Vavilov-Cherenkov radiation for electric and magnetic multipoles

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The energy of Vavilov-Cherenkov radiation (VCR) from arbitrary linear multipoles is found for a transparent medium with ε and μ differing from unity. For electric multipoles, two methods are used. The force, which for $\beta n > 1$ retards the motion of a system of rigidly bound charges, and the force perpendicular to the direction of motion are determined (Sec. 2). The second method consists of determining the emitted energy from the superposition of the VCR fields created by the individual charges in the moving system (Sec. 3). It is shown that both methods give identical results. The Vavilov-Cherenkov radiation from moving elementary electric and magnetic dipoles is examined in Sec. 4. The magnetic moment induced by a moving electric dipole and the electric moment induced by a moving magnetic dipole are taken into account. The formula for the VCR of an electric dipole coincides with the formula obtained in this particular case from the analysis of Secs. 2 and 3 and thus justifies the transformations relating the electric and magnetic moments. For the magnetic dipole (an elementary current loop), there is no simple analogy to the VCR of an electric dipole, especially in the case when the magnetic dipole is oriented perpendicular to its velocity. The formulas are an elementary generalization (to the case $\mu \neq 1$) of the formulas obtained by the author in 1942. The Vavilov-Cherenkov radiation of hypothetical magnetic multipoles consisting of magnetic charges is examined in Sec. 5. The results are based on the analogy between the usual Maxwell equations and the equations for magnetic charges and currents. For a medium with $\mu = 1$ and dipoles oriented parallel to the velocity, there is a deep analogy between the properties of the field of the usual magnetic dipole (current loop) and a hypothetical dipole consisting of magnetic charges. There is no such analogy in the general case. All formulas for magnetic charges and systems consisting of magnetic charges, including also the analog of the Lorentz force, are obtained, as expected, by interchanging μ and ε, **E** and **H**, and **H** and $-$ **E**. The Maxwell's equations for magnetic charges and currents in a medium with ε and μ not equal to unity are examined in the concluding Sec. 6.

CONTENTS

INTRODUCTION

The material included in this review is the result of work performed by the author over a period of forty years: in 1942, 1952, and 1982.

After the theory of Vavilov-Cherenkov radiation,[1] to which we shall refer below as VCR, appeared in 1937 and a more detailed discussion of a number of theoretical questions appeared in 1939,[2] a quantum treatment of the phenomenon was also given.[3,4] The theory was also generalized to anisotropic media.[5] There naturally arose the question of the general properties of the radiation accompanying uniform motion of different sources of light, both with and without a characteristic frequency. A step in this direction was taken in a paper on the Doppler effect in a refracting medium, published in 1942.[6] Aside from the properties of the radiation from a harmonic oscillator moving in a refracting medium, called the complex and anomalous Doppler effect, a foundation was also laid for the subsequent analysis of a number of other questions, such as threshold phenomena and the characteristic length (later called the coherence length), and the first step was taken toward the development of the theory of transition radiation. All the questions enumerated above fall outside the scope of the present review.

One of the results of Ref. 6 is, however, directly relevant to this review. The theory of VCR for an electric charge had

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been studied since 1937, and there, naturally, arose the question of the radiation of more complex systems, primarily, dipoles. This problem was apparently first considered in connection with the quantum theory of VCR.[4] The Vavilov-Cherenkov radiation for electric and magnetic dipoles oriented parallel and perpendicular to their velocity was studied in Ref. 6. The formulas obtained for electric and magnetic dipoles oriented parallel to their velocity are similar, and in addition, just as for a charge, the energy of the radiation is a function of the square of the sine of the characteristic angle of VCR, i.e.,

$$\sin^2\theta_0 = 1 - \frac{1}{\beta^2 n^2(\omega)}.$$

However, for dipoles oriented perpendicular to their velocity a different picture arises. For an electric dipole, nothing unusual happens, even in this case; but for a magnetic dipole oriented perpendicular to its velocity the emitted energy is a complicated function of n^2 and β^2 and does not approach 0 in the limit $\beta n \to 1$. The formula appeared unexpected and even anomalous from the very beginning. The question of this anomaly was discussed repeatedly and continues to be discussed even now, i.e., more than forty years later. This question is examined here primarily in Sec. 4.

A paper containing a theory of VCR for arbitrary linear electric and magnetic multipoles was published in 1952.[7] There the magnetic multipoles were viewed as a collection of magnetic charges. From this it follows that VCR must arise not only for a moving charge, but for any particle or system of particles carrying an electromagnetic field,[9] because the field can always be represented as a sum of the fields of charges and multipoles.

Reference 7 apparently also attracted attention to the question of the anomalies in the radiation from a magnetic dipole. The explanation of this question, however, required further discussion.

The choice of multipole radiation for the subject of Ref. 7 was not accidental: it was intended for a collection of articles in honor of the sixtieth birthday of S. I. Vavilov. As is well known, S. I. Vavilov was studying the effect of the nature of elementary radiators on the observed properties of radiation[8] and he was pleased that I undertook the analysis of this question in application to VCR. It happened, however, there was insufficient time for my paper, just as for other papers intended for the collection, to be published during S. I. Vavilov's lifetime, and they were published posthumously in a memorial volume.

I originally intended to include Ref. 7 in its entirety in this review, since the results obtained there are correct and apparently are still of value, and to generalize and give a new interpretation of the results in additional sections of this review. It turned out, however, to be more convenient to rework the paper somewhat, writing down at the outset the equations in a more general form. In Ref. 7 it was assumed that the magnetic permeability of the medium is $\mu = 1$, but it is reasonable to write all equations in the form suitable for the case of a medium with $\mu \neq 1$. This elementary generalization is of some value in itself; it is then evident when the index of refraction n, which with $\mu = 1$ satisfies $n^2 \equiv \varepsilon$, enters into the equation, and when ε does so. The assumption $\mu \neq 1$ is especially necessary when comparing the equations for the radiation from electric and magnetic multipoles.

Sections 1–3 of this paper primarily repeat the content of Ref. 7 with a generalization of the results to the case of a medium with $\mu \neq 1$. The remaining sections are essentially new.

The energy emitted by a moving multipole can be found if the force retarding the motion of the multipole is known. Finding the force acting on a moving charge is a very simple way to determine the energy emitted as VCR.[9] The force acting on two charges moving parallel with the same velocity was determined in Ref. 10 for the particular case of two charges lying in a plane perpendicular to the direction of motion. An analogous method is used here to find the energy emitted by an arbitrarily oriented linear electric multipole (Sec. 2), regarded as a system of moving rigidly coupled charges. It is shown here that, in addition to the component of the force that retards the motion of the multipole, there is also a force oriented perpendicular to the velocity (Secs. 2 and 3).

Another method for determining the emitted energy is based on the fact that the wave field of an electric multipole can be found by summing the waves emitted by the individual charges in the multipole (Sec. 3).

The theory of radiation from elementary electric and magnetic dipoles, which represents a generalization of Refs. 6 and 7, is studied within the framework of classical electrodynamics in Sec. 4.

In the next two sections (Secs. 5 and 6) the field equations and the energy of the radiation from moving magnetic charges and magnetic multipoles formed by magnetic charges are discussed. The results presented there essentially coincide with the results of Ref. 7. Their interpretation has, however, changed. In contrast to Ref. 7, it is shown that on the basis of the properties of the emitted radiation a dipole consisting of two magnetic charges is not always equivalent to the usual magnetic dipole created by an elementary current loop. The difference becomes especially evident in a medium with $\mu \neq 1$.

1. BASIC EQUATIONS OF ELECTRODYNAMICS FOR AN ELECTRIC CHARGE MOVING IN A MEDIUM

For what follows we shall require the basic equations used in the theory of the Vavilov-Cherenkov effect for a moving charge,[1,2] which we shall generalize to the case of a medium with $\mu \neq 1$.

The Fourier components of the vector and scalar potentials can be written as follows:

$$\nabla^2 \mathbf{A}_\omega + \frac{\omega^2}{c^2}\mu\varepsilon\mathbf{A}_\omega = -\frac{4\pi}{c}\mu\mathbf{j}_\omega, \tag{1.1}$$

$$\nabla^2 \varphi_\omega + \frac{\omega^2}{c^2}\mu\varepsilon\mathbf{A}_\omega = -\frac{4\pi}{\varepsilon}\rho_\omega, \tag{1.2}$$

$$\operatorname{div}\mathbf{A}_\omega + \frac{i\omega}{c}\mu\varepsilon\varphi_\omega = 0. \tag{1.3}$$

Using (1.3), the electric field vector can be expressed in terms of $\mathbf{A}_\omega$. Then, for $\mathbf{E}_\omega$ and $\mathbf{H}_\omega$, which satisfy Maxwell's equations, we have

$$\mathbf{E}_\omega = -\frac{ic}{\omega\varepsilon\mu}\,\mathrm{grad\,div}\,\mathbf{A}_\omega - \frac{i\omega}{c}\mathbf{A}_\omega, \tag{1.4}$$

$$\mathbf{H}_\omega = \frac{1}{\mu}\mathrm{curl}\mathbf{A}_\omega; \tag{1.5}$$

where ε and μ correspond to their values at the frequency ω. We assume that the medium is transparent, so that the quantities ε and μ and $n^2 = \varepsilon\mu$ do not contain an imaginary part.

Let us trace the relationship of (1.1)–(1.5) to Maxwell's equations. Substituting (1.5) in the equation

$$\mathrm{curl}\;\mathbf{E}_\omega = -\frac{1}{c}\frac{\partial \mathbf{B}_\omega}{\partial t}, \tag{1.6}$$

we obtain

$$\mathrm{curl}\left(\mathbf{E}_\omega + \frac{1}{c}\frac{\partial \mathbf{A}_\omega}{\partial t}\right) = 0, \tag{1.7}$$

and we can set

$$-\mathrm{grad}\;\varphi_\omega = \mathbf{E}_\omega + \frac{1}{c}\frac{\partial \mathbf{A}_\omega}{\partial t}. \tag{1.8}$$

In this case, using the fact that

$$\mathrm{div}\;\mathbf{D}_\omega = 4\pi\rho, \tag{1.9}$$

we obtain

$$\mathrm{div}\,\mathbf{D}_\omega = -\varepsilon\,\mathrm{div\,grad}\,\varphi_\omega - \frac{\varepsilon}{c}\mathrm{div}\frac{\partial \mathbf{A}_\omega}{\partial t} = 4\pi\rho_\omega. \tag{1.10}$$

Using Lorentz's relation (1.3), from (1.10) we indeed obtain (1.2). Finally, from (1.8) and (1.3) we have (1.4). As a check we verify that (1.1) is also written correctly. Substituting (1.5) into Maxwell's equations

$$\mathrm{curl}\,\mathbf{H}_\omega - \frac{1}{c}\frac{\partial \mathbf{D}_\omega}{\partial t} = \frac{4\pi}{c}\mathbf{j}_\omega, \tag{1.11}$$

and using the fact that

$$\mathrm{curl\;curl}\,\mathbf{A}_\omega = \mathrm{grad\;div}\;\mathbf{A}_\omega - \nabla^2\mathbf{A}_\omega, \tag{1.12}$$

as well as (1.4), we indeed obtain (1.1).

Proceeding to the solution of Eq. (1.1), we repeat the formulas used in the examination of the theory of Vavilov-Cherenkov radiation and we present them without a detailed explanation.[1)]

The current density created by a charge e moving along the z axis (its coordinates are $z = vt$, $x = y = 0$) can be set equal to

$$j_z = ev\delta(x)\,\delta(y)\,\delta(z - vt). \tag{1.13}$$

It is convenient to solve the problem in cylindrical coordinates, placing the z axis, along which the charge moves, at $\rho = 0$. Then, for the ω component of the current density we obtain

$$j_{\omega z} = \frac{e}{2\pi^2\rho}e^{-(i\omega z/v)}\delta(\rho), \quad j_{\omega\rho} = j_{\omega\varphi} = 0. \tag{1.14}$$

We shall write the solution of (1,1) with $j_{\omega z}$ equal to (1.14) and $\mu \neq 1$ as follows:

$$A_{\omega z} = \frac{e\mu}{2c}a(\rho,\omega)\,e^{-i\omega z/v}, \quad A_{\omega\rho} = A_{\omega\varphi} = 0. \tag{1.15}$$

Then the equation for $a(\rho,\omega)$ has the same form as in the case $\mu = 1$:

$$\frac{\partial^2 a}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial a}{\partial\rho} + s^2 a = -\frac{4}{\pi\rho}\delta(\rho); \tag{1.16}$$

[1)]The interested reader is referred to Ref. 1 and especially the 1939 paper by I. E. Tamm.[2]

where

$$s^2 = \frac{|\omega|^2 n^2}{c^2}\left(1 - \frac{1}{\beta^2 n^2}\right) = \frac{|\omega|^2}{v^2}(\beta^2 n^2 - 1), \tag{1.17}$$

and, in addition, of course, $n^2 = \varepsilon\mu$.

For $\rho > 0$, the function $a(\omega)$ satisfies Bessel's equation

$$\frac{\partial^2 a}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial a}{\partial\rho} + s^2 a = 0. \tag{1.18}$$

The solution of Eq. (1.18) must assume the appropriate value at the pole $\rho = 0$ and correspond to a wave outgoing from the z axis.

Taking these boundary conditions into account leads, as is well known, to the fact that the solution depends on the quantity βn.[1,2]

In the case $\beta n < 1$ we have

$$a(\rho,\omega) = iH_0^{(1)}(i\sigma\rho), \quad \sigma = \frac{|\omega|}{v}\sqrt{1-\beta^2 n^2}. \tag{1.19}$$

For what follows it is significant that $a(\rho,\omega)$ in (1.19) does not depend on the sign of ω.

If, on the other hand, the velocity of the charge is greater than the phase velocity of light, $v > c/n$, i.e., $\beta n > 1$, then

$$a(\rho,\omega) = -iH_0^{(2)}(s\rho) = -iJ_0(s\rho) - Y(s\rho) \quad \text{for} \quad \omega > 0, \tag{1.20}$$

$$a(\rho,\omega) = +iH_0^{(1)}(s\rho) = iJ_0(s\rho) - Y(s\rho) \quad \text{for} \quad \omega < 0. \tag{1.21}$$

Here $H_0^{(1)}$ and $H_0^{(2)}$ are zeroth-order Hankel functions of the first and second kind. They are expressed, as is well known, in terms of the zeroth-order Bessel J_0 and Weber Y_0 functions.

From (1.14), (1.4), and (1.5) we obtain the following expressions for the components of the field intensities

$$H_{\omega\varphi} = -\frac{e}{2c}\frac{\partial a}{\partial\rho}e^{-i\omega z/v}, \tag{1.22}$$

$$E_{\omega\rho} = -\frac{e\mu}{2vn^2}\frac{\partial a}{\partial\rho}e^{-i\omega z/v}, \tag{1.23}$$

$$E_{\omega z} = -\frac{ie\omega\mu}{2c^2}\left(1 - \frac{1}{\beta^2 n^2}\right)a e^{-i\omega z/v}. \tag{1.24}$$

The ρ-component of Poynting's vector is equal to

$$S = -\frac{c}{4\pi}\int_{-\infty}^{+\infty} E_{\omega z}e^{i\omega t}\,d\omega \int_{-\infty}^{+\infty} H_{\omega'\varphi}e^{i\omega' t}\,d\omega'. \tag{1.25}$$

The energy W emitted by a charge e over a path of length d can be obtained, as in the theory of VCR, by multiplying S by the area of the lateral surface of the cylinder $2\pi\rho d$ and integrating over time from $-\infty$ to $+\infty$.

Here, in contrast to a medium with $\mu = 1$, the quantity $E_{\omega z}$ (see (1.24)) contains an additional factor μ. From here, as is evident from (1.25), W must be a factor of μ greater than in the well-known case $\mu = 1$.

Thus

$$W = e^2 d\int W_\omega d\omega, \tag{1.26}$$

where

$$W_\omega = \frac{\mu}{c^2}\omega\left(1 - \frac{1}{\beta^2 n^2}\right) \tag{1.27}$$

is the energy radiated by a unit electric charge per unit path length and per unit frequency, for a medium with $\mu \neq 1$[11,12] (see Eq. (2.13)).

2. FORCE ACTING ON A MOVING ELECTRIC MULTIPOLE

The force acting on a charge e_i due to the field of the charge e_k is equal to

$$\mathbf{f}\,(e_i,\,k) = e_i\mathbf{E}\,(e_k) + \frac{e_i}{c}\,\mu\,[\mathbf{v}\,\mathbf{H}(e_k)]; \tag{2.1}$$

where $\mathbf{E}(e_k)$ and $\mathbf{H}(e_k)$ are the field intensities produced by the charge e_k at the location of the charge e_i and v is the velocity of the charge e_i.

In what follows we shall study the force $\mathbf{f}_\omega(e_i, k)$, determined by the sum of the field components with the frequencies $+\omega$ and $-\omega$, as follows:

$$\mathbf{f}_\omega\,(e_i,\,k) = e_i\,[\mathbf{E}_\omega\,(e_k)\,e^{i\omega t} + \mathbf{E}_{-\omega}\,(e_k)\,e^{-i\omega t}] + \frac{e_i}{c}\,\mu v\,[\mathbf{H}_\omega\,(e_k)\,e^{i\omega t} - \mathbf{H}_{-\omega}\,(e_k)\,e^{-i\omega t}]; \tag{2.2}$$

in addition,

$$\mathbf{f}\,(e_i,\,k) = \int\limits_0^\infty \mathbf{f}_\omega\,(e_i,\,k)\,d\omega. \tag{2.3}$$

We shall assume that the medium in which the motion occurs does not absorb light, i.e., its index of refraction n is a real quantity. In addition, we shall study only charges which are rigidly coupled with one another and move in the same direction with the same velocities ($v_i = v_k$). Assuming that the motion occurs along the z axis, setting

$$z_i - z_k = \Delta_{ik} \tag{2.4}$$

and choosing the origin of time such that $z_k = vt$, we have

$$z_i = vt + \Delta_{ik}. \tag{2.5}$$

From here, substituting into (2.2) the field components (1.22)–(1.24) with $e = e_k$, for the point z_i defined by (2.5) and $\rho = \rho_{ik}$ where ρ_{ik} is the distance between the trajectories of the charges e_i and e_k, we obtain

$$f_{\omega z}\,(e_{i,\,k}) = -\frac{i\omega\mu e_i e_k}{2c^2}\left(1-\frac{1}{\beta^2n^2}\right) \times [a\,(\rho_{ik},\,+\omega)\,e^{-i\omega\Delta_{ik}/v} - a\,(\rho_{ik},\,-\omega)\,e^{+i\omega\Delta_{ik}/v}], \tag{2.6}$$

$$f_{\omega\rho}\,(e_{i,\,k}) = +\frac{v\mu e_i e_k}{2c^2}\left(1-\frac{1}{\beta^2n^2}\right)\left[\frac{\partial}{\partial\rho}\,a\,(\rho,\,+\omega)\,e^{-i\omega\Delta_{ik}/v} + \frac{\partial}{\partial\rho}\,a\,(\rho,\,-\omega)\,e^{+i\omega\Delta_{ik}/v}\right]_{\rho=\rho_{ik}}; \tag{2.7}$$

where $f_{\omega z}$ is the component of the force acting on e_i and oriented along the z axis and $f_{\omega\rho}$ is the component of the force perpendicular to the z axis and oriented from the trajectory of the k th charge to the trajectory of the ith charge (Fig. 1).

To determine the total force acting on the system, we must also know the effect of the force produced by the charge e_k and acting on the charge e_i.

It is easy to verify that in order to obtain this force

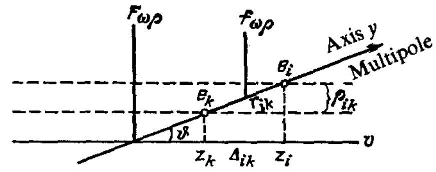

FIG. 1.

$f_\omega(e_k, i)$, acting on the particle k, it is sufficient to change (reverse) the sign in front of Δ_{ik} in (2.6) and (2.7). Here, a positive component of the force $f_\omega(e_k, i)$ corresponds to the direction from the trajectory of the ith particle to the trajectory of the k th particle, i.e., the direction opposite to (2.7).

As a result, for the sum of forces acting on both particles, we obtain

$$f_{\omega z}\,(ik) = f_{\omega z}\,(e_i,\,k) + f_{\omega z}\,(e_k,\,i) = -\frac{i\omega\mu e_i e_k}{c^2}\left(1-\frac{1}{\beta^2n^2}\right) \times [a\,(\rho_{ik},\,+\omega) - a\,(\rho_{ik},\,-\omega)]\cos\frac{\omega\Delta_{ik}}{v}, \tag{2.8}$$

$$f_{\omega\rho}\,(ik) = f_{\omega\rho}\,(e_i,\,k) - f_{\omega\rho}\,(e_k,\,i) = -\frac{iv\mu e_i e_k}{c^2}\left(1-\frac{1}{\beta^2n^2}\right) \times \left[\frac{\partial}{\partial\rho}\,a\,(\rho,\,+\omega) - \frac{\partial}{\partial\rho}\,a\,(\rho,\,-\omega)\right]_{\rho=\rho_{ik}}\sin\frac{\omega\Delta_{ik}}{v}. \tag{2.9}$$

The sign of $f_{\omega\rho}(ik)$ is chosen so that for positive $f_{\omega\rho}(ik)$ the force is oriented away from the k th trajectory to the ith trajectory (see Fig. 1).

We obtain the action of the self-field on the particle by setting in (2.6) $e_i = e_k$ and setting Δ_{ik} and ρ_{ik} to their limiting value of zero. Then we have[2)]

$$f_{\omega z}\,(e_i) = -\frac{i\omega\mu e_i^2}{2c^2}\,[a\,(0,\,+\omega) - a\,(0,\,-\omega)]. \tag{2.10}$$

It is evident from (2.8)–(2.10) that the forces acting on a system of two particles vanish identically if $a(\rho, +\omega) = a(\rho, -\omega)$. According to (1.19), this occurs when $\beta n < 1$.

The force acting on a system of point charges is equal to the sum of the retarding forces produced by the self-field (2.10) acting on each particle and the forces due to the interaction of pairs of separate particles (2.8) and (2.9). It follows from here that any system of rigidly coupled charges in uniform and rectilinear motion does not experience any forces retarding its motion or deflecting it from a rectilinear trajectory, if the velocity satisfies $v < c/n$. A different situation occurs when the velocity is greater than the phase velocity of light ($\beta n > 1$). In this case, substituting the value of $a(\rho,\omega)$ from (1.20) and (1.21) into (2.8) and (2.9), we obtain

$$f_{\omega z}\,(ik) = -\frac{2\omega\mu e_i e_k}{c^2}\left(1-\frac{1}{\beta^2n^2}\right) J_0\,(s\rho_{ik})\cos\frac{\omega\Delta_{ik}}{v}, \tag{2.11}$$

[2)]It is evident from (2.6) that the limiting value of $f_{\omega z}$, corresponding to $\Delta = 0$, does not depend on whether the limit is approached from positive or negative values of Δ. In contrast to this, the orientation of $f_{\omega\rho}$ is always associated with ρ. From symmetry considerations it is evident that the force acting on the particle must be identical for all orientations of the radius vector ρ. Thus the force $f_{\omega\rho}$ resulting from the action of the self-field of the particle must be set equal to zero.

$$f_{\omega\rho}(ik) = +\frac{2\omega\mu v n e_i e_k}{c^3}\left(1-\frac{1}{\beta^2 n^2}\right)^{3/2} J_1(s\rho_{ik}) \sin\frac{\omega\Delta_{ik}}{v}. \tag{2.12}$$

Equation (2.12) was obtained using the fact that

$$\frac{\partial}{\partial\rho} J_0(s\rho) = -sJ_1(s\rho).$$

For the forces $f_{\omega z}(e_i)$ and $f_{\omega z}(e_k)$, retarding the moving charges e_i and e_k, we obtain from (1.20) and (2.10) (since $J_0(s\rho)$ is equal to 1 at $\rho = 0$)

$$\begin{aligned} f_{\omega z}(e_i) &= -\frac{e_i^2\mu\omega}{c^2}\left(1-\frac{1}{\beta^2 n^2}\right) = -e_i^2 W_\omega, \\ f_{\omega z}(e_k) &= -e_k^2 W_\omega. \end{aligned} \tag{2.13}$$

The minus sign means that the force is oriented opposite to the direction of motion, i.e., that work is performed during the motion. Here, the magnitude of the force (2.13) is numerically equal to the energy emitted by the charge per unit path length (1.27).[3)]

The force $f_{\omega z}(ik)$, as is evident from (2.11), can be both positive and negative. Since, however, the absolute magnitudes of $J_0(s\rho_{ik})$ and $\cos(\omega\Delta_{ik}/v)$ are less than or equal to 1, the total force acting on the two moving rigidly coupled charges satisfies

$$[f_{\omega z}(e_i) + f_{\omega z}(ik) + f_{\omega z}(e_k)] \leqslant 0.$$

This sum, therefore, obviously cannot be positive. Otherwise the charges would be self-accelerated, i.e., the law of conservation of energy would be violated. Moreover, it is easy to verify that for $\beta n > 1$ this sum can vanish only for particular values of ω, but not for the integral over the frequency.[4)] Therefore, for $\beta n > 1$ the total force is always negative, i.e., the charges are retarded and, therefore, radiation is emitted. This result is also correct for a system consisting of any number of rigidly coupled charges moving with velocity $v > c/n$.

The force retarding the motion of a system of n particles is equal to

$$F_{z\omega} = \sum_{i=1}^{n} f_{\omega z}(e_i) + \sum_{i=1}^{n-1}\sum_{k=i+1}^{n} f_{\omega z}(ik); \tag{2.14}$$

where $\Sigma_{i=1}^{n} f_{\omega z}(e_i)$ is the sum of the forces (2.13), which would retard each of the charges separately in the absence of the other forces, and $f_{\omega z}(ik)$ is the force (2.11), caused by the interaction of the charges, which is summed over all different pairs of charges.

In what follows, we shall for simplicity restrict our attention to a collection of charges lying along one straight line and therefore equivalent to one or several linear multipoles whose axes coincide with this straight line. We shall assume that the line is positively oriented if it forms an acute angle with the z axis ($0 \leqslant \vartheta \leqslant \pi/2$). We shall assume that the positive orientation of the component of the forces acting perpendicular to $\mathbf{v}$ (the force $F_{\omega\rho}$) lies in the plane formed by the velocity and the axis of the multipoles and forms an acute angle with the axis of the multipoles[5)] (see Fig. 1).

It is evident that the projection of a positively oriented segment of the multipole axis on the v and $F_{\omega z}$ axes will be positive (since $\cos\vartheta \geqslant 0$ and $\sin\vartheta \geqslant 0$). Therefore, if the direction from the k th charge to the ith charge coincides with the positive orientation of the axis, then $r_{ik}\cos\vartheta > 0$, and in addition

$$\Delta_{ik} = z_i - z_k = r_{ik}\cos\vartheta.$$

In this case, the positive orientation of the force $f_{\omega\rho}(ik)$ (oriented from the k th trajectory to the ith trajectory) coincides with the positive orientation of $F_{\omega\rho}$ (see Fig. 1). If, on the other hand, r_{ik} is negative, then $\Delta_{ik} < 0$ also, and the positive orientations of $f_{\omega\rho}$ and $F_{\omega\rho}$ are opposite. The force $f_{\omega\rho}$, according to Eq. (2.12), is proportional to $\sin(\omega\Delta_{ik}/v)$, i.e., its sign changes when the sign of Δ_{ik} changes. Therefore, to make sure that the forces have the correct sign, it is sufficient in summing the forces acting on different pairs of charges to replace Δ_{ik} in (2.12) by $|\Delta_{ik}| = \Delta'_{ik}$. The force $F_{\omega\rho}$ is then equal to

$$F_{\omega\rho} = \sum_{i=1}^{n-1}\sum_{k=i+1}^{n} f'_{\omega\rho}(ik), \tag{2.15}$$

where $f'_{\omega\rho}(ik)$ differs from $f_{\omega\rho}(ik)$ in (2.12) in that Δ_{ik} is replaced by Δ'_{ik}.

If Δ_{ik} is also replaced by Δ'_{ik} in $f_{\omega z}(ik)$ of Eq. (2.11) (which does not change its magnitude, which depends on $\cos(\omega\Delta_{ik}/v)$), then the arguments of the functions $f_{\omega z}$ and $f'_{\omega z}$ will be $s\rho_{ik}$ and $\omega\Delta'_{ik}/v$. It is not difficult to express them in terms of the absolute distances between the charges, since

$$\Delta'_{ik} = |r_{ik}|\cos\vartheta, \quad \rho_{ik} = |r_{ik}|\sin\vartheta. \tag{2.16}$$

Setting

$$\alpha = \frac{\omega}{v}\cos\vartheta, \quad \delta = \frac{\omega}{v}\sqrt{\beta^2 n^2 - 1}\sin\vartheta, \tag{2.17}$$

we have

$$|r_{ik}|\alpha = \frac{\omega\Delta'_{ik}}{v}, \quad |r_{ik}|\delta = s\rho_{ik}. \tag{2.18}$$

To determine the magnitude of the force $F_{z\omega}$, retarding the motion of the system of charges, we expand (2.11) in a double power series in powers of $|r_{ik}|\alpha$ and $|r_{ik}|\delta$. Then, after substitution into (2.14), we obtain[6)]

$$F_{z\omega} = f_0 + f_1 + f_2 + f_3 + \ldots, \tag{2.19}$$

where

$$f_0 = -W_\omega\Big(\sum_{i=1}^{n} e_i^2 + 2\sum_{i=1}^{n-1}\sum_{k=i+1}^{n} e_i e_k\Big) = -W\Big(\sum_{i=1}^{n} e_i\Big)^2, \tag{2.20}$$

$$f_1 = +\frac{W_\omega}{2}(2\alpha^2 + \delta^2)\sum_{i=1}^{n-1}\sum_{k=i+1}^{n} e_i e_k r_{ik}^2, \tag{2.21}$$

$$f_2 = -\frac{W_\omega}{4}\left(\frac{1}{3}\alpha^4 + \alpha^2\delta^2 + \frac{1}{8}\delta^4\right)\sum_{i=1}^{n-1}\sum_{k=i+1}^{n} e_i e_k r_{ik}^4, \tag{2.22}$$

[3)]Just as for a moving charge, this result is correct only in the case when n is real, which is correct if the medium in which the motion occurs does not absorb light.

[4)]With the exception of the trivial case of total neutralization of the charges $e_i = -e_k$, $\Delta_{ik} = \rho_{ik} = 0$.

[5)]The choice of the positive orientation of the axis of the multipoles and, therefore, of $F_{\omega\rho}$ becomes indeterminate at $\vartheta = 0$ and $\vartheta = \pi/2$. This, however, is not significant, because, as will be evident from what follows, in both cases the force $F_{\omega\rho} = 0$.

[6)]We recall that $J_0(x)$ is equal to the sum of the series

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(\ 1)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots.$$

$$f_3=+\frac{W_\omega}{8}\left(\frac{1}{45}\alpha^6+\frac{1}{6}\alpha^4\delta^2+\frac{1}{8}\alpha^2\delta^4+\frac{1}{144}\delta^6\right)$$
$$\times\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}e_ie_kr_{ik}^6. \qquad (2.23)$$

Thus the first term of the expansion of f_0 in (2.20) is simply the force determined by the Vavilov-Cherenkov radiation for the total charge of the system (see (1.18) and (1.19)).

The quantities f_1, f_2, and so on, depend on the multipole moments of the system of charges. To find any of the f_l one must know the value of the double sum

$$S_l=\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}e_ie_kr_{ik}^{2l}. \qquad (2.24)$$

If in calculating S_l the summation over both indices i and k is performed from 1 to n, then each pair of charges will be counted not once, as in (2.24), but twice, i.e., the sum will be equal to $2S_l$ (the presence of terms with $i=k$ for $l>0$ does not alter the sum, since r_{ik}^{2l} is equal to zero for $i=k$. Thus

$$S_l=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}e_ie_kr_{ik}^{2l}=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}e_ie_k(r_i-r_k)^{2l}; \qquad (2.25)$$

where r_i and r_k are the coordinates of the ith and kth charges, measured along the axis on which the charges are positioned from the point on this axis adopted as the center of the system of charges.

Keeping in mind the fact that for the given collection of charges and for the center which we have chosen

$$p_a=\sum_{i=1}^{n}e_ir_i^a \qquad (2.26)$$

is the electric multipole moment of order a measured in the laboratory system, and (using the binomial expansion), we obtain from (2.25)

$$S_l=\sum_{m=0}^{m=l-1}\frac{(2l)!\,(-1)^m}{(2l-m)!\,m!}p_{2l-m}p_m+\frac{1}{2}\frac{(2l)!}{(l!)^2}(-1)^l p_l^2. \qquad (2.27)$$

For example, for the particular case $l=1$, we have

$$S_1=p_2p_0-p_1^2, \qquad (2.28)$$

where p_2 and p_1 are the quadrupole and dipole moments of the system, and the zeroth order moment p_0 denotes the algebraic sum of the charges $p_0=\Sigma_{i=0}^{n}e_i$.

If the system of charges has only an lth order multipole moment, then S_l will contain a single term proportional to p_l^2. Then f_l is proportional to p_l^2 and the remaining terms in the series (2.19) vanish, so that $F_{\omega z}=f_l$. Thus, in the case when only p_1 or p_2 or p_3 differs from zero, i.e., for a dipole, quadrupole, and octupole, we obtain from (2.27)

$$S_1=-p_1^2,\quad S_2=3p_2^2,\quad S_3=-10p_3^2.$$

Substituting the values of these sums into f_1, f_2, and f_3, respectively, in Eqs. (2.21)–(2.23) and using (2.17) and (1.19), we obtain the force retarding the motion of the multipole.

For a dipole

$$F_{\omega z}(p_1)=-\frac{\omega^3\mu p_1^2}{2c^2v^2}\left(1-\frac{1}{\beta^2n^2}\right)$$
$$\times\left[2\cos^2\vartheta+\beta^2n^2\left(1-\frac{1}{\beta^2n^2}\right)\sin^2\vartheta\right]. \qquad (2.29)$$

For a quadrupole

$$F_{\omega z}(p_2)=-\frac{3}{4}\frac{\omega^5\mu p_2^2}{c^2v^2}\left(1-\frac{1}{\beta^2n^2}\right)$$
$$\times\left[\frac{1}{3}\cos^4\vartheta+(\beta^2n^2-1)\cos^2\vartheta\sin^2\vartheta\right.$$
$$\left.+\frac{1}{8}(\beta^2n^2-1)^2\sin^4\vartheta\right]. \qquad (2.30)$$

For an octupole

$$F_{\omega z}(p_3)=-\frac{5}{4}\frac{\omega^7\mu p_3^2}{c^2v^6}\left(1-\frac{1}{\beta^2n^2}\right)$$
$$\times\left[\frac{1}{45}\cos^6\vartheta+\frac{1}{6}(\beta^2n^2-1)\cos^4\vartheta\sin^2\vartheta\right.$$
$$+\frac{1}{8}(\beta^2n^2-1)^2\cos^2\vartheta\sin^4\vartheta$$
$$\left.+\frac{1}{144}(\beta^2n^2-1)^3\sin^6\vartheta\right]. \qquad (2.31)$$

For $\mu=1$ Eqs. (2.29)–(2.31) coincide with the previously obtained formulas in Ref. 7.

If the system has more than one multipole moment, then interference terms, proportional to the product of these moments,[7)] can arise in the radiation (see (2.27)). Thus, for example, if the total charge and the quadrupole moment differ from zero, then, aside from the terms f_0 and f_2, corresponding to these moments, the term f_1, in which the product p_2p_0 plays the role of the dipole moment (see (2.28)), will differ from zero. Here, in contrast to a true dipole, the quantity f_1 can in this case be both negative and positive, i.e., it can increase or decrease the total energy of the radiation.

As already noted, the force $F_{\omega z}$, retarding the motion of the multipole, differs from zero only for frequencies which satisfy the condition $\beta n(\omega)>1$. We obtain the total force retarding the motion of the system by integrating over these frequencies. From here, the energy emitted by the multipole per unit path length is equal to

$$W(p_l)=-\int\limits_{\beta n(\omega)>1}F_{\omega z}(p_l)\,d\omega. \qquad (2.32)$$

Thus, for example, from (2.29) we obtain the following expressions for the energy of the radiation from a dipole with moment p, oriented parallel ($\vartheta=0$) and perpendicular ($\vartheta=\pi/2$) to the velocity:

$$W=\int\limits_{\beta n(\omega)>1}\frac{\omega^3\mu}{c^2v^2}p^2\left(1-\frac{1}{\beta^2n^2}\right)d\omega,\quad p\parallel\mathbf{v}, \qquad (2.33)$$

$$W=\int\limits_{\beta n(\omega)>1}\frac{\omega^3\mu n^2}{2c^4}p^2\left(1-\frac{1}{\beta^2n^2}\right)^2d\omega,\quad p\perp\mathbf{v}. \qquad (2.34)$$

These formulas differ from those obtained previously[5–7,8)] by the presence of the coefficient μ.

The force $F_{\omega\rho}$, oriented perpendicular to the velocity, can be found entirely analogously to the manner in which this was done for $F_{\omega z}$, namely, by expanding $f'_{\omega\rho}$, i.e., (2.12), replacing Δ_{ik} by Δ'_{ik}, in a series in powers of $|r_{ik}|\alpha$ and

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7)It is evident from (2.27) that the product of the moments p_a and p_b will appear in the radiation if $|a-b|$ is even. In this case, to within a numerical (positive or negative) factor, it is equivalent to

$p_l^2=p_ap_b$, where $l=(a+b)/2$.

8)In the formula for $\mathbf{p}\perp\mathbf{v}$, obtained in Ref. 6, as well as in the formula presented in Ref. 9, the factor b^2 was omitted.

$|r_{ik}|\delta$, and substituting into (2.15). In particular, for a dipole we obtain

$$F_{\omega\rho}(p) = \frac{\omega^2\mu n^2}{c^4} p^2\left(1-\frac{1}{\beta^2 n^2}\right)^2 \sin\vartheta\cos\vartheta. \qquad (2.35)$$

Thus, aside from the forces retarding the motion of the system, there exists a force which does not work with rectilinear motion, but which strives to deflect the dipole from its direction of motion. This force for dipoles, as well as for multipoles, vanishes only when the axis of the system is oriented either parallel or perpendicular to the velocity. As will be shown in the next section, it is directly related to the asymmetry of the angular distribution of the radiation.

It should be kept in mind that in all the equations presented above, both the electric multipole moment p and the angle ϑ are measured in a stationary coordinate system. If we keep in mind that the length scale along the z axis is contracted by a factor of $\alpha = \sqrt{(1-\beta^2)}$, then for the angle ϑ' with the axis of the multipole and the distance r_i' of the charge e_i from the center of the multipole, measured in a system of coordinates fixed to the charges, we have

$$\mathrm{tg}\,\vartheta = \frac{1}{\alpha}\,\mathrm{tg}\,\vartheta', \quad z_i = r_i'\sqrt{1-(1-\alpha^2)\cos^2\vartheta'}.$$

From here, by virtue of the invariance of the magnitude of the electric charge, we find that the electric multipole moments p_l', whose radiation is being studied, are related to p_l by the expression

$$p_l = p_l'\,[1-(1-\alpha^2)\cos^2\vartheta']^{l/2}. \qquad (2.36)$$

In the laboratory coordinate system, the moving system of charges is equivalent to a collection of not only electric multipoles p_l, but also magnetic multipoles. Thus the p_l are not the total moments of the system, although the radiation energy can be expressed, as done above, in terms of their magnitude. If, at the outset, we had examined the radiation of the multipoles p_l instead of the radiation of the moving system of charges, then we would have obtained a different result.[9)]

3. RADIATION FROM AN ELECTRIC MULTIPOLE

The same relations as obtained in Sec. 2 can be obtained by examining the magnitude of the energy emitted by a system of moving charges. It is well known that the Vavilov-Cherenkov radiation produced by a moving charge propagates in directions forming an acute angle θ with the direction of the velocity (z axis), and in addition

$$\cos\theta = \frac{1}{\beta n}, \quad \sin\theta = \sqrt{1-\frac{1}{\beta^2 n^2}}. \qquad (3.1)$$

The phase of the emitted wave is determined by the surface of the cone, whose vertex coincides with the moving charge and whose generatrices form an angle of $(\pi/2)+\theta$ with the z axis (Fig. 2a). If there are two moving charges, then the vertices of the two cones are displaced relative to one another and, therefore, the phases of the waves corresponding to both charges will be different (Fig. 2b). Let us assume that a dipole is moving, i.e., that the charges e_1 and e_2 have different signs and equal magnitudes e. We assume that

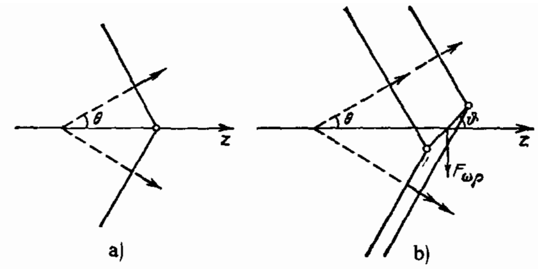

FIG. 2.

the distance r_{12} is much smaller than any wavelengths that can be emitted (i.e., for frequencies satisfying the condition $\beta n(\omega) > 1$). In this case, if the phases of the waves from both charges are identical, then the waves will completely cancel one another, and as the phase difference increases, $\Delta\omega \ll \pi/2$, the total amplitude will increase. Therefore, for the case shown in Fig. 2b, the intensity of the radiation oriented upwards from the z axis will be greater than that of the radiation oriented downwards. Therefore, the momentum carried away by the radiation will give rise to a recoil force, striving to deflect the dipole downwards. It is not difficult to verify that the sign of this force $F_{\omega\rho}$ agrees with (2.35).[10)]

An analysis of the interference permits finding both the angular distribution of the radiation of a linear multipole and the magnitude of the forces acting on it. We shall determine the intensity of the field at a point far away from the trajectory of the multipole. We assume that the component of the field intensity with frequency ω, which would be observed at time t if the multipole were replaced by a unit charge concentrated at its center, is equal to

$$Ae^{i\omega t}. \qquad (3.2)$$

Then the field determined by the collection of charges forming the multipole is obtained as a sum of waves with amplitudes proportional to

$$Ae^{i\omega t}\sum_{i=1}^{n} e_i e^{-(i\omega n/c) r_i \cos(p_l R)}, \qquad (3.3)$$

where r_i is, as before, the absolute displacement of the charge e_i from the center of the multipole (see Sec. 2), and $\cos(p_l R)$ is the cosine of the angle between the multipole axis and the ray for which the waves are summed.

We denote by φ the dihedral angle formed by the following planes: the plane passing through the z axis and the axis of the multipole and the plane formed by the z axis and the direction of the ray. Keeping in mind the fact that the ray forms an angle θ with the z axis (see (3.1)) and the angle between the axis of the multipole and the z axis is equal to ϑ, for $\cos(p_l R)$ we obtain

$$\cos(p_l R) = \sqrt{1-\frac{1}{\beta^2 n^2}}\sin\vartheta\cos\varphi + \frac{1}{\beta n}\cos\vartheta. \qquad (3.4)$$

Let us expand the quantity $\exp(-(i\omega n/c) r_i \cos(p_l R))$ on the right side of (3.3) under the summation sign in a power series. Keeping in mind the fact that for a multipole of order l we have

[9)]A particular case, namely, the transformation of dipole moments, is examined in greater detail in Sec. 4.

[10)]It is evident that the orientation of the force $F_{\omega\rho}$ for charges with the same signs is opposite to that for dipoles (see Fig. 2).

$$\sum_{i=1}^{n} e_i r_i^s = 0, \quad s < l,$$

since it does not have moments of order less than l, and the fact that by definition

$$\sum_{i=1}^{n} e_i r_i^l = p_l,$$

and assuming that the r_i are so small that the sums of the products containing r_i in powers greater than l can be neglected, from (3.3) we obtain the following expression for a multipole of order l

$$Ae^{i\omega t}\left[(-i)^l p_l \frac{\omega^l n^l \cos^l (p_l R)}{l!\, c^l}\right]. \quad (3.5)$$

Thus the radiation field of the multipole differs from the field of a moving unit charge by a factor equal in magnitude to the expression in brackets. Therefore, the energy emitted by a multipole differs from that of a charge by the square of the modulus of this quantity. Since the angular distribution of the radiation of a charge (2.27) is uniform along all generatrices of the cone, for an angular interval $d\varphi$ in the case of a unit charge we have

$$\frac{1}{2\pi} W_\omega \, d\varphi = \frac{\mu}{2\pi c^2}\omega\left(1 - \frac{1}{\beta^2 n^2}\right) d\varphi. \quad (3.6)$$

From here, for an electric multipole, keeping in mind (3.4), we obtain

$$W(p_l) = \int_0^{2\pi} d\varphi \int_{\beta n > 1} \frac{\omega^{2l+1}\mu n^{2l} p_l^2}{2\pi (l!)^2 c^{2l+2}}\left(1 - \frac{1}{\beta^2 n^2}\right)$$
$$\times\left[\sqrt{1 - \frac{1}{\beta^2 n^2}}\sin\vartheta\cos\varphi + \frac{1}{\beta n}\cos\vartheta\right]^{2l} d\omega. \quad (3.7)$$

Equation (3.7) gives the spectral distribution, as well as the angular distribution of radiation along different generatrices of the cone.

The energy emitted by the multipole can be found from (3.7) by integrating over φ.

Using the fact that the binomial expansion of the term in brackets in (3.7) is equal to

$$\sum_{k=0}^{2l} \frac{(2l)!}{(2l-k)!\,k!}\,\frac{1}{(\beta n)^{2l-k}}\left(1 - \frac{1}{\beta^2 n^2}\right)^{k/2} \cos^{2l-k}\vartheta\cos^k\varphi\sin^k\vartheta, \quad (3.8)$$

and that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2m}\varphi\, d\varphi = \frac{(2m)!}{2^{2m}(m!)^2},$$
$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2m+1}\varphi\, d\varphi = 0, \quad (3.9)$$

we obtain (setting $k = 2m$)

$$W(p_l) = \sum_{m=0}^{l} \frac{(2l)!}{(l!)^2 (2l-2m)!\,(m!)^2\, 2^{2m}}$$
$$\times \int_{\beta n>1} \frac{\omega^{2l+1}\mu n^{2m} p_l^2}{c^{2m+2} v^{2l-2m}}\left(1 - \frac{1}{\beta^2 n^2}\right)^{m+1}$$
$$\times \cos^{2l-2m}\vartheta\sin^{2m}\vartheta\, d\omega. \quad (3.10)$$

For particular cases of electric multipoles of order l oriented parallel ($\sin\vartheta = 0$) and perpendicular ($\cos\vartheta = 0$) to the velocity, we have, respectively,

$$W(p_l) = \frac{1}{(l!)^2}\int_{\beta n>1} \frac{\omega^{2l+1}\mu p_l^2}{c^2 v^{2l}}\left(1 - \frac{1}{\beta^2 n^2}\right) d\omega, \quad p_l \parallel \mathbf{v}, \quad (3.11)$$

$$W(p_l) = \frac{(2l)!}{(l!)^4\, 2^{2l}}\int_{\beta n>1} \frac{\omega^{2l+1} n^{2l}\mu}{c^{2l+2}} p_l^2$$
$$\times\left(1 - \frac{1}{\beta^2 n^2}\right)^{l+1} d\omega, \quad p_l \perp \mathbf{v}. \quad (3.12)$$

For a dipole ($l = 1$), a quadrupole ($l = 2$), and an octupole ($l = 3$) Eqs. (3.10)–(3.12) coincide with Eqs. (2.29)–(2.31) as well as Eqs. (2.33) and (3.24), obtained in Sec. 2. These results can be compared with the results obtained by Shirobokov,[13] who made a quantum analysis of Vavilov-Cherenkov radiation for a particle with spin 2. Separate terms of the expression obtained by him are interpreted as radiation from a quadrupole and an octupole and the remaining terms are interpreted as interference terms. Indeed, they differ from (3.12) for $l = 1$, 2, and 3 only by numerical factors, i.e., in the classical analysis they correspond to a transverse dipole, quadrupole and ocupole. Equation (3.7) also makes it possible to find the magnitude of the forces acting on a multipole.

We denote by $W_{\omega\varphi}$ the quantity in the integrand in Eq. (3.7), i.e., the energy emitted per unit path for angle φ and frequency ω. The momentum carried away by the radiation into the medium is equal to[4]

$$p_{\omega\varphi} = \frac{n}{c}\,\varepsilon_{\omega\varphi}, \quad (3.13)$$

where $\varepsilon_{\omega\varphi} = vW_{\omega\varphi}$ is the energy emitted per second. Thus a recoil force given by

$$F_{\omega\varphi} = -p_{\omega\varphi} = -\beta n W_{\omega\varphi} \quad (3.14)$$

must act on the multipole.

The projection of the force on any axis is obtained by multiplying (3.14) by the cosine of the angle between the direction of the ray and this axis and integrating over φ. The cosine of the angle between the direction of the radiation and the z axis is equal to $1/\beta n$. Therefore, the force retarding the multipole is equal to

$$F_z(p_l) = -\int_0^{2\pi} d\varphi \int_{\beta n>1} W_{\omega\varphi}\, d\omega = -W(p_l). \quad (3.15)$$

Conversely, the correctness of (3.15) is a simple proof of the fact that in a refracting medium the momentum and the energy of the radiation are related by Eq. (3.13).

The cosine of the angle between the direction of the ray and the direction chosen as the positive orientation of the force $F_{\omega\rho}$ is equal to

$$\cos(\varphi, \rho) = \sqrt{1 - \frac{1}{\beta^2 n^2}}\cos\varphi. \quad (3.16)$$

From here and from (3.14), and using also (3.8) and (3.9), we obtain

$$F_\rho(p_l) = -\sum_{m=1}^{l} \frac{(2l)!\, m}{(l!)^2 (2l-2m+1)!\,(m!)^2\, 2^{2m-1}} \int_{\beta n>1} \frac{\omega^{2l+1} n^{2m} p_l^2 \mu}{c^{2m+2} v^{2l-m}}$$
$$\times\left(1 - \frac{1}{\beta^2 n^2}\right)^{m+1} \cos^{(2l-2m+1)}\vartheta\sin^{2m-1}\vartheta\, d\omega. \quad (3.17)$$

As expected, in the particular case $l = 1$, i.e., for a dipole, Eq. (3.17) coincides with Eq. (2.35).

4. VAVILOV-CHERENKOV RADIATION FOR ELEMENTARY ELECTRIC AND MAGNETIC DIPOLES

To study magnetic dipole and magnetic multipole fields, we can use methods which are analogous to those used in the preceding sections of this paper for electric multipoles. To this end, we must write down the equation for a moving magnetic pole and then treat a magnetic multipole as a collection of magnetic poles (see Sec. 5). This is what was done in Ref. 7, which, aside from solving the problem of Vavilov-Cherenkov radiation for magnetic multipoles, addressed the problem of clarifying the anomalous radiation of a magnetic dipole, found in 1942, i.e., ten years earlier.[6] It was found that the results of Refs. 6 and 7 can be made to agree if a relationship differing from the one used in 1942 between the magnetic dipole and the electric dipole moment induced by the motion of the magnetic dipole is used. The problem is actually more complicated than assumed then and I believe that the situation is still not completely clear. The main point, however, becomes obvious, if as is done here (in contrast to Ref. 7), the analysis is performed for a medium with $\mu \neq 1$. A magnetic dipole is an elementary current loop. Reference 7 contains the assumption that there exists a complete identity between a magnetic moment created by a current loop (we shall call it simply a magnetic moment or the usual magnetic moment, since other types of magnetic moments have not yet been observed in nature) and a hypothetical dipole formed by two magnetic poles (we can conditionally call it a true magnetic dipole).

It is known, however, that when in motion both forms of magnetic dipoles must interact differently with the magnetic field in media with $\mu \neq 1$. This makes it possible, in particular, to prove that the magnetic moment of a neutron is not associated with magnetic charges and has the usual nature (for a discussion of this point, see, for example, Ref. 14).

It is therefore not obvious a priori whether or not the properties of the VCR from the different kinds of magnetic dipoles will be the same. A comparison of the results of this section with those of the next one (Sec. 5) shows that this is indeed not the case in general. At the same time, very profound analogies, which are incorporated in the symmetry of Maxwell's equations with respect to electric and magnetic charges, which are discussed in Sec. 5, do exist here. Although these analogies were clarified back in 1952,[7] the differences were not, strange as it may seem, noted and were never discussed either then or in the following 30 years.[11)]

These questions are analyzed primarily in the next section (Sec. 5). In this section, we shall show that for an electric dipole the same results are obtained regardless of whether the motion of two oppositely charged electric charges moving parallel to one another in a medium or the motion of an elementary electric dipole is studied. This result is not obvious beforehand, because, since different methods are used, the polarization of the medium may not have been treated in the same manner. The second question which is examined is the difference between the radiation from electric and magnetic dipoles. For what follows, we must briefly recapitulate the results obtained for VCR of electric and magnetic dipoles in Ref. 6, generalizing them to the case of a medium with $\mu \neq 1$. The starting equations were

$$\nabla^2 \mathbf{P}_\omega + \frac{\omega^2 n^2}{c^2}\mathbf{P}_\omega = -4\pi \mathbf{p}_\omega, \tag{4.1}$$

$$\nabla^2 \mathbf{M}_\omega + \frac{\omega^2 n^2}{c^2}\mathbf{M}_\omega = -4\pi \mathbf{m}_\omega, \tag{4.2}$$

where $\mathbf{P}_\omega$ is the component of the Hertz vector with frequency ω, $\mathbf{M}_\omega$ is the corresponding magnetic vector, and $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$ are the components of the electric and magnetic dipole moment densities.

If $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$ are obtained here by expanding the density of moving dipoles in a Fourier integral, then it should be kept in mind that their values measured in a stationary coordinate system, which are related to the intrinsic moments by Eq. (2.36), are being used.

Here, by $\mathbf{m}_\omega$ we shall mean, for the time being, the density of the usual (current) magnetic moments. It would have been more consistent to write these equations in a form containing in the denominator on the right side ε and μ, respectively; $\mathbf{P}_\omega$ and $\mathbf{M}_\omega$ would then be normalized differently. For our purposes, however, the form of Eqs. (4.1) and (4.2) is more convenient, because of their symmetry with respect to $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$, since the relationships between $\mathbf{p}_\omega$ and $\mathbf{P}_\omega$ and between $\mathbf{m}_\omega$ and $\mathbf{M}_\omega$ are the same in this case. To transform from the field equations containing $\mathbf{A}_\omega$ to equations written in terms of $\mathbf{P}_\omega$ and $\mathbf{M}_\omega$, we use the equations

$$i\omega \mathbf{p}_\omega + c\,\mathrm{curl}\,\mathbf{m}_\omega = \mathbf{j}_\omega, \tag{4.3}$$

$$\mathrm{div}\,\mathbf{p}_\omega = -\rho_\omega. \tag{4.4}$$

Comparing (4.1) and (4.2) with (4.3), we obtain[7]

$$\mathbf{A}_\omega = \mu\left(\frac{i\omega}{c}\mathbf{P}_\omega + \mathrm{curl}\,\mathbf{M}_\omega\right); \tag{4.5}$$

analogously, with the help of (4.4), we obtain

$$\varphi_\omega = \frac{1}{\varepsilon}\,\mathrm{div}\,\mathbf{P}_\omega. \tag{4.6}$$

Substituting (4.5) into (1.4) and (1.5), we obtain

$$\mathbf{E}_\omega = \frac{\mu}{n^2}\,\mathrm{grad\,div}\,\mathbf{P}_\omega + \frac{\omega^2}{c^2}\mu \mathbf{P}_\omega - \frac{i\omega\mu}{c}\mathrm{curl}\,\mathbf{M}_\omega, \tag{4.7}$$

$$\mathbf{H}_\omega = \mathrm{curl\,curl}\,\mathbf{M}_\omega + \frac{i\omega}{c}\mathrm{curl}\,\mathbf{P}_\omega. \tag{4.8}$$

In what follows, we shall examine the field in the radiation zone in the region where $\mathbf{p}_\omega = 0$ and $\mathbf{m}_\omega = 0$.

Using the formula from vector algebra (1.12) as well as (4.1) with $p_\omega = 0$, Eq. (4.7) can be written as follows:

$$\mathbf{E}_\omega = \frac{\mu}{n^2}\,\mathrm{curl\,curl}\,\mathbf{P}_\omega - \frac{i\omega\mu}{c}\mathrm{curl}\,\mathbf{M}_\omega. \tag{4.9}$$

Comparing these equations for $\mathbf{H}_\omega$ and $\mathbf{E}_\omega$ it is not difficult to verify that they cannot be obtained from one another by replacing ε by μ and $\mathbf{P}_\omega$ by $\mathbf{M}_\omega$. In what follows, for our purposes, the magnetic moments induced by the moving

[11)] V. L. Ginzburg and V. M. Tsytovich have informed me that they also considered them in the course of their work on the book "Transition Radiation and Transition Scattering".[15] They confirmed the equations known previously and independently obtained some of the results which I present in this paper.

electric dipole and the electric moments induced by a moving magnetic moment are significant. Whereas the equations relating the moving dipole to the induced dipole are analogous in both cases, the nonequivalence of $\mathbf{P}_\omega$ and $\mathbf{M}_\omega$ in Eqs. (4.8) and (4.9) necessarily leads to the fact that different results are obtained for VCR generated by $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$.

We shall assume that a point electric dipole p, whose orientation coincides with an arbitrarily oriented unit vector $\mathbf{p}_1$ (the magnitude and orientation of $\mathbf{p}$ are measured in a stationary coordinate system), moves with velocity v along the z axis ($z = vt$). Analogously to the manner in which this is done for finding the current density (see (1.13)), the component $\mathbf{p}_\omega$ of the dipole moment density must be assigned the value[6]

$$\mathbf{p}_\omega(z)\,e^{i\omega t} = \frac{\mathbf{p}_1}{2\pi v}\,p e^{i\omega\left(t-\frac{z}{v}\right)}\,\delta(x)\,\delta(y) \tag{4.10}$$

We shall assume below that the dipole traverses a finite path from $-z_0$ to $+z_0$. Then

$$\mathbf{P}_\omega e^{i\omega t} = \frac{\mathbf{p}_1}{R}\,C' p e^{i\omega\left(t-\frac{Rn(\omega)}{c}\right)}, \tag{4.11}$$

where R is the distance from the origin of coordinates $z = 0$ to the point of observation; in addition, we assume that $R \gg z_0$, i.e., that the radiation is studied in the radiation zone. Then

$$C' = \frac{1}{2\pi v}\int_{-z_0}^{+z_0} e^{-i\omega(z/v)(1-\beta n\cos\theta)}\,dz = \frac{1}{\pi\omega}\,q(\omega,\theta), \tag{4.12}$$

where, as done by Tamm,[2] $q(\omega,\theta)$ denotes

$$q(\omega,\theta) = \frac{\sin\omega(z_0/v)(1-\beta n\cos\theta)}{(1-\beta n\cos\theta)}. \tag{4.13}$$

Thus the Hertz vector in the radiation zone has the same form as for some stationary oscillator with an angle-dependent amplitude $q(\omega,\theta)p/\pi\omega$. Analogously, if a magnetic dipole, whose orientation coincides with the vector $\mathbf{m}_1$, is moving, then

$$\mathbf{M}_\omega e^{i\omega t} = \frac{\mathbf{m}_1}{2\pi\omega R}\,mq(\omega,\theta)\,e^{i\omega(t-Rn/c)}. \tag{4.14}$$

For electric and magnetic vectors in the radiation zone, it follows from (4.8) and (4.9) ($\mathbf{r}_1$ is the unit vector along the ray R) that

$$\mathbf{E}_\omega = -\frac{\omega^2\mu}{c^2}\,[\mathbf{r}_1[\mathbf{r}_1\mathbf{p}_1]]\,P_\omega - \frac{\omega^2 n\mu}{c^2}\,[\mathbf{r}_1\mathbf{m}_1]\,M_\omega, \tag{4.15}$$

$$\mathbf{H}_\omega = \frac{\omega^2 n^2}{c^2}\,[\mathbf{r}_1\mathbf{p}_1]\,P_\omega - \frac{\omega^2 n^2}{c^2}\,[\mathbf{r}_1[\mathbf{r}_1\mathbf{m}_1]]\,M_\omega; \tag{4.16}$$

Here, as it has to be,

$$\mathbf{H}_\omega = \sqrt{\frac{\varepsilon}{\mu}}\,[\mathbf{r}_1[\mathbf{r}_1\mathbf{E}_\omega]]. \tag{4.17}$$

If we are studying the motion of an electric dipole, then aside from $\mathbf{p}_\omega$, responsible for the Hertz vector $\mathbf{P}_\omega$, as already noted, there arises a current-induced magnetic moment $\mathbf{m}_\omega$. In this case, except for the case when $\mathbf{p}_1$ is parallel to z, $\mathbf{m}_\omega$ does not equal zero.

The relationship between $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$ follows uniquely from the relativistic transformations of the current.[6,7] If the polarization of the medium does not affect these transformations, then $\mathbf{p}_\omega$ induces a magnetic moment

$$\mathbf{m}_\omega = -\beta\,[\mathbf{z}_1\mathbf{p}_\omega] \tag{4.18}$$

(where $\mathbf{z}_1$ is a unit vector oriented along the velocity (the z axis)). Analogously, the relation

$$\mathbf{p}_\omega = +\beta\,[\mathbf{z}_1\mathbf{m}_\omega] \tag{4.19}$$

must hold for a moving magnetic moment. These formulas are indeed symmetrical with respect to $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$. From here, using Eqs. (4.1) and (4.2), we find that in the case of a moving electric dipole we can make the following substitution in Eq. (4.15):

$$[\mathbf{r}_1\mathbf{m}_1]\,M_\omega = -[\mathbf{r}_1[\mathbf{z}_1\mathbf{p}_1]]\beta\,P_\omega. \tag{4.20}$$

Then, instead of (4.15), we have

$$\mathbf{E}_\omega = -\frac{\omega^2\mu}{c^2}\,P_\omega\{[\mathbf{r}_1[\mathbf{r}_1\mathbf{p}_1]] - \beta n\,[\mathbf{r}_1[\mathbf{z}_1\mathbf{p}_1]]\}, \tag{4.21}$$

and, of course, Eq. (4.17) for the vector $\mathbf{H}_\omega$ remains correct. We note, and this will be proved below, that for limiting values of $\mathbf{P}_\omega$ the intensity of the VCR of an electric dipole obtained here coincides with (2.33) and (2.34). Let us suppose that a magnetic dipole is in motion. Using the transformation (4.19), we obtain in an analogous manner

$$\mathbf{H}_\omega = -\frac{\omega^2 n^2}{c^2}\,M_\omega\left\{[\mathbf{r}_1[\mathbf{r}_1\mathbf{m}_1]] - \frac{1}{n}\,\beta\,[\mathbf{r}_1[\mathbf{z}_1\mathbf{m}_1]]\right\} \tag{4.22}$$

$$\mathbf{E}_\omega = -\sqrt{\frac{\mu}{\varepsilon}}\,[\mathbf{r}_1\mathbf{H}_\omega]. \tag{4.23}$$

In order that the quantity in the braces in (4.22) be the analog of (4.21), the transformation (4.19) must be replaced by

$$\mathbf{p}_\omega = +\beta n^2\,[\mathbf{z}_1\mathbf{m}_\omega]. \tag{4.24}$$

This would indicate that the transformations of $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$ would differ significantly. The proposition that (4.24) must be used was stated in Ref. 7 (Eq. (4.29) of Ref. 7). Since a medium with $\mu = 1$ and $\varepsilon \equiv n^2 \neq 1$ was being studied at that time, the idea arose (see the remark for Eq. (4.29) in Ref. 7) that the difference between the transformations of $\mathbf{p}_\omega$ and $\mathbf{m}_\omega$ can be explained by this circumstance. From what follows it will be evident that this is not the case. An argument in favor of (4.24) in Ref. 7 was that an analogy does indeed exist, as it should, between an electric dipole and a dipole formed by magnetic charges. The extension of the results to a medium with $\mu \neq 1$ shows that this argument is unjustified.

V. L. Ginzburg[16] presents in his supplement to my paper[7] support for the correctness of (4.24).[12)] However, independently of whether this transformation should be adopted or not, the electric and magnetic moments do not behave equivalently in a medium: if (4.19) is correct, then a formula is obtained for the Vavilov-Cherenkov radiation that is not at all similar to the radiation of an electric dipole; if, on the other hand, (4.24) is assumed to be correct, then the interrelationship of moving electric and magnetic dipoles in the medium turns out to be different.

Let us examine in greater detail the consequences of Eqs. (4.21) and (4.22). We shall first complete the analysis of

[12)] V. L. Ginzburg has kindly informed me that he has reexamined this question. His remark on the applicability of (4.24) refers to the particular case when the volume inside the ring current creating the magnetic moment is filled with a medium which has the same value of n as the space outside the volume and for an elementary dipole moment the results following from (4.19) are correct. This work will be published in Izv. Vyssh. Uchebn. Zaved., Ser. Radiofiz.

the field of a moving electric dipole. Using the well-known formula of vector algebra

$$[\mathbf{a}\,[\mathbf{bc}]] = (\mathbf{ac})\,\mathbf{b} - (\mathbf{ab})\,\mathbf{c}, \tag{4.25}$$

from (4.21), using (4.10) and (4.13), we obtain

$$\mathbf{E}_\omega^A = -\frac{\omega^2\mu}{c^2} q(\omega,\ \theta) \frac{pe^{i\omega nR/c}}{\pi\omega R}$$
$$\times [\cos(r_1p_1)(\mathbf{r}_1 - \beta n\mathbf{z}_1) - (1-\beta n\cos\theta)\,\mathbf{p}_1]. \tag{4.26}$$

It can be shown that the integral of the energy flux at a frequency ω over time is equal to

$$S_\omega = c\sqrt{\frac{\varepsilon}{\mu}}\,|\mathbf{E}_\omega|^2$$
$$= \frac{\omega^2\mu^2\sqrt{\frac{\varepsilon}{\mu}}}{\pi^2R^2c^2} q^2(\omega,\ \theta)\,p^2\,|\cos(r_1p_1)(r_1 - \beta nz_1)$$
$$- (1-\beta n\cos\theta)\,p_1|^2. \tag{4.27}$$

Thus

$$S_\omega = \frac{\omega^2\mu n}{\pi^2c^3R^2} q^2p^2\{(1-\beta n\cos\theta)^2$$
$$+ 2\beta n\cos(r_1p_1)\cos(p_1z_1)(1-\beta n\cos\theta)$$
$$-(1-\beta^2n^2)\cos^2(r_1p_1)\}; \tag{4.28}$$

where $\cos(p_1z_1) = \cos\theta$ (see Sec. 3), and the quantity $\cos(r_1p_1) = \cos(pR)$ is given by Eq. (3.4).

To calculate the total magnitude of the energy flux, S_ω must be multiplied in (4.28) by the solid-angle element $d\Omega = R^2\sin\theta\, d\theta\, d\varphi$ and an integration must be carried out. We shall first integrate over θ. For this, we must know the integral of $q^2(\omega,\theta)\sin\theta$. This integral was discussed by Tamm for the limiting case of large z_0 for the VCR of an electric charge.[2] We shall also examine the limiting case of the integral[13)]

$$\int_0^\pi q^2(\omega,\ \theta)\sin\theta\, d\theta = \int_0^\pi \frac{\sin^2(\omega z/v)(1-\beta n\cos\theta)}{(1-\beta n\cos\theta)^2}\sin\theta\, d\theta. \tag{4.29}$$

Making a substitution of variables and setting

$$u = \frac{\omega z}{v}(1-\beta n\cos\theta), \tag{4.30}$$

we reduce (4.29) to the form

$$\frac{c\omega z_0}{v^2n}\int_{-\frac{\omega z_0}{v}(\beta n-1)}^{+\frac{\omega z_0}{v}(\beta n+1)} \frac{\sin^2 u}{u^2}\,du. \tag{4.31}$$

As is well known, the magnitude of the integral approaches π, if the limits of integration approach infinity. Therefore, the necessary condition for passing to the limit is that the magnitudes of the limits of integration must be large. This excludes, even for large z_0, from the analysis the region close to the VCR threshold, where $\beta n - 1$ is small. In the limiting case we can write $u = z_0x$ in the integrand and then

$$\left(\frac{\sin^2 z_0x}{z_0x^2}\right)_{z_0\to\infty} \to \pi\delta(x).$$

13)This integral, though elementary, with the exception of a limiting case is quite cumbersome (see Eqs. (14) and (15) in Ref. 17).

In this case, $x = 0$ corresponds to $\cos\theta = 1/\beta n$. Thus, in the limiting case, we have

$$\int_{-\infty}^{+\infty} q^2(\omega,\ \theta)\sin\theta f(\omega,\ \cos\theta)\,d\theta = \frac{c\pi\omega z_0}{v^2n} f\left(\omega,\ \frac{1}{\beta n}\right). \tag{4.32}$$

Applying this relation to (4.28), keeping in mind (3.4) and referring the energy to unit path length by dividing the result by $2z_0$, we obtain

$$W(p,\ \omega) = \frac{\omega^3\mu n^2}{2\pi c^4} p^2\left(1-\frac{1}{\beta^2n^2}\right)$$
$$\times\int_0^{2\pi} d\varphi\left(\sqrt{1-\frac{1}{\beta^2n^2}}\sin\vartheta\cos\varphi + \frac{1}{\beta n}\cos\vartheta\right). \tag{4.33}$$

This formula coincides with (3.7) with $l = 1$, and with (2.29) after integration over φ. In the particular cases of p parallel and perpendicular to the velocity, we obtain (2.33) and (2.34). From here, in particular, it is evident that the transformation (4.18) for an electric dipole can be used not only for a medium with $\mu = 1$, but also for arbitrary ε and μ. Thus, contrary to the assumption made in Ref. 7, the difference between (4.18) and (4.24) cannot be explained by the fact that $\mu = 1$; moreover, the validity of (4.18) is an argument in favor of the fact that (4.19) should be applicable to an elementary magnetic dipole.

Comparing (4.21) and (4.22) for a magnetic dipole, and taking (4.23) into account, we obtain in an analogous manner, instead of (4.27),

$$S_\omega = \frac{\omega^2n^4\sqrt{\frac{\mu}{\varepsilon}}}{\pi^2R^2c^3} q^2(\omega,\ \theta)\,m^2\left|\cos(r_1,\ \mathbf{m}_1)\left(\mathbf{r}_1 - \frac{\beta}{n}\mathbf{z}_1\right)\right.$$
$$\left. - \left(1-\frac{\beta}{n}\cos\theta\right)\mathbf{m}_1\right|^2.$$

From here it follows that in the formula analogous to (4.28) p^2 must be replaced by m^2, an additional factor of n^2 must be inserted in front of the braces, and within the braces p_1 and βn must be replaced by m_1 and β/n. As a result, applying (4.32), we obtain

$$W(m,\ \omega) = \frac{\omega^3\mu n^2}{2\pi c^2v^2} m^2\int_0^{2\pi} d\varphi\left[\left(1-\frac{1}{n^2}\right)^2\right.$$
$$+ 2\frac{\beta}{n}\left(\sqrt{1-\frac{1}{\beta^2n^2}}\sin\vartheta\cos\varphi + \frac{1}{\beta n}\cos\vartheta\right)$$
$$\times\cos\vartheta\left(1-\frac{1}{n^2}\right)$$
$$-\left(1-\frac{\beta^2}{n^2}\right)$$
$$\left.\times\left(\sqrt{1-\frac{1}{\beta^2n^2}}\sin\vartheta\cos\varphi + \frac{1}{\beta n}\cos\vartheta\right)^2\right]. \tag{4.34}$$

This quite complicated formula gives the angular distribution of the radiation from a magnetic dipole.

It is not difficult to verify that for a magnetic dipole oriented along the velocity ($\cos\vartheta = 1$), the energy flux does not depend on the angle φ, i.e., the radiation is the same along all generatrices of the cone. Its total magnitude is equal to

$$W(m,\ \omega)=\frac{m^2}{v^2c^2}\,\omega^3n^2\mu\left(1-\frac{1}{\beta^2n^2}\right),\quad m \parallel \mathbf{v}. \qquad (4.35)$$

This formula is similar to the one obtained for an electric dipole (see (2.33)). Moreover, for $\mu = 1$ it may be assumed to be the magnetic analog of (2.33), since $n^2 \equiv \varepsilon$ (see Eq. (5.18)). For this reason, Eq. (4.35), obtained for $\mu = 1$ in Refs. 4 and 6 and repeated in Ref. 7, did not give rise to any doubts. The situation is different for a magnetic dipole oriented perpendicular to the velocity ($\sin\vartheta = 1$).

The angular distribution, as is evident from (4.34), in this case is a complicated function of φ, and after integration we obtain

$$W(m,\ \omega)=\frac{m^2}{2v^2c^2}\,\omega^3n^2\mu\left[2\left(1-\frac{1}{n^2}\right)^2 - \left(1-\frac{\beta^2}{n^2}\right)\left(1-\frac{1}{\beta^2n^2}\right)\right],\quad m \perp \mathbf{v}. \qquad (4.36)$$

If, however, Eq. (4.24) and not Eq. (4.19) is used for the relationship between the moving magnetic dipole and the electric dipole it induces, then we obtain

$$W(m,\ \omega)=\frac{m^2}{2c^4}\,\omega^3n^4\,\mu\left(1-\frac{1}{\beta^2n^2}\right)^2,\quad m \perp \mathbf{v}. \qquad (4.37)$$

Several arguments in support of (4.37) with $\mu = 1$ and for considering Eq. (4.36) to be erroneous are given in Ref. 7. Indeed, Eq. (4.37) is similar to Eq. (2.34) for an electric dipole. Moreover, with $\mu = 1$ it may be regarded to be its magnetic analog, since the extra factor of n^2 in this case is equal to ε. Finally, $W(\omega)$ in Eq. (4.36), in contrast to all that we know for other cases of VCR, does not approach zero in the limit $\beta n \to 1$.[14)] To be sure, in contrast to what was said in Ref. 7, it was commonly believed that Eq. (4.36) is nevertheless correct and that (4.37) corresponds to a different case of polarization of the medium.[18] The generalization to the case $\mu \neq 1$ presented here invalidates the argument about the analogy of (4.35) and (4.37) to a dipole consisting of magnetic charges, for which μ must be replaced by ε (see the next section). We can see that an analogy does not necessarily exist here. There are therefore no grounds for regarding Eq. (4.36) as incorrect.

5. RADIATION FROM A MAGNETIC CHARGE AND FROM MAGNETIC DIPOLES AND MAGNETIC MULTIPOLES FORMED BY MAGNETIC CHARGES

The field of a magnetic charge or a system of magnetic charges can be analyzed in an elementary way, if we start from the symmetry of Maxwell's equations relative to the electric charges e and magnetic charges g. All the equations of the preceding Secs. 1–3 are applicable to magnetic charges with $\mathbf{E}_\omega$ replaced by $\mathbf{H}_\omega$ and $\mathbf{H}_\omega$ replaced by $-\mathbf{E}_\omega$, if ε and μ are interchanged at the same time.

However, the use of this symmetry, implicitly assumes that either only electric charges e or only magnetic charges g are being studied. The possibility of the coexistence of such charges is unclear and will not be discussed in this paper.

14) It should be noted, however, that the derivation of all the formulas presented here for dipoles, as already noted, cannot be used in the region near the threshold ($\beta n = 1$), since the limiting value of the integral (4.31) cannot be used in this case.

However, since the medium is characterized by only macroscopic quantities ε and μ, it makes no difference whether it is an ordinary medium or one made up of magnetic charges.

To obtain the field of magnetic charges, we shall use the method used in Ref. 7. In analogy to (4.3), we can write

$$i\omega m_\omega = j_{g\omega}; \qquad (5.1)$$

where $m_\omega = m_{g\omega}$ is the magnetic moment formed by the magnetic charges, so that $j_\omega = j_{g\omega}$ is the current of the magnetic charges. Formally, however, we can write (5.1) for an ordinary magnetic moment m_ω and then j_ω is some auxiliary quantity, which can be called a pseudocurrent.

We introduce the magnetic vector potential

$$\mathbf{K}_\omega=\frac{i\omega\varepsilon}{c}\,\mathbf{M}_\omega. \qquad (5.2)$$

It differs only by its normalization (additional cofactor of ε) from the one used in Ref. 7 and is the analog of the relationship between $\mathbf{A}_\omega$ and $\mathbf{P}_\omega$ (see Eq. (4.5)). We then obtain immediately from (4.2) the equation

$$\nabla^2\mathbf{K}_\omega+\frac{\omega^2n^2}{c^2}\,\mathbf{K}_\omega=-\frac{4\pi\varepsilon}{c}\,\mathbf{j}_{g\omega}, \qquad (5.3)$$

which is the magnetic analog of Eq. (1.1). We can also write an equation for the static magnetic potential χ_ω. For this, we set

$$\operatorname{div}\mathbf{m}_{g\omega} = -\rho_{g\omega} \qquad (5.4)$$

and assume that

$$-\frac{1}{\mu}\operatorname{div}\mathbf{M}_\omega=\chi_\omega. \qquad (5.5)$$

It then follows from (4.2) that

$$\nabla^2\chi_\omega+\frac{\omega^2\mu\varepsilon}{c^2}\,\chi_\omega=-\frac{4\pi}{\mu}\,\rho_{g\omega}. \qquad (5.6)$$

A consequence of (5.2) and (5.5) is that the Lorentz relation

$$\operatorname{div}\mathbf{K}_\omega+\frac{i\omega}{c}\,n^2\chi_\omega=0 \qquad (5.7)$$

is satisfied.

We now return to the proposition that m_ω is the usual magnetic moment. In this case, $\mathbf{M}_\omega$ is uniquely related to $\mathbf{A}_\omega$. If $\mu = 1$, then $\mathbf{A}_\omega = \operatorname{curl}\mathbf{M}_\omega$ (see (4.5)). Using Eq. (5.2), we obtain

$$\mathbf{A}_\omega=-\frac{ic}{\omega\varepsilon}\operatorname{curl}\mathbf{K}_\omega. \qquad (5.8)$$

Here, as already noted, it may be assumed that $\mathbf{K}_\omega$ satisfies (5.3), if $\mathbf{j}_{g\omega}$ is taken to mean $i\omega\mathbf{m}_\omega$. If this value of $\mathbf{K}_\omega$ is substituted into (1.4), then we obtain

$$\mathbf{E}_{g\omega}=-\frac{1}{\varepsilon}\operatorname{curl}\mathbf{K}_\omega, \qquad (5.9)$$

which indeed is the magnetic analog of (1.5). Substitution of this $\mathbf{K}_\omega$ into (1.5) gives

$$\mathbf{H}_\omega=-\frac{ic}{\omega\varepsilon\mu}\operatorname{curl}\operatorname{curl}\mathbf{K}_\omega. \qquad (5.10)$$

This equation can be written with the help of (1.12) and (5.3), in the region where $m_\omega = 0$, as

$$\mathbf{H}_\omega=-\frac{ic}{\omega\varepsilon\mu}\operatorname{grad}\operatorname{div}\mathbf{K}_\omega-\frac{i\omega}{c}\,\mathbf{K}_\omega. \qquad (5.11)$$

Equations (5.9) and (5.11) can be written immediately as the magnetic analog of (1.1), (1.4), and (1.5). Here they have been obtained from the representations of the usual magnetic moment, and this is an obvious consequence of the fact that the electromagnetic field of both types of magnetic moments in a medium with $\mu = 1$ have a far-reaching similarity. For this reason, in Ref. 7, where it is stated that these fields are identical, its validity was not doubted, although it was necessary to replace arbitrarily the transformation (4.19) by (4.24), which coincide only in vacuum.

In reality, however, in a medium differences do exist between the fields of both types of dipoles and they are immediately evident, if it is assumed that $\mu \neq 1$.

Indeed, then, as follows from (4.5), $\mathbf{A}_\omega = \mu$ curl $\mathbf{M}_\omega$ and, therefore, instead of (5.8) we obtain

$$\mathbf{A}_\omega = -\frac{ic}{\omega}\frac{\mu}{\varepsilon}\,\mathrm{curl}\mathbf{K}_\omega.$$

Substituting this quantity into (1.4), we obtain for $\mathbf{E}_\omega$ a value that is a factor of μ greater than (4.9) and is therefore not the magnetic analog of (1.5).

As already noted in Sec. 4, the quantities $\mathbf{P}_\omega$ and $\mathbf{M}_\omega$ do not enter symmetrically in Eqs. (4.8) and (4.9) and this leads to substantial differences in the results for the radiation from electric and magnetic dipoles.

To obtain the analog of the electric dipole, it must be assumed at the outset that the magnetic dipole consists of two magnetic charges, using the analogy between the equations for electric and magnetic charges. Thus it must be assumed that

$$i\omega\mathbf{m}_{g\omega} + c\,\mathrm{curl}\mathbf{p}_{g\omega} = \mathbf{j}_{g\omega}, \qquad (5.12)$$

$$\mathbf{K}_\omega = \varepsilon\left(\frac{i\omega}{c}\mathbf{M}_{g\omega} + \mathrm{curl}\mathbf{P}_{g\omega}\right). \qquad (5.13)$$

Without repeating everything that was said in Sec. 4, we immediately write down the analog of (4.15) and (4.16)

$$\mathbf{H}_\omega = -\frac{\omega^2\varepsilon}{c^2}[\mathbf{r}_1[\mathbf{r}_1\mathbf{m}_1]]\,M_{g\omega} - \frac{\omega^2 n\varepsilon}{c^2}[\mathbf{r}_1\mathbf{p}_1]\,P_{g\omega}, \qquad (5.14)$$

$$\mathbf{E}_\omega = -\frac{\omega^2 n}{c^2}[\mathbf{r}_1\mathbf{m}_1]\,M_{g\omega} + \frac{\omega^2 n^2}{c^2}[\mathbf{r}_1[\mathbf{r}_1\mathbf{p}_1]]\,P_{g\omega}, \qquad (5.15)$$

$$\mathbf{E}_\omega = -\sqrt{\frac{\mu}{\varepsilon}}[\mathbf{r}_1\mathbf{H}_{g\omega}]. \qquad (5.16)$$

Using (4.19) instead of (4.21), we obtain

$$\mathbf{H}_\omega = -\frac{\omega^2\varepsilon}{c^2}M_{g\omega}\{[\mathbf{r}_1[\mathbf{r}_1\mathbf{m}_1] - \beta n[\mathbf{r}_1[\mathbf{z}_1\mathbf{m}_1]]\}, \qquad (5.17)$$

which is its analog with $\mathbf{E}_\omega$ replaced by $\mathbf{H}_\omega$ and $\mathbf{P}_\omega$ replaced by $\mathbf{M}_\omega$.

From here, for the VCR from a dipole formed by two magnetic poles, equations analogous to (2.33) and (2.34) should be obtained, namely,

$$W = \int\limits_{\beta n(\omega)>1}\frac{\omega^3\varepsilon}{c^2v^2}m^2\left(1-\frac{1}{\beta^2n^2}\right)d\omega, \quad m\parallel\mathbf{v}, \qquad (5.18)$$

$$W = \int\limits_{\beta n(\omega)>1}\frac{\omega^3n^2\varepsilon}{2c^4}m^2\left(1-\frac{1}{\beta^2n^2}\right)^2d\omega, \quad m\perp\mathbf{v}. \qquad (5.19)$$

These formulas must be compared to the formulas for the usual magnetic moment. As already noted in the discussion of Eq. (4.35), i.e., for a dipole oriented parallel to the velocity with $\mu = 1$ and $n^2 \equiv \varepsilon$, it coincides with (5.18).

With regard to a magnetic dipole oriented perpendicular to the velocity, the assumption that the transformation (4.24) with $\mu = 1$ is valid indeed makes (4.37) agree with (5.19), since in this case $n^4 \equiv n^2\varepsilon$.

For $\mu \neq 1$, agreement cannot be achieved, even with an assumption such as (4.24).

The system of equations (5.3), (5.6), (5.7), (5.8), and (5.11), as already noted, is the magnetic analog of (1.1)–(1.5). Here, if (5.11) is viewed as the analog of (1.4), then there is no need to restrict its application to the region where $j_{g\omega}$ vanishes. The agreement between these equations and Maxwell's equations for magnetic charges and magnetic currents is demonstrated in Sec. 6.

Using these equations, we can obtain the equations for the VCR from a magnetic charge and from any multipole, analogous to the analysis of Secs. 1 and 2 for an electric charge and multipole.

We shall examine the motion of a point magnetic charge g with velocity v, as is done in Eq. (1.13) for the electric charge e. Comparing (1.1) and (5.3), in analogy to (1.15), we can set

$$K_{\omega z} = \frac{g\varepsilon}{2c}a(\rho,\ \omega)\,e^{-i\omega z/v}, \quad K_{\omega\rho} = K_{\omega\varphi} = 0; \qquad (5.20)$$

then $a(\omega,\rho)$ satisfies Eq. (1.16). From here, from (1.20), (5.10), and (5.11), analogously to how this was done in Sec. 1, we obtain

$$E_{\omega\varphi} = \frac{g}{2c}\frac{\partial a}{\partial\rho}e^{-i\omega z/v}, \qquad (5.21)$$

$$H_{\omega\rho} = -\frac{g\varepsilon}{2vn^2}\frac{\partial a}{\partial\rho}e^{-i\omega z/v}, \qquad (5.22)$$

$$H_{\omega z} = -\frac{ig\omega\varepsilon}{2c^2}\left(1-\frac{1}{\beta^2n^2}\right)a e^{-i\omega z/v}. \qquad (5.23)$$

These formulas can, of course, be written immediately as the magnetic analog of (1.22)–(1.24). In the same manner, by virtue of the obvious analogy to (1.26)–(1.27), for the VCR from a magnetic charge g we obtain[7]

$$W = g^2\int\limits_{\beta n(\omega)>1}\frac{\varepsilon\omega}{c^2}\left(1-\frac{1}{\beta^2n^2}\right)d\omega. \qquad (5.24)$$

As shown above (Sec. 3), knowing the radiation from a charged particle, it is not difficult to determine the radiation from any linear multipole by analyzing the interference of waves emitted by separate particles. It is evident that if the magnetic poles and electric charges are arranged in the same manner, then the result of interference will be the same. It follows from here that all formulas for the energy of the radiation and the magnitude of the forces acting on the electric multipoles (3.7), (3.10)–(3.12), (3.17) are also correct for a magnetic multipole, if in them μ is replaced by ε and the multipole moment p_l is replaced by the magnetic multipole moment m_l (in Ref. 7, as already repeatedly noted, it was assumed that $\mu = 1$, and therefore for magnetic multipoles, as compared to electric multipoles, an additional factor of $n^2 \equiv \varepsilon$ has appeared).

Further, it was shown in Sec. 3 that the formulas therein coincide with the results of Sec. 2, where the force retarding the system of charges was examined. To determine this force, it is necessary to know the action of the electric and magnetic fields on each of the charges. In the case of two electric charges, this force is equal to (2.1). In the case of magnetic charges, Eqs. (3.11), (3.12), and (3.17), under the condition that (3.13) holds, must also be correct for a magnetic charge with p_l replaced by m_l and μ replaced by ε. From here it follows in an elementary way that instead of (2.1), in the case of magnetic charges, the force acting on the charge is equal to

$$\mathbf{F}_\omega(g) = g\mathbf{H}_\omega - \frac{g\varepsilon}{c^2}[\mathbf{v}\mathbf{E}_\omega]. \qquad (5.25)$$

This formula is indeed the magnetic analog of (2.1) and, in addition, the second term in this equation is the analog of the Lorentz force for magnetic charges.

6. APPENDIX. MAXWELL'S EQUATIONS FOR MAGNETIC CHARGES AND CURRENTS

From the symmetry of the equations presented in the preceding section for magnetic charges ρ_g and currents j_g with respect to the usual equations, determining the field of electric charges ρ and currents j, it follows that Maxwell's equations must be satisfied for nonzero ρ_g and j_g.

We shall verify this, repeating mainly the discussion presented at the beginning of Sec. 1. If the electric current density is $j = 0$ and the electric charge density is $\rho = 0$, but magnetic charges g and magnetic currents j_g exist, then Maxwell's equations obviously have the form

$$\mathrm{curl}\,\mathbf{H}_\omega = \frac{1}{c}\frac{\partial \mathbf{D}_\omega}{\partial t}, \qquad (6.1)$$

$$-\mathrm{curl}\,\mathbf{E}_\omega = \frac{4\pi}{c}\mathbf{j}_{g\omega} + \frac{1}{c}\frac{\partial \mathbf{B}_\omega}{\partial t}, \qquad (6.2)$$

$$\mathrm{div}\;\mathbf{B}_\omega = 4\pi\rho_{g\omega}, \qquad (6.3)$$

$$\mathrm{div}\;\mathbf{D}_\omega = 0. \qquad (6.4)$$

For convenience of analysis, we repeat the formulas presented in the preceding section and show that they are equivalent to (6.1)–(6.4):

$$\mathbf{E}_\omega = -\frac{1}{\varepsilon}\mathrm{curl}\mathbf{K}_\omega, \qquad (6.5)$$

$$\mathbf{H}_\omega = -\frac{ic}{\omega}\frac{1}{\mu\varepsilon}\,\mathrm{grad\;div}\;\mathbf{K}_\omega - \frac{i\omega}{c}\mathbf{K}_\omega, \qquad (6.6)$$

$$\mathrm{div}\;\mathbf{K}_\omega + \frac{i\omega}{c}\varepsilon\mu\chi_\omega = 0, \qquad (6.7)$$

$$\nabla^2\mathbf{K}_\omega + \frac{\omega^2\mu\varepsilon}{c^2}\mathbf{K}_\omega = -\frac{4\pi}{c}\varepsilon\mathbf{j}_{g\omega}, \qquad (6.8)$$

$$\nabla^2\chi_\omega + \frac{\omega^2\mu\varepsilon}{c^2}\chi_\omega = -\frac{4\pi}{c}\rho_{g\omega}. \qquad (6.9)$$

First of all, it is evident that the assumption (6.5) means that (6.4) is satisfied. Further, from (6.1), substituting (6.5), we have

$$\mathrm{curl}\,\mathbf{H}_\omega - \frac{1}{c}\frac{\partial \mathbf{D}_\omega}{\partial t} = \mathrm{curl}\left(\mathbf{H}_\omega + \frac{1}{c}\frac{\partial \mathbf{K}_\omega}{\partial t}\right) = 0. \qquad (6.10)$$

Equation (6.10) is satisfied if the quantity operated on by the curl operator is the gradient of a scalar function. As usual, it is assumed that

$$-\mathrm{grad}\;\chi_\omega = \mathbf{H}_\omega + \frac{1}{c}\frac{\partial \mathbf{K}_\omega}{\partial t}. \qquad (6.11)$$

The validity of this assumption follows from the arguments given below. The field equation (6.6) for $\mathbf{H}_\omega$ follows from (6.11) and (6.7). It is evident that here Eq. (6.6) is not restricted to the region where $j_{g\omega} = 0$, as noted in going from (5.10) to (5.11). Thus with the help of the vector potential $\mathbf{K}_\omega$, it is possible to satisfy the field equations for $\mathbf{E}_\omega$ and $\mathbf{H}_\omega$ and the Maxwell equations (6.1) and (6.4), if the Lorentz condition (6.7) is adopted and the validity of (6.11) is assumed. We must now prove that the Maxwell equations (6.2) and (6.3) lead under the same conditions to the wave equations (6.8) and (6.9) for $\mathbf{K}_\omega$ and χ_ω. From Eqs. (6.3) and (6.11) we obtain

$$\mathrm{div}\;\mathbf{B}_\omega = -\mu\;\mathrm{div\;grad}\;\chi_\omega - \frac{\mu}{c}\,\mathrm{div}\frac{\partial \mathbf{K}_\omega}{\partial t} = -4\pi\rho_{g\omega}. \qquad (6.12)$$

Expressing $\mathrm{div}\,\partial\mathbf{K}_\omega/\partial t$ in terms of χ with the help of (6.7), we indeed obtain Eq. (6.9) for the scalar potential. Further, from Eqs. (6.2) and (6.5) we have

$$\mathrm{curl}\;\mathrm{curl}\,\mathbf{K}_\omega = \mathrm{grad\;div}\;\mathbf{K}_\omega - \nabla^2\mathbf{K}_\omega = \frac{4\pi\varepsilon}{c}\mathbf{j}_{g\omega} + \frac{\varepsilon}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

Using the quantity grad div $\mathbf{K}_\omega$, from (6.6) we arrive at Eq. (6.8). The system of equations (6.5)–(6.9) is thus indeed equivalent to Maxwell's equations (6.1)–(6.4).

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