Caustics, catastrophes, and wave fields

Yu. A. Kravtsov and Yu. I. Orlov

Institute of General Physics, Academy of Sciences of the USSR; Moscow Energetics Institute

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The modern view of caustics as the singularities of mappings performed by rays is set forth. The substantial progress which has recently been achieved in research on caustic fields can be credited to progress in the theory of the singularities of differentiable mappings (catastrophe theory). This theory has generated an exhaustive classification of the structurally stable caustics and of the corresponding standard diffraction integrals which describe the fields near caustics. The standard integrals can be used to construct both local and uniform asymptotic field representations in the presence of caustics. Asymptotic methods for describing the field have also been developed for penumbral caustics associated with edge catastrophes and for several other types of caustics which arise in wave problems in optics, acoustics, radio propagation, plasma physics, atomic physics, etc.

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CONTENTS

1. INTRODUCTION

Interest in caustics is a thread running throughout the history of physics, back to antiquity, stimulated by the significant concentration (or focusing) of a field which occurs on caustics. In many cases this concentration of a wave field—acoustic, optical, electromagnetic, seismic, etc.—can be detected by physical instruments, and in the case of light it can be detected visually. It is seen most clearly in the focal plane of an ordinary lens, when the caustic surface degenerates (ideally) to a point. Pronounced concentration of a field u can also be observed in several other situations, e.g., near a simple (nonsingular) caustic, as shown by the sketch in Fig. 1.

The problem of describing the focusing of wave fields in the presence of caustics is crucial to many branches of physics, but it is particularly acute in radio propagation and in the acoustics of natural media. Focusing and caustic phenomena must constantly be dealt with in the propagation of radio waves in the troposphere and ionosphere, in the propagation of radio waves through the interplanetary plasma and the plasma environment of the sun, in the propagation of light in a turbulent atmosphere, in the propagation of sound in the ocean, etc.

We will cite only three examples. Figure 2 shows a family of radio rays in the ionosphere excited by a source on the

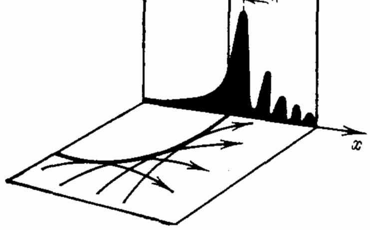

FIG. 1. Distribution of the field intensity near a nonsingular caustic.

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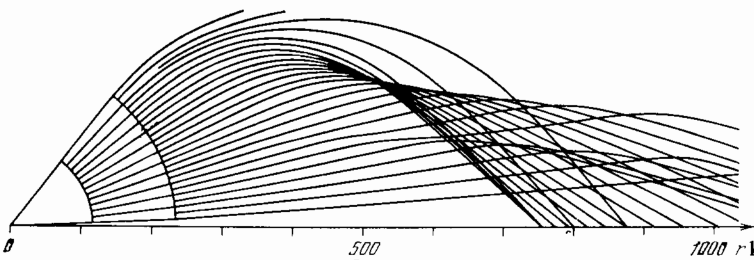

FIG. 2. Ray pattern when a ground-level point source of radio waves irradiates the ionosphere (from Ref. 1).

earth. In this figure, taken from Ref. 1, we can clearly see both the shadow region and the regions in which the electromagnetic field is focused. A focusing pattern no less complicated is observed in the propagation of sound in the deep ocean as shown in Fig. 3 (which is taken from Ref. 3; see also the ray patterns in Ref. 2). Figure 4 is a photograph taken by B. S. Agranovskiĭ and A. S. Gurvich of the intensity distribution of a light wave transmitted through a turbulent liquid. We can clearly see the caustic spots resulting from random focusings.[4]

Finding the fields near caustics is also important for problems involving the scattering of light and particles (the rainbows in optics and atomic physics, for example), for optical instrumentation, for the engineering of mirror radio antennas, for the theory of gravitational lenses, for nonlinear optics (self-focusing), and for many other applications.

Our purpose in this review is to report the present state of the problem of finding caustic fields. The problem has two aspects—geometric and field aspects—and substantial progress has been achieved recently in each. In the brief summary of the results which follows we will specify the particular sections of this review in which the corresponding topics are covered.

In speaking of the geometric aspect of the question we must first mention the development of a fundamentally new view of caustics—as singularities of certain mappings performed by a family of rays. The theory of the singularities of differentiable mappings is a new branch of mathematics which has yielded the general properties of the mappings of various types of manifolds onto spaces of various dimensionalities. Important contributions to this theory have been made by V. I. Arnol'd, H. Whitney, and R. Thom, among others.

The theory of the singularities of mappings has also become known as "catastrophe theory," in the terminology of Thom. This term was adopted because when the locus of singularities corresponding to a given system is crossed the state of the system undergoes an abrupt and qualitative change: a "catastrophe." In geometrical optics, this abrupt change in state at a caustic surface is seen as a change in the number of rays which arrive at a given point in space. Despite the well-known disadvantages of the term "catastrophe theory," which have stemmed primarily from its unjustifiably broad interpretation (cf. Refs. 5, 6 in this connection) we will certainly not tend to avoid it in the present review, in which we will use "catastrophe theory" as simply a synonym for the "theory of the singularities of differentiable mappings." The most important role which has been played by the theory of singularities in the questions which we will be discussing here has been to furnish a basis for a complete classification of the structurally stable caustics. Section 2 of this review discusses structurally stable caustics, i.e., the "zoology" of caustics. Certain examples of caustics which have been studied up to now are also discussed there. This part of the review was written primarily on the basis of Refs. 5–9, but it also draws from Refs. 10–12 and 134.

With regard to the field aspect of the caustic problem, which is our major concern here, there are several points to be made.

First, in addition to classifying caustics, catastrophe theory has also solved the problem of classifying the standard integrals which describe the diffraction of fields near

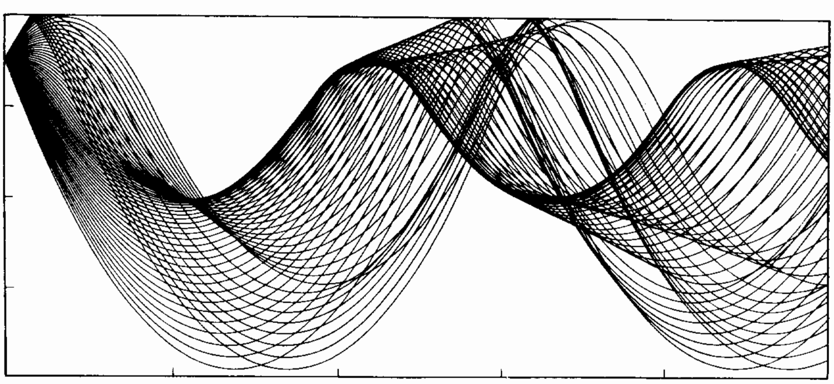

FIG. 3. An example of a ray pattern in an oceanic sound channel.[3]

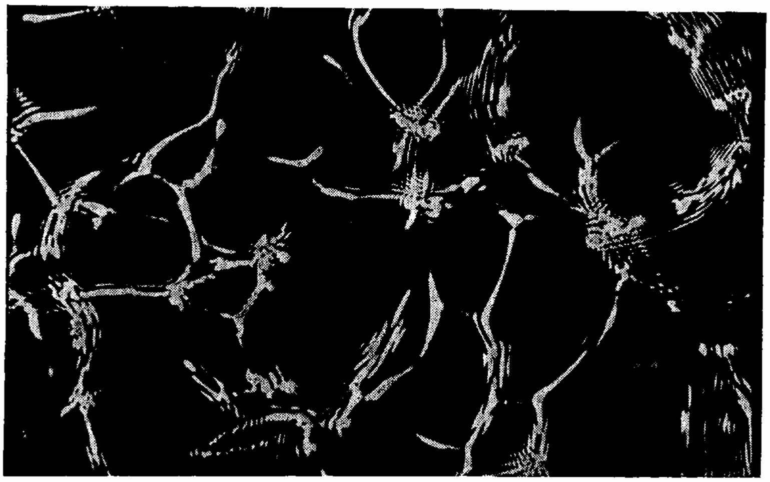

FIG. 4. Negative image of the intensity distribution in the cross section of a light beam emerging from a cell holding a turbulent liquid.[4] The bright spots with evidence of an interference pattern correspond to random caustics and foci formed in the light beam. (This photograph was graciously furnished by B. S. Agranovskiĭ and A. S. Gurvich.)

structurally stable caustics. This solution sets the stage for the fundamental solution of the problem of constructing local and uniform asymptotic representations of fields in the presence of caustics. These questions are discussed in Section 3.

Second, the list of standard functions and integrals intended for describing various types of caustic fields has recently been extended considerably. In particular, some standard integrals have been proposed for the penumbral caustics which are formed near light-shadow boundaries (Section 4) for certain types of structurally unstable caustics, e.g., for the case of axial symmetry, and also for several other caustics, which are listed in Section 5.

Third, simple methods have been proposed for evaluating the width of the caustic zone and the strength of the field directly on the caustics by working simply from the laws of geometrical optics.[12,51] The qualitative theory has also made it possible to resolve the question of the reality (observability) of caustics. The qualitative theory is outlined in Subsection 3b.

Fourth and finally, recent years have seen a substantial growth of the "geography" of caustics, by which we mean a lengthening of the list of problems in which caustics are encountered: space-time caustics, caustics in nonlinear and anisotropic media, caustics in media with a spatial dispersion, complex caustics, etc. There has also been an increase in the number of applications in which one must deal with the presence of caustics[1)] (Section 6).

In accordance with this outline, we turn now to the classification of caustics on the basis of catastrophe theory.

2. CAUSTICS AS SINGULARITIES OF DIFFERENTIABLE MAPPINGS (CATASTROPHES)

a) The ray surface and the Lagrange manifold

We first take up the question of how the theory of caustics is related to the theory of mappings. For this purpose we write the equation of a family of rays which emerge from starting surface S^0:

$$x = x\,(\xi,\ \eta,\ \tau),\quad y = y\,(\xi,\ \eta,\ \tau),\quad z = z\,(\xi,\ \eta,\ \tau), \tag{2.1$'$}$$

or, in more compact form,

$$\mathbf{r} = \mathbf{r}\,(\boldsymbol{\xi}),\quad \boldsymbol{\xi} = (\xi,\ \eta,\ \tau); \tag{2.1$''$}$$

here $\mathbf{r} = (x, y, z)$ are the Cartesian coordinates, ξ and η are parameters on S^0, and τ is a parameter along the ray (Fig. 5). The parameters $\boldsymbol{\xi} = (\xi, \eta, \tau)$ are called the "ray coordinates." In the case $\tau = 0$, Eqs. (2.1) describe the starting surface S^0. If all the functions characterizing the problem (the equation of the starting surface S^0 and the equation describing the behavior of the parameters of the medium over space) are continuous, along with all their derivatives, then we are dealing in Eqs. (2.1) with infinitely differentiable functions which describe a smooth three-dimensional hypersurface F in the expanded six-dimensional space $\{\mathbf{r}, \boldsymbol{\xi}\} = \{x, y, z; \xi, \eta, \tau\}$. We call F the "ray surface."

When ray surface F is projected (mapped) from the expanded space $\{\mathbf{r}, \boldsymbol{\xi}\}$ onto the physical three-dimensional (configuration) space $\mathbf{r} = \{x, y, z\}$, singularities may arise. These singularities are naturally identified with caustics, since the Jacobian of the transformation from the Cartesian coordinates to the ray coordinates vanishes, $\mathscr{D} = \partial(x, y, z)/\partial(\xi, \eta, \tau) = 0$, and this equation corresponds to the caustic equation.[12] The transition through the caustic, i.e., through the locus of points at which the mapping of F onto the space $\{x, y, z\}$ has singularities, corresponds to the increase or decrease by some even number in the number of rays which

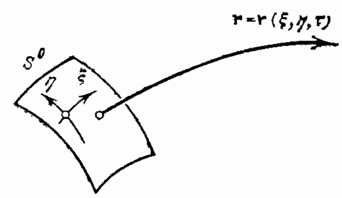

FIG. 5. Ray trajectory and ray coordinates.

[1)]We might also note that the "noncatastrophic" singularities of wave fields—in particular, dislocations—have recently attracted increased interest.[118–122]

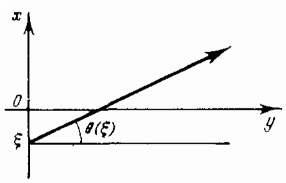

FIG. 6. Ray trajectory in the $\{x, y\}$ plane in a homogeneous medium.

reach the observation point $\mathbf{r} = (x, y, z)$. The abrupt appearance (or disappearance) of pairs of rays is interpreted in mapping theory as a catastrophe, in this case as a qualitative change in the ray pattern as we go from point to point. When we cross a simple caustic from the shadow region into the light region, for example, we find that two rays are created.

We cannot draw a picture of the six-dimensional space $\{\mathbf{r}, \boldsymbol{\xi}\}$, so we will consider a simple model of a ray surface in the three-dimensional space of the parameters $\{x, y, \xi\}$. We denote by ξ the exit point, while $\theta(\xi)$ is the angle at which the ray emerges from the surface $y = 0$ (Fig. 6). The equation of the family of rays in the $\{x, y\}$ plane is then written

$$x = \xi + y \operatorname{tg} \theta(\xi). \quad (2.2)$$

In the extended three-dimensional space $\{x, y, \xi\}$, this equation corresponds to a two-dimensional ray surface F: $\xi = \xi(x, y)$, described by Eq. (2.2). Figure 7a shows a representative form of this surface for the case in which the tangent of the angle at which the rays are inclined varies in accordance with $\operatorname{tg} \theta(\xi) = \beta\xi/(\xi^2 + a^2)$.

When the ray surface F is projected onto the physical plane $\{x, y\}$ we find caustics corresponding to singularities of the mapping. In the present case (Fig. 7b), the caustic has the shape of a "beak," and the corresponding singularity of the surface F is called a "cusp." When we go across the caustic into the beak, we find that the number of rays increases from one to three.

A similar interpretation of caustics arises when we consider the rays in the six-dimensional phase space $\{\mathbf{r}, \mathbf{p}\} = \{x, y, z; p_x, p_y, p_z\}$, where $\mathbf{p} = \nabla\psi$ is the gradient of the eikonal, which serves as a momentum. Instead of a ray surface, we would speak in this case of a Lagrange manifold.[1,10,11] The parametric equation of the Lagrange manifold, $\mathbf{r} = \mathbf{r}(\xi, \eta, \tau)$, $\mathbf{p} = \mathbf{p}(\xi, \eta, \tau)$, can be derived from the ray equations written in Hamiltonian form.

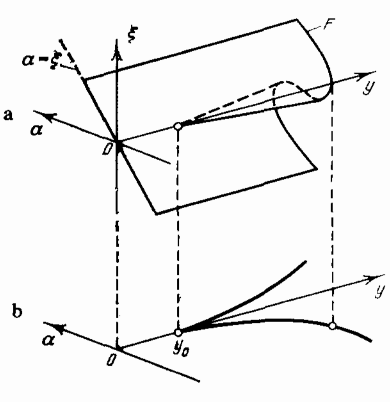

FIG. 7. a—The ray surface F in the expanded space $\{x, y, \xi\}$; b—singularities of its mapping of a "beak" onto the $\{x, y\}$ physical plane.

A more general view of caustics can be achieved by specifying the ray surface in the space $\{\mathbf{r}, \alpha\}$, whose dimensionality is increased by taking into account all the parameters α_j, $j = 1, \dots, N$, which are important to the given problem. The parameters α_j might characterize, for example, the position of the source, the point at which the ray emerges from S^0, arbitrary refractive indices along certain directions or others, the shape of the starting surface S^0, and so forth. The total number of varied parameters, N, along with the three Cartesian coordinates, forms the expanded space $\mathbf{w} = \{\mathbf{r}, \boldsymbol{\alpha}\}$ of dimensionality $N + 3$. The caustics in this case are singularities of the mapping of ray surface F onto some subspace of lower dimensionality. We would usually be interested in the mapping of F onto the physical space $\{x, y, z\}$ or onto some plane in this space, but in several cases we need to map onto planes of other parameters of importance to the given problem, especially if we suspect that slight changes in one of the parameters α_j might give rise to new singularities in the $\{x, y, z\}$ configuration space.

b) Classification principle for structurally stable caustics

As we have already mentioned, catastrophe theory makes it possible to classify the structurally stable caustics. The basis for the classification is that a local transformation of variables $\mathbf{w} \rightarrow \mathbf{w}'$ ("local" here means near some singularity of the mapping, $\mathbf{w}_0$) can be used to put the equation of the surface F into one of the typical (normal) polynomial forms by means of smooth transformations of variables, including the operations of rotations of axes, translations, and changes of scale.

The essential features can be illustrated by the simple example of a one-dimensional projection of a smooth curve $f(x, y) = 0$ onto the horizontal axis (x). We assume for simplicity that this curve passes through the origin, i.e., $f(0,0) = 0$. If the function $y = y(x)$ is monotonic near the origin (the upper part of Fig. 8a), then the smooth transformation $y_1 = \beta(y)$, $x_1 = \gamma(x)$, which changes only the scales along the x and y axes, can be used to "straighten" the curve $f(x, y) = 0$, i.e., to reduce it to a straight line $f_1(x_1, y_1) = y_1 - x_1 = 0$ (the lower part of Fig. 8a). In this case the mapping of the curve $f(x, y) = 0$ onto the x axis is mutually one-to-one, and there are no singularities.

If y is a double-valued function of x (the upper part of Fig. 8b) then the function $f(x, y)$ can be locally transformed to the form $f_2(x_1, y_1) = x_1 + y_1^2$ (the lower part of Fig. 8b). In this case we have only one singularity $(x' = 0)$ in the mapping of the curve $f(x, y) = 0$ onto the x axis. At this point we have $|dy/dx| = \infty$; i.e., the curve $f(x, y) = 0$ has a vertical tangent. If y is a triple-valued function of x (at the top in Fig. 8c), then the mapping has two singularities, x' and x'', at which $|dy/dx| = \infty$. When we cross these points we find catastrophies: abrupt changes in the number of branches which are projected in a one-to-one fashion onto the x axis. If there are two singularities, the function $f(x, y)$ can be reduced locally by a smooth transformation to two second-degree polynomi-

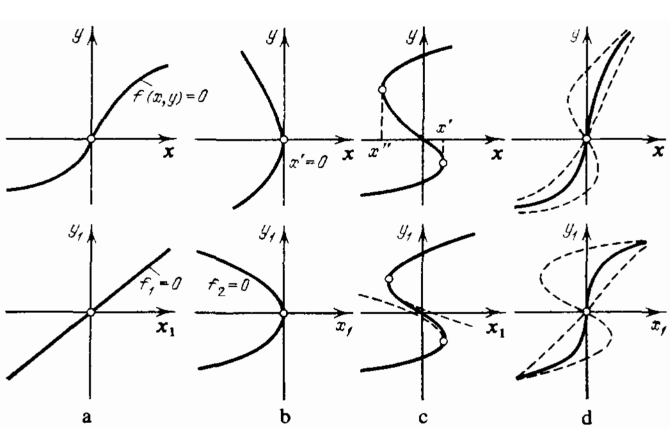

FIG. 8. The simplest typical situations which arise in the mapping of a smooth curve $f(x, y) = 0$ onto the horizontal axis. a—Mutually one-to-one mapping; b—mapping with a single singularity; c—mapping with two singularities; d—structurally unstable (atypical) situation (solid curves), which reduces as a result of a small disturbance (the dashed curves) to either case a or c.

als, which are "tied" to the singularities (the lower part of Fig. 8c).

The linear (f_1) and quadratic (f_2) polynomials discussed above are examples of the so-called normal (typical) forms to which essentially all curves in the $\{x, y\}$ plane can be reduced locally; alternatively, as one would say in catastrophe theory, all the curves have a "common position," or are "generic." The only exceptional cases are certain degenerate cases corresponding to structurally unstable mappings.

One example of a structurally unstable mapping is shown at the top in Fig. 8d. The point with the vertical tangent on this curve is simultaneously an inflection point. This curve reduces locally to the form $\tilde{f}_3 = x_1 - y_1^3 = 0$ (the lowest part of Fig. 8d). If, upon small variations in the parameters of the problem, the curve $f(x, y) = 0$ becomes one of the two dashed curves shown at the top of Fig. 8d, then this curve leads to either a linear function (Fig. 8a) or two quadratic polynomials which are locally fitted to the singularities, as shown at the bottom of Fig. 8c.

In mathematics one deals with entities of various natures: curves, functions, families of functions, mappings, caustics, etc. Back before the turn of the century, Poincaré suggested that instead of analyzing each separate entity, e.g., each curve or function, one could base the analysis on an equivalence concept introduced in an appropriate way for each class of entities. For example, two functions might be regarded as equivalent if one could be obtained from the other by a smooth change of variables. If we can classify entities with respect to this equivalence, we can obtain a considerable amount of information about all entities of a given nature by thoroughly studying simply one member of each equivalence class. These considerations underlie the theory of the singularities of differential mappings which has been worked out over the past 10–15 years.

In both physics and mathematics we are primarily interested in stable entities, which change only slightly in the face of perturbations of some sort or other. In catastrophe theory the entity is regarded as stable if all neighboring entities of the same nature are equivalent to it. The neighboring entities differ from the original one by small perturbations, which are called "small disturbances" in catastrophe theory. If the entities of interest have an analytic nature—if they are functions, mappings, etc.—then the concept of "small disturbances" means that not only the functions themselves but also their derivatives are neighbors.

The concept of structural stability, one of the fundamental concepts in catastrophe theory, is related to the concept of coarseness which was introduced in the theory of dynamical systems by Andronov.[13] Transitions from one normal form to another through a structurally unstable state correspond to bifurcations in the theory of dynamical systems. For example, the transition from the one-ray case to the three-ray case which is seen on the x axis as y is increased in Fig. 4 occurs at the bifurcation value $y = y_0$.

While all singularities of a mapping of the curve $f(x, y) = 0$ onto the x axis are classified within the framework of ordinary mathematical analysis, in the multidimensional case we find that the solution of the classification problem is significantly more complicated, because the typical polynomials are also multidimensional. Constructing a systematic classification of structurally stable singularities required borrowing ideas from topology and differential geometry. As was shown by Arnol'd, this classification is intimately related to the theory of Lie groups (see the articles of Refs. 5–9 and also the books of Refs. 14–16 and 134 for the history of the question and for references to the original papers).

All typical forms of the hypersurfaces F in multidimensional spaces of dimensionality up to $m = 10$ have now been identified. Analytic expressions have been found for structurally stable singularities of mappings of F onto spaces of other dimensionalities. In addition—a very important point—it has been shown that there are no singularities other than those which have been identified. We will restrict the treatment in this review to the "formula" aspects of the question, avoiding the mathematical subtleties, and we will adhere to the universal classification developed by Arnol'd.[5]

The only point which we feel has to be made here is the content of the term "small disturbances." The "small disturbances" which are used in catastrophe theory are related to the "small perturbations" used in physics but are not identical with them. The difference is that the term "small disturbance" implies that not only the perturbation $\tilde{f}$ itself but also its derivatives are small.

To illustrate this aspect of small disturbances, we note that if the sinusoidal perturbation $\tilde{f} = a \sin \varkappa y_1$ is to qualify as a small disturbance then not only a but also $\varkappa$ must be small, so that the derivative $|\tilde{f}'| \sim a\varkappa$, will be small. If the parameter $\varkappa a$ is too large, $\varkappa a \gg 1$, then the perturbation can no longer be regarded as a small disturbance. In this case the perturbed curve

$$f_2(x_1, y_1) + \tilde{f}(x_1, y_1) = x_1 + y_1^2 + a \sin \varkappa y_1 = 0$$

can acquire many additional singularities (Fig. 9), whose number can be estimated as $\varkappa a$. This example shows that small disturbances and the related ideas regarding structural stability in mathematics are treated in a fashion different

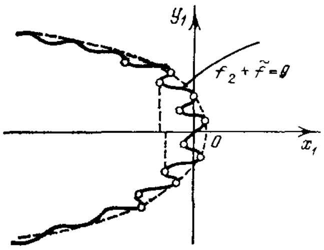

FIG. 9. The appearance of a set of singularities during the mapping onto the x_1 axis of the smooth curve $f_2 = x_1 + y_1^2 = 0$ perturbed by a small sinusoidal term $\tilde{f} = a \sin \mu y_1$.

from that in which small perturbations are treated in physics.

c) Caustic surfaces of low codimensionality (Thom's seven catastrophes)

The most economical method for classifying caustics is based on the introduction of generating functions of the type

$$\varphi(\zeta, \tau) = \varphi_0(\zeta) + \sum_{p=1}^{m} \zeta_p \varphi_p(\tau), \quad (2.3)$$

where $\zeta = \{\zeta_1,...,\zeta_m\}$ are the so-called external parameters of the problem, obtained as a result of the local transformation (rotations, translations, and changes in scale) of the original parameters $\mathbf{w} = \{\mathbf{r},\boldsymbol{\alpha}\}$ and $\boldsymbol{\tau} = \{\tau_1,...,\tau_l\}$ are the auxiliary internal parameters. The number of external parameters of importance to the given problem, m, is the "codimensionality" of the caustics, while the number of internal parameters, l, is the "corank" of the caustic.

The generating functions (2.3) are chosen in such a manner that they can serve as phase functions in diffraction integrals of the type $\int \exp(i\varphi) d^l \tau$, which we will discuss in Section 3. The corank l corresponds to the multiplicity of the diffraction integrals describing the field near the caustics; the internal variables τ_k serve as integration variables. The number of stationary points in such integrals determines the number of rays for the caustic of the given type. The sequential classification of caustics—purely geometric entities—thus embodies at the outset the prerequisites for the construction of a wave field.

The generating function (2.3) is linear in the external parameters ζ_p, and the functions φ_p are "monomials," i.e., products of powers of the auxiliary parameters τ_k:

$$\varphi_p(\tau) = c_p \prod_{k=1}^{l} (\tau_k)^{s_{pk}}, \quad s_{pk} \geqslant 0, \quad p \geqslant 1. \quad (2.4)$$

Each type of caustic has its own values of l and m and its own universal functions $\varphi_p(\tau)$.

The typical (normal) forms of the ray surface F can be found by differentiating the generating function φ with respect to the auxiliary parameters:

$$F: \frac{\partial \varphi}{\partial \tau_k} = 0, \quad k = 1, 2, \ldots, l, \quad (2.5)$$

The equation for the locus of singularities is found by constructing the determinant of second derivatives of φ and setting it equal to zero:

$$\det \left| \frac{\partial^2 \varphi}{\partial \tau_j \partial \tau_k} \right| = 0. \quad (2.6)$$

The projection (or mapping) is carried out from the space $\{\zeta_1,...,\zeta_m; \tau_1,...,\tau_l\}$ of all parameters of the problem onto the space $\{\zeta_1,...,\zeta_m\}$ of the external parameters.

Table I shows some typical generating functions φ and equations of the ray surface F for the simplest types of singularities ("Thom's seven elementary catastrophes"[14]), along with their Arnol'd classification, which is based on the relationship between the mappings and certain Lie groups. The appearance of singularities as a result of mappings can be

TABLE I. The simplest caustics ("Thom's seven catastrophes").

Designation and name	Generating function $\varphi(\zeta,\tau)$	Equation of ray surface F	Indices of caustic zones	Focusing index δ
A_2, fold	$\frac{1}{3}\tau_1^3 + \zeta_1\tau_1$	$\tau_1^2 + \zeta_1 = 0$	2/3	1/6
A_3, cusp	$\pm\frac{1}{4}\tau_1^4 + \zeta_1\tau_1 + \frac{1}{2}\tau_1^2\zeta_2$	$\pm\tau_1^3 + \zeta_1 + \zeta_2\tau_1 = 0$	3/4; 1/2	1/4
A_4, swallow-tail	$\frac{\tau_1^5}{5} + \zeta_1\tau_1 + \frac{\zeta_2\tau_2^2}{2} + \frac{\zeta_3\tau_1^3}{3}$	$\tau_1^4 + \zeta_1 + \zeta_2\tau_1 + \zeta_3\tau_1^2 = 0$	4/5; 3/5; 2/5	3/10
D_4^-, wave crest; D_4^+, hair	$\pm\tau_2^3\tau_1^2 + \zeta_1\tau_1 + \zeta_2\tau_2 + \zeta_3\tau_2^2$	$\mp 3\tau_2^2 + \zeta_2 + 2\zeta_3\tau_2 = 0$ $2\tau_1\tau_2 + \zeta_1 = 0$	2/3; 2/3; 1/3	1/3
A_5, butterfly	$\pm\frac{\tau_1^6}{6} + \zeta_1\tau_1 + \frac{\zeta_2\tau_1^2}{2} + \frac{\zeta_3\tau_1^3}{3} + \frac{\zeta_4\tau_1^4}{4}$	$\pm\tau_1^5 + \zeta_1 + \zeta_2\tau_1 + \zeta_3\tau_1^2 + \zeta_4\tau_1^3 = 0$	5/6; 2/3; 1/2; 1/3	1/3
A_5, mushroom (parabolic umbilic)	$\pm\tau_2^4 + \tau_1^2\tau_2 + \zeta_1\tau_1 + \zeta_2\tau_2 + \zeta_3\tau_2^2 + \zeta_4\tau_2^3$	$2\tau_1\tau_2 + \zeta_1 = 0$ $\pm 4\tau_2^3 + \tau_1^2 + \zeta_2 + 2\zeta_3\tau_2 + 3\zeta_4\tau_2^2 = 0$	5/8; 3/4 1/2; 1/4	3/8

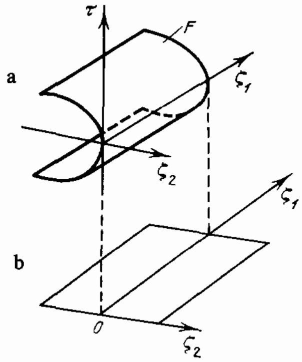

FIG. 10. The simplest singularity, A_2 (a fold; $m = 1, l = 1$), corresponding to a nonsingular caustic.

understood clearly by considering just two examples: folds ($m = 1, l = 1$; singularity A_2) and cusps ($m = 2, l = 1$; singularity A_3). The fold singularities correspond to a simple (nonsingular) caustic (Fig. 10), while a cusp corresponds to a caustic beak having a single cuspidal point (Fig. 11).

Continuing up the complexity scale we next find the swallowtail (type A_4; $m = 3$, $l = 1$), which is now a three-dimensional surface (Fig. 12). Three-dimensional surfaces also map caustics of the umbilical type: D_4^+, a wave crest or purse (the hyperbolic umbilic; Fig. 13) and D_4^-, the hair or pyramid (the elliptic umbilic; Fig. 14). These caustics have a codimensionality $m = 3$ and a corank $l = 2$. The caustics of higher codimensionality, $m > 3$, can be represented only by showing their intersections with certain planes. Figure 15a, for example, shows a three-dimensional section through a caustic of type A_5 ($m = 4, l = 1$; the butterfly), while Fig. 15b shows two-dimensional sections of this caustic. For all these caustics the maximum number of rays is determined by the codimensionality; specifically, this maximum number is $m + 1$.

d) Caustics of high codimensionality

The caustics of codimensionality $m \leqslant 4$ listed above, which correspond to $m + 1 \leqslant 5$ rays, include a large fraction

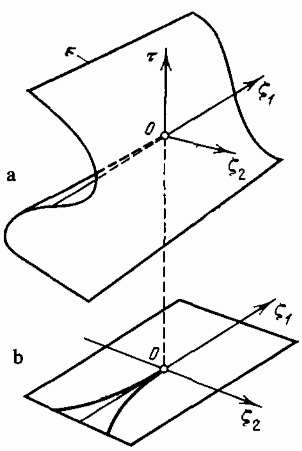

FIG. 11. The appearance of a caustic beak upon the projection of the ray surface F, which has a cusp singularity, of type A_3 ($m = 2, l = 1$).

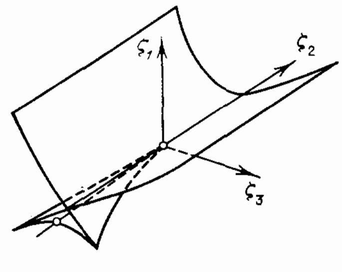

FIG. 12. The caustic surface of type A_4, the swallowtail ($m = 3$, $l = 1$).

of all cases of practical interest. We will nevertheless take a brief look at the "zoology" of caustics of higher codimensionality.

1) Simple (zero-modal) caustics of higher codimensionality. The caustics described in Subsection 2c are characteristic in that the corresponding generating functions $\varphi(\zeta,\tau)$ do not contain arbitrary constants (moduli). They are accordingly called "zero-modal" or "simple" caustics. The complete list of zero-modal caustics includes the two infinite series A_{m+1} ($m \geqslant 1$, $l = 1$) and D_{m+1} ($m \geqslant 3$, $l = 2$) and three others: E_6, E_7, and E_8 ($m = 5, 6, 7$, respectively; $l = 2$). The parameters of these caustics are given in Table II.

2) Unimodal caustics. Unimodal caustics are distinguished by the circumstance that the corresponding generating functions contain only a single nonremovable arbitrary constant. According to Ref. 5, the class of unimodal caustics is exhausted by one infinite three-index series (according to the complex classification), $T_{p,q,r}$ ($4 \leqslant p \leqslant q \leqslant r$, $m = p + q + r - 3 \geqslant 9$), and 14 exceptional types (not conforming to a series): $K_{12,13,14}$, $Z_{11,12,13}$, $W_{12,13}$, $Q_{10,12,12}$, $S_{11,12}$, and U_{12}. According to the real classification, we would distinguish the parabolic singularities $X_9 = T_{2,4,4}$, $J_{10} = T_{2,3,6}$, $P_8 = T_{3,3,3}$ and the series of hyperbolic singularities J_{m+2} ($m \geqslant 9$), X_{m+2} ($m \geqslant 8$), $Y_{p,q}$ ($5 \leqslant p \leqslant q$, $m = p + q - 1 \geqslant 9$), P_{m+2} ($m \geqslant 7$), $R_{p,q}$ ($4 \leqslant p \leqslant q$, $m = p + q \geqslant 8$). The parameters of the unimodal caustics of codimensionality $m \leqslant 1$ and corank $l = 2$ are listed in Table III.

The number of rays for unimodal caustics is $m + 2$; this number serves as an index for most of the caustics. The caustic J_{11}, for example, corresponds to 11 rays. In the general case of caustics of modality μ (i.e., for a generating function with μ nonremovable constants), the number of rays is $m + \mu + 1 \equiv M + 1$.

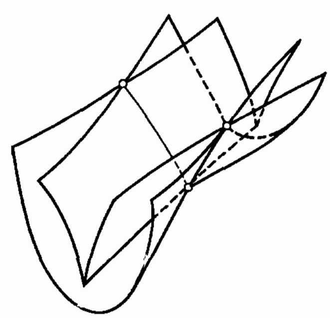

FIG. 13. The caustic surface of the umbilic type D_4^+, the purse ($m = 3$, $l = 2$).

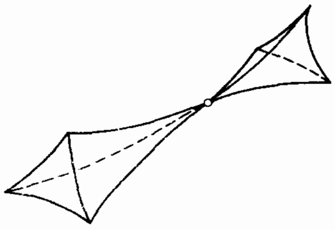

FIG. 14. The caustic surface of the umbilic type D_4^-, the pyramid ($m = 3$, $l = 2$).

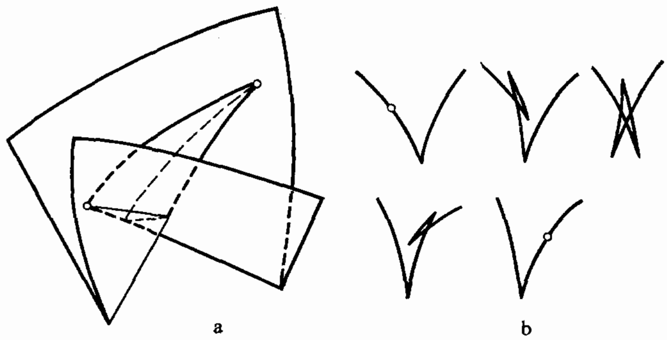

FIG. 15. Three-dimensional (a) and certain two-dimensional (b) sections through the caustic surface of type A_5 (butterfly, $m = 4$, $l = 1$).

3. Caustics of higher modality. Beginning with codimensionality $m = 10$, the list of singularities has the pentamodal caustic O_{16}, and at $m = 11$ we also see the bimodal caustics Q_{14}, S_{14}, U_{14} and the trimodal caustic V_{15}.

Table IV summarizes the caustics of codimensionality $m \leqslant 11$ and also shows the corank and modality of the caustics. According to this table, the lowest codimensionality at which caustics of corank $l = 3$ arise is $m = 6$. Unimodal caustics also appear beginning at $m = 6$, and caustics of higher modality begin at $m = 10$ (the numbers of rays, $M + 1 = m + \mu + 1$, are 8 and 16, respectively). Ray patterns with these numbers of rays are rather uncommon in specific studies and thus difficult to analyze. It is thus not surprising that practical studies have so far failed to identify caustics of corank $l = 3$, unimodal caustics, and, especially, caustics of higher modality. Table IV, which summarizes the situation, raises the hope for a more purposeful search for the caustics structures predicted by catastrophe theory. For a given codimensionality m, the number of types of caustics does not increase very rapidly, at least up to $m = 7$–8, as can be seen from Table V, which was constructed from Table IV. At $m = 7$ there are only five types of substantially different caustic structures, and at $m = 8$ there are only seven types of structures.

To conclude this subsection we note that there are definite relationships between the simpler and more complicated caustics, and these relationships can be expressed as graphs of the "subordination" or "adjacency" of caustics. These graphs can be used to predict how a caustic of a given type may degenerate upon a change in one or another of its parameters. The most important adjacencies are listed in Refs. 5, 9, 15, and 16.

e) Caustics in certain physical problems

Although caustics arise in many problems of wave theory, there have been surprisingly few thorough calculations demonstrating the changes in the shape of the caustic surface upon a variation in some parameter or other under realistic conditions. At this point we would like to illustrate the evolution of the shape of caustics upon changes in the param-

TABLE II. Classification of simple (zero-mode) caustics.

Type of caustic	Codimensionality	$\varphi(\zeta, \tau)$	Indices of caustic zones $\alpha_1, \alpha_2, \ldots, \alpha_m$	Focusing index δ
A_{m+1}	$m \geqslant 1$	$\pm \frac{1}{m+2} \tau^{m+2} + \sum_{p=1}^{m} \zeta_p \frac{\tau^p}{p}$	$\frac{m+1}{m+2}, \frac{m}{m+2}, \ldots, \frac{2}{m+2}$	$\frac{1}{2} - \frac{1}{m+2}$
D_{m+1}	$m \geqslant 3$	$\pm \tau_2^m + \tau_1^2 \tau_2 + \zeta_1 \tau_1 + \sum_{p=2}^{m} \zeta_p \tau_2^{p-1}$	$\frac{1}{2} + \frac{1}{2m}, \frac{m-1}{m}, \ldots, \frac{1}{m}$	$\frac{1}{2} - \frac{1}{2m}$
E_6	5	$\tau_1^3 \pm \tau_2^4 + \zeta_1 \tau_1 + \zeta_2 \tau_2 + \zeta_3 \tau_2^2 + \zeta_4 \tau_1 \tau_2 + \zeta_5 \tau_1 \tau_2^2$	$\frac{2}{3}, \frac{3}{4}, \frac{1}{2}, \frac{5}{12}, \frac{1}{6}$	$\frac{5}{12}$
E_7	6	$\tau_1^3 + \tau_1 \tau_2^3 + \zeta_1 \tau_1 + \sum_{p=2}^{5} \zeta_p \tau_2^{p-1} + \zeta_6 \tau_1 \tau_2$	$\frac{2}{3}, \frac{7}{9}, \frac{5}{9}, \frac{1}{3}, \frac{1}{9}, \frac{4}{9}$	$\frac{4}{9}$
E_8	7	$\tau_1^3 + \tau_2^5 + \zeta_1 \tau_1 + \sum_{p=2}^{4} \zeta_p \tau_2^{p-1} + \zeta_5 \tau_1 \tau_2 + \tau_6 \tau_1 \tau_2^2 + \zeta_7 \tau_1 \tau_2^3$	$\frac{2}{3}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{7}{15}, \frac{4}{15}, \frac{1}{15}$	$\frac{7}{15}$

TABLE III. Unimodal caustics of codimensionality $m \leqslant 11$ and corank 2.

Type of caustic	Codimensionality	$\varphi_0(\tau)$	$\varphi_p(\tau)$	Focusing index δ
X_{m+2}	$m \geqslant 8$	$\tau_1^4+\tau_1^2\tau_2^2+a\tau_2^{m-3}$	$\tau_1, \tau_1^2, \tau_1^3, \tau_1, \tau_2\tau_2, \ldots, \tau_2^{m-4}$	1/2
J_{m+2}	$m \geqslant 9$	$\tau_1^3+\tau_1^2\tau_2^2+a\tau_2^{m-2}$	$\tau_1, \tau_1\tau_2, \tau_1\tau_2^2, \tau_2, \ldots, \tau_2^{m-3}$	1/2
$Y_{p,q}$	$m=p+q-1 \geqslant 9$	$\tau_1^p+\tau_1^2\tau_2^2+a\tau_2^q$	$\tau_1, \ldots, \tau_1^{p-1}, \tau_1\tau_2, \tau_2, \ldots, \tau_2^{q-1}$	1/2
Z_{11}	9	$\tau_1^3\tau_2+\tau_2^5+a\tau_1\tau_2^4$	$\tau_1, \tau_1^2, \tau_1\tau_2, \ldots, \tau_1\tau_2^3, \tau_2, \ldots, \tau_2^4$	8/15
Z_{12}	10	$\tau_1^3\tau_2+\tau_1\tau_2^4+a\tau_1^2\tau_2^3$	$\tau_1, \tau_1^2, \tau_1\tau_2, \ldots, \tau_1\tau_2^3, \tau_1^2\tau_2, \tau_1^2\tau_2^2, \tau_2, \ldots, \tau_2^3$	6/11
Z_{13}	11	$\tau_1^3\tau_2+\tau_2^6+a\tau_1\tau_2^5$	$\tau_1, \tau_1^2, \tau_1\tau_2, \ldots, \tau_1\tau_2^4, \tau_2, \ldots, \tau_2^5$	5/9
W_{12}	10	$\tau_1^4+\tau_2^5+a\tau_1^2\tau_2^3$	$\tau, \tau_1^2, \tau_1\tau_2, \ldots, \tau_1\tau_2^3, \tau_1^2\tau_2, \tau_1^2\tau_2, \tau_2, \ldots, \tau_2^3$	11/20
W_{13}	11	$\tau_1^4+\tau_1\tau_2^4+a\tau_2^6$	$\tau_1, \tau_1^2, \tau_1\tau_2, \tau_1\tau_2^2, \tau_1^2\tau_2, \tau_1^2\tau_2^2, \tau_2, \ldots, \tau_2^5$	9/16
K_{12}	10	$\tau_1^3+\tau_2^7+a\tau_1\tau_2^5$	$\tau_1, \tau_1\tau_2, \ldots, \tau_1\tau_2^4, \tau_2, \ldots, \tau_2^5$	11/21
K_{13}	10	$\tau_1^3+\tau_1\tau_2^5+a\tau_2^8$	$\tau_1, \tau_1\tau_2, \ldots, \tau_1\tau_2^3, \tau_2, \ldots, \tau_2^7$	8/15

eters of the problem with the results of some detailed calculations[17–21] carried out for the most part before the advent of catastrophe theory (to some extent, these calculations are reflected in Ref. 12).

The first example describes the reflection of rays from an inhomogeneous half-space $z>0$ in which the dielectric permittivity varies linearly, $\varepsilon(z) = 1 - \varepsilon_1 z$ (Fig. 16). The family of caustics corresponding to various positions of the source z_0 has been found[17,18] to have a striking feature: In the interval $(-2/3, 1/9)$ of the parameter $\zeta = \varepsilon_1 z_0$, a loop appears on the caustic; four rays pass through each point in this loop (Fig. 17). Figure 18 shows the evolution of a caustic upon a change in the parameter ζ. This series of figures suggests that a caustic loop consisting of two closely spaced beaks may be the projection of a swallowtail in a space of higher dimensionality. If we expand this series of cross sections in the parameter ζ, we find the characteristic swallowtail in the expanded space $\{x,z,\zeta\}$ (see Fig. 19, which for clarity shows only that part of the resulting figure which corresponds to $\zeta>0$).

TABLE IV. Classification of caustics of codimensionality $m \leqslant 11$ (l is the corank of the caustic).

m																							
1	A_2																						
2	A_3																						
3	A_4	$D_4^{\pm}$																					
4	A_5	D_5																					
5	A_6	D_6	E_6																				
6	A_7	D_7	E_7								P_8												
7	A_8	D_8	E_8			X_9					P_9												
8	A_9	D_9		J_{10}		X_{10}					P_{10}	$R_{4,4}$				Q_{10}							
9	A_{10}	D_{10}		J_{11}		X_{11}	$Y_{5,5}$		Z_{11}		P_{11}	$R_{4,5}$		$T_{4,4,4}$		Q_{11}	S_{11}						
10	A_{11}	D_{11}		J_{12}	K_{12}	X_{12}	$Y_{5,6}$		Z_{12}	W_{12}	P_{12}	$R_{4,6}$	$R_{5,5}$	$T_{4,4,5}$		Q_{12}	S_{12}	U_{12}					O_{16}
11	A_{12}	D_{12}		J_{13}	K_{13}	X_{13}	$Y_{5,7}$	$Y_{6,6}$	Z_{13}	W_{13}	P_{13}	$R_{4,7}$	$R_{6,6}$	$T_{4,4,6}$	$T_{4,5,5}$				Q_{14}	S_{14}	U_{14}	U_{15}	
	Simple			Unimodal															Bimodal			3 modes	5 modes
	$l=1$	$l=2$									$l=3$												$l=4$

TABLE V. Number of types of caustics for a given codimensionality.

Codimensionality m	1	2	3	4	5	6	7	8	9	10	11
Number of types of caustics	1	1	2	2	3	4	5	7	11	16	18
Number with corank $l \leqslant 2$	1	1	2	2	3	3	4	4	6	8	9

The second example describes the five-ray situation in a parabolic plasma slab with dielectric permittivity

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}\left[1 - \left(\frac{2z - z_m}{z_m}\right)^2\right], \quad \omega_p^2 = \frac{4\pi e^2 N}{m}$$

in the interval $0 < z < z_m$ and $\varepsilon = 1$ at $z < 0$ and $z > z_m$.

Figure 20 shows various caustic configurations in the plane $\{|z_0/z_m|, \omega/\omega_{cr}\}$, where z_0 is the height of the source, and $\omega_{cr} = \sqrt{4\pi e^2 N/m}$ is the plasma frequency.[19,20] On the basis of the typical shape and the observed evolution of the branches, we should classify this caustic as a singularity of the A_5 type (the butterfly; Fig. 15).

The next example,[21] shown in Fig. 21, is the evolution of the caustic formed in a linear slab with dielectric permittivity $\varepsilon = 1 - \varepsilon_1 z$ by the radiation from a focused antenna whose phase varies quadratically along the x axis: $\varphi^0(x) = -\beta x^2$. The convex part of the caustic at $z > 0$ and the upward-oriented beak show the behavior as a function of the parameter β/ε_1 which we would expect in the case of a D_4^+ singularity (the purse; Fig. 13).

The literature has some other caustics which arise in studies of actual wave problems. For example, the analogy between optical caustics and the singularities in the distribution of matter during the evolution of the universe has recently been illustrated.[117]

These examples demonstrate the usefulness of the universal classification of caustics; i.e., they demonstrate that what had once appeared to be a set of unrelated exotic phenomena has now acquired an internal logic. After an appropriate procedure has been developed, we can work from fragmentary pieces of information to fill in the missing details and predict the evolution of the caustic pattern as a whole. This capability is particularly important for finding wave fields, which are the subjects of the following sections.

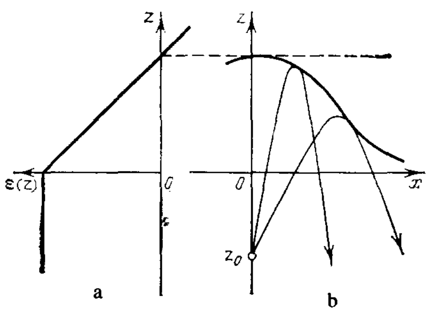

FIG. 16. a—Profile of the dielectric permittivity in the plasma slab; b—trajectories of rays describing the reflection of electromagnetic waves from this slab.

3. ASYMPTOTIC REPRESENTATIONS OF WAVE FIELDS FOR STRUCTURALLY STABLE CAUSTICS

a) Standard integrals; local asymptotic field representations

As we stated above, catastrophe theory has not only brought order to the geometric forms of caustics but has also led to an extremely comprehensive answer to the question of the typical diffraction integrals which describe the field near caustics. By analogy with the integral field representations in the case of simple caustics (which had been known previously), catastrophe theory has made it possible to classify the standard diffraction integrals of the type

$$I(\zeta) = \left(\frac{k}{2\pi}\right)^{l/2} \int_{-\infty}^{\infty} d^l\tau \exp[ik\varphi(\zeta, \tau)]. \tag{3.1}$$

Here the phase functions $\varphi(\zeta,\tau)$ are the same polynomial generating functions which were used in classifying the caustics; the order of the intergral, l, is determined by the corank of the caustic.[5,9,22-24,131-133] Here k is understood as the wave number, $k = \omega/c_0 = 2\pi/\lambda_0$, while ζ is the set of external parameters $\{\zeta_1, \zeta_2, \ldots, \zeta_m\}$, where m is the codimensionality of the caustic.

This is a natural field representation, since the stationary points of integral (3.1), which correspond to the geometrical-optics approximation, lie on ray surface (2.5),

$$F: \frac{\partial\varphi}{\partial\tau_k} = 0, \quad k = 1, 2, \ldots, l,$$

and the regions where two or more stationary points merge do in fact lie on caustics, on which the determinant (2.6) vanishes.

Essentially all the diffraction integrals with infinite limits with which we must deal in wave theory—the Kirchhoff

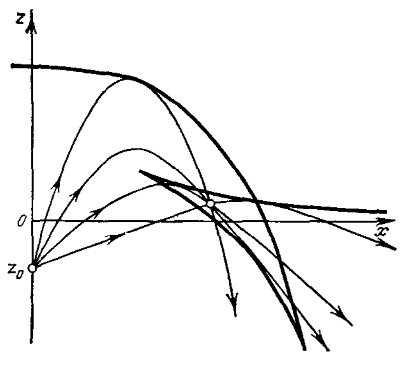

FIG. 17. Formation of a caustic loop (a projection of a swallowtail) upon the reflection of a spherical electromagnetic wave from a plasma slab with a linear profile of the dielectric permittivity.

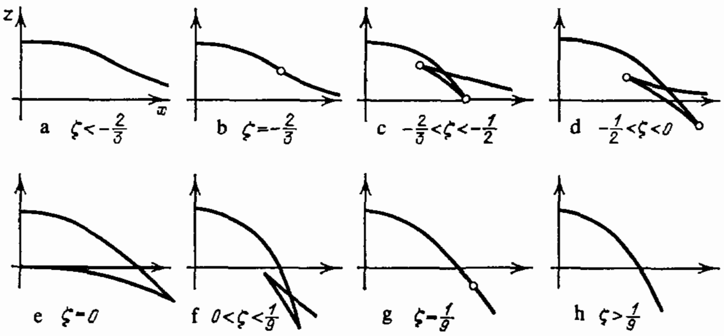

FIG. 18. Evolution of the shape of a caustic in a linear plasma slab.

integrals representing the fields as superpositions of spherical waves, integrals of the Rayleigh type, which perform a plane-wave expansion of fields, and the Maslov canonical integral representations,[10,11] among others—can be locally reduced to integrals of the type in (3.1). In all these cases, the order of the integration, l, is either one or two.

As an example we consider the local asymptotic behavior of the field near the caustic beak which arises in the case of cylindrical aberration (an analog of spherical aberration in the two-dimensional problem).

Near the focus ($z = F$, $x = 0$) the diffraction field $u(x,z)$ is expressed in terms of the initial field $u^0(x) \equiv u(x,0)$, specified in the $z = 0$ plane, by a Kirchhoff integral:

$$u\ (x,z)=\sqrt{\frac{ik}{2\pi z}}\int_{-\infty}^{\infty} u^0(x')\exp\left[ik\sqrt{z^2+(x-x')^2}\right]dx'. \tag{3.2}$$

If the initial field $u^0(x')$ is

$$u^0(x')=e^{-ikx'^2/2F},$$

then a caustic beak, i.e., a caustic of type A_3, is formed near the focus $z = F$. Introducing the variables $\zeta_1 = -x(2/F)^{1/4}$, $\zeta_2 = (z-F)(2/F)^{1/2}$, $\tau = x'(2F^3)^{-1/4}$ locally (near the tip of the beak), we can express the field in terms of the standard integral

$$I_P\ (\zeta,\ \zeta_2) = \sqrt{\frac{k}{2\pi}}\int_{-\infty}^{\infty}\exp\left[ik\left(-\frac{1}{4}\tau^4+\zeta_1\tau+\frac{1}{2}\zeta_2\tau^2\right)\right]d\tau, \tag{3.3}$$

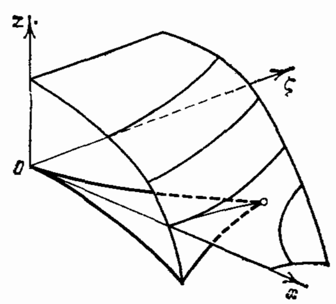

FIG. 19. Shape of the caustic surface (swallowtail) in the expanded space $\{x,z,\zeta\}$ in the case of a point source which is irradiating a linear plasma slab.

which corresponds to a caustic of type A_3 ($m = 2$, $l = 1$). In the physics literature this integral is called the Pearcey integral.[25]

We can thus write, locally,

$$u\ (x,\ z)=\left(\frac{i}{z}\right)^{1/2}\exp\left[ik\left(z+\frac{x^2}{2F}\right)\right](2F^3)^{1/4}I_P(\zeta_1,\ \zeta_2).$$

Other examples of the use of the Pearcey standard integral to describe the local asymptotic behavior of fields are discussed in Refs. 26–29.

The integral which arises most frequently in applications is the Airy integral

$$I_{Ai}\ (\zeta_1)=\sqrt{\frac{k}{2\pi}}\int_{-\infty}^{\infty}\exp\left[ik\left(\frac{1}{3}\tau^3+\zeta_1\tau\right)\right]d\tau = \sqrt{2}\,k^{1/6}v\ (k^{2/3}\zeta_1) \tag{3.4}$$

[$v(x)$ is the Airy function in Fok's notation[30]], which describes the field near a simple caustic (see Refs. 31–34 and 128, for example). In addition, local field asymptotic behavior has been studied for caustics of the A_{m+1} series and for

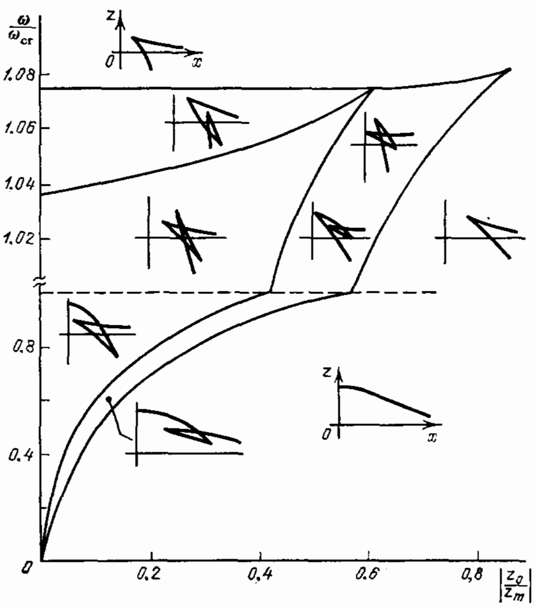

FIG. 20. Changes in the caustic pattern (two-dimensional sections of a butterfly) in the plane of the parameters ω/ω_{cr} and $|z_0/z_m|$ for a parabolic slab with the dielectric permittivity profile from Subsection 2e.

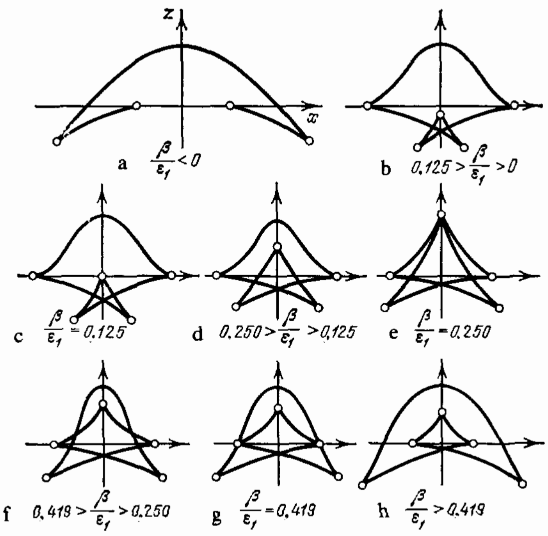

FIG. 21. Evolution of the shape of a caustic which arises in a linear plasma slab irradiated by a wave with a quadratic phase change in the $z = 0$ plane.

certain of the umbilical caustics.

The most important integrals—for the A_4, $D_4^{\pm}$, etc., caustics—are now being tabulated (see Refs. 48 and 123–127, for example; the Airy and Pearcey integrals were tabulated a long time ago).

Another example will demonstrate the usefulness of the general classification of standard integrals. Grikurov[49] has analyzed the asymptotic field representation for the case in which a simple caustic branch passes near a caustic beak. For this purpose, the new function

$$G(\alpha,\beta,\gamma)=\int_0^\infty \frac{dt}{\sqrt{t}}e^{-i\gamma t^2/2}\cos\left(\frac{t^3}{12}-\alpha t-\frac{\beta^2}{t}+\frac{\pi}{4}\right)$$

was introduced. Kryukovskiĭ *et al.*[50] pointed out that the caustic studied in Ref. 49 is classified as a hyperbolic umbilic D_4^+; if this is the case, then the field must be described by a corresponding standard integral with the phase function from Table I (we omit a factor of $k/2\pi$):

$$\Phi_+(\zeta_1,\zeta_2,\zeta_3)=\int\int_{-\infty}^{\infty} d\tau_1\, d\tau_2 \exp[i(\tau_2^2\tau_1+\tau_1^3+\zeta_1\tau_1^2+\zeta_3\tau_2)].$$

We could hardly assume that a given field is described by two distinct functions G and Φ_+. Indeed, a single-valued correspondence between these functions was established in Ref. 50:

$$\Phi(\zeta_1,\zeta_2,\zeta_3)$$
$$=2\sqrt{\pi}\,(12)^{-1/6}G\left[-\zeta_2(12)^{-1/3},\ \frac{1}{2}\zeta_1(12)^{1/6},\ \zeta_3(18)^{-1/3}\right].$$

b) Width of the caustic zone and degree of field focusing on caustics

Tables I–III specify two indices α and δ, which are calculated by certain rules which are too complicated for us to take the time to describe them here. The real meaning of these indices can be seen by examining the fields near caustics. The caustic-zone index α is a measure of the width of the caustic zones around various branches of a caustic of a given type.

Let us assume $\lambda_0 = 2\pi\lambda\!\!\!^{-}_0$ is the wavelength, while $\mu = (k_0L)^{-1} = \lambda\!\!\!^{-}_0/L$ is a small dimensionless parameter, equal to the ratio of the wavelength $\lambda\!\!\!^{-}_0$ to the characteristic dimension of the problem, L. In order of magnitude, we can then estimate the width of the caustic zone, Δl_k, to be

$$\Delta l_k \sim L\mu^{-\alpha} \sim L^{1-\alpha}k_0^{-\alpha}. \qquad (3.5)$$

In the case of the simplest caustic, A_2, for which we have $\alpha = 2/3$ (Table I), we can thus write

$$\Delta l_k \sim \left(\frac{L}{k_0^2}\right)^{1/3}.$$

This characteristic dimension is an order-of-magnitude measure of the distance from the caustic to the first zero of the Airy function (Fig. 1).

The width of the caustic zone can be found more accurately by working directly from an analysis of the diffraction integrals, but there is also another approach: the geometric approach proposed in Ref. 51 (see also Ref. 12). According to Ref. 51, the width of the caustic zone should be taken to be the distance over which the phase difference between any two rays is no less than π:

$$\min k\,|\psi_j-\psi_m| \geqslant \pi. \qquad (3.6)$$

Estimates from this expression agree well with the characteristic dimensions for the changes in the standard integrals, and they are consistent with cruder estimates from Eq. (3.5).

The ability to estimate the width of the caustic zone, Δl_k, is useful for solving several problems. First, if we treat the caustic as a physical entity, i.e., as a region in which the field is focused, then knowledge of Δl_k makes it possible to resolve the question of the "reality" of caustics,[12,51] i.e., the possibility of separately observing branches of caustics. Caustic branches can in practice be assumed distinguishable if they are separated by a distance greater than Δl_k.

Closely related to the question of the reality of caustics is the question of choosing a suitable standard integral. For the caustic A_3, for example (the caustic beak; Fig. 7), the wave field in the region where the caustic branches separate should be described by Airy functions; only where the branches converge do we have to resort to a complete description employing the Pearcey integral.

Finally, information on the width of the caustic zone can be used to estimate the field directly on the caustic from nothing more than the results of geometric calculations.[12,51] There are three approaches which can be taken here. The simplest approach is to get as close as possible to the caustic, i.e., to approach it to within a distance of order Δl_k, and make use of the value of the ray field at the boundary of the caustic zone. A second approach is based on conservation of the energy flux in a ray tube of finite cross section: If $\Delta S_{\rm caust}$ is the cross section of the ray tube around the caustic zone, and ΔS^0 is the corresponding initial cross section, then the field amplitude can be estimated from

$$A_{\rm caust} \sim A^0\sqrt{\frac{n^0\,\Delta S^0}{n_k\,\Delta S_{\rm caust}}}, \qquad (3.7)$$

where n^0 and n_k are the refractive indices.

TABLE VI. Values of the focusing index δ for the complicated caustics listed in Table IV.

Тип каустики	P_{m+2}	$R_{p,q}$	$T_{p,q,r}$	Q_{10}	Q_{11}	Q_{12}	Q_{14}	S_{11}	S_{12}	S_{14}	U_{12}	U_{14}	V_{15}	O_{16}
δ	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{13}{24}$	$\frac{5}{9}$	$\frac{17}{30}$	$\frac{7}{12}$	$\frac{9}{16}$	$\frac{15}{26}$	$\frac{3}{5}$	$\frac{7}{12}$	$\frac{11}{18}$	$\frac{5}{8}$	$\frac{2}{3}$

The third method is based on the concept of the Fresnel volume of a ray[51] and also uses energy considerations. We denote by $\Delta S_{\rm Fr}$ the initial cross section of the ray tube, which is equal to the Fresnel cross section of the ray, and we denote by $\Delta S_{\rm fin}$ the final cross section of the tube near the caustic. We then have

$$A_{\rm caust} \sim A^0 \sqrt{\frac{n^0\,\Delta S_{\rm fr}}{n_k\,\Delta S_{\rm fin}}}. \tag{3.8}$$

All three approaches lead to comparable values of the field, and these values have been shown by corresponding calculations[51] to differ from the exact values by no more than 30–50%.

Rougher estimates of the field can be found by using the focusing index[130] δ, whose values are given in Tables I–III. Specifically, this index can be used to compare the energy flux density on the caustic, S_k, with that of unfocused radiation, S^0:

$$\frac{S_k}{S^0} \sim \mu^{-2\delta} \sim (kL)^{2\delta}. \tag{3.9}$$

For a simple caustic we would have $\delta = 1/6$ according to Table I; for a caustic beak we would have $\delta = 1/4$; and for the tip of a swallowtail we would have $\delta = 3/10$. Using these values with Eq. (3.9), we can estimate the change in the degree of focusing when the caustic loop in Fig. 18 is compressed to a point. Let us assume, for example, $kL = 6\cdot 10^4$, which corresponds to the reflection of an electromagnetic wave of length $\lambda_0 = 10$ m from the ionosphere with a characteristic dimension $L = 100$ km. For the caustic loop (Fig. 18c) we then have $S_k/S^0 \approx 250$; for the loop compressed to a point (Fig. 18b) we have $S_k/S^0 \approx 740$; and for a simple caustic (Fig. 18a) we have $S_k/S^0 \approx 40$.

It can be seen from this example that, on the whole, the degree of focusing increases with increasing codimensionality. The limiting value of the focusing index is 1 (corresponding to a focused spherical wave); for ideal cylindrical focusing we would have $\delta = 1/2$. To supplement Tables I–III, we show in Table VI values of the focusing index for the other types of caustics in Table IV. Working from these results we can easily specify the caustic of a given codimensionality which has the maximum value of δ.

The increase in the focusing index for the more complicated types of caustics results from both the increase in the number of rays which converge at the caustics and the increase in the density of focused rays.

Now that we have described the results of the qualitative theory, we turn to the construction of uniform asymptotic expansions of the field in the presence of caustics.

c) Uniform asymptotic field representations through the use of standard integrals

A uniform asymptotic representation of a field can be constructed by using the ideas of the method of standard functions. For fields having a ray structure there is the hope that similar fields will correspond to qualitatively similar ray and caustic configurations. Since the diffraction integrals (3.1) describe completely definite caustics, these are the integrals which should be used as standard functions.

The method of standard functions was originally developed for one-dimensional problems.[52,53] A local field asymptotic behavior for the case of the simple caustic A_2 in a medium inhomogeneous in three dimensions was derived in Refs. 31–34. A uniform asymptotic behavior (for scalar and electromagnetic fields) using the Airy function (3.4) and its derivative was proposed in Refs. 54 and 55. The procedure is also described in Refs. 56 and 57. A uniform asymptotic representation for the "vacuum-like" caustics, which are now identified with the A_{m+1} series, was proposed by Ludwig[58] for homogeneous media and extended to inhomogeneous media by Kravtsov.[59]

The effectiveness of the method of standard functions formulated in Refs. 54, 55, 58, and 59 (the construction of a field through the use of not only the standard functions themselves but also their derivatives) was later demonstrated in many problems (some of which will be discussed below), but no general theory was derived because of a division of opinion regarding just which functions should be regarded as the set of standard functions. The matter has now been resolved [these functions are the integrals of the type in (3.1)]. We turn now to a brief description of the procedure for constructing a uniform asymptotic representation for caustics of arbitrary type in the general case of a smoothly inhomogeneous medium.

Let us assume that from the geometrical-optics equations

$$(\nabla\psi)^2 = n^2(\mathbf{r}), \quad 2\nabla U\nabla\psi + U\Delta\psi = 0 \tag{3.10}$$

we have found $M+1$ values of the eikonal ψ_j and $M+1$ values of the zeroth-order-approximation amplitude U_j. We assume that we have determined the type of caustic, by which we mean that we have determined its codimensionality m and its modality μ, whose sum gives us M: $m + \mu = M$. The geometric solution

$$u(\mathbf{r}) = \sum_{j=1}^{M+1} U_j(\mathbf{r}) \exp[ik\psi_j(\mathbf{r})] \tag{3.11}$$

becomes valid only far from the caustics, while the amplitudes U_j become infinite on the caustics themselves.

A uniform asymptotic representation of the field which yields finite field values everywhere, including on the caustics, can be expressed in terms of the standard integral (3.1) corresponding to the given type of caustic and in terms of its derivatives with respect to the arguments ζ_p. Among these arguments we include, for brevity $(m+1,...,m+\mu\equiv M)$, the nonremovable moduli $a_j, j=1,2,..., \mu$:

$$u(\mathbf{r}) = e^{ik\theta}\left[I(\zeta)\sum_{s=0}^{\infty}\frac{A^{(s)}}{(ik)^s}+\frac{1}{ik}\sum_{p=1}^{M}\frac{\partial I(\zeta)}{\partial\zeta_p}\sum_{s=0}^{\infty}\frac{B_p^{(s)}}{(ik)^s}\right]$$

$$=\left(\frac{k}{2\pi}\right)^{l/2}\sum_{s=0}^{\infty}\frac{1}{(ik)^s}\int_{-\infty}^{\infty}d^l\,\tau g^{(s)}(\mathbf{r},\,\boldsymbol{\tau})\,e^{ikf(\mathbf{r},\,\boldsymbol{\tau})}, \qquad (3.12)$$

where

$$g^{(s)}(\mathbf{r},\,\boldsymbol{\tau}) = A^{(s)}+\sum_{p=1}B_p^{(s)}\frac{\partial\varphi}{\partial\zeta_p}, \qquad (3.13)$$

$$f(\mathbf{r},\,\boldsymbol{\tau}) = \theta(\mathbf{r})+\varphi(\zeta,\,\boldsymbol{\tau}).$$

The $M+1$ unknown quantities $\theta, \zeta_p, p=1,2,...,M$, and the $M+1$ unknown amplitudes $A^s, B_p^{(s)}, p=1,2,...,M$ [the coefficients of $(ik)^{-s}$] are assumed to be functions of the coordinates and are to be determined.

Substituting (3.12) into the Helmholtz equation

$$\Delta u + k^2n^2(\mathbf{r})\,u = 0, \qquad (3.14)$$

we equate the coefficients of identical powers of k to quantities proportional to $\partial f/\partial\tau$ (this quantity will be set equal to zero below). A subsequent integration by parts leads to a system of equations for θ, ζ_p, $A^{(s)}$, and $B_p^{(s)}$, which we will write only in part, for the lowest orders in k:

$$[\nabla f(\mathbf{r},\,\boldsymbol{\tau})]^2-n^2(\mathbf{r}) = T(\boldsymbol{\tau})\frac{\partial f(\mathbf{r},\,\boldsymbol{\tau})}{\partial\boldsymbol{\tau}}, \qquad (3.15)$$

$$2\nabla g^{(0)}\,\nabla f+g^{(0)}\,\Delta f-\frac{\partial}{\partial\boldsymbol{\tau}}(g^{(0)}T) = R^0(\boldsymbol{\tau})\frac{\partial f(\mathbf{r},\,\boldsymbol{\tau})}{\partial\boldsymbol{\tau}}. \qquad (3.16)$$

The l-component quantities $T(\tau)$ and $R^0(\tau)$ here are found from the condition that Eqs. (3.15) and (3.16) are satisfied identically with respect to τ (see Refs. 58–61 for a description of some particular cases of this procedure).

Equations (3.15) and (3.16) are to be solved under the condition $\partial f/\partial\tau_p=0$ or the equivalent condition

$$\frac{\partial\varphi}{\partial\tau_p}=0,$$

which determines $M+1$ values of the parameters τ_j (which are generally complex), since the function $\varphi(\zeta,\tau)$ has $M+1=m+\mu+1$ stationary points for a caustic of codimensionality m and modality μ.

No matter how complicated Eqs. (3.15) and (3.16) might appear, their solutions can be expressed in an unexpected and even surprising manner in terms of the solutions ψ_j and U_j of the geometrical-optics equations. It turns out that if θ, ζ_p, $A^{(0)}$, and $B_p^{(0)}$ satisfy Eqs. (3.15) and (3.16), then the combinations of unknown functions

$$\psi_j = \theta(\mathbf{r})+\varphi(\zeta,\,\tau_j), \qquad (3.17)$$

$$U_j = g^{(0)}(r,\,\tau_j)\,|\det h(\tau_j)|^{-1/2}e^{i\pi\beta_j/4} \qquad (3.18)$$

satisfy the geometrical-optics equations (3.10). (Here $h=[h_{pq}]$ is a matrix with the elements $h_{pq}=\partial^2\varphi/\partial\tau_p\,\partial\tau_q$, and $\beta_j=\operatorname{sgn}h(\tau_j)$ is the signature of matrix h in the case $\tau=\tau_j$.) In principle, by transforming (3.17) and (3.18) we could express all $2(m+1)$ unknown functions $\theta, \zeta_p, A^{(0)}$, and $B_p^{(0)}$ in terms of the eikonals ψ_j and the amplitudes U_j corresponding to both real and complex rays. As a result, solutions can be found for Eqs. (3.15) and (3.16) essentially without solving these equations. It is very important to note that the amplitude factors $A^{(0)}$ and $B_p^{(0)}$ have finite values on the caustics, although the amplitudes U_j are singular there. Relations (3.17) and (3.18) could have been derived by joining asymptotic representation (3.12) with the ray asymptotic representation of field (3.11) far from the caustics, but this fact alone could hardly have served as a basis for assuming that relationships (3.17) and (3.18) would continue to hold directly on the caustics, where geometrical optics clearly breaks down.

In the particular case of a simple caustic with $\varphi(\zeta,\tau)=\zeta\tau+\tau^3/3$ and $\tau_{1,2}=\pm\sqrt{\zeta}$, we find from (3.17) and (3.18) the results of Ref. 54 (see also Refs. 55–58 and 129):

$$\theta=\frac{1}{2}(\psi_1+\psi_2),\qquad \zeta=\left[\frac{3}{4}(\psi_2-\psi_1)\right],$$

$$A^{(0)}=\zeta^{1/4}(A_2-iA_1)\left(\frac{i}{2}\right)^{1/2},$$

$$B_1^{(0)}=\zeta^{-1/4}(A_2+iA_1)\left(\frac{i}{2}\right)^{1/2}, \qquad (3.19)$$

where the field is expressed in terms of the Airy integral (3.4),

$$u\cong e^{ik\theta}[A^{(0)}I_{\mathrm{Ai}}(\zeta)+(ik)^{-1}B_1\partial I_{\mathrm{Ai}}(\zeta)/\partial\zeta]$$

$$=\sqrt{2}k^{1/6}e^{ik\theta}\,[A^{(0)}v(k^{2/3}\zeta)+ik^{-1/3}B_1^{(0)}v'(k^{2/3}\zeta)]. \quad (3.20)$$

A uniform asymptotic representation of the field for a caustic beak can be expressed in terms of the Pearcey integral (3.3):

$$u\approx e^{ik\theta}\left(A^{(0)}I_p+\frac{B_1^{(0)}}{ik}\frac{\partial I_p}{\partial\zeta_1}+\frac{E_2^{(0)}}{ik}\frac{\partial I_p}{\partial\zeta_2}\right), \qquad (3.21)$$

The relationship among $\theta, \zeta_{1,2}$ and $\psi_{1,2,3}$ in this case is not as simple as it is for the Airy asymptotic representation:

$$\theta+\zeta_1\tau_j+\frac{1}{2}\zeta_2\tau_j^2+\frac{1}{4}\tau_j^4=\psi_j,\qquad j=1,\,2,\,3,$$

where τ_j are the roots of the cubic equation

$$\frac{\partial\varphi}{\partial\tau}=\zeta_1+\zeta_2\tau+\tau^3=0.$$

The general algorithm (3.12)–(3.18) makes it possible to carry out an analytis for essentially any caustic, switching the thrust of the calculations to the solution of algebraic equations for θ, ζ_p, A, and B_p.

d) General comments regarding the local and uniform asymptotic representations of wave fields

In this subsection we take up certain aspects of the practical use of the results of catastrophe theory in calculating wave fields. For clarity we adopt a question-and-answer format.

Just how far can we pursue the approach of constructing caustic asymptotic representations of fields? Although the standard integrals generated by catastrophe theory provide a comprehensive solution ot the problem of constructing local and uniform asymptotic representations of wave fields for caustics of arbitrary complexity, the actual procedure of finding the fields is afflicted by serious difficulties. In the first place, even the geometrical-optics part of the problem—identifying all the rays associated with the given caustic and determining the type of caustic—is quite complicated. Second, even if the classification problem can be solved we must still deal with the difficulties in tabulating the corresponding standard integrals.

At present, this approach has not gone beyond caustics of codimensionality 4 and 5. It will take a major effort to deal with higher codimensionalities. In contrast, the problem becomes somewhat simpler when there is a very large number of rays, so that the resultant field can be assumed random and analyzed by methods from statistical radiophysics,[62,63] in particular, the methods for describing so-called speckle fields.[64] We should apparently put the nominal boundary between determinate and statistical descriptions of the complicated caustic fields where the fraction of the area occupied by the caustic zones is 10–20%.

Do swallowtails exist in two dimensions and butterflies in three? Up to this point we have cautiously avoided formulating the exact mathematical results regarding the types of caustics permitted in a space of a given dimensionality. One of such theorems, offered by Whitney, asserts that the projection of an arbitrary smooth surface onto a plane gives rise to singularities of only the fold type, A_2, and the cusp type, A_3, i.e., only lines and beaks (turning points in lines). According to another theorem, only the singularities A_2, A_3, A_4, and $D_4^{\pm}$ listed in Table I exist in three-dimensional space.

While acknowledging the importance of theorems of this type for classifying caustics, we would like to draw attention to some difficulties which arise when these theorems are used to calculate wave fields. Let us consider two closely spaced beaks (Fig. 22), whose branches are separated by a distance less than the width of the caustic zone (the dotted region in Fig. 22). If, in accordance with the Whitney theorem, we treat the caustic loop as a set of two A_3 beaks, then we must use Pearcey integrals (3.3) to describe the fields in their vicinity. The merging of the caustic zones of beaks makes this approach impossible, forcing us to appeal to the more complicated integral A_4, which corresponds to a swallowtail. According to the Whitney theorem, however, a tail cannot exist in two dimensions!

The contradiction is of course one of terminology: Under the conditions which we have just described, the caustic loop should be regarded as the projection of a swallowtail in an expanded parameter space: The two geometric coordinates in the plane must be supplemented with yet another parameter, which "unrolls" the loop into a swallowtail. An example of this unrolling procedure is shown in Fig. 19. The unrolling parameter in this case is the source coordinate $\zeta = \varepsilon_1 z_0$. In precisely the same manner, the butterflies in Fig. 20 and the purses in Fig. 21 should be interpreted as the projections of these singularities resulting from unrolling in the corresponding parameter.

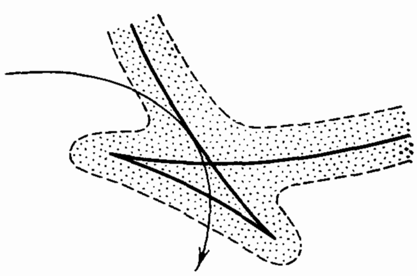

FIG. 22. Coalescence of the caustic zones of distinct branches of a caustic loop A_4 with increasing wavelength.

The addition of an unrolling parameter is particularly important when it is suspected that the system is near a bifurcation point. If, for example, there is some reason to expect the creation of a caustic loop (as in Fig. 18), it is expedient to carry out the unrolling in terms of an auxiliary parameter, having the foresight to work with the singularity A_4 (the swallowtail) and the corresponding standard integral, and not with the Airy function corresponding to the fold A_2. If we take these precautions we do not have to worry about making errors in determining the fields because of an inflexible adherence to the theorems of catastrophe theory.

Why are caustics of corank 3 and higher possible in physical (three-dimensional) space? As we mentioned above, in two and three physical dimensions we can speak of the existence of caustics of higher co-dimensionality ($m \geqslant 3$) only if we are dealing with auxiliary "unrolling" parameters. Slightly different factors explain why caustics of corank l equal to or greater than 3, $l \geqslant 3$, can exist in physical space (three dimensions). The integral field representations which are most commonly used have two-dimensional integrals: integrals over a surface (the Kirchhoff integral), integrals over two components of a wave vector (a Rayleigh plane-wave expansion), or integrals over two components of the momentum (the canonical Maslov operator). Where can the additional integrations come from?

One possible reason for the appearance of additional integrals is the complicated structure of the Green's functions which are used in the Kirchhoff integral. In an inhomogeneous medium, the Green's function itself can be represented by a diffraction integral, and this circumstance explains the existence of caustics with $l \geqslant 3$. If we work from the Freiman picture of field formation, we ultimately come to path integrals. For ray patterns of average complexity, there are of course integrals of some compromise multiplicity $2 \leqslant l < \infty$. If we use the method of standard functions described in Subsection 3c, we should first make use of integrals of the appropriate multiplicity.

4. PENUMBRAL CAUSTICS AND PENUMBRAL FIELDS

a) Penumbral caustics and edge catastrophes

Penumbral caustics arise much more frequently in applications than might be expected by the uninitiated. Figures 23 and 24 show some examples of the appearance of penumbral caustics. In the case in Fig. 23, the caustic formed by ordinary rays is interrupted by an opaque screen.

The penumbral caustics of diffraction rays are slightly more complicated. Figure 24 shows some edge diffraction rays which leave an illuminated edge and form a penumbral caustic (1) in an inhomogeneous reflecting medium. Boundary ray *PQ*, which separates the illuminated region from the

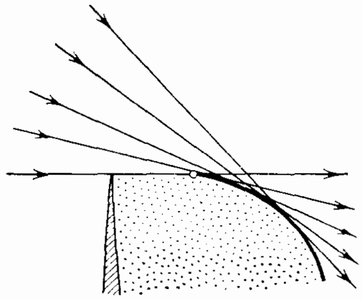

FIG. 23. Interrupted penumbral caustic which arises upon the screening of the primary wave in free space.

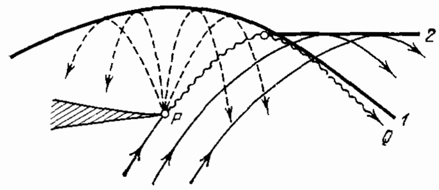

FIG. 24. Penumbral caustic 1 formed by diffraction waves (dashed curves) in a linear plasma slab. Boundary ray PQ touches both caustic 1 and the interrupted caustic 2, formed by ordinary rays.

shadow, also at the same time touches the caustic of the edge diffraction rays and the interrupted caustic of the ordinary rays (2).

It turns out that the penumbral caustics are intimately related with the theory of so-called edge singularities.[5,65] The list of these singularities differs from the list of catastrophes given in Section 2. Table VII shows some simple (zero-modal) caustics. This table contains two infinite series, B_{m+1} and C_{m+1}, and an additional singularity F_4 (as before, the indices on the caustics correspond to the associated Lie groups). Table VII also gives the canonical form of the generating function $\varphi(\zeta,\tau)$, which then arises in a natural way in the integral field representations.

At codimensionalities $m = 2$ and $m = 3$, only the simple penumbral caustics form. In particular, at $m = 3$ these are the caustics of the five types B_3, C_3, B_4, C_4, and F_4 (the sixth type, B_2, which is equivalent to C_2, corresponds to a shadow without caustics). In terms of the phase structure of the field, the series B_{m+1} and C_{m+1} are similar to the series A_{m+1} and D_{m+1}, respectively.

At $m \geqslant 4$ we find unimodal, bimodal, etc., penumbral caustics in addition to the simple ones. These new caustics are classified up to codimensionality $m = 8$ in Refs. 5 and 65.

b) Uniform asymptotic field representations for penumbral caustics

Uniform asymptotic expressions for interrupted penumbral caustics were constructed in Refs. 61, 66, and 67, even before the advent of the theory of edge catastrophes. The incomplete Airy function[69,70]

$$I_{\mathrm{Ai}}(\zeta,\eta)=\sqrt{\frac{k}{2\pi}}\int_{\eta}^{\infty}\exp\left[ik\left(\tau\zeta+\frac{\tau^3}{3}\right)\right]d\tau \tag{4.1}$$

was used as a standard integral in Refs. 61 and 67, and the unknown field (the leading term of the asymptotic series) was written in the form (the notation here is slightly different from that of Refs. 69 and 70):

$$u=e^{ik\theta}\left[AI_{\mathrm{Ai}}(\zeta,\eta)+\frac{B}{ik}\frac{\partial I_{\mathrm{Ai}}(\zeta,\eta)}{\partial\zeta}+\frac{C}{ik}\frac{\partial I_{\mathrm{Ai}}(\zeta,\eta)}{\partial\eta}\right]. \tag{4.2}$$

The integral $I_{\mathrm{Ai}}(\zeta,\eta)$ in a sense combines the properties of two simpler standard functions: the Fresnel integral near the penumbra (but far from the caustic) and the Airy function near the caustic (but far from the penumbra).

As for the asymptotic representation of the field found in Refs. 60 and 70 for the penumbral caustic of diffraction rays (Fig. 24), we note that the standard integral used here is yet another generalized Airy function: the "Airy-Fresnel function"

$$V(\zeta,\eta)=\sqrt{\frac{k}{2\pi}}\int_{-\infty+i\alpha^2}^{\infty+i\alpha^2}\frac{1}{(\tau-\eta)}\exp\left[ik\left(\frac{1}{3}\tau^3+\tau\zeta\right)\right]d\tau, \tag{4.3}$$

which becomes the Fresnel integral near the penumbra. The asymptotic field representation, on the other hand, is completely analogous to (4.2).

The development of a classification of edge catastrophes[65,5] made it clear that the results of Refs. 60, 61, 66, 67, and 70 can be generalized to penumbral caustics of more general form. The general calculation method remains the

TABLE VII. Classification of the simple (zero-modal) penumbral caustics.

Caustic type	Codimensionality	$\varphi(\zeta,\tau)$	Indices of caustic zones, α	Focusing indices δ
B_{m+1}	$m\geqslant 2$	$\pm\frac{1}{m+1}\tau_1^{m+1}\pm\tau_2^2+\sum_{p=1}^{m}\frac{1}{p}\zeta_p\tau_1^p$	$\frac{m}{m+1},\ \ldots,\ \frac{1}{m+1}$	$\frac{1}{2}-\frac{1}{m+1}$
C_{m+1}	$m\geqslant 2$	$\tau_1\tau_2\pm\frac{\tau_2^{m+1}}{m+1}+\sum_{p=1}^{m}\frac{1}{p}\zeta_p\tau_2^p$	$\frac{m}{m+1},\ \ldots,\ \frac{1}{m+1}$	0
F_4	3	$\pm\tau_1^2+\frac{1}{3}\tau_2^3+\zeta_1\tau_1+\zeta_2\tau_2+\zeta_3\tau_1\tau_2$	$\frac{1}{2},\ \frac{2}{3},\ \frac{1}{6}$	$\frac{1}{6}$

same as in Section 3, but the standard integral for the caustics of series B_{m+1} and C_{m+1} and for the caustic F_4 should be

$$I(\zeta)=\frac{k}{2\pi}\int_0^\infty d\tau_1\int_{-\infty}^{\infty} d\tau_2 \exp[ik\varphi(\zeta,\tau_1,\tau_2)], \qquad (4.4)$$

where $\varphi(\zeta,\tau_1,\tau_2)$ are the normal forms of the generating functions of the edge catastrophes. These functions are given in Table VII for the singularities of the B_{m+1}, C_{m+1}, and F_4 types.

In the case of the B_{m+1} series, the standard integral (4.4) reduces to the incomplete multidimensional Airy function

$$I(\zeta,\eta)=\int_\eta^\infty \exp\left[ik\left(\frac{\tau^{m+1}}{m+1}+\sum_{p=1}^{m-1}\zeta_p\frac{\tau^p}{p}\right)\right]d\tau, \qquad (4.5)$$

which describes the field near interrupted caustics.[61] For the C_{m+1} series, integral (4.4) becomes the multidimensional Airy-Fresnel function

$$V(\zeta,\eta)=\int_{-\infty}^{\infty}\frac{1}{\tau-\eta}\exp\left[ik\left(\frac{\tau^{m+1}}{m+1}+\sum_{p=1}^{m-1}\zeta_p\frac{\tau^p}{p}\right)\right]d\tau, \qquad (4.6)$$

which was introduced previously in Refs. 60, 61, and 70. Integrals of the type in (4.6) describe a rather slight focusing. Not accidentally, they correspond to a zero value of the index δ in Table VII.

The classification theorems for edge singularities have thus made it possible to "invent" standard integrals, and the conventional techniques of the method of standard functions have made it possible to construct the corresponding uniform asymptotic field representations.

5. OTHER TYPES OF CAUSTICS AND STANDARD INTEGRALS

In addition to the standard integrals which are associated with mapping theory and which are discussed in the two preceding sections, there are some other integrals and special functions which are used to describe wave fields. We will briefly outline the possibilities here.

a) Standard integrals with an arbitrary phase function

The reason for polynomials in the argument of the exponential function in standard integral (3.1) is more one of convenience for classification than necessity. Let us assume that the argument of the exponential function contains the smooth function $\Phi(\zeta,\tau)$, which has the required number of stationary points but is otherwise arbitrary. It is clear that the integral

$$\mathscr{Y}(\tau)=\left(\frac{k}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}\exp[ik\Phi(\zeta,\tau)]\,d\tau \qquad (5.1)$$

can asymptotically be reduced through a smooth change of variables to the integral (3.1), where $\varphi(\zeta,\tau)$ is one of the suitable normal forms of the generating function. For this reason, integrals of the type in (5.1) are equally as useful as the standard integrals (3.1) for constructing the asymptotic representations of fields.

Some particular standard caustic integrals with a nonpolynomial phase function $\Phi(\zeta,\tau)$ have been used repeatedly in wave theory. The best-known example is the use of the Hankel function $H_0^{(1)}(x)$ in place of the Airy function to describe the field near a nonsingular caustic (see, in particular, Ref. 71). The idea of using as a standard integral a one-dimensional integral with a phase function having the appropriate number of stationary points, equal to the number of rays, was raised back in 1972 by Permitin.[72] This idea can also be generalized to multidimensional integrals. Our calculations show that the use of nonpolynomial phase functions causes no essential change in the method of standard functions (Subsection 3b). All that changes is the particular form of the equations for determining the quantities ζ_p. At the same time, the use of nonpolynomial phase functions may be of definite practical interest if the corresponding integrals (5.1) are more amenable to study than the standard integrals of the type in (3.1) or if these integrals permit a higher-quality approximation.

b) Standard integrals for structurally unstable caustics

The perturbations with which we must deal in analyzing physical problems by no means always conform to the "small disturbance" category (Subsection 2b). We are thus forced to draw a distinction between the mathematical concept of "structural stability" and the concept of "physical stability," by which we mean the stability of the physical characteristics with respect to small perturbations of some type or other.

A question which naturally arises in this connection is that of a possible place for structurally stable singularities in physical theories. The same question should be raised regarding structurally unstable entities. In essence, the role played by entities of either type in physical theories is determined by how well they model the important aspects of the real entities, where the criterion "how well" must be determined by the physical formulation of the problem.

In the case of caustics we should regard a model as acceptable if the characteristic perturbations of the initial conditions and the parameters of the medium for the given problem lead to only small perturbations of the wave field, and not of the caustics themselves, since experimentally it is the values of the wave fields which are determined. We can be sure that the field perturbations are small if the perturbations of the caustic surface are small in comparison with the width of the caustic zone (Subsection 3b). If some perturbing factor or other causes the caustic to go beyond the original caustic zone, then the model (whether it has the property of structural stability or not is unimportant) should be refined by incorporating the perturbing factor in the number of independent parameters (inherent in the problem).

Although structurally unstable caustics are not typical from the standpoint of catastrophe theory, they can thus serve as acceptable models for various real entities. Structurally unstable formations are extremely common in physical problems; for example, plane, spherical, and cylindrical waves are all structurally unstable formations.

Structurally unstable caustics—point foci and singular caustics—generally require some special standard integrals, which cannot be reduced to (3.1). The theory has been pursued furthest for axial (sagittal) caustics. In the absence of an azimuthal dependence, the Bessel function of zero index, $J_0(x)$, serves as the standard function,[73–81] and when there is an azimuthal dependence one uses Bessel functions of higher order,[82] $J_m(x)$. The procedure for constructing the uniform asymptotic representation is basically the same as usual.

c) Contour standard integrals

An analysis shows that the method for constructing uniform asymptotic representations does not change if the standard integral is a contour integral:

$$I(\zeta) = \sqrt{\frac{k}{2\pi}} \int_C \exp[ik\Psi(\zeta, \tau)]\, d\tau. \qquad (5.2)$$

The phase function in (5.2) must have the required number of stationary points, and at the ends of the contour C, i.e., at $\tau = \tau_{\text{init}}$ and $\tau = \tau_{\text{fin}}$, the periodicity condition must hold. The asymptotic representation of the field is then given by the expressions from Section 3c, as in the general method of standard functions. The phase function ψ does not necessarily have to be a polynomial in τ.

In the particular case $\Psi(\zeta,\tau) = \zeta \sin\tau - a\tau, a > 0, \zeta > 0$, $-\pi < \tau < \pi$, the standard integral (5.2) can be expressed in terms of a Bessel function:

$$I(\zeta) = \sqrt{2\pi k}\, \Psi_{ka}(k\zeta). \qquad (5.3)$$

In this case, Eqs. (3.12) and (3.13) describe the asymptotic representation of an axisymmetric field, as mentioned in Subsection 4b.

d) Standard integrals with an amplitude correction; blurred caustics

All the standard integrals discussed above have only a rapidly oscillating function $\exp[ik\varphi(\zeta,\tau)]$ in the integrand. The introduction of an amplitude factor $B(\zeta,\tau)$ in the integrand in (3.1) or (5.2) would make it possible to perform an amplitude correction of the rays. This becomes necessary, for example, in a description of the near field of a nonuniformly excited aperture. If the field near the edge of the aperture ($\tau = 0$) has a τ^ν behavior ($\tau > 0$, $\nu > 0$), then integrals of the type[83–85]

$$F_\nu(\zeta, \eta) = \int_\eta^\infty (\tau - \eta)^\nu \exp\left[ik\left(\frac{\tau^3}{3} + \tau\zeta\right)\right] d\tau \qquad (5.4)$$

naturally arise in the presence of caustics. These integrals combine the properties of the parabolic cylinder function of fractional index, $D_{-\nu-1}(\zeta)$ (for integer values of ν, these functions are simply the Fresnel integral and its derivatives) and the Airy function.

The penumbral caustics which form near a blurred light-shadow boundary were analyzed in Refs. 83–85, where they were called "blurred" caustics. The field focusing on such caustics is less noticeable than at a sharp light-shadow boundary. The functions (5.4) are used not only for describing the field near blurred caustics but also for describing lateral waves (in the latter case, $\nu = 1/2$).

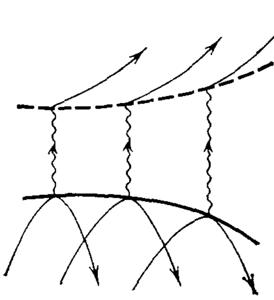

FIG. 25. Tunneling caustic. The lower branch (solid curve) corresponds to reflected rays, and the upper branch (dashed curve) corresponds to rays which tunnel under the barrier.

e) Tunneling caustics; waveguide caustics

Tunneling caustics arise when the energy of a wave tunnels under a potential barrier. Caustics of this sort have a "front" branch and a branch beyond the barrier (Fig. 25); this second branch may be complex, as mentioned in Ref. 87.

It was suggested in Ref. 86 that the fields can be described in the case of tunneling caustics through the use of the Weber function (a parabolic cylinder function), which satisfies the equation

$$w''(t) + (t^2 - b^2)\, w(t) = 0, \qquad (5.5)$$

which is a step up the complexity scale from the simple-harmonic-oscillator equation $w'' + w = 0$ and the Airy equation $w'' - tw = 0$. While the simple-harmonic-oscillator equation generates the sinusoidal function e^{it} which serves as a standard for geometrical optics, and the Airy equation generates the standard function for a nonsingular caustic, the Weber equation, (5.5), generates functions which correspond to partial reflection from the caustic and partial tunneling under the barrier. With a suitable change in parameters ($b \to ib$), the Weber functions can also describe above-barrier reflection,[86] in which case the front branch of the caustic also becomes complex.

Also closely related to tunneling caustics are the waveguide caustics which bound wave beams in waveguiding systems (for example, a deep-water sound channel or an ionospheric duct). In this case, t^2 and b^2 trade places in Eq. (5.5), and the Weber functions become the wave functions of a simple harmonic oscillator (Hermite polynomials with Gaussian coefficients). Since the appearance of Ref. 86, these functions have frequently been used as standard functions in solving problems of wave propagation in inhomogeneous waveguides (see Refs. 88–90, for example). The same functions arise in a description of the "bouncing-ball" natural oscillations of open and closed resonators.[91–93]

f) Standard functions generated by ordinary differential equations

If there are several regions separated by barriers, even more complications should be included in the standard equation (5.5). Gazazyan and Ivanyan[94] have derived a general theory for standard functions generated by the second-order equation

$$w''(t) + f_1(t)\, w'(t) + f_2(t)\, w(t) = 0 \qquad (5.6)$$

with arbitrary functions $f_1(t)$ and $f_2(t)$. Equation (5.6) is satisfied by many well-known special functions: Legendre polynomials, spherical Bessel functions, and Lamé functions, among others. The theory of Ref. 94 thus allows these well-studied functions to be used for describing caustic fields. A specific application of this theory is reported in Ref. 95, where the natural electromagnetic fields in a triaxial ellipsoid are analyzed.

In another particular case, with $f_1 = 0$ and $f_2(t) = t^\nu$, where $\nu > 1$, Eq. (5.6) allows a solution in terms of Hardy-Airy functions $H_\nu(t)$, which describe the reflection of waves from a turning point of arbitrary multiplicity, and not only integer ($\nu = 1,2,3,...$) but also fractional multiplicity.

Standard functions for caustics of fractional multiplicity and the corresponding uniform asymptotic representations are analyzed in Ref. 96.

6. ADDITIONAL QUESTIONS

In this section we will briefly discuss some caustics which arise in various problems in electrodynamics and acoustics.

a) Space-time caustics

Space-time caustics form in dispersive media because the parts of the wave packet corresponding to different frequencies move at different group velocities, and some may overtake others. In the simplest case, the wave field is described with Airy functions,[97] but it is a straightforward matter to construct asymptotic field representations for more complicated caustics by using the standard integrals in Section 3 (Ref. 98).

In the propagation of pulses of finite duration in dispersive media, penumbral caustics may also arise; these caustics (and other questions) are studied in Refs. 99 and 100.

b) Caustics of vector fields

The vector nature of the field does not affect the structure of caustics, but it does affect the form of the transport equations which the vector field amplitudes obey. In particular, in an anisotropic medium the field amplitudes satisfy not only energy-flux conservation but also the Rytov law regarding field-vector rotation.[12] In general, the approach for constructing a uniform asymptotic representation for an electromagnetic field or for a vector field of some other physical nature is the same as for a scalar field. This approach was first used in Ref. 55 for a simple caustic. More complicated caustics, including penumbral caustics, which are formed in electrodynamic problems were studied in Refs. 67, 70, and 101.

c) Caustics in anisotropic media and in media with a spatial dispersion

A distinctive feature of caustics in an anisotropic medium is that the rays are no longer orthogonal to the wavefronts, as they are in an isotropic medium. Furthermore, the caustics may be formed by rays corresponding to an arbitrary normal wave in the given medium. Otherwise, the caustic fields in anisotropic media are described by the mathematical approaches described above.

The same is true of media with spatial dispersion: Despite the nonlocal term in the wave equation, the caustic fields have the same structure as in a medium with a local response.[102]

d) Complex caustics

Complex caustics, in contrast with ordinary caustics, are formed by complex rays, i.e., rays which arrive at the real observation point from complex points on the starting surface.[103] We have already mentioned one example of a complex caustic (Subsection 5e).

Complex caustics serve as the place where complex rays converge, and therefore the field concentration on these caustics is usually not very noticeable.

e) Caustics with an anomalous phase shift

As the observation point is withdrawn from the caustic, the reflected field acquires an additional ("caustic") phase shift, which has the value $-\pi/2$ for a simple caustic. And yet, it has been found[104] that a different phase shift, of $+\pi/2$, arises. This change in the phase shift has turned out to be associated with the group velocity of wave propagation; the anomalous phase shift corresponds to the case in which a longer path is traversed in a shorter time.

f) Random caustics

If light, sound, or radio waves propagate through a randomly inhomogeneous medium with inhomogeneities which are smooth on the scale of the wavelength, random caustics similar to those in Fig. 4 form in the medium. These caustics arise where strong field fluctuations are observed.[105] Random caustics are also formed upon the reflection and refraction of waves by an irregular surface,[29,106–108] in scattering by liquid droplets,[109] and in other physical situations.

g) Caustics in quantum mechanics

In quantum-mechanical problems, caustics arise in two cases. First, a variety of caustics form in the scattering of particles by atoms and molecules. The intensified scattering in the caustic directions has been called "rainbow scattering" by analogy with the corresponding optical phenomenon. Scattering calculations have been carried out in a variety of approximations[27,28,32–36,38–44,71,73,78–80,110] (in the last of these papers, the range of applicability of the semiclassical approximation in three-dimensional problems was established).

The other type of problem involving caustics is that in which we wish to find eigenfunctions which are concentrated within a caustic surface. The state of the research on this question is reported in Ref. 111.

The caustic problems which we have mentioned here of course do not constitute an exhaustive list. We might also mention the caustics in the general theory of relativity,[112–114] in the theory of nonlinear hydrodynamic waves,[115,116] in the theory of gravitational lenses, etc. Even without taking up these other areas, however, we can see the substantial progress which has been achieved on the problem of caustics in recent years.

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