B. V. Chirikov, *Dynamic chaos in classical and quantum systems*. Dynamic chaos (d. c.) is defined as the random motion (usually oscillatory) of a fully determinate classical system (syn.: dynamic stochasticity). The paper considers one of the typical problems—a small perturbation (parameter $\varepsilon \ll 1$) of a fully integrable Hamiltonian system. The existence of a critical perturbation ($\varepsilon_{cr} > 0$) below which quasiperiodic oscillations persist for most paths assuming nonlinearity of the unperturbed system and sufficient smoothness of the perturbation, has been established by the Kolmogorov-Arnol'd-Mozer theorem. An effective estimate of ε_{cr} is obtained by analyzing the structure of nonlinear resonances and their interaction.[2] It has been possible in this way to solve the familiar Fermi-Past-Ulam problem of randomization of oscillations of a nonlinear string,[3] Budker's problem of retention of a particle in an adiabatic magnetic trap,[4] and others (see Refs. 2, 5–8).

If $\varepsilon \gg \varepsilon_{cr}$, the motion becomes chaotic for most initial conditions and admits of simple statistical description with a diffusion equation.[2] The mechanism by which d. c. arises involves strong (exponential) local instability, which is characterized by the average divergence rate of nearby paths or by the metric entropy h [sec^{-1}].[9] The latter is a convenient practical criterion of d. c. in computer modeling.[2] According to the algorithmic theory of dynamic systems, the condition $h > 0$ is necessary and sufficient for randomness of nearly all paths,[10] and the randomness concept is introduced on the basis of the complexity and unpredictability of the individual paths in accordance with intuitive "true"-randomness conceptions. We note the complexity of the chaotic path consists not in the equations of motion, but in the initial conditions of the path, and reflects the continuity of the phase space (p. s.) in classical mechanics.[11] In the author's opinion it is possible and useful in view of these results to proceed in a contrary manner, *defining* the random process as the motion of a classical deterministic system with $h > 0$.

When $\varepsilon \ll \varepsilon_{cr}$, d. c. is preserved in narrow layers around the separatrices of the resonances.[2,12] For multidimensional oscillations, this results in universal (for all $\varepsilon \to 0$) instability—Arnol'd diffusion throughout the entire system of stochastic layers intersecting in the p. s.[2,8,13] Despite the lower velocity, it may play a definite role in magnetic traps,[4] colliding-beam systems,[14] and the Solar System.[15]

Dynamic chaos is impossible in quantum mechanics owing to the discrete nature of a closed system bounded in p. s.[16] and the discreteness of the p. s. itself. However, the correspondence principle requires a transition to classical mechanics, including to d. c. This contradiction is resolved in Ref. 7 with the aid of the notion of temporary or transitory chaos, i.e., by introducing various time scales on which various statistical properties of d. c. are simulated.

The shortest, dynamic scale τ_d is obtained directly from Ehrenfest's theorem and, with consideration of the exponentially rapid spreading of the packets, is equal[7,17] to $\tau_d \sim \ln(n)/h$, where $n \gg 1$ is the characteristic quasiclassic parameter. Quantum dynamics agrees fully with classical dynamics on this scale.

The diffusion scale $\tau_D \gg \tau_d$, which was observed in Ref. 18 and explained in Ref. 7, is more interesting and unexpected. Consider, for example, a classical system specified by the mapping $(I, \theta) \to (\bar{I}, \bar{\theta})$:

$$\bar{I} = I + k \sin\theta, \quad \bar{\theta} = \theta + T\bar{I}, \tag{1}$$

i.e., a rotator acted upon by short "bushes" ($k \sin\theta$) with period T. Many nonlinear-oscillation problems can be reduced to such a model including motion in the vicinity of a separatrix.[2] For $k > k_{cr} = 1/T$, we have unlimited diffusion with respect to I at a rate $E/t = \langle I^2\rangle/2t \approx k^2/4k_{cr}$, and t is the number of iterations of the mapping). In the quantum case ($I = n$ is the "rotator" level number, k is the number of quanta per push, and $\hbar = 1$), the same diffusion is preserved on the interval $t \lesssim \tau_D$ with the additional condition $k \gg 1$ ($k \sim 1$ is the quantum limit of stability[19]). At $t \gtrsim \tau_D$, the diffusion rate drops sharply (Fig. 1, curve 2 at $t < 150$; the straight line corresponds to classical diffusion). In this example, $k = 20$, $T = 1/4$, $\tau_D \sim k^2/8 = 50$, $\tau_{cl} \sim \ln(1/T)/2h_{cl} \approx 0.8$. At $t > t_d$ there is no local instability in the quantum system ($h_{qu} = 0$). This is illustrated in Fig. 1[20] by the reversal of the velocities ($\psi \to \psi^*$) at time $t = 150$, followed by "antidiffusion," i.e., a return to the initial state (with accuracy better than 10^{-6}). Curve 1 characterizes the classical system with the same parameters ($E(t)$ is the average over 10^3 paths, the scale is unchanged). Here $h_{cl} \approx \ln(kT/2) > 0$ and the computing errors ($\sim 10^{-12}$) "restore" diffusion after ~30 iterations following velocity reversal.

The relaxation toward the equilibrium state at $t \gg \tau_D$ (open circles in the left-side of Fig. 2, $k = 40$) is described satisfactorily by the expression $4E(t)/k^2 \approx \tau_D \ln(1.5t/\tau_D)$ (solid line; $\tau_D \approx 140$) up to $t \sim \tau_R \sim \tau_D N \ln N$, where $N \sim k^2/6 \sim \tau_D$ is the number of excited eigenfunctions. During relaxation, the diffusion Gaussian distribution becomes an equilibrium distribution $f_s(n)$, which is compatible with a simple exponential at $n \gg 1$ (Fig. 2; the dark circles represent $t = 10^3$, the open circles $t = 10^4$, and the straight line $f_s(n) = \exp(-n/21)$; $k = 10$). By virtue of the discreteness of the spectrum (this is not mandatory for the nonclosed system (1), but in the present case it proceeds from the boundedness of diffusion), the fluctuations of $E(t)$ are clearly regular (upper curve in Fig. 2; $k = 10$, $\tau_D \sim 13$,

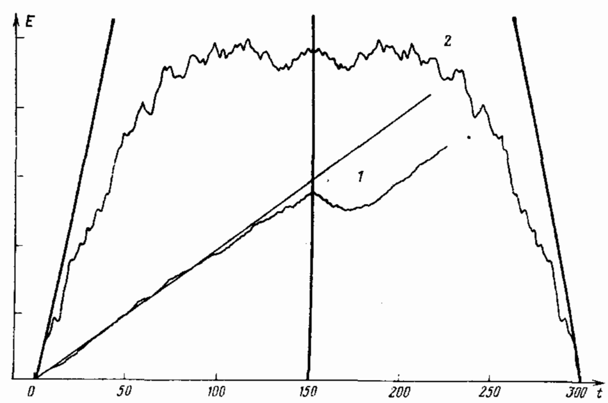

FIG. 1.

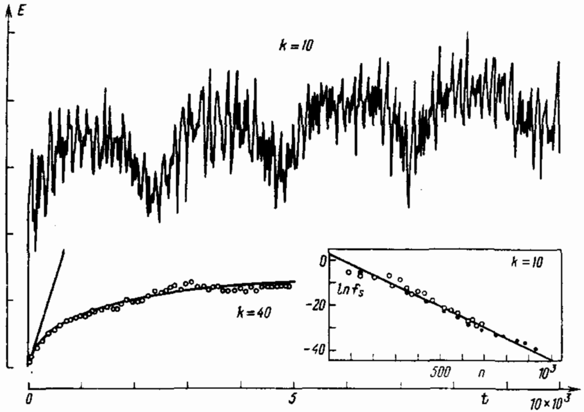

FIG. 2.

$\tau_R \sim 600$). Nevertheless, the eigenfunctions are not only Shnirel'man ergodic[21] (i.e., their Wigner functions are, on the average, uniformly distributed in the accessible region of p.s.[22]), but are also, apparently, Gaussian random functions (Shnirel'man's hypothesis). Numerical modeling indicates that the spatial structure of $|\psi|^2$ resembles that of $N \sim k^2/6$ randomly distributed classical particles.

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