Singularities, bifurcations, and catastrophes

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The theories of smooth-mapping singularities and dynamical-system bifurcations are reviewed. Mention is made of the applications to optics (caustic and wave-front metamorphoses) and to theories of short-wave asymptotics, the origin of large-scale structure in the universe, and loss of equilibrium and self-oscillation stability ("catastrophe theory").

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INTRODUCTION

The theory of the singularities of smooth mappings is a young branch of mathematics, a far-reaching generalization of studies of functional behavior at maximum and minimum points. Functions are replaced in this theory by mappings—sets comprising an arbitrary number of functions of any number of variables.

For a typical function of one variable, the critical points are maxima and minima, and are nondegenerate: the second derivative of the function does not vanish there. In the neighborhood of a nondegenerate maximum or minimum a function of a single variable can be put into the form $y = \pm x^2 + c$ by a suitable *smooth* change of the independent variable. By the word "smooth" we will mean "differentiable as many times as required." Deformations that have little effect on a function or on its derivatives up to the required order will be called small disturbances, or "wobbles."

Similarly, a typical smooth function of n variables can be reduced to the form

$$y = \pm x_1^2 \pm \cdots \pm x_n^2 + c$$

by an appropriate smooth change of independent variables. An "atypical" function, say $y = x^3$, can be converted to a typical function by introducing an arbitrarily small disturbance: $y = x^3 - \varepsilon x$, for example.

The term "typical" function here refers to a function which does not belong to a certain meager, or thin, set in the space of all functions. In functional space a meager set of atypical functions is singled out by an algebraic relation among the Taylor coefficients of a function that does *not* reduce to an identity. Accordingly the meager set will form a hypersurface in functional space—a submanifold of codimension one, analogous to a surface in ordinary 3-space, or to a curve in a plane.

The set complementary to this meager set of atypical functions is called the set of functions "in general position," or *generic* functions. Any function, then, can be put into general position by an arbitrarily small disturbance.

In other situations as well we can single out a thin set of "atypical" objects and a thick set of generic objects. For instance, a plane curve in general position will not contact any straight line at more than two points; a surface will not do so at more than four points. The Gibbs phase rule states that in an n-parameter generic system no more than $n + 1$ phases can occur simultaneously. A generic vector field in a plane can have node-, saddle-, or focus-type singular points, but not a center-type singularity.

By limiting the discussion to generic objects one can construct very simple models of phenomena, which nevertheless do exhibit the effects being studied. In mathematical language, such models are usually described as *normal forms*, like those given above for the critical points of functions.

This article offers the nonspecialist a concise review of several applications of modern singularity theory. Among these applications are the analysis of typical wave-front and caustic singularities, solutions of the eikonal or Hamilton-Jacobi equations, and their metamorphosis into single-parameter generic families. From the wave or quantum point of view we shall consider singularities in the asymptotic behavior of multidimensional oscillating or saddle-point integrals as several stationary-phase points or several saddle points merge. By classifying the singularities one can investigate the asymptotic properties of n-dimensional generaliza-

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tions of the Airy functions and kindred integrals.

The asymptotic analysis of these integrals involves the geometry of multidimensional "kaleidoscopes"—crystallographic groups generated by reflections. Some particularly simple wave-front and caustic singularities and integral asymptotics are described by simple Lie groups ($A_k \approx SU_{k+1}$, $D_k \approx O_{2k}$), which turn up in these problems in an altogether unexpected way.

To interpret the relationship between the stationary-phase integral asymptotics and the root systems of simple Lie groups, we turn to the topology of complex level manifolds of the phase function (Riemann surfaces and their n-dimensional generalizations). This topology is described by the theories of vanishing cycles and monodromy, which study how cycles on the Riemann surfaces and integrals over them behave as the complex phase approaches a critical value and circles around it.

Caustic bifurcation theory serves as the mathematical foundation of the "pancake" theory which Yakov B. Zel'dovich has invoked as a model for the development of large-scale structure in the universe.

The paper concludes by describing some applications of singularity theory to the theory of bifurcations of dynamical systems, including the theory of the loss of self-oscillation and equilibrium stability, with "strange attractors" being formed.

"Catastrophe theory," the term which the French mathematician René Thom introduced to designate the theories of singularities and bifurcations and their applications, is concerned with situations of this last type, wherein a smooth change in the parameters of a system is capable of triggering a sudden, abrupt change in its state or its regime of motion.

Although the "general position" arguments on which this theory relies have consciously and persistently been used by many physicists (such as A. A. Andronov and L. D. Landau as well as Ya. B. Zel'dovich), a systematic mathematical theory of singularities dates only from 1955, when the American mathematician Hassler Whitney[1] classified the singularities of mappings of generic surfaces into the plane.

This new mathematical theory interacts with its applications in much the same way as is true of analysis: in simple practical work the methods of differential and integral calculus can be replaced by elementary Huygens-type procedures, but in more complicated contexts only an intensive use of analysis will serve the purpose.

We begin by considering a problem in singularity theory which cannot be handled without some serious mathematical apparatus: the study of the singularities and metamorphoses of wave fronts and caustics.

1. WAVE-FRONT SINGULARITIES

Let us take a very simple example. Suppose a disturbance (light, sound, perhaps an epidemic) is propagating along a plane at unit velocity and at starting time exists on the outside of an ellipse (Fig. 1). To construct the wave front at elapsed time t, we have to lay off a segment of length t along each normal interior to the ellipse. So long as t remains

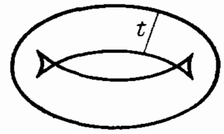

FIG. 1. How a wave front develops singularities.

small, the resultant wave front will form a smooth curve, but after a certain time interval the front will acquire a pair of cusps. These cusps will be stable: if we smoothly wiggle the original front (the ellipse), the singularities will not disappear but will merely shift slightly.

The generic front in the neighborhood of a cusp may by a smooth change of coordinates be reduced to the standard "normal form" $x^2 = y^3$. That front will have no singularities in the plane other than such standard cusps and crossing points of the two branches. Any more complicated singularities will resolve into singularities of these types when the initial front is given a small general wobble.

To illustrate, the front of a disturbance propagating into the interior of a circle will collapse to a point; but if we replace the circle by a slightly different generic curve, then instead of collapsing to a point the front will collapse to a curve which remains close to the point but has an even number of cusps (Fig. 2).

Now one of the fundamental results of singularity theory is that in generic situations only certain standard, universal transforms will be encountered; they can be thoroughly studied once and for all, and have to be recognized in various guises. The simplest of these universal images is a semicubical-type cusp ($x^2 = y^3$). Next in complexity comes the *swallowtail*, a singular surface in 3-space (Fig. 3).

Swallowtails develop in many contexts of singularity theory. Let us return, for example, to the propagation of a wave front along a plane. We can describe the process in terms of spacetime (three-dimensional, in our case). As it travels over the plane, the front will form an image surface in three-dimensional spacetime. Sections of that surface by isochrones $t =$ const will represent instantaneous fronts.

A pair of cusps formed on an instantaneous wave front will then be imaged as a swallowtail singularity on our spacetime surface. This singularity will be stable: if we wobble the original ellipse, replacing it by a slightly different smooth curve, the swallowtail will merely shift a bit in spacetime; it will not disappear.

The same surface in spacetime can also be described as the graph of a multivalued function—the distance from the original front along the normals, or as the graph expressing the solution of the equation $(\nabla u)^2 = 1$, that is, the eikonal equation in optics or the Hamilton-Jacobi equation in mechanics. Thus the generic solutions of eikonal or Hamilton-

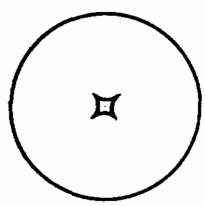

FIG. 2. The cusped wave front from a near-circular curve.

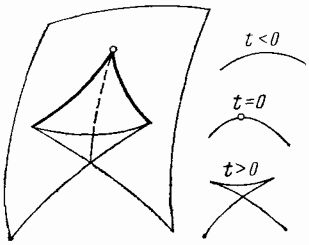

FIG. 3. A swallowtail, and three sections across it.

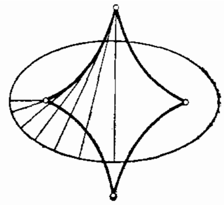

FIG. 4. An astroid, the caustic of the set of normals to an ellipse.

Jacobi equations in two independent variables will have singularities that cannot be eliminated by a small disturbance of the initial conditions: near a singular point the graph of the solution will have a swallowtail singularity or a cusp-type edge.

A swallowtail is a typical singularity on generic wave fronts in 3-space. One can obtain such a front, for example, by laying off a segment of length t on each inward normal to a triaxial ellipsoid (or a smooth surface differing slightly from an ellipsoid).

Generic wave-front singularities are equivalent to singularities in the Legendre transforms of smooth generic functions. Thus the theory of wave-front singularities, also called the theory of Legendre singularities, concurrently supplements the classification of the singularities of thermodynamic potentials (assuming that one of them is a smooth, although not a convex, function).

In mathematics, swallowtails occur chiefly as surfaces in $x^4 + ax^2 + bx + c$ polynomial space that are formed by multiple-root polynomials. As sketched in Fig. 3, one can easily visualize such a surface by cutting it with planes $a = \text{const}$.

There is another mathematical description of a swallowtail that is often useful: the surface formed in 3-space by all tangents to the curve $A = t^2$, $B = t^3$, $C = t^4$ at the point (A, B, C).

2. CAUSTIC SINGULARITIES

Disturbances that are being propagated can be described not only by their wave fronts but also by a system of rays. The rays will not intersect next to a smooth front, but a certain distance away they will produce focal points—intersections of infinitely close rays. Taken together, the focal points will form a *caustic* of the ray system. In the case of a disturbance advancing at constant velocity through Euclidean space, the focal points will be the centers of curvature of the fronts. The system of normals to an ellipse, for instance, forms an astroid caustic (Fig. 4), a semicubical curve with four cusps.

These singularities again are stable: if the ellipse is replaced by a slightly different smooth curve, the cusps will not disappear but will merely shift slightly. The caustics of generic ray systems in a plane have no singularities other than semicubical-type cusps (and self-crossing points).

The caustic comprising the set of normals to a circle consists of a single point, the center. On stretching the circle slightly into some generic curve, say an ellipse, the central caustic point will spread into a curve (for an ellipse, into a tiny astroid).

In 3-space as well, generic caustics have only standard singularities: cusp (semicubical type) edges, swallowtails, and two new types of point singularities called *pyramid* and *pucker* points (Fig. 5). At a pyramid singularity itself, three (smooth) cusp edges are mutually tangent. In the neighborhood of a pucker point, the caustic comprises two surfaces with similar singularities, intersecting along a pair of curves; the cusp edges of the two surfaces continue each other, forming a single smooth curve.

All these singularities are stable. For example, the tangency of the three caustic cusp edges at the vertex of a pyramid singularity cannot be eliminated by wobbling the initial ray system. Every more complicated type of caustic singularity in 3-space will break down into singularities of the types mentioned above if the generic position is disturbed slightly.

3. SHORT-WAVE ASYMPTOTIC SINGULARITIES

Caustic singularities can also be described as singularities in the asymptotic behavior of integrals of rapidly oscillating functions (such as the Fresnel and Airy integrals):

$$I_\lambda(h) = \int e^{iF(x,\lambda)/h} a(x,\lambda)\,dx,\ h \to 0;$$

here the small parameter h represents a wavelength, the *phase*[1)] F is a real function, and the smooth function a, the *amplitude*, differs from zero only within some restricted region.

According to the stationary-phase principle, the main contribution to the asymptotic behavior as $h \to 0$ will come from the stationary points of the phase F (at which $\partial F/\partial x = 0$). For generic λ, these stationary points will be nonde-

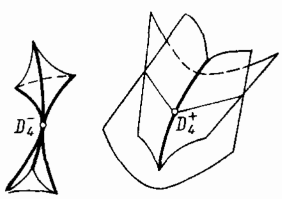

FIG. 5. Pyramid- and pucker-type caustic singularities.

[1)]Think of $F(x, \lambda)$ as the optical path length from a light source at point x to an observer at point λ.

generate: $\det \partial^2 F/\partial x_i \partial x_j \neq 0$. A change of variable will reduce the integral in the neighborhood of a stationary point to a Fresnel-type integral, and it will therefore fall off like $h^{n/2}$ as $h \to 0$, where n is the dimensionality of the x-space.

A typical phase function, regarded as a function of the point x, will have no degenerate critical points. If, however, we consider not an individual phase function of point x but a whole family, depending on one or more parameters λ, then for certain exceptional values of those parameters we may indeed encounter degenerate critical points.

For example, the family of x-functions $x^3 + \lambda x$ with a zero value for the parameter λ has a degenerate critical point: $x = 0$. Any single-parameter family of functions close to this one will, for some near-zero value of that parameter, have a degenerate critical point of the same kind.

The values of the parameter λ for which the phase has a degenerate critical point are called the *caustic* values. As $h \to 0$ the integral falls off for the caustic values more slowly than usual, like $h^{(n/2)-\beta}$. The exponents β defined in this way measure the degree to which light, for example, is concentrated toward the caustic, and they depend on the character of the caustic singularity. An ordinary caustic point will have $\beta = 1/6$ (reflecting the asymptotic behavior of the Airy integral); for ordinary points along a caustic cusp edge, $\beta = 1/4$; for swallowtails, $\beta = 3/10$; and for pyramid and pucker points,[2] $\beta = 1/3$.

4. FRONT AND CAUSTIC METAMORPHOSES

As a wave front advances through 3-space, its singularities (cusp edges) will slide along a caustic. When the front passes through a *singular* point of the caustic (a swallowtail vertex, a pyramid or pucker point), it will undergo a metamorphosis. The metamorphoses of moving generic fronts are stable, and take the standard forms illustrated in Fig. 6. All more complicated metamorphoses can be dispensed with, because if the initial front is wobbled, such metamorphoses will break down into the small number of standard ones.[3,4]

Caustics can also metamorphose if they depend on parameters, say upon time, An example of time-dependent caustics is furnished by Zel'dovich's theory,[5] wherein density singularities would develop in a dustlike medium. Although this theory is designed to interpret the formation of large-scale irregularities in the distribution of matter in an expanding universe from small fluctuations in the primordial velocity field of the gravitating particles, the essential features of the phenomenon can best be understood by considering a very simple model of noninteracting particles that move by inertia through Euclidean 3-space.

In time t a particle will travel from point $\mathbf{q}$ to point $\mathbf{q} + t\mathbf{v}(\mathbf{q})$, where $\mathbf{v}$ designates the initial velocity field. Assume that the field $\mathbf{v}$ can be derived from a potential: $\mathbf{v} = \partial S/\partial \mathbf{q}$ (such an assumption is physically prompted by the fact that in an expanding universe irrotational perturbations will grow faster than vortex perturbations). The graph of the velocity field $\mathbf{p} = \mathbf{v}(\mathbf{q})$ will form a three-dimensional submanifold in six-dimensional phase space $\{(\mathbf{p}, \mathbf{q})\}$ (Fig. 7).

In view of the potential property of the field, the action integral $\int \mathbf{p}\, d\mathbf{q}$ will depend only on the ends of the integration path and will not change if the path itself is deformed, provided the path remains on the manifold referred to. Submanifolds of phase space that possess this property are said to be Lagrangian. Thus an initial potential velocity field will be imaged by a Lagrangian manifold in phase space.

Transformations of the phase flow of any Hamiltonian differential equation will carry Lagrangian phase-space submanifolds into Lagrangian submanifolds. This will be true, in particular, for the phase flow of the equation of a free particle. Accordingly, the velocity field of a medium will remain irrotational if it was so at initial time, and if the forces have that property as well.

The initial velocity field will be represented by a phase-space submanifold nondegenerately projected onto configuration space (a nonvertical tangent to the graph in Fig. 7). If the Hamiltonian phase flow undergoes transformations, an initially Lagrangian manifold will remain Lagrangian but, generally speaking, will (after a long enough time t) lose its property of nonverticality: the projection onto configuration space will cease to be one–one. The points where verticality occurs are called the *singularities* of the projection of the Lagrangian manifold onto configuration space. The projections of these singularities onto configuration space are called the *caustics*.

Caustics will be formed if fluctuations in the original velocity field make the particles begin to overtake one another. At time t the caustic will consist of the points where particles issuing from infinitely close points of space come into collision. The particle distribution along the caustic will have infinite density.

The term "caustic," borrowed from optics, is employed here because in a mathematical sense we are dealing with the same phenomenon in both cases—with the critical values of

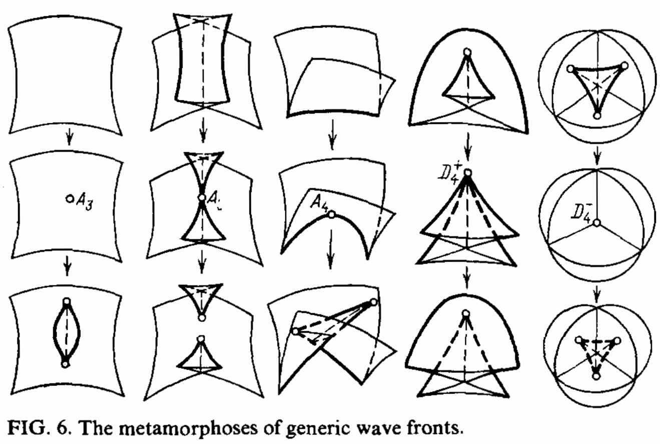

FIG. 6. The metamorphoses of generic wave fronts.

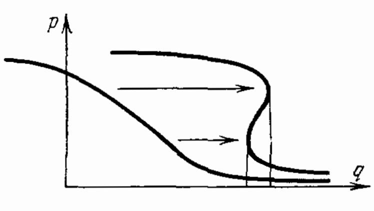

FIG. 7. The development of Lagrangian mapping singularities as a Lagrangian manifold is transported by phase flow.

a Lagrangian mapping, that is, a mapping of the projection of a Lagrangian phase-space submanifold onto configuration space. In the optical context, a Lagrangian manifold is defined by the condition $p = \partial u/\partial q$, where the solution u of the eikonal equation $(\nabla u)^2 = 1$ may be multivalued. In other words, the Lagrangian manifold corresponding to a given ray system (say the rays emerging from a point source, a line, or a surface) will consist of the "front delay vectors" at all points of the medium.

Once the caustic has formed, the particle motion will become multistream: several streams (initially three, as a rule) will pass through a single point of space. The Lagrangian-manifold concept is the mathematical equivalent of a multistream irrotational flow in physics.

From the mathematical standpoint, then, the particle motion specifies a single-parameter family of Lagrangian mappings of 3-spaces: the value of the parameter t stipulates a map taking the point q into the point $Q = q + tv(q)$. The critical points of this mapping are determined by the condition that $\det \partial Q/\partial q = 0$. The critical values (that is, the images of the critical points) will form a caustic. At any definite instant the caustic will form a surface in configuration space.

As the parameter t changes, the caustic will move, and it may metamorphose. The first metamorphosis will be the creation of the caustic itself. Zel'dovich calls the newborn caustic a "pancake" (*blin*). Indeed, a short time ε after it is born the caustic will be shaped like an elliptical dish roughly $\varepsilon^{3/2}$ thick, ε deep, and with axes of the order of $\sqrt{\varepsilon}$ (Fig. 8). In a scattering medium, optical caustics could become visible; V. M. Zakalyukin suggests that an observer might perceive them as flying saucers.

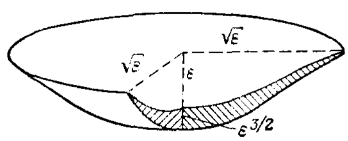

FIG. 8. A newborn caustic—a Zel'dovich pancake.

After the pancakes have formed, the caustics may experience further metamorphoses (Fig. 9). To sort them out one has to apply fairly elaborate mathematical apparatus from the theory of Lagrangian and Legendre singularities[4,6,7] (the singularities of Legendre transformations).

5. THE *A, D, E* CLASSIFICATION

The classification of the singularities that can occur in all the problems we have mentioned (and in many others) turns out to be related in a remarkable way to the classification of objects which would seem to have nothing to do with such problems: regular polyhedra in Euclidean space, for instance, and simple Lie groups (thus, swallowtails on a caustic correspond to the SU_5 group; pucker points, to the orthogonal-matrix groups O_8).

Although the reasons why such disparate theories should be interrelated are not yet fully understood, these relationships have proved to be an exceptionally powerful tool for studying singularities. Only by devising this technique has it been feasible to work out the classification of front and caustic singularities and metamorphoses outlined above.

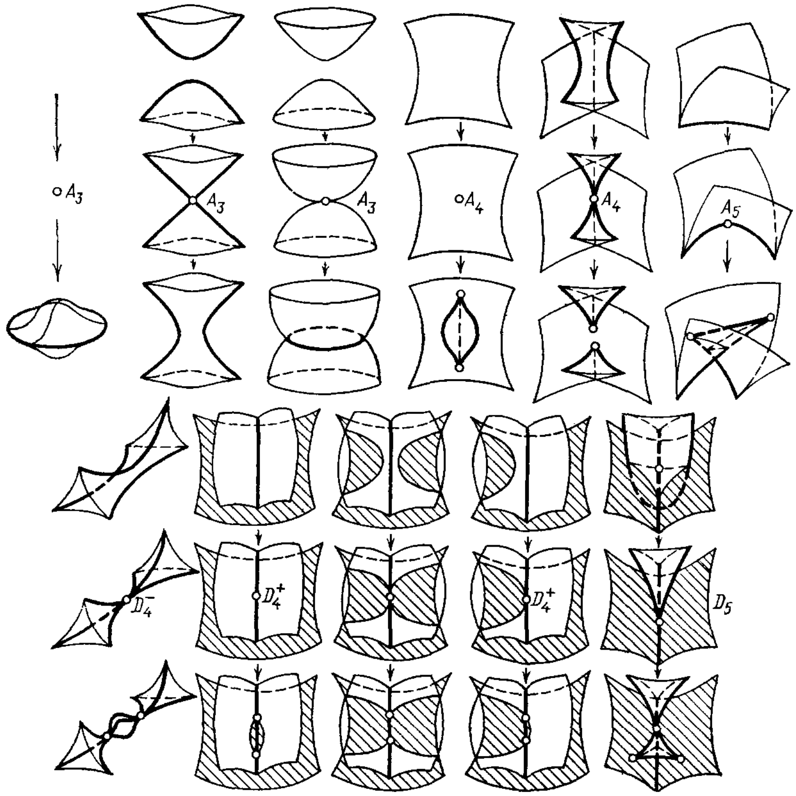

FIG. 9. Some post-pancake metamorphoses of a caustic.

An unexpected benefit of all these classifications has been the finding that a short, universal list of simple objects repeats itself in various guises (Lie groups, regular polyhedra, caustics, wave fronts, Riemann surfaces, ...). The list comprises two infinite sequences (A_k, D_k) and three exceptional objects (E_6, E_7, E_8), as depicted in Fig. 10. In the theory of regular polyhedra the A_k represent the regular polygons; the D_k, the dihedra; E_6, the tetrahedron; E_7, the octahedron; E_8, the icosahedron. Corresponding to each regular polyhedron is a particular wave-front singularity, caustic singularity, oscillating Airy-type integral, and so on.

In Fig. 10 all these objects are represented graphically, joined by lines in *Dynkin diagrams*. These diagrams may be interpreted in entirely different ways, depending on the character of the objects under study. But whatever approach is adopted, the diagrams are always present and play an important role. For example, all the possible decays of wave-front singularities may be exactly described by decompositions of corresponding Dynkin diagrams.

In order to explain how complicated singularities pertaining to wave fronts, caustics, and stationary-phase and saddle-point integrals are composed of elementary, very simple singularities, we need certain information from the geometry of complex manifolds. In fact, most of the roads from caustics, fronts, and oscillating integrals to Lie groups, Dynkin diagrams, and regular polyhedra take us through the theory of the critical points of holomorphic functions—that is, through the geometry of Riemann surfaces and their n-dimensional generalizations. In applications to optics, the holomorphic function that comes into play is the phase—the optical path length. But in other circumstances, such as in thermodynamics, the holomorphic function may have a completely different physical meaning.

We therefore consider at the outset a holomorphic function (a polynomial, say) of several complex variables: $f(z_1,...,z_n)$. It turns out that the asymptotic behavior of the integrals associated with such a function (for example, integrals of the form $\int e^{if/h}\,dz$) and the geometry of the associated singularities (fronts, caustics, ...) will be fundamentally influenced by the topology of the complex level manifold $f(z_1,...,z_n) = c$ [or, in the case of a function of only two variables, by the topology of the Riemann surface $f(z_1, z_2) = c$].

6. MONODROMY

The level manifold (a line, a surface) of a real function will undergo topological rearrangements when the function takes on critical values (called Morse rearrangements, or Morsifications). For example, the hyperbola $x^2 - y^2 = c$ will interchange branches as c passes through the critical value zero. In the complex case, the plane of function values is not divided by critical values, so a complex level manifold will be altered only at the instant of passage through a critical value (the manifold will become singular); after passage the level

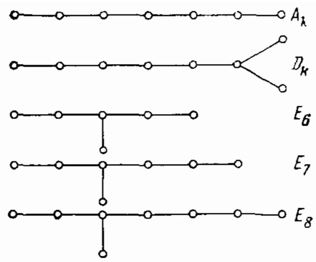

FIG. 10. Dynkin diagrams.

manifold will instantly revert to its former state.

On the other hand, in the complex case instead of passing *through* a critical value, a circuit can be taken *around* it. As the function values undergo a continuous change, so also will the level manifold. After the value has returned to the original point in the function-value plane, the level manifold, continuously changing, will similarly return to its original position. But along the way it will be capable of "flipping over," so that each individual point of the level manifold will in general find itself returning to a different point of the same manifold (like the reversal that occurs when a rectangle is spliced into a Möbius strip).

Such a mapping of the nonsingular level manifold of a function into itself is called a *monodromy*. Along with the topology of the level manifold, a monodromy conveys much information on the singularities that may arise as a critical value is approached (on the asymptotics of integrals, for instance). With present-day mathematical techniques one can study all this topology in detail and extract from it any information needed on the asymptotics (this approach is known as the Hodge theory of mixed structures, an n-dimensional generalization of the Legendre theory of elliptic integrals of the first, second, and third kinds).

Some idea of the geometry of the complex objects that may develop can be gained from the following example (which in this theory plays the role of $1/z$ in the elementary theory of residues).

Let $f = z_1^2 + z_2^2$. The level manifold $z_1^2 + z_2^2 = c$ will be a two-sheet Riemann surface, $z_2 = (c - z_1^2)^{1/2}$, with two branch points A, B. (Fig. 11). After a complete circuit of the noncritical value c around the critical value $c = 0$, the pair of branch points will make a half revolution, and the Riemann surface $f = c$ will revert to its former position.

Topologically, the "complex circle" $z_1^2 + z_2^2 = c (c \neq 0)$ represents a cylinder (the profiles joining the branch points on the two sheets are drawn in Fig. 12 as circles going once around the cylinder). The progressive change in the Riemann surface as c varies is depicted in Fig. 11. This process may occur in such a way that outside a sufficiently large (compared with $\sqrt{c}$) sphere the surface points remain almost motionless. As a result the path Δ_0 on the original surface, gradually deforming, will ultimately become the path Δ_1. Figure 12 shows that the monodromy will twist our level

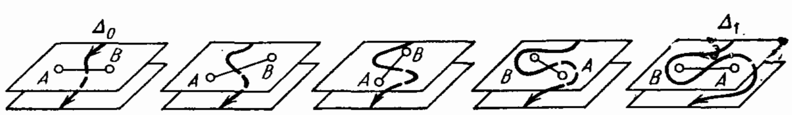

FIG. 11. A monodromy: the path Δ_0 is transformed into Δ_1 as the pair of branch points executes a half revolution.

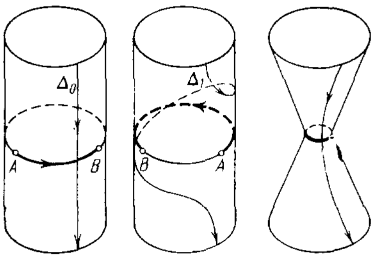

FIG. 12. The twist of a cylinder by monodromy and the constriction of a vanishing cycle as a critical value is circuited or approached.

manifold (the cylinder) such that one of its bases turns by 2π relative to the other (with intermediate parallel planes turning by intermediate angles).

7. VANISHING CYCLES

In our example as the value c approaches the critical value zero, the central parallel ABA of the cylinder $z_1^2 + z_2^2 = c$ (Fig. 12) will constrict to a point (the circle ABA is formed by the real roots of the equation $z_1^2 + z_2^2 = c > 0$). A cycle on a level manifold that constricts to a point as the function value approaches a critical value is termed a *vanishing* cycle at that critical point (in our case, the point $x = 0$).

For a function of two variables with several critical points (assuming that each critical point is nondegenerate), several vanishing cycles will develop on the nonsingular level Riemann surface. To determine them we have to move along the function-value plane from an initial noncritical value to each of the critical values. But if instead of coming in close to the critical values we circuit around them (in little circles, say), then the nonsingular level manifold each time will return to its old position, turning around anew. In place of a single monodromy transformation we will therefore have a whole *monodromy group*.

Consider, for example, the function $f = z_1^3 - 3z_1 + z_2^2$. Apart from the two points $c = \pm 2$, the level manifold $f = c$ will be nonsingular in the plane of the complex variable c. Let us begin, say, with the nonsingular level manifold $f = 0$. This two-sheet covering of the plane, $z_2 = (3z_1 - z_1^3)^{1/2}$, is topologically equivalent to a torus with a side hole (as at a in Fig. 13). As c approaches one of its critical values, say $c = 2$, one pair of branch points will come together (the pair AB) until the torus meridian vanishes (ξ in Fig. 13). If c approaches its other critical value, $c = -2$, the other branch-point pair (BC) will coalesce; in this case the vanishing cycle will be the torus parallel (η in Fig. 13).

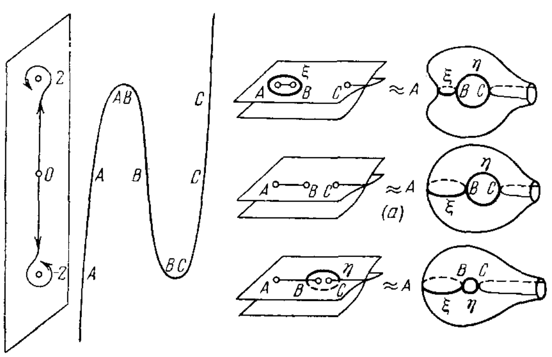

FIG. 13. The vanishing-cycle basis (ξ, η) on a Riemann surface.

We can also readily compute the monodromy group for this example. Any cycle on our level manifold $c = 0$ (a torus with a hole) will be a homologous[2)] linear combination $p\xi + q\eta$ of the meridan ξ and the parallel η, with integer coefficients. As the variable c circuits around the point $c = 2$, the torus will twist, experiencing a monodromy. In the basis (ξ, η) this monodromy operator will have a matrix of the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. A circuit of c around the point $c = -2$ will correspond to the monodromy matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. In our example the full monodromy group comprises all possible products of these two matrices and their inverses. This group is the group $SL_2(z)$ of all integer-valued second-order matrices with determinant unity.

In the multidimensional case a level manifold $f(z_1, \ldots, z_n) = c$ will have a real dimension of $2n - 2$ in n-dimensional complex (that is, $2n$-dimensional real) space. The vanishing cycles will have half that dimension: $n - 1$. For instance, at the nondegenerate critical point O of the function $z_1^2 + \ldots + z_n^2$ the real sphere S^{n-1} formed by the real solutions of the equation $z_1^2 + \ldots + z_n^2 = c > 0$ will vanish[8–10] as $c \to 0$.

8. VERSAL DEFORMATIONS

Now let us consider a *degenerate* critical point of a function, say $f = z^4$. Singularities of this type do not occur either in families of generic functions or in general one-parameter families of functions. However, in the two-parameter family

$$F(z, \lambda) = z^4 + \lambda_1 z^2 + \lambda_2 z$$

such a singularity is encountered for zero values of the parameters λ, and any family nearly the same as this one will experience the same type of degeneracy for some nearby value of the parameters. Properly speaking, any family $G(z, \lambda)$ almost identical with the family F may be put into the form

$$G(z, \lambda) \equiv F(h(z, \lambda), \varphi(\lambda)) + c(\lambda),$$

that is, it can be reduced to the family F by a smooth change of variable depending smoothly on the parameters, and by a smooth change of parameters.[11] Such a deformation, to which any other deformation of the function f can be reduced by the substitutions mentioned, is said to be *versal*. The term "versal" comes from the words "universal" and "transversal": the prefix "uni-" is dropped because such a deformation usually is not unique, but versal deformations do prove to be transversal, in the following sense.

Suppose that we have classified in some manner a set of objects (say functions, differential equations, vector fields, matrices, tensors, caustics, wave fronts, ...). We divide the

---

2) Two one-dimensional oriented cycles are said to be *homologous* if the difference between them serves as the oriented boundary of a two-dimensional sheet. This definition is motivated by the fact that closed-form integrals (the circulations of locally irrotational fields) will have equal values along such cycles. Similar definitions apply in the n-dimensional case.

manifold of all possible objects of this kind (say the function space or the matrix space) into parts corresponding to objects of differing classes (such as matrices of differing rank). The bounding hypersurfaces will have codimension one, being determined by a single equation in the space (generally speaking, infinite-dimensional functional space) of all objects of interest to us (Fig. 14).

Generic objects will not lie on the bounding hypersurfaces (generic functions, say, have no degenerate critical points). In the single-parameter families of objects, however, very simple degenerate objects may indeed occur nonremovably. In fact, a single-parameter family will be imaged by a curve in functional space, and this curve may transversally (at a nonzero angle) intersect the bounding hypersurface. In that event, any nearly identical family will contain a similar degeneracy, for any nearby curve will still intersect the bounding hypersurface.

Functions with critical points analogous to $f = z^4$ cannot occur even in single-parameter generic families. In fact, such functions with critical points form in the functional space of all functions a submanifold of codimension *two*, whereas a single-parameter family is imaged in functional space by a curve. A small perturbation can remove this curve from the submanifold of codimension two, just as a curve in 3-space can escape interesecting another curve by means of a suitable wiggle. In typical *two*-parameter families, however, z^4-type functions with singularities are no longer removable, for the two-dimensional surface imaging such a family in functional space will transversally intersect the (codimension 2) manifold of functions with a singularity of the type z^4.

It is this transversality property of a family with respect to a manifold in functional space which ensures that the deformation of a function will be versal—that any small deformation of the function can be reduced to a versal deformation by suitable changes of the coordinates and parameters.

Although physicists have long made use of particularly simple special cases of arguments along these lines, their systematic development represents a mathematical achievement of just the past few years.

It has been found that finite-parameter versal deformations exist for functions of any number of variables having finite-multiplicity critical points, that is, critical points formed through coalescence of a finite number of elementary nondegenerate critical points. For example, the critical points of z^3, a function of one variable, has multiplicity 2, as it is formed by coalescence of two elementary critical points (see Fig. 13). This function has the single-parameter versal deformation $F(z, \lambda) = z^3 + \lambda z$ (the fact that this is the Airy phase integral is no accident). In general, the number of parameters in a versal deformation at a critical point of a function is one smaller than the multiplicity (the number of elementary critical points into which a given compound critical point can be decomposed).

The multiplicity of a critical point is also called the Milnor number, and is commonly designated by letter μ. Examples are:

$$\mu(z^a) = a - 1, \quad \mu(z_1^{a_1} + \ldots + z_n^{a_n}) = (a_1 - 1) \ldots (a_n - 1).$$

9. THE HIERARCHY OF SINGULARITIES

The compound critical points of functions form a hierarchy: simpler points can be produced by decomposing more complicated ones. At the present time this hierarchy has been calculated as far as multiplicty $\mu = 16$; that is, all the critical points that can be formed by coalescence of up to 16 elementary critical points have been described.[11]

The starting point for the hierarchy of compound critical points is a hierarchy of Dynkin diagrams (Fig. 15). The letters here designate the functions

$$A_k = x^{k+1}, \quad D_k = x^2y + y^{k-1}, \quad E_6 = x^3 + y^4,$$
$$E_7 = x^3 + xy^3, \quad E_8 = x^3 + y^5$$

or functions of a greater number of variables obtained by writing out nondegenerate quadratic forms for additional variables (for example,[6] $z_1^4 + z_2^2 + z_3^2$ is of type A_3).

Dynkin diagrams can describe various methods of building compound critical points by combining elementary ones. I shall explain how this can be done by taking a very simple example, the versal deformation of the function A_3:

$$F(x, y, \lambda) = x^4 + y^2 + \lambda_1 x^2 + \lambda_2 x.$$

A value of the parameters [that is, a point in the (λ_1, λ_2) plane] is called a *bifurcation* value if the corresponding function of the variables x, y has nonelementary critical points. In our example the bifurcation values form the semicubical parabola $8\lambda_1^3 + 27\lambda_2^2 = 0$ (Fig. 16). Let us select a generic, nonbifurcation point in the λ plane. The corresponding function will have $\mu = 3$ critical points. We take a noncritical level manifold (topologically equivalent, in our case, to a torus with two holes), and successively allow the function value to approach the three critical values. As a result three different cycles will vanish on the noncritical level manifold. To construct the Dynkin diagram we assign a point to each vanishing cycle and join these points whenever the corresponding cycles intersect (see a review by Husein-Zade[10] for further details). In our example we arrive in this way at the Dynkin diagram for A_3: ·——·——·.

It is the bifurcation values of the versal-deformation parameters for the functions listed above that form the caus-

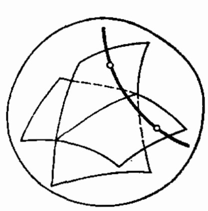

FIG. 14. A single-parameter generic family transversally intersecting a manifold of degeneracies.

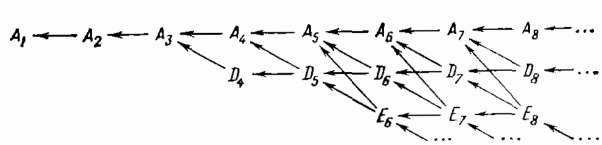

FIG. 15. A hierarchy of function critical points, caustic singularities, and wave-front singularities.

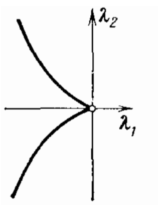

FIG. 16. A semicubical parabola, representing a manifold of cubic, multiple-root polynomials.

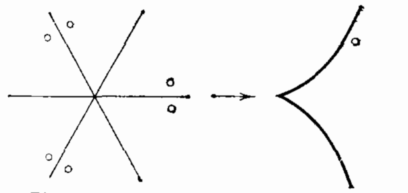

FIG. 17. A Vieta mapping, taking each point into its orbit.

tics of the corresponding Lagrangian singularities. For instance, A_2 is an ordinary point on a caustic; A_3 represents a semicubical parabola for a plane caustic or a cusp edge for a three-dimensional caustic, A_4 is a swallowtail, and D_4 is a pyramid or pucker point. In the complex domain, the last two caustics are equivalent.

To obtain the corresponding wave fronts, one need only expand the space of versal-deformation parameters by an axis of values and note all the critical values of the functions in the family. For example, the critical values of the versal-deformation functions A_3 form a swallowtail [in $(\lambda_1, \lambda_2, F)$ space].

10. REFLECTION GROUPS AND WAVE FRONTS

Wave-front singularities can be derived directly from the Dynkin diagrams by the following construction. To each diagram there corresponds a "kaleidoscope," the term used in mathematics for *a group generated by reflections in Euclidean space*, or a "Coxeter group."[12] In particular, to each vertex in a Dynkin diagram we can assign a vector of length $\sqrt{2}$. If two vertices are joined by an edge, then the angle between the two corresponding vectors is 120°; if no edge is present, the vectors will be orthogonal.

For example, the diagram · — · represents the vectors $(1, -1, 0)$, $(0, 1, -1)$ generating the plane $x + y + z = 0$ in Euclidean 3-space. Similarly, every Dynkin diagram consisting of μ points will determine a "hedgehog" of μ (nonorthogonal) basis vectors in μ-dimensional Euclidean space.

A kaleidoscope is made up of mirrors passing through the origin and orthogonal to the vectors comprising the hedgehog. For instance, the diagram · — · corresponds to a pair of mirrors in the plane, subtending a 60° angle. The group generated by reflections in the mirrors of any of the kaleidoscopes A, D, E is finite. This finiteness condition seriously restricts the position of the mirrors. All mirror positions for which the group is finite have been determined. The positions such that all angles between mirror pairs are 90° or 120° exhaust the mirror positions generating the reflection groups A, D, E and their products.

For each reflection group A, D, E we now construct a wave front. We consider the set of all images of a point in the kaleidoscope mirrors, including multiple reflections. This set is called the *orbit* of the point. The orbit is said to be *regular* if it comprises the maximum possible number of points (equal to the number of elements in the group).

To illustrate, the regular orbit of a point under the reflection group A_2 consists of six points (Fig. 17). If the initial point had been located on a mirror, its orbit would have comprised only three points (a nonregular orbit).

A similar construction can be carried out in the complex domain (by complexifying the initial Euclidean space). Reflections will act on complex points as well, and most complex points have regular orbits.

We now consider the set of all orbits of the group generated by reflections in complex space. This set itself turns out to be a complex space, of the same dimensions.

The effect of the group A_2, for example, may be regarded as a permutation of the three coordinates z_1, z_2, z_3, on the plane $z_1 + z_2 + z_3 = 0$ in three-dimensional (complex) space. Orbits will consist of nonordered triads of complex numbers adding to zero. Such a triad will be uniquely determined by the polynomial $z^3 + \lambda_2 z + \lambda_3$ as its triad of roots. Hence the orbit manifold will be identified with the (λ_2, λ_3) plane of two (complex) variables.

In the manifold of all orbits, nonregular orbits will form a hypersurface. For the example given above, the orbit of the point (z_1, z_2, z_3) will be nonregular if not all the numbers z_k are distinct. Hence the nonregular-orbit manifold will comprise the set of points of the λ plane for which the polynomial with coefficients λ has multiple roots. This set forms a semicubical parabola. We have thereby obtained the A_2 wave front from the corresponding reflection group.

In the general case, an analogous transition can be made from the reflection group to the orbit manifolds and the corresponding wave fronts. We have here a deep generalization of the Vieta theorem and the theory of symmetric functions. A swallowtail, for instance, is obtained from the A_3 reflection group. Since the corresponding algebraic apparatus is quite well developed, one can study the wave-front geometry and rearrangements in detail, and accordingly the geometry of the resultant caustics.

The reflection groups A_k, D_k, E_k themselves can simply be described in terms of phase-function singularities or the asymptotics of oscillating or saddle-point integrals. They represent monodromy groups for the versal deformations (see end of Sec. 8) of phase functions of an odd number of variables (E_8, for example, corresponds to the function $x^3 + y^5 + z^2$).

In the language of Lie-group theory, the reflection groups are obtained in the form of "Weyl groups"; the vanishing cycles correspond to simple roots, the wave fronts describe bifurcations of the Jordan normal forms of matrices, and so on.

The extensive theory that relates caustic and wave-front singularities and integral asymptotics to Lie groups and the groups generated by reflections cannot be surveyed at all adequately in this review.[3,11,13] I would merely mention that the singularities have now been found which correspond to all groups generated by reflections, also including noncrystallographic groups such as the icosahedron symmetry group H_3.

It has been discovered, for example, that the icosahedron controls the singularities for the family of involutes of a plane curve near its inflection point (Fig. 18). That the icosahedron should make an appearance in this situation seems just as surprising as its occurrence in Kepler's mystical law of planetary distances—only here we are talking of a rigorously proved theorem.

The involutes of a plane curve represent the wave fronts of a disturbance propagating in the region bounded by that curve. Consider the graph of the corresponding (multivalue) function of time. To construct such a graph we need merely lift each involute as a whole, along with its analytic continuation, to a height equal to the length of a thread which, if unwound, would form that involute. Figure 19 shows the graph of a multivalued function of time. This surface has two cusp edges (of order 3/2 above the curve itself and order 5/2 above the tangent at the inflection). The surface turns out to be none other than the nonregular-orbit manifold of the icosahedron symmetry group (apart from a smooth change of variables). One can also describe this surface[14] as the ensemble of all tangents to the curve (t, t^3, t^5).

I should emphasize that there seem to be no *a priori* reasons for this relationship between involutes and the icosahedron (between wave fronts and reflection groups). What we have here is a mathematical marvel, just as much so as, say, the connection between problems concerning tangents and those regarding areas. That is why the theory of caustic and wave-front singularities was developed not in the day of Huygens and Newton, who had already begun to look into it, but only in the past decade.

11. BIFURCATION THEORY

Another large arena for applying the concepts of singularity theory is the theory of bifurcations of dynamical systems. Let us begin with the problem of bifurcations in equilibrium positions. Consider the system of differential equations

$$\dot{x} = v(x, \varepsilon),$$

specified by a vector field v in the n-dimensional phase space of points x, the field depending on the parameter ε [or, more generally, on the l-dimensional parameter $\varepsilon = (\varepsilon_1,...,\varepsilon_l)$].

For fixed values of the parameters, the equilibrium positions will be determined by a system of n equations in n unknowns: $v(x,\varepsilon) = 0$. Consider the space of pairs (x,ε), that is, the product of phase space by parameter space. This product has dimensions $n + l$. In the case of a one-parameter system with a one-dimensional phase space, the product will constitute the (x,ε) plane (Fig. 20). On this plane the equation of equilibrium states, $v(x,\varepsilon) = 0$, will specify a curve (or, in general, a submanifold whose dimensionality is equal to the number of parameters). For generic systems this curve will be *smooth*, and will be in "general position" relative to the projection of the parameter on the axis of values. If that is not the case for some particular system, then an arbitrarily small disturbance of the system (that is, of a model of the phenomenon under study) will be capable of qualitatively altering the behavior of the system (Fig. 21).

Accordingly, if at a nongeneral position in some model a bifurcation should occur, like that shown at the left in Fig. 21, one would have to ascertain what special circumstances are responsible. The most common circumstances triggering nongeneral bifurcations are symmetries and a Hamiltonian character for the equations of the problem.[3)] If departures from those special circumstances (symmetries, Hamiltonicity) are indeed insignificant, the problem should be treated in terms of bifurcation theory for systems of that class (symmetric systems, say), with the bifurcations observed being compared against those occurring in generic systems having the same symmetry group.

If on the other hand the degeneracy in the bifurcation process is a chance feature of the model, then the model has to be improved by including some of the small quantities neglected in setting up the equations—and bifurcation theory will suggest which quantities to consider.

The general arguments outlined above, largely due to Henri Poincaré, demonstrate that comprehensive analysis of degeneracies should be accompanied by a study of bifurcations in families have a number of parameters such that a

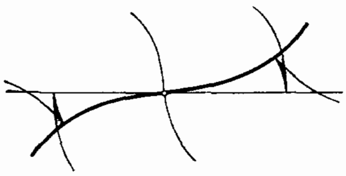

FIG. 18. The family of involutes to a plane curve having an inflection point.

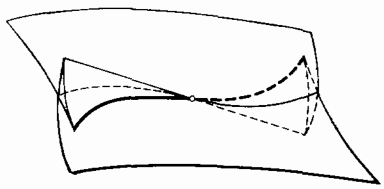

FIG. 19. The nonregular-orbit surface for the icosahedron symmetry group H_3.

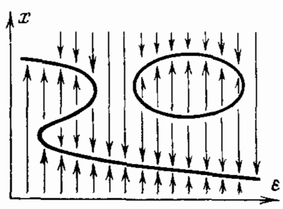

FIG. 20. Bifurcations of equilibrium positions for a generic system.

[3)]An instructive example: for fixed $A \approx 2$ the equation $u_{xxxx} + Au_{xx} + u_x^2 + u = 0$ has whole families of periodic solutions. The reason lies in the symmetry $x \to -x$.

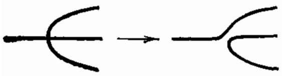

FIG. 21. A small disturbance takes a bifurcation into general position.

given degeneracy is rendered nonremovable by a small disturbance of the family. Thus when studying individual systems it is sufficient to discuss generic cases, neglecting all degeneracies. If there is a single parameter, one should investigate degeneracies of codimension one, that is, those corresponding to a submanifold specified by a single equation in the functional space of all systems. There is no point in studying degeneracies of codimension two except in connection with bifurcations in two-parameter generic families; and so on.

12. EQUILIBRIUM BIFURCATIONS

Although Poincaré's program as described above has long been familiar (in particular, it received wide currency in the schools of A. A. Andronov and L. I. Mandel'shtam), not too much real progress has been made in putting it into practice. Even in the case of just two parameters we are not yet able to cope with all the mathematical difficulties.

The main achievement of singularity theory in this regard is that of sorting out the behavior of equilibrium-position bifurcations (but, unfortunately, not the corresponding phase portraits) in generic systems with an arbitrary finite number of parameters.

As mentioned in Sec. 11, the manifold of equilibrium positions of a generic system forms in (x, ε) space a smooth manifold whose dimensionality is equal to the number l of parameters ε_k (see Fig. 20). Let us consider the projection of that l-dimensional manifold onto the l-dimensional parameter space (along the "x axis," that is, along phase space).

Singularity theory specifies the normal forms to which this projection is brought by a smooth local change of coordinates (depending on the parameters ε) and a smooth change of the parameters ε near each point of (x, ε) space for a generic system.

If, for example, phase space and parameter space have dimension one, as in Fig. 20, then the normal form will become $v(x, \varepsilon) = \pm x^2 + \varepsilon$. As a result, when the single parameter in the generic system varies, only the following bifurcations can occur: coalescence of a stable equilibrium position with an unstable one followed by destruction of both, or creation of paired stable and unstable equilibria. This very simple case had, of course, already been studied by Poincaré in his early work. Modern singularity theory has the advantage of being able to handle analogous problems in higher dimensions.

For instance, if the number l of parameters is arbitrary but, as before, the phase variable x is one-dimensional, then by a theorem of Whitney[1,11] the normal form will be

$$v(x, \varepsilon) = \pm x^{k+1} + \varepsilon_1 x^{k-1} + \ldots + \varepsilon_k, \quad k \leqslant l. \qquad (*)$$

Figure 22 sketches an equilibrium surface $v(x, \varepsilon) = 0$ for the case of two parameters. The projection of this smooth surface on the parameter plane $(\varepsilon_1, \varepsilon_2)$ has a singularity on the

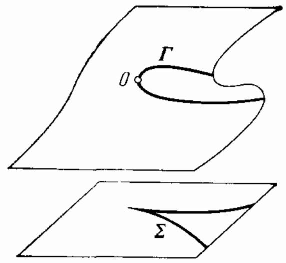

FIG. 22. Map of a Whitney fold.

fold line (Γ in Fig. 22). The projection of the line Γ onto the parameter plane (the set of critical values of the mapping) is a cusped curve (curve Σ in Fig. 22). This curve in the parameter plane is called the *bifurcation curve* or the *catastrophe line*.

As a point in the parameter plane approaches the catastrophe line from inside the cusp, two of the three equilibrium positions will coalesce (one stable and one unstable). At that juncture the system will be obliged to jump from the dying equilibrium position, where stability is being lost, to a third and remote equilibrium position. If the model describes, say, an economic system, then the corresponding stable equilibrium position of the economic system should jump to a new state distant from the original one. It was sudden jumps of this kind that prompted Thom[15] to invent the term "catastrophe theory," combining the singularity theory developed by Whitney, the bifurcation theory of Poincaré and Andronov, and their applications (a detailed account will be found in my recent book[16]). As the catastrophe line is approached the stable and unstable equilibria will come together ultimately at infinite speed. That explains why it is so hard to fight against a catastrophe as soon as it seems imminent.

In the neighborhood of the singular point O the singularity of the projected surface (Fig. 22), is called a pleat. This singularity is stable, for any slightly different mapping will have the same type of singularity at some nearby point. Corresponding to this projected pleat in the image plane is the cusp contour. Whitney[1] proved that such folds or pleats are the only types of singularities of projected generic surfaces. Thus the development of a cusp point on the corresponding image curve is a generic phenomenon, not destroyed by a small disturbance of the original projection.

The projection of a generic curve from 3-space onto a plane cannot have cusps, but only self-crossing points. The fact that the image contour does develop a cusp manifests a general principle, according to which "singularities attract singularities"; for the image contour is the projection not of an arbitrary curve but of a curve comprising projected singular points. This principle also accounts for the special character of the singularities in generic caustics and wave fronts (by comparison with the singularities that develop in sets of critical values of generic projections, that is, the singularities in the bifurcation diagrams of the equilibrium positions for general systems of differential equations).

For example, the bifurcation diagrams of three-parameter generic families exhibit only cusp- and swallowtail-type

FIG. 23. Growth of a cycle when an equilibrium position experiences a soft loss of stability.

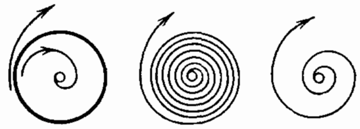

FIG. 24. Constriction of an equilibrium-regime basin in a hard loss of stability.

singularities, whereas generic caustics can also develop pyramid and pucker points (Fig. 5). Equation (*) implies that the singularities in the bifurcation diagrams of, more generally, l-parameter families of differential equations with a one-dimensional phase space will be of the same type as those for the series A wave fronts (Fig. 6).

13. STABILITY LOSS

If phase space is multidimensional, an equilibrium state can lose its stability without the equilibrium itself having to undergo a bifurcation. There is an an alternative possibility: a pair of eigenvalues linearizing an equation at a singular point may make a transition from the stable (left) half-plane, $\mathrm{Re}\lambda < 0$, to the right half-plane.

Andronov had studied the corresponding phase-portrait bifurcation in the plane, publishing the results in the original 1937 edition of his textbook written with Khaĭkin.[17] In 1942 Eberhard Hopf[18] extended part of Andronov's theory to the n-dimensional case, and for that reason in the West this type of bifurcation is usually called the Hopf bifurcation.

In this case the topological normal form becomes $\dot{z} = (i + \varepsilon)z + cz|z|^2$. Depending on the sign of Re c, the bifurcation will entail either a "soft" growth of self-oscillations, with an amplitude proportional to the square root of the supercriticality, $\sqrt{\varepsilon}$ (Fig. 23), or a shrinkage to zero of the attraction zone of the equilibrium regime, followed by a "hard" loss of stability and a jump to a different regime (Fig. 24).

This new regime of motion may constitute a new equilibrium position or self-oscillation regime (with strictly periodic oscillations being established), or some more complicated type of motion may arise, described by an attracting set more intricate than a cycle in phase space (that is, by a *strange attractor*).

That complicated "stochastic" motions such as these in determinate dynamical systems could be stable was recognized in the early 1960s by Smale,[19] Sinaĭ,[20] and Anosov.[21] The first efforts to apply these concepts to a description of turbulence date from this same period, or even from 1958-1959, when Andreĭ N. Kolmogorov combined into a single seminar research on the ergodic theory of dynamical systems and on hydrodynamic instability. Yet even today there still do not seem to be any rigorous results concerning the stochasticity of attractor motion for the Navier-Stokes equations. Computer experiments, first performed by Lorenz[22] in 1963, suggest that for at least some attractors the motion will be exponentially unstable.

So far as I am aware, the only definitive results in this field set *upper* limits on the attractor dimension for two-dimensional Navier-Stokes equations in a compact region. Such limits were recently obtained for the first time by Il'yashenko[23] for periodic boundary conditions; M. I. Vishik and A. V. Babin have subsequently extended them to more general attachment conditions. The dimension limits take the form: $\dim \lessdot \mathrm{Re}^p$ (in Vishik and Babin's latest theorems, $p = 4$). It has neither been proved nor refuted that as $\mathrm{Re} \to \infty$ the dimension of *all* attractors for the Navier-Stokes equation will increase without bound. Most specialists feel that no attractor of dimension bounded as $\mathrm{Re} \to \infty$ exists which would attract solutions having almost any initial conditions; but this has not been proved either.

In the case of a soft loss of equilibrium-state stability (such as an interruption of laminar flow), a stable periodic motion (a self-oscillation) will develop. As the parameter continues to vary, the self-oscillatory motion may itself lose stability. The theory of the loss of self-oscillation stability represents another large branch of general bifurcation theory. Some universal singularities have been discovered, not depending on the particular type of system. I cannot describe this theory here in any detail at all (see my book on differential equations[24]), but merely wish to point out the important role of resonances between the self-oscillatory motion occurring at the very outset and small oscillations close to that

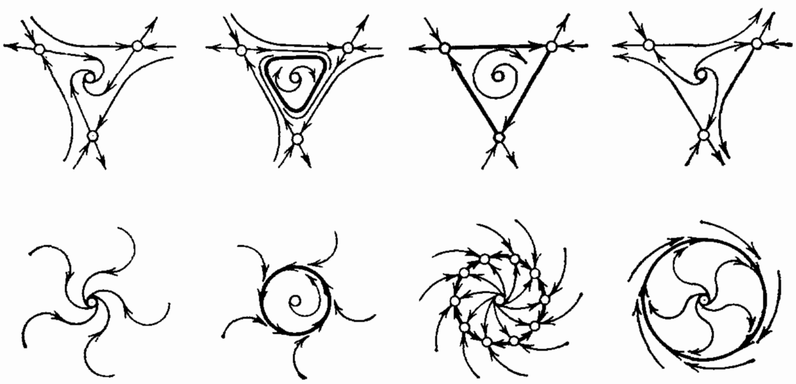

FIG. 25. Typical sequences of phase-oscillation metamorphoses as periodic motions lose their stability near third- and fifth-order resonances.

motion.

It turns out that the chief contribution comes from resonances of the first four orders—the *strong* resonances. Rational numbers with denominator $\geqslant 5$, corresponding to *weak* resonances, may in this context be considered "almost irrational" numbers, whereas the strong resonances will induce qualitatively different "phase oscillations." Figure 25 illustrates the difference between the phase oscillations for the 1:3 and the 1:5 resonances. If self-oscillation stability is lost near a 1:2 resonance, the cycle will double, and usually (although not necessarily) a sequence of universal Feigenbaum doublings[25,26] will be generated.

The path from laminar to turbulent flow may lead either through a chain of soft stability-loss bifurcations [such as the Landau twinnings or bifurcations[27] when n-dimensional tori grow out of $(n-1)$-dimensional ones] or through a hard chain, with the attractor developing far from the laminar flow. And from a physical point of view, the laminar flow need not even lose its stability: it would be enough for its attraction zone to shrink as the Reynolds number increases.

Comprehensive bibliographies of singularity, bifurcation, and catastrophe research will be found in standard texts.[11,16,28–31]

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