METHODOLOGICAL NOTES

Charge and current electrostatics. Nonstationary sources of static fields

M. A. Miller

Institute of Applied Physics, Academy of Sciences of the USSR, Gor'kiĭ
Usp. Fiz. Nauk **142**, 147–158 (January 1984)

The problem of determination of the sources of an electrostatic field within the framework of Maxwellian electrodynamics has two forms of solution: stationary charges and nonstationary rotational current. For example, an external current which depends on the time t according to a linear law and which flows along the surface of a toroidal solenoid creates in the external region a field identical to the field of an external charge dipole. From currents which increase as even powers of t it is possible to construct magnetostatic multipoles, and from currents which increase as odd powers of t it is possible to construct electrostatic multipoles. Following Ya. B. Zel'dovich one can call toroidal current sources anapoles, and therefore a system of static fields produced by nonstationary toroidal currents can be classified as anapole static fields (anapole-statics). Such current sources have certain strange features. For example, when the current charges are placed in an electrostatic field they do not experience the usual Coulomb forces, but obtain from the field or give it an electromagnetic momentum, experiencing thereby a corresponding recoil. Taking into account this recoil re-establishes the equivalence of current and charge multipoles for any interactions.

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1. Classification of our remarks as methodological is to a certain degree forced upon us by the lack of a category more appropriate to their contents. Purely methodological remarks arise when it becomes possible to understand in some new sense something which is already quite well known. Here, however, we are rather dealing with the construction of exotic electromagnetic arrangements with unusual properties, and not with a new interpretation of something which is known. However, any pretension to novelty in a field which has been well studied for a long time calls for caution. And so, writing these remarks the author kept feeling that somewhere at the dawn of Maxwellian electrodynamics something of this sort must have been reliably thought out.

Any general statement of an initial, embryonic stage can be prefaced by a problem which is simple to formulate, but not necessarily simple to solve, and which therefore could appropriately be called stimulating or productive. As such a problem we shall take one of the standard exercises from a course in electrodynamics.

2. The inverse problem of electrostatics consists of searching for sources being given a specified field. Suppose that in a uniform medium (vacuum), on being given an electrostatic field $\mathbf{E}(\mathbf{r})$ which is known everywhere, we are required to find a distribution of stationary electric charges with density $\rho(\mathbf{r})$. The solution of the problem is unique and reduces to simple differentiation:

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 4\pi\rho, \\ \operatorname{curl}\mathbf{E} &= 0. \end{aligned} \tag{1}$$

But if the sources are concentrated inside some region (for example, inside a sphere of radius a) and the field (for example, a dipole field) is given only outside the region, then the solution will already by "slightly" ambiguous, but nevertheless it forms a class of mutually compatible distributions $\rho(\mathbf{r})$ inside $r < a$ or on the surface $r = a$ which have only a dipole moment $\mathbf{p}^e$ and no other moments; therefore if we do not go into the fine structure of the source the ambiguity turns out to be unimportant.

The electrostatics described by the equations (1) could be called ordinary *charge electrostatics*, if it were possible to contrast it to some other kind.

3. A rotational electric field can be created by means of closed currnts which are variable and which satisfy the solenoidal field condition

$$\operatorname{div}\mathbf{j}^\bullet = 0. \tag{2}$$

The simplest and best known way of obtaining such fields involves use of currents which rise or fall linearly with the time t:

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$$\mathbf{j}^e = \mathbf{j}_1^e \frac{t}{\tau}; \qquad (3)$$

here $\mathbf{j}^e$ is the volume density of current, τ is some characteristic time of the process, $\mathbf{j}_1^e$ is the value of current at $t = \tau$, and the subscript indicates the power of the dependence on t. In this case the magnetic field **H** will have a magnetostatic structure.

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{j}^e, \qquad (4)$$

which changes with time in step with (3),

$$\mathbf{H} = \mathbf{H}_1 \frac{t}{\tau}. \qquad (5)$$

The electric field **E** turns out to be strictly static ($\partial \mathbf{E}/\partial t = 0$) and to obey the equations

$$\operatorname{div} \mathbf{E} = 0, \quad \operatorname{curl} \mathbf{E} = -\frac{1}{c\tau} \mathbf{H}_1 = -\frac{4\pi}{c} \mathbf{j}_0^m, \qquad (6)$$

where the role of the sources of the curl is played by the time-independent derivative of a magnetic field (5) specified externally (and not self-consistent with **E**); therefore these sources can be identified with some constant magnetic current

$$\mathbf{j}_0^m = \frac{1}{4\pi\tau} \mathbf{H}_1. \qquad (7)$$

The term "magnetic current" is frequently accompanied by the adjective "effective," "equivalent," or "fictitious," in an attempt to emphasize that this source of a field is not the result of averaging of *microscopic* sources which have some magnetic charges (isolated poles) but arises only as the result of redefinition of *macroscopic* electric currents. Such a macroscopic magnetic current, of course, is not in any way inferior to an electric current, since it is uniquely defined algorithmically in terms of the latter and its magnetic designation arises simply from the functions which are imposed on it in the equations (6).

The system of sources and fields (2)–(6) is in principle magnetoelectric; we have resorted specifically to this word order in order to make a distinction from the usual combination "electromagnetic," in which one understands in a stereotyped way the mutual relation between **E** and **H**, whereas here one of the inverse relations is cut off by the identical absence of a displacement current; this is a dgenerate solution—the only form of variable field which is rigorously described by pre-Maxwellian electrodynamics. And although in principle one cannot exclude a magnetic field from it, the latter can, however, be concentrated inside some bounded region of space, preserving in other regions the static field **E** completely free of **H**.

4. The field of an ideal solenoid, in which ring-shaped purely transverse *external* currents (3) are flowing, is shown in Fig. 1. This is the simplest example of the purging of the external region of **H**. Indeed, inside the solenoid ($r < a$) a magnetoelectric field is excited: a uniform longitudinal field $\mathbf{H} = H\mathbf{1}_z$ $\mathbf{1}_z$ is a unit vector) and an azimuthal transverse field (as in a betatron) $\mathbf{E}_\theta$:

$$\mathbf{E}_\theta = -\frac{r}{2c\tau} H_1 \mathbf{1}_\theta,$$

while outside at $r > a$ there remains only the electric field

$$E_\theta = -\frac{1}{2c\tau} H_1 \frac{a^2}{r}, \qquad (8)$$

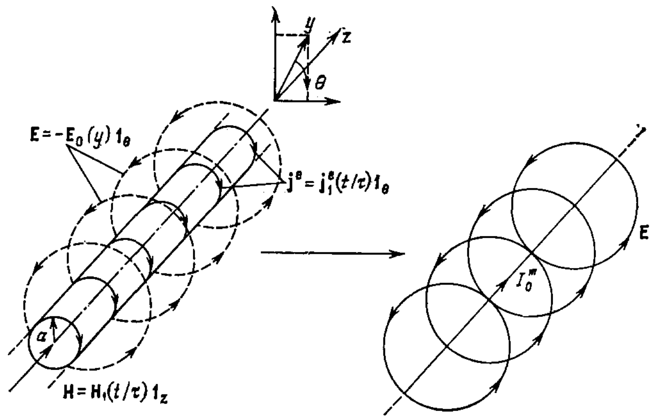

FIG. 1. The field of an ideal solenoid over which are flowing azimuthal external electric currents varying linearly with time. In the external region it is equivalent to the field of a rectilinear constant magnetic current ($y \equiv r$).

which coincides formally with the field of the magnetic current

$$I_0^m = \int j_0^m \, dS = \frac{a^2}{4\tau} H_1, \qquad (9)$$

which has been set up along the axis of the cylinder. We can now compress the solenoid into a line, leaving unchanged the product

$$H_1 \frac{a^2}{\tau} = \text{const},$$

and obtain rotational electric field devoid of magnetic effects over all space $r > 0$ except the source region $r = 0$.

In "technological"-solenoid language[1)] this means $j^e \Delta l a^2/\tau = \text{const}$, where Δl is the thickness of the winding, and $J^e \Delta l = j^e_{\text{surf}}$ is the surface current flowing in it and which is equal to nI^e (I^e is the total current in one turn and n is the number of turns per unit length in the z direction). And of course "winding" must be bifilar (there and back) in order to exclude a longitudinal component of the electric current.

5. Description by means of potentials of such a field appears to be extremely simple. For $\rho = 0$ the scalar potential can without loss of generality be set equal to zero (having in mind only the solution which has been established) and the fields **E** and **H** can be represented in terms of a vector potential **A** satisfying the inhomogeneous wave equation

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}^e$$

$$\left(\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H} = \operatorname{curl} \mathbf{A} \right). \qquad (10)$$

In the case of sources linear in t of the type (2) we have

[1)]Here we are considering some model solenoid formed by external currents which are not flowing along conducting coils (a current sheet). Otherwise in view of the discontinuity of $E_{\tan}$ the field outside the solenoid will not be completely free of a component rising linearly with time. We note incidentally that although the "constructions" given here are in a sense speculative, in principle they also permit direct laboratory demonstrations: ferrite rods with high permeability in the regions of linear rise of $B(t)$ or charged cylinders undergoing rotational acceleration with a compensated radial field will behave as linear constant magnetic currents.

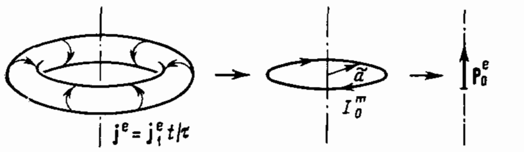

FIG. 2. A straight solenoid with a linear current, folded into a torus, is equivalent to a ring-shaped magnetic current constant in time whose external field on contraction of the ring to a point coincides with the field of an elementary electric dipole.

$$\Delta \mathbf{A}_1 \frac{t}{\tau} = -\frac{4\pi}{c} \mathbf{j}_1^{\mathrm{e}} \frac{t}{\tau}, \quad \mathbf{E} = -\frac{1}{c\tau} \mathbf{A}_1,$$

$$\mathbf{H} = \frac{t}{\tau} \operatorname{curl} \mathbf{A}_1. \tag{11}$$

This means that the field **E** coincides up to a constant factor (dimension of reciprocal length) with the field $\mathbf{A}_1$, while the latter has the structure of a magnetostatic vector potential. There follows from this, by the way, a curious recommendation for measurment of the magnetostatic vector potential $A(\mathbf{r})$ in any region of space, even where $\mathbf{H} = 0$, which consists of replacing the constant-current system by a system of currents with the same spatial distribution but varying linearly with time.[2)]

6. A current electric dipole is obtained by folding a straight solenoid (Fig. 1) into a torus (Fig. 2),[3)] i.e., by means of a ring-shaped magnetic current ($\operatorname{div} \mathbf{j}^{\mathrm{m}} = 0$), as should be suggested by the principle of permutational duality, in view of which the Maxwell equations are invariant with respect to the substitutions $\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$ and $\mathbf{j}^{\mathrm{e}} \to \mathbf{j}^{\mathrm{m}}$, $\mathbf{j}^{\mathrm{m}} \to -\mathbf{j}^{\mathrm{e}}$, $\rho^{\mathrm{e}} \to \rho^{\mathrm{m}}, \to -\rho^{\mathrm{e}}$ (as before all quantities refer to vacuum). Here the electric dipole moment of the torus $\mathbf{p}^{\mathrm{e}}$ is expressed in terms of its radii a and $\tilde{a}$ and in terms of the surface current density $j^{\mathrm{e}}_{\mathrm{surf}_1}$, which flows in the winding, as follows:

$$p_0^{\mathrm{e}} = \frac{1}{\tau} j^{\mathrm{e}}_{\mathrm{surf}_1} \left(\frac{\pi a \tilde{a}}{c} \right)^2. \tag{12}$$

From Eq. (12) we can understand how it is necessary to shrink this torus to a point in order to maintain p_0^{e} constant.

Thus, or starting-point "textbook" exercise of the inverse problem of searching for the sources of a dipole electrostatic field has at least two independent sets of solutions: either distribtutions of fixed charges $j^{\mathrm{e}} = 0, \rho^{\mathrm{e}} \neq 0$ or a distribution of rotational currents ($\rho = 0, j^{\mathrm{e}} \neq 0$) linearly changing with time. And it seems that it is not possible to unravel their nature by any external measurements without intruding into the interior of the sources. The initial problem indeed has turned out to be "productive."

7. Current electrostatics—we can use this term to describe a system of fields created by various distributions of current sources. In contrast to ordinary electrostatics (now

[2)]It does not follow, however, that we should perceive this as an attack on the gauge ambiguity of the potential. In the description (10) the vector **A** has already been normalized and is a completely determined quantity. The procedure given in the text permits the search for a field **A** created by constant currents to be reduced to solution of Eq. (11), i.e., to some similar system where this field can be measured directly.

[3)]Here the external region remains as before free of the field **H**. It can be shown that screening **E** (or **H**), permit arbitrary deformations without destruction of the screening effect.

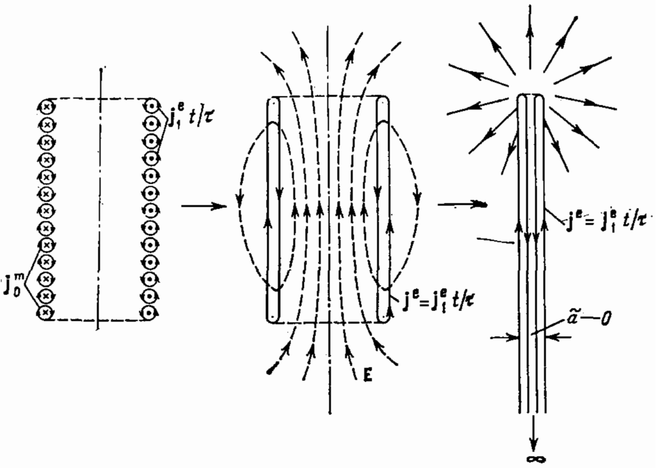

FIG. 3. Schematic construction of an isolated electric pole (charge) in current electrostatics: the set of toruses threaded on a common axis is equivalent to a single torus. If one of the ends is removed to infinity, and the cross section is compressed to a cylinder of zero cross section, then the field near the other end will coincide with the field of a point electric charge ($\tilde{a} \to 0$).

our use of this term is already completely motivated), in it, as in magnetostatics, there is no free monopole; of course, its sources are fundamentally nonstationary—their lifetime is always limited. However, the first incompatibiity is rather simply avoided—in any case, for purely theoretical constructions— by alignment (as in magnetostatics) of the set of dipoles into a semibound chain and thus removing one of the poles to infinity. Actually one obtains a set of toruses which are densely strung on a common axis (Fig. 3), which as the result of Amperian compensation of the currents reduces to a single torus with an extended oval cross section (in tokamaks such configurations are called ring-shaped).[4)] This means that a current electric monopole, in contrast to a charge electric monople, does not belong to a number of electrically autonomous, isolated sources or sinks of the field **E**: necessarily associated with it is a "flux guide" along which $\int \mathbf{E}\, d\mathbf{S}$ is fed or removed.

In regard to the finiteness of the lifetime of the sources, this is an unavoidable physical property of any stationary states and processes—the difference is only in scale. In charge electrostatics it is also necessary initially to wait until the perturbations due to initial transient processes die out and then to deal only with intervals less than the Maxwellian relaxation time of the charges due to the conducting properties of the medium.

If we understand these limitations and if we observe a certain reasonable caution, we can make the following general statement: any electrostatic field created by any system of stationary charges and described with in the framework of phenomenological electrodynamics can be exactly reproduced with use only of current sources.

[4)]Such devices find application as linear induction accelerators, in which the main attention is devoted to creation of an accelerating field of the necessary structure in the paraxial region of an extended torus, and in order to obtain a field **E** constant in time the operation of the accelerator usually occurs just in periods of linearly rising functions $H(t)$ or $B(t)$.[4]

In principle it is even permissible to imagine an infinite set of electrodynamics embedded one within the other (like nesting dolls or *matreshki*): stationary charges⟶ moving charges⟶ currents⟶ stationary current charges⟶ moving current charges⟶ currents again, and so forth. This sequence has "hierarchial-retranslation" symmetry: a primary nature of the sources can be postulated at any stage. However, these electrodymanics already belong to the category of what we might call concocted physics, i.e., to sciences which study phenomena permitted by the laws of nature but distinct from their manifestations under natural conditions.

8. Current dipoles of arbitrary rank. There are sources with currents arbitrarily varying with time:

$$\mathbf{j}^e = \mathbf{j}_n^e \frac{t^n}{\tau^n}, \tag{13}$$

which as the result of an appropriately constructed distribution in space have only static dipole momnts. The procedure of constructing these distributions is in itself so remarkable that one can extract even aesthetic benefits from it. Therefore we shall attempt to deal with it systematically.

First regarding terminology; it is well known that the order of a source is characterized by the number of poles represented in it—monopole, dipole, . . . multipole. It turns out that any multipole of order m can be achieved by means of currents varying as the n-th power of t; the term "order" is already preempted, and the available word closest in meaning is "rank." Thus one obtains "multiple of order m and rank n."

Let us begin with sources of zero rank, i.e., with closed currents constant in time. They create a magnetostatic field described by Eq. (4). We shall rewrite it in a form more convenient for the recurrence relations which will follow:

$$\mathrm{in}] \rightarrow \frac{4\pi}{c}\mathbf{j}_0^e = \operatorname{curl}\mathbf{H}_0] \rightarrow \mathrm{out}. \tag{14}$$

The arrows indicate the order of the action: at the input (in) the current $\mathbf{j}_0^e$ is specified, and at the output (out) a field $\mathbf{H}_0$ is obtained. The simplest two-dimensional source is a direct current, and the simplest three-dimensional source is a coil with direct current, in the limit—an elementary magnetic dipole.[5)]

Next come sources of first rank, which we have been dealing with up to this time. For unity of the general presentation we shall repeat (2)–(6), rewriting them in the form already used in (14):

$$\begin{array}{l} \mathrm{in}) \rightarrow \dfrac{4\pi}{c}\mathbf{j}_1^e \dfrac{t}{\tau} = \operatorname{curl}\mathbf{H}_1 \dfrac{t}{\tau} \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad -\dfrac{1}{c\tau}\mathbf{H}_1 = \operatorname{curl}\mathbf{E}_0) \rightarrow \mathrm{out}, \end{array} \tag{15}$$

[5)]Clearly about a third of all physical effects must be preserved in one-dimensional models. All the systems considered here belong to this third. The distribution of plane-parallel currents in sources of any rank, described below, coincides with the distribution in the cross sections of a cylindrical solenoid which pass through the axis.

where the arrows also make clear the procedure (both analytical and topological) of construction of the solution: the horizontal arrows with right angles preserve the powers of t, and the vertical arrows reduce them by 1. In the spatial pictures it is different: vertical operations do not change the directions of the field vectors, while horizontal operations rotate them: for example, in the two-dimensional case an axially polarized field is transformed into an azimuthally polarized field and vice versa. Therefore the chain of operations (15) can be provided with the following symbolic scheme:

$$\begin{array}{l} in) \longrightarrow \bigcirc \longrightarrow x \\ \qquad\qquad\quad \downarrow \\ \qquad\qquad\quad x \longrightarrow \bigcirc) \longrightarrow out. \end{array} \tag{16}$$

The circles with arrows denote azimuthally polarized fields, and the crosses (x) denote axially polarized fields. This scheme suggests a general procedure for construction of elementary sources of any rank: since at the output it is convenient (for the next formation of a dipole) to distinguish closed ring-shaped "purefied" (unmixed) fields, sources of even rank are produced from axial currents and lead to magnetostatic fields (an even number of vertical operations), and sources of odd rank are produced from ring-shaped currents and lead to electrostatic fields (an odd number of vertical operations).

Let us apply this reasoning to sources first of second rank and then of third rank:

$$\begin{array}{l} \mathrm{in}) \rightarrow \dfrac{4\pi}{c}\mathbf{j}_2^e \dfrac{t^2}{\tau^2} = \operatorname{curl}\mathbf{H}_2 \dfrac{t^2}{\tau^2} - \dfrac{1}{c\tau}E_1 + \operatorname{curl}\mathbf{H}_0) \rightarrow \mathrm{out}, \\ \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\quad \downarrow \\ \qquad\qquad -\dfrac{2}{c\tau}\mathbf{H}_2 \dfrac{t}{\tau} = \operatorname{curl}\mathbf{E}_1 \dfrac{t}{\tau}, \end{array} \tag{17}$$

$$\begin{array}{l} in \longrightarrow x \longrightarrow \bigcirc \qquad x \longrightarrow \bigcirc \longrightarrow out. \\ \qquad\qquad\qquad \downarrow \qquad\quad \uparrow \\ \qquad\qquad\qquad \bigcirc \longrightarrow x \end{array} \tag{18}$$

The corresponding system of currents together with the field distributions is shown in Figs. 4–7. the current $\mathbf{j}^e$ flows along the cylinder $r = a$, and the current opposite ot it $\mathbf{j}_2^e$ flows along $r = b$. (Compensating longitudinal currents are necessary to avoid variable fields in the external regions). The field

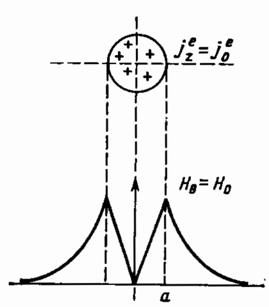

FIG. 4. Current static source of zero rank: an axial (longitudinal) constant current creates an azimuthal magnetostatic field.

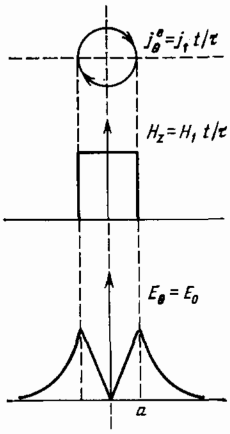

FIG. 5. Current static source of first rank: an azimuthal current linear in t creates in the external region a pure electrostatic azimuthal field.

H_2 is azimuthal, and different from zero only inside the region $b>r>a$; however, an azimuthally twisting flux H_2 of this type is produced by a solenoid with magnetic currents flowing around it. Consequently the further procedure reduces to (16) to the accuracy of a twofold permutation of the fields $E \to H$, $H \to -E$); as a result for $r>b$ there arises a pure magnetstatic field H_0 which in no way differs from the field of an axial electric current which is constant in time. However, we recall that it is actually produced by variable currents—quadratic in t. The last operation—folding into a torus—is carried out in a standard manner for sources of any rank.

The last illustration is of sources of third rank. For the sake of variety we shall explain the analytical procedure for reduction in this case with the aid of vector potentials:

$$\text{in)}\to -\frac{4\pi}{c}\,\mathbf{j}_3^e\,\frac{t^3}{\tau^3}=\Delta\mathbf{A}_3\,\frac{t^3}{\tau^3}-\frac{6}{c\tau^2}\,\mathbf{A}_3\,\frac{t}{\tau}+\Delta\mathbf{A}_1\,\frac{t}{\tau}\Big)\to \text{out},$$

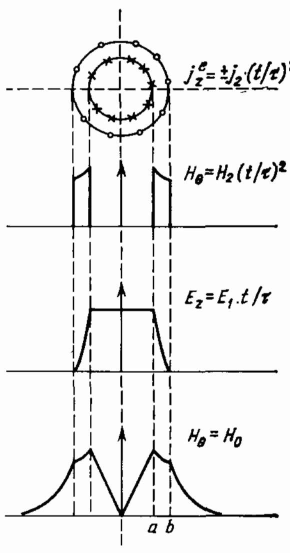

FIG. 6. Current static source of second rank: two longitudinal coaxial currents varying quadratically with time create in the external region an azimuthal pure magnetostatic field.

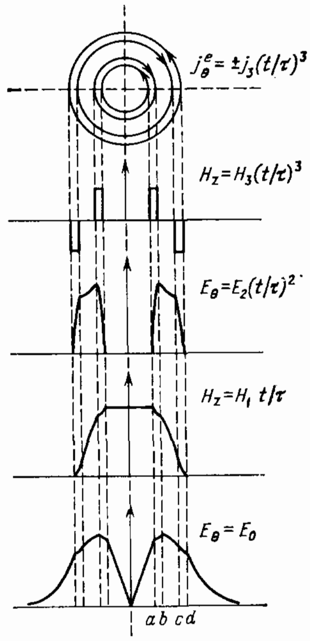

FIG. 7. Current static source of third rank: four coaxial solenoids with azimuthal currents varying cubically with t create in the external region an azimuthal pure electrostatic field.

$$\mathbf{H}=\operatorname{curl}\mathbf{A}_3\,\frac{t^3}{\tau^3}+\operatorname{curl}\mathbf{A}_1\,\frac{t}{\tau}, \tag{19}$$

$$\mathbf{E}=-\frac{3}{c\tau}\,\mathbf{A}_3\,\frac{t^2}{\tau^2}-\frac{1}{c\tau}\,\mathbf{A}_1, \tag{20}$$

$$\begin{array}{ccccccc} in)\to\bigcirc\to x & & \bigcirc\to x & & & & \\ \downarrow & & \uparrow & \downarrow & & & \\ x\to\bigcirc & & & x\to\bigcirc\to out. & & & \end{array} \tag{21}$$

Azimuthal input currents flow around two pairs of coaxial solenoids (Fig. 7): $\mathbf{j}_3^e(r=a)$ $-\mathbf{j}_3^e(r=b)$, $-\mathbf{j}_3^e(r=c)$, $\mathbf{j}_3^e(r=d)$. Therefore the cross section $a<r<b$ is penetrated by a longitudinal magnetic current in one direction, and the cross section $c<r<d$ is penetrated by a longitudinal magnetic current in the other direction. The power of variation of this current with t is already reduced by unity—it is quadratic. By forming a coaxial line with opposite magnetic currents, i.e., with accuracy to a twofold permutation of the fields ($E \to H$, $H \to -E$), we obtain the input system of a source of second rank. This means that now in the external regon there will be again a pure electrostatic field.

There is no doubt that this rule of recursion of sources of rank $n-1$ with application of the dualization operation or to sources of rank $n-2$ without this operation is valid for any n.

It comes out that the solution of any inverse problem with currents as sources is still infinitely multivalued, and this applies both to electrostatics and to magnetostatis.[6)]

[6)] So that our remarks would also permit the following heading: "On the possibility of creating electrostatic and magnetostatic fields by means only of currents varying in time in some rather arbitrary manner". The formulation of the general theorem is apparently possible, but not simple: as can be seen from the constructions described above, the time and space dependences of the currents are functionally related in different ways for sources of each rank.

We are by no means calling for a complication, for this reason, of the means of creation (or description) of static fields, but shall turn our attention to the fundamental unlimited nature of such possibilities. We recall that everywhere here we are considering only established processes, assuming the initial conditions to be fitted to them appropriately. Otherwise the time of establishment will depend on the rank of the source as the result of partial capture of the initial radiation in the web of the current distribution, which becomes more complicated with increase of the rank.

9. Interaction of current sources of higher ranks has some unusual manifestations which definitely can form the content of a number of paradoxes which are perplexing but of pedogogically high quality.

Let us now consider the simplest example. Assume that inside, $r<a$, there is a source which creates in the outer region $r>a$ a centrally symmetric electrostatic field. In the framework of the alternative studied here, this can be either a charge or current monopole characterized by identical electric charges q_α^e (not a very fortunate but unavoidable juxtaposition of words). In the external field $\mathbf{E}$ there must act on the charge a force

$$\mathbf{F}=q_\alpha^e\mathbf{E}. \tag{22}$$

If $\mathbf{E}$ is created by another charge q_β^e we obtain the Coulomb interaction law

$$F_{\alpha\rightleftarrows\beta}=\frac{q_\alpha^e q_\beta^e}{r_{\alpha\beta}^2}. \tag{23}$$

This is necessarily correct, regardless of the means of specification, creation, or maintenance of actual, fictitious, effective, etc. charges q_α^e and q_β^e. However, derivation of Eq. (23) for current sources is impossible without overcoming certain confusions. Indeed, in this case an electric monopole is produced by means of rotational nonstationary currents.

$$\mathbf{j}^e=\mathbf{j}^e_{2n+1}\left(\frac{t}{\tau}\right)^{2n+1},\quad n=0,\ 1,\ \ldots,\ \operatorname{div}\mathbf{j}^e=0$$

with completely compensated charges $\rho^e=0$. This means that the Lorentz force which acts in an electrostatic field ($\mathbf{E}\neq 0$, $\mathbf{H}=0$) on a pre current distribution ($\rho^e=0$, $\mathbf{j}^e\neq 0$), will be identically equal to zero

$$\mathbf{f}=\rho^e\mathbf{E}+\frac{1}{c}\,[\mathbf{j}^e,\ \mathbf{H}]=0. \tag{24}$$

(Here the square brackets indicate a vector product.) One can of course suspect a trick in the form of a thin solenoid which maintains a current charge by a field $\mathbf{E}$, but it is clear that similar discussions could be carried out also for dipoles ($\mathbf{F}=(\mathbf{p}^e\nabla)\mathbf{E}$), and there all provision of p^e occurs in a limited region of space.

This "paradox of losing strength" can be aggravated by resorting to the Maxwell stress tensor T. Since in electrostatics

$$\operatorname{div}\,T=\mathbf{f}, \tag{25}$$

and the fields entering into T outside the sources—both charge and current monopoles—are identical, then integration of (25) over any volume containing them must give an identical result:

$$\int \operatorname{div}\,T\,dV=\oint T\,d\mathbf{S}=\int \mathbf{f}\,dV=\mathbf{F}, \tag{26}$$

which does not depend on the nature of the source, i.e., on the manner of realization.

Like almost all correct paradoxes, this one is also due to the extension of a correct statement to the region in which it is not applicable. Current sources of an electrostatic field are nonstationary: the field $\mathbf{H}$ inside them changes linearly (in sources of the first rank) with time, and therefore the abbreviated static ($\partial/\partial t=0$) formula (25) must be replaced by the complete dynamical law of momentum conservation in a magnetoelectric system

$$\operatorname{div}T=\mathbf{f}+\frac{\partial\mathbf{g}}{\partial t}, \tag{27}$$

where

$$\mathbf{g}=\frac{1}{4\pi c}\,[\mathbf{EH}]=\frac{t}{4\pi c\tau}\,[\mathbf{E}_0\mathbf{H}_1]$$

is the electromagnetic momentum (its volume density) accumulated or expended by the source. Consequently instead of (25) it is necessary to write

$$\oint T\,d\mathbf{S}=\int \mathbf{f}\,dV+\frac{\partial}{\partial t}\int \mathbf{g}\,dV. \tag{28}$$

Thus, the identity of the left-hand sides of (27) or (28) for monopoles of the two types (one the assumption, of course, that the surface of integration encloses the region of the source without intersecting it) is due to different terms: in the case of charge monopole $\mathbf{g}=0$ and the force takes the form (22); in the case of a current monopole $\mathbf{f}=0$ but inside there is an increase or decrease of the electromagnetic momentum $\partial g/\partial t \gtrless 0$.[7)]

The paradox is not yet exhausted. It is necessary to rehabilitate Coulomb's law (23) for current monopoles. At first glance it might appear that current electrostatic monopoles (and of course also multipoles) do not interact with each other at all, because the force $\mathbf{f}$ in this case disappears in accordance with Eq. (24). Still more surprising is the case of interaction of an ordinary charge monopole q_α^e with a current monopole q_β^3, in which the forces entering into (27) and (28) turn out not be be mutual: $\mathbf{F}_{\alpha\to\beta}\neq\mathbf{F}_{\beta\to\alpha}$, which contradicts the law of conservation of momentum. But this law is valid only for closed systems, which generally do not include external, electric currents $\mathbf{J}^e$ flowing in a field $\mathbf{E}$, whether the field is its own field or is produced by a different source. The system which provides the current $\mathbf{J}_\beta^e$, i.e., devices of any nature (possibly also including electromagntic devices), by means of which the values of $\mathbf{j}^e$ are maintained in a specified manner which does not depend on the fields, must supply to these fields definite amounts of energy (power) $\int \mathbf{j}^e\ \mathbf{E}\,dV$ and momentum $\mathbf{G}=\int \mathbf{g}\,dV$, and consequently this device will

[7)] In the absence of an external field a linearly rising closed current (for example, a current flowing around a solenoid as shown in Fig. 1) radiates energy into the interior of the cylinder in such a way that the total momentum supplied to the field by it is zero. Such a system can serve—with understandable reservations—as a macroscopic model of ring beams of uniformly accelerated charged particles. Although the radiation process usually involves removal of energy *by waves*, in regions close to the source in comparison with a wavelength this can be manifested only through quasistationary fluxes.

experience a corresponding "recoil." A "field + source" system with its supply is closed and autonomous and therefore the total flux of momentum (which is proportional to the *flow of energy flux*) must be exactly equal to the change in the recoil momentum; the source behaves only as a transformer. As a result the equality (28) can be extended:

$$\oint T\,d\mathbf{S} = \int \mathbf{f}\,dV + \frac{\partial}{\partial t}\int \mathbf{g}\,dV = \frac{d\mathbf{G}_\Sigma}{dt} = F_\Sigma. \tag{29}$$

Thus, the interaction of sources which have an arbitrary internal structure is in the last analysis determined by their multipole moments (in the case of monopoles—by their charges). However, if the means of measurement permit the contribution of the change of accumulated or expended momentum to be distinguished from the Lorentz force, this can be taken into account by introduction of an additional force:

$$\mathbf{f}_{\mathrm{add}} = \frac{\partial \mathbf{g}}{\partial t} = \frac{1}{4\pi c}\frac{\partial}{\partial t}[\mathbf{E},\ \mathbf{H}] = \frac{1}{c}[\mathbf{j}_{\mathrm{disp}}\mathbf{H}] - \frac{1}{c}[\mathbf{j}^{\mathrm{m}},\ \mathbf{E}], \tag{30}$$

where $\mathbf{j}^{\mathrm{e}}_{\mathrm{disp}} = (1/4\pi)\partial\mathbf{E}/\partial t$ is the Maxwell displacement current $\mathbf{j}^{\mathrm{m}} = (1/4\pi)\times\partial\mathbf{H}/\partial t$ is the effective magnetic current inside the source. The expression (30) is invariant with respect to permutational duality. It already appeared in Maxwell's Treatise on Electricity and Magnetism.[1] It would be appropriate to call it the Abraham force for an external source, in analogy with the Abraham source for a medium, i.e., by analogy with the force which actually acts on self-consistent polarization currents.

10. Our concluding remarks probably should contain an evaluation of the significance of the solutions which have been described. Evidently they are not only interesting in their unusual nature but also can serve as a good pedagogic demonstration of the fact that even in the well-mastered macroscopic classical domain the conservation laws work more reliably than the particular dynamics which are responsible for their actual fulfillment, including dynamics which are concealed from measurements invented, or better to say, concocted on a legal basis. In addition, all our discussions have been carried out for the simplest models (such physics is sometimes called "sterile"); in more realistic situations, however, more complicated assemblies of charges, currents, and momenta circulating inside sources are permissible, where discovery of the force relations requires an individual approach with examination of Eqs. (29) and (30).

Finally, some of the distributions of fields and currents presented above can fill the need of standard solutions which are valid—for brief periods of time—for processes which are arbitrary functions of time.

In this connection it is interesting to note that toroidal currents always turns out to be set apart from a number of other distributions. This fact was noted at onetime by Ya. B. Zel'dovich, who named such sources (on the suggestion of A. S. Kompaneets) anapoles (pole-like).[2] Their properties have been studied in detail in research on quantum electrodynamics.[3] One also encounters them on a large scale in studying fields produced by toroidal antennas. It is well known that only the rotational part (div $\mathbf{j}^{\mathrm{e}}_r = 0$) and not the potential part (curl$\mathbf{j}^{\mathrm{e}}_p = 0$) of the current $\mathbf{j}^{\mathrm{e}}(\mathbf{r},t) = \mathbf{j}^{\mathrm{e}}_p + \mathbf{J}^{\mathrm{e}}_r$ is responsible for the radiation field. Consequently the true, so to say, "pure" sources of radiation must be considered to be just ring-shaped, closed, charge-free currents, which raises the torus into the rank of elite geometrical figures in the theory of radiators, since by means of pure rotational currents which flow over it one can create any radiation field: longitudinal azimuthal currents, flowing coaxially around the large circumferences of a torus radiate in the same manner as an elementary magnetic dipole (for the case $a\tilde{a}/\lambda^2 \to 0$); transverse poloidal currents flowing around the small circumferences of a torus radiate as an electrical dipole, . . . , and so forth. Therefore in some sense the anapole is a more elementary electric dipole source than even a Hertzian oscillator. From these positions one could approach current electrostatics not by construction as was done above, but in a rigorously formalized way, by the limiting transitions from arbitrarily variable fields to fields of zero frequency ω, which is well known leads to two solutions—constant and linearly rising:

$$(\mathbf{j}^{\mathrm{e}}_{\mathrm{c}}\cos\omega t + \mathbf{j}^{\mathrm{e}}_{\mathrm{s}}\sin\omega t)_{\omega\to 0} \to \mathbf{j}^{\mathrm{e}}_{\mathrm{c}} + (\mathbf{j}^{\mathrm{e}}_{\mathrm{s}}\omega)\,t.$$

This type of degenerate behavior is encountered freqeuntly in various problems: the fields near a caustic, critical modes in waveguides, the limiting case of the tunnel effect, and many more.

The author is grateful to many of his colleagues for their enthusiastic help. The article was written in this style at the friendly urging of L. S. Dolin. Observations by M. L. Levin, G. V. Permitin, and E. I. Yakubovich permitted certain inaccuracies to be avoided, and those of V. B. Gil'denburg in addition suppressed certain liberties of interpretation.

This work was presented at the seminar organized by V. L. Ginzburg on June 30, 1982; the author is indebted to the participants in this seminar, who drew attention to the connection between current statics and anapole concepts.

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Translated by Clark S. Robinson