Dissipative equations for many-particle distribution functions

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This article deals with the papers by M. A. Leontovich on the subject of thermodynamics and statistical physics. Primary attention is given to the paper entitled "The fundamental equations of the kinetic theory of gases from the standpoint of the theory of random processes," which was published in 1935 in *Zhurnal Eksperimental'noĭ i Teoreticheskoĭ Fiziki*, No. 5. This paper is one of the basic works in the modern statistical theory of nonequilibrium processes.

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Even a simple list of the papers by Mikhail Aleksandrovich Leontovich on the subject of thermodynamics and statistical physics[1-13] shows that his interests were very broad. But what is striking is not the breadth so much as the originality and depth of his analysis of these problems. The questions that occupied Leontovich many years ago remain current to this day.

For example, in an annotation to his article "On kinetics of fluctuations," which was published more than fifty years ago, we read:

> A method is presented for determining the spectrum of fluctuations i.e., for determining the statistical averages of the squares of the spatiotemporal Fourier expansion of the fluctuations. The method is applied to concentration and density fluctuations in liquids. The consequences of the theory which bear upon the fine structure of the lines of the scattering spectrum are discussed in connection with the available experimental data.

This said it all, with the utmost clarity and precision.

Another paper by Leontovich, entitled "On the free energy of nonequilibrium states," might be carried over almost in its entirety to a contemporary course in statistical physics. Here also the goal and result are formulated with extreme clarity:

> The free energy of a nonequilibrium state can be defined more generally than is ordinarily done by taking into consideration the additional potential energy in the presence of which the nonequilibrium state becomes as equilibrium state. The relationship of this definition to the Boltzmann principle is analyzed.

The method proposed in this article for describing a wide range of nonequilibrium states was given a more detailed exposition in Leontovich's book *Statistical Physics*.[8]

It is impossible here to analyze and evaluate from a modern point of view all of Leontovich's papers on thermodynamics and statistical physics.[1-13] Let us examine just one of them[6] in greater detail—"The fundamental equations of the statistical theory of gases from the standpoint of the theory of random processes." The choice of this particular paper, of course, was not accidental, as will become clear from this article.

This truly remarkable paper appeared in No. 5 of the journal *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* for 1935. Its ideas represented a significant departure from the existing level of the statistical theory of nonequilibrium processes.

Up to the time that this paper was written the statistical theory of nonequilibrium processes was based on the famous Boltzmann equation. This equation implied the law of increasing entropy (the H theorem of Boltzmann). It served as a basis for the equations of gas dynamics and the equations describing the free-molecular flow of a gas. This was indeed a triumph of the kinetic theory. It seemed that this theory was close to completion. Only a few prominant physicists of the time understood that this was only the first stage in the development of the statistical theory of nonequilibrium processes. In Leontovich's paper we read:

> The kinetic theory treats processes in gases. It is a statistical theory, inasmuch as equation (1) [the Boltzmann equation] is based on the statistical proposition of the Stosszahlansatz. The structure of this theory, however, is undoubtedly very imperfect. The quantity $f\,d\omega do (d\omega = dv_x\,dv_y\,dv_z,\ do = dxdydz)$ must be assigned the value of some statistical average (mathematical expectation) of the number of particles in a volume $d\omega do$ of phase space μ—only then can the irreversible character of equation (1) and its consequences be understood. However, in the framework of the theory itself the meaning of this mathematical expectation remains extremely unclear, since one does not consider the probabilities which are used to generate these mathematical expectations. The theory is therefore also unable to give any information on the fluctuations in the gas and how they change in time.[1] (Ref. 6, p. 211).

In fact, the Boltzmann kinetic equation was treated as an equation for a determinate (not random) distribution function. Accordingly, in the transition from the kinetic equation to the gasdynamic equation, the gasdynamic functions—the density $\rho(\mathbf{r}, t)$, the velocity $\mathbf{U}(\mathbf{r}, t)$, and the temperature $T(\mathbf{r}, t)$—turned out to be determinate. As a result, effects due to fluctuations of the distribution function (kinetic fluctuations) and to fluctuations of the gasdynamic functions dropped out of consideration.

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The situation was as follows: In the classical papers of Rayleigh, Planck, Einstein, and Smoluchowski it was shown that even in an equilibrium state fluctuations play a fundamental role in many phenomena. For example, density fluctuations govern the scattering of light, and fluctuations of the electromagnetic fields are manifested in thermal radiation. Without allowance for the fluctuations of the medium in which the Brownian particles move one cannot explain this "perpetual motion." This list could, of course, be extended. For example, in recent years we have come to understand the fundamental role of fluctuations in second-order phase transitions. Nevertheless, for many years nonequilibrium fluctuations remained outside the sphere of interest of kinetic theory. There were several reasons for this.

Working from Smoluchowski's equation, Leontovich obtained for the case of a rarefied gas a Markov-type equation for the most general distribution function f_N of a system of N particles. The equation which he obtained was irreversible right from the start. Here the question of the reasons for the irreversibility did not come up. Leontovich wrote:

> It should be noted that I have not touched upon the *basic physical question* of the extent to which the statistical description of the processes in terms of transition probabilities can be related to or put into correspondence with the description of quantum (or classical) mechanics. I think only that this statistical scheme is the most expedient for setting forth in a more complete form the *actual content* of the kinetic theory (Ref. 6, p. 213).

Thus, the matter of how the reversible equations of mechanics are related to the irreversible equations of the statistical theory of nonequilibrium processes remained an open question. This problem was promoted to the forefront. The problem of constructing a systematic fluctuational theory of nonequilibrium processes withdrew to the background and for a long time did not attract the attention of researchers.

Important contributions toward solving the problem of substantiating the kinetic theory were given in the papers by N.N. Bogolyubov, M. Born and H.S. Green, and J.G. Kirkwood. In Bogolyubov's now-classical monograph *Problems of Dynamical Theory in Statistical Physics* (1946) a method is developed for obtaining the Boltzmann kinetic equations (for a rarefied gas) and the kinetic equations of Landau and Vlasov (for systems of charged particles). Thanks to these papers, it became clear in what way and *at what cost* one can obtain from the reversible equations of mechanics the irreversible equations of kinetic theory.

As a result, many questions which had disturbed investigators simply went away. However, new questions, also of a difficult nature, arose. One of these was the question of nonequilibrium fluctuations.

In Bogolyubov's derivation of the kinetic equations—the closed equations for single-particle distribution functions—the hypothesis (*principle*) of total attenuation of the original correlations played an important role. Here it was (*implicitly*) assumed that the long-lived correlations (with a time τ_{cor} of the order of, or larger than, the relaxation time of the single-particle distribution functions) do not play an appreciable role. The kinetic and hydrodynamic fluctuations thereby dropped out of consideration.

We note that Bogolyubov[14] cites Leontovich's article,[6] but not in connection with the question of fluctuations of the distribution functions. In §2 of this book[14] is the comment: "The study of the functions F_s in many cases can be simplified appreciably by the introduction of a special functional, the generalization of the generating functions employed by Leontovich[6] in the theory of stochastic processes with a discrete phase space."

Leontovich's paper[6] was subsequently almost forgotten, and it did not have an appreciable influence on the development of the theory of nonequilibrium fluctuations. The dissipative equations for the many-particle distribution functions were discovered anew by other authors—for example, in the works of I. Prigogine, R. Brout, and M. Kac (see Chs. 4 and 11 in Ref. 15, Ch. 10 in Ref. 16, Ch. 2 in Ref. 17, and Ch. 24 in Ref. 18).

For example, the study of the kinetic equations for many-particle distribution functions (master equations) occupies many pages in the splendid book[19] by the American mathematician M. Kac, which is a record of lectures on a number of problems in statistical theory. Kac, unfortunately, was not acquainted with Leontovich's paper at the time he was working on these lectures.[1)] On the basis of Liouville's equation, V.N. Zhigulev[20] established a chain of dissipative equations for the sequence of distribution functions of a rarefied gas which is a direct consequence of Leontovich's equation. Attempts at approximate solution of this chain of equations for the purpose of studying the effect of turbulent fluctuations on the velocity distribution of the particles of a rarefied gas have recently been undertaken by Japensee investigators (see Ref. 21).

We shall return later to the discussion of dissipative equations for many-particle distribution functions. For now we shall only point out the following.

When large-scale and long-lived fluctuations are taken into account in kinetic theory, new contributions appear in addition to those incorporated in the Leontovich equation. These contributions are governed by fluctuations which have lifetimes much longer than the mean free time and so cannot be taken into account in the Boltzmann scheme. These additional contributions are particularly large for states which are far from equilibrium, such as in the presence of well-developed

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[1)]During a school on statistical physics in Jadwisin, Poland, Kac told me that after his book appeared in Russian, a physicist from Leningrad had sent him a copy of Leontovich's article (Ref. 6). Kac asked me, "How could he [Leontovich] have known and understood all this back in 1935?" I felt that Kac's pride had been wounded. When later in the conversation I mentioned Leontovich's friendship and collaboration with A. N. Kolmogorov, Kac immediately replied, "Oh It was Kolmogorov who taught him this."

turbulence. Here both the thermodynamic functions and the kinetic coefficients are appreciably altered.

Let us now return to the question of fluctuations in the distribution function of a rarefied gas.

The first step in the kinetic theory of fluctuations was taken by B.B. Kadomtsev[22] in a calculation of the fluctuations of the distribution function of an equilibrium rarefied gas. The result was obtained by using a linearized Boltzmann kinetic equation as the relaxation equation in the theory of equilibrium fluctuations developed in the papers of H.B. Callen and T.A. Welton,[23] S.M. Rytov,[24] and L.D. Landau and E.M. Lifshits.[25] In an analogous way L.P. Gor'kov, I.E. Dzyaloshinskiĭ, and L.P. Pitaevskiĭ[26] calculated the equilibrium fluctuations for the Fokker-Planck equation and the linearized Landau equation.

Kadomtsev's formula was generalized to nonequilibrium states by various methods in the papers of Sh. M. Kogan and A. Ya. Shul'man, S.V. Gantsevich, V. L. Gurevich, R. Katilus, the present author, and others (see the review of Ref. 27, Chs. 5 and 11 in Ref. 15, and §19.20 in Ref. 28).

One of the ways of constructing a theory of nonequilibrium fluctuations is based on the use of the dissipative equation for a many-particle distribution function (§18 and Ch. 4 in Ref. 15). However, the starting point here is different from that of Leontovich's paper.

The initial equation in Ref. 15 is the Liouville equation—a reversible equation for the distribution function f_N. The transition from this equation to a dissipative equation for the smoothed many-particle distribution function $\tilde{f}_N$ is accomplished by averaging over a physically infinitesimal volume V_p. To make such a transition possible, Bogolybov's principle of total attenuation of the initial correlations is replaced by the condition of partial attenuation of the initial correlations: Only the small-scale correlations, for which

$$\tau_{\text{cor}} < \tau_{\text{p}}, \quad r_{\text{cor}} < l_{\text{p}}, \tag{1}$$

are attenuated: here τ_p and l_p are the intervals of time and length that are adopted as physically infinitesimal For a rarefied gas, when the density parameter $\varepsilon = nr_0^3$ is much smaller than unity, the quantities τ_p and l_p in the kinetic stage of relaxation can be defined in the following way (§18 in Ref. 15 and Ch. 7 in Ref. 18):

$$\tau_{\text{p}} \sim \sqrt{\varepsilon\tau} \ll \tau, \quad l_{\text{p}} \sim \sqrt{\varepsilon}\, l \ll 1, \quad N_{\text{p}} \sim \frac{1}{\sqrt{\varepsilon}} \gg 1 \quad \text{for} \quad \varepsilon = nr_0^3 \ll 1. \tag{2}$$

The introduction of the quantities τ_p and l_p enables one to separate correlations into large-scale and small-scale. As a result, on can write for $\tilde{f}_N$ the equation

$$\frac{\partial \tilde{f}_N}{\partial t} + \sum_{1\leqslant i\leqslant N}\left(\mathbf{v}_i \frac{\partial \tilde{f}_N}{\partial \mathbf{r}_i} + \mathbf{F}_0 \frac{\partial \tilde{f}_N}{\partial \mathbf{p}_i} - \sum_{1\leqslant j\leqslant N} \frac{\partial \tilde{\Phi}_{ij}}{\partial \mathbf{r}_i}\frac{\partial \tilde{f}_N}{\partial \mathbf{p}_i}\right) = I_N(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N, t); \tag{3}$$

here we have introduced the notation I_N for the corresponding collision integral. This integral can be written either in the Bogolyubov representation [as in (18.10) of Ref. 15]

$$I_N = \sum_{1\leqslant i,\, j\leqslant N} \delta(\mathbf{r}_i - \mathbf{r}_j) \int_{V_\Phi} \mathrm{d}(\mathbf{r}_i - \mathbf{r}_j) \frac{\partial \Phi_{ij}}{\partial \mathbf{r}_i} \times [\tilde{f}_N(\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_i\mathbf{P}_i(-\infty), \ldots, \mathbf{r}_j, \mathbf{P}_j(-\infty), \ldots, \mathbf{r}_N, \mathbf{p}_N, t) - \tilde{f}_N(\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_i, \mathbf{p}_i, \ldots, \mathbf{r}_j, \mathbf{p}_j, \ldots, \mathbf{r}_N, \mathbf{p}_N, t)], \tag{4}$$

or, more conveniently for comparison with Leontovich's equation, in the Boltzmann representation

$$I_N = \sum_{1\leqslant i,\, j\leqslant N} \int_0^{2\pi} \mathrm{d}\varphi_{ij} \int_0^\infty \rho_{ij}\, \mathrm{d}\rho_{ij}\, |\mathbf{v}_i - \mathbf{v}_j|\, \delta(\mathbf{r}_i - \mathbf{r}_j) \times [\tilde{f}_N(\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_i, \mathbf{p}_i', \ldots, \mathbf{r}_j, \mathbf{p}_j', \ldots, \mathbf{r}_N, \mathbf{p}_N, t) - \tilde{f}_N(\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_i, \mathbf{p}_i, \ldots, \mathbf{r}_j, \mathbf{p}_j, \ldots, \mathbf{r}_N, \mathbf{p}_N, t)]. \tag{5}$$

In expressions (4) and (5) the "width" of the function $\delta(\mathbf{r}_i - \mathbf{r}_j)$ is characterized by l_p.

Equation (3) with collision integral (5) corresponds to equation (42), (43) of Leontovich's paper (the quantity I_N in (42) is defined in Ref. 6 on p. 231). The difference lies in the following:

In equation (42) of Ref. 6 the collision integral lacks the second term in the square brackets in (5) [with the function $\tilde{f}_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N, t)$], which is important, for example, in proving the law of increasing entropy of the entire system (see below). However, in going over from equation (42) to the Boltzmann equation, Leontovich[6] does, of course, include the contribution of this term.

On the left-hand side of equation (3) there is an additional term that does not appear in (43) of Ref. 6; this term takes into account the interaction of the particles. This term, as we shall see, is important in the study of the contribution of the large-scale fluctuations.

Let us consider the most important consequences of equation (3) with collision integral (5) [or (4)].

With the aid of equation (3), we shall find an equation for the single-particle distribution function:

$$f_1(\mathbf{r}_1, \mathbf{p}_1, t) \equiv \tilde{f}_1 = V \int \tilde{f}_N(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N, t)\, \mathrm{d}\mathbf{r}_2 \ldots \mathrm{d}\mathbf{r}_N\, \mathrm{d}\mathbf{p}_2 \ldots \mathrm{d}\mathbf{p}_N. \tag{6}$$

In the integration over $\mathbf{r}_2, \ldots, \mathbf{r}_N, \mathbf{p}_2, \ldots, \mathbf{p}_N$, all the terms with $i \neq 1$ drop out. All the terms in the sum $\sum_{1<j\leqslant N}$ under the integral over $\mathbf{r}_j, \mathbf{p}_j$ are on equal footing; one may therefore set $j = 2$ and replace the summation by a factor $N-1$, which is the number of terms in the sum over j. Finally, we let $(N-1)/V \to N/V = n$. This results in the following equation:

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}_1} + \mathbf{F}_0 \frac{\partial f_1}{\partial \mathbf{p}_1} = n \int \frac{\partial \Phi_{12}}{\partial \mathbf{r}_1}\frac{\partial}{\partial \mathbf{p}_1} \tilde{f}_2(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2, t)\, \mathrm{d}\mathbf{r}_2\, \mathrm{d}\mathbf{p}_2 + n \int_0^{2\pi} \mathrm{d}\varphi_{12} \int_0^\infty \rho_{12}\, \mathrm{d}\rho_{12} \int \mathrm{d}\mathbf{p}_2\, |\mathbf{v}_1 - \mathbf{v}_2|\, [\tilde{f}_2(\mathbf{r}_1, \mathbf{p}_2', \mathbf{r}_1, \mathbf{p}_1', t) - \tilde{f}_2(\mathbf{r}_1, \mathbf{p}_2, \mathbf{r}_1, \mathbf{p}_1, t)], \tag{7}$$

which corresponds to equation (63) in the paper by Leonotovich. The only difference is that on the right-hand side of equation (7) there is an additional term (the first term on the right-hand side) which takes into account the contribution from large-scale fluctuations (see below).

Let us introduce the two-particle correlation function. By definition, with allowance for the fact that $f_1 \equiv \tilde{f}_1$, we have

$$\tilde{f}_2 = f_1 f_1 + \tilde{g}_2. \tag{8}$$

It can then be said that equation (7) is not closed, since it contains the correlation function $\tilde{g}_2$ in addition to the function f_1. For this reason, even when the first term on the right-hand side is neglected this equation differs from the Boltzmann equation. It is the first equation of a chain of coupled equations for the smoothed (over a physically infinitesimal volume V_p) functions $f_1, \tilde{g}_2, \tilde{g}_3, \ldots$ (see Ref. 15). In contrast to the Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations for the ordinary distribution functions, the system under study is approximate because of the smoothing over the volume, and for this reason is dissipative.

In equation (7) the correlation function $\tilde{g}_2$ enters in two ways which are, in a sense, complementary. In the second term on the right-hand side of equation (7) the function $\tilde{g}_2$ appears inside the integral that takes into account the contribution of binary collisions. In order to obtain the Boltzmann collision integral, one must set $\tilde{g}_2 = 0$ inside it. Now how is one to justify such an approximation? In this regard Leontovich writes:

> This relation [relation (7) without the first term on the right-hand side, auth.] will have the same form as the "fundamental equation of the theory of gases" [the Boltzmann equation, auth.] if one replaces [in our notation, auth.] $\tilde{f}_2(p_2', p_1')$ by $f_1(p_2')f_1(p_1')$ and, correspondingly, $\tilde{f}_2(p_2, p_1)$ by $f_1(p_2)f_1(p_1)$. Such a replacement might be justified if it were proved that as the total number of particles goes to infinity, the quantities giving the dispersion of the numbers of particles in definite states increase in proportion to N. By analogy with the "limit theorem" proved for a discrete series of states, such a behavior of the dispersion and, hence, the validity of such a limit theorem in this case as well seem probable to me, although I have not been able to prove it. As a result of the indicated replacement, equation (63) [our Eq. (7) without the first term on the right-hand side, auth.] goes over to equation (1) [the Boltzmann equation, auth.]

Thus Leontovich assumed $\tilde{g}_2 = 0$ in the collision integral only in the thermodynamic limit: $N \to \infty$, $V \to \infty$, but N/V finite. In Bogolyubov's book[14] the Boltzmann equation is a closed equation for a single-particle distribution function obtained on the basis of the principle of total attenuation of the initial correlations. One can discern a certain connection between these approaches. In both cases we arrive at a closed equation for a determinate (not random) distribution function. For this reason, as we have already mentioned, all phenomena governed by kinetic and hydrodynamic fluctuations drop out of consideration.

To estimate the role of the function $\tilde{g}_2$ in the collision integral, let us use the condition of partial attenuation of the initial correlations [condition (1)]. Assuming (for small-scale correlations) that the correlation length $r_{cor} \sim r_0$ (r_0 is the diameter of an atomic sphere), we obtain the estimate

$$\tilde{g}_2 = \int g_2(\mathbf{r}) \frac{d\mathbf{r}}{V_p} \sim \frac{r_0^3}{V_p} \sim \frac{\varepsilon}{N_p} \sim \varepsilon^{3/2}, \quad \text{since} \quad N_p \sim \frac{1}{\sqrt{\varepsilon}}. \tag{9}$$

This is what provides the grounds for dropping the functions $\tilde{g}_2$ inside the collision integral. The large-scale correlations are included in the first term on the right-hand side of (7). As a result, the equation for the function f_1 assumes the form [Eq. (18.6) in Ref. 15)

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \frac{\partial f_1}{\partial \mathbf{r}_1} + \mathbf{F} \frac{\partial f_1}{\partial \mathbf{p}_1} = I_B(\mathbf{r}_1, \mathbf{p}_1, t) + \tilde{I}(\mathbf{r}_1, \mathbf{p}_1, t); \tag{10}$$

here

$$I_B = n \int_0^{2\pi} d\varphi_{12} \int_0^{\infty} \rho_{12}\, d\rho_{12} \int d\mathbf{p}_2 \, |\mathbf{v}_1 - \mathbf{v}_2| \times [f_1(\mathbf{r}_1, \mathbf{p}_2', t) f_1(\mathbf{r}_1, \mathbf{p}_1', t) - f_1(\mathbf{r}_1, \mathbf{p}_2, t) f_1(\mathbf{r}_1, \mathbf{p}_1, t)] \tag{11}$$

is the Boltzmann collision integral, and

$$\tilde{I} = n \int \frac{\partial \Phi_{12}}{\partial \mathbf{r}_1} \frac{\partial}{\partial \mathbf{p}_1} \tilde{g}_2(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2 t)\, d\mathbf{r}_2\, d\mathbf{p}_2 \tag{12}$$

is an additional integral determined by the large-scale fluctuations. The force F in Eq. (10) is given by the expression

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{F}_0 - n \int \frac{\partial \Phi(|\mathbf{r} - \mathbf{r}'|)}{\partial \mathbf{r}} f_1(\mathbf{r}', \mathbf{p}', t)\, d\mathbf{r}'\, d\mathbf{p}'. \tag{13}$$

Thus, under the condition of attenuation of small-scale correlations, the equation for f_1 is of the form (10). This equation is dissipative. Here the dissipation due to the exclusion of small-scale correlations enters explicitly through the Boltzmann collision integral. Additional dissipation due to the functions $\tilde{g}_2$ (the integral $\tilde{I}$) is also possible. Before discussing this matter, let us make the following remark.

Instead of using Eq. (3) for the function $\tilde{f}_N$ as the initial equation, one can use the equation for the microscopic phase density smoothed over a volume V_p:

$$N(\mathbf{r}, \mathbf{p}, t) = \sum_{1 \leqslant i \leqslant N} \delta(\mathbf{r} - \mathbf{r}_i(t))\, \delta(\mathbf{p} - \mathbf{p}_i(t)). \tag{14}$$

Let us denote this by $\tilde{N}$. Averaging this equation (see §22 in Ref. 15) and using the equality $\langle \tilde{N} \rangle = nf_1$ and the condition of attenuation of the small-scale correlations, we again arrive at Eq. (10). Now, however, the integral $\tilde{I}$ is represented in a different, but equivalent form:

$$\tilde{I}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{n} \int \frac{\partial \Phi(|\mathbf{r} - \mathbf{r}'|)}{\partial \mathbf{r}} \frac{\partial \langle \delta\tilde{N} \delta\tilde{N} \rangle_{\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}', t}}{\partial \mathbf{p}}\, d\mathbf{r}'\, d\mathbf{p}' \equiv -\frac{1}{n} \frac{\partial \langle \delta\tilde{\mathbf{F}} \delta\tilde{N} \rangle}{\partial \mathbf{p}}; \tag{15}$$

here $\delta\tilde{N} = \tilde{N} - nf_1$ and

$$\delta\tilde{\mathbf{F}} = -\int \frac{\partial \Phi(|\mathbf{r} - \mathbf{r}'|)}{\partial \mathbf{r}} \delta\tilde{N}(\mathbf{r}', \mathbf{p}', t)\, d\mathbf{r}'\, d\mathbf{p}'. \tag{16}$$

In this approach Eq. (10) must be supplemented by an equation for the correlator of the fluctuations of the phase density $\tilde{N}$.

Of course, the equation for such a correlator, by virtue of the nonlinearity of the system, will contain a more complex, ternary correlator. This sequence of equations can be closed under the condition that the fluctuations are small. This condition is valid for a wide class of problems if the averaging volume V_{p} is suitably chosen so that it contains many particles ($N_{\mathrm{p}} \gg 1$).

One may, of course, turn not to the equations for the moments of the fluctuations $\delta\tilde{N}$, but instead use Eq. (12), which expresses the integral $\tilde{I}$ in terms of $\tilde{g}_2$. The equation for $\tilde{g}_2$ is of the form given by (18.25) in Ref. 15. In (18.25) the condition that the fluctuations $\delta\tilde{N}$ be small corresponds to the approximation $\tilde{g}_3 = 0$, $\tilde{g}_2 \ll f_1/f_2$.

In the zeroth order approximation in the fluctuations, the integral $\tilde{I}$ in (10) is zero, and we return to the kinetic equation of Boltzmann. In the next approximation the equation for the correlator $\langle\delta\tilde{N}\delta\tilde{N}\rangle$ can be written in the form of an equation with a source $A(x, x', t)$—a function determined by the single-particle distribution function [Eq. (22, 21) in Ref. 15]. The source is represented as a sum of two terms:

$$A(x, x', t) = A_{\mathrm{B}}(x, x', t) + \tilde{A}(x, x', t). \tag{17}$$

The first term A_{B} is determined by the atomic structure of the subsystem in the physically infinitesimal volume V_{p}. The collisions of the particles of volume V_{p} are not a continuous process. A shot effect is present.

The second term on the right-hand side of (17) is due to the dissipative effect of the large-scale fluctuations $\delta\tilde{N}$, and so the function $\tilde{A}$ can be expressed in terms of the integral $\tilde{I}$ [(22.23) in Ref. 15].

In summary, there is a sort of dual superstructure on top of the level of description by means of the Boltzmann equation.

The "first level" is governed by the molecular structure, which leads to a shot effect in the collision processes. For this reason the source $A_{\mathrm{B}}(x, x', t)$ is called *molecular*.

The "second level" is governed by the large-scale fluctuations and is not directly related to the molecular structure of the system. The source $\tilde{A}$ might therefore be called *turbulent*.

In specific situations one of these two factors may be dominant. Then one can distinguish two more particular generalizations of the Boltzmann equation. Let us first consider the case in which source $\tilde{A} = 0$. In this approximation one has $\tilde{I} = 0$ in (10), and equation (10) coincides with the Boltzmann equation. Thus the function $f_1(\mathbf{r}, \mathbf{p}, t)$ can be determined independently of the problem of calculating the fluctuations of the distribution function.

However, since the function f_1 is introduced, as was proposed in Leontovich's paper, in the form of a "mathematical expectation" ($nf_1 = \langle\tilde{N}\rangle$), there exists fluctuations $\delta\tilde{N} = \tilde{N} - \langle\tilde{N}\rangle$. In the present case the source $A(x, x', t)$ in the equation for the correlator $\langle\delta\tilde{N}\delta\tilde{N}\rangle$ is specified completely by the function $A_{\mathrm{B}}(x, x', t)$. This function can be expressed as follows in terms of the function f_1 [(10.12) in Ch. 11 of Ref. 18]:

$$A_{\mathrm{B}} = [(\delta\hat{I}_{\mathbf{p}} + \delta\hat{I}_{\mathbf{p}'}) - (\delta\hat{I}_{\mathbf{p}} + \delta\hat{I}_{\mathbf{p}'})_0]\, n\delta(\mathbf{r} - \mathbf{r}')\,\delta(\mathbf{p} - \mathbf{p}')\, f_1(\mathbf{r}, \mathbf{p}, t); \tag{18}$$

here $\delta\hat{I}_{\mathbf{p}}$ is an operator determined by the linearized Boltzmann collision integral. The subscript zero on the second term in the square brackets indicates that the collision operators act only on the distribution function [and not on the function $\delta(\mathbf{p} - \mathbf{p}')$].

The correlator $\langle\delta\tilde{N}\delta\tilde{N}\rangle_{x,x',t}$ can serve as an initial condition ($t = t'$) for calculating the double-time correlator, which satisfies the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \delta\hat{I}_{\mathbf{p}} + \mathbf{F}\frac{\partial}{\partial \mathbf{p}}\right)\langle\delta\tilde{N}\delta\tilde{N}\rangle_{x,t,x',t'} = 0, \quad t > t'. \tag{19}$$

The system of equations for the single-time and double-time correlators is equivalent to the Langevin equation for the function $\delta\tilde{N}$:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \delta\hat{I}_{\mathbf{p}} + \mathbf{F}\frac{\partial}{\partial \mathbf{p}}\right)\delta\tilde{N}(\mathbf{r}, \mathbf{p}, t) = y(\mathbf{r}, \mathbf{p}, t). \tag{20}$$

The left-hand side of this equation is of the same form as the linearized Boltzmann equation. The moments of the Langevin source are given by the formulas

$$\begin{gathered}\langle y(\mathbf{r}, \mathbf{p}, t)\rangle = 0,\\ \langle y(\mathbf{r}, \mathbf{p}, t)\, y(\mathbf{r}', \mathbf{p}', t)\rangle = A_{\mathrm{B}}(x, x', t)\,\delta(t - t').\end{gathered} \tag{21}$$

Thus the intensity of the Langevin source in the linearized Boltzmann equation for the fluctuations $\delta\tilde{N}$ is given by expression (18). The latter in turn serves as a source in the equation for the single-time correlator of the fluctuations $\delta\tilde{N}$ for $\tilde{A} = 0$.

For an equilibrium state the second term (with subscript zero) in formula (18) drops out, and we arrive at the result of Kadomtsev's paper on kinetic fluctuations in an ideal gas under equilibrium conditions. For a nonequilibrium state formulas (21) and (18) are equivalent to those given in Refs. 27, 28, and 15.

It follows from what we have said that results (18)–(21) can be obtained on the basis of the dissipative equation introduced by Leontovich, together with collision integral (5).

Let us now establish the connection between formula (5) [or (4)] and the well-known Prigogine-Brout expression for the collision integral in the equation for the many-particle function of a gas which is spatially homogeneous with respect to the coordinates of all the particles. These distribution functions are defined by

$$\tilde{f}_N(\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_N, \mathbf{p}_N, t) = \frac{1}{V^N}\tilde{f}_N(\mathbf{p}_1, \ldots, \mathbf{p}_N, t). \tag{22}$$

The expression for the integral $I_N(\mathbf{p}_1, \ldots, \mathbf{p}_N, t)$ follows from (5) and is of the form:

$$\begin{aligned}I_N(\mathbf{p}_1, \ldots, \mathbf{p}_N, t) = \frac{1}{V}\sum_{1\leqslant i,\, j\leqslant N}\int_0^{2\pi} d\varphi_{ij}\int_0^{\infty}\rho_{ij}\, d\rho_{ij}|\mathbf{v}_i - \mathbf{v}_j|\\ \times[\tilde{f}_N(\mathbf{p}_1, \ldots, \mathbf{p}_i', \ldots, \mathbf{p}_j', \ldots, \mathbf{p}_N, t) - \tilde{f}_N(\mathbf{p}_1, \ldots, \mathbf{p}_i, \ldots, \mathbf{p}_j, \ldots, \mathbf{p}_N, t)].\end{aligned} \tag{23}$$

By using Eq. (4), one can also write this integral in the Bogolyubov representation.

Formula (23) incorporates binary (Boltzmann model), but strong interactions. In the approximation of a perturbation theory in the interaction, it implies the result of Prigogine and Brout (Ch. 2 in Ref. 17). The corresponding equation for the single-particle distribution function (neglecting large-scale correlations) coincides with the kinetic equation of Landau.

Let us turn now to the other limiting case, in which the turbulent source $\tilde{A}$ is dominant on the right-hand side of (17). In this case, as we know, equation (10) does not reduce to the Boltzmann equation.

The additional contribution to the dissipative characteristics that is determined by the integral $\tilde{I}$ can be significant. In particular, this contribution governs the anomalous electrical conductivity of a plasma.[29,18]

Let us now return to equation (3) for the many-particle distribution function $\tilde{f}_N$ and consider some of its properties.

In an equilibrium state the collision integrals (4) and (5) go to zero when the multidimensional Maxwellian distribution

$$\tilde{f}_N(\mathbf{p}_1, \ldots, \mathbf{p}_N, t) = \frac{1}{V^N} \frac{1}{(2\pi mkT)^{3N/2}} \exp\Big(-\sum_{1\leqslant i\leqslant N} \frac{\mathbf{p}_i^2}{2mkT}\Big) \quad (24)$$

is substituted into them.

If the collision integral (5) [or (4)] is multiplied by the function $-k \ln \tilde{f}_N(\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_N, \mathbf{p}_N, t)$ and integrated over all the variables, we then have

$$-k \int \ln \tilde{f}_N \cdot I_N \, d\mathbf{r}_1 \, d\mathbf{p}_1 \ldots d\mathbf{r}_N \, d\mathbf{p}_N \geqslant 0. \quad (25)$$

This property provides for the undiminishing of entropy of the insulated system

$$S(t) = -k \int \ln \tilde{f}_N \cdot \tilde{f}_N \, d\mathbf{r}_1 \, d\mathbf{p}_1 \ldots d\mathbf{r}_N \, d\mathbf{p}_N, \quad (26)$$

i.e.,

$$\frac{dS}{dt} \geqslant 0. \quad (27)$$

The equal sign corresponds to an equilibrium state.

With the aid of equation (3) together with the collision integrals (4) and (5) [or (23) for the spatially homogeneous case] one can estimate the relaxation times in the various stages of the time evolution. For example, in the kinetic stage, which is described by the Boltzmann equation, we obtain for the relaxation time the familiar expression

$$\tau_{\mathrm{rel}} \sim \frac{l_{\mathrm{rel}}}{V_T} \sim \frac{1}{V_T n r_0^2}. \quad (28)$$

The relaxation time is thus determined by the mean free time of some *given* particle [such as particle number 1 in Eq. (10)].

With the aid of expression (23), for example, one can estimate the minimum relaxation time $(\tau_{\mathrm{rel}})_{\mathrm{min}}$—the time in which *any* single particle of the system "forgets its path." This is sufficient for the system as a whole to be unable to return to the initial state when the signs of the velocities of all the particles of the system are reversed. It follows from (23) that

$$(\tau_{\mathrm{rel}})_{\mathrm{min}} \sim \frac{1}{N} \tau_{\mathrm{rel}} \sim \frac{1}{N V_T n r_0^2}. \quad (29)$$

Thus, the minimum relaxation time is smaller by a factor of N than the mean free time. This time characterizes the initial stage in the onset of irreversibility.

One can see from what we have said that Leontovich's paper had great potential. The author regrets that he first read this paper when Leontovich was already gravely ill and unable to discuss any questions. One can only be amazed at how much Leontovich was ahead of his time in his understanding of the fundamental questions of statistical physics.

In conclusion, the author wishes to point out that other types of dissipative equations are possible for many-particle distribution functions.

Dissipative equation (3) is written for the distribution function of the complete set of variables $\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_N, \mathbf{p}_N$ of the N-particle system under consideration. Dissipation arises in this system when small-scale correlations are excluded. This is what dictates an incompleteness in the description.

A different situation is considered in Ch. 10 of Ref. 16 (see also Ch. 24 of Ref. 24). An equation is sought for the distribution function of the variables of the main system, which consists of N particles with an arbitrarily strong interaction between them. Here dissipation is established by the incompleteness of the description in the auxiliary variables of the extended system. In contrast to the case of (3) with the collision integral (5), here one obtains an equation for the many-particle distribution function which is nonlinear in the function $\tilde{f}_N$.

In the equilibrium state the "collision" integral in this equation goes to zero when the Gibbs canonical distribution with the Hamiltonian of the main system is substituted into it. Here the H theorem of Boltzmann is also valid. The minimum relaxation time, as in (29), is proportional to $1/N$.

The kinetic equations for the many-particle distribution functions are too complicated to solve. However, they can prove extremely efficient for constructing approximate equations corresponding to different levels of description and are well suited for describing the kinetics of coherent states during nonequilibrium phase transitions.

Basic to these studies is Leontovich's paper[6] on "The fundamental equations of the kinetic theory of gases from the standpoint of the theory of random processes." This paper will no doubt be studied for many years to come by students of the development of the statistical theory of nonequilibrium processes.

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