The Landau-Lifshitz equation revisited

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1. The Landau-Lifshitz equation

$$\frac{d\mathbf{M}}{dt}=\gamma[\mathbf{M},\ \mathbf{H}]-\alpha\frac{\gamma}{M}[\mathbf{M},\ [\mathbf{M},\ \mathbf{H}]] \tag{1}$$

was introduced[1] in order to describe the change in the magnetization **M** of a single-domain ferromagnetic sample as a function of time. It was obtained from simple phenomenological considerations. The first term on the right side of the equation describes the precession of the vector **M** in a uniformly magnetized sample in a magnetic field **H**. The second (relaxation) term determines its approach to equilibrium. It represents the simplest nonlinear combination with the correct tensor dimensionality which can be formed from the axial vectors **M** and **H**.

In molecular-field theory the effective field **H** is determined from the free-energy density F

$$\mathbf{H}=-\frac{\partial F}{\partial \mathbf{M}}, \tag{2}$$

whose dependence on the components of the vector **M** is known for samples of definite shape and with a fixed magnetic anisotropy.[2]

Equation (1) forms the foundation of the theory of magnetic resonance in ferromagnetic dielectrics[3] and has found numerous and diverse applications in the phenomenological theory of dynamic processes in ferromagnets.[3–5] It provides not only a qualitative but also an entirely satisfactory quantitative description of the evolution of the electronic magnetization of ferromagnetic samples with different shapes in an appropriately selected effective magnetic field. The factor ν in this case is negative and is very close to the gyromagnetic ratio of free electrons. The dimensionless constant $\alpha>0$, characterizing the relaxation rate of the magnetization vector, is determined by the spin-spin and spin-orbital interactions.

Together with the canonical form of Eq. (1), a different, equivalent form is also used:

$$\frac{d\mathbf{M}}{dt}=\tilde{\gamma}[\mathbf{M},\ \mathbf{H}]-\frac{\alpha}{M}\left[\mathbf{M},\ \frac{d\mathbf{M}}{dt}\right], \tag{3}$$

where $\gamma\text{-}\tilde{\gamma}(1+\alpha^2)$. The equivalence of (3) and (1) can be easily verified by eliminated $d\mathbf{M}/dt$ from the relaxation term[3] and taking into account the fact that the length of the vector **M** remains constant as it evolves: $(\mathbf{M}d\mathbf{M}/dt)=0$. In terms of the unit vector $\mathbf{m}=\mathbf{M}/M$, Eq. (1) can be written in the form

$$\frac{d\mathbf{m}}{dt}=[\mathbf{m},\ \boldsymbol{\omega}]-\alpha[\mathbf{m}[\mathbf{m},\ \boldsymbol{\omega}]], \tag{4}$$

where $\boldsymbol{\omega}=\gamma\mathbf{H}$.

In a spherical coordinate system

$$m_x=\sin\vartheta\cos\varphi,\quad m_y=\sin\vartheta\sin\varphi,\quad m_z=\cos\vartheta$$

the vector equation (1) decomposes into two equations[7]:

$$\frac{d\varphi}{dt}\sin\vartheta-\omega_\vartheta=\alpha\omega_\varphi,\quad \frac{d\vartheta}{dt}+\omega_\varphi=\alpha\omega_\vartheta, \tag{5}$$

which determine the instantaneous position of the vector **m** in terms of the independent angles of precession φ and nutation ϑ. Here

$$\omega_\vartheta=\omega_x\cos\vartheta\cos\varphi+\omega_y\cos\vartheta\sin\varphi-\omega_z\sin\vartheta,$$
$$\omega_\varphi=-\omega_x\sin\varphi+\omega_y\cos\varphi. \tag{6}$$

It is especially convenient to write the Landau-Lifshitz equation in the form (5) because the free energy of ferromagnetic samples can usually be expressed explicitly in terms of the angles ϑ and φ.[2,5]

We note that in the simplest case $F=-(\mathbf{M}\cdot\mathbf{H})$, when the spherically shaped ferromagnetic sample is in a constant magnetic field $H_z=H_0$, $\omega_\varphi=0$, $\omega_\vartheta=-\omega_0\sin\vartheta$, and $\omega_0=\gamma H_0$, Eqs. (5) assume the form

$$\frac{d\varphi}{dt}=-\omega_0,\quad \frac{d\vartheta}{dt}=-\alpha\omega_0\sin\vartheta. \tag{7}$$

2. The constancy of the length of the vector **M** of a ferromagnetic sample magnetized up to saturation expresses one of the basic properties of ferromagnetism. The position of the vector **m** is determined uniquely by any two of its components. It can be determined with the help of the single complex function ξ[6]:

$$\xi(t)=\frac{m_x-im_y}{1-m_z}=\frac{1+m_z}{m_x+im_y}. \tag{8}$$

Indeed, it follows from (8) that

$$m_x=\frac{\xi^*+\xi}{|\xi^2|+1},\quad im_y=\frac{\xi^*-\xi}{|\xi|^2+1},\quad m_z=\frac{|\xi|^2-1}{|\xi|^2+1}. \tag{9}$$

The complex variable $\xi(t)$ uniquely determines the position of a point on the surface of a unit sphere.

The vector equation (4) for the three real variables m_x, m_y, and m_z can be reduced with the help of (9) to a single equation for the scalar complex function $\xi(t)$. Using (8), we obtain

$$\frac{dm_x}{dt}-i\frac{dm_y}{dt}=(1-m_z)\frac{d\xi}{dt}-\frac{dm_z}{dt}\xi,$$
$$\frac{dm_x}{dt}+i\frac{dm_y}{dt}=\frac{(dm_z/dt)\xi-(1+m_z)d\xi/dt}{\xi^2}$$

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Now, according to (9), after simple algebraic transformations we find the equation[1)]

$$\frac{i}{1-i\alpha}\frac{d\xi}{dt}-\frac{\omega_{12}}{2}\xi^2+\omega_z\xi+\frac{\omega_{12}^*}{2}=0; \tag{10}$$

where $\omega_{12}=\omega_x+i\omega_y$.

The variable $\xi(t)$, determined by relations (8), was first introduced by Darboux[8] for the solution of a system of equations of the form (4) (without the relaxation term). The Landau-Lifshitz equation (1) for the variable $\xi(t)$ was reduced with the help of the substitution (8) to a single first-order linear differential equation with, generally speaking, time-dependent coefficients.[9]

Remarkably, taking into account the relaxation of the magnetization in the form (1) or (3) merely adds the factor $(1-i\alpha)^{-1}$ in Eq. (10) in front of the derivative $d\xi/dt$, i.e., it leads to the substitution $t\to\tau=(1-i\alpha)t$. The form of Eq. (10) does not change if the relaxation is ignored, i.e., if $\alpha=0$.

By substitutions the nonlinear Riccati equation (10) can be transformed into a second-order linear equation with time-dependent coefficients.[7,9]

Having found the solution of Eq. (10), using (9) we find the components of the vector $\mathbf{m}(t)$ as a function of time. Thus, in the simplest case of a spherically shaped magnetic sample in a quite strong, uniform, magnetic field $H_0=H_z$, we have

$$i\frac{d\xi}{dt}+\omega_0\xi=0,\quad \omega_0=\gamma H_0. \tag{11}$$

Equations (7) and (11), of course, are equivalent, since they describe the same situation in different coordinate systems. Their solutions

$$\varphi=-\omega_0 t,\quad \mathrm{tg}\frac{\vartheta}{2}=\mathrm{tg}\frac{\vartheta_0}{2}e^{-\alpha\omega_0 t}\quad \text{and}\quad \xi(t)=\xi_0 e^{\alpha\omega_0 t}e^{i\omega_0 t} \tag{12}$$

reduce to one another, since according to (8) we have

$$\xi(t)=\frac{m_x-im_y}{1-m_z}=\frac{\sin\vartheta e^{-i\varphi}}{2\sin^2(\vartheta/2)}=\mathrm{ctg}\frac{\vartheta}{2}\cdot e^{-i\varphi},$$

where $\vartheta_0=\vartheta(0)$ and $\xi_0=\xi(0)$.

It is easy to see that

$$\mathrm{tg}\,\vartheta(t)=\frac{\sin\vartheta_0}{\cos^2(\vartheta_0/2)\cdot e^{\alpha\omega_0 t}-\sin^2(\vartheta_0/2)\cdot e^{-\alpha\omega_0 t}}.$$

If $m_z(0)=0$ at $t=0$, i.e., the populations of the spin states in the field H_0 are the same, $\vartheta_0=\pi/2$, then

$$\sin\vartheta(t)=\frac{1}{\mathrm{ch}\,\alpha\omega_0 t},\quad \cos\vartheta=\mathrm{th}\,\alpha\omega_0 t. \tag{13}$$

In this case, according to (8) and (12) or (13) the solutions of the Landau-Lifshitz equations have the form

$$m_x=\frac{\cos\omega_0 t}{\mathrm{ch}\,\alpha\omega_0 t},\quad m_y=\frac{\sin\omega_0 t}{\mathrm{ch}\,\alpha\omega_0 t},\quad m_z=\frac{\mathrm{sh}\,\alpha\omega_0 t}{\mathrm{ch}\,\alpha\omega_0 t}. \tag{14}$$

[1)]Equation (3) in terms of the variable $\xi(t)$ assumes the form

$$i(1+i\alpha)\dot{\xi}-\frac{\tilde{\omega}_{12}}{2}\xi^2+\tilde{\omega}_z\xi+\frac{\tilde{\omega}_{12}^*}{2}=0,$$

where $\tilde{\omega}=\gamma H$.

These solutions describe the free precession of the magnetic moment of a uniformly magnetized ferromagnetic sample in a constant external magnetic field $\mathbf{H}_0$. As is evident from (14), the damping of the transverse components of the magnetic field, which determine the shape of the absorption lines, follows the law

$$m_x+im_y=\frac{2e^{i\omega_0 t}}{e^{\alpha\omega_0 t}+e^{-\alpha\omega_0 t}}.$$

At the beginning of the process $t\leqslant(\alpha\omega_0)^{-1}$ this dependence is far from being a simple exponential and the line shape is far from being Lorentzian.

These are the well-known, not so much in the theory of electronic spin magnetic resonance as in the theory of superradiance, solutions for the free precession of a pseudospin electric dipole moment interacting with the intrinsic field of the radiation.[10,11] They describe the decay of the coherent state of the dipole moments induced by the external (optical) field.[12,13] They lie at the foundation of the semiclassical theory of radiation which is widely used to describe the photon echo, induced transparency, and other fast transient processes.[13,14] These same dependences describe the damped precession of the polarization vector of a beam of slow neutrons, passing through the pseudomagnetic field formed by an oriented nuclear target.[15,16] The equations (1) describe nuclear magnetic resonance in nuclear ferromagnets.

3. The examples listed above suggest that the Landau-Lifshitz equations describe a much wider range of phenomena than usually presumed. It appears that they describe the behavior of an ensemble of arbitrary two-level systems which interact in some prescribed manner. We shall try to clarify the form that this interaction must assume. To this end, we set $\xi(t)=a_1(t)/a_2(t)$. The expressions (9) will then assume the form

$$m_x=\frac{a_1^*a_2+a_1a_2^*}{|a_1|^2+|a_2|^2},\quad im_y=\frac{a_1^*a_2-a_1a_2^*}{|a_1|^2+|a_2|^2},$$
$$m_z=\frac{|a_1|^2-|a_2|^2}{|a_1|^2+|a_2|^2}. \tag{15}$$

If we now let the column matrix $\binom{a_1}{a_2}=|a\rangle$ denote the state of a two-level quantum system, then the expressions (7) determine the average values of the Pauli spin vector-operator[17] $\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$:

$$\mathbf{m}=\langle\boldsymbol{\sigma}\rangle=\frac{\langle a|\boldsymbol{\sigma}|a\rangle}{\langle a|a\rangle}. \tag{16}$$

4. Equation (10) in the variables $a_1(t)$ and $a_2(t)$ decomposes into two Pauli equations

$$-i\frac{da_1}{d\tau}=\frac{\omega_z}{2}a_1+\frac{\omega_{12}^*}{2}a_2,$$
$$-i\frac{da_2}{d\tau}=\frac{\omega_{12}}{2}a_1-\frac{\omega_z}{2}a_2, \tag{17}$$

which describe the behavior of a two-level system in an external field. This is easily verified by noting that

$$i\frac{d\xi}{d\tau}=\frac{i}{a_2}\frac{da_1}{d\tau}-\frac{i}{a_2}\frac{da_2}{d\tau}\cdot\frac{a_1}{a_2}.$$

We write (17) in the form of the Schrödinger equation

$$i\hbar|\dot{a}\rangle=\widehat{\mathscr{H}}|a\rangle, \tag{18}$$

where $|\dot{a}\rangle=d|a\rangle/dt$, and

$$\widehat{\mathscr{H}}=-\frac{\hbar}{2}\begin{pmatrix}\omega_z & \omega_x-i\omega_y\\ \omega_x+i\omega_y & -\omega_z\end{pmatrix}(1-i\alpha). \tag{19}$$

Thus the relaxation term in the Landau-Lifshitz equation appears as the result of the simple multiplication of the Hamiltonian of a two-level system of noninteracting spins

$$\mathscr{H}_0=-\frac{\hbar}{2}(\sigma,\ \omega) \tag{20}$$

by the complex factor $1-i\alpha$.

Equation (18) describes the behavior of an arbitrary spin (or pseudospin) moment. The coefficients ω_{12} and ω_z in (17) represent the transition matrix elements. The diagonal elements $\pm\omega_z$ are proportional to the energy of the levels between which these transitions occur.

It would appear to be more reasonable and more consistent not to multiply all terms of the Hamiltonian matrix by $1-i\alpha$ but only the diagonal terms:

$$\widehat{\mathscr{H}}=-\frac{\hbar}{2}\begin{pmatrix}\omega_0(1-i\alpha) & \omega_{12}\\ \omega_{12}^* & -\omega_0(1-i\alpha)\end{pmatrix}, \tag{21}$$

i.e., to regard only the energy of the levels as a complex quantity. A transverse alternating field $\omega_{12}(t)$ in this case gives rise only to transitions between sublevels, without affecting their width. The equations (1) and (3) are now no longer equivalent. The transverse components of the field will no longer appear in the relaxation term of Eq. (1). It contains in this case only the component of the field or pseudofield that splits the levels.

In the notation adopted, the density matrix of the spin system is expressed in a simple manner in terms of the variable

$$\rho\equiv\frac{|a\rangle\langle a|}{\langle a\,|\,a\rangle}=\frac{1}{|\xi|^2+1}\begin{pmatrix}|\xi|^2 & \xi\\ \xi^* & 1\end{pmatrix}, \tag{22}$$

and in addition

$$\rho_{12}+\rho_{21}=m_x,\ \rho_{21}-\rho_{12}=im_y,\ \rho_{11}-\rho_{22}=m_z. \tag{23}$$

The coefficients $a_1(t)$ and $a_2(t)$ are expressed in terms of the angles ϑ and φ and, as should be expected, are equal to

$$a_1(t)=\cos\frac{\vartheta}{2}\cdot e^{-i\frac{\varphi}{2}}e^{-\alpha\omega_0 t},\quad a_2(t)=\sin\frac{\vartheta}{2}\cdot e^{i\frac{\varphi}{2}}e^{-\alpha\omega_0 t}.$$

The nonhermitean nature of the Hamiltonian does not permit normalizing the coefficients z_1 and a_2 in the usual manner $|a_1|^2+|a_2|^2=1$. Therefore

$$\frac{d\langle\sigma\rangle}{dt}=\frac{\langle\dot{a}\,|\,\sigma\,|\,a\rangle+\langle a\,|\,\sigma\,|\,\dot{a}\rangle}{\langle a\,|\,a\rangle}-\langle\sigma\rangle\frac{\langle\dot{a}\,|\,a\rangle+\langle a\,|\,\dot{a}\rangle}{\langle a\,|\,a\rangle}. \tag{24}$$

Eliminating $|\dot{a}\rangle$ and $\langle\dot{a}|$ with the help of the equation of motion (18) and its conjugate and separating the Hamiltonian (19) into hermitean and antihermitean parts, we once again obtain the Landau-Lifshitz equation (4) for $\langle\sigma\rangle=\mathbf{m}$.

Thus the Landau-Lifshitz equations of the form (1) or (4) describe the superposition of the states of an arbitrary two-level system, whose levels have the constant width $\alpha\omega_0$, as a function of time. This procedure can be satisfactorily justified for the case when $\alpha\omega_0\ll\omega_0$.[18] This was essentially stated in words in the classical work[1] and demonstrated for an ensemble of two-level systems interacting via the general radiation field.[10]

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