ for $u = 5.79$ and $j_2 = 0.291$ for $u = 12$) the superconducting current state decays, converting into an expanding normal domain. At $j > j_2$ the superconducting state is thus unstable. This condition is the same as the stability condition for the interface between superconducting and normal phases in the current state studied by Likharev and Yakobson.[21,22]

3. In the current interval between $j_{\min}$ and j_2, there is a solution which corresponds to phase slippage. This solution is constructed in the following way: Over most of its length the channel remains superconducting, but in a certain region there are local oscillations of the modulus of the order parameter. At the time when Δ vanishes, the phase jumps by 2π. As $j \to j_{\min}$ the oscillation period tends toward infinity, and at $t \to \pm\infty$ the solution asymptotically becomes the solution of Langer and Ambegaokar,[42] (5.12). As the current is increased, the amplitude of the oscillations decreases. Figure 15 shows Kramer and Baratoff's solution for $u = 5.79$ and $j = j_{\min}$.

These results constitute the first direct evidence that there exists a solution with phase slippage, and they demonstrate that the system (6.1)–(6.2) has a limiting cycle which gives rise to oscillations of the appropriate type.

The system (6.1)–(6.2) was studied later by Ivlev *et al.*[16,55] They used the equations of gap-free superconductivity to simulate the processes which occur in gap superconductors. The basic idea of their approach can be summarized as follows: In the gap situation, which corresponds to small values $\Gamma \ll 1$, the electric-field penetration depth is large, $l_E \sim \Gamma^{-1/2} \gg 1$. In terms of Eqs. (6.1) and (6.2) this circumstance can be modeled by assigning the parameter u small values: $u \ll 1$. The very simple equations (6.1)–(6.2) can thus describe that property of real superconductors which is the most important from the standpoint of the resistive state: the large electric-field penetration depth.

System (6.1)–(6.2) was solved numerically for the value $u = 0.01$ in Refs. 16 and 55. The interval $0 < x < L$, where $L = 40$, was selected, and at the boundaries of this interval the conditions $\partial\Delta^2/\partial x = 0$ and $\Phi = \varphi + (\partial\chi/\partial t) = 0$ were imposed. These conditions follow from the periodicity of the structure along x and from the symmetry of the problem with respect to the points $x = 0$ and L; the even parity of Δ and j_s and the odd parity of Φ are taken into account. The length L is thus the distance between adjacent phase-slippage centers, which form a periodic structure along the sample. Figure 16 shows a solution which describes oscillations of the modulus of the order parameter for $j = 0.4$. Curves 1–3 correspond to the times $t_1 = 0$, $t_2 = 0.053$, and $t_3 = 2.188$; the oscillation period is $t_0 = 2.52$.

An extremely important point to be noted in these re-

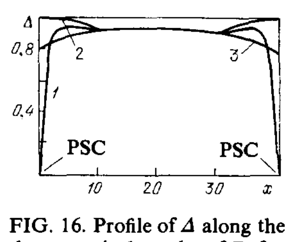

FIG. 16. Profile of Δ along the coordinate at various times according to the numerical results of Refs. 16 and 55 for the equations of the time-dependent Ginzburg-Landau theory. $u = 5.79$, $j = 0.4$.

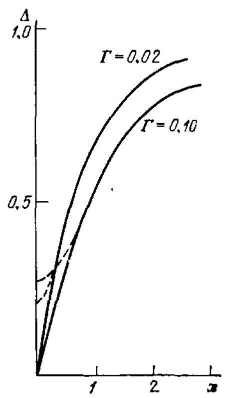

FIG. 17. Profile of Δ along the coordinate for various values of the parameter Γ. The solid curves correspond to a minimum of Δ ($x = 0$), while the dashed curves correspond to a maximum of Δ ($x = 0$). These are results of a numerical solution[4] of Eqs. (4.9)–(4.12).

sults is that the oscillations of the order parameter are confined exclusively to a rather small neighborhood of the phase-slippage centers, while Δ, j_s, and Φ remain essentially constant in time over most of the distance between two centers. The reasons for this behavior will be discussed in the following section.

Kramer and Watts-Tobin[3] and Watts-Tobin *et al.*[4] pursued the numerical studies directly for Eqs. (2.18)–(2.19) for various values of the parameter Γ. They found that the current interval in which a phase-slippage solution exists expands with decreasing value of the parameter Γ; the current j_2 increases in proportion to Γ^{-1} at small values $\Gamma \ll 1$. The oscillations of the order parameter are of qualitatively the same nature as in the case of the simple time-dependent Ginzburg-Landau equation. Figure 17 reproduces the results of Ref. 4 for Δ with $L = 6$ and a current $j = 0.4$ for several values of Γ. Watts-Tobin *et al.*[4] pointed out that with decreasing Γ the $\Delta(x,t)$ curves approach that corresponding to a static dependence of the order parameter near a boundary with a normal phase, i.e., the curve which satisfies the boundary condition $\Delta(x = 0) = 0$. This result is explained in Subsection 7a.

These numerical results are extremely important. In the first place, they establish that solutions corresponding to phase slippage do in fact exist. They thus confirm the basic concepts on which our understanding of the nature of the resistive state rests. Because of their particular nature, however, numerical methods cannot give us a comprehensive description of the properties of the resistive state as functions of various parameters of the sample and of the conditions in an actual experiment. It would thus be extremely important to attempt to derive (where possible) an analytic solution of the corresponding equations. The following sections of this review describe recent progress in this direction in the theory of the resistive state.

7. MICROSCOPIC THEORY OF THE RESISTIVE STATE

We turn now to a description of a more rigorous theory of the resistive state, based on an analysis of the time-varying microscopic equations of superconductivity.

a) Structure of phase slippage centers

Let us consider the dynamic equations of superconductivity, (2.13)–(2.16), in the case of a gap superconductor, $\Gamma \ll 1$, which is the most interesting case both experimentally and theoretically. This case corresponds to temperatures $(\hbar/\tau_{ph} T_c)^2 \ll 1 - (T/T_c)$. We recall that in this case the electric-field penetration depth is large, $l_E/\xi \sim \Gamma^{-1/2}$. The discussion below is based on the results of Refs. 16, 55, and 56.

We recall in this connection that according to quantization rule (4.7) the electric field averaged over time and the coordinate is related to the periods of the structure of phase slippage centers. In the dimensionless units of Section 2, the relationship is

$$\langle E\rangle = \frac{2\pi n}{L t_0}. \tag{7.1}$$

The integer n is the multiple of 2π by which the phase slips at the time when a slippage center forms. We will set $n = 1$ here, working by analogy with the ordinary vortices in type II superconductors.

Considering a homogeneous sample (without defects), we clearly see that the distance between phase slippage centers is determined by the relaxation rate of the potential Φ, i.e., by the relaxation rate of that deviation from equilibrium between the chemical potential of the pairs, μ_p, and the chemical potential of the quasiparticles μ_e, which is created by the phase slippage. It follows that this distance must be of the order of the electric-field penetration depth l_E. If the current through the sample is not very large, i.e., if $j \sim \sigma E \sim j_s$ ($j \sim 1$ in our units), then the period of the oscillations in time, t_0, will be of order $t_0 \sim L^{-1} \sim \Gamma^{1/2}$, according to (7.1).

At sufficiently large distances from a slippage center, where $\Delta \sim 1$, we can write Eqs. (2.14) and (2.15) as

$$j = -\frac{\partial Q}{\partial t} + \frac{1}{u\Gamma}\frac{\partial}{\partial x}\left[\frac{1}{\Delta}\frac{\partial(\Delta^2 Q)}{\partial x}\right] - \Delta^2 Q. \tag{7.2}$$

Let us estimate the terms on the right side of (7.2). The first is of order $Q_\sim t_0^{-1} \sim Q_\sim \Gamma^{-1/2}$, where $Q_\sim$ is the variable part of Q: $Q = \bar{Q} + Q_\sim$. The superior bar denotes the time average. The second and third terms are of order Q at distances of order l_E. It follows that the variable part $Q_\sim$ must be small, $Q_\sim \Gamma^{1/2}\bar{Q}$, so that we have $Q \approx \bar{Q}$. The variable part Q is comparable in magnitude to $\bar{Q}$ only at very short distances from a slippage center, where the first and second terms in (7.2) are comparable in magnitude. Clearly, this situation occurs at distances of order $x_1 \sim \Gamma^{-1/4} \ll L$ from a slippage center. In ordinary units, we would have $x_1 = (\xi l_E)^{1/2}$. We wish to emphasize that the oscillations of the parameter over these distances are still small, as can be seen by evaluating the corresponding terms in Eq. (2.13). At distances $x \gtrsim 1$ we find $(\Delta/\Gamma)\partial\Delta/\partial t \sim \Delta$ from (2.13), from which it follows that $\Delta_\sim \Gamma t_0 \sim \Gamma^{3/2}$. At $\Delta \sim 1$, this oscillation amplitude is insufficient for the vanishing of Δ at certain times. It is thus clear that near a slippage center the order parameter must be suppressed to an extremely small value, $\Delta_2 \ll 1$. This region of very small values of Δ determines the characteristic dimension of the slippage center proper, i.e., the dimension of that part of the sample in which the order parameter oscillates markedly. We denote by x_2 the dimension of this region.

Since we have $\Delta_2 \sim (\partial\Delta/\partial x) x_2$ and $\partial\Delta/\partial x \sim 1$ in this region, we conclude that $x_2 \sim \Delta_2 \sim 1$.

We first consider the problem at a large distance from a slippage center, $x \gg x_1$, where all the quantities are essentially independent of the time. In this case, the system (2.13)–(2.15) can be written

$$\frac{\partial^2\Delta}{\partial x^2} + \left(1 - \Delta^2 - \frac{\overline{j_s^2}}{\Delta^4}\right)\Delta = 0, \qquad (7.3)$$

$$j = -\frac{\partial\Phi}{\partial x} + j_s, \qquad (7.4)$$

$$u\Gamma\Delta\Phi = \frac{\partial j_s}{\partial x}, \qquad (7.5)$$

where an average has been taken over time and where we have introduced the superconducting current $j_s = -\Delta^2 Q$. At distances $x \gg 1$ we can omit the term with $\partial^2\Delta/\partial x^2$ in (7.3), so that

$$j_s = \Delta^2\sqrt{1-\Delta^2}. \qquad (7.6)$$

The system (7.4)–(7.6) can be integrated easily:

$$\Phi^2 = \frac{2}{u\Gamma}\int_{j_s}^{j_{s0}} dj_s \frac{j - j_s}{\Delta(j_s)} = \frac{2}{u\Gamma}\int_{\Delta_0}^{\Delta} d\Delta \frac{(j - \Delta^2\sqrt{1-\Delta^2})(3\Delta^2 - 2)}{\sqrt{1-\Delta^2}}, \qquad (7.7)$$

$$\frac{L}{2} - x = \frac{1}{u\Gamma}\int_{j_s}^{j_{s0}} \frac{dj_s}{\Delta(j_s)\Phi(j_s)} = \frac{1}{u\Gamma}\int_{\Delta_0}^{\Delta} d\Delta \frac{3\Delta^2 - 2}{\sqrt{1-\Delta^2}\,\Phi(\Delta)}; \qquad (7.8)$$

here j_{s_0} and Δ_0 are the values of j_s and Δ halfway between slippage centers, i.e., at $x = L/2$. The integration limits have been chosen on the basis that, by virtue of the symmetry of the structure with the point $x = L/2$, the potential Φ, which is odd, vanishes at this point. Consequently, j_{s_0} is the maximum value of j_s, reached halfway between centers. In the integration in (7.7) and (7.8) we should choose that branch of the $\Delta(j_s)$ dependence from (7.6) which is thermodynamically stable, i.e., the branch with $0 < j_s < j_c$, $1 > \Delta > \sqrt{2/3}$ (Fig. 18).

We turn now to distances $x \ll l_E \sim \Gamma^{-1/2}$, and we seek the value of $\partial j_s/\partial x$. From (7.5) we have $\partial j_s/\partial x \sim \Gamma\Phi$. Since $\Phi = \pi t_0^{-1} \sim \Gamma^{-1/2}$, we can write

$$\frac{\partial j_s}{\partial x} \sim \Gamma^{1/2}. \qquad (7.9)$$

Since the derivatives $\partial j_s/\partial x$ is small, we may assume that j_s is independent of x at $x \ll \Gamma^{-1/2}$. We average (2.13) over the time and write it in the region where Δ is static, i.e., at $x \gg x_2$. The derivative $\partial\Delta/\partial t$ drops out when the average is taken, and we find

$$\frac{\partial^2\Delta}{\partial x^2} + \left(1 - \Delta^2 - \frac{\overline{j_s^2}}{\Delta^4}\right)\Delta = 0. \qquad (7.10)$$

By virtue of condition (7.9), the solution of Eq. (7.10) can be

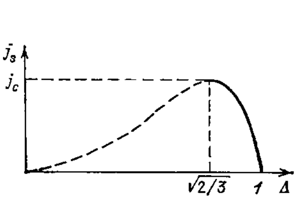

FIG. 18. The function $j_s(\Delta)$. Solid curve—thermodynamically stable branch; dashed curve—unstable branch.

found by the same approach which was taken to derive the solution of Langer and Ambegaokar[42] (Subsection 5a); the result is (5.12), where $u(\Delta) = \Delta^2 - (\Delta^4/2) + (\overline{j_s^2}/\Delta^2)$ (Figs. 13 and 14). This solution must be joined at $x \gg 1$ with solution (7.8), which contains the small derivative $\partial\Delta/\partial x \ll 1$. A solution (5.12) which has a vanishing derivative at $x \gg 1$ is found by choosing $C = u_{max}$. The solution we need is thus of the form in (5.13), where $x = 0$ corresponds to some minimum value of Δ_2. As was found above, as we approach a phase slippage center in the static region along x the order parameter must decrease to values far smaller than unity, which correspond to $\Delta_2 \ll 1$. A small value of Δ is possible only if $\overline{j_s^2} \ll 1$. It is clear from the form of the potential $u(\Delta)$ that in this case we have $\Delta_2 = \sqrt{2\overline{j_s^2}}$. In this case, (5.13) reduces to

$$\Delta = \text{th}\,\frac{|x|}{\sqrt{2}} \qquad (7.11)$$

in the region with $\Delta \gg \sqrt{\overline{j_s^2}}$.

The quantity Δ_2 is of the order of the average value of Δ in the oscillation region. In this region we evidently have $\Delta \sim j_s$, so that it is a simple matter to estimate Δ from (2.13). Since $Q \sim j_s/\Delta^2 \sim \Delta^{-1}$, we have $(\Delta/\Gamma)\partial\Delta/\partial t \sim \Delta^{-1}$, and we conclude that in the oscillation region we have $\Delta \sim (\Gamma t_0)^{1/3} \sim \Gamma^{1/2}$. For the width of the oscillation region we find $x_2 \sim \Gamma^{1/2}$ or $x_2 = \xi\Gamma^{1/2}$ in ordinary units.

It is interesting to examine the behavior of the potential Φ. Its time average, $\bar{\Phi}$, is essentially independent of x at $x \ll l_E \sim \Gamma^{-1/2}$, as follows from Eq. (2.16), averaged over the time:

$$\frac{\partial^2\bar{\Phi}}{\partial x^2} = u\Gamma\Delta\bar{\Phi}.$$

We thus see that the change $(\delta\bar{\Phi})$ in the potential over distances $x \ll \Gamma^{-1/2}$ is small: $\delta\bar{\Phi} \ll \bar{\Phi}$. Consequently, in the limit $x \to 0$ the potential Φ remains finite: $\Phi(x \to +0) = \Phi_0$, where Φ_0 is determined from (7.7), which holds at $x \gg x_1$. On the other hand, by virtue of the symmetry of the problem with respect to the point $x = 0$ and the odd parity of Φ, the potential $\Phi(x = 0)$ must vanish at all times except at times of phase slippage, when $\Phi(x = 0)$ becomes infinite. It is easy to find from Eq. (2.15) that the growth of Φ from zero to a value of order Φ_0 occurs over distances of order x_2.

We have thus found the following results.

1. In a neighborhood of a phase slippage center with a width of order $\xi\Gamma^{1/2}$, all quantities undergo large oscillations. When Δ vanishes, the phase χ jumps by 2π, while $\Phi(x = 0)$ becomes infinite. The oscillation amplitude Δ is of the order of $\Delta_{GL}\Gamma^{1/2}$.

2. The oscillations of Δ die out rapidly with distance from a phase slippage center, and at $x \gg x_2 = \xi\Gamma^{1/2}$ the order parameter is essentially independent of the time. Its behavior in this region is described by (7.11). At $x \gg \xi$ the order parameter reaches an equilibrium value ($\Delta = 1$ in our units). The superconducting current oscillates but remains small, so that the entire current is carried by normal excitations.

3. At distances $x_1 \ll x \lesssim l_E$ the oscillations in all the quantities are negligibly small. In this region there is a relaxation of the deviation of Φ from equilibrium which is created in the vicinity of the slippage center. The behavior of Φ and

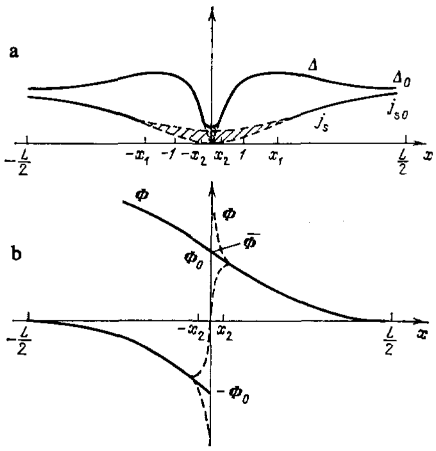

FIG. 19. Behavior of Δ and j_s (a) and of the potential Φ (b) in the static region (solid curves) and in the dynamic region (dashed curves). The hatching shows the region of values which Δ and j_s can take on in the course of the oscillations.

of the currents is determined by (7.7) and (7.8). With decreasing Φ, the normal current decreases, the superconducting current increases, and Δ decreases.

Figure 19 shows the behavior of the order parameter, the potential, and the currents. The behavior of the order parameter near a slippage center agrees well with the results found by numerical calculations.[4]

To conclude this subsection we find the range of applicability of these results along the temperature scale. The Josephson oscillation frequency ω_J is of order $(\Delta^2/T)\Gamma^{-1/2}$. Since the conditions $\Gamma \ll 1$ and $\omega_J \ll \tau_{ph}^{-1}$ must be satisfied for Eqs. (21.)–(2.3) to be applicable, the temperature must satisfy the condition $(\hbar/\tau_{ph} T_c)^2 \ll 1-(T/T_c) \ll (\hbar/\tau_{ph} T_c)^{6/5}$.

b. Voltage-current characteristic [16,56]

Taking the time average of the expression for the electric field,

$$E=-\frac{\partial Q}{\partial t}-\frac{\partial \Phi}{\partial x},$$

we find $\bar{E}=-\partial\bar{\Phi}/\partial x$. It follows that the field averaged over time and the coordinate, which we again denote by E, is

$$E=\frac{2\Phi_0}{L}, \tag{7.12}$$

where Φ_0 and L are determined by (7.7)–(7.8), with allowance for the boundary conditions established above; that for $1 \ll x \ll l_E$ the order parameter is $\Delta=1$. Using (7.8) we find the period of the structure to be

$$L=\frac{2}{u\Gamma}\int_{\Delta_0}^{1} d\Delta \frac{3\Delta^2-2}{\sqrt{1-\Delta^2}\,\Phi(\Delta)}, \tag{7.13}$$

where $\Phi(\Delta)$ is determined by (7.7). For the voltage-current characteristic we finally find[16,56]

$$E\int_{\Delta_0}^{1} d\Delta \frac{3\Delta^2-2}{(1-\Delta^2)^{1/2} f(\Delta, \Delta_0)}=2f(\Delta=1, \Delta_0), \tag{7.14}$$

where

$$f^2(\Delta, \Delta_0)=\int_{\Delta_0}^{\Delta} dx \frac{(j-x^2\sqrt{1-x^2})(3x^2-2)}{\sqrt{1-x^2}}. \tag{7.15}$$

To find the voltage-current characteristic it is thus sufficient to derive a solution of the equations for the potential only in the static region. In this regard the results of the dynamic theory of the resistive state[16,43,55,56] converge on those of the static model developed by Galaĭko *et al.*[26,35–40] Comparison of Fig. 10, which illustrates the static model, with Fig. 19 reveals that the behavior of the potential Φ is nearly identical in the two cases over almost the entire distance between phase slippage centers, except in a narrow region directly near a center. In the static model the process by which the voltage at a slippage center is formed is of the same physical nature as in the dynamic theory, and it results from a relaxation of the difference between the chemical potentials, $\Phi=(\mu_p-\mu_e)/e$, at distances large in comparison with the coherent length. We should point out, however, that when taken literally the results of the static model[26,35–40] have an important shortcoming: In integrating the equations for the potential [equations like (7.4) and (7.5)], Galaĭko *et al.* used the thermodynamically unstable branch of the $\Delta(j_s)$ dependence, which has Δ vanishing in the limit $j_s \to 0$ (Fig. 18). A solution of this type could not exist in a real physical system. The results of these papers must therefore be corrected.

An important point for the discussion below is the relationship between the length of the superconducting channel, L_0, and the distance between adjacent phase slippage centers, L. If $l_E \sim L_0$, there is a finite number of phase slippage centers in the channel, and the behavior of the voltage-current characteristic is determined by the appearance of new slippage centers as the current increases. This question will be discussed in Subsection 7d. At this point we consider the opposite limit: $l_E \ll L_0$. In this case the number of phase slippage centers in the channel is large, so we may assume that their density increases smoothly with increasing current. Equations (7.14) and (7.15) describe a family of curves determined by the adjustable parameter Δ_0 (or j_{s_0}). These curves are shown in Fig. 20. To choose the parameter Δ_0 we need to appeal to some additional physical considerations. Here we can make use of the principle of minimum entropy

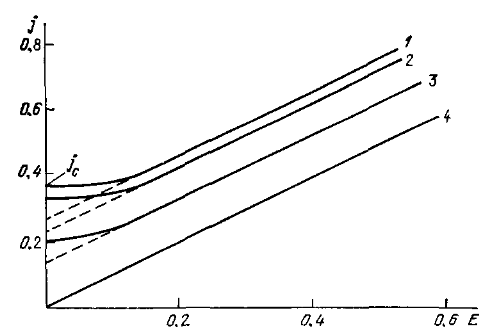

FIG. 20. Voltage-current characteristics calculated from Eqs. (5.19) and (5.20) for various values of the parameter Δ_0. Visible here are regions with an excess current $j_{exc}=aj_c$. 1—$\Delta_0=\sqrt{2/3}$, $a=0.68$; 2—$\Delta_0=0.9$, $a=0.61$; 3—$\Delta_0=0.975$, $a=0.35$; 4—normal state.

tion (minimum dissipation). According to this principle, at a given current in the system there must be a regime with a minimum possible field E, so that the dissipative function jE is minimized. Such a regime corresponds to the upper curve in Fig. 20, which in turn corresponds to the parameter value $\Delta_0 = \sqrt{2/3}$ or ($j_{s_0} = j_c$). This choice is equivalent to adopting the condition that the superconducting current must reach its maximum possible value. We now assume that we have adopted this value of Δ_0.

In order to transform to ordinary units in (7.14) and (7.15) we need to replace E by $2\sigma E/3\sqrt{3j_c}$ and to replace j by $2j/3j_c\sqrt{3}$. As a result we find

$$\frac{\sigma E}{c}\int_{\sqrt{2/3}}^{1} dx \frac{3x^2-2}{\sqrt{1-x^2}\, f(x, \sqrt{2/3})} = 3\sqrt{3}\, f\left(1, \sqrt{\frac{2}{3}}\right). \quad (7.16)$$

The function $f(x, \sqrt{2/3})$ is determined by (7.15).

On the volage-current characteristic corresponding to (7.16) we can see two characteristic regions (Fig. 20).

1) *Initial region.* $j - j_c \ll j_c$. In this case the period of the structure is logarithmically large:

$$L = 2l_E \sqrt{\frac{\Delta_{GL}}{\Delta_0}} \ln \frac{cj_c}{j-j_c},$$

where $c \sim 1$. The voltage-current characteristic is

$$j - j_c = j_c \exp\left(-\frac{0{,}91 j_c}{\sigma E}\right).$$

In the limit $E \to 0$, the characteristic has a vanishing slope, which is a consequence of the infinite length of the sample.

2) *High-current region.* $j \gg j_c$. Here the voltage-current characteristic runs parallel to an Ohm's-law line: $j = \sigma E + j_{exc}$ with an excess current $j_{exc} = aj_c$, where

$$a = \frac{3\sqrt{3}}{4} p^{-1}\left(1, \sqrt{\frac{2}{3}}\right) \int_{\sqrt{2/3}}^{1} dx \frac{x^2(3x^2-2)}{p(x, \sqrt{2/3})},$$

while

$$p^2(x, x_0) = \int_{x_0}^{x} dy \frac{3y^2-2}{\sqrt{1-y^2}}.$$

The voltage-current characteristic (7.16) was derived in Ref. 16, where the characteristic was also calculated for the more complicated case in which the relaxation of Φ is affected by the velocity of the superconducting condensate, i.e., j_s. As mentioned above, to find the voltage-current characteristic it is sufficient to solve the problem of the potential distribution in the static region alone. In this region, the equation for the potential is analogous to Eq. (2.6), but now the right side contains a factor which reflects the effect of the superconducting current[57]:

$$D\nabla^2\Phi = \frac{\pi\Delta}{4T\tau_{ph}} q(z)\Phi,$$

where

$$q(z) = 1 + \frac{2z}{\pi}\int_1^\infty dx \frac{\sqrt{x^2-1}}{x(x\sqrt{x^2-1}+z)} = \begin{cases} 1, & z \ll 1, \\ \sqrt{z}, & 1 \ll z \ll T^2/\Delta_{GL}^2 \end{cases}$$

and $z = 4De^2Q^2\tau_{ph}/\hbar^2c^2$. In the units of (2.9)–(2.11) this equation can be written in a form analogous to (7.5):

$$\frac{\partial j_s}{\partial x} = u\Gamma\Delta q(z)\Phi, \quad (7.17)$$

where now $z = 8(T_c - T)\tau_{ph}Q^2/\pi\hbar$. Equations (7.3), (7.4), and (7.17) can also be integrated easily. As a result, we find the following replacements for Eqs. (7.7) and (7.8) ($Q^2 = 1 - \Delta^2$):

$$\Phi^2 = \frac{2}{u\Gamma}\int_{\Delta_0}^{\Delta} d\Delta \frac{(j-\Delta^2\sqrt{1-\Delta^2})(3\Delta^2-2)}{\sqrt{1-\Delta^2}\, q(z)},$$

$$\frac{L}{2} - x = \frac{1}{u\Gamma}\int_{\Delta_0}^{\Delta} d\Delta \frac{3\Delta^2-2}{\sqrt{1-\Delta^2}\,\Phi(\Delta)\, q(z)}.$$

The voltage-current characteristic is determined by expression (7.12); in this case, it looks much like that discussed above, and it is qualitatively the same as in Fig. 20.

These results span a broader temperature interval:

$$\left(\frac{\hbar}{\tau_{ph}T_c}\right)^2 \ll 1 - \frac{T}{T_c} \ll \sqrt{\frac{\hbar}{\tau_{ph}T_c}}.$$

Looking back at Table I, we see that this temperature interval corresponds to temperatures at which experiments can actually be carried out. In principle, it is thus possible to compare these voltage-current characteristics with experiment both qualitatively and quantitatively (Section 8).

c) Upper boundary on the resistive state

The resistive state of narrow superconducting channels extends over a rather broad current range experimentally (Figs. 8a and 8b). As we have seen, the lower boundary of this range is of the order of the Ginzburg-Landau critical current j_c and has a temperature dependence $(T_c - T)^{3/2}$. We are also interested in the upper boundary of the resistive state, by which we mean the current at which the deviations from Ohm's law first become apparent. Strictly speaking, there must always be deviations from Ohm's law because of the fluctuational formation of the Cooper pairs, which subsequently disappear as a result of acceleration by the electric field.[17,18] However, at those high currents at which the deviations are actually observed experimentally these fluctuational corrections must still be exceedingly small. Here we can speak only in terms of a change in the structure of the entire state of the current-carrying channel and the appearance of a resistive state of this type.

The first attempt to determine the point at which the normal state of the channel converts into a resistive state was made by Galaĭko,[35,39,40] who worked using the static model of the resistive state. Galaĭko studied the appearance of nuclei of the superconducting state against the background of the normal state of the current-carrying channel. According to the results of Refs. 35, 39, 40 the resistive state appears at a current below the critical value

$$\frac{4\sigma(T_c-T)}{e}\sqrt{\frac{2\pi T_c}{7\zeta(3)\hbar D}}. \quad (7.18)$$

This result, however, suffers from the same shortcoming as does the entire static model of the resistive state: The unavoidable oscillations of the order parameter in the regions where nuclei overlap are ignored. The static equation for the modulus of the order parameter, used in deriving (7.18), is therefore violated, and the result in (7.18) does not solve the problem of determining the upper boundary for the resistive state.

As is clear from the results of Section 3, the resistive state must appear and disappear through a transition which is of the nature of a first-order phase transition. As the current is raised above a certain critical j_2', the dynamic state described in Subsection 7a should be suddenly disrupted, and the sample should go into the normal state. Let us find the upper boundary of this dynamic state.[16] According to (7.7) and (7.13), at high currents the period of the structure along the coordinate decreases and is of order $L \sim \xi\sqrt{j_c/j\Gamma}$. At a current $j \sim j_c/\Gamma$ the distance between phase slippage centers is of order ξ. The order parameter Δ must be small at phase slippage centers and must reach values of order unity halfway between centers. Clearly, a solution of this type must disappear if the distance between phase slippage centers is considerably smaller than ξ. We thus conclude that a resistive state of this type will disappear at currents above

$$j_2' \sim \frac{j_c}{\Gamma} \sim \frac{\sigma\sqrt{T}\,\tau_{ph}}{e\sqrt{D}\,\hbar^{3/2}}(T_c - T)^2. \tag{7.19}$$

This estimate of the upper critical current is of the order of the result found for the upper critical current j_2 associated with the formation of superconducting nuclei against the background of the normal state of the channel.[4] Although the functional dependences in (3.2) and (7.19) are the same, the corresponding numerical coefficients may differ. The reason, as already mentioned, is that the resistive state appears from the normal state, and the transition from the resistive state to the normal state occurs through a first-order phase transition. In this situation there can be a hysteresis, which will be manifested by different values of the currents in (3.2) and (7.19).

The heating which results from the power dissipated by the current plays an important role in the destruction of superconductivity in the resistive state. This heating intensifies with distance from the critical temperature, as the critical currents increase, and it depends strongly on the cooling conditions for the sample. Even with an ideal heat removal, which keeps the temperature of the crystal lattice of the sample equal to the temperature of the surrounding medium, however, the electron system will still be overheated with respect to phonons. As a result, there will be an effective lowering of T_c, and the superconductivity will disappear altogether at sufficiently high currents. The corresponding current can be evaluated by considering in Eq. (2.1), along with the term Δ^3, a correction to the distribution function for the overheating of electrons. In the τ approximation, this correction is proportional to $\tau_{ph} De^2E^2\partial^2 f/\partial\varepsilon^2$, where $f(\varepsilon)$ is the equilibrium distribution function. As a result, a term

$$\frac{7\zeta(3)}{2\pi^2}\,\frac{\tau_{ph} De^2E^2}{T^2}\,\Delta$$

also appears in Eq. (2.1). The critical field is found by equating it to the term $\Delta(T_c - T)/T$; as a result we find

$$j_2'' = \frac{\pi\sigma}{e}\sqrt{\frac{2T_c(T_c - T)}{7\zeta(3)\,\tau_{ph} D}}. \tag{7.20}$$

This result, derived in Refs. 39 and 40, is valid at $\hbar/T_c\tau_{ph} \ll 1 - (T/T_c)$. At higher temperatures, $1 - (T/T_c) \ll \hbar/T_c\tau_{ph}$, heating becomes less important, and the current j_2 is determined by (7.19).

We should point out that these results on j_2 are basically just estimates, and the problem of the upper critical current still awaits solution.

d) Voltage jumps on the voltage-current characteristics of superconducting channels of finite length

As was mentioned in Section 4, the voltage-current characteristics of superconducting channels which are not very long exhibit voltage jumps. These jumps are related to the appearance of new phase slippage centers in the channel; these new centers appear as the current is raised. A qualitative picture of the voltage jumps, based on the Skocpol-Beasley-Tinkham model,[26,52] was drawn in Subsection 5b. At this point we approach the subject more systematically, working from the microscopic theory. The voltage-jump problem was first solved by Bezuglyĭ *et al.*,[41] and it is their results that we will be discussing here.

Here, as in the derivation of the voltage-current characteristics of long channels in Subsection 7b, we can use the static equations since the dynamic region is narrow. We assume that a superconducting channel of length L_0 is connected to a current source by means of normal contacts (Fig. 21). At currents $j < j^{(1)}$, where $j^{(1)}$ corresponds to the appearance of the first phase slippage center, the voltage across the channel is determined by the penetration of the electric field from the normal contacts into the channel (see Refs. 10 and 58, for example). This state is shown in Fig. 21a. Near the SN interface the order parameter drops sharply from equilibrium value to zero over a distance of order ξ. With distance from this interface, Φ and j_n decrease; j_s increases; and Δ falls off over distances of order l_E. As the current is raised above $j^{(1)}$, the first phase slippage center appears at the center of the channel (Fig. 21b). This state prevails over the current

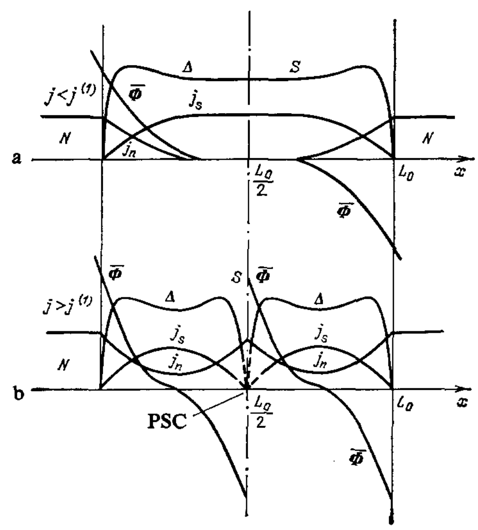

FIG. 21. Behavior of Δ and the currents in a superconducting channel with normal contacts. a—A current $j < j^{(1)}$, no phase slippage centers; b—$j^{(1)} < j < j^{2}$, with a single phase slippage center at the center of the channel.

interval $j^{(1)} < j < j^{(2)}$; then a second phase slippage center appears, etc. The conditions near the boundary with the normal metal,[58] i.e., at distances large in comparison with ξ but small in comparison with l_E, are identical to those at the boundary of the static region near a phase slippage center (Subsection 7a): $j_s = 0$, $\Delta = 1$. The voltage across the channel can thus be found by using Eq. (7.7) directly. If we assume that there are n slippage centers in the channel, the potential formed at one center is

$$\Phi_n^2(\Delta) = \frac{2}{u\Gamma} \int_{\Delta_{0n}}^{\Delta} d\Delta \frac{(j - \Delta^2 \sqrt{1-\Delta^2})(3\Delta^2 - 2)}{\sqrt{1-\Delta^2}}; \qquad (7.21)$$

here Δ_{0n} is the value of Δ halfway between neighboring phase slippage centers (or halfway between the SN boundary and the nearest center). The parameter Δ_{0n} is now related to the current and to the total length of the channel, L_0, by

$$\frac{L_0}{2(n+1)} = \frac{1}{u\Gamma} \int_{\Delta_{0n}}^{1} d\Delta \frac{3\Delta^2 - 2}{\sqrt{1-\Delta^2}\,\Phi(\Delta)}. \qquad (7.22)$$

The total voltage drop across the channel is

$$V = 2(n+1)\,\Phi_n(\Delta = 1). \qquad (7.23)$$

Expressions (7.21)–(7.23) give a complete description of the voltage-current characteristic of a channel of length L_0. These expressions are extremely cumbersome, however, so we will restrict the discussion to the case in which the current is just slightly above the critical current, $j - j_c \ll j_c$, $n \sim 1$, and we will assume $L_0 \gg \Gamma^{-1/2}$ (i.e., $L_0 \gg l_E$). From (7.21)–)7.22) we then find

$$\frac{L_0}{2(n+1)} = \frac{1}{\sqrt{u\Gamma\Delta_{0n}}} \ln \frac{c j_c}{j - j_{s0}^{(n)}}, \qquad (7.24)$$

where $c \sim 1$, and $j_{s_0}^{(n)} = \Delta_{0n}^2 \sqrt{1 - \Delta_{0n}^2}$ is the superconducting current halfway between slippage centers. If the external current j is increased at a fixed number of centers in the channel (n = const), then the current $j_{s_0}^{(n)}$ must also increase, according to (7.24). When $j_{s_0}^{(n)}$ reaches the value j_c, a solution with n slippage centers becomes impossible, and the $(n+1)$-st center appears. The current $j^{(n)}$, at which the n-th center forms—i.e., the current corresponding to the n-th voltage jump— is found from (7.24):

$$j^{(n)} - j_c = j_c \exp\left[-\frac{(u\Gamma)^{1/2} L_0}{2n}\left(\frac{2}{3}\right)^{1/4}\right], \quad n = 1, 2, \ldots \qquad (7.25)$$

The magnitude of the jump is essentially independent of n at small n and equal to the voltage $2\Phi_0$ across a single slippage center, where

$$\Phi_0^2 = \frac{2}{u\Gamma} \int_{\sqrt{2/3}}^{1} d\Delta \frac{(j_c - \Delta^2\sqrt{1-\Delta^2})(3\Delta^2 - 2)}{\sqrt{1-\Delta^2}} \approx \frac{0{,}150}{u\Gamma}. \qquad (7.26)$$

The dependence of the voltage Φ_n across one slippage center on the current between the n-th and $(n+1)$-st jumps is

$$\Phi_n^2 = \Phi_0^2 + \frac{2}{u\Gamma}(j - j_c) \int_{\sqrt{2/3}}^{1} d\Delta \frac{3\Delta^2 - 2}{\sqrt{1-\Delta^2}}. \qquad (7.27)$$

In the interval between the n-th and $(n+1)$-st jumps, the voltage-current characteristic thus has a constant slope with a differential resistance

$$\left(\frac{\partial V}{\partial j}\right)_n = \frac{2(n+1)}{u\Gamma\Phi_0} p^2\left(1, \sqrt{\frac{2}{3}}\right) \approx 2{,}06\,\frac{n+1}{u\Gamma}. \qquad (7.28)$$

This voltage-current characteristic is shown in Fig. 22.

In ordinary units, the differential resistance is

$$\left(\frac{\partial V}{\partial j}\right)_n = 2{,}06\,(n+1)\frac{l_E}{\sigma}, \qquad (7.29)$$

where l_E is determined by (2.8). There is an extremely interesting circumstance here: The numerical value of the differential resistance introduced by a single isolated phase slippage center according to the exact microscopic theory, 2.06 l_E/σ, agrees surprisingly well with the phenomenological Skocpol-Beasley-Tinkham result,[26] $2l_E/\sigma$ (Subsection 5b).

The initial part of the voltage-current characteristic, from $j = 0$ to $j = j^{(1)}$, corresponds to the superconducting state of the channel and is determined exclusively by the penetration of the field from the normal contacts. In general, this part of the voltage-current characteristic is described by the dependence derived in Ref. 58. If the channel is connected to supeconducting contacts, then no voltage will be observed in the initial region, and $n + 1$ must be replaced by n in Eqs. (7.22)–(7.28).

8. CONCLUSION

In general, the basic features of the voltage-current characteristics of narrow superconducting channels are described by the theory as it exists today. It would apparently be premature, however, to speak in terms of numerical agreement. The theoretical results have been derived under extremely restrictive conditions on the parameters, e.g., on the distance from the critical temperature, the width of the sample, and the heat-removal conditions. All these conditions can be met in practice, but only at the cost of taking special measures which have not yet been taken experimentally. Furthermore, such an important parameter of the theory as the characteristic time for the inelastic electron-phonon relaxation has not yet been derived reliably (the data in Table I are correct only in order of magnitude). This quantity must therefore be regarded as an adjustable parameter in a comparison of theory and experiment. Nevertheless,it is undoubtedly true that the theory gives a correct description of the basic qualitative features of the resistive state: primarily, the general form of the voltage-current characteristic and the voltage jumps on it. That this is true can be seen from a comparison of the experimental curves in Fig. 8 with the theoretical predictions in Figs. 20 and 22. The voltage jumps

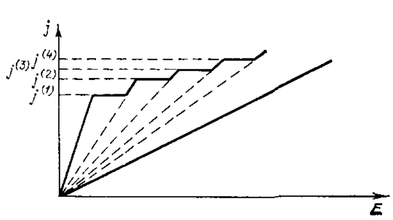

FIG. 22. Sketch of the voltage-current characteristic of a channel of finite length.

on the voltage-current characteristics are the most convenient entities for a comparison. It was shown in Ref. 27 that the differential resistance in the regions between neighboring jumps satisfies $\partial V/\partial j = R_0 N$, where R_0 is a constant and N is an integer. This result agrees with expression (7.18). From the differential resistance we can also extract information on the behavior of l_E and compare it with theoretical expressions (2.7) and (2.8). Experiments confirm the basic tendencies in the dependence of l_E on the temperature[27,60] and the magnetic field.[60] Experiments also cofirm the spatial profiles of the potentials μ_p and μ_e near phase slippage centers.[51] The time-dependent Josephson effect observed experimentally[26] is undoubtedly related to the oscillations of phase slippage centers which occur at the Josephson frequency.[59] The generation of lower frequencies,[28,30] on the other hand, has yet to find a reliable explanation. This explanation can apparently be sought in the motion of the entire structure of phase slippage centers as a whole,[28] although it is difficult to draw a definite conclusion at this point. We also lack the data required for testing the temperature dependence of the upper critical current j_2. The only experimental data on this question are from Ref. 28, where a linear dependence $j_2 \propto T_c - T$ was found; this dependence agrees with neither (7.19) nor (7.20).

Despite these and other open questions, we can assert that the general picture of the resistive state as a structure of phase slippage centers has been established quite reliably.

The ideas dealing with phase slippage in resistive states of various types have proved extremely fruitful. The motion of vortices in a type II superconductor (and in a superfluid liquid) and the motion of flux tubes in type I superconductors are also phase-slippage mechanisms. The very term "phase slippage" was introduced[61] to describe these other processes. In narrow superconducting channels, however, we find a qualitatively new type of phase slippage: In superconductors of massive dimensions, phase slippage occurs as a result of the motion of defects of the superconducting structure; such defects exist in the system even in the absence of dissipation. In the case of superconducting channels, in contrast, the slippage centers exist for only a short time interval and only if there is dissipation. A system with phase slippage centers is definitely a dissipative system, and the methods based on a thermodynamic approach cannot be used.

The concept of phase slippage has also proved extremely useful for describing the two-dimensional mixed state and the intermediate state of a current-carrying superconducting wire (see the review by Landau and Dolgopolov[62]). It has been shown [63,64] that again in these states the existence of an electric field against the background of superconductivity results from a phase slippage mechanism similar to that which operates in the resistive state of narrow superconducting channels. The inhomogeneous distribution of the order parameter and of the chemical potential of Cooper pairs along a superconducting channel in the resistive state can also be used to explain the results found by Iguchi *et al.*[65] on tunneling injection. They observed an inhomogeneous state at a superconducting tunnel contact, which they interpreted as a state with several values of the gap over the length of the contact. An inhomogeneous state of this type can be explained as follows[66]: The tunneling current flowing through the contact then flows along a superconducting film, increasing along the coordinate. In an actual experimental situation, the average current density along the film can exceed the Ginzburg-Landau critical current, giving rise to a resistive state characterized by jumps in the chemical potential of Cooper pairs. As a result of these jumps, the origin for the threshold voltage on the voltage-current characteristic of the detector shifts, giving rise to a picture similar to that observed experimentally. The mechanism for this effect may be the formation of phase slippage centers, if the film is narrow, or the motion of eddies, if the film is wide.

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